ON THE GEOMETRIC IMAGE OF $\mathbb{F}_\ell$-LINEAR REPRESENTATIONS OF ETALE FUNDAMENTAL GROUPS

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Abstract. Let X be a connected scheme, smooth and separated over an algebraically closed field k and let $\rho_\ell : \pi_1(X) \to \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$ be a family (indexed by an infinite set L of primes) of continuous $\mathbb{F}_\ell$-linear representations of the étale fundamental group of X of bounded degree $r_\ell \leq r$. The most important example of such families are those arising from the étale cohomology with $\mathbb{F}_\ell$-coefficients of the geometric generic fiber of a smooth proper scheme over X. The main result of this paper asserts that, under a mild finiteness assumption, the image G_ℓ of $\rho_\ell : \pi_1(X) \to \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$ is 'almost algebraic' for $\ell \gg 0$. This is the analogue for $\mathbb{F}_\ell$-coefficients of Grothendieck's unipotency theorem - a crucial step in the proof of Deligne's semisimplicity theorem in Weil II. Just as for $\mathbb{Q}_\ell$-coefficients, our result is a crucial step to establish the analogue of Deligne's semisimplicity theorem for $\mathbb{F}_\ell$-coefficients (proved by Chun Yin Hui and the authors in a subsequent paper). Our result also has a wide range of other applications - in particular to the variation of invariants in 1-dimensional families of varieties and the existence of closed Galois-generic points for motivic representations. We give a first simple example of such applications in the final section of this paper.

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1. Introduction

The étale fundamental group of a scheme ([SGA1]) is one of the most elaborated tool in arithmetic geometry. It enables one to translate intricate geometric problems into representation-theoretic ones, easier to handle. As a result, representations of the étale fundamental group - especially the ones arising from étale cohomology - are ubiquitous. For instance, those with $\mathbb{Q}_\ell$-coefficients play a crucial part in the proof of the Weil conjectures ([D74], [D80]) and are at the heart of the Langlands program for function fields ([L02], [L16]) or of the Grothendieck-Serre-Tate conjectures ([T65]). They also are the natural tool to study the variation of motivic invariants in families ([CT12], [CT13], [C13], [C17b]).

One remarkable consequence of Deligne's weight theory is the semisimplicity of the Zariski-closure of the image of the geometric monodromy acting on ℓ-adic cohomology [D80, Cor. 3.4.13] - a significant step towards the Grothendieck-Serre semisimplicity conjecture. More precisely, Deligne's semisimplicity theorem is the combination of two independent results:

(1) the radical of the image is unipotent (namely, Grothendieck's unipotency theorem - [D80, Thm. 1.3.8, Cor. 1.3.9]), which is deduced from class field theory ([D80, Thm. 1.3.1, Prop. 1.3.4]) by a purely group-theoretic argument;
(2) the representation is semisimple (equivalently, the radical of the image is a torus) ([D80, Cor. 3.4.13]), which is deduced from Deligne's weight theory.

Representations with $\mathbb{F}_\ell$-coefficients are less understood. They contain additional information but this information is more difficult to capture. One reason is the lack of a 'well-behaved Zariski-closure' for finite subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$. However, for finite subgroups $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ which are *generated by their ℓ-Sylow subgroups*, Nori constructed an *algebraic envelope*, which is a right substitute for the 'well-behaved Zariski-closure' [N87]. More recently, Larsen and Pink developed a different approach leading to a variant of the notion of algebraic envelope, also suitable for finite subgroups $G \subset \mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$ [LP11].

The main result of this paper - Theorem 1.1 below - is the analogue for $\mathbb{F}_\ell$-coefficients of Grothendieck's unipotency theorem (1) in the proof of Deligne's semisimplicity theorem. The proof of the semisimplicity theorem for $\mathbb{F}_\ell$-coefficients is completed in subsequent papers: in [CT17], we establish that (1') the images are 'almost' perfect and, in [CHT17], we prove that (2) the representations are semisimple. Contrary to the proof for $\mathbb{Q}_\ell$-coefficients, where (2) is independent from (1), for $\mathbb{F}_\ell$-coefficients, the order in which we prove (1), (1') and (2) is crucial. Namely, (2) uses both (1') and (1), and (1') uses (1). The reason why, for $\mathbb{F}_\ell$-coefficients, (1) has to come first is that it establishes both the *algebraicity* of the image of the geometric monodromy - which is 'automatic' for $\mathbb{Q}_\ell$-coefficients - and the fact that it is generated by its unipotent subgroups.

A first kind of applications of (1) and (1') for $\mathbb{F}_\ell$-coefficients is to the variation of invariants in 1-dimensional families of algebraic varieties, where the key-point is to obtain lower bounds for the genus ([CT17]) and gonality ([CT16]) of abstract modular curves with level-ℓ structures. These are the analogues for $\mathbb{F}_\ell$-coefficients of the results of [CT12], [CT13]. Another kind of application of Theorem 1.1 is the existence of closed Galois-generic points ([C15], [CK16], [C17a]). We give a first simple example of such applications in Section 4.

Let us now state our main result and describe more precisely the content of this paper.

Let k be a field of characteristic $p \geq 0$ and let X be a scheme geometrically connected, smooth and separated over k. Let L be an infinite set of primes and let

$$\pi_1(X) \xrightarrow{\rho_\ell} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell),\ \ell \in L$$

be a family of continuous $\mathbb{F}_\ell$-linear representations of the étale fundamental group of X with $r_\ell \leq r$ bounded as ℓ varies. Typical examples of such families are those arising from the étale cohomology with coefficients in $\mathbb{F}_\ell$ of the geometric generic fiber of a smooth proper morphism $Y \to X$.

Assume that k is algebraically closed. Given a subgroup $G \subset \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, write $G[\ell] \subset G$ for the set of all $g \in G$ such that $g^\ell = 1$ (note that, for $\ell \geq r$, there is no element of order ℓ^2 in $\mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$) and $G^+ \subset G$ for the (normal) subgroup generated by $G[\ell]$. Then the following main result of this paper asserts that under mild finiteness assumption - Condition (F) in Subsection 3.1 - the images of the $\pi_1(X) \xrightarrow{\rho_\ell} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell),\ \ell \in L$ are generated by their ℓ-Sylow and 'almost algebraic' for $\ell \gg 0$.

Theorem 1.1. *Assume that Condition (F) holds. Then there exists an open subgroup $\Pi \subset \pi_1(X)$ such that*

$$\rho_\ell(\Pi) = \rho_\ell(\Pi)^+$$

for $\ell \gg 0$ (depending on Π). Furthermore, for any such Π and every open subgroup $\Pi' \subset \Pi$, one also has $\rho_\ell(\Pi') = \rho_\ell(\Pi')^+ (= \rho_\ell(\Pi))$ for $\ell \gg 0$ (depending on Π').

The terminology 'almost algebraic' comes from the following. Write $\mathcal{G} \hookrightarrow \mathrm{GL}_{r_\ell/\mathbb{F}_\ell}$ for the *algebraic envelope* of G that is the algebraic subgroup generated by the 1-parameter subgroups

$$\begin{array}{llll} e_g\colon & \mathbb{A}^1_{\mathbb{F}_\ell} & \to & \mathrm{GL}_{r_\ell/\mathbb{F}_\ell} \\ & t & \mapsto & \exp(t\log(g)) \end{array},\ g \in G[\ell].$$

Then, we have

Theorem 1.2. ([N87, Thm. B and Rem. 3.6]) *For $\ell \gg 0$ (depending only on r) and for every subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$, one has*

$$G \supset G^+ = \mathcal{G}(\mathbb{F}_\ell)^+ \subset \mathcal{G}(\mathbb{F}_\ell)$$

and $\mathcal{G}(\mathbb{F}_\ell)/\mathcal{G}(\mathbb{F}_\ell)^+$ is an abelian group of order $\leq 2^{r-1}$.

The proof of Theorem 1.1 reduces easily to the case where the $\pi_1(X) \xrightarrow{\rho_\ell} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell),\ \ell \in L$ are semisimple and then, decomposes into two steps - Claim 1 and Claim 2 in Subsection 3.2. Condition (F) may seem technical at first glance but just says that we may always assume that our base scheme is a curve and that the representations are almost tame (which, again, is automatic in the setting of $\mathbb{Q}_\ell$-coefficients). It is there to make the arguments in the proof of Claim 2 work. Claim 2 has to be thought of as the analogue of class field theory in Grothendieck's unipotency theorem (note that class field theory is only available for étale fundamental group of *curves*!) while Condition (F) is the substitute of the specialization step in Deligne's argument (See [D80, (1.11.4) and *Variante* after Prop. 1.3.4]). The fact that it is satisfied by representations arising from étale cohomology is non-trivial and is established in Proposition 3.2. Eventually, Claim 1 is the analogue of the group-theoretic argument in Grothendieck's unipotency theorem and just as it, is purely group-theoretic. More precisely, it is based on a refinement (Theorem 2.2) of a theorem of Nori ([N87, Thm. C]) about the structure of subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$. We should mention that the full strength of Theorem 2.2 is not used in the proof of Claim 1. Indeed, for semisimple representations, Theorem 2.2 is probably well-known to specialists and, as pointed out by a referee, could be reconstructed from ingredients in the existing litterature (such as those in [KLS08]; see also Remark 2.10). However, to keep the exposition elementary and self-contained and also because we feel Theorem 2.2 is interesting in itself and might have further applications, we give a full detailed proof, based only on Nori theory.

The paper is organized as follows. Section 2 is devoted to the statements and proofs of the group-theoretical preliminaries required for Claim 1. Section 3 is devoted to the structure of the geometric images of the $\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$. In Subsection 3.1, we formulate Condition (F) and show that it is satisfied by families of $\mathbb{F}_\ell$-linear representations arising from the étale cohomology with coefficients in $\mathbb{F}_\ell$ of the geometric generic fiber of a smooth proper morphism $Y \to X$. The proof of Theorem 1.1 (including the proofs of Claim 1 and Claim 2) is carried out in Subsection 3.2. We also state there two corollaries (Corollary 3.3 and Corollary 3.5) which refine Theorem 1.1 when the $\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$ are semisimple. Corollary 3.3 enables one to extend, for instance, the main result of [EHK12] (for families of representations arising from the ℓ-torsion of abelian schemes) to families of representations arising from étale cohomology in characteristic 0. In Section 4, we give another application of Theorem 1.1, to the problem of almost ℓ-independence (in the sense of Serre - [S13]) for families of ℓ-adic representations. This is strengthened and developed in subsequent works of the first author ([C15], [CK16], [C17a]).

2. Group-theoretical preliminaries

2.1. **Statements.** An old theorem of C. Jordan [J1878] asserts that there exists an integer $\delta(r) \geq 1$ such that every finite subgroup of $\mathrm{GL}_r(\mathbb{C})$ has a normal abelian subgroup of index $\leq \delta(r)$. M. Nori [N87] extended this result to subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ as follows.

Theorem 2.1. ([N87, Thm. C]) *For every integer $r \geq 1$ there exists an integer $\delta(r) \geq 1$ such that for every prime ℓ and for every subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ there exists a subgroup $T \subset G$ such that*

- *T is abelian of prime-to-ℓ order;*
- *G^+T is normal in G ;*
- *$[G : G^+T] \leq \delta(r)$.*

In particular, there exist normal subgroups $G_2 \subset G_1 \subset G$ satisfying the following properties

- $[G : G_1] \leq \delta(r)$;
- G_1/G_2 is abelian of prime-to-ℓ order;
- $G_2 = G_2^+ (= G^+)$.

We improve Theorem 2.1 by showing that T can be chosen in such a way that it centralizes G^+ provided G^+ acts semisimply on $\mathbb{F}_\ell^{\oplus r} =: H_\ell$. This will be a consequence of the following slightly more general result.

Theorem 2.2. *Fix an integer $d \geq 1$. Then, for every integer $r \geq 1$ there exists an integer $\delta(d,r) \geq 1$ such that for every prime ℓ and for every subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ and subgroups $\Delta, T \subset G$ satisfying the following properties*

1. *$\Delta = \Delta^+$ and it acts semisimply on H_ℓ;*
2. *Δ is normal in G;*
3. *T is abelian (resp. abelian of prime-to-ℓ order);*
4. *ΔT is normal in G;*
5. *$[G : \Delta T] \leq d$,*

there exists a subgroup $\Theta \subset G$ satisfying the following properties

- *Θ is abelian (resp. abelian of prime-to-ℓ order);*
- *Θ centralizes Δ;*
- *Θ (hence $\Delta\Theta$) is normal in G;*
- *$[G : \Delta\Theta] \leq \delta(d,r)$.*

In particular, combining Theorem 2.2 and Theorem 2.1 we obtain the following general structure result for subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$.

Corollary 2.3. *For every integer $r \geq 1$ there exists an (explicit) integer $\delta(r) \geq 1$ such that for every prime ℓ and for every subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ such that G^+ acts semisimply on H_ℓ, there exists a subgroup $T \subset G$ satisfying the following properties*

- *T is abelian of prime-to-ℓ order;*
- *T centralizes G^+;*
- *T (hence G^+T) is normal in G;*
- *$[G : G^+T] \leq \delta(r)$.*

From which, in turn, one recovers a slight variant of the filtration of [LP11, Th. 0.2] for $k = \mathbb{F}_\ell$.

Corollary 2.4. *For every integer $r \geq 1$ there exists an (explicit) integer $\delta(r) \geq 1$ such that for every prime ℓ and for every subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ there exist normal subgroups $G_3 \subset G_2 \subset G_1 \subset G$ satisfying the following properties*

- *$[G : G_1] \leq \delta(r)$;*
- *$G_1/G_2 \twoheadleftarrow \mathcal{G}(\mathbb{F}_\ell)^+ (= [\mathcal{G}(\mathbb{F}_\ell), \mathcal{G}(\mathbb{F}_\ell)])$ for some semisimple algebraic subgroup $\mathcal{G} \hookrightarrow \mathrm{GL}_{r/\mathbb{F}_\ell}$;*
- *G_2/G_3 is abelian of prime-to-ℓ order;*
- *G_3 is an ℓ-group.*

Proof. First, let us observe that it is enough to prove Corollary 2.4 for $\ell \gg 0$ (depending only on r). Indeed, assume there exists a prime $\ell(r)$ such that Corollary 2.4 holds for every prime $\ell \geq \ell(r)$ with a given constant $\delta^0(r)$. Then, up to replacing $\delta^0(r)$ with $\delta(r) := \delta^0(r)|\mathrm{GL}_r(\mathbb{F}_{\ell(r)})|$, the statement of Corollary 2.4 still holds for primes $\ell < \ell(r)$ (with $G_3 = G_2 = G_1 = 1$). So, in the proof of Corollary 2.4, we may consider only primes $\ell \gg 0$ (depending only on r).

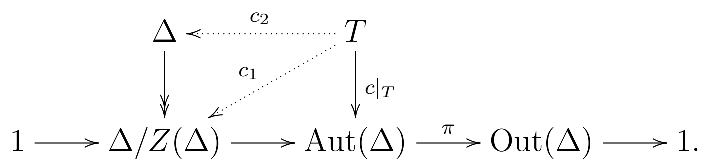
Given a subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$, consider the semisimplification H_ℓ^{ss} of H_ℓ as a G-module and let $G \twoheadrightarrow G^{ss}$ denote the image of G acting on H_ℓ^{ss}. Let G_3 denote the kernel of $G \twoheadrightarrow G^{ss}$. Let $T \subset G^{ss}$ denote any subgroup satisfying the conclusions of Corollary 2.3 (applied to $G^{ss} \subset \mathrm{GL}_r(\mathbb{F}_\ell)$) and let G_2 and G_1 be the inverse images of T and $G^{ss,+}T$ by $G \twoheadrightarrow G^{ss}$ respectively. By construction, $G_3 \subset G_2 \subset G_1 \subset G$ are normal subgroups. Also, G_3 is a unipotent group hence an ℓ-group; $T \simeq G_2/G_1$ is abelian of prime-to-ℓ order; $[G : G_1] = [G^{ss} : G^{ss,+}T] \leq \delta(r)$; $G^{ss,+}$ surjects onto G_1/G_2 and, by Theorem 1.2 and Lemma 2.6 below, $G^{ss,+} = \mathcal{G}(\mathbb{F}_\ell)^+$ for some semisimple algebraic subgroup $\mathcal{G} \hookrightarrow \mathrm{GL}_{r/\mathbb{F}_\ell}$ and for $\ell \gg 0$ (depending on r). Also, from Theorem 1.2 again, $[\mathcal{G}(\mathbb{F}_\ell), \mathcal{G}(\mathbb{F}_\ell)] \subset G^{ss,+}$. But from Lemma 3.4 below, $G^{ss,+} = [G^{ss,+}, G^{ss,+}] (\subset [\mathcal{G}(\mathbb{F}_\ell), \mathcal{G}(\mathbb{F}_\ell)])$ for $\ell \gg 0$ (depending on r). □

2.2. **Proof of Theorem 2.2.** Before turning to the proof of Theorem 2.2 itself, let us observe that it is enough to prove it for $\ell \gg 0$ (depending only on r). Indeed, assume there exists a prime

$\ell(r)$ such that Theorem 2.2 holds for every prime $\ell \geq \ell(r)$ with a given constant $\delta^0(d,r)$. Then, up to replacing $\delta^0(d,r)$ with $\delta(d,r) := \max\{\delta^0(d,r), |\mathrm{GL}_r(\mathbb{F}_{\ell(r)})|\}$, the statement of Theorem 2.2 still holds for primes $\ell < \ell(r)$ (with $\Theta = 1$). So, in the proof of Theorem 2.2, we may consider only primes $\ell \gg 0$ (depending only on r).

Also, it is enough to prove Theorem 2.2 with the seemingly weaker conclusion that $\Delta\Theta$ is normal in G (instead of Θ is normal in G). Indeed, then the center of $\Delta\Theta$ (resp. the prime-to-ℓ part of the center of $\Delta\Theta$) will satisfy the conclusions of Theorem 2.2.

2.2.1. *Strategy.* The basic idea is to show that T can be 'adjusted' in such a way that it centralizes Δ. Roughly, let $c : G \to \mathrm{Aut}(\Delta)$ denote the morphism induced by conjugation and consider the following diagram, whose lower sequence is exact.



Here, $Z(\Delta)$, $\mathrm{Aut}(\Delta)$ and $\mathrm{Out}(\Delta)$ denote the center, group of automorphisms and group of outer automorphisms of Δ respectively. Assume that up to replacing T by a subgroup of index bounded only in terms of r and still satisfying properties (4), (5) (with d replaced) of Theorem 2.2 and c by the nth power of c for some $n > 0$ bounded only in terms of r, one can lift successively c to c_1 and c_2. Then it is not difficult to see that

$$\Theta := \{t^n c_2(t)^{-1} \mid t \in T\} \subset G$$

will satisfy the conclusion of Theorem 2.2.

Lifting c_1 to c_2 is a purely group-theoretical problem, which can be solved elementarily (see Paragraph 2.2.2.2). Lifting c to c_1 is more delicate. (In this process, we may have to replace $Z(\Delta)$ by a possibly smaller subgroup $Z \subset Z(\Delta)$, but this will not affect the final purely group-theoretical lifting step.) It would be straightforward if $|\mathrm{Out}(\Delta)|$ were bounded above by a constant depending only on r. Such a constant does not exist *a priori* for abstract finite groups Δ. But it does for semisimple algebraic subgroups of $\mathrm{GL}_{r/\mathbb{F}_\ell}$. More precisely, one has (see *e.g.* [Co11, Prop. 7.1.6, Rem. 7.1.7 and Thm. 7.1.9]):

Theorem 2.5. *Let $\mathcal{G}$ be a semisimple algebraic group over a field k. Then the functor of automorphisms $\underline{\mathrm{Aut}}_k(\mathcal{G})$ is represented by a smooth algebraic group $\mathrm{Aut}(\mathcal{G})$ over k, which fits into a short exact sequence of algebraic groups over k*

$$1 \to \mathcal{G}/Z(\mathcal{G}) \to \mathrm{Aut}(\mathcal{G}) \to \mathrm{Out}(\mathcal{G}) \to 1.$$

Furthermore $\mathrm{Out}(\mathcal{G})$ is finite, étale over k of degree $\leq 6^{\mathrm{rank}(\mathcal{G})}$.

The idea is thus to replace Δ by its algebraic envelope $\mathcal{D} \hookrightarrow \mathrm{GL}_{r/\mathbb{F}_\ell}$, which is a semisimple algebraic group as soon as Δ acts semisimply on H_ℓ.

Lemma 2.6. *Let $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ be a subgroup. Assume that H_ℓ is a semisimple G^+-module. Then the algebraic envelope $\mathcal{G} \hookrightarrow \mathrm{GL}_{r/\mathbb{F}_\ell}$ of G is a connected semisimple algebraic group.*

Proof. From [Bor69, Prop. 2.2 and its proof], one already knows that $\mathcal{G}$ connected.

- Reductivity: By definition, $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$ is the algebraic group generated by the set

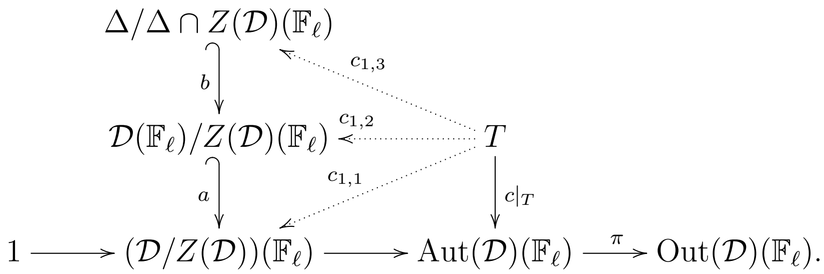
$$Y := \left\{\exp(t\log(g)) \mid t \in \overline{\mathbb{F}}_\ell,\ g \in G[\ell]\right\}$$

in $\mathrm{GL}_{r/\overline{\mathbb{F}}_\ell}$. Observe first that $H_{\ell\overline{\mathbb{F}}_\ell} := H_\ell \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$ is a semisimple G^+-module; this follows from the facts that $\mathbb{F}_\ell$ is a perfect field and that H_ℓ is a semisimple G^+-module by assumption. Now any $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$-stable subspace $V \subset H_{\ell\overline{\mathbb{F}}_\ell}$ is a G^+-submodule hence there exists a G^+-stable submodule

$V' \subset H_{\ell\overline{\mathbb{F}}_\ell}$ such that $H_{\ell\overline{\mathbb{F}}_\ell} = V \oplus V'$. But V' is then Y-stable by construction hence $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$-stable. This shows that H_ℓ is a faithful semisimple representation of $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$ hence, in particular, that $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$ is reductive.

- Semisimplicity: By construction $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$ is generated by unipotent elements hence has no quotient isomorphic to a torus. This shows that $\mathcal{G}_{/\overline{\mathbb{F}}_\ell}$ is semisimple. □

2.2.2. *Proof of Theorem 2.2.* By definition of the algebraic envelope $\mathcal{D}$, the action of G by conjugation on Δ induces a morphism $c : G \to \mathrm{Aut}(\mathcal{D})(\mathbb{F}_\ell)$. Consider the following diagram, whose lower sequence is exact.



We are going to prove that up to replacing T by a subgroup of index bounded only in terms of r and still satisfying properties (4), (5) (with d replaced) of Theorem 2.2 and c by the nth power of c for some $n > 0$ bounded in terms of r, we can lift successively c to $c_{1,1}$, $c_{1,2}$ and $c_{1,3}(= c_1)$ (and it is the group $Z := \Delta \cap Z(\mathcal{D})(\mathbb{F}_\ell) \subset Z(\Delta)$ that we will consider in the remaining purely group-theoretical lifting step). Actually, we are going to consider only characteristic subgroups

$$T^n := \{t^n \mid t \in T\} \subset T,\ n > 0.$$

Lemma 2.7. *Let $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ be a subgroup and $\Delta, T \subset G$ be two subgroups such that $\Delta = \Delta^+$ is normal in G and*

(1) *T is abelian (resp. abelian of prime-to-ℓ order);*
(2) *ΔT is normal in G.*

Then, for any integer $n < \ell$ the subgroup $T^n \subset T$ still satisfies properties (1), (2) and

$$[G : \Delta T^n] \leq [G : \Delta T] n^r.$$

Proof. First, let us observe that for $\ell > n$ one has $[T : T^n] \leq n^r$. Indeed, write $T = T^{(\ell)} \times T^{(\ell')}$ as the direct product of its ℓ-part and prime-to-ℓ part. As $T^{(\ell')}$ acts semisimply, it is contained in a $\mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$-conjugate of the torus $\mathbb{G}^r_{m/\mathbb{F}_\ell}$. For $\ell > n$, T/T^n has prime-to-ℓ order hence $T^{(\ell')}/(T^{(\ell')})^n \tilde{\to} T/T^n$ is a quotient of exponent $\leq n$ of $T^{(\ell')}$, which implies $[T : T^n] \leq n^r$. Then, it remains to show that ΔT^n is normal in G. As ΔT is normal in G, it is actually enough to show that ΔT^n is characteristic in ΔT. But this follows from the fact that for $\ell > n$ one has

$$\Delta T^n = \ker(\Delta T \twoheadrightarrow (\Delta T)^{ab}/n)$$

(use that $\Delta = \Delta^+$). □

2.2.2.1. From c to $c_{1,3} = c_1$.

From c to $c_{1,1}$. Set $N := \ker(\pi \circ c) \lhd G$. From Theorem 2.5, one has $[G : N] \leq 6^{r-1}$ hence *a fortiori* $[T : T \cap N] \leq 6^{r-1}$. Thus, up to replacing T with $T^{6^{r-1}!} \subset N \cap T \subset T$ and the constant d with $(6^{r-1}!)^r d$, one may assume that $c : T \to \mathrm{Aut}(\mathcal{D})(\mathbb{F}_\ell)$ factors through

$$c_{1,1} : T \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell).$$

From $c_{1,1}$ to $c_{1,2}$. We are to bound from above the cokernel of a by a constant depending only on r. From the short exact sequence of flat sheaves

$$1 \to Z(\mathcal{D}) \to \mathcal{D} \to \mathcal{D}/Z(\mathcal{D}) \to 1$$

one obtains the exact sequence

$$1 \to Z(\mathcal{D})(\mathbb{F}_\ell) \to \mathcal{D}(\mathbb{F}_\ell) \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell).$$

Claim: *The image of* $\mathcal{D}(\mathbb{F}_\ell) \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell)$ *has index* $\leq 2^{r-1}$.

Proof of the claim. As $Z(\mathcal{D})$ is a finite commutative group scheme over $\mathbb{F}_\ell$ it decomposes as the direct product

$$Z(\mathcal{D}) \simeq \mathcal{Z}^\circ \times \mathcal{Z}^{et}$$

of its connected and étale parts.

Then, on the one hand, one has a short exact sequence of flat sheaves

$$0 \to \mathcal{Z}^{et} \to \mathcal{D}/\mathcal{Z}^\circ \to \mathcal{D}/Z(\mathcal{D}) \to 0$$

hence an exact sequence

$$0 \to \mathcal{Z}^{et}(\mathbb{F}_\ell) \to (\mathcal{D}/\mathcal{Z}^\circ)(\mathbb{F}_\ell) \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell) \to \mathrm{H}^1_{fl}(\mathbb{F}_\ell, \mathcal{Z}^{et}).$$

Since $\mathcal{Z}^{et}$ is étale over $\mathbb{F}_\ell$, the right-hand term of this exact sequence can be computed in terms of Galois cohomology:

$$\mathrm{H}^1_{fl}(\mathbb{F}_\ell, \mathcal{Z}^{et}) = \mathrm{H}^1(\mathbb{F}_\ell, \mathcal{Z}^{et}(\overline{\mathbb{F}}_\ell)) = \mathcal{Z}^{et}(\overline{\mathbb{F}}_\ell)/(F - Id),$$

where the second equality follows from [S68, Chap. XIII, §1, Prop. 1 and Rem.] (here F denotes the Frobenius of the absolute Galois group of $\mathbb{F}_\ell$). This shows that the order of $\mathrm{H}^1_{fl}(\mathbb{F}_\ell, \mathcal{Z}^{et})$ divides $|\mathcal{Z}^{et}(\overline{\mathbb{F}}_\ell)|$. So, as the center of a rank ρ semisimple algebraic group has order at most 2^ρ (see for instance [N87, p. 270]), the image of $(\mathcal{D}/\mathcal{Z}^\circ)(\mathbb{F}_\ell) \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell)$ has index $\leq 2^{r-1}$.

On the other hand, the morphism $\mathcal{D} \to \mathcal{D}/\mathcal{Z}^\circ$ is radicial hence induces an isomorphism

$$\mathcal{D}(\mathbb{F}_\ell) \tilde{\to} (\mathcal{D}/\mathcal{Z}^\circ)(\mathbb{F}_\ell).$$

This shows that $\mathcal{D}(\mathbb{F}_\ell) \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell)$ and $\mathcal{D}/\mathcal{Z}^\circ(\mathbb{F}_\ell) \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell)$ have the same image. □

So, up to replacing again T with $T^{2^{r-1}!} \subset T$ and the constant $(6^{r-1}!)^r d$ with $(2^{r-1}!)^r(6^{r-1}!)^r d$, one may assume that $c_{1,1} : T \to (\mathcal{D}/Z(\mathcal{D}))(\mathbb{F}_\ell)$ factors through

$$c_{1,2} : T \to \mathcal{D}(\mathbb{F}_\ell)/Z(\mathcal{D})(\mathbb{F}_\ell).$$

From $c_{1,2}$ to $c_{1,3}$. We are to bound from above the cokernel of b by a constant depending only on r. But this follows from Theorem 1.2 since for $\ell \gg 0$ (depending on r) one has $\Delta = \mathcal{D}(\mathbb{F}_\ell)^+$ hence

$$[\mathcal{D}(\mathbb{F}_\ell) : \Delta Z(\mathcal{D}(\mathbb{F}_\ell))] \leq [\mathcal{D}(\mathbb{F}_\ell) : \mathcal{D}(\mathbb{F}_\ell)^+] \leq 2^{r-1}.$$

So, up to replacing again T with $T^{2^{r-1}!} \subset T$ and the constant $(2^{r-1}!)^r(6^{r-1}!)^r d$ with $(2^{r-1}!)^{2r}(6^{r-1}!)^r d$, one may assume that $c_{1,2} : T \to \mathcal{D}(\mathbb{F}_\ell)/Z(\mathcal{D})(\mathbb{F}_\ell)$ factors through

$$c_{1,3} : T \to \Delta/\Delta \cap Z(\mathcal{D}(\mathbb{F}_\ell))$$

Note that $Z := Z(\mathcal{D})(\mathbb{F}_\ell) \cap \Delta \subset Z(\mathcal{D})(\mathbb{F}_\ell)$ has order $\leq 2^{r-1}$.

2.2.2.2. End of the proof. The end of the proof relies on the following two elementary group-theoretical lemmas.

Lemma 2.8. *Let G be a finite group, let $Z \subset Z(G)$ be a central subgroup and let T be an abelian group. Consider an embedding problem of the form*

$$\begin{array}{ccccccccc} & & & & & & T & & \\ & & & & & \swarrow & \downarrow{\scriptstyle \overline{\psi}} & & \\ 1 & \longrightarrow & Z & \longrightarrow & G & \xrightarrow[\pi]{} & G/Z & \longrightarrow & 1. \end{array}$$

Then the induced embedding problem

$$\begin{array}{ccccccccc} & & & & T & \xrightarrow{(-)^{2|Z|}} & T & & \\ & & & & \downarrow & & \downarrow{\scriptstyle \overline{\psi}} & & \\ 1 & \longrightarrow & Z & \longrightarrow & G & \xrightarrow[\pi]{} & G/Z & \longrightarrow & 1 \end{array}$$

has a solution.

Proof. Fix a set-theoretic lift $\tilde{\psi} : T \to G$ of $\overline{\psi} : T \to G/Z$ and write

$$\begin{array}{rccc} \gamma: & T \times T & \to & Z \\ & (t, t') & \to & \tilde{\psi}(tt')\tilde{\psi}(t')^{-1}\tilde{\psi}(t)^{-1} \end{array}$$

for the corresponding 2-cocycle. Then, by definition one has

$$\begin{aligned} \tilde{\psi}(tt') &= \gamma(t, t')\tilde{\psi}(t)\tilde{\psi}(t') \\ &= \tilde{\psi}(t't) = \gamma(t', t)\tilde{\psi}(t')\tilde{\psi}(t) \end{aligned}$$

hence, with $\omega(t, t') := \gamma(t, t')^{-1}\gamma(t', t)$ one has

$$\tilde{\psi}(t)\tilde{\psi}(t') = \omega(t, t')\tilde{\psi}(t')\tilde{\psi}(t).$$

With these notation, one obtains by a straightforward induction that

$$\tilde{\psi}(tt')^n = \gamma(t, t')^n\omega(t', t)^{\frac{n(n-1)}{2}}\tilde{\psi}(t)^n\tilde{\psi}(t')^n.$$

Hence, setting $N := |Z|$, one obtains that $\psi := (-)^{2N} \circ \tilde{\psi} : T \to G$ is a group homomorphism lifting $(-)^{2N} \circ \overline{\psi} = \overline{\psi} \circ (-)^{2N} : T \to G/Z$. □

Lemma 2.9. *Let G be a finite group, $\Delta \subset G$ a subgroup and T an abelian group together with a group homomorphism $\phi : T \to G$. Assume that we have a group homomorphism $\psi : T \to \Delta$ such that $\phi(t)\psi(t)^{-1} \in \mathrm{Cen}_G(\Delta)$ for all $t \in T$. Then the set*

$$\Theta := \{\phi(t)\psi(t)^{-1} \mid t \in T\} \subset G$$

is an abelian subgroup.

Proof. Trivial. □

It follows from Lemma 2.8 applied to the embedding problem

$$\begin{array}{ccccccccc} & & & & & & T & & , \\ & & & & & \swarrow & \downarrow{\scriptstyle c_3} & & \\ 1 & \longrightarrow & Z & \longrightarrow & \Delta & \longrightarrow & \Delta/Z & \longrightarrow & 1 \end{array}$$

that $c_3\circ(-)^{2(2^{r-1}!)}:T\to\Delta/Z$ lifts to a morphism $\psi:T\to\Delta$. By definition of c, for every $t\in T$ the element $t^{2(2^{r-1}!)}\psi(t)^{-1}$ lies in $\mathrm{Cen}_G(\Delta)$ hence one can apply Lemma 2.9 to produce an abelian subgroup

$$\Theta:=\{t^{2(2^{r-1}!)}\psi(t)^{-1}\mid t\in T\}\subset G,$$

which, by construction, lies in $\mathrm{Cen}_G(\Delta)$. Also, as $\Delta\Theta=\Delta T^{2(2^{r-1}!)}$ one has

$$[G:\Delta\Theta]=[G:\Delta T^{2(2^{r-1}!)}]\leq\delta(d,r):=2^r(2^{r-1}!)^{3r}(6^{r-1}!)^r d$$

and $\Delta\Theta$ is normal in G by Lemma 2.7 (for $\ell>2(2^{r-1}!)^3 6^{r-1}!$).

Remark 2.10. (Comparison with [LP11]) In order to remain in the rather elementary setting of [N87], we deduce Corollary 2.3 from Theorem 2.2 and Theorem 2.1. Alternatively, we could also have deduced it from [LP11], which uses much more of the sophisticated machinery of the theory of algebraic groups. More precisely, [LP11, Thm. 02 (and its proof)] implies that there exists an integer $\delta(r)\geq 1$ such that for every prime ℓ and every finite subgroup $G\subset\mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$, there exists an algebraic subgroup $\mathcal{G}\hookrightarrow\mathrm{GL}_{r/\overline{\mathbb{F}}_\ell}$ such that $[\mathcal{G}:\mathcal{G}^\circ]\leq\delta(r)$ and $G\subset\mathcal{G}(\overline{\mathbb{F}}_\ell)$, and there exist normal subgroups $G_3\subset G_2\subset G_1\subset G$ with the following properties

- $G_1=G\cap\mathcal{G}^\circ(\overline{\mathbb{F}}_\ell)$ (in particular $[G:G_1]\leq\delta(r)$);
- G_1/G_2 is a direct product of simple groups of the form $D(\mathcal{S}^F)$, where $\mathcal{G}^\circ\twoheadrightarrow\mathcal{S}$ is a simple quotient and $F:\mathcal{S}\to\mathcal{S}$ a Frobenius map so that the derived subgroup $D(\mathcal{S}^F)$ be simple;
- G_2/G_3 embeds into the center of $(\mathcal{G}^\circ/R_u(\mathcal{G}^\circ))(\overline{\mathbb{F}}_\ell)$;
- $G_3=G\cap R_u(\mathcal{G}^\circ)(\overline{\mathbb{F}}_\ell)$.

Let $G^+\subset G$ denote the (normal) subgroup generated by the ℓ-Sylow subgroups. One has $G_3\subset G^+$ and, for $\ell>\delta(r)$, one also has $G^+\subset G_1$. As $D(\mathcal{S}^F)$ is simple of order divisible by ℓ, one obtains that G^+ surjects onto G_1/G_2. Assume that G^+ acts semisimply on $\overline{\mathbb{F}}_\ell^{\oplus r}$. Then as G_3 is normal in G^+, it also acts semisimply on $\overline{\mathbb{F}}_\ell^{\oplus r}$ hence is trivial. Thus we may assume that $\mathcal{G}^\circ$ is a connected reductive group and, then, the prime-to-ℓ part $T(\supset G_2)$ of $Z(G_1)$ satisfies the conclusion of Corollary 2.3. Note that [LP11] actually implies Corollary 2.3 for finite subgroups of $\mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$ (not only for subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$). On the contrary, it is not clear whether the machinery of [LP11] can be expoited to show that Theorem 2.2 holds (when ℓ divides $|T|$) with H_ℓ replaced by $H_\ell\otimes_{\mathbb{F}_\ell}\overline{\mathbb{F}}_\ell$: the point is that the action of T by conjugation on Δ will not extend, *a priori*, to an action on $\mathcal{G}^\circ$.

3. Application to representations of the étale fundamental group

In this section, we carry out the proof of Theorem 1.1. So, let k be an algebraically closed field of characteristic $p\geq 0$ and let X be a connected scheme, smooth and separated over k. Let L be an infinite set of primes and let

$$\pi_1(X)\overset{\rho_\ell}{\to}\mathrm{GL}_{r_\ell}(\mathbb{F}_\ell),\ \ell\in L$$

be a family of $\mathbb{F}_\ell$-linear representations with $r_\ell\leq r$ bounded as ℓ varies.

3.1. Condition (F).

Consider the following condition.

(F) There exist a finitely generated field K, an algebraically closed field Ω containing K, a curve C geometrically connected, smooth and separated over K and a morphism of profinite groups $f:\pi_1(C_\Omega)\to\pi_1(X)$ with open image such that for any $\ell\in L$, $\rho_\ell\circ f$ factors through the tame fundamental group $\pi_1(C_\Omega)\to\pi_1(C)\twoheadrightarrow\pi_1^t(C)$, that is, there exists a representation $\pi_1^t(C)\to\mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$ (again denoted by ρ_ℓ) such that the following diagram commutes

$$\begin{array}{ccccc} \pi_1(C_\Omega) & \xrightarrow{f} & \pi_1(X) & \xrightarrow{\rho_\ell} & \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell) \, . \\ \downarrow & & & \nearrow & \\ \pi_1(C) & \twoheadrightarrow & \pi_1^t(C) & & \end{array}$$

Note that the morphism $\pi_1(C_\Omega) \to \pi_1^t(C)$ can also be decomposed as $\pi_1(C_\Omega) \twoheadrightarrow \pi_1(C_{\overline{K}}) \twoheadrightarrow \pi_1^t(C_{\overline{K}}) \hookrightarrow \pi_1^t(C)$.

Condition (F) may look technical but it is rather natural. It is there to ensure, on the one hand, that the ρ_ℓ, $\ell \in L$ all factor through the same topologically finitely generated quotient (more precisely, setting $K := \cap \ker(\rho_\ell) \lhd \pi_1(X)$, then $\pi_1(C_\Omega) \to \pi_1(X)/K$ has open image and factors through $\pi_1(C_\Omega) \twoheadrightarrow \pi_1^t(C_\Omega)$, which is a topologically finitely generated profinite group) and, on the other hand, to reduce to the case where the base scheme is a curve (a situation where the étale fundamental group is better understood) over a finitely generated field (which will enable us to use arguments of arithmetic nature).

Example 3.1. Condition (F) is always satisfied by families of representations arising from étale cohomology. More precisely, we have the following. Let k be a field, X a smooth and geometrically connected scheme over k and $Y \stackrel{f}{\to} X$ a smooth, projective morphism. (In fact, f may be more general. For this generalization, see the proof of Corollary 4.6.) By the proper-smooth base change theorem, for any point $x \in X$ one gets a family of $\mathbb{F}_\ell$-linear representations

$$\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{F}_\ell)), \ \ell \gg 0,$$

with $r_\ell = \dim_{\mathbb{F}_\ell}(H^i(Y_{\overline{x}}, \mathbb{F}_\ell))$ independent of ℓ for $\ell \gg 0$. This follows from the fact that

- $H^i(Y_{\overline{x}}, \mathbb{Z}_\ell)$ is finitely generated over $\mathbb{Z}_\ell$, and torsion free for $\ell \gg 0$ [G83], which, in turn, implies that $H^i(Y_{\overline{x}}, \mathbb{F}_\ell) = H^i(Y_{\overline{x}}, \mathbb{Z}_\ell)/\ell$ for $\ell \gg 0$ and;
- the $\mathbb{Q}_\ell$-dimension of $H^i(Y_{\overline{x}}, \mathbb{Q}_\ell)$ is independent of ℓ.

(See [CT17, §2.4.1] for more details). Then,

Proposition 3.2. *The family $\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{F}_\ell))$, $\ell \gg 0$ satisfies condition (F).*

Proof. As all the objects are of finite type over k, we may assume that $f : Y \to X$ admits a model $f_0 : Y_0 \to X_0$ over a finitely generated field k_0. Up to replacing X_0 by a non-empty open subscheme, one may assume that X_0 is affine, hence quasi-projective. Fix an embedding $X_0 \hookrightarrow \mathbb{P}^n_{k_0}$ and let d denote the dimension of X. Write $Gr(d-1, n)$ for the Grassmanian of codimension $d-1$ linear subspaces of $\mathbb{P}^n_{k_0}$ and ξ for its generic point. Let Ω be any algebraically closed field containing the residue field $k(\xi_k)$ of the generic point ξ_k of $Gr(d-1,n)_k$. Let

$$Z := \{(x, V) \in X_0 \times_{k_0} Gr(d-1, n) \mid x \in V\} \hookrightarrow X_0 \times_{k_0} Gr(d-1, n)$$

denote the incidence variety and $p : Z \to Gr(d-1, n)$ the canonical projection. Note that $Z_\xi \hookrightarrow X_{0,k_0(\xi)}$ is, by construction and Bertini's theorem [Jou83, Thm. 6.10, 2), 3)], a smooth, separated, geometrically connected curve over the residue field $k_0(\xi)$ of ξ, which is a finitely generated field.

Claim: *The morphism of étale fundamental groups*

$$\pi_1(Z_{\xi,\Omega}) \to \pi_1(X)$$

induced by $Z_{\xi,\Omega} \stackrel{p_\Omega}{\to} X_\Omega \to X$ is an epimorphism.

Proof of the claim. Indeed, for every connected étale cover $\pi : X' \to X$ let again

$$Z' := \{(x', V) \in X' \times_k Gr(d-1, n)_k \mid \pi(x') \in V\} = Z_k \times_X X' \hookrightarrow X' \times_k Gr(d-1, n)_k$$

denote the corresponding incidence variety. Then, from Bertini's theorem, the generic fiber of the canonical projection $p: Z' \to Gr(d-1,n)_k$ is smooth and geometrically connected. In particular,

$$Z' \times_{Gr(d-1,n)_k} \Omega \to Z_k \times_{Gr(d-1,n)_k} \Omega = Z_{\xi,\Omega}$$

is a connected étale cover. This implies that

$$\pi_1(Z_{\xi,\Omega}) \to \pi_1(X)$$

is an epimorphism ([SGA1, Exp. V, Prop. 6.9]).
Now, write $K := k_0(\xi)$ and $C := Z_\xi$. Fix $c \in C_\Omega$, and let x be the image of c in X and $\overline{x}$ a geometric point above c. The family of $\mathbb{F}_\ell$-linear representations

$$\pi_1(C_\Omega) \to \pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{F}_\ell)),\ \ell \gg 0,$$

then corresponds to the locally constant constructible sheaves $R^i f_{C_\Omega *}\mathbb{F}_\ell$, where $Y_{C_\Omega} \stackrel{f_{C_\Omega}}{\to} C_\Omega$ denote the pull-back of $Y \stackrel{f}{\to} X$ *via* $C_\Omega \to X$. But $Y_{C_\Omega} \stackrel{f_{C_\Omega}}{\to} C_\Omega$ is also the pullback of $Y_{0,K} \stackrel{f_{0,K}}{\to} X_{0,K}$ *via* $C_\Omega \to C \to X_{0,K}$, which shows that the ρ_ℓ, $\ell \gg 0$ factor through the arithmetic fundamental group $\pi_1(C_\Omega) \to \pi_1(C)$. The fact that up to replacing C with a connected étale cover the ρ_ℓ, $\ell \gg 0$ factor through the tame fundamental group follows from de Jong's alteration Theorem [B96, Prop. 6.3.2] together with the facts that $H^i(Y_{\overline{x}}, \mathbb{Z}_\ell)$ is torsion free and that $H^i(Y_{\overline{x}}, \mathbb{F}_\ell) = H^i(Y_{\overline{x}}, \mathbb{Z}_\ell)/\ell$ for $\ell \gg 0$. □

3.2. **Proof of Theorem 1.1.** Observe first that the second assertion of Theorem 1.1 follows from the fact that for a subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$, every proper subgroup $H \subset G^+$ has index $\geq \ell$.

For the first assertion, let $\pi_1(X) \stackrel{\rho_\ell^{ss}}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$ denote the semisimplification of ρ_ℓ and set $G_\ell :=\mathrm{im}(\rho_\ell) \subset \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $G_\ell^{ss} :=\mathrm{im}(\rho_\ell^{ss}) \subset \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$. As the kernel of $G_\ell \twoheadrightarrow G_\ell^{ss}$ is an ℓ-group, it is enough to show that the conclusion of Theorem 1.1 holds for the

$$\pi_1(X) \stackrel{\rho_\ell^{ss}}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell),\ \ell \in L.$$

Furthermore, it follows from $\ker(\rho_\ell) \subset \ker(\rho_\ell^{ss})$ that condition (F) is still satisfied by the $\pi_1(X) \stackrel{\rho_\ell^{ss}}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$. So, without loss of generality, we may assume that $\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$ is semisimple for every $\ell \in L$.

By (and with the notation of) Condition (F), without loss of generality one may assume that $k = \overline{K}$, that $X = C_{\overline{K}}$ is a curve and that the $\rho_\ell : \pi_1(C_{\overline{K}}) \to \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$ all factor through $\pi_1(C_{\overline{K}}) \twoheadrightarrow \pi_1^t(C_{\overline{K}})$. We proceed in two steps.

Claim 1: *There exists an open subgroup $\Pi \subset \pi_1(C_{\overline{K}})$ such that for every open subgroup $\Pi' \subset \Pi$*

$$\rho_\ell(\Pi') = \rho_\ell(\Pi')^+ Z(\rho_\ell(\Pi'))$$

for $\ell \gg 0$ (depending on Π').

For every $\ell \in L$, let $T_\ell \subset G_\ell$ be an abelian subgroup of prime-to-ℓ order as in Corollary 2.3 that is, T_ℓ centralizes G_ℓ^+, $G_\ell^+ T_\ell$ is normal in G_ℓ and $[G_\ell : G_\ell^+ T_\ell] \leq \delta(r)$. (Note that one has $G_\ell \subset \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell) \hookrightarrow \mathrm{GL}_r(\mathbb{F}_\ell)$.) As a topologically finitely generated profinite group has only finitely many open subgroups of bounded index, the set of the inverse images of the $G_\ell^+ T_\ell$, $\ell \in L$ in $\pi_1^t(C_{\overline{K}})$ is finite hence their intersection Π^t is again an open subgroup of $\pi_1^t(C_{\overline{K}})$. Then, the inverse image Π of Π^t in $\pi_1(C_{\overline{K}})$ is an open subgroup of $\pi_1(C_{\overline{K}})$ and, by construction, it satisfies the conclusion of Claim 1.

Claim 2: *For every open subgroup $\Pi \subset \pi_1(C_{\overline{K}})$ there exists an integer $B_\Pi \geq 1$ such that for every prime ℓ one has*

$$|Z(\rho_\ell(\Pi))| \leq B_\Pi.$$

The proof of Claim 2 can be reconstructed from the arguments involved in the proofs of [CT11, Prop. 3.1] and [CT17, Thm. 2.8]. For the convenience of the reader, we recall the main steps.

Without loss of generality, one may assume that $\Pi = \pi_1(C_{\overline{K}})$. Write $Z(G_\ell) =: Z_\ell$. Then Z_ℓ can be decomposed as the direct product of its ℓ and prime-to-ℓ part

$$Z_\ell = Z_\ell^{(\ell)} \times Z_\ell^{(\ell')}.$$

As $Z_\ell^{(\ell)}$ is characteristic in Z_ℓ and Z_ℓ is normal in G_ℓ, $Z_\ell^{(\ell)}$ is normal in G_ℓ and, in particular, it acts semisimply on H_ℓ. As it is an ℓ-group, it also acts unipotently hence is necessarily trivial. So $Z_\ell = Z_\ell^{(\ell')}$ is of prime-to-ℓ order. Write $F := \mathbb{F}_\ell[Z_\ell] \subset \mathrm{End}(H_\ell)$ for the (semisimple) $\mathbb{F}_\ell$-subalgebra generated by Z_ℓ and set $G_\ell^{ar} := \rho_\ell(\pi_1(C)) \rhd G_\ell$. The action by conjugation of G_ℓ^{ar} on G_ℓ restricts to an action on the characteristic subgroup Z_ℓ, which in turn extends by $\mathbb{F}_\ell$-linearity to an action on F. As G_ℓ acts trivially on Z_ℓ, we thus get

$$\begin{array}{ccc} G_\ell^{ar} & \longrightarrow & \mathrm{Aut}_{\mathbb{F}_\ell\text{-}Alg}(F) \subset \mathrm{Aut}_{\overline{\mathbb{F}}_\ell\text{-}Alg}(F \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell) \simeq \mathcal{S}_t \ , \\ \downarrow & \nearrow & \\ G_\ell^{ar}/G_\ell & \twoheadleftarrow & \pi_1(C)/\pi_1(C_{\overline{K}}) \simeq \Gamma_K \end{array}$$

where $F \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \simeq \overline{\mathbb{F}}_\ell^t$ as an $\overline{\mathbb{F}}_\ell$-algebra (since $\mathbb{F}_\ell$ is perfect). So, there exists a finite extension K_ℓ of K with degree $[K_\ell : K] \leq t! \leq r_\ell! \leq r!$ such that $Z_\ell \subset Z(\rho_\ell(\pi_1(C_{K_\ell})))$. Write $\overline{Z}_\ell$ for the image of Z_ℓ in the abelianization $G_\ell \twoheadrightarrow G_\ell^{ab}$. One has

$$Z_\ell \twoheadrightarrow \overline{Z}_\ell \hookrightarrow G_\ell^{ab} \twoheadleftarrow \pi_1^t(C_{\overline{K}})^{ab} \twoheadleftarrow \pi_1(C_{\overline{K}})^{ab}$$

as Γ_K-modules. Note that Z_ℓ and $\overline{Z}_\ell$ are trivial as Γ_{K_ℓ}-modules.

From the arguments used in Claim 3.4 in the proof of [CT11, Prop. 3.1], it is enough to bound $|\overline{Z}_\ell|$ independently of ℓ. For this, we reduce by specialization to the case where K is finite. Up to enlarging K, we may assume that C admits a (unique) smooth compactification $C \subset C^{cpt}$ with $C^{cpt} \setminus C$ étale over K and that C has a K-rational point c. Consider a model $(\mathrm{spec}(R) \overset{c}{\to} \mathcal{C} \subset \mathcal{C}^{cpt} \to \mathrm{spec}(R))$ of $(\mathrm{spec}(K) \overset{c}{\to} C \subset C^{cpt} \to \mathrm{spec}(K))$. More precisely, R is a finitely generated normal integral $\mathbb{Z}$-algebra with fraction field K; $\mathcal{C}^{cpt} \to \mathrm{spec}(R)$ is a proper, smooth, geometrically connected curve over R and $\mathcal{C}^{cpt} \setminus \mathcal{C}$ is a relatively finite étale divisor, such that $\mathcal{C}^{cpt} \times_R K$ and $\mathcal{C} \times_R K$ are isomorphic to (and will be identified with) C^{cpt} and C respectively over K; and $c : \mathrm{spec}(R) \to \mathcal{C}$ is an (a unique) extension of $c : \mathrm{spec}(K) \to C$ (under the identification $\mathcal{C} \times_R K = C$). Fix any closed point $v \in \mathrm{spec}(R)$ and let $p > 0$ denote its residue characteristic. Then one gets a specialization isomorphism for the prime-to-p part of the étale fundamental groups (see [SGA1])

$$\pi_1^t(C_{\overline{K}})^{(p')} = \pi_1(C_{\overline{K}})^{(p')} \tilde{\to} \pi_1(\mathcal{C}_{\overline{v}})^{(p')} = \pi_1^t(\mathcal{C}_{\overline{v}})^{(p')},$$

which induces an isomorphism on the abelianization

$$\pi_1^t(C_{\overline{K}})^{ab,(p')} = \pi_1(C_{\overline{K}})^{ab,(p')} \tilde{\to} \pi_1(\mathcal{C}_{\overline{v}})^{ab,(p')} = \pi_1^t(\mathcal{C}_{\overline{v}})^{ab,(p')}.$$

This isomorphism is compatible with the actions of

$$\Gamma_K \supset D_v \twoheadrightarrow \Gamma_{\kappa(v)},$$

where D_v stands for the decomposition group at v and $\kappa(v)$ for the residue field at v. Further, let R_ℓ be the integral closure of R in K_ℓ and let v_ℓ be the closed point of $\mathrm{spec}(R_\ell)$ above v such that $D_{v_\ell} \subset D_v$. Now, one gets homomorphisms

$$\overline{Z}_\ell^{(p')} \hookrightarrow G_\ell^{ab,(p')} \twoheadleftarrow \pi_1^t(C_{\overline{K}})^{ab,(p')} \tilde{\to} \pi_1^t(\mathcal{C}_{\overline{v}})^{ab,(p')},$$

which are compatible with the actions of $\Gamma_{K_\ell} \supset D_{v_\ell} \twoheadrightarrow \Gamma_{\kappa(v_\ell)}$. In particular, the action of D_{v_ℓ} on $G_\ell^{ab,(p')}$ factors through $\Gamma_{\kappa(v_\ell)}$, as $G_\ell^{ab,(p')}$ is a quotient of the $\Gamma_{\kappa(v_\ell)}$-module $\pi_1^t(\mathcal{C}_{\overline{v}})^{ab,(p')}$. Note that

$$[\Gamma_{\kappa(v)} : \Gamma_{\kappa(v_\ell)}] \leq [D_v : D_{v_\ell}] \leq [\Gamma_K : \Gamma_{K_\ell}] \leq r!.$$

Since $\Gamma_{\kappa(v)} \simeq \hat{\mathbb{Z}}$ is a finitely generated profinite group, the intersection Γ of all open subgroups $\Gamma' \subset \Gamma_{\kappa(v)}$ with $[\Gamma_{\kappa(v)} : \Gamma'] \leq r!$ is again an open subgroup. (The index $[\Gamma_{\kappa(v)} : \Gamma]$ is equal to the least common multiple of $1, \ldots, r!$, which is independent of ℓ.) Write κ for the finite extension of $\kappa(v)$ corresponding to $\Gamma \subset \Gamma_{\kappa(v)}$, and let ϕ denote the $|\kappa|$-th power Frobenius element, which is a generator of $\Gamma = \Gamma_\kappa$. By construction, ϕ acts trivially on $\overline{Z}_\ell^{(p')}$. This implies that $|\overline{Z}_\ell^{(p')}|$ is bounded from above by $|P_\phi(1)|$, where $P_\phi(T) \in \mathbb{Z}[T](\subset \prod_{a \neq p} \mathbb{Z}_a[T])$ is the characteristic polynomial of ϕ acting on $\pi_1^t(\mathcal{C}_{\overline{v}})^{ab,(p')}$ by conjugation. (The proof of this fact can be seen in the last part of [CT11, Prop. 3.1].) Note that $P_\phi(1)$ is a nonzero integer which is independent of ℓ.

To treat the p-part $\overline{Z}_\ell^{(p)}$, we proceed as follows. If K has characteristic 0, consider the nonempty open subscheme $V := \mathrm{spec}(R[1/p]) \subset \mathrm{spec}(R)$. If K has characteristic > 0, then the characteristic of K must coincide with p. In this case, consider the p-rank of (jacobian varieties of) curves obtained as fibers of the family $\mathcal{C}^{cpt} \to \mathrm{spec}(R)$, and let $V \subset \mathrm{spec}(R)$ denote the nonempty open subscheme on which the fiber has maximal p-rank. Now, fix any closed point $w \in V \subset \mathrm{spec}(R)$ and let $q > 0$ denote its residue characteristic. (Thus, if the characteristic of K is 0 (resp. > 0), one has $q \neq p$ (resp. $q = p$).) Then one gets a specialization isomorphism for the pro-p part of the étale fundamental groups

$$\pi_1^t(C_{\overline{K}})^{(p)} = \pi_1(C_{\overline{K}}^{cpt})^{(p)} \tilde{\to} \pi_1(\mathcal{C}_{\overline{w}}^{cpt})^{(p)} = \pi_1^t(\mathcal{C}_{\overline{w}})^{(p)}.$$

(When the characteristic of K is 0, again this can be seen in [SGA1]. When the characteristic of K is $p > 0$, see, for example, [Bou00].) Now, we can bound the p-part $|\overline{Z}_\ell^{(p)}|$ just as in the case of the prime-to-p part $|\overline{Z}_\ell^{(p')}|$. (When the characteristic of K is $p > 0$, we resort to the argument appearing in the last part of [CT17, Thm. 2.8], instead of that appearing in the last part of [CT11, Prop. 3.1].)

To conclude the proof of (the first assertion of) Theorem 1.1, one just applies again the finiteness argument in the proof of Claim 1 with G_ℓ^+ instead of $G_\ell^+ T_\ell$. □

3.3. **The semisimple case.** We end this section with the following refinements of Theorem 1.1 when the $\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell \in L$ are furthermore assumed to be semisimple.

Corollary 3.3. *Assume that Condition (F) is satisfied and that $\pi_1(X) \stackrel{\rho_\ell}{\to} \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$ is semisimple for every $\ell \in L$. Then there exists an open subgroup $\Pi \subset \pi_1(X)$ such that for every open subgroup $\Pi' \subset \Pi$*

$$\rho_\ell(\Pi') = \rho_\ell(\Pi')^+ \text{ and } \rho_\ell(\Pi')^{ab} = 0$$

for $\ell \gg 0$ (depending on Π').

Proof. This follows directly from Theorem 1.1 and Lemma 3.4 below. □

Lemma 3.4. *For $\ell \gg 0$ (depending only on r) and every subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ such that $G = G^+$ and G acts semisimply on $H_\ell = \mathbb{F}_\ell^{\oplus r}$ one has*

$$G^{ab} = 0.$$

Proof. As $G = G^+$ and for $\ell > r$, G contains no element of order ℓ^2, one can identify G^{ab} with the dual of $\mathrm{Hom}(G, \mathbb{F}_\ell) = \mathrm{H}^1(G, \mathbb{F}_\ell)$ (where $\mathbb{F}_\ell$ denotes the trivial G-module). From the split short exact sequence of G-modules

$$0 \to \mathbb{F}_\ell \to \mathbb{F}_\ell \oplus H_\ell \to H_\ell \to 0,$$

one obtains an embedding of abelian groups

$$\mathrm{H}^1(G,\mathbb{F}_\ell)\hookrightarrow \mathrm{H}^1(G,\mathbb{F}_\ell\oplus H_\ell).$$

But as $\mathbb{F}_\ell\oplus H_\ell$ is a faithful semisimple G-module, it follows from [N87, Thm. E] that $\mathrm{H}^1(G,\mathbb{F}_\ell\oplus H_\ell)=0$ for $\ell\gg 0$ (depending only on r). □

Corollary 3.5. *Assume that $k=\overline{k}_0$ with k_0 a finite field and that $X=X_{0,k}$ with X_0 a scheme geometrically connected, smooth and separated over k_0. Also, assume that the $\pi_1(X)\stackrel{\rho_\ell}{\to}\mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell\in L$ are semisimple and satisfy condition (F). Then there exists an open subgroup $\Pi\subset\pi_1(X_0)$ such that for every open subgroup $\Pi'\subset\Pi$ and prime $\ell\gg 0$ (depending on Π'), one has*

$$Z(\rho_\ell(\Pi'))\rho_\ell(\Pi'\cap\pi_1(X))=\rho_\ell(\Pi')$$

and $\rho_\ell(\Pi'\cap\pi_1(X))=\rho_\ell(\Pi'\cap\pi_1(X))^+$.

Proof. From Theorem 1.1, up to replacing X with a connected étale cover one may assume that $\rho_\ell(\pi_1(X))=\rho_\ell(\pi_1(X))^+$ for $\ell\gg 0$ and that for all open subgroup $\Pi\subset\pi_1(X_0)$ and all prime $\ell\gg 0$ (depending on Π), one has $\rho_\ell(\Pi\cap\pi_1(X))=\rho_\ell(\pi_1(X))$. Set $G_\ell:=\rho_\ell(\pi_1(X_0))$, $\Delta_\ell:=\rho_\ell(\pi_1(X))$ and $C_\ell:=G_\ell/\Delta_\ell$. Fix $\phi_\ell\in G_\ell$ lifting any generator of C_ℓ and set $T_\ell:=\langle\phi_\ell\rangle$. Then Theorem 2.2 applied to $G=G_\ell$, $\Delta=\Delta_\ell$ and $T=T_\ell$ provides an element $\varphi_\ell\in G_\ell$ commuting with Δ_ℓ and mapping to a generator of a subgroup of index $\leq\delta(1,r)$ of the cyclic group C_ℓ. So, up to replacing k_0 by its degree $\delta(1,r)!$ extension, we are done. □

Corollary 3.3 extends [EHK12, Prop. 16] (which resorts to the delicate techniques developed by J.-P. Serre to describe the algebraic envelope of the Galois image on ℓ-torsion points of abelian varieties over number fields [S86a] - See [EHK12, Thm. 17, Thm. 18 and App. B]) to arbitrary base fields k and arbitrary families of semisimple representations $\pi_1(X)\stackrel{\rho_\ell}{\to}\mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$, $\ell\in L$. In particular, the results of [EHK12] essentially extend as they are to families of representations arising from étale cohomology in characteristic 0. We refer to [CT16] (where Corollary 3.5 is used) for extensions of the results of [EHK12] to families of representations arising from étale cohomology in arbitrary characteristic.

Remark 3.6.

(1) (About Claim 2 in the proof of Theorem 1.1) Though we will not need it in the remaining part of this paper, let us mention that the arguments used in the proof of 'Claim 3.2 implies Proposition 3.1' in the proof of [CT11, Prop. 3.1] yield the following seemingly stronger version of Claim 2. *Assume that condition (F) is satisfied and that $\pi_1(X)\stackrel{\rho_\ell}{\to}\mathrm{GL}_{r_\ell}(\mathbb{F}_\ell)$ is semisimple for every $\ell\in L$. Then, for every open subgroup $\Pi\subset\pi_1(X)$ there exists an integer $B_\Pi\geq 1$ such that for every prime ℓ and abelian normal subgroup $Z_\ell\lhd\rho_\ell(\Pi)$ one has*

$$|Z_\ell|\leq B_\Pi.$$

(2) From Remark 2.10, Theorem 1.1 (resp. Corollary 3.3) remains true for families of continuous (resp. semisimple continuous) representations $\pi_1(X)\stackrel{\rho_\ell}{\to}\mathrm{GL}_{r_\ell}(\overline{\mathbb{F}}_\ell)$, $\ell\in L$ satisfying Condition (F).

4. Applications to almost ℓ-independence in the sense of [S13]

4.1. **Almost ℓ-independence in the sense of** [S13]**.** Let us first recall the definition of almost ℓ-independence for a family of ℓ-adic representations of a profinite group. Let Γ be a profinite group, L an infinite set of primes, Γ_ℓ, $\ell\in L$ a family of ℓ-adic Lie groups and

$$\rho_\ell:\Gamma\to\Gamma_\ell,\ \ell\in L$$

a family of continuous representations. (We will refer to such a family as a family of ℓ-adic representations of Γ for short). One says that the ρ_ℓ, $\ell \in L$ are *(ℓ-)independent* if the resulting product representation

$$\rho := (\rho_\ell)_{\ell \in L} : \Gamma \to \prod_{\ell \in L} \Gamma_\ell$$

satisfies

$$\rho(\Gamma) = \prod_{\ell \in L} \rho_\ell(\Gamma).$$

and that the ρ_ℓ, $\ell \in L$ are *almost (ℓ-)independent* if there exists an open subgroup $\Pi \subset \Gamma$ such that the $\rho_\ell|_\Pi$, $\ell \in L$ are independent. The notion of (almost) ℓ-independence was introduced by Serre [S13] (see also [S86b]); it corresponds to the case where the image of the product representation $\rho := (\rho_\ell)_{\ell \in L} : \Gamma \to \prod_{\ell \in L} \Gamma_\ell$ is as large as one can (reasonably) expect. The main result of [S13] is a criterion for almost ℓ-independence when Γ is the absolute Galois group of a number field; this criterion applies in particular to show

Theorem 4.1. ([S13, §3.2]) *Let k be a number field and let Y be a scheme separated and of finite type over k. Then the representations*

$$\rho_\ell : \Gamma_k \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{k}}, \mathbb{Q}_\ell)),\ \ell : \text{ prime}$$

are almost-independent (here $\mathrm{H}^i(-, \mathbb{Q}_\ell)$ may refer either to the usual ℓ-adic cohomology or to the ℓ-adic cohomology with compact support).

The proof of Serre's criterion is built on several intermediate technical results, some of which we will re-use below and recall now.

Lemma 4.2. ([S13, Lemma 1 and Lemma 3])

(1) *If for $\ell \neq \ell'$ no simple quotient of $\rho_\ell(\Gamma)$ is isomorphic to a simple quotient of $\rho_{\ell'}(\Gamma)$ then the ρ_ℓ, $\ell \in L$ are independent.*
(2) *If there exists a finite subset $F \subset L$ such that the ρ_ℓ, $\ell \in L \setminus F$ are independent then the ρ_ℓ, $\ell \in L$ are almost independent.*

For every prime ℓ, let Σ_ℓ denote the set of all (isomorphism classes of) finite groups which are either a simple group of Lie type in characteristic ℓ (see [S13, §6.1]) or $\mathbb{Z}/\ell$.

Theorem 4.3. ([S13, Thm. 4 and Thm. 5])

(1) *Every finite simple subquotient of $\mathrm{GL}_r(\mathbb{Z}_\ell)$ of order divisible by ℓ is in Σ_ℓ for $\ell \gg 0$ (depending on r).*
(2) *For $\ell, \ell' \geq 5$, $\ell \neq \ell'$ one has $\Sigma_\ell \cap \Sigma_{\ell'} = \emptyset$.*

4.2. **Notation.** Let k be a field of characteristic $p \geq 0$ and let X be a scheme geometrically connected, smooth and separated over k. Let L be an infinite set of primes and consider a family of ℓ-adic representations together with their reduction modulo ℓ

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\rho_{\ell^\infty}} & \mathrm{GL}_{r_\ell}(\mathbb{Z}_\ell)\ ,\ \ell \in L \\ & \underset{\rho_\ell}{\searrow} & \downarrow \text{mod } \ell \\ & & \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell) \end{array}$$

with $r_\ell \leq r$ bounded as ℓ varies. Write

$$\begin{array}{llll} G_{\ell^\infty} & := \mathrm{im}(\rho_{\ell^\infty}); & G_\ell & := \mathrm{im}(\rho_\ell); \\ G_{\ell^\infty}^{geo} & := \rho_{\ell^\infty}(\pi_1(X_{\overline{k}})) \lhd G_{\ell^\infty}; & G_\ell^{geo} & := \rho_\ell(\pi_1(X_{\overline{k}})) \lhd G_\ell. \end{array}$$

4.3. Almost ℓ-independence for families of ℓ-adic representations of the étale fundamental group.

Corollary 4.4. *Assume that k is algebraically closed and that the ρ_ℓ, $\ell \in L$ satisfy condition (F). Then the ρ_{ℓ^∞}, $\ell \in L$ are almost independent.*

Proof. From Theorem 1.1, Theorem 4.3 (1) and Lemma 4.2 (2), up to replacing X by a connected étale cover and excluding finitely many $\ell \in L$, one may assume that

(1) $2, 3 \notin L$;
(2) for every $\ell \in L$, every finite simple subquotient of $\mathrm{GL}_r(\mathbb{Z}_\ell)$ of order divisible by ℓ is in Σ_ℓ;
(3) $G_\ell = G_\ell^+$.

In particular, from (3) every simple quotient of G_ℓ has order divisible by ℓ. Since G_{ℓ^∞} is an extension of G_ℓ by the kernel of reduction modulo ℓ, which is a pro-ℓ group, it follows from (2) that every finite simple quotient of G_{ℓ^∞} lies in Σ_ℓ. The conclusion thus follows from Theorem 4.3 (2) and Lemma 4.2 (1). $\square$

Recall that every closed point $x \in X$ viewed as a morphism $x : \mathrm{spec}(k(x)) \to X$ induces a quasi-section of the fundamental short exact sequence for $\pi_1(X)$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{k}}) & \longrightarrow & \pi_1(X) & \longrightarrow & \Gamma_k & \longrightarrow & 1. \\ & & & & \nwarrow^{\sigma_x} & & \uparrow & & \\ & & & & & & \Gamma_{k(x)} & & \end{array}$$

Write

$$\begin{array}{llll} \rho_{\ell^\infty,x} & := \rho_{\ell^\infty} \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}_{r_\ell}(\mathbb{Z}_\ell); & \rho_{\ell,x} & := \rho_\ell \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell); \\ G_{\ell^\infty,x} & := \mathrm{im}(\rho_{\ell^\infty,x}); & G_{\ell,x} & := \mathrm{im}(\rho_{\ell,x}). \end{array}$$

The following extends [GP13, Thm. 3.4], which only works for k a number field, to an arbitrary base field k.

Corollary 4.5. *Assume that the $\rho_\ell|_{\pi_1(X_{\overline{k}})}$, $\ell \in L$ satisfy condition (F) and that there exists a closed point $x \in X$ such that the $\rho_{\ell^\infty,x}$, $\ell \in L$ are almost independent then the ρ_{ℓ^∞}, $\ell \in L$ are almost independent.*

Proof. Up to replacing k by a finite extension, one may assume that $x \in X(k)$. From Corollary 4.4, the $\rho_{\ell^\infty}|_{\pi_1(X_{\overline{k}})}$, $\ell \in L$ are almost independent. Hence, up to replacing X by a connected étale cover one may assume that both the $\rho_{\ell^\infty}|_{\pi_1(X_{\overline{k}})}$, $\ell \in L$ and the $\rho_{\ell^\infty,x}$, $\ell \in L$ are independent. As $\pi_1(X) = \pi_1(X_{\overline{k}}) \rtimes_{\sigma_x} \Gamma_k$ it straightforwardly follows that the ρ_ℓ, $\ell \in L$ are independent as well. $\square$

4.4. Almost ℓ-independence for motivic families of ℓ-adic Galois representations.

Let K be a field of characteristic $p \geq 0$ and Y a scheme separated and of finite type over K. Consider the resulting family of ℓ-adic Galois representations

$$\rho_{\ell^\infty} : \Gamma_K \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{K}}, \mathbb{Q}_\ell)),\ \ell : \text{ prime} \neq p,$$

where $\mathrm{H}^i(-, \mathbb{Q}_\ell)$ may refer either to the usual ℓ-adic cohomology or to the ℓ-adic cohomology with compact support.

As in [GP13], one can apply the uniformity results of [I10] (see also [KL85]) to reduce the case of ℓ-adic Galois representations to the case of ℓ-adic representations of the étale fundamental group.

Corollary 4.6. *Assume that K is finitely generated over k and that k is either $\mathbb{Q}$ or an algebraically closed field. Then the ℓ-adic Galois representations*

$$\rho_{\ell^\infty} : \Gamma_K \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{K}}, \mathbb{Q}_\ell))$$

are almost independent.

Proof. Let X be a scheme smooth, separated and geometrically connected over k and whose generic point η has residue field $k(\eta) = K$ and let $\mathcal{Y} \xrightarrow{f} X$ be a morphism of k-schemes, separated and of finite type such that

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & \square & \downarrow f \\ \mathrm{spec}(K) & \xrightarrow[\eta]{} & X. \end{array}$$

Let $R^i f\mathbb{Z}_\ell$ denote either $R^i f_*\mathbb{Z}_\ell$ or $R^i f_!\mathbb{Z}_\ell$, corresponding to the choice that $\mathrm{H}^i(-,\mathbb{Q}_\ell)$ is the usual ℓ-adic cohomology or the ℓ-adic cohomology with compact support. From [I10, Cor. 2.6], there exists a dense open subscheme $U \hookrightarrow X$ such that $R^i f\mathbb{Z}_\ell|_U$ is lisse and of formation compatible with any base change. In particular $\mathrm{H}^i(Y_{\overline{K}},\mathbb{Q}_\ell) = \mathrm{H}^i(\mathcal{Y}_{\overline{\eta}},\mathbb{Q}_\ell) = (R^i f\mathbb{Q}_\ell)_{\overline{\eta}}$ and ρ_{ℓ^∞} factors through $\Gamma_K \twoheadrightarrow \pi_1(U)$. As in Example 3.1, resorting to [Jou83, Thm. 6.10], one can construct a finitely generated field extension L of k and a curve $C \subset U_L$ smooth, separated and geometrically connected over L, such that the induced morphism $\pi_1(C_{\overline{L}}) \twoheadrightarrow \pi_1(U_{\overline{k}})$ is surjective and that

$$\pi_1(C_{\overline{L}}) \twoheadrightarrow \pi_1(U_{\overline{k}}) \stackrel{\rho_{\ell^\infty}}{\to} \mathrm{GL}(\mathrm{H}^i(\mathcal{Y}_{\overline{\eta}},\mathbb{Q}_\ell))$$

factors through $\pi_1(C_{\overline{L}}) \to \pi_1(C)$. From de Jong's alteration theorem [B96, Prop. 6.3.2], up to replacing C with a connected étale cover, we may assume that the resulting representation $\pi_1(C) \to \mathrm{GL}(\mathrm{H}^i(\mathcal{Y}_{\overline{\eta}},\mathbb{Q}_\ell))$ factors through the tame fundamental group $\pi_1(C) \twoheadrightarrow \pi_1^t(C)$. Now, let ρ_ℓ denote the modulo ℓ representation obtained as the reduction of the ℓ-adic representation $\rho_{\ell^\infty} : \Gamma_K \to \mathrm{GL}(\overline{\mathrm{H}}^i(Y_{\overline{K}},\mathbb{Z}_\ell)) \subset \mathrm{GL}(\mathrm{H}^i(Y_{\overline{K}},\mathbb{Q}_\ell))$, where $\overline{\mathrm{H}}^i(Y_{\overline{K}},\mathbb{Z}_\ell) := \mathrm{H}^i(Y_{\overline{K}},\mathbb{Z}_\ell)/(\text{torsion})$. Again, as in Example 3.1, this implies that the ρ_ℓ satisfy condition (F) for $\ell \gg 0$. (Note that $r_\ell := \dim_{\mathbb{F}_\ell}\overline{\mathrm{H}}^i(Y_{\overline{K}},\mathbb{Z}_\ell)/\ell = \dim_{\mathbb{Q}_\ell}\mathrm{H}^i(Y_{\overline{K}},\mathbb{Q}_\ell)$ is bounded as ℓ varies. See [I10, Cor. 1.3].) The conclusion now follows from Corollary 4.5 and Theorem 4.1 (when $k = \mathbb{Q}$) or Corollary 4.4 (when k is algebraically closed). □

For a different approach (following more closely Serre's original strategy in [S13]) of Corollary 4.6 see [GP13] (when $k = \mathbb{Q}$) and the preprint [BGP13] (when k is algebraically closed).

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