GONALITY OF ABSTRACT MODULAR CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let C be a smooth, separated and geometrically connected curve over a finitely generated field k of characteristic $p \geq 0$, η the generic point of C, and $\pi_1(C)$ its étale fundamental group. Let $f: X \to C$ be a smooth proper morphism, and $i \geq 0$, j integers. To the family of continuous $\mathbb{F}_\ell$-linear representations $\pi_1(C) \to \mathrm{GL}(R^i f_* \mathbb{F}_\ell(j)_{\overline{\eta}})$ (where ℓ runs over primes $\neq p$), one can attach families of abstract modular curves $C_0(\ell)$ and $C_1(\ell)$ which, in this setting, are the analogues of the usual modular curves $Y_0(\ell)$ and $Y_1(\ell)$. If $i \neq 2j$, it is conjectured that the geometric and arithmetic gonalities of these abstract modular curves go to infinity with ℓ (for the geometric gonality, under a certain necessary condition). We prove the conjecture for the arithmetic gonality of the abstract modular curves $C_1(\ell)$. We also obtain partial results for the growth of the geometric gonality of $C_0(\ell)$ and $C_1(\ell)$. The common strategy underlying these results consists in reducing by specialization theory to the case where the base field k is finite in order to apply techniques of counting rational points.

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1. Introduction

Let k be a field of characteristic $p \geq 0$.

Let C be a smooth, separated curve over k. Write $C_1, \dots, C_r$ for the connected components of C and $C_i^{(1)}, \dots, C_i^{(n_i)}$ for the connected components of $C_i \times_k \overline{k}$, $i = 1, \dots, r$. Set

$$g(C) := \min\{g(C_i^{(j)}) \mid j = 1, \dots, n_i,\ i = 1, \dots, r\},$$

$$\gamma_{\overline{k}}(C) := \min\{\gamma_{\overline{k}}(C_i^{(j)}) \mid j = 1, \dots, n_i,\ i = 1, \dots, r\},$$

$$\gamma_k(C) := \min\{\gamma_k(C_i) \mid i = 1, \dots, r\},$$

where $g(C_i^{(j)})$ (resp. $\gamma_{\overline{k}}(C_i^{(j)})$, $\gamma_k(C_i)$) denotes the geometric genus (resp. the $\overline{k}$-gonality, the k-gonality) of (the smooth compactification - if any - of) $C_i^{(j)}$ (resp. $C_i^{(j)}$, C_i). See Subsection 2.1 for the definition of $\overline{k}$- and k-gonalities. Also, for every integer $d \geq 1$, write

$$C(k, \leq d) := \{c \in C \mid [k(c) : k] \leq d\}.$$

Thus, $C(k) = C(k, \leq 1)$ and

$$|C| = \bigcup_{d \geq 1} C(k, \leq d),$$

where, for a scheme S, $|S|$ denotes the set of closed points of S.

Let C be a smooth, separated and geometrically connected curve over k with generic point η and étale fundamental group $\pi_1(C)$. Fix an integer $n \geq 1$, an infinite set of primes L and a family $H_\bullet = (H_\ell \simeq \mathbb{F}_\ell^{\oplus n})_{\ell \in L}$ of n-dimensional $\mathbb{F}_\ell$-vector spaces equipped with a continuous action of $\pi_1(C)$ or, equivalently, a family of n-dimensional $\mathbb{F}_\ell$-linear continuous representations

$$\rho_\bullet = (\rho_\ell : \pi_1(C) \to \mathrm{GL}(H_\ell) \simeq \mathrm{GL}_n(\mathbb{F}_\ell))_{\ell \in L}.$$

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Typical examples of such families are those arising from étale cohomology with coefficients in $\mathbb{F}_\ell$. More precisely, let $f : X \to C$ be a smooth, proper scheme over C. By the smooth proper-base change theorem for étale cohomology, for every prime $\ell \neq p$ and every pair of integers $i \geq 0$, j, the étale sheaf $R^i f_* \mathbb{F}_\ell(j)$ is locally constant constructible hence defines a representation

$$\rho_\ell^i(j) : \pi_1(C) \to \mathrm{GL}(R^i f_* \mathbb{F}_\ell(j)_{\overline{\eta}})$$

and the $\mathbb{F}_\ell$-dimension of

$$R^i f_* \mathbb{F}_\ell(j)_{\overline{\eta}} = \mathrm{H}^i(X_{\overline{\eta}}, \mathbb{F}_\ell)(j)$$

is finite and constant for $\ell \gg 0$ [Or13, Rem. 3.1.5]. We will call such families the *motivic families of* $X \to C$. When $X \to C$ is an abelian scheme with $d = \dim(X_\eta)$ and

$$H_\ell = X_{\overline{\eta}}[\ell] = \mathrm{H}^1(X_{\overline{\eta}}^\vee, \mathbb{F}_\ell)(1) \simeq H^{2d-1}(X_{\overline{\eta}}, \mathbb{F}_\ell)(d)$$

is the group of ℓ-torsion points of the generic fiber, we will call the resulting motivic family the *motivic torsion family* of $X \to C$.

To such a family, one can associate families of *abstract modular curves* (see Subsection 2.2) $C_1(\ell) \to C$ and $C_0(\ell) \to C$, which, in this setting, are the analogues of the classical modular curves $Y_1(\ell) \to Y(1)$ and $Y_0(\ell) \to Y(1)$ classifying respectively ℓ-torsion points and order ℓ cyclic subgroups of elliptic curves.

In general, one expects that the arithmetico-geometric complexity of the abstract modular curves $C_i(\ell)$ increases with ℓ. More precisely, under mild assumptions on $\rho_\bullet$ one expects that the following properties hold

$$\text{(G-1)} \quad \lim_{\ell \to +\infty} g(C_i(\ell)) = +\infty$$

$$\text{(G-2)} \quad \lim_{\ell \to +\infty} \gamma_{\overline{k}}(C_i(\ell)) = +\infty$$

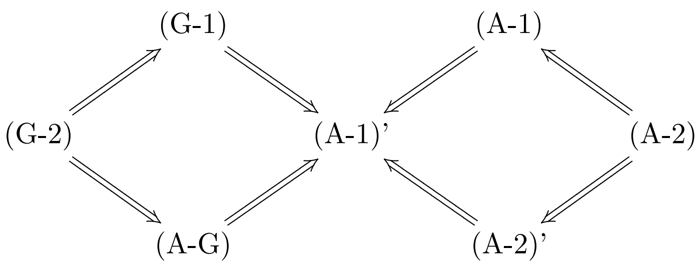
$$\text{(A-G)} \quad \lim_{\ell \to +\infty} \gamma_k(C_i(\ell)) = +\infty$$

and, if k is finitely generated,

(A-1)' $|C_i(\ell)(k)| < +\infty$, $\ell \gg 0$ (A-1) $C_i(\ell)(k) = \emptyset$, $\ell \gg 0$

(A-2)' $|C_i(\ell)(k, \leq d)| < +\infty$, $\ell \gg 0$, $d \geq 1$ (A-2) $C_i(\ell)(k, \leq d) = \emptyset$, $\ell \gg 0$, $d \geq 1$

Note that



and, if $p = 0$,

$$\text{(A-G)} \Longrightarrow \text{(A-2)'}.$$

(See Subsection 2.1 for more details.)

Properties (G-1) and (G-2) have been investigated thoroughly in the classical setting of Shimura curves[1] (See for instance [A96] for (G-2) when $p = 0$, [Po07] for (G-2) when $p > 0$) and more recently in the setting of abstract modular curves (See [CT14b] for (G-1) when $p \geq 0$, [EHK12] (and [CT14a]) for (G-2) when $p = 0$). Properties (A-1), (A-2) are only known for the modular curves $Y_1(\ell)$, $Y_0(\ell)$ ([Ma77], [K92], [Me96], [Ma78], [Mo95]) and for abstract modular curves in the ℓ-adic setting ([CT12a], [CT09], [CT12b], [CT13]). Proving properties (A-1), (A-2) for abstract modular curves in the modulo ℓ setting at stake in this paper is a challenging widely open problem. Proving properties (G-1), (G-2), (A-G) is a first significant step towards it.

In this paper, we investigate properties (G-2) and (A-G) when $p \geq 0$ and prove them under mild assumptions. Precise statements require some technical preliminaries gathered in Section 2 so we postpone them untill Subsections 2.3.2 (when k is finite) and 5.1 (when k is finitely generated). Let us however state simplified versions of our main results in the case of finitely generated fields:

- (Special case of Corollary 5.1): Start with
$$\mathcal{X} \to \mathcal{C} \hookrightarrow \mathcal{C}^{cpt} \to T \to U,$$
where U is a non-empty open subscheme of $\mathrm{spec}(\mathbb{Z})$ (resp. $U = \mathrm{spec}(\mathbb{F}_p)$) when $p = 0$ (resp. $p > 0$), T is an integral scheme with generic point ζ, $T \to U$ is a dominant morphism of finite type, $\mathcal{C}^{cpt} \to T$ is a smooth, proper, geometrically connected curve over T, $\mathcal{C}^{cpt} \setminus \mathcal{C}$ is a relatively finite étale divisor and $\mathcal{X} \to \mathcal{C}$ is an abelian scheme. Assume that
  - The generic fiber of $\mathcal{X}_\zeta \to \mathcal{C}_\zeta$ contains no non-trivial abelian subvariety isogenous to a $k(\zeta)$-isotrivial abelian variety.
  - There exists a closed point $t \in |T|$ (resp. a Zariski-dense set of closed points $t \in |T|$) such that $\mathcal{X}_t \to \mathcal{C}_t$ has a closed fiber which is supersingular[2].

  Then the motivic torsion family of $\mathcal{X}_\zeta \to \mathcal{C}_\zeta$ satisfies
$$\lim_{\ell \to +\infty} \gamma_{\overline{k(\zeta)}}(C_{\zeta,0}(\ell)) = +\infty.$$
  In particular, if $p = 0$ (resp. $p > 0$), for every integer $d \geq 1$ (resp. $d = 1$) and for $\ell \gg 0$, there are only finitely many $c \in C(k(\zeta), \leq d)$ (resp. $c \in C(k(\zeta))$) such that $\mathcal{X}_{\overline{c}}[\ell]$ admits a 1-dimensional $\Gamma_{k(c)}$-submodule.

- (Corollary 5.2 (2)) Let $\rho^i_\bullet(j)$ be a motivic family attached to a smooth proper morphism $X \to C$. Assume $2i \neq j$. Then
$$\lim_{\ell \to +\infty} \gamma_k(C_1(\ell)) = +\infty.$$
  In particular, if $p = 0$ (resp. $p > 0$), for every integer $d \geq 1$ (resp. $d = 1$) and for $\ell \gg 0$, there are only finitely many $c \in C(k(\zeta), \leq d)$ (resp. $c \in C(k(\zeta))$) such that
$$\mathrm{H}^i(X_{\overline{c}}, \mathbb{F}_\ell)(j)^{\Gamma_{k(c)}} \neq 0.$$

The main new contribution of this paper to the above questions is to deal with the case of positive characteristic under rather general hypotheses. It seems also that the definition and study of arithmetic gonality is new. Let us however point out that, in positive characteristic, the implication (A-G) $\Rightarrow$ (A-2)' no longer holds. In the appendix, which can be read independently, we introduce the notion of (geometric) isogonality and show that it is the right invariant to measure the finiteness of $C(k, \leq d)$ in positive characteristic. Unfortunately, geometric isogonality seems very difficult to control in general and it is not clear if one can reasonably expect Property (A-G) to hold with isogonality instead

[1] For a review of the problem of and a graph-theoretic approach to the gonality of Drinfeld modular curves, see for instance [CoKKo15].

[2] A special case of the notion of closed fiber of supersingular type introduced in Paragraph 2.3.2.1. See also Paragraph 3.3.2 for an example of a closed fiber of supersingular type which is not a supersingular abelian variety.

of gonality.

The paper is organized as follows. In Section 2 we make a short review of gonalities (Subsection 2.1), define abstract modular curves (Subsection 2.2) and state precisely our main results over finite fields (Subsection 2.3). The proof of the statements over finite fields are carried out in Section 3 (geometric gonality) and Section 4 (arithmetic gonality). In Section 5, we derive several corollaries over finitely generated fields (Subsection 5.1) from the statements over finite fields. This is done - unsurprisingly - by a specialization argument which is detailed in Subsection 5.2. The appendix introduces the notion of (geometric) isogonality and discusses its relation with the finiteness of points of bounded degree on curves in positive characteristic.

2. Preliminaries and statements

2.1. **Geometric and arithmetic gonalities.** Let k be a field. We define below the geometric and arithmetic gonalities of a connected curve C smooth and *proper* over k. Later on, when C is separated but not proper, the geometric and arithmetic gonalities of C will implicitly refer to the geometric and arithmetic gonalities of its smooth compactification - if any.

2.1.1. *Geometric gonality.* Assume furthermore that C is geometrically connected over k. For a field extension K of k the *K-gonality* of C is defined as $\gamma_K(C) := d+1$, where d is the largest integer ≥ 1 satisfying the equivalent conditions (i)-(v) below.

- (i) For every $d' \leq d$ the map $C^{(d')}(K) \to J^{(d')}(K)$ is injective;
- (ii) For every $d' \leq d$ there is no non-constant K-morphism $\mathbb{P}^1_K \to C_K^{(d')}$;
- (iii) $L(D) = K$ for every effective degree $d' \leq d$ (Cartier) divisor D on C_K;
- (iv) There is no $f \in K(C) \setminus K$ such that $[K(C) : K(f)] \leq d$;
- (v) There is no non-constant K-morphism $C_K \to \mathbb{P}^1_K$ of degree $\leq d$.

Here, $J^{(d)}$ denotes the degree d part of the Picard scheme of C over k, $C^{(d)}$ the dth symmetric product of C and, for a divisor D on C_K, $L(D)$ the K-vector space of all $f \in K(C)$ such that $D + div(f)$ is effective.

We will use several times the following fact describing the behaviour of gonality under ground field extension.

Fact 2.1. ([Po07, Thm. 2.5 (iii)])[3] *Assume k is perfect and C carries a degree one k-rational divisor. Then for every algebraic extension $L \supset k$, one has*

$$\gamma_L(C) \geq \sqrt{\gamma_k(C)}.$$

When the ground field k is finite, the assumption that C carries a degree one k-rational divisor is automatically satisfied.

When $K = \overline{k}$ is an algebraic closure of k, we will call $\gamma_{\overline{k}}(C)$ the *geometric gonality* of C. The geometric gonality is a subtle invariant, which encodes both geometric and arithmetic information about C. For instance, one always has [Po07, Prop. A.1 (v)]

$$\gamma_{\overline{k}}(C) \leq \lfloor \frac{g(C)+3}{2} \rfloor.$$

So the growth of the geometric gonality implies the growth of the genus. In particular,

[3] Note that the proof of [Po07, Thm. 2.5 (iii)] only uses the fact that X carries a degree one k-rational divisor (and that a proper smooth curve of genus 0 over a finite field is $\mathbb{P}^1$).

Fact 2.2. *When k is finitely generated there exists an integer $\gamma(k)$ (≥ 3) such that*

$$\gamma_{\overline{k}}(C) \geq \gamma(k) \Longrightarrow |C(k)| < +\infty.$$

(This is the Mordell conjecture when $p = 0$ [FW92] and a result of Voloch when $p > 0$ - see [EElHK09, Thm. 3]).

Fact 2.3. *When $p = 0$, one has*

$$\gamma_{\overline{k}}(C) \geq 2d + 1 \Longrightarrow |C(k, \leq d)| < +\infty.$$

(This is a consequence of the Mordell-Lang conjecture [FW92], [Fr94]. See Corollary A.4.)

However, in practice, the geometric gonality is usually difficult to estimate (especially when $p > 0$), compared to the genus. This motivates the introduction of a weakened arithmetic variant, which is easier to handle than the geometric gonality (and, sometimes, than the genus) but, still, can be used to test the finiteness of rational points when k is finitely generated (see Lemma 2.4 below).

2.1.2. *Arithmetic gonality.* We no longer assume that C is geometrically connected over k. Let k_C denote the algebraic closure of k in the function field of C (thus, C is geometrically connected over k_C).

The *k-gonality* of C is defined as

$$\gamma_k(C) := \min\{\deg(f) \mid f : C \to \mathbb{P}^1_k \text{ non-constant } k\text{-morphism}\} = [k_C : k]\gamma_{k_C}(C).$$

Note that if C is geometrically connected over k (*i.e.*, if $k_C = k$), then the above definition of $\gamma_k(C)$ coincides with the one in Subsection 2.1.1, and, by definition, one always has

$$\gamma_{\overline{k}}(C) \leq \gamma_k(C).$$

Lemma 2.4. *Let k be a finitely generated field of characteristic $p \geq 0$ and let C be a connected curve, smooth, proper (but not necessarily geometrically connected) over k.*

(1) There exists an integer $\gamma(k)$ (≥ 3) such that

$$\gamma_k(C) \geq \gamma(k) \Longrightarrow |C(k)| < +\infty.$$

(2) If $p = 0$, then, for any integer $d \geq 1$, one has

$$\gamma_k(C) \geq 4d^3 + 1 \Longrightarrow |C(k, \leq d)| < +\infty.$$

Proof. Assertion (1) follows from the fact that either $C(k) = \emptyset$ or $C(k) \neq \emptyset$ and $k_C = k$, in which case [Po07, Prop.1.1 (iv)]

$$g(C) \geq \gamma_{k_C}(C) - 1 = \gamma_k(C) - 1$$

and the conclusion follows from Fact 2.2. For assertion (2), assume that $\gamma_k(C) \geq 4d^3 + 1$. If $C(k, \leq d) = \emptyset$, there is nothing to say. Otherwise, pick $c \in C(k, \leq d)$. Then

$$d \geq [k(c) : k] = [k(c) : k_C][k_C : k]$$

and

$$\gamma_{\overline{k}}(C) \geq \sqrt{\gamma_{k(c)}(C)} \geq \sqrt{\frac{\gamma_{k_C}(C)}{[k(c) : k_C]}} \geq \sqrt{\frac{\gamma_k(C)}{d}} \geq \sqrt{\frac{4d^3 + 1}{d}} > 2d,$$

where the first inequality follows from Fact 2.1 and the second one from the fact that for a finite extension L/K (with $k_C \subset K$), which is automatically separable as $p = 0$, one always has

$$\gamma_L(C) \geq \frac{\gamma_K(C)}{[L : K]}.$$

Indeed, let $\widehat{L}/K$ denote the Galois closure of L/K. Then, starting from a non-constant L-morphism $f : \mathbb{P}^1_L \to C^{(d)}_L$, just observe that the induced morphism

$$t \in \mathbb{P}^1_{\widehat{L}} \to \sum_{\sigma \in \mathrm{Gal}(\widehat{L}|K)/\mathrm{Gal}(\widehat{L}|L)} ({}^{\sigma} f_{\widehat{L}})(t) \in C^{(d[L:K])}_{\widehat{L}} \simeq (C^{(d)}_{\widehat{L}})^{[L:K]}/\mathcal{S}_{[L:K]}$$

is non-constant and descends to K, hence $\gamma_K(C) \leq [L:K]\gamma_L(C)$. □

2.2. **The abstract modular curves $C_1(\ell)$, $C_0(\ell)$.** Let C be a connected scheme. Fix a point $c \in C$ and recall that, by definition, the étale fundamental group $\pi_1(C;\overline{c})$ of C with base point any geometric point $\overline{c}$ over c is the automorphism group of the fiber functor $F_{\overline{c}}$ sending an étale cover $C' \to C$ to the finite set $C'_{\overline{c}}(k(\overline{c}))$. The group $\pi_1(C;\overline{c})$ is naturally endowed with a profinite topology and the fiber functor $F_{\overline{c}}$ induces an equivalence of categories between the category of étale covers of C and the category of finite discrete $\pi_1(C;\overline{c})$-sets. In this equivalence, connected étale covers correspond to transitive $\pi_1(C;\overline{c})$-sets or, equivalently, to open subgroups of $\pi_1(C;\overline{c})$. Also, given any two points $c_0, c_1 \in C$, the set of étale paths $\alpha : F_{\overline{c}_0} \to F_{\overline{c}_1}$ is non-empty. Every choice of an étale path $\alpha : F_{\overline{c}_0} \to F_{\overline{c}_1}$ induces an isomorphism of étale fundamental groups

$$\begin{array}{ccc} \pi_1(C;\overline{c}_0) & \tilde{\to} & \pi_1(C;\overline{c}_1) \\ \gamma & \to & \alpha\gamma\alpha^{-1} \end{array}$$

Thus, unless it helps understand the situation, we will omit the base point from our notation.

If we assume furthermore that C is of finite type and geometrically connected over a field k, the sequence of morphisms $C_{\overline{k}} := C \times_k \overline{k} \to C \to \mathrm{spec}(k)$ induces by functoriality a sequence of fundamental groups

$$1 \to \pi_1(C_{\overline{k}}) \to \pi_1(C) \stackrel{pr}{\to} \Gamma_k \to 1,$$

which is exact, and every closed point $c \in C$, viewed as a morphism $c : \mathrm{spec}(k(c)) \to C$, induces a quasi-splitting

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(C_{\overline{k}}) & \longrightarrow & \pi_1(C) & \xrightarrow{pr} & \Gamma_k & \longrightarrow & 1 \\ & & & & & {\scriptstyle \sigma_c} \nwarrow & \uparrow & & \\ & & & & & & \Gamma_{k(c)} & & \end{array}$$

of this short exact sequence. (In the above, given a field k, we identify $\pi_1(\mathrm{spec}(k))$ with the absolute Galois group Γ_k of k).

Given an open subgroup $U \subset \pi_1(C)$, let $C_U \to C$ denote the corresponding connected étale cover and $k \subset k_U \subset \overline{k}$ the subfield defined by $pr(U) = \Gamma_{k_U} \subset \Gamma_k$. Then k_U is the field of definition for C_U and C_U is geometrically connected over k_U. We call $C_U \to C$ the *abstract modular curve* attached to U. Let us also recall that

Fact 2.5.
(1) The connected étale cover $C_U \times_{k_U} \overline{k} \to C_{\overline{k}}$ corresponds to the open subgroup $U \cap \pi_1(C_{\overline{k}}) \subset \pi_1(C_{\overline{k}})$;
(2) For any closed point $c \in C$, the image of σ_c is contained in U if and only if c lifts to a $k(c)$-rational point on C_U [SGA1, V, Prop. 6.4].

Property (2) is at the origin of the terminology ‘modular’.

Let ℓ be a prime $\neq p$ and H_ℓ a finite-dimensional $\mathbb{F}_\ell$-vector space equipped with a continuous action of $\pi_1(C)$. For $v \in H_\ell$, we will write $C_{1,v} \to C_{0,v} \to C$ for the connected étale covers corresponding to the inclusion of open subgroups

$$\mathrm{Stab}_{\pi_1(C)}(v) \subset \mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_\ell v) \subset \pi_1(C)$$

and $k_{1,v}$ and $k_{0,v}$ for the fields of definition of $C_{1,v}$ and $C_{0,v}$ respectively.

We will also write

$$C_1(\ell) := \bigsqcup_{0\neq v\in H_\ell} C_{1,v}, \ \ C_0(\ell) := \bigsqcup_{0\neq v\in H_\ell} C_{0,v}.$$

By construction $C_1(\ell) \to C_0(\ell) \to C$ are (possibly disconnected) étale covers with the property that a closed point $c \in C$ lifts to a $k(c)$-rational point on $C_1(\ell)$ (respectively, $C_0(\ell)$) if and only if $H_\ell^{\Gamma_{k(c)}} \neq 0$ (respectively, $\mathbb{P}(H_\ell)^{\Gamma_{k(c)}} \neq \emptyset$, where $\mathbb{P}(H_\ell) := (H_\ell \setminus \{0\})/\mathbb{F}_\ell^\times$). Here, the action of $\Gamma_{k(c)}$ on H_ℓ is *via* $\Gamma_{k(c)} \stackrel{\sigma_c}{\to} \pi_1(C) \stackrel{\rho_\ell}{\to} \mathrm{GL}(H_\ell)$.

2.3. **Statements.** Let C be a smooth, separated and geometrically connected curve over k with generic point η and étale fundamental group $\pi_1(C)$. Fix an integer $n \geq 1$, an infinite set of primes L and a family $H_\bullet = (H_\ell \simeq \mathbb{F}_\ell^{\oplus n})_{\ell\in L}$ of n-dimensional $\mathbb{F}_\ell$-vector spaces equipped with a continuous action of $\pi_1(C)$ or, equivalently, a family of n-dimensional $\mathbb{F}_\ell$-linear continuous representations

$$\rho_\bullet = (\rho_\ell : \pi_1(C) \to \mathrm{GL}(H_\ell) \simeq \mathrm{GL}_n(\mathbb{F}_\ell))_{\ell\in L}.$$

2.3.1. *Characteristic polynomial, finiteness and semisimplicity assumptions.* Our general strategy will be to reduce by a specialization argument (See Subsection 5.2) to the case where the base field k is finite in order to exploit the fact that, over finite fields, elementary numerical invariants attached to closed points usually encode enough data to estimate the geometric or arithmetic gonality from below.

So, *from now on and till the end of Section 4, we assume* $k = \mathbb{F}_q$ *with* $q = p^r$. We refer to Subsection 5.1 for the statements over finitely generated fields. Let C^{cpt} denote the smooth compactification of C.

2.3.1.1. *Characteristic polynomial assumptions*. The conditions on the characteristic polynomials of Frobenius elements we formulate below will be used to control the numerical invariants attached to $C_1(\ell)$ in order to estimate its arithmetic gonality.

Write $\varphi \in \Gamma_k \simeq \widehat{\mathbb{Z}}$ for a generator of Γ_k.

For every point $c \in |C^{cpt}|$, let D_c and I_c denote respectively 'the' decomposition group and inertia group at c in $\pi_1(C)$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_c & \longrightarrow & D_c & \longrightarrow & \Gamma_{k(c)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(C_{\overline{k}}) & \longrightarrow & \pi_1(C) & \longrightarrow & \Gamma_k & \longrightarrow & 1
\end{array}$$

Write

$$\rho_{\ell,c} := \rho_\ell|_{D_c} : D_c \to \mathrm{GL}(H_\ell)$$

for the resulting local representation at c.

Case 1: $c \in |C|$. Then I_c is trivial and the morphism $\Gamma_{k(c)} \tilde{\to} D_c \hookrightarrow \pi_1(C)$ can be identified with the morphism

$$\sigma_c : \Gamma_{k(c)} = \pi_1(\mathrm{spec}(k(c))) \to \pi_1(C).$$

Set $\varphi_c := \varphi^{[k(c):k]} \in \Gamma_{k(c)}$ and $\varphi_{\ell,c} := \rho_{\ell,c}(\varphi_c)$. We will say that $\rho_\bullet$ is *weakly φ-$\mathbb{Z}$-integral*[4] *at c* if there exists a polynomial $P_c \in \mathbb{Z}[T]$ whose image in $\mathbb{F}_\ell[T]$ is an $\mathbb{F}_\ell^\times$-multiple of the characteristic polynomial of $\varphi_{\ell,c}$ for $\ell \gg 0$ and that $\rho_\bullet$ is *φ-$\mathbb{Z}$-integral at c* if there exists a monic polynomial $P_c \in \mathbb{Z}[T]$ whose image in $\mathbb{F}_\ell[T]$ is the characteristic polynomial of $\varphi_{\ell,c}$ for $\ell \gg 0$.

Actually, when $\rho_\bullet$ is weakly φ-$\mathbb{Z}$-integral or φ-$\mathbb{Z}$-integral, we will also need to control the roots $\alpha_{c,1}, \ldots, \alpha_{c,n}$ of P_c. This will be ensured by the following two conditions. Fix an integer $N \geq 1$.

φ-(R,c) $\rho_\bullet$ is weakly φ-$\mathbb{Z}$-integral at c and none of the products $\alpha_{c,i_1} \cdots \alpha_{c,i_t}$, $1 \leq i_1 < \cdots < i_t \leq n$, $1 \leq t \leq n$, is a root of unity;

φ-(R,N,c) $\rho_\bullet$ is φ-$\mathbb{Z}$-integral at c and $1 \neq |\alpha_{c,i}| \leq |k(c)|^N$, $i = 1, \ldots, n$.

Case 2: $c \in |C^{cpt}| \setminus |C|$**.** Then the inertia group I_c is no longer trivial so $\rho_{\ell,c} : D_c \to \mathrm{GL}(H_\ell)$ is not identified with a representation of $\Gamma_{k(c)}$ as above. However, assume that

(U) For every $c \in |C^{cpt}| \setminus |C|$ there exists an open subgroup $D'_c \subset D_c$ such that $I'_c := D'_c \cap I_c$ acts unipotently on H_ℓ for $\ell \gg 0$.

Then (for $\ell \gg 0$) I'_c lies in the kernel of the semisimplification

$$\rho^{ss}_{\ell,c} : D_c \to \mathrm{GL}(H^{ss}_\ell)$$

of $\rho_{\ell,c} : D_c \to \mathrm{GL}(H_\ell)$ and $\rho^{ss}_{\ell,c}|_{D'_c} : D'_c \to \mathrm{GL}(H^{ss}_\ell)$ factors through

$$\rho^{ss'}_{\ell,c} : D'_c/I'_c \to \mathrm{GL}(H^{ss}_\ell).$$

Let $k(c)'$ denote the finite field extension of $k(c)$ fixed by $D'_c/I'_c \hookrightarrow D_c/I_c \simeq \Gamma_{k(c)}$ and set $\varphi'_c := \varphi^{[k(c)':k]} \in \Gamma_{k(c)'}$, $\varphi'_{\ell,c} := \rho^{ss'}_{\ell,c}(\varphi'_c)$. We will say that $\rho_\bullet$ is *weakly φ-$\mathbb{Z}$-integral*[4] *at c* if there exists a non-zero polynomial $Q_c \in \mathbb{Z}[T]$ whose image in $\mathbb{F}_\ell[T]$ is divisible by the characteristic polynomial of $\varphi'_{\ell,c}$ for $\ell \gg 0$.

Theorem 2.6. *Let $f : X \to C$ be a smooth, proper scheme over C with $d := \dim(X_\eta)$, and consider the corresponding motivic family*

$$\rho^i_\bullet(j) = (\rho^i_\ell(j) : \pi_1(C) \to \mathrm{GL}(R^i f_* \mathbb{F}_\ell(j)_{\overline{\eta}}))_{\ell \gg 0}.$$

(1) Assume that $i \neq 2j$. Then there exists [5] *a generator $\varphi \in \Gamma_k$ such that $\rho^i_\bullet(j)$ satisfies Condition φ-(R,c) at every $c \in |C|$.*
(2) Assume that $i \neq 2j$ and that $j \leq \max\{0, i-d\}$ or $j \geq \min\{i, d\}$. Then there exists [6] *a generator $\varphi \in \Gamma_k$ such that $\rho^i_\bullet(j)$ satisfies Condition φ-(R,$\frac{|i-2j|}{2}$,c) at every $c \in |C|$.*
(3) $\rho^i_\bullet(j)$ satisfies Condition (U) and is weakly $\mathbb{Z}$-integral at every $c \in |C^{cpt}| \setminus |C|$.

Proof. Recall first that the assumption that $X \to C$ is proper ensures that for $i \geq 0$ and $\ell \gg 0$, the $\mathbb{Z}_\ell$-module $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Z}_\ell)$ is torsion-free (hence a $\mathbb{Z}_\ell$-lattices in $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$) [Or13, Rem. 3.1.5] and $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Z}_\ell)/\ell \simeq \mathrm{H}^i(X_{\overline{\eta}}, \mathbb{F}_\ell)$ (see *e.g.* [CT14b, (1) in Proof of Fact 5.1]). Thus, it is enough to prove that the conditions at stake in Theorem 2.6 (with $\mathbb{F}_\ell$ replaced by $\mathbb{Q}_\ell$) are satisfied by the family

$$\pi_1(C) \to \mathrm{GL}(R^i f_* \mathbb{Q}_\ell(j)_{\overline{\eta}}).$$

[4] Note that the definition of being weakly φ-$\mathbb{Z}$-integral at c does not depend on the choice of $\varphi^{\pm 1}$ since we do not require P_c (if $c \in |C|$) or Q_c (if $c \in |C^{cpt}| \setminus |C|$) to be monic.

[5] More precisely, one can take φ to be the geometric Frobenius or the arithmetic Frobenius.

[6] More precisely, if $j \leq \max\{0, i-d\}$, one can take φ to be the geometric Frobenius and if $j \geq \min\{i, d\}$, one can take φ to be the arithmetic Frobenius.

Then (1) directly follows from the Weil Conjectures [D80]. Assertion (2) also follows from the Weil Conjectures together with Poincaré duality and [P98, Thm. 3.3] (the restrictions on i, j are there to ensure the $\mathbb{Z}$-integrality condition).

In (3), Condition (U) and the weakly φ-$\mathbb{Z}$-integral condition at every $c \in |C^{cpt}| \setminus |C|$ follow essentially from de Jong's alteration theorem and the Rapoport-Zink weight spectral sequence. Indeed, Condition (U) is [B96, Prop. 6.3.2]. So, up to replacing C by a connected étale cover (independent of ℓ), one may assume that I_c acts unipotently on $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)(j)$ for every prime $\ell \neq p$ and $c \in |C^{cpt}| \setminus |C|$. As the question is local at c, we may replace C with $\mathrm{spec}(\widehat{\mathcal{O}}_{C,c}) = \{\eta, c\}$. Set $\tilde{D}_c := \Gamma_{k(\eta)}$ and let $\tilde{I}_c \subset \tilde{D}_c$ denote the inertia group. By the argument in the second paragraph of the proof of [B96, Prop. 6.3.2] (here we use that $X \to C$ is proper and smooth), up to replacing C by a connected, generically étale cover, there exists a strictly semistable (that is, proper, flat, generically smooth with simple normal crossing special fiber) morphism $X' \to C$ such that $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)(j)$ embeds as a $\tilde{D}_c$-module into $\mathrm{H}^i(X'_{\overline{\eta}}, \mathbb{Q}_\ell)(j)$ for all primes $\ell \neq p$. As $\tilde{I}_c$ acts unipotently on $\mathrm{H}^i(X'_{\overline{\eta}}, \mathbb{Q}_\ell)(j)$ the $\tilde{D}_c$-action on the semisimplification as $\tilde{D}_c$-module factors through $\tilde{D}_c \to \Gamma_{k(c)} = \tilde{D}_c/\tilde{I}_c$ hence $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)(j)^{ss}$ embeds as a $\Gamma_{k(c)}$-module into $\mathrm{H}^i(X'_{\overline{\eta}}, \mathbb{Q}_\ell)(j)^{ss}$. In particular, the characteristic polynomial of the Frobenius acting on $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)(j)^{ss}$ divides the characteristic polynomial of the Frobenius acting on $\mathrm{H}^i(X'_{\overline{\eta}}, \mathbb{Q}_\ell)(j)^{ss}$. Thus, we may assume that $X \to C$ is strictly semistable. Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_t$ denote the irreducible components of X_c and for each integer $1 \leq s \leq t$ set

$$\mathfrak{X}^{(s)} := \bigsqcup_{1 \leq i_1 < \cdots < i_s \leq t} \mathfrak{X}_{i_1} \cap \cdots \cap \mathfrak{X}_{i_s}.$$

The $\mathfrak{X}^{(s)}$ are smooth proper schemes over $k(c)$. Write $\mathfrak{X}^{(s)}_{\overline{c}}$ for their geometric fibers. The Rapoport-Zink weight spectral sequence [RZ82] is a $\tilde{D}_c$-equivariant spectral sequence:

$$E_1^{p,q} = \bigoplus_{a \geq \max\{0,p\}} \mathrm{H}^{2p+q-2a}(\mathfrak{X}_{\overline{c}}^{(2a-p+1)}, \mathbb{Q}_\ell)(j+p-a) \Rightarrow \mathrm{H}^{p+q}(X_{\overline{\eta}}, \mathbb{Q}_\ell)(j).$$

By the Weil conjectures, (the geometric Frobenius of) $\Gamma_{k(c)}$ acts with weight $q-2j$ on $E_r^{p,q}$, $r \geq 1$. As the differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are $\tilde{D}_c$-equivariant, they are trivial for $r \geq 2$ and the spectral sequence degenerates at E_2. In particular,

$$Gr_p^n := \mathrm{Gr}_p(\mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)) = E_2^{p,n-p} = \frac{\ker(d_1^{p,n-p} : E_1^{p,n-p} \to E_1^{p+1,n-p})}{\mathrm{im}(d_1^{p-1,n-p} : E_1^{p-1,n-p} \to E_1^{p,n-p})}.$$

(Observe that $E_1^{p,n-p} = 0$, hence $Gr_p^n = 0$, unless $-n \leq p \leq n$.) Thus,

$$\chi_{\mathrm{im}(d_1^{p-1,n-p})} \chi_{Gr_p^n} = \chi_{\ker(d_1^{p,n-p})},$$

where χ_- denotes the characteristic polynomial of the Frobenius acting on $-$. This implies that $\chi_{Gr_p^n}$ divides $\chi_{E_1^{p,n-p}}$. Now, the Weil conjectures for the $\mathrm{H}^i(\mathfrak{X}^{(s)}_{\overline{c}}, \mathbb{Q}_\ell)(j)$ ensure that $\chi_{E_1^{p,n-p}} \in \mathbb{Q}[T]$. So, take for Q_c

$$\lambda \prod_{-i \leq p \leq i} \chi_{E_1^{i,n-i}},$$

where $0 \neq \lambda \in \mathbb{Z}$ is chosen in such a way that $Q_c \in \mathbb{Z}[T]$. □

2.3.1.2. *Finiteness and semisimplicity assumptions.* Condition (U) above will be used not only to define the notion of weak $\mathbb{Z}$-integrality for points at infinity but also to ensure that the group

$$\pi_1(C_{\overline{k}})/K(\rho_\bullet)$$

is topologically finitely generated [CT14b, Lemma 4.3] (note that condition (U) implies Condition (F) of [CT14b]), where

$$K(\rho_\bullet) := \bigcap_\ell \ker(\rho_\ell) \cap \pi_1(C_{\overline{k}}).$$

Apart from this finiteness condition, we will also consider the following semisimplicity condition.

(SSgeo) For every open subgroup $\Pi \subset \pi_1(C_{\overline{k}})$, H_ℓ is a semisimple Π-module for $\ell \gg 0$.

Condition (SSgeo) is satisfied by motivic families when $X \to C \to \mathrm{spec}(k)$ lifts to characteristic 0 ([D71] - see [CT11, Lemma 2.5] for details), and by motivic torsion families of abelian schemes ([Ta66], [Z77], [Sz85]).

2.3.2. *Statements.*

2.3.2.1. *Growth of the geometric gonality.* Write

$$\begin{array}{ll} G_\ell & := \rho_\ell((\pi_1(C)) \subset \mathrm{GL}(H_\ell); \\ G_\ell^{geo} & := \rho_\ell(\pi_1(C_{\overline{k}})) \lhd G_\ell; \\ G_{\ell,c} & := \langle \varphi_{\ell,c} \rangle \subset G_\ell \ \ (c \in |C|). \end{array}$$

For each prime ℓ, fix a commutative $\mathbb{F}_\ell$-subalgebra

$$F_\ell \subset \mathrm{End}_{G_\ell}(H_\ell) \subset \mathrm{End}_{\mathbb{F}_\ell}(H_\ell).$$

(An example: $F_\ell = \mathbb{F}_\ell[Z(G_\ell)]$.) Also, given $0 \neq v \in H_\ell$ and a subgroup $G_\ell^\# \subset G_\ell$, define the subgroups:

$$\begin{array}{l} G^\#_{\langle\langle v \rangle\rangle} := \mathrm{Stab}_{G_\ell^\#}(F_\ell v) \ ; \\ G^\#_{\langle v \rangle} := \mathrm{Stab}_{G_\ell^\#}(\mathbb{F}_\ell v); \\ G^\#_v := \mathrm{Stab}_{G_\ell^\#}(v). \end{array}$$

Note that $G^\#_v \subset G^\#_{\langle v \rangle} \subset G^\#_{\langle\langle v \rangle\rangle}$ and that $G^\#_v, G^\#_{\langle v \rangle} \lhd G^\#_{\langle\langle v \rangle\rangle}$. We will also sometimes consider the representation obtained from H_ℓ after tensoring by $\mathbb{F}_{\ell^r}$. In that case, given $0 \neq v \in H_\ell \otimes \mathbb{F}_{\ell^r}$, we will write $G^\#_{\mathbb{F}_{\ell^r} v}$ for the group $\mathrm{Stab}_{G_\ell^\#}(\mathbb{F}_{\ell^r} v)$.

A point $c \in |C|$ is said to be *of supersingular type* for $\rho_\bullet$ if there exist integers $B_c, N_c \geq 1$ such that for every prime ℓ one has

$$[G_\ell : Nor_{G_\ell}(G_{\ell,c}^{N_c})] \leq B_c.$$

The notion of point of supersingular type generalizes the notion of supersingular abelian variety to the abstract setting we consider, in the following sense. Assume that $\rho_\bullet$ is the motivic torsion family arising from an abelian scheme $X \to C$. Then every $c \in |C|$ such that the fiber X_c is a supersingular abelian variety is of supersingular type. Indeed, X_c is then isogenous (over a finite extension of $k(c)$) to a product of supersingular elliptic curves thus there exists an integer $n \geq 1$ (independent of ℓ) such that for $\ell \neq p$, $\rho_{\ell,c}(\varphi_c^{2n}) = |k(c)|^n Id$ (where φ_c is the arithmetic Frobenius) lies in the center of G_ℓ.

Proposition 2.7. *Assume that Condition (U) holds, that there exists a point $c \in |C|$ of supersingular type for $\rho_\bullet$, and that*

$$\lim_{\ell \to +\infty} \min\{[G_\ell^{geo} : G_v^{geo}] \mid 0 \neq v \in H_\ell\} = +\infty.$$

Then

$$\lim_{\ell \to +\infty} \gamma_{\overline{k}}(C_0(\ell)) = +\infty,$$

and, in particular,

$$\lim_{\ell \to +\infty} \gamma_{\overline{k}}(C_1(\ell)) = +\infty.$$

Proposition 2.7 is well-adapted to the case where $X \to C$ is a universal abelian scheme over a Shimura curve, since the moduli problem a Shimura curve represents usually ensures the existence of a supersingular point (see Subsection 3.3).

Under Condition (U), the condition that

$$\lim_{\ell \to +\infty} \min\{[G_\ell^{geo} : G_v^{geo}] \mid 0 \neq v \in H_\ell\} = +\infty$$

is equivalent to the condition that

$$\lim_{\ell \to +\infty} \min\{[G_\ell^{geo} : G_{\langle v \rangle}^{geo}] \mid 0 \neq v \in H_\ell\} = +\infty$$

(see Lemma 2.8 below), which is necessary for $\lim_{\ell \to +\infty} \gamma_{\overline{k}}(C_0(\ell)) = +\infty$.

For every integer $r \geq 1$ and prime ℓ, set $H_{\ell^r} := H_\ell \otimes \mathbb{F}_{\ell^r}$ and define

$$d_0(\ell^r) := \min\{[G_\ell^{geo} : G_{\mathbb{F}_{\ell^r} v}^{geo}] \mid 0 \neq v \in H_{\ell^r}\}$$

and

$$d_1(\ell^r) := \min\{[G_\ell^{geo} : G_v^{geo}] \mid 0 \neq v \in H_{\ell^r}\}.$$

Lemma 2.8. *Assume that Condition (U) holds. Then, the following conditions are equivalent.*

(1) *For every open subgroup $\Pi \subset \pi_1(C_{\overline{k}})$, $H_\ell^\Pi = 0$ for $\ell \gg 0$;*
(2) $\lim_{\ell \to +\infty} d_0(\ell) = +\infty$*;*
(2′) $\lim_{\ell \to +\infty} d_1(\ell) = +\infty$*;*
(3) *For every integer $r \geq 1$, $\lim_{\ell \to +\infty} d_0(\ell^r) = +\infty$;*
(3′) *For every integer $r \geq 1$, $\lim_{\ell \to +\infty} d_1(\ell^r) = +\infty$.*

Proof. Condition (U) ensures that Condition (F) of [CT14a, Thm. 1.1] holds thus, up to replacing C by a connected étale cover, one may assume that for every open subgroup $\Pi \subset \pi_1(C_{\overline{k}})$, the group $\rho_\ell(\Pi)$ is generated by its order ℓ elements for $\ell \gg 0$.

The implications (3) ⇒ (2) ⇒ (2′) and (3) ⇒ (3′) ⇒ (2′) are straightforward. It remains to prove (1) ⇒ (3) and (2′) ⇒ (1).

(1) ⇒ (3): Since $H_{\ell^r} = H_\ell \otimes \mathbb{F}_{\ell^r}$ is (non-canonically) isomorphic to $H_\ell^{\oplus r}$ as a $\pi_1(C)$-module, both Conditions (U) and (1) hold for H_{ℓ^r}. Assume that there exists an integer $d \geq 1$ such that for infinitely many ℓ one has $d_0(\ell^r) \leq d$ that is, there exists $0 \neq v_\ell \in H_{\ell^r}$ such that $\deg(C_{\mathbb{F}_{\ell^r} v_\ell, \overline{k}} \to C_{\overline{k}}) \leq d$ (here, $C_{\mathbb{F}_{\ell^r} v_\ell} \to C$ denotes the connected étale cover corresponding to $\rho_\ell^{-1}(G_{\mathbb{F}_{\ell^r} v}) \subset \pi_1(C)$). As $\pi_1(C_{\overline{k}})$ acts through a topologically finitely generated quotient, there are only finitely many possibilities for the covers $C_{\mathbb{F}_{\ell^r} v_\ell, \overline{k}} \to C_{\overline{k}}$ of degree $\leq d$. Thus, up to replacing C by a connected étale cover (*viz* $\pi_1(C)$ by an open subgroup), one may assume that for infinitely many ℓ there exists $0 \neq v_\ell \in H_{\ell^r}$ such that $D_\ell := \mathbb{F}_{\ell^r} v_\ell$ is a $\pi_1(C_{\overline{k}})$-submodule. But then, the image of $\pi_1(C_{\overline{k}})$ acting on D_ℓ is both a quotient of G_ℓ^{geo} (hence generated by its order ℓ elements for $\ell \gg 0$) and a subgroup of $\mathbb{F}_{\ell^r}^\times$ (hence of prime-to-ℓ order). So $\pi_1(C_{\overline{k}})$ acts trivially on D_ℓ for $\ell \gg 0$, which contradicts (1).

(2′) ⇒ (1): Assume that there exists an open subgroup $\Pi \subset \pi_1(C_{\overline{k}})$ such that for infinitely many primes ℓ there exists $0 \neq v_\ell \in H_\ell$ such that

$$\rho_\ell(\Pi) \subset G_{v_\ell}^{geo}.$$

Then

$$d_1(\ell) \leq [G^{geo} : G_{v_\ell}^{geo}] \leq [\pi_1(C_{\overline{k}}) : \Pi],$$

which contradicts (2′). □

For instance, if $\rho_\bullet$ is the motivic torsion family arising from an abelian scheme $X \to C$, the geometric Lang-Néron Theorem [LN59] ensures that $\rho_\bullet$ satisfies Condition (1) of Lemma 2.8 as soon as X_η contains no non-trivial abelian subvariety isogenous to a k-isotrivial abelian variety.

The basic idea behind the proof of Proposition 2.7 is that a point of supersingular type for $\rho_\bullet$ provides 'enough k-rational points' on the abstract modular curves $C_0(\ell)$. Refining this idea and up to strengthening the condition on the growth of the degree, one can get rid of the condition that there exists a point of supersingular type for $\rho_\bullet$.

Proposition 2.9. *Assume that Conditions (U) and (SSgeo) hold and that*

$$\lim_{\ell\to+\infty} \min\{[G^{geo}_{\langle\langle v\rangle\rangle} : G^{geo}_v] \mid 0 \neq v \in H_\ell\} = +\infty.$$

Then

$$\lim_{\ell\to+\infty} \gamma_{\overline{k}}(C_1(\ell)) = +\infty.$$

The conditions of Proposition 2.9 are satisfied in the case of 'big symplectic geometric monodromy', that is, $\rho_\ell(\pi_1(C_{\overline{k}})) = \mathrm{Sp}(H_\ell)$ (with respect to some non-degenerate alternating bilinear form on H_ℓ) for $\ell \gg 0$. See [H08, §5] for an example of an abelian scheme $X \to C$ whose associated family of $\mathbb{F}_\ell$-linear representations has big symplectic geometric monodromy. See also [EElHK09, Prop. 5] for results about the growth of the genus.

2.3.2.2. *Growth of the arithmetic gonality.* Our results about the growth of arithmetic gonality only involve Condition (U) and the purely arithmetic conditions on the characteristic polynomials of Frobenius we formulated in Subsection 2.3.1.1.

Theorem 2.10. *Assume that Condition (U) holds and that at least one of the following two conditions holds:*
(1) There exist $c \in |C|$ and a generator $\varphi \in \Gamma_k$ such that $\rho_\bullet$ satisfies Condition φ-(R,c);
(2) There exist a generator $\varphi \in \Gamma_k$ and an integer $N \geq 1$ such that $\rho_\bullet$ is weakly φ-$\mathbb{Z}$-integral at every $c \in |C^{cpt}| \setminus |C|$ and that $\rho_\bullet$ satisfies[7] Condition φ-(R,N,c) at every $c \in |C|$.
Then

$$\lim_{\ell\to+\infty} \gamma_k(C_1(\ell)) = +\infty.$$

Let us mention that (1) and (2) of Theorem 2.10 rely on rather different arguments. More precisely, (1) is a corollary of Proposition 2.9 and, as such, its proof is based on a rather standard 'counting of rational points' method, whereas the proof of (2)[8] uses the arithmetic property of the set

$$D_C := \{[k(c) : k] \mid c \in |C|\} \subset \mathbb{Z}_{\geq 1}.$$

Also, the proof of (1) works under weaker assumptions but is not effective, whereas the proof of (2) is effective and gives a lower bound in $\ln(\ell)$.

[7] Here, we need to impose the $\mathbb{Z}$-integrality condition - not only the weak $\mathbb{Z}$-integrality one - because, in the proof, we have to consider the reduction modulo ℓ of the characteristic polynomials of the (geometric or arithmetic) Frobenius at the *infinitely many* c in $|C|$.

[8] The proof of (2) works (and is, actually, simpler) in the ℓ-adic setting as well.

3. Growth of the geometric gonality over finite fields: Proofs and applications

Before performing the proofs of Proposition 2.7 and Proposition 2.9, let us recall that finite base changes do not affect the growth of geometric gonality. More precisely, let $f : C' \to C$ be a non-constant morphism. By functoriality of étale fundamental groups, it induces a morphism $\pi_1(f) : \pi_1(C') \to \pi_1(C)$ hence a family $\rho'_\bullet$ of n-dimensional $\mathbb{F}_\ell$-linear continuous representations

$$\rho'_\ell := \rho_\ell \circ \pi_1(f) : \pi_1(C') \to \mathrm{GL}(H_\ell),\ \ell \in L$$

and for every $0 \neq v \in H_\ell$ and $i = 0, 1$ one has a commutative square of non-constant morphism of curves

$$\begin{array}{ccc} C'_{i,v} & \longrightarrow & C_{i,v} \\ \downarrow & & \downarrow \\ C' & \xrightarrow[f]{} & C \end{array}$$

with $\deg(C'_{i,v} \to C_{i,v}) \leq \deg(f)$ (more precisely, $C'_{i,v}$ can be identified with a connected component of $C_{i,v} \times_C C'$). From Fact 2.1, this implies

$$\gamma_{\overline{k}}(C_{i,v})\deg(f) \geq \gamma_{\overline{k}}(C'_{i,v}) \geq \gamma_{\overline{k}}(C_{i,v}).$$

Thus, to prove Proposition 2.7 and Proposition 2.9, we may perform finitely many finite base changes.

3.1. Proof of Proposition 2.7.

Step 1: As $\pi_1(C_{\overline{k}})$ acts through a topologically finitely generated quotient, up to replacing C by a connected étale cover, *one may assume that $G_{\ell,c} = \langle \varphi_{\ell,c} \rangle$ is normal in G_ℓ.* Let $G_{\ell,c} = G_{\ell,c}^{(\ell')} \times G_{\ell,c}^{(\ell)}$ be the unique decomposition into the direct product of a (cyclic) group $G_{\ell,c}^{(\ell')}$ of order prime to ℓ and a (cyclic) group $G_{\ell,c}^{(\ell)}$ of ℓ-power order, and $\varphi_{\ell,c}^{(\ell)}$ the image of $\varphi_{\ell,c}$ in $G_{\ell,c}^{(\ell)}$. Thus, $G_{\ell,c}^{(\ell)} = \langle \varphi_{\ell,c}^{(\ell)} \rangle$. As $G_{\ell,c}^{(\ell)}$ is characteristic in $G_{\ell,c}$, it is normal in G_ℓ. It follows from this that there exists a character $\chi : G_\ell \to (\mathbb{Z}/\ell^e\mathbb{Z})^\times$, such that, for each $\sigma \in G_\ell$, $\sigma\varphi_{\ell,c}^{(\ell)}\sigma^{-1} = (\varphi_{\ell,c}^{(\ell)})^{\chi(\sigma)}$ holds, where ℓ^e is the order of $G_{\ell,c}^{(\ell)}$. For each element $a \in \mathbb{Z}/\ell^e\mathbb{Z}$, write $\widetilde{a} \in \{0, 1, \ldots, \ell^e - 1\} \subset \mathbb{Z}$ for the unique lift of a.

Step 2: In this step, we are going to reduce to the case where $\varphi_{\ell,c}$ acts semisimply on H_ℓ. Set $\varepsilon := \varphi_{\ell,c}^{(\ell)} - 1 \in \mathbb{F}_\ell[G_\ell]$. Observe:

$$\sigma\varepsilon\sigma^{-1} = (1+\varepsilon)^{\chi(\sigma)} - 1 = \sum_{j=1}^{\widetilde{\chi(\sigma)}} \binom{\widetilde{\chi(\sigma)}}{j} \varepsilon^j = u_\sigma \varepsilon,$$

where $u_\sigma := \sum_{j=1}^{\widetilde{\chi(\sigma)}} \binom{\widetilde{\chi(\sigma)}}{j} \varepsilon^{j-1}$. As the element $\varphi_{\ell,c}^{(\ell)}$ of ℓ-power order acts unipotently on H_ℓ, ε acts nilpotently on H_ℓ. Let $m = m_{\ell,c}$ be the minimal integer such that $\varepsilon^m H_\ell = 0$. Consider the filtration $H_\ell = F_m \supset F_{m-1} \supset \cdots \supset F_1 \supset F_0 = 0$, where $F_i := H_\ell[\varepsilon^i](:= \ker(\varepsilon^i : H_\ell \to H_\ell))$.

Claim 1: *$F_i \subset H_\ell$ is G_ℓ-stable.*

Proof of Claim 1. For each $\sigma \in G_\ell$, $\sigma(H_\ell[\varepsilon^i]) = H_\ell[\sigma\varepsilon^i\sigma^{-1}]$. But, as $\sigma\varepsilon^i\sigma^{-1} = (\sigma\varepsilon\sigma^{-1})^i = (u_\sigma\epsilon)^i = u_\sigma^i\varepsilon^i$, one has $\sigma(H_\ell[\varepsilon^i]) = H_\ell[u_\sigma^i\varepsilon^i] \supset H_\ell[\varepsilon^i]$. Considering the $\mathbb{F}_\ell$-dimension, one gets $\sigma(H_\ell[\varepsilon^i]) = H_\ell[\varepsilon^i]$, as desired. □

Claim 2: *For each $0 < i < m$, the $\mathbb{F}_\ell$-linear map $F_{i+1}/F_i \to F_i/F_{i-1}$ given by the ε-multiplication is injective and induces an injective G_ℓ-equivariant map $\mathbb{P}(F_{i+1}/F_i) \to \mathbb{P}(F_i/F_{i-1})$ of projective spaces* *over* $\mathbb{F}_\ell$.

Proof of Claim 2. By definition, $F_{i+1}/F_i \to F_i/F_{i-1}$ is injective. Take any $x \in F_{i+1}/F_i$. Then, for each $\sigma \in G_\ell$,

$$\varepsilon(\sigma x) = \sigma(\sigma^{-1}\varepsilon\sigma x) = \sigma(u_{\sigma^{-1}}\varepsilon x) = \sigma(\widetilde{\chi(\sigma^{-1})}\varepsilon x) = \widetilde{\chi(\sigma^{-1})}\sigma\varepsilon(x) = \chi(\sigma)^{-1}\sigma\varepsilon(x) \in \mathbb{F}_\ell^\times \sigma\varepsilon(x),$$

where the third equality follows from the fact that $\varepsilon^2 F_{i+1} \subset F_{i-1}$. Thus, the injective map $\mathbb{P}(F_{i+1}/F_i) \to \mathbb{P}(F_i/F_{i-1})$ in question is G_ℓ-equivariant. □

For each $0 \neq v \in H_\ell$, there exists a unique $0 < i \leq m$ such that $v \in F_i \setminus F_{i-1}$. Let v_i denote the image of v in F_i/F_{i-1}. Then, by Claim 2, one has

$$\mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_\ell v) \subset \mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_\ell v_i) = \mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_\ell \varepsilon v_i) = \cdots = \mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_\ell \varepsilon^{i-1} v_i),$$

where $\varepsilon^k v_i \in F_{i-k}/F_{i-k-1}$ and, in particular, $\varepsilon^{i-1}v_i = \varepsilon^{i-1}v \in F_1$. Thus, $C_{0,v} \to C$ factors through $C_{0,\varepsilon^{i-1}v_i} \to C$. As a result, it is enough to consider the $\overline{k}$-gonality of $C_{0,v}$ for those $0 \neq v \in H_\ell$ lying in $F_1 = H_\ell[\varepsilon] \subset H_\ell$. Note that, as $\varphi_{\ell,c}^{(\ell)}$ acts trivially on F_1, the $G_{\ell,c}$-submodule $F_1 \subset H_\ell$ is semisimple. Thus, up to replacing H_ℓ with F_1, one may assume that φ_c acts semisimply on H_ℓ.

Step 3: Set $r := n!$. In this step, we are going to replace H_ℓ with $H_{\ell^r} := H_\ell \otimes_{\mathbb{F}_\ell} \mathbb{F}_{\ell^r}$ but, as already observed in (the proof of) Lemma 2.8, since H_{ℓ^r} is (non-canonically) isomorphic to $H_\ell^{\oplus r}$ as a $\pi_1(C)$-module, both Condition (U) for H_{ℓ^r} and the fact that

$$\lim_{\ell \to +\infty} d_0(\ell^r) = +\infty$$

still hold.

As the minimal polynomial of $\varphi_{\ell,c}$ has degree $\leq n$, $\varphi_{\ell,c}$ is diagonalizable over $\mathbb{F}_{\ell^r}$. As $G_{\ell,c}$ is normal in G_ℓ, the elements of G_ℓ permute the eigenspaces of $\varphi_{\ell,c}$ in H_{ℓ^r}. In particular, there exists a subgroup $S_\ell \subset G_\ell$ with $[G_\ell : S_\ell] \leq r$ whose elements preserve each eigenspace of $\varphi_{\ell,c}$ in H_{ℓ^r}. Again, as $\pi_1(C_{\overline{k}})$ acts through a topologically finitely generated quotient, up to replacing C by a connected étale cover, one may assume that each eigenspace of $\varphi_{\ell,c}$ in H_{ℓ^r} is a G_ℓ-submodule.

Write

$$H_{\ell^r} = \bigoplus_{1 \leq i \leq t} \ker(\varphi_{\ell,c} - \lambda_{\ell,i} Id)$$

as a direct sum of the eigenspaces of $\varphi_{\ell,c}$ each of which, by assumption, is a G_ℓ-submodule. For every $0 \neq v \in H_{\ell^r}$ and $1 \leq i \leq t$, let v_i denote the projection of v onto $\ker(\varphi_{\ell,c} - \lambda_{\ell,i} Id)$. Then the inclusion

$$\mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_\ell v) \subset \mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_{\ell^r} v) \subset \mathrm{Stab}_{\pi_1(C)}(\mathbb{F}_{\ell^r} v_i)$$

shows that $C_{0,v} \to C$ factors through $C_{\mathbb{F}_{\ell^r} v_i} \to C$. In particular, $\gamma_{\overline{k}}(C_{0,v}) \geq \gamma_{\overline{k}}(C_{\mathbb{F}_{\ell^r} v_i})$. As a result, it is enough to consider the $\overline{k}$-gonality of $C_{\mathbb{F}_{\ell^r} v}$ for those $0 \neq v \in H_{\ell^r}$ lying in an eigenspace of $\varphi_{\ell,c}$.

Step 4: From Step 3, for $0 \neq v_\ell \in H_{\ell^r}$ and every $\gamma \in \pi_1(C)$ one has $\gamma\sigma_c(\varphi_c)\gamma^{-1} \in \pi_1(C_{\mathbb{F}_{\ell^r} v_\ell})$, which implies that the fiber of $C_{\mathbb{F}_{\ell^r} v_\ell} \to C$ over c is totally $k(c)$-rational (Fact 2.5 (2)). Thus,

$$|C_{\mathbb{F}_{\ell^r} v_\ell}(k(c))| \geq \deg(C_{\mathbb{F}_{\ell^r} v_\ell, \overline{k}} \to C_{\overline{k}}) \geq d_0(\ell^r).$$

Hence

$$\gamma_{\overline{k}}(C_{\mathbb{F}_{\ell^r} v_\ell}) \geq \sqrt{\gamma_{k(c)}(C_{\mathbb{F}_{\ell^r} v_\ell})} \geq \sqrt{\frac{|C_{\mathbb{F}_{\ell^r} v_\ell}(k(c))|}{|k(c)|+1}} \geq \sqrt{\frac{d_0(\ell^r)}{|k(c)|+1}} \to +\infty.$$

Here, the first inequality follows from Fact 2.1 and the second one from the general fact that if C is a smooth, separated and geometrically connected curve over a finite field k with k-gonality γ, one has $|C(k)| \leq \gamma|\mathbb{P}^1(k)|$ (as can be seen by considering a degree γ non-constant rational map $f : C \to \mathbb{P}^1_k$).

3.2. **Proof of Proposition 2.9.**

Step 1: Let H_ℓ^{ss} denote the semisimplification of H_ℓ as a G_ℓ-module. From Condition (SSgeo), one has $H_\ell \tilde{\to} H_\ell^{ss}$ as a G_ℓ^{geo}-module. Thus, one may freely replace H_ℓ with H_ℓ^{ss} hence *assume that H_ℓ is a semisimple G_ℓ-module.*

Step 2: As already mentioned, Condition (U) ensures that Condition (F) of [CT14a, Thm. 1.1] holds. As Condition (SSgeo) holds by assumption, it follows from [CT14a, Cor. 3.5] that, up to replacing C by a connected étale cover, *one may assume that for every prime $\ell \gg 0$, G_ℓ^{geo} is generated by its order ℓ elements and $Z(G_\ell)G_\ell^{geo} = G_\ell$.*

Step 3: Up to replacing F_ℓ by $F_\ell[Z(G_\ell)] \subset \mathrm{End}_{G_\ell}(H_\ell)$, one may assume that F_ℓ contains $Z(G_\ell)$. For $0 \neq v \in H_\ell$, take any irreducible F_ℓ-submodule $H \subset F_\ell v$ and any $0 \neq w \in H$, so that $H = F_\ell w$. As $w \in H \subset F_\ell v$, there exists an $a \in F_\ell$, such that $w = av$. As a-multiplication $H_\ell \to H_\ell$ is G_ℓ-equivariant, one has

$$\mathrm{Stab}_{\pi_1(C)}(v) \subset \mathrm{Stab}_{\pi_1(C)}(w),$$

hence $C_{1,v} \to C$ factors through $C_{1,w} \to C$. As a result, it is enough to consider the $\overline{k}$-gonality of $C_{1,v}$ for $0 \neq v \in H_\ell$ such that $F_\ell v$ is an irreducible F_ℓ-module. Then, in particular, the image F_v of F_ℓ in $\mathrm{End}_{\mathbb{F}_\ell}(F_\ell v)$ is a finite field extension - say of degree s_v - of $\mathbb{F}_\ell$. Let H_ℓ^* denote the set of $v \in H_\ell$ such that $F_\ell v$ is an irreducible F_ℓ-module.

Step 4: Fix $\varphi_\ell \in Z(G_\ell)$ lifting a generator of G_ℓ/G_ℓ^{geo}. Write

$$\gamma_k^{\langle\langle\rangle\rangle}(\ell) := \min\{\gamma_k(C_{\langle\langle v\rangle\rangle}) \mid v \in H_\ell^*\}.$$

and

$$\gamma_{\overline{k}}^{\langle\langle\rangle\rangle}(\ell) := \min\{\gamma_{\overline{k}}(C_{\langle\langle v\rangle\rangle}) \mid v \in H_\ell^*\}.$$

Fix $v_\ell \in H_\ell^*$. As $\varphi_\ell \in G_{\langle\langle v_\ell\rangle\rangle}$ and φ_ℓ generates G_ℓ/G_ℓ^{geo}, the étale cover $C_{\langle\langle v_\ell\rangle\rangle} \to C$ corresponding to the inclusion $G_{\langle\langle v_\ell\rangle\rangle} \subset G_\ell$ is defined over k. As $C_{\langle\langle v_\ell\rangle\rangle}$ is defined over k and k is finite, Fact 2.1 yields

$$\gamma_k(C_{\langle\langle v_\ell\rangle\rangle}) \geq \gamma_{\overline{k}}(C_{\langle\langle v_\ell\rangle\rangle}) \geq \sqrt{\gamma_k(C_{\langle\langle v_\ell\rangle\rangle})}.$$

Thus, $\gamma_k^{\langle\langle\rangle\rangle}(\ell) \to +\infty$ if and only if $\gamma_{\overline{k}}^{\langle\langle\rangle\rangle}(\ell) \to +\infty$. If $\gamma_{\overline{k}}^{\langle\langle\rangle\rangle}(\ell) \to +\infty$, we are done.

Step 5: Otherwise, there exists an integer $\gamma \geq 1$ and infinitely many primes ℓ such that there exists $v_\ell \in H_\ell^*$ with $\gamma_k(C_{\langle\langle v_\ell\rangle\rangle}) \leq \gamma$. In particular, one has a degree $\leq \gamma$ non-constant rational map

$$f_{v_\ell} : C^{cpt}_{\langle\langle v_\ell\rangle\rangle} \to \mathbb{P}^1_k.$$

So, replacing k by its degree $\gamma!$ extension, one may assume that $C^{cpt}_{\langle\langle v_\ell\rangle\rangle}(k) \neq \emptyset$.

Also, from Condition (U), up to replacing C with a connected étale cover, one may assume that the images of the inertia groups $I_{c,\ell}$ are unipotent of order dividing ℓ. But, on the other hand, the cover $C_{1,v_\ell} \to C_{\langle\langle v_\ell\rangle\rangle}$ is Galois with group a subgroup of $F_{v_\ell}^\times \simeq \mathbb{Z}/(\ell^{s_{v_\ell}} - 1)$. Thus, $C_{1,v_\ell} \to C_{\langle\langle v_\ell\rangle\rangle}$ actually extends to an étale cover $C^{cpt}_{1,v_\ell} \to C^{cpt}_{\langle\langle v_\ell\rangle\rangle}$. Similarly, the morphism

$$\begin{array}{ccc} \pi_1(C_{\langle\langle v_\ell\rangle\rangle}) & \to & \mathrm{Aut}(F_\ell v_\ell) \\ \gamma & \to & v_\ell \mapsto \gamma \cdot v_\ell \end{array}$$

factors through

$$\pi_1(C_{\langle\langle v_\ell\rangle\rangle}) \twoheadrightarrow \pi_1(C^{cpt}_{\langle\langle v_\ell\rangle\rangle}).$$

Let $c \in C^{cpt}_{\langle\langle v_\ell\rangle\rangle}(k)$ and let $\sigma_c : \Gamma_k \to \pi_1(C^{cpt}_{\langle\langle v_\ell\rangle\rangle})$ denote the section it induces. Define a twisted morphism

$$\begin{array}{rccl} \alpha_{v_\ell,c}: & \pi_1(C^{cpt}_{\langle\langle v_\ell\rangle\rangle}) & \to & \mathrm{Aut}(F_\ell v_\ell) \\ & \gamma & \to & v_\ell \mapsto \sigma_c(pr(\gamma))^{-1}\gamma\cdot v_\ell \end{array}$$

and let $T_{v_\ell,c} \to C^{cpt}_{\langle\langle v_\ell\rangle\rangle}$ denote the connected abelian étale cover corresponding to $\ker(\alpha_{v_\ell,c}) \lhd \pi_1(C^{cpt}_{\langle\langle v_\ell\rangle\rangle})$. By construction

(1) $T_{v_\ell,c,\overline{k}} \simeq C^{cpt}_{1,v_\ell,\overline{k}}$ (Fact 2.5 (1), observing that $\pi_1(C^{cpt}_{\langle\langle v_\ell\rangle\rangle,\overline{k}}) \cap \ker(\alpha_{v_\ell,c}) = \mathrm{Stab}_{\pi_1(C^{cpt}_{\langle\langle v_\ell\rangle\rangle,\overline{k}})}(v_\ell) = \pi_1(C^{cpt}_{1,v_\ell,\overline{k}})$) hence $\gamma_{\overline{k}}(C_{1,v_\ell}) = \gamma_{\overline{k}}(T_{v_\ell,c})$ and

$$\deg(T_{v_\ell,c} \to C^{cpt}_{\langle\langle v_\ell\rangle\rangle}) \geq \deg(C_{1,v_\ell,\overline{k}} \to C_{\langle\langle v_\ell\rangle\rangle,\overline{k}}) = [G^{geo}_{\langle\langle v_\ell\rangle\rangle} : G^{geo}_{v_\ell}].$$

(2) The fiber of $T_{v_\ell,c} \to C^{cpt}_{\langle\langle v_\ell\rangle\rangle}$ above c is totally k-rational.

In particular,

$$\gamma_{\overline{k}}(C_{1,v_\ell}) = \gamma_{\overline{k}}(T_{v_\ell,c}) \geq \sqrt{\gamma_k(T_{v_\ell,c})} \geq \sqrt{\frac{|T_{v_\ell,c}(k)|}{|k|+1}} \geq \sqrt{\frac{[G^{geo}_{\langle\langle v_\ell\rangle\rangle} : G^{geo}_{v_\ell}]}{|k|+1}}.$$

So the conclusion follows from the last assumption in Proposition 2.9. □

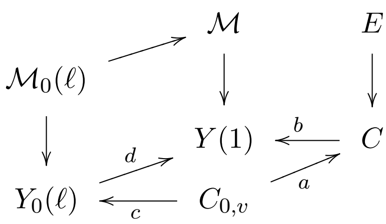
3.3. **Applications of Proposition 2.7.**

3.3.1. *Growth of the gonality of Shimura curves.* Let $Y(1)$ and $Y_0(\ell)$ denote respectively the coarse moduli schemes for the moduli stack $\mathcal{M}$ of elliptic curves and for the moduli stack $\mathcal{M}_0(\ell)$ of elliptic curves with a cyclic subgroup of order exactly ℓ as stacks over $\mathbb{Z}$. Proposition 2.7 gives a new proof of the following standard result.

Corollary 3.1. *For every prime p one has*

$$\lim_{\ell\to+\infty} \gamma_{\overline{\mathbb{F}}_p}(Y_0(\ell)_{\mathbb{F}_p}) = +\infty.$$

Proof. (See also [C12, Proof of Cor. 3].) Fix an elliptic scheme (*i.e.*, abelian scheme of relative dimension 1) $E \to C$, where C is a smooth, separated and geometrically connected curve over a finite extension F of $\mathbb{F}_p$ with generic point η. Assume that E_η is not (isogenous to) an F-isotrivial elliptic curve and that it has a supersingular fiber[9]. For every prime $\ell \neq p$ and $0 \neq v \in E_{\overline{\eta}}[\ell]$, one has the following commutative diagram (over $\overline{F}$)



where $a : C_{0,v} \to C$ (resp. $d : Y_0(\ell) \to Y(1)$) is the natural covering morphism associated to v (resp. ℓ), and $b : C \to Y(1)$ (resp. $c : C_{0,v} \to Y_0(\ell)$) is the (coarse) classifying morphism for $E \to C$ (resp. $(E \times_C C_{0,v} \to C_{0,v}, \langle v\rangle)$), which is independent of (resp. dependent on) ℓ and v.

[9]Such an elliptic scheme always exists. For instance, since $Y(N)$ is a fine moduli scheme for $N \geq 3$ ($p \nmid N$), one can take for $E \to C$ the underlying elliptic scheme of the universal object over $Y(3)$ (if $p \neq 3$) or $Y(5)$ (if $p = 3$).

In particular we can estimate the gonality as

$$\gamma_{\overline{F}}(Y_0(\ell)) \geq \frac{\gamma_{\overline{F}}(C_{0,v})}{\deg(c)} = \frac{\gamma_{\overline{F}}(C_{0,v})\deg(d)}{\deg(a)\deg(b)} = \frac{\gamma_{\overline{F}}(C_{0,v})(\ell+1)}{|G_\ell^{geo}\cdot\langle v\rangle|\deg(b)} \geq \frac{\gamma_{\overline{F}}(C_{0,v})}{\deg(b)}$$

with $\deg(b)$ independent of ℓ and v. Applying Proposition 2.7 to $E \to C$ yields the desired result. □

The standard proof of Corollary 3.1 (See for instance [Po07]) also relies on a rational points counting argument but it requires to know exactly how many supersingular elliptic curves one has over $\mathbb{F}_{p^2}$ whereas our argument only uses the existence of one supersingular elliptic curve.

The argument in the proof of Corollary 3.1 extends to more general Shimura curves (for instance, to those classifying abelian surfaces with quaternionic multiplication).

3.3.2. *An example of abelian scheme with a fiber of supersingular type which is not a supersingular abelian variety.* Let ω be a primitive third root of unity and consider the smooth, affine and geometrically connected relative curve $\mathcal{U} \to \mathcal{C}_0 := \mathbb{P}^1_{\mathbb{Q}[\omega],\lambda} \setminus \{0,1,\infty\}$ defined by the equation

$$y^3 = x^2(x-1)(x-\lambda)$$

Let $\mathcal{U} \hookrightarrow \mathcal{X}_0 \to \mathcal{C}_0$ denote a relative smooth compactification of $\mathcal{U} \to \mathcal{C}_0$ and let $S \subset \mathrm{spec}(\mathbb{Z}[\omega][\frac{1}{\ell}])$ denote a non-empty open subscheme over which $\mathcal{X}_0 \to \mathcal{C}_0$ extends to a smooth, proper and geometrically connected curve $\mathcal{X} \to \mathcal{C} := \mathbb{P}^1_{S,\lambda} \setminus \{0,1,\infty\}$ (of genus 2). Let $\mathcal{J} \to \mathcal{C}$ denote the Jacobian scheme of $\mathcal{X} \to \mathcal{C}$.

Proposition 3.2. *For every $s \in |S|$ the closed point $-1 \in |\mathcal{C}_s|$ is of supersingular type for the motivic torsion family arising from $\mathcal{J}_s \to \mathcal{C}_s$ but for every $s \in |S|$ of residue characteristic $p \equiv 1 \pmod 3$, $\mathcal{J}_{s,-1}$ is not a supersingular abelian variety.*

Proof. Write η for the generic point of $\mathcal{C}$ and for every closed point $s \in S$ let η_s denote the generic point of $\mathcal{C}_s$. Note that for every $s \in |S|$ one has $\mathcal{J}_{s,-1} \simeq \mathcal{J}_{-1,s}$, where $\mathcal{J}_{-1} \to S$ denotes the pull-back of $\mathcal{J} \to \mathcal{C}$ by the (-1)-section $S \to \mathcal{C} = \mathbb{P}^1_{S,\lambda} \setminus \{0,1,\infty\}$.

For every $s \in S$, one has a natural faithful action of $\mathbb{Z}/3$ over $\mathcal{X}_s$ which fixes $(0,0)$ given by

$$t : (x,y) \mapsto (x, \omega y).$$

Whence a group morphism $\mathbb{Z}/3 \to \mathrm{Aut}(\mathcal{J}_s)$, inducing a morphism of rings

$$\mathbb{Z}[\mathbb{Z}/3] \to \mathrm{End}(\mathcal{J}_s).$$

Claim 1: *The morphism of rings $\mathbb{Z}[\mathbb{Z}/3] \to \mathrm{End}(\mathcal{J}_{s,\eta_s})$ factors through a monomorphism of rings*

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/3] & \longrightarrow & \mathrm{End}(\mathcal{J}_{s,\eta_s}) \\ \downarrow & \nearrow & \\ \mathbb{Z}[\omega] & & \end{array}$$

Here, we write

$$\mathbb{Z}[\mathbb{Z}/3] \simeq \mathbb{Z}[T]/(T^3-1) = \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2$$

for the group ring of $\mathbb{Z}/3 = \langle t \rangle$ and $\mathbb{Z}[\omega] \simeq \mathbb{Z}[T]/(T^2+T+1) = \mathbb{Z} \oplus \mathbb{Z}\omega \subset \mathbb{Q}(\omega)$ for the ring of integers of the quadratic imaginary field $\mathbb{Q}(\omega)$.

As $\mathcal{J}_s \to \mathcal{C}_s$ is the Néron model of its generic fiber $\mathcal{J}_{s,\eta_s} \to \eta_s$, Claim 1 implies that the morphism of ring $\mathbb{Z}[\mathbb{Z}/3] \to \mathrm{End}(\mathcal{J}_s)$ also factors through a monomorphism of rings $\mathbb{Z}[\omega] \hookrightarrow \mathrm{End}(\mathcal{J}_s)$.

Proof of Claim 1. As the canonical morphism

$$\mathrm{End}(\mathcal{J}_{s,\eta_s}) \to \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(\mathcal{J}_{s,\eta_s}))$$

is a monomorphism, it is enough to show that the composite

$$\mathbb{Z}[\mathbb{Z}/3] \to \mathrm{End}(\mathcal{J}_{s,\eta_s}) \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(\mathcal{J}_{s,\eta_s}))$$

factors through a monomorphism $\mathbb{Z}[\omega] \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(\mathcal{J}_{s,\eta_s}))$. First, observe that the quotient curve $Y_s := \mathcal{X}_{s,\eta_s}/(\mathbb{Z}/3)$ has genus 0. From [Se68, VI, §4], this implies $V_\ell(\mathcal{J}_{s,\eta_s})^{\mathbb{Z}/3} = V_\ell(J_{Y_s|k(\eta_s)}) = 0$. For every $0 \neq v \in T_\ell(\mathcal{J}_{s,\eta_s})$ one has $(1+t+t^2)v \in T_\ell(\mathcal{J}_{s,\eta_s})^{\mathbb{Z}/3}$, hence $(1+t+t^2) \subset \ker(\mathbb{Z}[\mathbb{Z}/3] \to \mathrm{End}(\mathcal{J}_{s,\eta_s}))$. This inclusion is actually an equality. Otherwise, the image of $\mathbb{Z}[\mathbb{Z}/3]$ in $\mathrm{End}(\mathcal{J}_{s,\eta_s})$ would be a proper quotient of $\mathbb{Z}[\omega]$, hence finite (and non-trivial), while $\mathrm{End}(\mathcal{J}_{s,\eta_s})$ is torsion-free. □

We now specialize to $\lambda = -1$. Then $\mathcal{X}_{-1}$ has an extra order 2 automorphism τ given by $(x,y) \mapsto (-x,y)$. The quotient curve $E_{-1} := \mathcal{X}_{-1}/\langle\tau\rangle$ is given by the equation $y^3 = z(z-1) = (z-\frac{1}{2})^2 - \frac{1}{4}$ or, setting $z' := z - \frac{1}{2}$ by

$$y^3 = (z')^2 - \frac{1}{4}.$$

Thus, E_{-1} is an elliptic curve having CM by $\omega : (z',y) \mapsto (z',\omega y)$. The induced morphism $\mathcal{J}_{-1} \to E_{-1}$ is $\mathbb{Z}/3$-equivariant hence the connected component E of $\ker(\mathcal{J}_{-1} \to E_{-1})$ is also an elliptic curve with CM by ω. As the class number of $\mathbb{Z}[\omega]$ is 1, E and E_{-1} become isomorphic over a finite extension k of $\mathbb{Q}(\omega)$. Let T denote the inverse image of S in $\mathrm{spec}(\mathcal{O}_k)$, where $\mathcal{O}_k$ is the ring of integers of k. Then $\mathcal{J}_{-1,T}$ is isogenous to $E_{-1} \times_T E_{-1}$ over T.

Now, fix a closed point $s \in T$ with residue field of characteristic $p \equiv 1 \pmod 3$. As p splits in $\mathbb{Z}[\omega]$, the elliptic curve $E_{-1,s}$ is ordinary. In particular, $\mathcal{J}_{-1,s}$ is *not supersingular*. However, considering the family

$$\rho_\ell : \pi_1(\mathcal{C}_s) \to \mathrm{GL}(\mathcal{J}_{\eta_s}[\ell]),\ \ell \neq p,$$

Claim 2: *For every prime $\ell \neq p$ one has $G_{\ell,-1} \subset Z(G_\ell)$ (hence, in particular, $-1 \in \mathcal{C}_{\lambda,s}$ is of supersingular type).*

Proof of Claim 2. Let F denote the Frobenius of $E_{-1,s}$. Then $\mathbb{Z}[F] \subset \mathrm{End}(E_{-1,s})$ is a rank-2 free $\mathbb{Z}$-module and F has characteristic polynomial of the form $\chi_F = T^2 - aT + |k(s)| \in \mathbb{Z}[T]$ with $\Delta = a^2 - 4|k(s)| < 0$ (ordinary case). In particular, F is integral over $\mathbb{Z}$ and $\mathbb{Q}[F]$ is 2-dimensional. As $\mathbb{Q}[F] = \mathrm{End}(E_{-1,s}) \otimes_{\mathbb{Z}} \mathbb{Q} \supset \mathbb{Q}[\omega]$, this implies

$$\mathbb{Q}[F] = \mathbb{Q}[\omega]$$

hence $\mathbb{Z}[F] \subset \mathbb{Z}[\omega]$ so $F \in \mathbb{Z}[\omega]$.

But the image of the representation

$$\rho_{\ell^\infty} : \pi_1(\mathcal{C}_s) \to \mathrm{GL}(T_\ell(\mathcal{J}_{s,\overline{\eta}_s}))$$

commutes with $\mathbb{Z}[\omega] \subset \mathrm{End}(\mathcal{J}_{s,\eta_s})$, hence with $F(= \rho_{\ell^\infty} \circ \sigma_{-1}(\varphi_{-1}))$ (where φ denotes the arithmetic Frobenius). The conclusion follows by reduction modulo ℓ. □

4. Growth of the arithmetic gonality over finite fields: Proofs

4.1. **Proof of Theorem 2.10 (1).**

4.1.1. *Proof of Theorem 2.10 (1) under Condition (SSgeo).* We prove first Theorem 2.10 (1) under Condition (SSgeo). Recall (Subsection 3.2, Step 2) that, under this assumption, up to replacing C by a connected étale cover, one may assume that for every prime $\ell \geq 0$, G_ℓ^{geo} is generated by its order ℓ elements and $Z(G_\ell)G_\ell^{geo} = G_\ell$. For simplicity, write $Z_\ell := Z(G_\ell)$.

Fix $\varphi_\ell \in Z_\ell$ lifting a generator of G_ℓ/G_ℓ^{geo}. For every $0 \neq v \in H_\ell$ and subgroup $G_\ell^\# \subset G_\ell$, set

$$G^\#_{Z_\ell v} := \{g \in G^\# \mid gv \in Z_\ell v\}.$$

$G^\#_{Z_\ell v}$ is a subgroup of G_ℓ such that $G_v^\# \lhd G^\#_{Z_\ell v} \lhd G^\#_{\langle\langle v\rangle\rangle}$, where $G^\#_{\langle\langle v\rangle\rangle}$ is considered with respect to $F_\ell := \mathbb{F}_\ell[Z_\ell]$. Let $C_{Z_\ell v} \to C$ denote the connected étale cover corresponding to the inclusion of subgroup $G_{Z_\ell v} \subset G_\ell$ and let $k_{Z_\ell v}$ denote its field of definition. One has

$$\begin{aligned} [G_{Z_\ell v} : G_v] &= [G_{Z_\ell v} : G^{geo}_{Z_\ell v} G_v][G^{geo}_{Z_\ell v} G_v : G_v] \\ &= [G_{Z_\ell v}/G^{geo}_{Z_\ell v} : G^{geo}_{Z_\ell v} G_v / G^{geo}_{Z_\ell v}][G^{geo}_{Z_\ell v} : G^{geo}_v] \\ &= \frac{[k_{1,v} : k]}{[k_{Z_\ell v} : k]}[G^{geo}_{Z_\ell v} : G^{geo}_v]. \end{aligned}$$

As $\varphi_\ell \in G_{Z_\ell v}$ and φ_ℓ generates G_ℓ/G_ℓ^{geo}, one has $k_{Z_\ell v} = k$. Hence

$$[G_{Z_\ell v} : G_v] = [k_{1,v} : k][G^{geo}_{Z_\ell v} : G^{geo}_v].$$

Write

$$d(\ell) := \min\{[G_{Z_\ell v} : G_v] \mid 0 \neq v \in H_\ell\}.$$

Claim: $\lim_{\ell \to +\infty} d(\ell) = +\infty$.

Proof of: Claim $\Longrightarrow$ Theorem 2.10 (1). For every integer $d \geq 1$ define

$$\mathcal{H}_\ell(d,1) := \{0 \neq v \in H_\ell \mid [k_{1,v} : k] \leq d\}, \quad \mathcal{H}_\ell(d,2) := \{0 \neq v \in H_\ell \mid \gamma_{\overline{k}}(C_{1,v}) \leq d\}.$$

It is enough to show that $\mathcal{H}_\ell(d,1) \cap \mathcal{H}_\ell(d,2) = \emptyset$ for $\ell \gg 0$. Otherwise, there would exist an infinite subset $L' \subset L$ and for each $\ell \in L'$ an element $0 \neq v_\ell \in H_\ell$ such that $[k_{1,v_\ell} : k] \leq d$ and $\gamma_{\overline{k}}(C_{v_\ell}) \leq d$. Then it follows from the claim that

$$[G^{geo}_{\langle\langle v_\ell\rangle\rangle} : G^{geo}_{v_\ell}] \geq [G^{geo}_{Z_\ell v_\ell} : G^{geo}_{v_\ell}] = \frac{[G_{Z_\ell v_\ell} : G_{v_\ell}]}{[k_{1,v_\ell} : k]} \geq \frac{d(\ell)}{d} \to +\infty.$$

But this contradicts Proposition 2.9.

Proof of the Claim. Assume that ℓ is large enough so that the reduction modulo ℓ of P_c is non-zero and choose $0 \neq v_\ell \in H_\ell$ such that $[G_{Z_\ell v_\ell} : G_{v_\ell}] = d(\ell)$. One has a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{v_\ell} & \longrightarrow & G_{Z_\ell v_\ell} & \longrightarrow & Z_\ell v_\ell & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & Z_{\ell,v_\ell} & \longrightarrow & Z_\ell & \longrightarrow & Z_\ell v_\ell & \longrightarrow & 1 \end{array}$$

(Here, we implicitly equip $Z_\ell v_\ell$ with the group structure induced by $Z_\ell/Z_{\ell,v_\ell} \tilde{\to} Z_\ell v_\ell$). In particular

$$d(\ell) = [G_{Z_\ell v_\ell} : G_{v_\ell}] = |Z_\ell v_\ell| = [Z_\ell : Z_{\ell,v_\ell}].$$

Write $M_\ell := \mathbb{F}_\ell[G_\ell v_\ell] \subset H_\ell$ for the G_ℓ-submodule generated by v_ℓ and r_ℓ for its $\mathbb{F}_\ell$-dimension. Let

$$\det{}_\ell : G_\ell \overset{|_{M_\ell}}{\to} \mathrm{GL}(M_\ell) \overset{\det}{\to} \mathrm{GL}(\Lambda^{r_\ell} M_\ell) \simeq \mathbb{F}_\ell^\times$$

the determinant representation of G_ℓ acting on M_ℓ. As G_ℓ^{geo} is generated by order ℓ elements, $\det_\ell : G_\ell \to \mathbb{F}_\ell^\times$ factors through G_ℓ/G_ℓ^{geo} hence $\mathrm{im}(\det_\ell) = \det_\ell(Z_\ell)$. As Z_{ℓ,v_ℓ} acts trivially on M_ℓ, one has

$$d(\ell) = [Z_\ell : Z_{\ell,v_\ell}] \geq |\mathrm{im}(\det_\ell)| =: f_\ell.$$

Thus, it is enough to show that $\lim_{\ell\to+\infty} f_\ell = +\infty$.

Up to replacing k with a finite extension, one may assume that $k(c) = k$. By definition $\varphi_{\ell,c}^{f_\ell}$ acts trivially on $\Lambda^{r_\ell} M_\ell$, which is a sub-vector space of $\Lambda^{r_\ell} H_\ell$. Recall that, with the notation of Subsection 2.3.1.1, $\varphi_{\ell,c}$ acts on H_ℓ with characteristic polynomial the monic polynomial associated to the reduction modulo ℓ of

$$P_c = a_c \prod_{1\leq i\leq n} (T - \alpha_{c,i}) \in \mathbb{Z}[T].$$

Lemma 4.1. *Let P be a degree n polynomial in $\mathbb{Z}[T]$, and write $P(T) = a \prod_{1\leq i\leq n} (T - \alpha_i)$ with $a \in \mathbb{Z}$, $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$. Then, for every pair of positive integers r and f,*

$$P(r, f, T) := a^{\binom{n}{r}rf} \prod_{1\leq i_1<\cdots<i_r\leq n} (T - (\alpha_{i_1} \cdots \alpha_{i_r})^f) \in \mathbb{Z}[T].$$

As $\varphi_{\ell,c}^{f_\ell}$ acts on $\Lambda^{r_\ell} H_\ell$ with characteristic polynomial the monic polynomial associated to the reduction modulo ℓ of $P_c(r_\ell, f_\ell, T)$, the prime ℓ divides $|P_c(r_\ell, f_\ell, T)(1)|$, which is an integer ≥ 1 (by our assumptions on $P_c(T)$) and less than

$$|a_c|^{\binom{n}{r_\ell}r_\ell f_\ell} \prod_{1\leq i_1<\cdots<i_{r_\ell}\leq n} (1 + |\alpha_{c,i_1} \cdots \alpha_{c,i_{r_\ell}}|^{f_\ell}) \leq |a_c|^{\binom{n}{r_\ell}r_\ell f_\ell}(1 + b_c^{f_\ell r_\ell})^{\binom{n}{r_\ell}} \leq |a_c|^{n!nf_\ell}(1 + b_c^{f_\ell n})^{n!},$$

where $b_c = \max\{|\alpha_{c,i}| \mid i = 1, \dots, n\}$. Hence $\ell \leq |a_c|^{n!nf_\ell}(1 + b_c^{f_\ell n})^{n!}$, which implies

$$\lim_{\ell\to+\infty} f_\ell = +\infty.$$

Proof of Lemma 4.1. The assertion of Lemma 4.1 follows from the combination of (1) and (2) below.

(1) $a^{\binom{n}{r}r} \prod_{1\leq i_1<\cdots<i_r\leq n} (T - \alpha_{i_1} \cdots \alpha_{i_r}) \in \mathbb{Z}[T]$. Let $\sigma_j := \sigma_j^n(\alpha_1, \dots, \alpha_n)$ denote the jth fundamental symmetric polynomial in $\alpha_1, \dots, \alpha_n$. By assumption $a\sigma_j \in \mathbb{Z}$, $j = 1, \dots, n$. Write

$$\prod_{1\leq i_1<\cdots<i_r\leq n} (T - \alpha_{i_1} \cdots \alpha_{i_r}) =: T^N + A_1T^{N-1} + \cdots + A_N,$$

where $N := \binom{n}{r}$. Then A_k can be written as a $\mathbb{Z}$-linear combination of $\sigma_1^{m_1} \cdots \sigma_n^{m_n}$, $m_1, \dots, m_n \geq 0$. Regarded as polynomials in the $\alpha_1, \dots, \alpha_n$, A_k is homogeneous of degree kr whereas $\sigma_1^{m_1} \cdots \sigma_n^{m_n}$ has degree $\sum_{i=1}^n im_i$. Thus,

$$kr = \sum_{i=1}^n im_i \geq \sum_{i=1}^n m_i.$$

This shows that $a^{kr}A_k \in \mathbb{Z}$, $k = 1, \dots, N$, hence $a^{Nr}A_k \in \mathbb{Z}$, $k = 1, \dots, N$, as claimed.

(2) $a^f \prod_{1\leq i\leq n} (T - \alpha_i^f) \in \mathbb{Z}[T]$. Write

$$P_0 := \prod_{1\leq i\leq n} (T - \alpha_i) = T^n - \sigma_1T^{n-1} + \cdots + (-1)^n\sigma_n.$$

For every prime p let v_p denote the (additive) p-adic valuation on $\overline{\mathbb{Q}}_p$ with $v_p(p) = 1$. Recall that the Newton polygon of P_0 associated to v_p is the convex hull of

$$\{(i, v_p(\sigma_i)) \mid i = 0, \ldots, n\} \cup ([0, n] \times \{+\infty\}) \subset \mathbb{R} \times (\mathbb{R} \cup \{+\infty\})$$

(where we set $\sigma_0 := 1$) and that the successive slopes $\lambda_1, \ldots, \lambda_n$ are given by the $v_p(\alpha_i)$. Thus, the condition that $a \prod_{1 \leq i \leq n}(T - \alpha_i)$ has coefficients in $\mathbb{Z}$ is equivalent to the condition that for every prime p one has

$$(v_p(a) + v_p(\sigma_i) =)\ v_p(a) + \sum_{1 \leq j \leq i} \lambda_j \geq 0,\ i = 1, \ldots, n.$$

Since $f \geq 1$, this implies that $v_p(a^f) + \sum_{1 \leq j \leq i} f\lambda_j \geq 0$, $i = 1, \ldots, n$. As the $f\lambda_1, \ldots, f\lambda_n$ are the slopes of the Newton polygon of $\prod_{1 \leq i \leq n}(T - \alpha_i^f)$, the conclusion follows. □

4.1.2. *End of the proof of Theorem 2.10 (1).* We now prove the general case of Theorem 2.10 (1).

Let H_ℓ^{ss} denote the semisimplification of H_ℓ as a $\pi_1(C)$-module. As $\pi_1(C_{\overline{k}})$ is normal in $\pi_1(C)$, H_ℓ^{ss} is also a semisimple $\pi_1(C_{\overline{k}})$-module. Thus, the resulting family

$$\rho_\bullet^{ss} := (\rho_\ell^{ss} : \pi_1(C) \to \mathrm{GL}(H_\ell^{ss})),\ \ell \in L$$

satisfies Condition (SSgeo). Also, $\rho_\bullet^{ss}$ satisfies Condition (U) and Condition φ-(R,c) as $\rho_\bullet$ does. Hence, setting

$$C_1^{ss}(\ell) := \bigsqcup_{0 \neq v \in H_\ell^{ss}} C_{1,v},$$

it follows from Subsection 4.1.1 that

$$\lim_{\ell \to +\infty} \gamma_k(C_1^{ss}(\ell)) = +\infty.$$

For every prime ℓ, let again $0 \neq v_\ell \in H_\ell$ such that $\gamma_k(C_{v_\ell}) = \gamma_k(C_1(\ell))$. Considering the socle filtration

$$H_\ell = S_{n_{\ell,c},c}(H_\ell) \supset S_{n_{\ell,c}-1,c}(H_\ell) \supset \cdots \supset S_{0,c}(H_\ell) = 0$$

for H_ℓ as a $\pi_1(C)$-module, there exists a unique $1 \leq i \leq n_\ell$ such that $v_\ell \in S_{i,c}(H_\ell) \setminus S_{i-1,c}(H_\ell)$. Let $v_{\ell,i}$ denote the projection of v_ℓ onto $S_{i,c}(H_\ell)/S_{i-1,c}(H_\ell)(\hookrightarrow H_\ell^{ss})$. It then follows from the inclusion $\mathrm{Stab}_{\pi_1(C)}(v_\ell) \subset \mathrm{Stab}_{\pi_1(C)}(v_{\ell,i})$ (which corresponds to an étale cover $C_{1,v_\ell} \to C_{1,v_{\ell,i}}$) that

$$\gamma_k(C_1(\ell)) = \gamma_k(C_{1,v_\ell}) \geq \gamma_k(C_{1,v_{\ell,i}}) \geq \gamma_k(C_1^{ss}(\ell)) \to +\infty.$$

4.2. **Proof of Theorem 2.10 (2).**

4.2.1. *An elementary lemma on k-gonality.* Given a field Q and a smooth, proper, geometrically connected curve C over a finite extension Q_C of Q, set

$$D_C := \{[Q(c) : Q] \mid c \in |C|\} \subset \mathbb{Z}_{\geq 1}.$$

Lemma 4.2. *Assume that $D_{\mathbb{P}^1_Q} = \mathbb{Z}_{\geq 1}$. Then for every integer $d \geq \gamma_Q(C)$ and $m \geq 1$ one has*

$$D_C \cap \{m, 2m, \ldots, dm\} \neq \emptyset.$$

Proof. Let $f : C \to \mathbb{P}^1_Q$ be a non-constant (hence surjective) morphism of degree $\leq d$. As $D_{\mathbb{P}^1_Q} = \mathbb{Z}_{\geq 1}$, one can find $t_m \in \mathbb{P}^1_Q$ such that $[Q(t_m) : Q] = m$. Pick any $c \in f^{-1}(t_m)$ and just observe that

$$[Q(c) : Q] = [Q(c) : Q(t_m)][Q(t_m) : Q] = [Q(c) : Q(t_m)]m \in D_C \cap \{m, 2m, \ldots, dm\}. \ \square$$

Remark 4.3. The assumption $D_{\mathbb{P}^1_Q} = \mathbb{Z}_{\geq 1}$ simply means that $Q[T]$ contains degree d irreducible polynomials for every integer $d \geq 1$. This holds for instance if Q is finitely generated or finitely generated over $\mathbb{Q}_p$.

In particular (take $m = 2$ in Lemma 4.2), if $D_C \subset \{1\} \cup \mathbb{Z}_{\geq A}$ then $\gamma_Q(C) \geq \frac{A}{2}$. So, to exploit Lemma 4.2, we will have to estimate the degree of the residue field of closed points; this can be done group-theoretically as follows. For every prime ℓ and $0 \neq v \in H_\ell$ write $\pi_v : C^{cpt}_{1,v} \to C^{cpt}$. For every $c \in |C^{cpt}|$, write $I_{\ell,c}$ and $G_{\ell,c}$ for the image of I_c and D_c in G_ℓ respectively. For any $c_v \in \pi_v^{-1}(c)$, one can choose D_c in such a way that $D_{c_v} := D_c \cap \pi_1(C_{1,v})$ be a decomposition group of c_v in $\pi_1(C_{1,v})$. Then one can recover $[k(c_v) : k(c)]$ group-theoretically as

$$[k(c_v) : k(c)] = [\Gamma_{k(c)} : \Gamma_{k(c_v)}] = [D_{\ell,c}/I_{\ell,c} : D_{\ell,c_v} I_{\ell,c}/I_{\ell,c}] = [D_{\ell,c} : D_{\ell,c_v} I_{\ell,c}].$$

4.2.2. *Proof of Theorem 2.10 (2).* Up to replacing C with a connected étale cover one may assume that

- The points in $|C^{cpt}| \setminus |C|$ are all k-rational;
- For every $c \in |C^{cpt}| \setminus |C|$ and prime ℓ, I_c acts unipotently on H_ℓ. In particular, in condition (U) of Subsection 2.3.1.1, one can take $D'_c = D_c$ and $I'_c = I_c$;
- For every $c \in |C^{cpt}| \setminus |C|$, Q_c decomposes as

  $$Q_c(T) = (T-1)^{r_c - t_c} Q'_c(T)$$

  where none of the roots $\beta_{c,1}, \dots \beta_{c,t_c}$ of Q'_c is a root of unity or 0. Here, $0 \neq Q_c \in \mathbb{Z}[T]$ is the polynomial introduced in Subsection 2.3.1.1, whose image in $\mathbb{F}_\ell[T]$ is divisible by the characteristic polynomial of $\varphi_{\ell,c} := \rho^{ss}_{\ell,c}(\varphi_c)$.

For every prime $\ell \in L$, fix $0 \neq v_\ell \in H_\ell$ such that

$$\gamma_k(C_{1,v_\ell}) = \gamma_k(C_1(\ell)).$$

Our aim is to prove that there exists a sequence $(A_\ell)_{\ell \in L}$ of positive integers such that $\lim_{\ell \to +\infty} A_\ell = +\infty$ and

$$D_{C^{cpt}_{1,v_\ell}} \subset \{1\} \cup \mathbb{Z}_{\geq A_\ell}.$$

To estimate A_ℓ, fix $c_\ell \in |C^{cpt}_{1,v_\ell}|$ lying over $c \in |C^{cpt}|$. We distinguish between two cases:

Case 1: $c \in |C|$**.** Then one has $v_\ell \in H_\ell^{\Gamma_{k(c_\ell)}}$ so 1 is an eigenvalue for $\varphi_{\ell,c}^{[k(c_\ell):k(c)]}$. This implies

$$\ell | \prod_{1 \leq i \leq n} (1 - \alpha_{c,i}^{[k(c_\ell):k(c)]})$$

hence

$$\ell \leq \prod_{1 \leq i \leq n} (1 + |\alpha_{c,i}|^{[k(c_\ell):k(c)]}) \leq (1 + |k|^{N[k(c_\ell):k]})^n.$$

So

$$[k(c_\ell) : k] \geq \frac{\ln(\ell^{\frac{1}{n}} - 1)}{N \ln(|k|)} =: B_\ell.$$

Case 2: $c \in |C^{cpt}| \setminus |C|$**.** Then $k(c) = k$ by assumption.

Assume that $\ell \neq p$, $\ell \geq n$ (so that $\mathrm{GL}(H_\ell)$ contains no element of order ℓ^2) and that ℓ is large enough so that the reduction modulo ℓ of Q_c is non-zero. As I_c is an extension of $\widehat{\mathbb{Z}}^{(p')}(1)$ (as a $\Gamma_{k(c)}$-module) by a pro-p group and I_c acts unipotently on H_ℓ, this action factors through its unique order ℓ quotient

$I_{c,\ell} = \langle\gamma\rangle \simeq \mathbb{F}_\ell(1)$. Let $I_c(\ell)$ denote the (characteristic) kernel of $I_c \twoheadrightarrow I_{c,\ell}$ and write $D_{c,\ell} := D_c/I_c(\ell)$. Thus, $D_{c,\ell}$ fits into the exact sequence (recall that $k(c) = k$)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_{c,\ell} & \longrightarrow & D_{c,\ell} & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\ && \| &&&& \| && \\ && \mathbb{F}_\ell(1) = \langle\gamma\rangle &&&& \widehat{\mathbb{Z}} = \langle\varphi\rangle && \end{array}$$

Set

$$I_{c,\ell,v_\ell} := \mathrm{Stab}_{I_{c,\ell}}(v_\ell) \subset I_{c,\ell} \ \text{ and } \ D_{c,\ell,v_\ell} := \mathrm{Stab}_{D_{c,\ell}}(v_\ell) \subset D_{c,\ell}.$$

By definition, one has the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_{c,\ell} & \longrightarrow & D_{c,\ell} & \xrightarrow{pr} & \Gamma_k & \longrightarrow & 1 \\ && \uparrow && \uparrow && \uparrow && \\ 1 & \longrightarrow & I_{c,\ell,v_\ell} & \longrightarrow & D_{c,\ell,v_\ell} & \xrightarrow{pr} & \Gamma_{k(c_\ell)} & \longrightarrow & 1 \end{array}$$

and the two exact sequences split (say, because Γ_k and $\Gamma_{k(c_\ell)}$ are free profinite groups). To estimate $[k(c_\ell) : k] = [\Gamma_k : \Gamma_{k(c_\ell)}]$, we are going to choose appropriate complements Φ and Φ_{v_ℓ} of $I_{c,\ell}$ and I_{c,ℓ,v_ℓ} in $D_{c,\ell}$ and D_{c,ℓ,v_ℓ}, respectively.

Let $W_\ell := \mathbb{F}_\ell[D_{c,\ell}v_\ell] \subset H_\ell$ denote the $D_{c,\ell}$-submodule generated by v_ℓ. We distinguish between two subcases

(1) $I_{c,\ell,v_\ell} = 1$. Set $\tilde{D}_{c,\ell,v_\ell} := pr^{-1}(\Gamma_{k(c_\ell)}) \subset D_{c,\ell}$. Then one has by construction:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_{c,\ell} & \longrightarrow & D_{c,\ell} & \xrightarrow{pr} & \Gamma_k & \longrightarrow & 1 \\ && \| && \uparrow & \square & \uparrow && \\ 1 & \longrightarrow & I_{c,\ell} & \longrightarrow & \tilde{D}_{c,\ell,v_\ell} & \xrightarrow{pr} & \Gamma_{k(c_\ell)} & \longrightarrow & 1 \\ && \uparrow && \uparrow && \| && \\ 1 & \longrightarrow & 1 & \longrightarrow & D_{c,\ell,v_\ell} & \xrightarrow{pr} & \Gamma_{k(c_\ell)} & \longrightarrow & 1 \end{array}$$

In particular, D_{c,ℓ,v_ℓ} provides a complement to $I_{c,\ell}$ in $\tilde{D}_{c,\ell,v}$; let

$$\sigma : \Gamma_{k(c_\ell)} \overset{pr^{-1}}{\tilde{\to}} D_{c,\ell,v_\ell} \hookrightarrow \tilde{D}_{c,\ell,v_\ell}$$

denote the corresponding section. Consider the following commutative diagram describing how the sections of the two first rows are related.

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma_k, I_{c,\ell}) & \xrightarrow{res} & \mathrm{H}^1(\Gamma_{k(c_\ell)}, I_{c,\ell}) \\ \| && \| \\ k^\times/(k^\times)^\ell & \longrightarrow & k(c_\ell)^\times/(k(c_\ell)^\times)^\ell \end{array} \quad ,$$

where the vertical arrows are the Kummer isomorphisms (recall that $I_{c,\ell} \simeq \mathbb{F}_\ell(1)$ as a Γ_k-module), the upper horizontal arrow is the restriction and the lower horizontal arrow is induced by the inclusion.

We distinguish again between two subcases:

(a) $k(c_\ell)^\times/(k(c_\ell)^\times)^\ell \neq 1$. Then $\ell||k(c_\ell)^\times|$ hence $[k(c_\ell) : k] \geq \frac{\ln(\ell+1)}{\ln(|k|)}$.

(b) $k(c_\ell)^\times/(k(c_\ell)^\times)^\ell = 1$. Then the section $\sigma : \Gamma_{k(c_\ell)} \hookrightarrow \tilde{D}_{c,\ell,v_\ell}$ extends to a section $\Gamma_k \hookrightarrow D_{c,\ell}$ of $pr : D_{c,\ell} \twoheadrightarrow \Gamma_k$. Let $\Phi \subset_{cl} D_{c,\ell}$ denote the corresponding complement of $I_{c,\ell}$. By construction $\Phi_{v_\ell} := \tilde{D}_{c,\ell,v_\ell} \cap \Phi = D_{c,\ell,v_\ell}$.

(2) $I_{c,\ell,v_\ell} = I_{c,\ell}$. Then, as $I_{c,\ell}$ is normal in $D_{c,\ell}$ and acts trivially on v_ℓ, it also acts trivially on the whole W_ℓ. Hence the action of $D_{c,\ell}$ on W_ℓ factors through $D_{c,\ell} \twoheadrightarrow \Gamma_k \simeq D_{c,\ell}/I_{c,\ell}$. Let $\Phi \subset D_{c,\ell}$ be any complement of $I_{c,\ell}$ and set $\Phi_{v_\ell} := D_{c,\ell,v_\ell} \cap \Phi$. Then $D_{c,\ell,v_\ell} = I_{c,\ell}\Phi_{v_\ell}$ and $pr(\Phi_{v_\ell}) = pr(D_{c,\ell,v_\ell}) = \Gamma_{k(c_\ell)}$.

In cases (1) (b) and (2), we have

$$[\Phi : \Phi_{v_\ell}] = [k(c_\ell) : k].$$

Let

$$V_\ell := \mathbb{F}_\ell[\Phi v_\ell] \subset W_\ell$$

denote the Φ-submodule generated by v_ℓ. As Φ is abelian (in fact, even cyclic), one has

$$\Phi_{v_\ell} = \ker(\Phi \to \mathrm{GL}(V_\ell))$$

hence

$$[k(c_\ell) : k] = |\mathrm{im}\ (\Phi \to \mathrm{GL}(V_\ell))|.$$

Then, either ℓ divides $|\mathrm{im}\ (\Phi \to \mathrm{GL}(V_\ell))|$ (hence $[k(c_\ell) : k] \geq \ell$) or Φ acts semisimply on V_ℓ. In the latter case, writing H_ℓ^{ss} for the semisimplification of H_ℓ as a $D_{c,\ell}$-module, one has a monomorphism $V_\ell \hookrightarrow H_\ell^{ss}$ of Φ-modules.

But as a $\Phi \simeq \Gamma_k$-module, H_ℓ^{ss} decomposes as

$$H_\ell^{ss} = \underbrace{\ker(Q'_c(\rho^{ss}_{\ell,c}(\varphi))}_{:=H_\ell^{ss\prime}} \oplus \underbrace{\ker(\rho^{ss}_{\ell,c}(\varphi) - Id)}_{:=H_\ell^{ss\circ}}.$$

Since $v_\ell \in V_\ell \subset H_\ell^{ss}$, one can write $v_\ell = v'_\ell + v^\circ_\ell$ with $v'_\ell \in H_\ell^{ss\prime}$ and $v^\circ_\ell \in H_\ell^{ss\circ}$. If $v'_\ell = 0$, then $[\Phi : \Phi_{v_\ell}] = 1$ *i.e.* $k(c_\ell) = k$. If $v'_\ell \neq 0$, then

$$[k(c_\ell) : k] = [\Phi : \Phi_{v_\ell}] \geq [\Phi : \Phi_{v'_\ell}] = [k(c'_\ell) : k],$$

where c'_ℓ is the image of c_ℓ *via* $C^{cpt}_{1,v_\ell} \to C^{cpt}_{1,v'_\ell}$. But then (see (2) in the proof of Lemma 4.1),

$$\ell \leq |a_c|^{[k(c'_\ell):k]} \prod_{1 \leq i \leq t_c} (1 + |\beta_{c,i}|^{[k(c'_\ell):k]}) \leq |a_c|^{[k(c'_\ell):k]}(1 + b^{[k(c'_\ell):k]})^{t_c} \leq a^{[k(c'_\ell):k]}(1 + b^{[k(c'_\ell):k]})^t,$$

where a_c denotes the leading coefficient of Q'_c and

$$a := \max\{|a_c| \mid c \in |C^{cpt}|\backslash|C|\},\ b := \min\{|\beta_{c,i}| \mid c \in |C^{cpt}|\backslash|C|,\ i = 1, \dots, t_c\},\ \ t := \max\{t_c \mid c \in |C^{cpt}|\backslash|C|\}.$$

As a result

$$[k(c_\ell) : k] \geq \frac{\ln(\ell)}{\ln(a) + t\ln(1 + b)}.$$

To sum it up, we have shown that there exists a sequence B_ℓ^{cusp} such that $\lim_{\ell \to +\infty} B_\ell^{cusp} = +\infty$ and for every $c_\ell \in |C^{cpt}_{1,v_\ell}|$ lying over $c \in |C^{cpt}| \setminus |C|$ one has $[k(c_\ell) : k] = 1$ or $[k(c_\ell) : k] \geq B_\ell^{cusp}$.

Setting $A_\ell := \min\{B_\ell, B_\ell^{cusp}\}$ one has

$$D_{C^{cpt}_{1,v_\ell}} \subset \{1\} \cup \mathbb{Z}_{\geq A_\ell}$$

hence

$$\gamma_k(C_1(\ell)) \geq \frac{A_\ell}{2} \to +\infty.$$

5. Results over finitely generated fields

We now turn to the general situation where k is a finitely generated field of characteristic $p \geq 0$ with prime field F. Let $\mathbb{F}$ denote $\mathbb{Z}$ if $p = 0$ and $\mathbb{F}_p$ if $p > 0$.

5.1. **Statements.** We give here a sample of statements which can be derived from the statements of Subsection 2.3 by the specialization method we explain in Subsection 5.2 below.

Let $X \to C$ be a smooth proper morphism. Up to enlarging k, we may assume that C admits a (unique) smooth compactification $C \subset C^{cpt}$ with $C^{cpt} \setminus C$ étale over k. We call *an $\mathbb{F}$-model of* $X \to C \hookrightarrow C^{cpt} \to \mathrm{spec}(k) \to \mathrm{spec}(F)$ the data of

$$\mathcal{X} \to \mathcal{C} \hookrightarrow \mathcal{C}^{cpt} \to T \to U,$$

where U is a non-empty open subscheme of $\mathrm{spec}(\mathbb{F})$, T is an integral scheme with generic point ζ and function field $k(\zeta) = k$, $T \to U$ is a dominant morphism of finite type, $\mathcal{C}^{cpt} \to T$ is a smooth, proper, geometrically connected curve over T and $\mathcal{C}^{cpt} \setminus \mathcal{C}$ is a relatively finite étale divisor, such that $\mathcal{C}^{cpt}_\zeta$ and $\mathcal{C}_\zeta$ are isomorphic to (and will be identified with) C^{cpt} and C respectively over k and $\mathcal{X} \to \mathcal{C}$ is a smooth proper morphism whose generic fiber $\mathcal{X}_\zeta \to \mathcal{C}_\zeta$ is isomorphic to (and will be identified with) $X \to C$ over k. When $X \to C$ is an abelian scheme, we require furthermore that $\mathcal{X} \to \mathcal{C}$ be an abelian scheme. Also, we will say that an $\mathbb{F}$-model is regular if T is.

Up to shrinking T, $\mathbb{F}$-models always exist and can be chosen to be regular. However, in order to apply Proposition 2.7, one has to work with a model given *a priori* with closed points $t \in |T|$ which cannot all be removed and may be non-regular (for instance the closed points corresponding to the supersingular fibers in Corollary 5.1 below).

Corollary 5.1. *Let $\rho_\bullet$ be the motivic torsion family attached to an abelian scheme $X \to C$ such that X_η contains no non-trivial abelian subvariety isogenous to a k-isotrivial abelian variety. Assume that $p = 0$ (resp. $p > 0$) and that $X \to C$ admits an $\mathbb{F}$-model $\mathcal{X} \to \mathcal{C} \hookrightarrow \mathcal{C}^{cpt} \to T \to U$, with a closed point $t \in |T|$ (resp. a Zariski-dense set of closed points $t \in |T|$) such that $\mathcal{X}_t \to \mathcal{C}_t$ has a point of supersingular type for the motivic torsion family attached to it. Then*

$$\lim_{\ell \to +\infty} \gamma_{\overline{k}}(C_0(\ell)) = +\infty.$$

In particular, for $\ell \gg 0$, there are only finitely many $c \in C(k)$ such that $X_{\overline{c}}[\ell]$ admits a 1-dimensional Γ_k-submodule and, if $p = 0$, for every integer $d \geq 1$ and for $\ell \gg 0$, there are only finitely many $c \in C(k, \leq d)$ such that $X_{\overline{c}}[\ell]$ admits a 1-dimensional $\Gamma_{k(c)}$-submodule.

Corollary 5.2. *Let $\rho^i_\bullet(j)$ be a motivic family attached to a smooth proper morphism $X \to C$.*
(1) Assume that $\rho^i_\bullet(j)$ has big symplectic geometric monodromy. Then

$$\lim_{\ell \to +\infty} \gamma_{\overline{k}}(C_1(\ell)) = +\infty.$$

(2) In any case and provided $2i \neq j$, one has

$$\lim_{\ell \to +\infty} \gamma_k(C_1(\ell)) = +\infty.$$

In particular, for $\ell \gg 0$, there are only finitely many $c \in C(k)$ such that

$$\mathrm{H}^i(X_{\overline{c}}, \mathbb{F}_\ell)(j)^{\Gamma_k} \neq 0$$

and, if $p = 0$, for every integer $d \geq 1$ and for $\ell \gg 0$, there are only finitely many $c \in C(k, \leq d)$ such that

$$\mathrm{H}^i(X_{\overline{c}}, \mathbb{F}_\ell)(j)^{\Gamma_{k(c)}} \neq 0.$$

The finiteness statements about k-rational points (resp. points of bounded degree $\leq d$) in Corollary 5.1 and Corollary 5.2 (2) follow from Lemma 2.4 (1) (resp. Lemma 2.4 (2)) and the definition of $C_i(\ell)$, $i = 0, 1$ (see Fact 2.5 (2)). In the remaining part of this subsection, we prove the statements about the growth of gonality.

5.2. **Specialization.** In this section, our aim is to deduce the statements of Corollary 5.1 and Corollary 5.2 from those of Proposition 2.7, Proposition 2.9 and Theorem 2.10, reducing by specialization to the case where k is finite. Consider an $\mathbb{F}$-model

$$\mathcal{X} \to \mathcal{C} \hookrightarrow \mathcal{C}^{cpt} \to T \to U$$

of $X \to C \hookrightarrow C^{cpt} \to \mathrm{spec}(k) \to \mathrm{spec}(F)$ and a closed point $t \in |T|$; the residue field $k(t)$ is finite of characteristic p_t ($p_t = p$ if $p > 0$). Let η_t denote the generic point of $\mathcal{C}_t$. More precisely,

A) In the proof of Corollary 5.1, the $\mathbb{F}$-model is given *a priori* and we take for $t \in |T|$ a closed point such that there exists $c_t \in \mathcal{C}_t$ of supersingular type for the motivic torsion family attached to $\mathcal{X}_t \to \mathcal{C}_t$.
B) In the proof of Corollary 5.2, the $\mathbb{F}$-model is not given *a priori*, so we will take a *regular* $\mathbb{F}$-model and any closed point $t \in |T|$.

For every prime $\ell \neq p_t$, any choice of an étale path from $\overline{\eta}$ to $\overline{\eta}_t$ defines an isomorphism

$$H_\ell = \mathrm{H}^i(X_{\overline{\eta}}, \mathbb{F}_\ell)(j) \tilde{\to} \mathrm{H}^i(\mathcal{X}_{\overline{\eta}}, \mathbb{F}_\ell)(j) \tilde{\to} \mathrm{H}^i(\mathcal{X}_{\overline{\eta}_t}, \mathbb{F}_\ell)(j)$$

which is compatible with the isomorphism of étale fundamental groups

$$\pi_1(\mathcal{C}[\frac{1}{\ell}], \overline{\eta}) \tilde{\to} \pi_1(\mathcal{C}[\frac{1}{\ell}], \overline{\eta}_t)$$

and such that the induced representation

$$\rho_{\ell,t}^{mod} : \pi_1(\mathcal{C}_t, \overline{\eta}_t) \to \pi_1(\mathcal{C}[\frac{1}{\ell}], \overline{\eta}_t) \stackrel{\rho_\ell^{mod}}{\to} \mathrm{GL}(\mathrm{H}^i(\mathcal{X}_{\overline{\eta}_t}, \mathbb{F}_\ell)(j))$$

identifies with the motivic representation attached to $\mathcal{X}_t \to \mathcal{C}_t$. In order to deduce the growth of the gonality of the abstract modular curves attached to $\rho_\bullet$ from the growth of the gonality of those attached to $\rho_{\bullet,t}$, we have to check that

(i) Up to replacing $\mathcal{C}$ by a finite cover $\rho_\ell(\pi_1(C_{\overline{k}})) = \rho_{\ell,t}^{mod}(\pi_1(\mathcal{C}_{\overline{t}}))$ for $\ell \gg 0$;
(ii) For every $0 \neq v_\ell \in H_\ell$ one has $\gamma_{\overline{k}}(C_{i,v_\ell}) \geq \gamma_{\overline{k(t)}}(\mathcal{C}_{t,i,v_\ell})$ and $\gamma_k(C_{i,v_\ell}) \geq \gamma_{k(t)}(\mathcal{C}_{t,i,v_\ell})$ for $\ell \gg 0$ and $i = 0, 1$.

First, up to replacing $\mathcal{C}$ with a finite cover, *we may and will assume that the representations*

$$\rho_\ell^{mod} : \pi_1(\mathcal{C}[\frac{1}{\ell}]) \to \mathrm{GL}(H_\ell)$$

all factor through the tame fundamental group $\pi_1(\mathcal{C}[\frac{1}{\ell}]) \to \pi_1^t(\mathcal{C}[\frac{1}{\ell}])$ *along* $\mathcal{C}^{cpt} \setminus \mathcal{C} \subset \mathcal{C}^{cpt}$. Indeed, if $p = 0$, this is straightforward since $\pi_1(\mathcal{C}[\frac{1}{\ell}]) \simeq \pi_1^t(\mathcal{C}[\frac{1}{\ell}])$. If $p > 0$, for $\ell \neq p$ one has $\mathcal{C} = \mathcal{C}[\frac{1}{\ell}]$ and $\rho_\ell : \pi_1(C) \to \mathrm{GL}(\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{F}_\ell))$ factors through $\pi_1(C) \to \pi_1(\mathcal{C})$. In particular, from de Jong's alteration theorem [B96, Prop. 6.3.2], there exists a finite cover $\mathcal{C}^{cpt'} \to \mathcal{C}^{cpt}$ (with $\mathcal{C}^{cpt'}$ normal), whose restriction to $\mathcal{C}$ is étale and such that up to base changing $\mathcal{X} \to \mathcal{C}$ *via* $\mathcal{C}' := \mathcal{C} \times_{\mathcal{C}^{cpt}} \mathcal{C}^{cpt'} \to \mathcal{C}$ one may assume that the image of the inertia groups at the generic points of the irreducible components of $\mathcal{C}^{cpt} \setminus \mathcal{C}$ are unipotent as subgroups of $\mathrm{GL}(\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell))$. It then follows from the fact that $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Z}_\ell)$ is torsion-free and $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{F}_\ell) = \mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Z}_\ell)/\ell$ for $\ell \gg 0$ that the image of the inertia groups at the generic points of the irreducible components of $\mathcal{C}^{cpt} \setminus \mathcal{C}$ in $\mathrm{GL}(H_\ell)$ are of order a power of ℓ for $\ell \gg 0$ (See [CT14b, (3) Proof of Fact 5.1] for more details). Note that, *a priori*, $\mathcal{C}^{cpt'}$ is no longer smooth over T and $\mathcal{C}^{cpt'} \setminus \mathcal{C}'$ is no longer étale over T. But recall that, when $p > 0$, we assume that there is a *Zariski-dense* set of closed points $t \in |T|$ whose corresponding fibers have a closed point of supersingular type. So, first, we can replace T by a purely inseparable cover T' and $\mathcal{C}^{cpt'}$ by the normalization of $\mathcal{C}^{cpt'} \times_T T'$ hence assume that $\mathcal{C}^{cpt'}$ is generically smooth over T and $\mathcal{C}^{cpt'} \setminus \mathcal{C}'$ is generically étale over T. And, next, we can replace T with a non-empty open subscheme hence assume that $\mathcal{C}^{cpt'}$ is smooth over T and that $\mathcal{C}^{cpt'} \setminus \mathcal{C}'$ is étale over T (and that there still exists a closed point $t \in |T|$ whose corresponding fiber has a closed point of supersingular type). The smoothness of $\mathcal{C}^{cpt'}$ and étaleness of $\mathcal{C}^{cpt'} \setminus \mathcal{C}'$ over T are necessary to apply the theory of specialization of tame fundamental group (see Subsection 5.2.2).

5.2.1. *Dévissage.* We reduce to the case where T is the spectrum of a discrete valuation ring with generic point ζ and closed point t. This will be required to apply Lemma 5.3 below.

For this, up to replacing T by an open subscheme containing t, one may assume that $T = \mathrm{spec}(\mathcal{O})$ is affine. Let $\mathfrak{m}$ denote the maximal ideal corresponding to $t \in |T|$ and consider a chain of prime ideals

$$\mathfrak{p}_r = \mathfrak{m} \supset \mathfrak{p}_{r-1} \supset \cdots \supset \mathfrak{p}_0 = (0)$$

with $\mathfrak{p}_i$ of height i. Then

$$\mathcal{O}_i := (\mathcal{O}/\mathfrak{p}_i)_{\mathfrak{p}_{i+1}}$$

is a 1-dimensional excellent local ring. Set

$$T_i := \mathrm{spec}(\mathcal{O}_i)$$

and let ζ_i and t_i denote respectively the generic and closed point of T_i. In case B), where T is regular, one can choose the $\mathfrak{p}_i$ in such a way that $\mathcal{O}_i$ is a discrete valuation ring. In that case, set $\tilde{\mathcal{O}}_i := \mathcal{O}_i$. In case A), where T is *a priori* not regular, let $\mathcal{O}_i^\circ$ denote the normalization of $\mathcal{O}_i$. As $\mathcal{O}_i$ is excellent, $\mathcal{O}_i^\circ$ is a semilocal ring and localizing it at any of its maximal ideals yields a discrete valuation ring $\tilde{\mathcal{O}}_i$. In both cases, set

$$\tilde{T}_i := \mathrm{spec}(\tilde{\mathcal{O}}_i)$$

and let $\tilde{\zeta}_i$ and $\tilde{t}_i$ denote respectively the generic and closed point of $\tilde{T}_i$. By construction $k(\zeta_i)\tilde{\rightarrow}k(\tilde{\zeta}_i)$, $k(t_i)\tilde{\rightarrow}k(\zeta_{i+1})$ and, in case B) $k(t_i)\tilde{\rightarrow}k(\tilde{t}_i)$ but, in case A), $k(t_i) \hookrightarrow k(\tilde{t}_i)$ is a finite field extension. Set $k_i := k(\zeta_i)$ $(= k(t_{i-1})$ for $i > 0)$ and $\tilde{k}_i := k(\tilde{t}_i)$ for simplicity.

As a result, one obtains the following commutative diagram of cartesian squares

$$\begin{array}{ccccccccccccccc}
X & \longrightarrow & \mathcal{X}_0 & \longleftarrow & X_0 & \longrightarrow & \mathcal{X}_1 & \longleftarrow & X_1 & \longrightarrow & \mathcal{X}_2 & \longleftarrow \cdots \longrightarrow & \mathcal{X}_r & \longleftarrow & X_r \\
\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square \quad \square & \downarrow & \square & \downarrow \\
C & \longrightarrow & \mathcal{C}_0 & \longleftarrow & C_0 & \longrightarrow & \mathcal{C}_1 & \longleftarrow & C_1 & \longrightarrow & \mathcal{C}_2 & \longleftarrow \cdots \longrightarrow & \mathcal{C}_r & \longleftarrow & C_r \\
\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square \quad \square & \downarrow & \square & \downarrow \\
C^{cpt} & \longrightarrow & \mathcal{C}_0^{cpt} & \longleftarrow & C_0^{cpt} & \longrightarrow & \mathcal{C}_1^{cpt} & \longleftarrow & C_1^{cpt} & \longrightarrow & \mathcal{C}_2^{cpt} & \longleftarrow \cdots \longrightarrow & \mathcal{C}_r^{cpt} & \longleftarrow & C_r^{cpt} \\
\Big\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square \quad \square & \downarrow & \square & \downarrow \\
 & & \tilde{T}_0 & \xleftarrow{\tilde{t}_0} & \mathrm{spec}(\tilde{k}_0) & \longrightarrow & \tilde{T}_1 & \xleftarrow{\tilde{t}_1} & \mathrm{spec}(\tilde{k}_1) & \longrightarrow & \tilde{T}_2 & \longleftarrow \cdots \longrightarrow & \tilde{T}_r & \xleftarrow{\tilde{t}_r} & \mathrm{spec}(\tilde{k}_r) \\
\Big\downarrow & \nearrow^{\tilde{\zeta}_0} & \downarrow & & \downarrow & \nearrow^{\tilde{\zeta}_1} & \downarrow & & \downarrow & \nearrow^{\tilde{\zeta}_2} & \downarrow & \nearrow^{\tilde{\zeta}_r} & \downarrow & & \downarrow \\
\mathrm{spec}(k) & \xrightarrow{\zeta_0} & T_0 & \xleftarrow{t_0} & \mathrm{spec}(k_0) & \xrightarrow{\zeta_1} & T_1 & \xleftarrow{t_1} & \mathrm{spec}(k_1) & \xrightarrow{\zeta_2} & T_2 & \longleftarrow \cdots \xrightarrow{\zeta_r} & T_r & \xleftarrow{t_r = t} & \mathrm{spec}(k(t))
\end{array}$$

where $\mathcal{X}_i \to \mathcal{C}_i \hookrightarrow \mathcal{C}_i^{cpt} \to \tilde{T}_i$ (resp. $X_i \to C_i \hookrightarrow C_i^{cpt} \to \mathrm{spec}(\tilde{k}_i)$) is the pull-back of $\mathcal{X} \to \mathcal{C} \hookrightarrow \mathcal{C}^{cpt} \to T$ by the natural morphism $\tilde{T}_i \to T$ (resp. $\mathrm{spec}(\tilde{k}_i) \to T$). Thus, in particular, $X_r \to C_r \hookrightarrow C_r^{cpt} \to \mathrm{spec}(\tilde{k}_r)$ coincides with the base change of $\mathcal{X}_t \to \mathcal{C}_t \hookrightarrow \mathcal{C}_t^{cpt} \to \mathrm{spec}(k(t))$ by the natural inclusion $k(t) = k_r \hookrightarrow \tilde{k}_r$. Also, since $\rho_\ell^{mod} : \pi_1(\mathcal{C}[\frac{1}{\ell}]) \to \mathrm{GL}(H_\ell)$ factors through

$\pi_1(\mathcal{C}[\frac{1}{\ell}]) \to \pi_1^t(\mathcal{C}[\frac{1}{\ell}])$, the induced representations $\rho_{\ell,i}^{mod} : \pi_1(\mathcal{C}_i[\frac{1}{\ell}]) \to \mathrm{GL}(H_\ell)$ also factor through $\pi_1(\mathcal{C}_i[\frac{1}{\ell}]) \to \pi_1^t(\mathcal{C}_i[\frac{1}{\ell}])$.

So, from now on, we will assume that T is the spectrum of a discrete valuation ring with generic point ζ and closed point t with $k(t)$ of characteristic $p \geq 0$ and that $\rho_\ell^{mod} : \pi_1(\mathcal{C}) \to \mathrm{GL}(H_\ell)$ factors through $\pi_1(\mathcal{C}) \to \pi_1^t(\mathcal{C})$ for $\ell \neq p$.

5.2.2. *End of the proofs of (i) and (ii).* Let η_t denote the generic point of $\mathcal{C}_t$. For every prime $\ell \neq p$, recall that any choice of an étale path from $\overline{\eta}$ to $\overline{\eta}_t$ defines an isomorphism

$$H_\ell = \mathrm{H}^i(X_{\overline{\eta}}, \mathbb{F}_\ell)(j) \tilde{\to} \mathrm{H}^i(\mathcal{X}_{\overline{\eta}}, \mathbb{F}_\ell)(j) \tilde{\to} \mathrm{H}^i(\mathcal{X}_{\overline{\eta}_t}, \mathbb{F}_\ell)(j)$$

which is compatible with the isomorphism of étale fundamental groups $\pi_1(\mathcal{C}, \overline{\eta}) \tilde{\to} \pi_1(\mathcal{C}, \overline{\eta}_t)$ and such that the induced representation

$$\rho_{\ell,t} : \pi_1(\mathcal{C}_t, \overline{\eta}_t) \to \pi_1(\mathcal{C}, \overline{\eta}) \to \mathrm{GL}(\mathrm{H}^i(\mathcal{X}_{\overline{\eta}}, \mathbb{F}_\ell)(j))$$

identifies with the motivic representation attached to $\mathcal{X}_t \to \mathcal{C}_t$.

Now, (i) follows from the specialization theory for the tame fundamental group (of curves). More precisely, we have the following commutative diagram

$$\begin{array}{ccccc} \pi_1^t(C_{\overline{k}}) & \hookrightarrow & \pi_1^t(C) & & \\ & & & \searrow^{\rho_\ell} & \\ \downarrow & & & & \mathrm{GL}(H_\ell) \\ & & & \nearrow_{\rho_{\ell,t}} & \\ \pi_1^t(\mathcal{C}_{\overline{t}}) & \hookrightarrow & \pi_1^t(\mathcal{C}_t) & & \end{array},$$

where the vertical arrow is the specialization morphism for the tame fundamental group [SGA1, Exp.XIII, 2.10] (see also [OrV00, §4]), which, by definition of an $\mathbb{F}$-model, is surjective [SGA1, Exp. XIII, 2.8] (the 0-acyclicity assumption [SGA4, Exp. XV, 1.11] follows from the smoothness of $\mathcal{C}^{cpt} \to T$ - see [SGA4, Exp. XV, Thm. 2.1]) (see also [OrV00, §Thm. 4.4]).

As for (ii), let us recall that gonality decreases under specialization.

Lemma 5.3. *Let T be the spectrum of a (discrete) valuation ring with generic point ζ and closed point t and let $\mathcal{C} \to T$ be a smooth, proper and geometrically connected curve over T. Then one has:*

$$\gamma_{k(\zeta)}(\mathcal{C}_\zeta) \geq \gamma_{k(t)}(\mathcal{C}_t) \text{ and } \gamma_{\overline{k(\zeta)}}(\mathcal{C}_\zeta) \geq \gamma_{\overline{k(t)}}(\mathcal{C}_t).$$

Proof. See for instance [ACG11, Chap. XXI, §3]. There it is shown that the sublocus $\mathcal{C}_r^{(d)} \subset \mathcal{C}^{(d)}$ classifying degree d effective divisors whose linear series has dimension $\geq r$ is a closed subscheme of the dth symmetric power $\mathcal{C}^{(d)}$ of $\mathcal{C}$. In particular, $\mathcal{C}_2^{(d)}$ is proper over T and the conclusion follows from the valuative criterion of properness. □

For every $0 \neq v_\ell \in H_\ell$, let $\mathcal{C}_{i,v_\ell} \to \mathcal{C}$ denote the connected étale cover corresponding to the open subgroup $\mathrm{Stab}_{\pi_1(\mathcal{C})}(v_\ell) \subset \pi_1(\mathcal{C})$ if $i = 1$ and $\mathrm{Stab}_{\pi_1(\mathcal{C})}(\mathbb{F}_\ell v) \subset \pi_1(\mathcal{C})$ if $i = 0$. Since the morphism $\pi_1^t(C) \to \pi_1^t(\mathcal{C})$ is surjective, one has $\mathcal{C}_{i,v_\ell,\eta} \simeq C_{i,v_\ell}$. Also, considering the morphism $\pi_1^t(\mathcal{C}_t) \to \pi_1^t(\mathcal{C})$, we deduce that $\mathcal{C}_{i,v_\ell,t}$ maps surjectively onto $\mathcal{C}_{t,i,v_\ell}$. In particular

$$\gamma_{k(t)}(\mathcal{C}_{i,v_\ell,t}) \geq \gamma_{k(t)}(\mathcal{C}_{t,i,v_\ell}) \text{ and } \gamma_{\overline{k(t)}}(\mathcal{C}_{i,v_\ell,t}) \geq \gamma_{\overline{k(t)}}(\mathcal{C}_{t,i,v_\ell})$$

As T is the spectrum of a discrete valuation ring, Lemma 5.3 implies $\gamma_k(C_{i,v_\ell}) \geq \gamma_{k(t)}(\mathcal{C}_{i,v_\ell,t})$ and $\gamma_{\overline{k}}(C_{i,v_\ell}) \geq \gamma_{\overline{k(t)}}(\mathcal{C}_{i,v_\ell,t})$. This already concludes the proof of (ii) for geometric gonality. As for arithmetic gonality, just observe that, by construction, the degree of the field of definition of $\mathcal{C}_{t,i,v_\ell}$ over $k(t)$ is smaller than the degree of the field of definition of $\mathcal{C}_{i,v_\ell,t}$ over $k(t)$ which, in turn, is always smaller than the degree of the field of definition of C_{i,v_ℓ} over k.

5.2.3. *End of the proofs.*

- Corollary 5.2 (2): This follows directly from (ii), Theorem 2.6, Theorem 2.10 and Lemma 2.4.
- Corollary 5.2 (1): From (i), the assumption that $\rho_\bullet$ has big symplectic geometric monodromy transfers to $\rho_{\bullet,t}$. In particular $\rho_{\bullet,t}$ satisfies
$$\lim_{\ell\to+\infty} \min\{[G^{geo}_{\langle\langle v\rangle\rangle} : G^{geo}_v] \mid 0 \neq v \in H_\ell\} = +\infty.$$
As it always satisfies Condition (U) by Theorem 2.6, the conclusion follows from Proposition 2.9.
- Corollary 5.1: From (i), the assumption that
$$\lim_{\ell\to+\infty} \min\{[G^{geo}_\ell : G^{geo}_v] \mid 0 \neq v \in H_\ell\} = +\infty$$
transfers to $\rho_{\bullet,t}$. As $\rho_{\bullet,t}$ always satisfies Condition (U) and, by construction, has a point of supersingular type, the conclusion follows from Proposition 2.7.

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Appendix A. Gonality, isogonality and points of bounded degree on curves

Let k be a field of characteristic $p \geq 0$. A curve over k will always mean a smooth, separated and geometrically integral scheme of dimension 1 over k. We fix once for all a proper curve C over k. Given an integer $d \geq 1$, we use again the notation

$$C(k, \leq d) := \{c \in C \mid [k(c) : k] \leq d\}.$$

The purpose of this appendix is to discuss the implication

$$\begin{array}{c}\text{Mordell-Lang conjecture}\\ + \\ \text{assumptions on } C,\ d\end{array} \quad \stackrel{(*)}{\Longrightarrow} \quad \text{finiteness of } C(k, \leq d)$$

both for $p = 0$ and $p > 0$ and under the assumption that k is finitely generated. We are looking for 'assumptions on C, d' which are minimal. When $p = 0$, the observation goes back to [Fr94] that the less restrictive assumption is that $2d + 1$ does not exceed the geometric gonality of C (See Corollary A.4). When $p > 0$, this is no longer enough (See Subsection A.3) because of the appearance of isotriviality phenomena. To take these phenomena into account, one has to introduce a new strengthened variant of gonality - the isogonality. Then the less restrictive assumptions are that $2d + 1$ does not exceed the geometric gonality of C and $d + 1$ does not exceed the geometric isogonality of C (See Corollary A.7).

A.1. **Review of gonality.** The basic material for this section can be found in [Mi86].

Let d be a positive integer and D a degree d effective (Cartier) divisor on C. Consider the sequence of morphisms of k-schemes

(1) $$\mathbb{P}(L(D)) \stackrel{\iota_D}{\to} C^{(d)} \stackrel{j}{\to} J^{(d)},$$

where $J^{(d)}$ denotes the degree d part of the Picard scheme of C over k (thus, in particular, $J := J^{(0)}$ is the Jacobian variety of C over k), $C^{(d)}$ the dth symmetric power of C and $\mathbb{P}(L(D))$ the projective space on

$$L(D) := \{f \in k(C) \mid D + div(f) \geq 0\}.$$

Note that $C^{(d)}$ can be regarded as the (fine) moduli space Div^d_C of degree d effective divisors on C. Then $\iota_D : \mathbb{P}(L(D)) \to C^{(d)}$ sends the class of $f \in L(D)\setminus\{0\}$ to the degree d effective divisor $D+div(f)$, and $j : C^{(d)} \to J^{(d)}$ sends an effective degree d divisor D' on C to the class of corresponding degree d invertible sheaf on C. Furthermore, $\iota_D : \mathbb{P}(L(D)) \to C^{(d)}$ is a closed immersion which identifies with the scheme-theoretic fiber $C^{(d)}_{j(D)} \hookrightarrow C^{(d)}$ of $j : C^{(d)} \to J^{(d)}$ at $j(D)$. As $J^{(d)}$ is a J-torsor, $J^{(d)} \simeq J$ if and only if $J^{(d)}(k) \neq \emptyset$. In particular, $J^{(d)}_{\overline{k}} \simeq J_{\overline{k}}$.

As there is no non-trivial morphism from a projective space to an abelian variety, any morphism $\mathbb{P}^n_k \to C^{(d)}$ factors through $\iota_D : \mathbb{P}(L(D)) \to C^{(d)}$ for some $D \in \mathrm{Div}^d_C(k)$.

The above shows that, given an integer $d \geq 1$, the following assertions are equivalent:

(i) For every $d' \leq d$ the map $C^{(d')}(k) \to J^{(d')}(k)$ is injective;
(ii) For every $d' \leq d$ there is no non-constant morphism $\mathbb{P}^1_k \to C^{(d')}$;
(iii) For every $d' \leq d$, $L(D) = k$ for every $D \in \mathrm{Div}^{d'}_C(k)$;
(iv) There is no $f \in k(C) \setminus k$ such that $[k(C) : k(f)] \leq d$;
(v) There is no non-constant morphism $C \to \mathbb{P}^1_k$ of degree $\leq d$.

The *k-gonality* $\gamma_k(C)$ of C is then defined as $\gamma_k(C) := d + 1$, where d is the largest integer ≥ 1 satisfying the equivalent conditions (i)-(v).

The k-gonality is an arithmetic invariant. However, over an algebraically closed field, it has the following more geometric interpretation.

Lemma A.1. *If $d + 1 \leq \gamma_{\overline{k}}(C)$, then $j : C^{(d)} \to J^{(d)}$ is a closed immersion.*

Proof. By [Mi86, Lemma 2.4 and Thm. 5.1 (b)] plus the fact that being a closed immersion descends along $\mathrm{spec}(\overline{k}) \to \mathrm{spec}(k)$, it is enough to show that $j : C^{(d)} \to J^{(d)}$ induces an injective map on $\overline{k}$-points. But this is precisely what the assumption $d + 1 \leq \gamma_{\overline{k}}(C)$ says. □

When k is infinite, from the definition of k-gonality, a necessary condition for $C(k, \leq d)$ to be finite is that $d + 1 \leq \gamma_k(C)$. In the sequel, our purpose will be to find sufficient conditions for $C(k, \leq d)$ to be finite, when k is finitely generated. Showing the finiteness of $C(k, \leq d)$ amounts to showing the finiteness of $C^{(d')}(k)$ for every $d' \leq d$, hence, under the assumption that $d + 1 \leq \gamma_k(C)$, the finiteness of the image of $C^{(d')}(k)$ in $J^{(d')}(k)$. Actually, we will need a stronger assumption, namely that $2d + 1 \leq \gamma_{\overline{k}}(C)$. Let $W_d \hookrightarrow J$ denote the image of the closed immersion $j : C^{(d)} \hookrightarrow J^{(d)}$. We are to show that $W_d(k)$ is finite or, equivalently, that the Zariski-closure $\overline{W}_d$ of $W_d(k)$ in W_d is. This will follow from the Mordell-Lang conjectures and the following elementary lemma.

Lemma A.2. *If $2d + 1 \leq \gamma_{\overline{k}}(C)$, then $W_d(\overline{k})$ contains no subset of the form $a + \Delta$ with $a \in J^{(d)}(\overline{k})$ and $\Delta \subset J(\overline{k})$ an infinite subgroup.*

Proof. Let $a \in J^{(d)}(\overline{k})$ and $\Delta \subset J(\overline{k})$ a subgroup such that $a + \Delta \subset W_d(\overline{k})$. As $d + 1 \leq \gamma_{\overline{k}}(C)$, for every $\delta \in \Delta$ there exists a unique $D_\delta \in \mathrm{Div}^d_C(\overline{k})$ such that $j(D_\delta) = a + \delta$. In particular

$j(D_\delta + D_{-\delta}) = 2a = j(2D_0)$ in $J^{(2d)}$. But as $2d+1 \leq \gamma_{\overline{k}}(C)$ and $D_\delta + D_{-\delta}, 2D_0 \in \mathrm{Div}^{2d}_C(\overline{k})$, this forces $D_\delta + D_{-\delta} = 2D_0$. As $D_\delta + D_{-\delta}$ and $2D_0$ are both effective, the equality $D_\delta + D_{-\delta} = 2D_0$ is possible for only finitely many values of D_δ (hence of δ). □

A.2. **Points of bounded degree.**

A.2.1. *Statement of Mordell-Lang conjectures.* The statement of Mordell-Lang conjectures differs slightly when $p = 0$ and $p > 0$; this is due to the appearance of isotriviality phenomena when $p > 0$. More precisely, let F be an algebraically closed field of characteristic $p \geq 0$, A an abelian variety over F, $W \hookrightarrow A$ an integral closed subscheme and $\Gamma \subset A(F)$ a subgroup. Let $A_W \hookrightarrow A$ denote the reduced translation stabilizer of W (note that the identity component A_W° of A_W is automatically an abelian variety) [SGA3, Exp. VIII, §6] and, if $p > 0$, let

$$\Gamma_{(p')} := \bigcup_{n \geq 1,\, p \nmid n} [n]^{-1}(\Gamma)$$

denote the prime-to-p divisible hull of Γ.

Theorem A.3. (Mordell-Lang Conjectures)

(1) ([F91]) *If $p = 0$, $\dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q})$ is finite and $\Gamma \cap W$ is Zariski-dense in W, then there exists $\gamma \in \Gamma \cap W$ such that*

$$W = \gamma + A_W$$

(or equivalently, W/A_W is reduced to one point)[10].

(2) ([H96]) *If $p > 0$, Γ is finitely generated and $\Gamma_{(p')} \cap W$ is Zariski-dense in W, then W/A_W is 'almost isotrivial', that is, there exist $\gamma \in \Gamma \cap W$, a finite field $F_0 \subset F$, an abelian subvariety $A' \hookrightarrow A$, containing A_W, an abelian variety B_0 over F_0, a closed geometrically irreducible subscheme $Z_0 \hookrightarrow B_0$ and a purely inseparable isogeny $\varphi : A'/A_W \to B_0 \times_{F_0} F$, such that*

$$W = \gamma + (\varphi \circ p_{A_W})^{-1}(Z_0 \times_{F_0} F),$$

where $p_{A_W} : A' \to A'/A_W$ is the natural surjective homomorphism of abelian varieties.

Before going further, observe that for every $w \in W$ on has $A_W \subset W - w$. In particular, assuming that $2d+1 \leq \gamma_{\overline{k}}(C)$, it follows from Lemma A.2 that for every irreducible component $\mathcal{W} \hookrightarrow W_{d,\overline{k}}$ the reduced stabilizer of $\mathcal{W}$ in $J_{\overline{k}}$ is finite.

If we assume furthermore that k is finitely generated then the Lang-Néron Theorem [LN59] asserts that the group of k-rational points of an abelian variety over k is always finitely generated. Also, by construction $\overline{W}_d(k)$ is Zariski-dense in $\overline{W}_d$. This implies that every irreducible component $\mathcal{W} \hookrightarrow \overline{W}_d$ is geometrically irreducible and that $\mathcal{W}(k) = \mathcal{W}_{\overline{k}} \cap J(k)$ is Zariski-dense in $\mathcal{W}_{\overline{k}}$ [11].

[10] In particular, A_W is connected.

[11] Indeed, if $\mathcal{W}_1, \dots, \mathcal{W}_r$ denote the irreducible components of $\overline{W}_d$ then

$$\overline{W}_d(k) = \bigcup_{1 \leq i \leq r} \mathcal{W}_i(k)$$

hence

$$\bigcup_{1 \leq i \leq r} \mathcal{W}_i = \overline{W}_d = \overline{\overline{W}_d(k)} = \overline{\bigcup_{1 \leq i \leq r} \mathcal{W}_i(k)} = \bigcup_{1 \leq i \leq r} \overline{\mathcal{W}_i(k)}.$$

So, $\mathcal{W}_i = \overline{\mathcal{W}_i(k)} \bigcup_{1 \leq j \neq i \leq r} \overline{\mathcal{W}_j(k)} \cap \mathcal{W}_i$. As $\overline{\mathcal{W}_j(k)} \cap \mathcal{W}_i \subset \mathcal{W}_j \cap \mathcal{W}_i$, it has dimension strictly smaller than the dimension of $\mathcal{W}_i$. So $\overline{\mathcal{W}_i(k)} \subset \mathcal{W}_i$ has the same dimension as $\mathcal{W}_i$, which forces $\overline{\mathcal{W}_i(k)} = \mathcal{W}_i$. As the set of singular points is a proper closed subset in $\mathcal{W}_i$, one can find a regular point $c \in \mathcal{W}_i(k)$, which is automatically a smooth (*i.e.*, geometrically regular) point. Let $\mathcal{W}_{i,1}, \dots, \mathcal{W}_{i,s}$ denote the irreducible components of $\mathcal{W}_{i,\overline{k}}$. As c is a regular point in $\mathcal{W}_{i,\overline{k}}$, it belongs to a unique irreducible component - say $\mathcal{W}_{i,1}$ of $\mathcal{W}_{i,\overline{k}}$, which is then automatically defined over k. Thus, $\mathcal{W}_{i,\overline{k}} = \mathcal{W}_{i,1}$.

A.2.1.1. *Finiteness of* $C(k, \leq d)$ *when* $p = 0$. If $p = 0$, then Theorem A.3 (1) implies that $\mathcal{W}_{\overline{k}} = \gamma + A_{\mathcal{W}_{\overline{k}}}$ (for some $\gamma \in \mathcal{W}(k)$). In particular $A_{\mathcal{W}_{\overline{k}}}$ is connected, hence trivial from the above. Since $\overline{W}_d$ has only finitely many irreducible components, we have just shown the following.

Corollary A.4. *Assume that k is a finitely generated field of characteristic* 0 *and that C is a proper curve over k. Then*

$$2d + 1 \leq \gamma_{\overline{k}}(C) \Longrightarrow |C(k, \leq d)| < +\infty.$$

A.2.1.2. *Finiteness of* $C(k, \leq d)$ *when* $p > 0$. If $p > 0$ and without any further assumption on C, Theorem A.3 (2) only implies that $\mathcal{W}_{\overline{k}}$ is 'almost isotrivial', which is not enough to ensure the finiteness of $\mathcal{W}(k)$. To obtain an analogue of Corollary A.4 when $p > 0$ we introduce a new strengthened notion of gonality - the k-isogonality, which is in general smaller than the k-gonality.

- *k-isogonality.* Let k be a field of characteristic $p > 0$. We say that a scheme S over k is *isotrivial over k* if there exists a finite field $F_0 \subset \overline{k}$ and a scheme S_0 over F_0 such that $S_0 \times_{F_0} \overline{k}$ and $S \times_k \overline{k}$ are isomorphic over $\overline{k}$.

Lemma A.5. *Let K be any field extension of k and let S be a scheme of finite type over k. Then S is isotrivial over k if and only if $S \times_k K$ is isotrivial over K.*

Proof. The 'only if' implication is straightforward. For the 'if' implication, let $F_0(\subset \overline{k} \subset \overline{K})$ be a finite subfield and let S_0 be a scheme over F_0 such that $S_0 \times_{F_0} \overline{K}$ and $S \times_k \overline{K}$ are isomorphic over $\overline{K}$. Then $\overline{K}$ can be written as the inductive limit of its finitely generated $\overline{k}$-subalgebras. From [EGAIV-3, Thm. (8.8.2)] there exists a finitely generated $\overline{k}$-subalgebra $R \subset \overline{K}$ such that the $\overline{K}$-isomorphism between $S_0 \times_{F_0} \overline{K}$ and $S \times_k \overline{K}$ descends to an R-isomorphism between $S_0 \times_{F_0} R$ and $S \times_k R$. This isomorphism specializes to a $\overline{k}$-isomorphism between $S_0 \times_{F_0} \overline{k}$ and $S \times_k \overline{k}$ at any closed point of spec(R). □

Lemma A.6. *Given an integer $d \geq 1$, consider the following assertions:*
(1) For every $d' \leq d$, there is no non-constant k-morphism $B \to C^{(d')}$ with B an isotrivial curve over k;
(2) There is no diagram $C \leftarrow C' \stackrel{f}{\to} B$ of non-constant k-morphisms of proper curves over k with B an isotrivial curve over k and $deg(f) \leq d$.
Then one always has (1) $\Longrightarrow$ (2). If, moreover, $k = \overline{k}$, then one also has (2) $\Longrightarrow$ (1).

Proof. More precisely, given an integer $d \geq 1$ and a curve B over k, we prove $(2, B) \Longrightarrow (1, B)$ and, if moreover $k = \overline{k}$, $(1, B) \Longrightarrow (2, B)$, where
- (1,B) There is an integer $d' \leq d$ and a non-constant k-morphism $B \to C^{(d')}$;
- (2,B) There is a diagram $C \leftarrow C' \stackrel{f}{\to} B$ of non-constant k-morphisms of proper curves over k with $deg(f) \leq d$.

To prove $(2, B) \Longrightarrow (1, B)$, let $C \leftarrow C' \to B$ be a diagram of non-constant k-morphisms of proper curves over k with B an isotrivial curve over k and $d' := deg(f) \leq d$. Then $graph(f) \hookrightarrow C' \times_k B$ defines an injective k-morphism $B \to (C')^{(d')}$. As the natural k-morphism $(C')^{(d')} \to C^{(d')}$ is finite, the composite k-morphism $B \to C^{(d')}$ is also non-constant.

To prove $(1, B) \Longrightarrow (2, B)$ under the extra assumption that $k = \overline{k}$, let $B \to C^{(d')}$ be a non-constant k-morphism with $d' \leq d$ and B an isotrivial curve over k. We take d' to be minimal. Consider the natural k-morphism $C \times C^{(d'-1)} \to C^{(d')}$ corresponding to the addition map $\mathrm{Div}^1_C \times \mathrm{Div}^{d'-1}_C \to \mathrm{Div}^{d'}_C$, which is finite flat of degree d'. Let $C_1 \to C$ be the pull-back of $C \times C^{(d'-1)} \to C^{(d')}$ by $B \to C^{(d')}$.

Take any irreducible component C_2 of C_1 and let C' be the normalization of C_2. As $C_1 \to B$ is finite flat of degree d', $C' \to B$ is finite flat of degree $\leq d'$. On the other hand, consider the composite of the natural k-morphism $C' \to C \times C^{(d'-1)}$ and the projection $C \times C^{(d'-1)} \to C$. If this composite morphism is constant and its image is denoted by $c \in C$, then the image of the non-constant morphism $B \to C^{(d')}$ is contained in the injective image of $C^{(d'-1)} \tilde{\to} \{c\} \times C^{(d'-1)}$ in $C^{(d')}$. It follows from this that there exists a non-constant morphism $B \to C^{(d'-1)}$, which contradicts the minimality of d'. □

The *k-isogonality* $\gamma_k^{iso}(C)$ of C is then defined as $\gamma_k^{iso}(C) := d+1$, where d is the largest integer ≥ 1 satisfying condition (1) of Lemma A.6.

- *Finiteness of $C(k, \leq d)$ when $p > 0$.* Having introduced the k-isogonality, we can now state an analogue of Corollary A.4.

Corollary A.7. *Assume that k is a finitely generated field of characteristic $p > 0$ and that C is a proper curve over k. Then*

$$2d + 1 \leq \gamma_{\overline{k}}(C) \text{ and } d + 1 \leq \gamma_{\overline{k}}^{iso}(C) \Longrightarrow |C(k, \leq d)| < +\infty.$$

Proof. Otherwise, $\overline{W}_d$ has at least one (geometrically) irreducible component $\mathcal{W} \hookrightarrow \overline{W}_d$ of dimension ≥ 1. Then Theorem A.3 (2) implies that there exists an abelian subvariety $J' \subset J_{\overline{k}}$, containing $A_{\mathcal{W}_{\overline{k}}}$, a finite subfield $F_0 \subset \overline{k}$, an abelian variety J'_0 over F_0, a geometrically irreducible closed subscheme $Z_0 \hookrightarrow J'_0$, a purely inseparable isogeny $\varphi : J'/A_{\mathcal{W}_{\overline{k}}} \to J'_0 \times_{F_0} \overline{k}$ and a $\overline{k}$-point $\overline{a} \in \mathcal{W}(\overline{k})$ such that

$$\mathcal{W}_{\overline{k}} = a + (p_{A_{\mathcal{W}_{\overline{k}}}} \circ \varphi)^{-1}(Z_0 \times_{F_0} \overline{k}).$$

Furter, $A_{\mathcal{W}_{\overline{k}}}$ is finite by Lemma A.2, hence $A_{\mathcal{W}_{\overline{k}}} = A_{\mathcal{W}_{\overline{k}}}/A^{\circ}_{\mathcal{W}_{\overline{k}}}$ is finite étale. As $\dim(Z_0)$ $=\dim(\mathcal{W}_{\overline{k}}) \geq 1$, one can fix a 1-dimensional geometrically integral closed subscheme $S_0 \hookrightarrow Z_0$. Set $S := a + (p_{A_{\mathcal{W}_{\overline{k}}}} \circ \varphi)^{-1}(S_0 \times_{F_0} \overline{k}) \hookrightarrow \mathcal{W}_{\overline{k}}$ and $S_1 := p_{A_{\mathcal{W}_{\overline{k}}}}(S) = p_{A_{\mathcal{W}_{\overline{k}}}}(a) + \varphi^{-1}(S_0 \times_{F_0} \overline{k}) \hookrightarrow p_{A_{\mathcal{W}_{\overline{k}}}}(\mathcal{W}_{\overline{k}}) = p_{A_{\mathcal{W}_{\overline{k}}}}(a) + \varphi^{-1}(Z_0 \times_{F_0} \overline{k})$, and write $\tilde{S}_0 \to S_0$, $\tilde{S}_1 \to S_1$ and $\tilde{S} \to S$ for the normalizations of S_0, S_1 and S, respectively. By universal property of normalization, $S \to S_1 \to S_0 \times_{F_0} k$ lifts to finite morphisms $\tilde{S} \to \tilde{S}_1 \to \tilde{S}_0 \times_{F_0} k$. Further, $\tilde{S} \to \tilde{S}_1$ is finite étale and $\tilde{S}_1 \to \tilde{S}_0 \times_{F_0} k$ is purely inseparable. By construction, $\tilde{S}_1$ and $\tilde{S}_0 \times_{F_0} k$ are projective, normal 1-dimensional geometrically integral schemes, hence every purely inseparable morphism between them is a composition of Frobenius iterates. In particular, $\tilde{S}_1$ is isotrivial. Further, as $\tilde{S} \to \tilde{S}_1$ is finite étale, every connected component $\tilde{S}_2$ of $\tilde{S}$ is an isotrivial curve over k. As $d + 1 \leq \gamma_{\overline{k}}(C)$, the morphism $j : C^{(d)} \to W_d$ is an isomorphism. Thus, we obtain a non-constant morphism $\tilde{S}_2 \to C^{(d)}$, which contradicts the assumption $d+1 \leq \gamma_{\overline{k}}^{iso}(C)$. □

A.3. **Concluding remarks about isogonality.**

A.3.1. *The 'good arithmetic invariant'.* $\overline{k}$-isogonality is the 'good' arithmetic invariant to measure the finiteness of $C(k, \leq d)$ in the sense that for any curve C over a finitely generated field k of characteristic $p > 0$ with $\overline{k}$-isogonality d, one can always construct a finitely generated field extension K of k such that $C(K, \leq d)$ is infinite. Indeed, consider any curve B_0 over a finite field $F_0 \subset \overline{k}$ such that one has a non-constant morphism $B_0 \times_{F_0} \overline{k} \to (C \times_k \overline{k})^{(d)}$. Up to replacing k by a finite extension, one may assume that $F_0 \subset k$ and $B_0 \times_{F_0} \overline{k} \to (C \times_k \overline{k})^{(d)}$ is defined over k. Let K denote the function field of $B_0 \times_{F_0} k$ and let $\eta \in B_0 \times_{F_0} k$ denote its generic point. Then, writing Fr for the base-change of the Frobenius endomorphism from B_0 to $B_0 \times_{F_0} k$, one obtains an infinite[12] sequence of K-rational points $Fr^n(\eta)$, $n \geq 0$ in $B_0 \times_{F_0} k$ hence in $C^{(d)}(K)$.

[12]Indeed, just embed $B_0 \times_{F_0} k$ into some affine space $\mathbb{A}_k^n$ such that η has coordinates $(x_1, \ldots, x_n)$ with at least one of the x_i transcendental over k. But then, $Fr^n(\eta)$ has coordinates $(x_1^{q^n}, \ldots, x_n^{q^n})$ (where $q = |F_0|$) hence cannot be equal to $Fr^m(\eta)$ for $n \neq m$.

A.3.2. *Gonality and isogonality.* By definition, $\gamma_k^{iso}(C) \leq \gamma_k(C)$, and a sufficient condition for $\gamma_k^{iso}(C) = \gamma_k(C)$ is that J admits no non-trivial isotrivial isogeny factor[13]. Also, an upper bound for $\gamma_k^{iso}(C)$ is given by

$$\tilde{\gamma}_k^{iso}(C) := \min\{deg(f) \mid C \xrightarrow{f} B \text{ non-constant, } B \text{ isotrivial curve over } k\},$$

which also does not exceed $\gamma_k(C)$. But it is not clear whether, in general, the inequality $\gamma_k^{iso}(C) \leq \tilde{\gamma}_k^{iso}(C)$ is strict or not[14]. Note that showing the equality $\gamma_k^{iso}(C) = \tilde{\gamma}_k^{iso}(C)$ amounts to showing that for every finite morphism of curves $C' \to C$ one has $\tilde{\gamma}_k^{iso}(C) \leq \tilde{\gamma}_k^{iso}(C')$.

It is easy to construct examples where $\gamma_k^{iso}(C)(\leq \tilde{\gamma}_k^{iso}(C)) \ll \gamma_k(C)$. First, as the set of curves with maximal (geometric) gonality $\lfloor \frac{g+3}{2} \rfloor$ in the moduli space $\mathcal{M}_{g,\mathbb{F}_p}$ is open, for any integer $\gamma \geq 1$, one can always find a finite field F_0 of characteristic $p > 0$ and a curve C_0 over F_0 with $\overline{\mathbb{F}}_p$-gonality $\geq \gamma$. Thus, C_0 has $\overline{\mathbb{F}}_p$-gonality $\geq \gamma$ and $\overline{\mathbb{F}}_p$-isogonality 1. More generally, one can construct for any integer $d \geq 2$ a non-isotrivial curve C over a finitely generated field k of characteristic $p > 0$ with $\overline{k}$-gonality $\geq 2d + 1$ and $\overline{k}$-isogonality $\leq d$. Indeed, consider any curve C_0 over some finite field $F_0 \subset \overline{\mathbb{F}}_p$ with $\overline{\mathbb{F}}_p$-gonality $\geq 2d+1$ and elliptic curve E_1 over some finitely generated field k_1 of characteritsic p with j-invariant $j_{E_1} \notin \overline{\mathbb{F}}_p$. Take a finitely generated field k containing both F_0 and k_0 and consider the canonical degree 2 cover $E_1 \to \mathbb{P}^1_{k_1}$ and any non-constant morphism $C_0 \to \mathbb{P}^1_{F_0}$ (necessarily of degree $\geq 2d + 1$). Write

$$\begin{array}{ccc} C & \longrightarrow & C_0 \times_{F_0} k \\ \downarrow & \square & \downarrow \\ E_1 \times_{k_1} k & \longrightarrow & \mathbb{P}^1_k. \end{array}$$

Then, since E_1 is non-isotrivial, the normalization of C (which is smooth over k, up to replacing k by a finite extension if necessary) is non-isotrivial as well [Ta02, Lemma 1.32]. Now, C has $\overline{k}$-gonality $\geq 2d + 1$ but $\overline{k}$-isogonality $2 \leq d$.

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[13] In particular, one recovers [S03, Thm. 2.1] as a special case of Corollary A.7.

[14] Let B be an isotrivial curve over k and d an integer $\leq \gamma_k(C) - 1$. Then the existence of a non-constant morphism $B \to C^{(d)}$ implies that J has an isogeny factor which is isotrivial over k whereas the existence of a non-constant morphism (of arbitrary degree d) $C \to B$ implies that J has an isogeny factor which is an isotrivial *jacobian* over k.

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