Ultraproduct Weil II for curves and $\mathbb{Z}_\ell$-compagnons

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Disclaimer : This talk has nothing to do with model theory...

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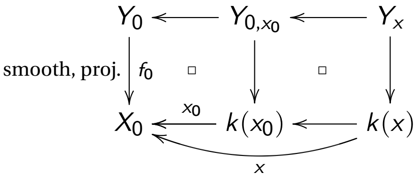

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$$x : k(x) \to X_0$$

- **Grothendieck**'s standard conjectures : Pure isomotives (semisimple Tannakian category) $/\overline{\mathbb{Q}}$

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$$G_\ell(Y_{0,x_0}) := \overline{\mathrm{im}(\pi_1(x_0) \rightharpoonup H(Y_x, \overline{\mathbb{Q}}_\ell))^{Zar}}$$

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Metaconjecture : All !

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- **Weil**'s conjectures (Deligne, Weil I - 1974) : $f_0 : Y_0 \to X_0 = \mathrm{spec}(k_0)$ smooth proper. Then the eigenvalues α of φ acting on $H^i(Y, \mathbb{Q}_\ell)$ are algebraic and pure of weight i : for every $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ $|\iota\alpha| = |k_0|^{\frac{i}{2}}$
  In part., $\det(Id - T\varphi | H^i(Y, \mathbb{Q}_\ell)) \in \mathbb{Q}[T]$, independent of ℓ

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Conj. (Compagnons)

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Rem : $\mathscr{F}_{\ell'}$ is then automatically irreducible with finite determinant

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- For higer dimensional X_0, reduce to the case of curves by geometric methods (no motives...)

Integral motives ? ?

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Thm. B (C., 2018)

For $\ell \gg 0$ and every choice of a torsion-free $\overline{\mathbb{Z}}_\ell$-model $\mathscr{H}_\ell$ of $\mathscr{F}_\ell$,

1. For $? = \emptyset, c$, $i \geqslant 0$ and if X_0 is proper over k_0 or if X_0 is a curve or, For $? = \emptyset$ and $i = 0$ or $? = c$ and $i = 2dim(X_0)$
   - $dim(H^i_?(X, \mathscr{F}_\ell)) = dim(H^i_?(X, \mathscr{H}_\ell \otimes \overline{\mathbb{F}}_\ell))$
   - $H^j_?(X, \mathscr{H}_\ell)[\ell] = 0$, $j = i, i+1$
   - $H^i_?(X, \mathscr{H}_\ell) \otimes \overline{\mathbb{F}}_\ell = H^i_?(X, \mathscr{H}_\ell \otimes \overline{\mathbb{F}}_\ell)$
2. $\mathscr{H}_\ell \otimes \overline{\mathbb{F}}_\ell|_X$ is semisimple and if $\mathscr{F}_\ell|_X$ is irreducible (resp. $\mathscr{F}_\ell$ is semisimple, resp. $\mathscr{F}_\ell$ is irreducible) then $\mathscr{H}_\ell \otimes \overline{\mathbb{F}}_\ell|_X$ is irreducible (resp. $\mathscr{H}_\ell \otimes \overline{\mathbb{F}}_\ell$ is semisimple resp. $\mathscr{H}_\ell \otimes \overline{\mathbb{F}}_\ell$ is irreducible).
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4. (Resp. If $\mathscr{F}_\ell$ is semisimple) the Zariski-closure of the image of $\pi_1(X)$ (resp. of $\pi_1(X_0)$) acting on the stalks of $\mathscr{H}_\ell$ is a semisimple (resp. a reductive) group scheme over $\overline{\mathbb{Z}}_\ell$.

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Key technical ingredient

Introduction of an *ad hoc* category of ultra product coefficients (almost $\mathfrak{u}$-tame local systems) and develop a (partial) theory of Frobenius weights in this setting

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$\mathscr{U}$: set of *non-principal* ultrafilters on $\mathscr{L}$

Ultraproducts

Fact

- $\bigcap_{\mathfrak{u}\in\mathscr{U}} \mathfrak{u} = \{S \subset \mathscr{L} \mid |\mathscr{L}\backslash S| < +\infty\}$ Fréchet filter

$$0 \to \oplus_{\ell\in\mathscr{L}}\overline{\mathbb{F}}_\ell \to \underline{\mathbb{F}} \to \prod_{\mathfrak{u}\in\mathscr{U}} \overline{\mathbb{Q}}_{\mathfrak{u}}$$

For $\mathfrak{u} \in \mathscr{U}$

- $\overline{\mathbb{Q}}_{\mathfrak{u}} \simeq \mathbb{C}$
- $\underline{\mathbb{F}} \twoheadrightarrow \overline{\mathbb{Q}}_{\mathfrak{u}}$ flat

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$$\implies H^1_{\mathfrak{u}}(X, \underline{\mathcal{M}}' \otimes \underline{\mathcal{M}}''^{\vee})^{\varphi} = 0$$

As this holds *for every* $\mathfrak{u} \in \mathscr{U}$, $H^1(X, \mathcal{M}'_\ell \otimes \mathcal{M}''^{\vee}_\ell)^{\varphi=1} = 0$, $\ell \gg 0$

Almost $\mathfrak{u}$-tame local systems

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One has to force these properties by imposing **tameness condition**

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  $\mathscr{M}_\ell$ is tame if

  - (C) For every $C \to X$ with C a smooth, separated, connected curve over k, $\mathscr{M}_\ell|_C$ is tame
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  $\underline{\mathscr{M}}$ is $\mathfrak{u}$-tame if the set of all $\ell \in \mathscr{L}$ such that $\mathscr{M}_\ell$ is tame is in $\mathfrak{u}$

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1. (de Jong, Orgogozo) $\dim(H^i_{?,\mathfrak{u}}(X, \underline{\mathscr{M}})) < +\infty$, $? = \emptyset, c$, $i \geqslant 0$
   $\Rightarrow$ cohomological interpretation of L-function
2. $\exists$ a connected étale cover $X_0' \to X_0$ such that the action of $\pi_1(X_0)$ on $\underline{\mathscr{M}}_x$ factors through

$$\pi_1(X_0) \twoheadrightarrow \Pi := \pi_1(X_0)/\ker(\pi_1(X_0') \twoheadrightarrow \pi_1^t(X_0'))$$

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- Local monodromy : if X_0 is a curve with smooth compactification $X_0 \hookrightarrow \overline{X}_0$, the monodromy at $x_0 \in \overline{X}_0 \backslash X_0$ acts quasi-unipotently on $\underline{\mathscr{M}}_{x,\mathfrak{u}}$

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In particular

$$\prod_{i\geqslant 0} \det(Id - T\varphi | H^i_{c,\mathfrak{u}}(X, \underline{\mathscr{M}}))^{(-1)^{i+1}} = \prod_{i \geqslant 0} \det(Id - T\varphi | H^i_c(X, \mathscr{F}_\ell))^{(-1)^{i+1}}$$

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Examples :

- (Deligne, Weil I) $\mathscr{F}_\ell = R^i f_{0,*}\mathbb{Q}_\ell$, $\ell \nmid p$ for $f_0 : Y_0 \to X_0$ smooth proper
- (L. Lafforgue, Deligne, Drinfeld) $\mathscr{F}_\ell$ irreducible with finite determinant $\rightsquigarrow$ automatically algebraic, pure of weight 0 and lies in a unique compatible family of semisimple $\overline{\mathbb{Q}}_\ell$-local systems

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Assume $\underline{\mathscr{M}}$ ι-pure of weight w : for every $x_0 \in |X_0|$ and every eigenvalue α of φ_{x_0} acting on $\underline{\mathscr{M}}_{x,\mathfrak{u}}$, $|\iota(\alpha)| = |k(x_0)|^{w/2}$

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Thm. A (Weil II ultraproduct for curves - C., 2018)

For $i \geqslant 0$ $\quad H^i_{c,\mathfrak{u}}(X, \underline{\mathscr{M}})$ is ι-mixed of weights $\leqslant w+i$

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- For most applications, one can reduce to the case of curves *via* geometric arguments : Lefschetz pencils, elementary fibrations, Bertini theorem *etc.*

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(Almost tame Bertini theorem, Drinfeld, Tamagawa 2018)

$X_0' \to X_0$ connected étale cover, $K(X_0') := \ker(\pi_1(X_0') \to \pi_1^t(X_0'))$. There exists a smooth, separated, geo. connected curve C_0 over k_0 and a morphism $C_0 \to X_0$ such that $\pi_1(C_0) \to \pi_1(X_0)/K(X_0')$ is surjective and factors through $\pi_1(C_0) \twoheadrightarrow \pi_1^t(C_0)$. Furthermore, given any finite set $S \subset |X_0|$, one may assume $C_0 \to X_0$ admits a section $S \to C_0$

Corollaries 'à la Weil II'

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1. (Purity) If X_0 is proper and $\underline{\mathscr{M}}$ is ι-pure of weight w, $H^i_{\mathfrak{u}}(X, \underline{\mathscr{M}})$ is ι-pure of weights $w+i$, $i \geqslant 0$.
2. (Geometric semisimplicity) If $\underline{\mathscr{M}}$ is ι-pure, $\pi_1(X,x)$ acts semisimply on $\underline{\mathscr{M}}_{x,\mathfrak{u}}$ (equivalently, the set of primes $\ell \in \mathscr{L}$ such that $\mathscr{M}_\ell|_X$ is semisimple is in $\mathfrak{u}$).
3. (Weak Cebotarev) Let $\underline{\mathscr{M}}'$ such that
$$\det(Id - T\varphi_{x_0}|\underline{\mathscr{M}}_{x,\mathfrak{u}}) = \det(Id - T\varphi_{x_0}|\underline{\mathscr{M}}'_{x,\mathfrak{u}}),\ x_0 \in |X_0|$$
Then $\underline{\mathscr{M}}^{ss}_{x,\mathfrak{u}} \simeq \underline{\mathscr{M}}'^{ss}_{x,\mathfrak{u}}$ as $\pi_1(X_0)$-modules (equivalently, the set of primes $\ell \in \mathscr{L}$ such that $\mathscr{M}_\ell$ and $\mathscr{M}'_\ell$ have isomorphic semisimplifications is in $\mathfrak{u}$).

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$? \sim ??$ if $L_{x_0}(?) = L_{x_0}(??)$, $x_0 \in U_{?,0} \cap U_{??,0}$

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Conj. (Langlands correspondance (L, r, $\dagger$))

There exists maps

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such that $V_{\dagger,-} \circ \pi_{\dagger,-} = id$, $\pi_{\dagger,-} \circ V_{\dagger,-} = Id$ and

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- For $\dagger = \ell$ L. Lafforgue 2002 (Drinfeld, Deligne, Laumon *etc.*) + Ramanujan-Peterson conjecture every $\mathscr{F}_\ell \in \mathscr{I}_{r,\ell}(\eta_0)$ is pure of weight 0 with field of coefficients a number field.
- For $\dagger = \mathfrak{u}$ C., 2018

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- Strong multiplicity one theorem of Piatetski-Shapiro

Applications

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- Compagnons
- Mixity
- Finiteness (with ramification constraints)
- Lifting (asymptotic de Jong's conjecture)
- (Strong) Tannakian Cebotarev

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(Finiteness and asymptotic de Jong conjecture)

For $\ell \gg 0$, the reduction modulo-ℓ map $\mathscr{I}_{r,\ell}(\leqslant \alpha, \chi) \to \overline{\mathscr{I}}_{r,\ell}(\leqslant \alpha, \chi)$ is bijective. In particular, $\overline{\mathscr{I}}_{r,\ell}(\leqslant \alpha, \chi)$ is finite and every $\mathscr{M}_\ell \in \overline{\mathscr{I}}_{r,\ell}(\leqslant \alpha, \chi)$ lifts uniquely to a $\overline{\mathbb{Z}}_\ell$-model of some $\mathscr{F}_\ell \in \mathscr{I}_{r,\ell}(\leqslant \alpha, \chi)$.

Thank you !