Integral and adelic aspects of the Mumford-Tate conjecture

by

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Abstract. Let Y be an abelian variety over a subfield $k \subset \mathbb{C}$ that is of finite type over $\mathbb{Q}$. We prove that if the Mumford-Tate conjecture for Y is true, then also some refined integral and adelic conjectures due to Serre are true for Y. In particular, if a certain Hodge-maximality condition is satisfied, we obtain an adelic open image theorem for the Galois representation on the (full) Tate module of Y. We also obtain an (unconditional) adelic open image theorem for K3 surfaces. These results are special cases of a more general statement for the image of a natural adelic representation of the fundamental group of a Shimura variety.

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Introduction

To a Shimura datum (G, X) and a neat compact open subgroup $K_0 \subset G(\mathbb{A}_f)$ one can associate (see Section 3) a representation

$$\phi\colon \pi_1(S_0) \to K_0 \subset G(\mathbb{A}_f)\,,$$

where $S_0 \subset \mathrm{Sh}_{K_0}(G, X)$ (over some number field F) is a geometrically irreducible component of the Shimura variety $\mathrm{Sh}_{K_0}(G, X)$. This reflects the fact that we have a tower of finite étale covers $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K_0}(G, X)$ indexed by the open subgroups $K \subset K_0$. For ℓ a prime number, let $\phi_\ell\colon \pi_1(S_0) \to K_{0,\ell}$ be the ℓ-primary component of ϕ. The main result of this paper is about the image of ϕ.

Main Theorem. *With notation as above, fix a free $\mathbb{Z}$-module H of finite rank and a closed embedding $i\colon G \hookrightarrow \mathrm{GL}(H \otimes \mathbb{Q})$, and let $\mathscr{G} \subset \mathrm{GL}(H)$ be the Zariski closure of G in $\mathrm{GL}(H)$.*

(i) *There exists a positive integer N, depending only on $\mathscr{G}$ and X, such that $\bigl[\mathscr{G}(\mathbb{Z}_\ell) : \mathrm{Im}(\phi_\ell)\bigr] \leq N$ for all ℓ.*

(ii) *For almost all ℓ the image of ϕ_ℓ contains the commutator subgroup of $\mathscr{G}(\mathbb{Z}_\ell)$.*

(iii) *If (G, X) is maximal (see* Def. 2.7*), $\mathrm{Im}(\phi)$ is an open subgroup of $G(\mathbb{A}_f)$.*

The proof of this result only involves abstract theory of Shimura varieties. It relies on Deligne's group-theoretic description of the reciprocity law that gives the Galois action on the set of geometric connected components of $\mathrm{Sh}(G, X)$.

Using that the moduli spaces of abelian varieties and K3 surfaces are (essentially) Shimura varieties, we obtain as consequence of our main theorem that the usual Mumford-Tate conjecture implies refined integral and adelic forms of it that were conjectured by Serre. To describe these results in more detail, let Y be either an abelian variety or a K3 surface over a subfield $k \subset \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Write $H = H_1\bigl(Y(\mathbb{C}), \mathbb{Z}\bigr)$ in the first case and $H = H^2\bigl(Y(\mathbb{C}), \mathbb{Z}\bigr)(1)$ if Y is a K3 surface. Let $G_{\mathrm{B}} \subset \mathrm{GL}(H)$ be the Mumford-Tate group. We may identify $H \otimes \hat{\mathbb{Z}}$ with the étale cohomology of Y with $\hat{\mathbb{Z}}$-coefficients (H_1 or $H^2(1)$); this gives us a Galois representation

$$\rho_Y\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(H)\bigl(\hat{\mathbb{Z}}\bigr)\,.$$

It is known that, possibly after replacing k with a finite extension, the image of ρ_Y is contained in $G_{\mathrm{B}}(\hat{\mathbb{Z}})$. From now on, we assume this is the case. If ℓ is a prime number, let $\rho_{Y,\ell}\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(H)\big(\mathbb{Z}_\ell\big)$ be the ℓ-primary component of ρ_Y.

In the case of an abelian variety, our main theorem implies the following.

Theorem A. *Let Y be an abelian variety over k for which the Mumford-Tate conjecture is true.*

(i) *The index $\big[G_{\mathrm{B}}(\mathbb{Z}_\ell) : \mathrm{Im}(\rho_{Y,\ell})\big]$ is bounded when ℓ varies. Moreover, for almost all ℓ the image of $\rho_{Y,\ell}$ contains the commutator subgroup of $G_{\mathrm{B}}(\mathbb{Z}_\ell)$, as well as the integral homotheties $\mathbb{Z}_\ell^* \cdot \mathrm{id}$.*

(ii) *If the Hodge structure $H_1\big(Y(\mathbb{C}), \mathbb{Q}\big)$ is Hodge-maximal (see* Def. 2.3*), the image of ρ_Y is an open subgroup of $G_{\mathrm{B}}(\mathbb{A}_f)$.*

By a result of Larsen and Pink (see [14], Thm. 4.3), if the Mumford-Tate conjecture for an abelian variety is true for one prime number ℓ, it is true for all ℓ. The assumption that the Mumford-Tate conjecture for Y is true is therefore unambiguous. Let us also note that Hodge-maximality is a necessary condition for the image of ρ_Y to be open in $G_{\mathrm{B}}(\mathbb{A}_f)$; see Remark 2.6.

In the case of a K3 surface, the Mumford-Tate conjecture is known (due to Tankeev [29] and, independently, André [1]), and we prove that the Hodge-maximality assumption is always satisfied. In this case we obtain the following adelic open image theorem.

Theorem B. *If Y is a K3 surface, the image of ρ_Y is an open subgroup of $G_{\mathrm{B}}(\mathbb{A}_f)$.*

To understand how our main theorem leads to Theorems A and B, we have to link the representation ϕ to the Galois representation ρ_Y on the cohomology of Y. Taking G to be the Mumford-Tate group of Y, we obtain a Shimura variety that has an interpretation as a moduli space of abelian varieties or K3 surfaces with additional structures. If $y \in S_0(k)$ is the point corresponding to Y, we obtain a homomorphism $\sigma_y\colon \pi_1(y) \to \pi_1(S_0)$ (with $\pi_1(y) \cong \mathrm{Gal}(\bar{k}/k)$), and the composition $\phi \circ \sigma_y$ is isomorphic to the Galois representation ρ_Y. Using a classical result of Bogomolov, the assumption that the Mumford-Tate conjecture for Y is true, for some prime number ℓ, implies that the image of $\phi_\ell \circ \sigma_y$ is open in the image of ϕ_ℓ. Points $y \in S_0(k)$ for which this holds are said to be ℓ-Galois-generic with respect to ϕ. For points on a Shimura variety of abelian type, a result of the first author and Kret says that being ℓ-Galois-generic for some ℓ implies something that is a priori much stronger, namely that the image of the adelic representation $\phi \circ \sigma_y$ is open in the image of ϕ. (See [6], Thm. A. This result is a consequence of the open adelic image theorem for abelian schemes proven by the first author in [5].) Theorem A and Theorem B are obtained by combining this with our main Theorem on the image of ϕ.

Following the pioneering work of Serre, several authors have investigated the integral and (some variants of) the adelic forms of the Mumford-Tate conjecture. After we completed a first version of this paper, it was brought to our attention by Hindry and Ratazzi that they have recently also obtained Theorem A(i); see Thm. 10.1 in [9]. We refer to Remark 5.4(ii) for a brief overview of how our results relate to the work of other people.

The first two sections of the paper are of a preliminary nature. We recall some conjectures due to Serre that refine the Mumford-Tate conjecture. Also we discuss the notion of (Hodge-)maximality, which is a necessary condition for an adelic open image theorem to hold. Section 3 forms the core of the paper. We define the representation ϕ associated with a Shimura variety; further we state and prove the main results, Theorem 3.6 and its Corollary 3.7 (our main Theorem above), about the image of ϕ. In Section 4 we briefly recall various notions of Galois-genericity, and we state the result of Cadoret-Kret that we need. In Section 5, which is devoted to abelian varieties, we prove Theorem A. Also we give examples of abelian varieties for which the H_1 is not Hodge-maximal. These examples suggest that Hodge-maximality depends in a rather subtle way on the structure of the Mumford-Tate group. Finally, in Section 6 we discuss K3 surfaces. We prove that the $H^2(1)$ of a K3 surface is always Hodge-maximal, and we deduce Theorem B.

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1. An integral variant of the Mumford-Tate conjecture

1.1 Let Y be a smooth proper scheme of finite type over a subfield k of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Fix integers i and n.

Let $H = H^i\big(Y(\mathbb{C}),\mathbb{Z}\big)(n)/(\mathrm{torsion})$, which is a polarizable Hodge structure of weight $i-2n$. We denote by $G_{\mathrm{B}} \subset \mathrm{GL}(H)$ the Mumford-Tate group. By this we mean that the generic fibre $G_{\mathrm{B},\mathbb{Q}} \subset \mathrm{GL}(H_{\mathbb{Q}})$ is the Mumford-Tate group of $H_{\mathbb{Q}}$ in the usual sense, and that G_{B} is the Zariski closure of $G_{\mathrm{B},\mathbb{Q}}$ inside $\mathrm{GL}(H)$. If the context requires it, we include Y in the notation, writing $G_{\mathrm{B},Y}$, etc.

For a prime number ℓ, let $H_\ell = H^i\big(Y_{\bar{k}},\mathbb{Z}_\ell\big)(n)/(\mathrm{torsion})$, which is a free $\mathbb{Z}_\ell$-module of finite rank on which we have a continuous Galois representation

$$\rho_\ell \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(H_\ell)\,.$$

We denote by $G_\ell \subset \mathrm{GL}(H_\ell)$ the Zariski closure of the image of ρ_ℓ. The generic fibre $G_{\ell,\mathbb{Q}_\ell} \subset \mathrm{GL}(H_{\ell,\mathbb{Q}_\ell})$ is the Zariski closure of the image of the Galois representation on $H_{\ell,\mathbb{Q}_\ell}$. We define $G_\ell^0 \subset G_\ell$ to be the Zariski closure of the identity component $(G_{\ell,\mathbb{Q}_\ell})^0$.

If we replace k by a finitely generated extension, G_ℓ may become smaller, but its identity component G_ℓ^0 does not change. By a result of Serre (see [24] or [13], Prop. 6.14), there exists a finite field extension $k \subset k^{\mathrm{conn}}$ in $\mathbb{C}$ (depending on Y, i and n) such that for every field K that contains k^{conn} and every prime number ℓ, the generic fibre of G_{ℓ,Y_K} is connected.

Via the comparison isomorphism $H \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H_\ell$, we may view $G_{\mathrm{B}} \otimes \mathbb{Z}_\ell$ as a subgroup scheme of $\mathrm{GL}(H_\ell)$.

1.2 Mumford-Tate Conjecture. *With notation as above, $G_{\mathrm{B}} \otimes \mathbb{Z}_\ell = G_\ell^0$ as subgroup schemes of $\mathrm{GL}(H_\ell)$.*

Note that, though we have stated the conjecture using group schemes over $\mathbb{Z}_\ell$, the Mumford-Tate conjecture in this form is equivalent to the conjecture that $G_{\mathrm{B}} \otimes \mathbb{Q}_\ell$ equals $G^0_{\ell,\mathbb{Q}_\ell}$ as algebraic subgroups of $\mathrm{GL}(H_{\ell,\mathbb{Q}_\ell})$, which is the Mumford-Tate conjecture as it is usually stated. As H_ℓ is Hodge-Tate, it follows from a result of Bogomolov [2] (with some extensions due to Serre; see also [24]) that the image of ρ_ℓ is open in $G_\ell(\mathbb{Q}_\ell)$. Hence the Mumford-Tate conjecture is equivalent to the assertion that $\mathrm{Im}(\rho_\ell)$ is an open subgroup of $G_{\mathrm{B}}(\mathbb{Z}_\ell)$ (assuming $k = k^{\mathrm{conn}}$). Further note that the Mumford-Tate conjecture depends, a priori, on ℓ and also on the chosen complex embedding of k.

The following strengthening of the Mumford-Tate conjecture was proposed by Serre; see Conjecture C.3.7 in [23] and cf. [28].

1.3 Integral Mumford-Tate Conjecture (Serre)**.** *Retain the notation of* 1.1*, and assume $k = k^{\mathrm{conn}}$. Then for all ℓ the image $\mathrm{Im}(\rho_\ell)$ is contained in $G_{\mathrm{B}}(\mathbb{Z}_\ell)$ as an open subgroup, and the index $\big[G_{\mathrm{B}}(\mathbb{Z}_\ell) : \mathrm{Im}(\rho_\ell)\big]$ is bounded when ℓ varies. Further, for almost all ℓ the image of ρ_ℓ contains the commutator subgroup of $G_{\mathrm{B}}(\mathbb{Z}_\ell)$ and all homotheties of the form $c^{i-2n}\cdot \mathrm{id}$, for $c \in \mathbb{Z}_\ell^*$.*

Compared with the usual Mumford-Tate conjecture, the main point in the above conjecture is that it should be possible to bound the index of $\mathrm{Im}(\rho_\ell)$ in $G_{\mathrm{B}}(\mathbb{Z}_\ell)$ by a constant independent of ℓ.

2. Maximality, and an adelic form of the Mumford-Tate conjecture

2.1 Definition. Let M be a connected algebraic group over a field k of characteristic 0. Let $k \subset F$ be a field extension, S an algebraic group over F, and $h\colon S \to M_F$ a homomorphism. Then we say that h is *maximal* if there is no non-trivial isogeny of connected k-groups $M' \to M$ such that h lifts to a homomorphism $S \to M'_F$.

Note that if F is algebraically closed, maximality of h only depends on its $M(F)$-conjugacy class.

2.2 The following remarks closely follow [34], 0.2. Let M be a connected reductive group over a subfield $k \subset \mathbb{C}$. Let $\mathscr{C}$ be a conjugacy class of complex cocharacters $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to M_{\mathbb{C}}$. Let $\pi_1(M)$ denote the fundamental group of M as defined by Borovoi in [3]. This is a finitely generated $\mathbb{Z}$-module with a continuous action of $\Gamma = \mathrm{Gal}(\bar{k}/k)$. If $(X^*, R, X_*, R^\vee)$ is the root datum of $M_{\bar{k}}$ and $Q(R^\vee) = \langle R^\vee \rangle \subset X_*$ is the coroot lattice, $\pi_1(M) \cong X_*/Q(R^\vee)$.

The conjugacy class $\mathscr{C}$ of complex cocharacters corresponds to an orbit $\mathscr{C} \subset X_*$ under the Weyl group W. As the induced W-action on $\pi_1(M)$ is trivial, any two elements in $\mathscr{C}$ have the same image in $\pi_1(M)$; call it $[\mathscr{C}] \in \pi_1(M)$.

If M' is a connected reductive k-group and $f\colon M' \to M$ is an isogeny, the map induced by f identifies $\pi_1(M')$ with a $\mathbb{Z}[\Gamma]$-submodule of finite index in $\pi_1(M)$. Conversely, every such submodule comes from an isogeny of connected k-groups, which is unique up to isomorphism over M. A conjugacy class $\mathscr{C}$ as above lifts to M' if and only if $[\mathscr{C}] \in \pi_1(M')$.

We shall usually be in a situation where the $\mathbb{Z}[\Gamma]$-submodule spanned by $[\mathscr{C}]$ has finite index in $\pi_1(M)$. (See below.) In this case, there is a uniquely determined maximal isogeny $M' \to M$ of connected k-groups such that the $\mu \in \mathscr{C}$ lift to complex cocharacters of M'. Further, the cocharacters $\mu \in \mathscr{C}$ are maximal in the sense of Definition 2.1 if and only if $[\mathscr{C}]$ generates $\pi_1(M)$ as a $\mathbb{Z}[\Gamma]$-module.

2.3 Definition. Let V be a $\mathbb{Q}$-Hodge structure, given by the homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. Let $M \subset \mathrm{GL}(V)$ be the Mumford-Tate group. Then V is said to be *Hodge-maximal* if $h\colon \mathbb{S} \to M_{\mathbb{R}}$ is maximal in the sense of Definition 2.1.

Hodge-maximality of V is equivalent to the condition that the associated cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to M_{\mathbb{C}}$ is maximal. This allows us to apply 2.2, taking $k = \mathbb{Q}$ and $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If $\mathscr{C}$ is the $M(\mathbb{C})$-conjugacy class of μ, the assumption that M is the Mumford-Tate group of V implies that the $\mathbb{Z}[\Gamma]$-submodule of $X_*(M)$ generated by $\mathscr{C}$ has finite index. Hence also the $\mathbb{Z}[\Gamma]$-submodule of $\pi_1(M)$ generated by $[\mathscr{C}]$ has finite index. By what was explained in 2.2, V is Hodge-maximal if and only if $\mathbb{Z}[\Gamma] \cdot [\mathscr{C}] = \pi_1(M)$.

2.4 Retaining the notation and assumptions of 1.1, let $\hat{H} = \prod_\ell H_\ell$, where the product is taken over all prime numbers ℓ. We then have a continuous Galois representation

$$\rho\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(\hat{H})$$

whose ℓ-primary component is the representation ρ_ℓ defined in 1.1. The comparison isomorphism between singular and étale cohomology gives an isomorphism $\hat{H} \cong H \otimes \hat{\mathbb{Z}}$; via this we may view ρ as a representation taking values in $\mathrm{GL}(H)\big(\hat{\mathbb{Z}}\big)$.

The following adelic version of the Mumford-Tate conjecture was proposed by Serre; see Conjecture C.3.8 in [23].

2.5 Adelic Mumford-Tate Conjecture (Serre). *With notation as in* 1.1*, suppose that* $k = k^{\mathrm{conn}}$ *and that the Hodge structure* H *is Hodge-maximal. Then* $\mathrm{Im}(\rho)$ *is an open subgroup of* $G_{\mathrm{B}}(\mathbb{A}_f)$.

2.6 Remark. It follows from a result of Wintenberger that the Hodge-maximality of H is essential. (For simplicity we shall assume here that the ground field k is a number field.) Indeed, suppose there exists a non-trivial isogeny of $\mathbb{Q}$-groups $M' \to G_{\mathrm{B}}$ with M' connected, such that $h\colon \mathbb{S} \to G_{\mathrm{B},\mathbb{R}}$ lifts to a homomorphism $\mathbb{S} \to M'_{\mathbb{R}}$. By [33], Théorème 2.1.7, and possibly after replacing the ground field k with a finite extension, the ℓ-adic representations ρ_ℓ lift to Galois representations with values in M'. On the other hand, it follows from [18], Proposition 6.4, that the image of $M'(\mathbb{A}_f) \to G_{\mathrm{B}}(\mathbb{A}_f)$ is not open in $G_{\mathrm{B}}(\mathbb{A}_f)$; hence $\mathrm{Im}(\rho)$ cannot be open in $G_{\mathrm{B}}(\mathbb{A}_f)$.

2.7 Let (G, X) be a Shimura datum such that G is the generic Mumford-Tate group on X. By definition, this means that there exist points $h \in X$ for which there is no proper subgroup $G' \subset G$ such that $h\colon \mathbb{S} \to G_{\mathbb{R}}$ factors through $G'_{\mathbb{R}}$. The locus of points h for which this holds forms a subset $X^{\mathrm{Hgen}} \subset X$ called the Hodge-generic locus.

Similar to the definition in 2.3, we say that (G, X) is maximal if there is no non-trivial isogeny of Shimura data $f\colon (G', X') \to (G, X)$. (Note that G' is necessarily connected, as it is part of a Shimura datum.) Clearly, if (G, X) is maximal then all $h\colon \mathbb{S} \to G_{\mathbb{R}}$ in X^{Hgen} are maximal in the sense of Definition 2.1. Conversely, if some $h \in X^{\mathrm{Hgen}}$ is maximal then (G, X) is maximal. (If we have f as above, $f(X') \subset X$ is a union of connected components but need not be the whole X; however, changing f by an inner automorphism of G we can always ensure that some given $h \in X$ lies in $f(X')$.)

To each $h \in X$ corresponds a complex cocharacter μ_h of G, and the μ_h thus obtained all lie in a single $G(\mathbb{C})$-conjugacy class $\mathscr{C}(G, X)$. It follows from the previous remarks that (G, X) is maximal if and only if the associated class $\big[\mathscr{C}(G, X)\big]$ generates $\pi_1(G)$ as a $\mathbb{Z}[\Gamma]$-module.

2.8 Remarks. (i) Let $f\colon (G_1, X_1) \to (G_2, X_2)$ be a morphism of Shimura data with $f\colon G_1 \to G_2$ surjective. If G_1 is the generic Mumford-Tate group on X_1 then G_2 is the generic Mumford-Tate group on X_2. If $f\colon G_1 \to G_2$ is an isogeny then also the converse is true.

(ii) If in (i) $\mathrm{Ker}(f)$ is semisimple then also maximality is preserved: if (G_1, X_1) is maximal, so is (G_2, X_2). Indeed, in this case $\pi_1(G_2)$ is a quotient of $\pi_1(G_1)$ in such a way that $\big[\mathscr{C}(G_2, X_2)\big]$ is the image of $\big[\mathscr{C}(G_1, X_1)\big]$.

(iii) Given a Shimura datum (G, X), it follows from the remarks in 2.2 that, up to isomorphism, there exists a unique isogeny of Shimura data $f\colon (\tilde{G}, \tilde{X}) \to (G, X)$ such that $(\tilde{G}, \tilde{X})$ is maximal. By (i), G is the generic Mumford-Tate group on X if and only if $\tilde{G}$ is the generic Mumford-Tate group on $\tilde{X}$.

3. Adelic representations associated with Shimura varieties

3.1 Let (G, X) be a Shimura datum. Throughout we assume that G is the generic Mumford-Tate group on X. (See 2.7.) In this case, conditions (2.1.1.1–5) of [8] are satisfied and $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$. (Cf. [8], 2.1.11; for details see also [31], Lemma 5.13.)

If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup, we have $\mathrm{Sh}_K(G, X)\big(\mathbb{C}\big) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$. For $h \in X$ and $\gamma K \in G(\mathbb{A}_f)/K$, let $[h, \gamma K]$ denote the corresponding $\mathbb{C}$-valued point of $\mathrm{Sh}_K(G, X)$.

Let $K_0 \subset G(\mathbb{A}_f)$ be a neat compact open subgroup. If $K \subset K_0$ is an open subgroup, the induced morphism on Shimura varieties $\mathrm{Sh}_{K,K_0}\colon \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K_0}(G, X)$ is finite étale. If, moreover, K is normal in K_0, this morphism is Galois with group K_0/K.

Choose a point $h_0 \in X$. Let $S_{0,\mathbb{C}}$ be the irreducible component of $\mathrm{Sh}_{K_0}(G, X)_{\mathbb{C}}$ that contains the point $[h_0, eK_0]$. Let F be the field of definition of this component, which is a finite extension of the reflex field $E(G, X)$. To simplify notation, we write Sh_K for $\mathrm{Sh}_K(G, X)_F$. By construction, we have a geometrically irreducible component $S_0 \subset \mathrm{Sh}_{K_0}$.

For K an open normal subgroup of K_0, let $S_K \subset \mathrm{Sh}_K$ be the inverse image of $S_0 \subset \mathrm{Sh}_{K_0}$ under the transition morphism Sh_{K,K_0}. Then $S_K \to S_0$ is étale Galois with group K_0/K.

Let $\bar{s}_K = [h_0, eK] \in S_K(\mathbb{C})$. The system of points $\bar{s} = (\bar{s}_K)$ thus obtained is compatible in the sense that $\mathrm{Sh}_{K_2,K_1}(\bar{s}_{K_2}) = \bar{s}_{K_1}$ for $K_2 \subset K_1 \subset K_0$. We abbreviate $\bar{s}_{K_0}$ to $\bar{s}_0$. With this choice of base points, $S_K \to S_0$ corresponds to a homomorphism $\phi_K \colon \pi_1(S_0, \bar{s}_0) \to K_0/K$. If $K_2 \subset K_1$ are open normal subgroups of K_0, the homomorphism ϕ_{K_1} equals the composition of $\phi_{K_2} \colon \pi_1(S_0, \bar{s}_0) \to K_0/K_2$ and the canonical map $K_0/K_2 \to K_0/K_1$. We may therefore pass to the limit; as the intersection of all open normal subgroups $K \subset K_0$ is trivial, this gives a continuous homomorphism

$$\phi_{\bar{s}} \colon \pi_1(S_0, \bar{s}_0) \to K_0\,. \tag{3.1.1}$$

3.2 Remarks. (i) The homomorphism ϕ is functorial in the following sense. Let $f \colon (G, X) \to (G', X')$ be a morphism of Shimura data. On reflex fields we have $E(G', X') \subset E = E(G, X)$. Let $K_0 \subset G(\mathbb{A}_f)$ and $K'_0 \subset G'(\mathbb{A}_f)$ be neat compact open subgroups with $f(K_0) \subset K'_0$. Choose $h_0 \in X$ and let $h'_0 = f(h_0) \in X'$. As in 3.1, this gives rise to geometrically irreducible components $S_0 \subset \mathrm{Sh}_{K_0}(G, X)_F$ and $S'_0 \subset \mathrm{Sh}_{K'_0}(G', X')_{F'}$, and it is easy to see that $EF' \subset F$. Further, h_0 and h'_0 give rise to compatible systems of base points $\bar{s} = (\bar{s}_K)$ and $\bar{s}' = (\bar{s}'_{K'})$. The morphism $\mathrm{Sh}(f) \colon \mathrm{Sh}_{K_0}(G, X) \to \mathrm{Sh}_{K'_0}(G', X')_E$ restricts to a morphism $S_0 \to S'_{0,F}$ over F with $\bar{s}_0 \mapsto \bar{s}'_0$. We then have a commutative diagram

$$\begin{array}{ccc}
\pi_1(S_0, \bar{s}_0) & \xrightarrow{\ \mathrm{Sh}(f)_*\ } & \pi_1(S'_{0,F}, \bar{s}'_0) \subset \pi_1(S'_0, \bar{s}'_0) \\
\Big\downarrow{\scriptstyle \phi_{\bar{s}}} & & \Big\downarrow{\scriptstyle \phi'_{\bar{s}'}} \\
K_0 & \xrightarrow{\quad\quad f \quad\quad} & K'_0
\end{array}$$

(ii) The homomorphism ϕ is essentially independent of the choice of $h_0 \in X$ and the resulting system of base points $\bar{s}$. If we choose another point $h'_0 \in X$ that lies in the same connected component as h_0, this gives rise to a different collection of base points $\bar{s}'$. There is a canonically determined conjugacy class of isomorphisms $\alpha \colon \pi_1(S_0, \bar{s}_0) \xrightarrow{\sim} \pi_1(S_0, \bar{s}'_0)$. For α in this class, the homomorphisms $\phi_{\bar{s}}$ and $\phi_{\bar{s}'} \circ \alpha$ differ by an inner automorphism of K_0.

If h'_0 lies in a different connected component of X, there exists an inner automorphism $\alpha = \mathrm{Inn}(g)$ of (G, X) such that $\alpha(h_0)$ and h'_0 lie in the same component of X. By functoriality together with the previous case, it follows that the associated representations $\phi_{\bar{s}}$ and $\phi_{\bar{s}'}$ are conjugate when we restrict to suitable subgroups of finite index in the respective π_1's.

In view of the above remarks, we shall from now on omit the base point $\bar{s}_0$ from the notation, unless it plays a role in the discussion.

3.3 Our main goal in this section is to describe the image of the representation $\phi = \phi_{\bar{s}}$ defined in 3.1. In order to do this, we need to recall some definitions and results from the theory of Shimura varieties. For proofs of the stated results we refer to [8], Section 2. Throughout, (G, X) is a Shimura datum as in 3.1. Let $G(\mathbb{R})_+ \subset G(\mathbb{R})$ be the subgroup of elements that are mapped into the identity component $G^{\mathrm{ad}}(\mathbb{R})^+ \subset G^{\mathrm{ad}}(\mathbb{R})$ (for the Euclidean topology) under the adjoint map.

Let $\gamma \colon \tilde{G} \to G^{\mathrm{der}}$ denote the simply connected cover of the derived group of G. Then $G(\mathbb{Q})\gamma\tilde{G}(\mathbb{A})$ is a closed normal subgroup of $G(\mathbb{A})$, and we define $\pi(G) = G(\mathbb{A})/G(\mathbb{Q})\gamma\tilde{G}(\mathbb{A})$. Next define $\bar{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0 G(\mathbb{R})_+$, where π_0 means the group of connected components. This $\bar{\pi}_0\pi(G)$ is an abelian profinite group.

Define $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$, and let $G(\mathbb{Q})_+^-$ denote its closure inside $G(\mathbb{A}_f)$. The natural homomorphism $G(\mathbb{A}_f) \to \bar{\pi}_0\pi(G)$ induces an isomorphism $G(\mathbb{A}_f)/G(\mathbb{Q})_+^- \xrightarrow{\sim} \bar{\pi}_0\pi(G)$. If there is no risk of confusion we identify the two groups.

The group $G(\mathbb{A}_f)$ acts on the Shimura variety $\mathrm{Sh}(G, X)$ from the right. This action makes the set $\pi_0\bigl(\mathrm{Sh}(G, X)_{\mathbb{C}}\bigr)$ a torsor under $\bar{\pi}_0\pi(G)$.

3.4 Let $E = E(G, X)$ be the reflex field and E^{ab} its maximal abelian extension. As $\pi_0\bigl(\mathrm{Sh}(G, X)_{\mathbb{C}}\bigr)$ is a torsor under $\bar{\pi}_0\pi(G)$, which is abelian, the action of $\mathrm{Gal}(\overline{E}/E)$ on $\pi_0\bigl(\mathrm{Sh}(G, X)_{\mathbb{C}}\bigr)$ gives rise to a well-determined homomorphism

$$\mathrm{rec}\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to \bar{\pi}_0\pi(G) \cong G(\mathbb{A}_f)/G(\mathbb{Q})_+^-\,, \tag{3.4.1}$$

called the reciprocity homomorphism.

Let $q\colon G(\mathbb{A}_f) \to \bar{\pi}_0\pi(G)$ be the canonical map. For $K \subset G(\mathbb{A}_f)$ a compact open subgroup, we have an induced action of $\bar{\pi}_0\pi(G)$ on the set of irreducible components of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. All these components have the same stabilizer in $\bar{\pi}_0\pi(G)$, namely $q(K)$. Let $\mathrm{rec}_K\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to \bar{\pi}_0\pi(G)/q(K)$ denote the reciprocity map modulo $q(K)$.

3.5 Proposition. *Retain the notation and assumptions of* 3.1*. Then the image of the homomorphism* $\phi\colon \pi_1(S_0) \to K_0$ *is the subgroup* $q^{-1}\bigl(\mathrm{Im}(\mathrm{rec})\bigr) \cap K_0$ *of* K_0*.*

Proof. Since F is defined to be the field of definition of the irreducible component $S_{0,\mathbb{C}} \subset \mathrm{Sh}_{K_0}(G, X)_{\mathbb{C}}$, we have

$$\mathrm{Gal}(E^{\mathrm{ab}}/F) = \mathrm{rec}^{-1}\bigl(q(K_0)\bigr)$$

as subgroups of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. For K a normal open subgroup of K_0, the set of geometric irreducible components of S_K is a torsor under $q(K_0)/q(K) \subset \bar{\pi}_0\pi(G)/q(K)$, and the irreducible components of S_K (over F) correspond to the orbits under the action of $\mathrm{Gal}(E^{\mathrm{ab}}/F)$ via rec_K. Hence the image of $\phi_K\colon \pi_1(S_0) \to K_0/K$ is the inverse image under $q\colon K_0/K \to q(K_0)/q(K)$ of $\mathrm{rec}_K\bigl(\mathrm{Gal}(E^{\mathrm{ab}}/F)\bigr)$. The latter group is the image of $\bigl(\mathrm{Im}(\mathrm{rec}) \cap q(K_0)\bigr)$ in $q(K_0)/q(K)$, and so we find that $\mathrm{Im}(\phi_K)$ is the image of $q^{-1}\bigl(\mathrm{Im}(\mathrm{rec})\bigr) \cap K_0$ in K_0/K. The proposition follows by passing to the limit. □

3.6 Theorem. *With assumptions as in* 3.1*, the cokernel of the reciprocity map* (3.4.1) *has finite exponent, and it is a finite discrete group (i.e.,* $\mathrm{Im}(\mathrm{rec}) \subset \bar{\pi}_0\pi(G)$ *is an open subgroup) if* (G, X) *is maximal.*

Before we start discussing the proof, let us give the main corollary of this result.

3.7 Corollary. *With assumptions as in* 3.1*, consider the homomorphism* $\phi\colon \pi_1(S_0) \to K_0$*, and for a prime number* ℓ*, let* $\phi_\ell\colon \pi_1(S_0) \to K_{0,\ell}$ *be its* ℓ*-primary component. Fix a free* $\mathbb{Z}$*-module* H *of finite rank and a closed embedding* $i\colon G \hookrightarrow \mathrm{GL}(H \otimes \mathbb{Q})$*, and let* $\mathscr{G} \subset \mathrm{GL}(H)$ *be the Zariski closure of* G *in* $\mathrm{GL}(H)$*.*

- (i) *There exists a positive integer* N*, depending only on* $\mathscr{G}$ *and* X*, such that* $\bigl[\mathscr{G}(\mathbb{Z}_\ell) : \mathrm{Im}(\phi_\ell)\bigr] \leq N$ *for all* ℓ*.*
- (ii) *For almost all* ℓ *the image of* ϕ_ℓ *contains the commutator subgroup of* $\mathscr{G}(\mathbb{Z}_\ell)$*.*
- (iii) *If* (G, X) *is maximal in the sense defined in* 2.7*,* $\mathrm{Im}(\phi)$ *is an open subgroup of* $G(\mathbb{A}_f)$*.*

Proof. In view of Proposition 3.5, (iii) of the corollary is immediate from the second assertion in the theorem. For (ii) we only have to note that $\mathrm{Im}(\phi)$ contains the commutator subgroup of K_0 and that $K_{0,\ell} = \mathscr{G}(\mathbb{Z}_\ell)$ for almost all ℓ.

It remains to deduce (i) from the theorem. With the notation of 3.4, $q^{-1}\big(\mathrm{Im}(\mathrm{rec})\big)$ is a normal subgroup of $G(\mathbb{A}_f)$, with profinite abelian quotient $G(\mathbb{A}_f)/q^{-1}\big(\mathrm{Im}(\mathrm{rec})\big) \cong \mathrm{Coker}(\mathrm{rec})$. If m is the exponent of Coker(rec), it follows that $K_{0,\ell}/\mathrm{Im}(\phi_\ell)$ is a compact abelian ℓ-adic analytic group that is killed by m. Hence it is finite. It follows that $\mathrm{Im}(\phi_\ell)$ has finite index in $\mathscr{G}(\mathbb{Z}_\ell)$ for all ℓ, and so it suffices to prove (i) for all ℓ sufficiently large.

Choose a multiple M of m such that $\mathscr{G}$ is reductive over $\mathbb{Z}[1/M]$ and $K_{0,\ell} = \mathscr{G}(\mathbb{Z}_\ell)$ for all $\ell > M$. Fix an $\ell > M$, and let $\mathscr{G}_0 = \mathscr{G} \otimes \mathbb{F}_\ell$ denote the characteristic ℓ fibre of $\mathscr{G}$. Let $\tilde{\mathscr{G}_0} \to \mathscr{G}_0^{\mathrm{der}}$ be the simply connected cover of the derived subgroup. The image of $\tilde{\mathscr{G}_0}(\mathbb{F}_\ell) \to \mathscr{G}_0^{\mathrm{der}}(\mathbb{F}_\ell)$ is the normal subgroup $\mathscr{G}_0(\mathbb{F}_\ell)^+ \triangleleft \mathscr{G}_0(\mathbb{F}_\ell)$ that is generated by the ℓ-Sylow subgroups.

Still with $\ell > M$, the image of ϕ_ℓ contains the subgroup of $\mathscr{G}(\mathbb{Z}_\ell)$ generated by the ℓ-Sylow groups, and hence contains all elements $g \in \mathscr{G}(\mathbb{Z}_\ell)$ whose reduction modulo ℓ lies in $\mathscr{G}_0(\mathbb{F}_\ell)^+$. It therefore suffices to bound the m-torsion in $\mathscr{G}_0(\mathbb{F}_\ell)/\mathscr{G}_0(\mathbb{F}_\ell)^+$ by a constant independent of ℓ. As a first step, let μ be the kernel of $\tilde{\mathscr{G}_0} \to \mathscr{G}_0^{\mathrm{der}}$; this is a group of multiplicative type whose rank $|\mu|$ only depends on G. (If R is the absolute rank of G, it is known that μ has rank at most 2^R.) The quotient $\mathscr{G}_0^{\mathrm{der}}(\mathbb{F}_\ell)/\mathscr{G}_0(\mathbb{F}_\ell)^+$ injects into $H^1(\mathbb{F}_\ell,\mu)$, which is a quotient of $\mu(\overline{\mathbb{F}}_\ell)$ (cohomology of procyclic groups). In particular, $\big[\mathscr{G}_0^{\mathrm{der}}(\mathbb{F}_\ell) : \mathscr{G}_0(\mathbb{F}_\ell)^+\big]$ divides $|\mu|$. It therefore suffices to bound the m-torsion in $\mathscr{G}_0(\mathbb{F}_\ell)/\mathscr{G}_0^{\mathrm{der}}(\mathbb{F}_\ell)$ by a constant independent of ℓ. Writing $\mathscr{G}_0^{\mathrm{ab}} = \mathscr{G}_0/\mathscr{G}_0^{\mathrm{der}}$, the group $\mathscr{G}_0(\mathbb{F}_\ell)/\mathscr{G}_0^{\mathrm{der}}(\mathbb{F}_\ell)$ is a subgroup of $\mathscr{G}_0^{\mathrm{ab}}(\mathbb{F}_\ell)$. Further, if r is the rank of the torus G^{ab}, the kernel of multiplication by m on $\mathscr{G}_0^{\mathrm{ab}}$ is a finite group scheme of rank r^m; hence the m-torsion in $\mathscr{G}_0^{\mathrm{ab}}(\mathbb{F}_\ell)$ has cardinality at most r^m. □

We now turn to the proof of Theorem 3.6. In 3.9 and 3.11 we first prove the result in two special cases; the general case is then deduced in 3.13. To prove that Coker(rec) has finite exponent (resp., is finite), we use Deligne's group-theoretic description of the reciprocity map. (See [8], Théorème 2.6.3.) If $E = E(G,X)$ is the reflex field, we simply write E^* for the $\mathbb{Q}$-torus $\mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},E}$. Class field theory gives an isomorphism $\pi_0\pi(E^*) \xrightarrow{\sim} \mathrm{Gal}(E^{\mathrm{ab}}/E)$, which we normalize as in [8], 0.8.

We start with a general remark that is useful to us.

3.8 Remark. Let $f\colon G_1 \to G_2$ be a surjective homomorphism of reductive $\mathbb{Q}$-groups. Factor f as

$$G_1 \twoheadrightarrow G_2' \xrightarrow{\psi} G_2\,,$$

where $G_2' = G_1/\mathrm{Ker}(f)^0$. We make the simplifying assumption that $\mathrm{Ker}(\psi)$ (which is the finite étale group scheme $\pi_0\big(\mathrm{Ker}(f)\big)$) is commutative, as this is the only case we need.

By [18], Proposition 6.5, $G_1(\mathbb{A}) \to G_2'(\mathbb{A})$ has open image. The commutativity of $\mathrm{Ker}(\psi)$ implies that the image of $\psi(\mathbb{A})\colon G_2'(\mathbb{A}) \to G_2(\mathbb{A})$ is a normal subgroup. Further, if M is the rank of $\mathrm{Ker}(\psi)$ then $H^1\big(\mathrm{Spec}(\mathbb{A})_{\mathrm{\acute{e}t}}, \mathrm{Ker}(\psi)\big)$, and hence also the cokernel of $\psi(\mathbb{A})$, is killed by M.

3.9 The case of a torus. Suppose $G = T$ is a torus, in which case $X = \{h\}$ is a singleton. By definition of the reflex field E, the corresponding cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to T_{\mathbb{C}}$ is defined over E; so we have a homomorphism $\mu\colon \mathbb{G}_{\mathrm{m},E} \to T_E$. Restricting scalars to $\mathbb{Q}$ and composing with the norm map then gives a homomorphism of algebraic tori

$$\nu\colon E^* \xrightarrow{\mathrm{Res}(\mu)} \mathrm{Res}_{E/\mathbb{Q}} T_E \xrightarrow{\mathrm{Norm}_{E/\mathbb{Q}}} T\,.$$

Via the class field isomorphism, the reciprocity map (3.4.1) is the inverse of the composition

$$\pi_0\pi(E^*) \xrightarrow{\pi_0\pi(\nu)} \pi_0\pi(T) \xrightarrow{\mathrm{pr}} \bar{\pi}_0\pi(T)\,.$$

We apply what was explained in 2.2, taking $M = T$. In this case $\pi_1(T)$ is just the cocharacter group $X_*(T)$. With $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the assumption that T is the Mumford-Tate group of h means that $\mathbb{Z}[\Gamma] \cdot \mu$ has finite index in $X_*(T)$. Further, $(T, \{h\})$ is maximal if and only if $X_*(T)$ is generated by μ as a $\mathbb{Z}[\Gamma]$-module.

On the other hand, by definition of the reflex field E, the stabilizer of $\mu \in X_*(T)$ in Γ is precisely the subgroup $\Gamma_E = \mathrm{Gal}(\overline{\mathbb{Q}}/E) \subset \Gamma$. If $\mathfrak{a} \subset \mathbb{Z}[\Gamma]$ is the left ideal generated by the augmentation ideal of $\mathbb{Z}[\Gamma_E]$, we have $X_*(E^*) \cong \mathbb{Z}[\Gamma]/\mathfrak{a}$ as Galois modules, and $X_*(\nu)\colon X_*(E^*) \to X_*(T)$ is the map given by $(\gamma \bmod \mathfrak{a}) \mapsto \gamma \cdot \mu$. From the preceding remarks it therefore follows that ν, as a homomorphism of algebraic tori, is surjective, and that $\mathrm{Ker}(\nu)$ is connected if $(T, \{h\})$ is Hodge-maximal. The assertion of 3.6 now follows from Remark 3.8.

3.10 Let $f\colon (G_1, X_1) \to (G_2, X_2)$ be a morphism of Shimura data. Let E_i $(i = 1, 2)$ be the reflex field of (G_i, X_i), and denote by E_i^{ab} its maximal abelian extension. Then E_1 is a finite extension of E_2 in $\mathbb{C}$ and we have a commutative diagram

(3.10.1)
$$\begin{array}{ccc} \mathrm{Gal}(E_1^{\mathrm{ab}}/E_1) & \xrightarrow{\mathrm{rec}_{(G_1,X_1)}} & \bar{\pi}_0\pi(G_1) \cong G_1(\mathbb{A}_f)/G_1(\mathbb{Q})_+^- \\ \downarrow & & \downarrow \bar{f} \\ \mathrm{Gal}(E_2^{\mathrm{ab}}/E_2) & \xrightarrow{\mathrm{rec}_{(G_2,X_2)}} & \bar{\pi}_0\pi(G_2) \cong G_2(\mathbb{A}_f)/G_2(\mathbb{Q})_+^- \end{array}$$

in which $\bar{f}$ denotes the map induced by f. The image of the left vertical map is a subgroup of finite index in $\mathrm{Gal}(E_2^{\mathrm{ab}}/E_2)$.

3.11 The case when G^{der} is simply connected. Next we treat the case when the derived group G^{der} is simply connected. Let $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ and let $p\colon G \to G^{\mathrm{ab}}$ be the canonical map. Then $h^{\mathrm{ab}} = p \circ h$ is independent of $h \in X$, and $(G^{\mathrm{ab}}, \{h^{\mathrm{ab}}\})$ is a Shimura datum. By Remark 2.8(ii), if (G, X) is maximal, so is $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$. We apply 3.10 to the morphism $p\colon (G, X) \to (G^{\mathrm{ab}}, \{h^{\mathrm{ab}}\})$. By [7], Théorème 2.4, the right vertical map in the diagram is surjective with finite kernel. (Deligne's result says that p induces an isomorphism on the groups that we denote by $\pi_0\pi$; the groups $\bar{\pi}_0\pi$ are quotients of these by finite subgroups.) The theorem for (G, X) therefore follows from the result for $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$, which was proven in 3.9.

3.12 Lemma. *Let (G, X) be a Shimura datum as in* 3.1. *Then there exists a Shimura datum $(\tilde{G}, \tilde{X})$ and a morphism $f\colon (\tilde{G}, \tilde{X}) \to (G, X)$ such that*
(a) *the group $\tilde{G}$ is the generic Mumford-Tate group on $\tilde{X}$;*
(b) *the homomorphism $f\colon \tilde{G} \to G$ is surjective, and the induced $f^{\mathrm{der}}\colon \tilde{G}^{\mathrm{der}} \to G^{\mathrm{der}}$ is the simply connected cover of G^{der}.*
Moreover, if (G, X) is maximal, we can choose $(\tilde{G}, \tilde{X})$ such that it is maximal, too, and such that the kernel of $f\colon \tilde{G} \to G$ is connected.

Proof. By [16], Application 3.4, there exists a morphism of Shimura data $f_1\colon (G_1, \tilde{X}) \to (G, X)$ such that $f_1\colon G_1 \to G$ is surjective, $\mathrm{Ker}(f_1)$ is a torus, and $f_1^{\mathrm{der}}\colon G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ is the simply connected cover. Let $\tilde{G} \subset G_1$ be the generic Mumford-Tate group on $\tilde{X}$, and let $f\colon \tilde{G} \to G$ be the restriction of f_1 to $\tilde{G}$. The assumption that G is the generic Mumford-Tate group on X implies that f is surjective, and as $\tilde{G}$ is normal in G_1, this implies that $\tilde{G}^{\mathrm{der}} = G_1^{\mathrm{der}}$. So $(\tilde{G}, \tilde{X})$ and f satisfy (a) and (b).

Next assume (G, X) is maximal. We claim that for any $f\colon (\tilde{G}, \tilde{X}) \to (G, X)$ such that (a) and (b) hold, $\mathrm{Ker}(f)$ is connected (hence a torus). Indeed, let (G_2, X_2) be the quotient of $(\tilde{G}, \tilde{X})$ by $\mathrm{Ker}(f)^0$.

We then have an induced morphism of Shimura data $(G_2, X_2) \to (G, X)$. The map $G_2 \to G$ is an isogeny with kernel the group scheme $\pi_0\bigl(\mathrm{Ker}(f)\bigr)$ of connected components of $\mathrm{Ker}(f)$. By maximality of (G, X) this implies that $\pi_0\bigl(\mathrm{Ker}(f)\bigr)$ is trivial, which proves the claim.

The Shimura datum $(\tilde{G}, \tilde{X})$ we have obtained need not be maximal. By Remark 2.8(iii), there exists an isogeny $(\hat{G}, \hat{X}) \to (\tilde{G}, \tilde{X})$ with $(\hat{G}, \hat{X})$ maximal, and by 2.8(i) $\hat{G}$ is the generic Mumford-Tate group on $\hat{X}$. As $\hat{G} \to \tilde{G}$ is an isogeny and $\tilde{G}^{\mathrm{der}}$ is simply connected, we have $\hat{G}^{\mathrm{der}} \xrightarrow{\sim} \tilde{G}^{\mathrm{der}}$, and as we have already seen, the maximality of (G, X) implies that the kernel of $\hat{G} \to G$ is connected. ☐

3.13 To complete the proof of Theorem 3.6, let (G, X) be the Shimura datum considered in the assertion, and take $f\colon (\tilde{G}, \tilde{X}) \to (G, X)$ as in Lemma 3.12. As shown in 3.11, the theorem is true for $(\tilde{G}, \tilde{X})$. In view of diagram (3.10.1), it suffices to show that the induced map $\tilde{G}(\mathbb{A}_f) \to G(\mathbb{A}_f)$ has cokernel of finite exponent, and that the image of this map is open if (G, X) is maximal. This follows from Remark 3.8.

4. Galois-generic points

4.1 Let S be a geometrically connected scheme of finite type over a field k. Assume given a continuous representation $\psi\colon \pi_1(S) \to G(\mathbb{A}_f)$, where G is an algebraic group over $\mathbb{Q}$. (Throughout, we omit the choice of a geometric base point of S from the notation.) If ℓ is a prime number, we denote by $\psi_\ell\colon \pi_1(S) \to G(\mathbb{Q}_\ell)$ the ℓ-primary component of ψ.

If y is a point of S, we have a homomorphism $\sigma_y\colon \pi_1(y) \to \pi_1(S)$, well-determined up to conjugation. (Recall that $\pi_1(y)$ is the absolute Galois group of the residue field $k(y)$.)

4.2 Definition. (i) A point $y \in S$ is said to be *Galois-generic* with respect to ψ if the image of $\psi \circ \sigma_y$ is open in the image of ψ.

(ii) A point $y \in S$ is said to be *ℓ-Galois-generic* with respect to ψ if the image of $\psi_\ell \circ \sigma_y$ is open in the image of ψ_ℓ.

If it is clear which ψ we mean, we omit the phrase "with respect to ψ".

Clearly, if a point y is Galois-generic, it is ℓ-Galois generic for all ℓ. The following theorem by the first author and Kret (see [6], Theorem A) shows that for the representation associated with a Shimura variety of abelian type, the converse holds.

4.3 Theorem. *Let (G, X) be a Shimura datum of abelian type, $K_0 \subset G(\mathbb{A}_f)$ a neat compact open subgroup, $h_0 \in X$ a base point. Let $\phi_{\bar{s}}\colon \pi_1(S_0) \to K_0$ be the associated representation, as defined in* 3.1. *If a point $y \in S$ is ℓ-Galois-generic for some prime number ℓ then y is Galois-generic.*

5. Application to abelian varieties

5.1 For $g \geq 1$, equip $\mathbb{Z}^{2g}$ with the standard symplectic form, and let GSp_{2g} be the reductive group over $\mathbb{Z}$ of symplectic similitudes of $\mathbb{Z}^{2g}$. Let $\mathfrak{H}_g^{\pm}$ be the set of homomorphisms $h\colon \mathbb{S} \to \mathrm{GSp}_{2g,\mathbb{R}}$ that define a Hodge structure of type $(-1,0)+(0,-1)$ on $\mathbb{Z}^{2g}$ for which $\pm 2\pi i \cdot \psi$ is a polarization. The real group $\mathrm{GSp}_{2g}(\mathbb{R})$ acts transitively on $\mathfrak{H}_g^{\pm}$, and the pair $(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$ is a Shimura datum with reflex field $\mathbb{Q}$.

Let $K(3) \subset \mathrm{GSp}_{2g}(\hat{\mathbb{Z}})$ be the subgroup of elements that reduce to the identity modulo 3. Then $\mathrm{Sh}_{K(3)}(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$ is isomorphic (over $\mathbb{Q}$) to the moduli space $\mathscr{A}_{g,3}$ of g-dimensional principally polarized abelian varieties with a (Jacobi) level 3 structure. See for instance [7], Section 4. In what follows we identify the two schemes.

Let (B, λ) be a principally polarized abelian variety of dimension g over a subfield $k \subset \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Assume that all 3-torsion points of B are k-rational. (This implies that $\mathbb{Q}(\zeta_3) \subset k$.) Choose a similitude $i\colon H_1\bigl(B(\mathbb{C}), \mathbb{Z}\bigr) \xrightarrow{\sim} \mathbb{Z}^{2g}$. This gives a Hodge structure on $\mathbb{Z}^{2g}$; let $h_0 \in \mathfrak{H}_g^{\pm}$ be the corresponding point. As in section 3.1, $[h_0, eK(3)]$ defines a $\mathbb{C}$-valued point $\bar{t}_0 \in \mathscr{A}_{g,3}(\mathbb{C})$. The corresponding level 3 structure on $(B_{\mathbb{C}}, \lambda)$ is defined over k, which means that $\bar{t}_0$ comes from a k-valued point $t_0 \in \mathscr{A}_{g,3}(k)$ by composing it with the given embedding $k \hookrightarrow \mathbb{C}$.

Let $\mathscr{A}_0 \subset \mathscr{A}_{g,3} \otimes \mathbb{Q}(\zeta_3)$ be the irreducible component such that $t_0 \in \mathscr{A}_0(k)$. This component is geometrically irreducible. The construction of 3.1 gives a representation

$$\phi_{\bar{t}}\colon \pi_1(\mathscr{A}_0, \bar{t}_0) \to K(3) \subset \mathrm{GL}_{2g}(\hat{\mathbb{Z}})\,.$$

(Here, as in 3.1, $\bar{t}$ refers to the compatible system of base points $(\bar{t}_K)$ obtained from h_0.) The point t_0 gives rise to a homomorphism $\sigma_{t_0}\colon \mathrm{Gal}(\bar{k}/k) \to \pi_1(\mathscr{A}_0, \bar{t}_0)$ such that the composition with the projection $\pi_1(\mathscr{A}_0, \bar{t}_0) \to \mathrm{Gal}\bigl(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_3)\bigr)$ is the natural homomorphism $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}\bigl(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_3)\bigr)$. Note that σ_{t_0} is canonically defined, not only up to conjugation.

On the other hand, if we let $H = H_1\bigl(B(\mathbb{C}), \mathbb{Z}\bigr)$ then we may identify the full Tate module $\hat{H} = \varprojlim_n B[n]\bigl(\bar{k}\bigr)$ (limit over all positive integers n, partially ordered by divisibility) with $H \otimes \hat{\mathbb{Z}}$. Via the chosen similitude i we obtain an isomorphism $\hat{H} \xrightarrow{\sim} \hat{\mathbb{Z}}^{2g}$. The natural Galois action on $\hat{H}$ therefore gives a representation

$$\rho_B\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}_{2g}\bigl(\hat{\mathbb{Z}}\bigr)\,.$$

The following result is an immediate consequence of the definitions and the modular interpretation of $\mathrm{Sh}(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$. See also [31], Remark 2.8, as well as the next section, where we discuss the analogous (but slightly more involved) case of K3 surfaces.

5.2 Proposition. *The representations $\phi_{\bar{t}} \circ \sigma_{t_0}$ and ρ_B are the same.*

The main result of this section is that the usual form of the Mumford-Tate conjecture implies the integral and adelic versions of the Mumford-Tate conjecture as formulated in 1.3 and 2.5.

5.3 Theorem. *Let B be an abelian variety over a subfield $k \subset \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Assume that for some prime number ℓ the Mumford-Tate conjecture for B is true. Then the integral and adelic Mumford-Tate conjectures for B are true as well.*

Note that in this case the last part of Conjecture 1.3 says that the image of ρ_B contains an open subgroup of $\hat{\mathbb{Z}}^* \cdot \mathrm{id}$. This is in fact a result proven by Wintenberger [35]. (The result is stated in loc. cit. only for k a number field; this implies the same result over finitely generated fields, as we can specialize B to an abelian variety over a number field in such a way that the Mumford-Tate group does not change.)

Proof. In proving the theorem, we may replace the ground field k with a finite extension and B with an isogenous abelian variety. Hence we may assume that B admits a principal polarization and that all 3-torsion points of B are k-rational. This puts us in the situation of 5.1. We retain the notation and choices introduced there.

Via the chosen similitude $H \xrightarrow{\sim} \mathbb{Z}^{2g}$ we may view the Mumford-Tate group G_{B} of B as an algebraic subgroup of $\mathrm{GSp}_{2g,\mathbb{Q}}$. We take its Zariski closure $G_{\mathrm{B}} \subset \mathrm{GSp}_{2g}$ as integral model. (There is no need to introduce new notation for this integral form.) With $h_0 \in \mathfrak{H}_g^{\pm}$ as in 5.1, let $X \subset \mathfrak{H}_g^{\pm}$ be the $G_{\mathrm{B}}(\mathbb{R})$-orbit of h_0. The pair (G_{B}, X) is a Shimura datum, and by construction we have a morphism

$f\colon (G_{\mathrm{B}}, X) \to (\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$. Let $K_0 = f^{-1}\bigl(K(3)\bigr)$, which is a neat compact open subgroup of $G_{\mathrm{B}}(\mathbb{A}_f)$, and, with E the reflex field of (G_{B}, X), let $\mathrm{Sh}(f)\colon \mathrm{Sh}_{K_0}(G_{\mathrm{B}}, X) \to \mathscr{A}_{g,3} \otimes E$ be the morphism induced by f.

Let $\bar{s}_0 = [h_0, eK_0]$, which is a $\mathbb{C}$-valued point of $\mathrm{Sh}_{K_0}(G_{\mathrm{B}}, X)$ whose image under $\mathrm{Sh}(f)$ is $\bar{t}_0$. As in 3.1, let $S_{0,\mathbb{C}} \subset \mathrm{Sh}_{K_0}(G_{\mathrm{B}}, X)_{\mathbb{C}}$ be the irreducible component containing $\bar{s}_0$, let $F \subset \mathbb{C}$ be its field of definition, and let $S_0 \subset \mathrm{Sh}_{K_0}$ be the geometrically irreducible component thus obtained. Possibly after replacing k with a finite extension, $\bar{s}_0$ comes from a point $s_0 \in S_0(k)$, whose image in $\mathscr{A}_{g,3}(k)$ is the point t_0 that corresponds to (B, λ) equipped with a suitable level 3 structure. This point s_0 gives rise to a homomorphism $\sigma_{s_0}\colon \mathrm{Gal}(\bar{k}/k) \to \pi_1(S_0, \bar{s}_0)$. With $\phi_{\bar{s}}\colon \pi_1(S_0, \bar{s}_0) \to K_0 \subset G_{\mathrm{B}}(\hat{\mathbb{Z}})$ the homomorphism (3.1.1), it follows from the functoriality explained in Remark 3.2 together with Proposition 5.2 that $\phi_{\bar{s}} \circ \sigma_{s_0}$ is the Galois representation on the (full) Tate module of B.

Let ℓ be a prime number and $\phi_\ell\colon \pi_1(S_0, \bar{s}_0) \to G_{\mathrm{B}}(\mathbb{Z}_\ell)$ the ℓ-primary component of ϕ. If the Mumford-Tate conjecture for B is true at ℓ, s_0 is ℓ-Galois-generic with respect to the representation ϕ. (See the remark after 1.3.) By Theorem 4.3, s_0 is then Galois-generic, and in view of the description of $\mathrm{Im}(\phi)$ given in Corollary 3.7, this implies that Conjecture 1.3 and, if the Hodge structure H is maximal, Conjecture 2.5 are true for B. □

5.4 Remarks. (i) There are many special classes of abelian varieties for which the Mumford-Tate conjecture is known. For a sample of what is known, see for instance [17], Section 5, or the more recent paper [15]. These results build upon earlier work of Larsen, Pink, Ribet, Serre, and Tankeev, among others. On the other hand, already for abelian varieties of dimension 4 the Mumford-Tate conjecture remains open.

(ii) As mentioned in the introduction, several authors have investigated the integral and (some variants of) the adelic forms of the Mumford-Tate conjecture. Their approaches combine and develop the number-theoretic and group-theoretic techniques introduced by Serre in [22], [25] (see also the letters to Vigneras [26] and Ribet [24], [27]). More precisely, let A be an abelian variety over a finitely generated field $k \subset \mathbb{C}$, let G_{B} be the (integral) Mumford-Tate group of $A_{\mathbb{C}}$, and for a prime number ℓ, let G_ℓ be the Zariski-closure of the image of the Galois representation $\rho_\ell\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}_{2g}\bigl(T_\ell A\bigr)$ on the ℓ-adic Tate module of A. Concerning the comparison of $\mathrm{Im}(\rho_\ell)$ and $G(\mathbb{Z}_\ell)$ the first main result is the celebrated open image theorem of Serre; see [25], Thm. 3 and its Corollary (p. 35).

Serre also proved an "integral Mumford-Tate result" for the center:

(1) If C is the connected component of the center of G then the index $\bigl[C(\mathbb{Z}_\ell) : C(\mathbb{Z}_\ell) \cap \mathrm{Im}(\rho_\ell)\bigr]$ is bounded independently of ℓ.

See [25], Thm. 2 on p. 34, and [24], the Theorem on p. 60. The proof uses the Tate conjecture for abelian varieties, due to Faltings, and Serre's theory of abelian ℓ-adic representations [22]. An expanded exposition of Serre's argument can be found in [9], Section 10.

Refining Serre's ideas, and using inputs from p-adic Hodge theory, Wintenberger showed:

(2) Let $\tilde{G}_\ell$ be the simply connected cover of the derived subgroup of G_ℓ. Then for all $\ell \gg 0$ the image of ρ_ℓ contains the image of $\tilde{G}_\ell(\mathbb{Z}_\ell)$ in $G_\ell^{\mathrm{der}}(\mathbb{Z}_\ell)$. In particular, the index $\bigl[G_\ell^{\mathrm{der}}(\mathbb{Z}_\ell) : G_\ell^{\mathrm{der}}(\mathbb{Z}_\ell) \cap \mathrm{Im}(\rho_\ell)\bigr]$ is bounded independently of ℓ.

See [35], Thm. 2.

Combining (1) and (2), Hindry and Ratazzi have recently also obtained a proof of the integral Mumford-Tate conjecture assuming the usual Mumford-Tate conjecture; see [9], Thm. 10.1. (Their argument in fact shows that (1) and (2) imply that $[G_\ell(\mathbb{Z}_\ell) : \mathrm{Im}(\rho_\ell)]$ is bounded independently of ℓ.)

For arbitrary compatible systems of semisimple Galois representations, using group-theoretic arguments, Larsen [12] obtained a result similar to the one of Wintenberger, but only for a set of primes of density 1. Still in this setting, Larsen and Hui formulated a weak avatar of the adelic form of the Mumford-Tate conjectures (see [11], Conjecture 1.3) and they proved it for compatible systems coming from geometry under the assumption that the G_ℓ only have simple factors of Lie type A ([11], Thm. 4.1). Their arguments are mostly group-theoretic.

5.5 Example. As a first application we recover the result given in [28], 11.11. If B is a g-dimensional abelian variety with g odd (or $g = 2$, or $g = 6$) and $\mathrm{End}(B_{\bar{k}}) = \mathbb{Z}$, it is known that the Mumford-Tate conjecture for B is true and that the Mumford-Tate group is the full GSp_{2g}. (See [17], Theorem 5.14, for a more general result.) It is easily seen that the Shimura datum $(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$ is maximal; the conclusion therefore is that in this case the image of the representation ρ_B is open in $\mathrm{GSp}_{2g}(\mathbb{A}_f)$.

Next we want to give examples of abelian varieties B over finitely generated subfields of $\mathbb{C}$ for which the Hodge structure $H_1\bigl(B(\mathbb{C}), \mathbb{Q}\bigr)$ is not Hodge-maximal. Such examples of course also give us Shimura data of Hodge type that are not maximal. The first examples we discuss are of CM type; after that, we discuss an example in which the Mumford-Tate group is the almost direct product of the homotheties and a semisimple group.

5.6 Example. For our first construction, we start with a totally real field E_0 of degree g over $\mathbb{Q}$. Let $\sigma_1, \ldots, \sigma_g$ be the complex embeddings of E_0. Let k be an imaginary quadratic field. Then $E = k \cdot E_0$ is a CM field. Fix an embedding $\alpha \colon k \to \mathbb{C}$, and let τ_i $(i = 1, \ldots, g)$ be the complex embedding of E that extends σ_i and such that $\tau_i|_k = \alpha$. Thus, $T = \bigl\{\tau_1, \ldots, \tau_g, \bar{\tau}_1, \ldots, \bar{\tau}_g\bigr\}$ is the set of complex embeddings of E.

Consider the CM type Φ on E given by

$$\Phi = \{\tau_1, \bar{\tau}_2, \ldots, \bar{\tau}_g\}\,.$$

The pair (E, Φ) gives rise to an isogeny class of g-dimensional complex abelian varieties B, determined by the rule that $H_1\bigl(B(\mathbb{C}), \mathbb{Q}\bigr) \cong E$ as a $\mathbb{Q}$-vector space, with Hodge decomposition of $H_1\bigl(B(\mathbb{C}), \mathbb{C}\bigr) \cong \oplus_{\tau \in T}\, \mathbb{C}^{(\tau)}$ given by the rule that $\mathbb{C}^{(\tau)}$ is of type $(-1, 0)$ if $\tau \in \Phi$ and of type $(0, -1)$ otherwise. If $g > 1$ then Φ is a primitive CM type; in this case B is simple. As any abelian variety of CM type, B is defined over a number field, and by a result of Pohlmann [19] the Mumford-Tate conjecture is true for B.

As in Section 3, if F is a number field we simply write F^* for the torus $\mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_{\mathrm{m},F}$. Let $\mathrm{Norm} \colon E^* \to E_0^*$ be the norm homomorphism, and let $U \subset E^*$ be the subtorus given by $U = \mathrm{Norm}^{-1}(\mathbb{Q}^*)$. The cocharacter group $X_*(E^*)$ is the free $\mathbb{Z}$-module on the set T. The cocharacter group of U is given by

$$X_*(U) = \left\{ \sum_{i=1}^{g} a_i\tau_i + \sum_{i=1}^{g} b_i\bar{\tau}_i \in X_*(E^*) \;\middle|\; a_i + b_i \text{ is independent of } i \right\}.$$

The elements $f_i = \tau_i - \bar{\tau}_i$ $(i = 1, \ldots, g)$ together with $f_{g+1} = \sum_{i=1}^{g} \bar{\tau}_i$ form a basis for $X_*(U)$.

The cocharacter $\mu \colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to E^*$ corresponding to the Hodge structure $H_1\bigl(B(\mathbb{C}), \mathbb{Q}\bigr)$ is given by $\mu = \tau_1 + \bar{\tau}_2 + \cdots + \bar{\tau}_g = f_1 + f_{g+1}$. The Galois conjugates of μ are the elements $f_i + f_{g+1}$ for $i = 1, \ldots, g$ together with their complex conjugates $f_1 + \cdots + \hat{f}_i + \cdots + f_g + f_{g+1}$, for $i = 1, \ldots, g$. These are cocharacters in $X_*(U)$, and for $g > 2$ they span a submodule of index $g - 2$ in $X_*(U)$. The conclusion, therefore, is that U is the Mumford-Tate group of B if $g > 2$, and that $H_1\bigl(B(\mathbb{C}), \mathbb{Q}\bigr)$ is not Hodge-maximal if $g > 3$. In this last case, the image of the adelic Galois representation is therefore not open in the adelic points of the Mumford-Tate group.

5.7 Example. For our final example, we consider a Shimura datum (G, X) such that
(a) G is an inner form of a split group;
(b) $\pi_1(G)$ is non-cyclic.
Note that (b) holds if G^{ab} has dimension at least 2, or if $\dim(G^{\mathrm{ab}}) = 1$ and G^{der} is not simply-connected. (In the latter case this follows using [3], Cor. 1.7.) If (a) holds, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on $\pi_1(G)$ (see [3], Lemma 1.8), and by what was explained in 2.2 and 2.7, we conclude that any (G, X) satisfying (a) and (b) is non-maximal.

To obtain a concrete example, let D be a quaternion algebra over $\mathbb{Q}$ that is non-split at infinity, i.e., $D \otimes_{\mathbb{Q}} \mathbb{R}$ is Hamilton's quaternion algebra $\mathbb{H}$. The canonical involution $*$ on D is then a positive involution. Let $n = 2r$ be an even positive integer with $n \geq 6$, and consider a free (left-) D-module V of rank n equipped with a (-1)-hermitian form Ψ of discriminant 1 and Witt index r. Let $G' = \mathrm{U}_D(V, \Psi)$ be the corresponding unitary group, which we view as an algebraic subgroup of $\mathrm{GL}_{\mathbb{Q}}(V)$, and let $G \subset \mathrm{GL}_{\mathbb{Q}}(V)$ be the algebraic group generated by G' together with the homotheties $\mathbb{G}_{\mathrm{m}} \cdot \mathrm{id}$. The group $G' \otimes \mathbb{R}$ is isomorphic to the identity component of $\mathrm{U}_n^*(\mathbb{H})$ (which in some literature is denoted by $\mathrm{SO}^*(2n)$), and there is a unique $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to G_{\mathbb{R}}$ that make the pair (G, X) a Shimura datum of PEL type. (Cf. [8], Section 1.3.) The corresponding Shimura variety parametrizes polarized abelian varieties of dimension $2n$ with an action by (an order in) D, which is of Albert Type III. As G' is a $\mathbb{Q}$-simple group, G is the generic Mumford-Tate group on X.

Our assumption that Ψ has trivial discriminant implies that the index of G' is ${}^1\mathrm{D}_n$; see [30], Table II, pages 56–57. This means that G' (and hence also G) is an inner form of the split form, i.e., condition (a) is satisfied. On the other hand, $\dim(G^{\mathrm{ab}}) = 1$ and G^{der} is not simply-connected, so also (b) is satisfied. (In fact, $\pi_1(G) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.) We conclude that (G, X) is not maximal. If B is a complex abelian variety that corresponds to a Hodge-generic point of the Shimura variety defined by (G, X), the Hodge structure $H_1\bigl(B(\mathbb{C}), \mathbb{Q}\bigr)$ is not Hodge-maximal.

6. Application to K3 surfaces

6.1 Let L be a number field, G a connected reductive group over L, and let $M = \mathrm{Res}_{L/\mathbb{Q}}\, G$. Then $M_{\mathbb{C}} \cong \prod_{\sigma \in \Sigma} G_\sigma$, where Σ is the set of complex embeddings of L and $G_\sigma = G \otimes_{L,\sigma} \mathbb{C}$. With $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma_L = \mathrm{Gal}(\overline{\mathbb{Q}}/L)$, the fundamental group $\pi_1(M)$ is the Γ-module obtained from the Γ_L-module $\pi_1(G)$ by induction.

Let μ be a complex cocharacter of M. As in 2.2, its conjugacy class $\mathscr{C}$ defines an element $[\mathscr{C}] \in \pi_1(M)$. Let $W \subset \pi_1(M)$ be the $\mathbb{Z}[\Gamma]$-submodule generated by $[\mathscr{C}]$.

Suppose there is a unique $\tau \in \Sigma$ such that the projection μ_τ of μ onto the factor G_τ is non-trivial. View L as a subfield of $\mathbb{C}$ via τ; then μ_τ is a complex cocharacter of G. Its conjugacy class $\mathscr{C}_\tau$ defines an element $[\mathscr{C}_\tau]$ in $\pi_1(G)$. Let $W_\tau \subset \pi_1(G)$ be the $\mathbb{Z}[\Gamma_L]$-submodule that it generates. In this situation we have $W = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_L]} W_\tau$ as submodule of $\pi_1(M) = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_L]} \pi_1(G)$. Consequently, μ is maximal as a cocharacter of M if and only if μ_τ is maximal as a complex cocharacter of G.

6.2 Proposition. *Let V be a polarizable $\mathbb{Q}$-Hodge structure of K3 type, by which we mean that V is of type $(-1, 1) + (0, 0) + (1, -1)$ with Hodge numbers 1–n–1 for some n. Then V is Hodge-maximal.*

Proof. Without loss of generality we may assume that V is simple as a Hodge structure, which in this case means that there are no non-zero Hodge classes in V. (By definition, Hodge-maximality only depends on the Mumford-Tate group M and the defining homomorphism $h\colon \mathbb{S} \to M_{\mathbb{R}}$; these do not change if we replace V with $V \oplus \mathbb{Q}(0)$.) Let $L = \mathrm{End}_{\mathbb{Q}\mathsf{HS}}(V)$ be the endomorphism algebra of V as a $\mathbb{Q}$-Hodge structure, and choose a polarization form $\psi\colon V \times V \to \mathbb{Q}$. As shown by Zarhin in [36], Theorem 1.6, L is a field which is either totally real or a CM field.

First suppose L is totally real. By [32], Lemma 3.2, $\dim_L(V) \geq 3$. By [36], Theorem 2.2.1, $M = \mathrm{Res}_{L/\mathbb{Q}}\,\mathrm{SO}_L(V, \Psi)$, where $\Psi\colon V \times V \to L$ is the unique symmetric L-bilinear form on V such that $\mathrm{trace}_{L/\mathbb{Q}} \circ \Psi = \psi$. In particular, M is semisimple. Let Σ be the set of complex embeddings of L. Write $G = \mathrm{SO}_L(V, \Psi)$ and let $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to M_\mathbb{C} \cong \prod_{\sigma\in\Sigma} G_\sigma$ be the cocharacter that gives the Hodge structure. There is a unique $\tau \in \Sigma$ such that $\mu_\tau \neq 1$, so we are in the situation of 6.1. If $\dim_L(V) = 2l+1$ is odd (resp. $\dim_L(V) = 2l$ is even), the root system of $G_\mathbb{C}$ is of type B_l (resp. D_l). We follow the notation of [4], Planches II and IV. In the even case the calculation that follows goes through without changes if $l = 2$. With respect to the basis $\varepsilon_1, \ldots, \varepsilon_l$ for $\mathbb{R}^l = X_*(G) \otimes \mathbb{R}$, we have $X_*(G) = \mathbb{Z}^l$, and the coroot lattice $Q(R^\vee)$ consists of the vectors $(m_1, \ldots, m_l) \in \mathbb{Z}^l$ for which $\sum m_j$ is even. On the other hand, the cocharacter μ_τ corresponds to the vector $(1, 0, \ldots, 0)$; its image in $\pi_1(G) = X_*(G)/Q(R^\vee) \cong \mathbb{Z}/2\mathbb{Z}$ is therefore the non-trivial class. By what was explained in 2.2 this implies the assertion.

Next suppose L is a CM field. Let $L_0 \subset L$ be the totally real subfield. There is a unique symmetric hermitian form $\Psi\colon V \times V \to L$ (with respect to complex conjugation on L) such that $\mathrm{trace}_{L/\mathbb{Q}} \circ \Psi = \psi$, and by [36], Theorem 2.3.1, $M = \mathrm{Res}_{L_0/\mathbb{Q}}\,\mathrm{U}_L(V, \Psi)$. Write $G = \mathrm{U}_L(V, \Psi)$, and let Σ be the set of complex embeddings of L_0. As in the totally real case there is a unique $\tau \in \Sigma$ such that the cocharacter μ is non-trivial on the factor G_τ. If $n = \dim_L(V)$, we have $G_\mathbb{C} \cong \mathrm{GL}_n$ in such a way that μ_τ is conjugate to the cocharacter $\mathbb{G}_\mathrm{m} \to \mathrm{GL}_n$ given by $z \mapsto \mathrm{diag}(z, 1, \ldots, 1)$. It is straightforward to check that the corresponding class in $\pi_1(\mathrm{GL}_n) \cong \mathbb{Z}$ is a generator, and again by 2.2 and 6.1 this implies the assertion. □

6.3 Remark. In the proposition it is essential that we work with a Hodge structure of weight 0. As is well-known, if Y is a complex K3 surface, the Hodge structure $H = H^2_{\mathrm{prim}}\big(Y(\mathbb{C}), \mathbb{Q}\big)$ is not, in general, Hodge-maximal; but $H(1) = H \otimes \mathbb{Q}(1)$ is. For instance, if $\mathrm{End}_{\mathbb{Q}\mathsf{HS}}(H) = \mathbb{Q}$, the Mumford-Tate group of H is the group $\mathrm{GO}(H, \phi)$ of orthogonal similitudes, where ϕ is a polarization form. We have a non-trivial isogeny $\mathrm{CSpin}(H, \phi) \to \mathrm{GO}(H, \phi)$, such that the homomorphism $h\colon \mathbb{S} \to \mathrm{GO}(H, \phi)_\mathbb{R}$ that defines the Hodge structure on H lifts to a homomorphism $\tilde{h}\colon \mathbb{S} \to \mathrm{CSpin}(H, \phi)_\mathbb{R}$. Cf. [33], 2.2.3–4. By contrast, the Mumford-Tate group of $H(1)$ is the special orthogonal group $\mathrm{SO}(H, \phi)$. We can still lift to $\mathrm{CSpin}(H, \phi)$, but the homomorphism $\mathrm{CSpin}(H, \phi) \to \mathrm{SO}(H, \phi)$ is not an isogeny.

6.4 As a preparation for the main result of this section, we need to recall some facts about the moduli of polarized K3 surfaces. We closely follow Rizov [20], [21].

Fix a natural number d. Let (L_0, ψ) be the quadratic lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ (with U the hyperbolic lattice). With $\{e_1, f_1\}$ the standard basis of the first copy of U, let (L_{2d}, ψ_{2d}) be the sublattice $\langle e_1 + df_1\rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$ of L_0. In what follows we write SO for the $\mathbb{Z}$-group scheme $\mathrm{SO}(L_{2d}, \psi_{2d})$. For $n \geq 1$, let $K(n) \subset \mathrm{SO}(\mathbb{A}_f)$ be the compact open subgroup of elements in $\mathrm{SO}(\hat{\mathbb{Z}})$ that reduce to the identity modulo n. If K is an open subgroup of $K(n)$ for some $n \geq 3$, Rizov defines in [20], Section 6, a moduli stack $\mathscr{F}_{2d,K}$ over $\mathbb{Q}$ of K3 surfaces with a primitive polarization of degree $2d$ and a level K structure. (In fact, Rizov does this over open parts of $\mathrm{Spec}(\mathbb{Z})$, but for our purposes it suffices to work over $\mathbb{Q}$.) By [21], Cor. 2.4.3, $\mathscr{F}_{2d,K}$ is a scheme. If (Y, λ) is a K3 surface over a field k of characteristic 0 equipped with a primitive polarization of degree $2d$, a level $K(n)$-structure on (Y, λ) is an isometry $H^2_{\mathrm{prim}}(Y_{\bar{k}}, \mathbb{Z}/n\mathbb{Z})\big(1\big) \xrightarrow{\sim} L_{2d}/nL_{2d}$. (See [20], Example 5.1.3.)

The construction of 3.1 has an analogue in this setting. Let $\mathscr{F}_{0,\mathbb{C}}$ be an irreducible component of $\mathscr{F}_{2d,K(3)} \otimes \mathbb{C}$, and let $F \subset \mathbb{C}$ be its field of definition, so that we have a geometrically irreducible component $\mathscr{F}_0 \subset \mathscr{F}_{2d,K(3)} \otimes F$. For $K \subset K(3)$ we have an étale morphism $\mathscr{F}_{K,K(3)}\colon \mathscr{F}_{2d,K} \to \mathscr{F}_{2d,K(3)}$, which for K normal in $K(3)$ is Galois with group $K(3)/K$. Let $\mathscr{F}_K \subset \mathscr{F}_{2d,K} \otimes F$ be the inverse image of $\mathscr{F}_0$. Suppose we are given a compatible collection $\bar{y} = (\bar{y}_K)$ of geometric points of the $\mathscr{F}_K$, for K open and normal in $K(3)$. We write $\bar{y}_0$ for $\bar{y}_{K(3)}$. We then have and associated homomorphism

$$\Phi_{\bar{y}} \colon \pi_1(\mathscr{F}_0, \bar{y}_0) \to K(3) \subset \mathrm{SO}(\hat{\mathbb{Z}})\,. \tag{6.4.1}$$

6.5 With SO as above, let $\Omega^\pm$ be the space of homomorphisms $h\colon \mathbb{S} \to \mathrm{SO}_\mathbb{R}$ that give $L_{2d} \otimes \mathbb{Q}$ a Hodge structure of type $(-1,1) + (0,0) + (1,-1)$ with Hodge numbers 1–19–1, such that $\pm\psi_{2d}$ is a polarization. The group $\mathrm{SO}(\mathbb{R})$ acts transitively on $\Omega^\pm$, and the pair $(\mathrm{SO}_\mathbb{Q}, \Omega^\pm)$ is a Shimura datum of abelian type with reflex field $\mathbb{Q}$.

On of the main results of [21] (loc. cit., Thm. 3.9.1) is that for an open subgroup $K \subset K(3)$ there is an étale morphism of $\mathbb{Q}$-schemes

$$j_K \colon \mathscr{F}_{2d,K} \to \mathrm{Sh}_K(\mathrm{SO}_\mathbb{Q}, \Omega^\pm) \tag{6.5.1}$$

in such a way that for $K_2 \subset K_1$ the diagram

$$\begin{array}{ccc} \mathscr{F}_{2d,K_2} & \xrightarrow{j_{K_2}} & \mathrm{Sh}_{K_2}(\mathrm{SO}_\mathbb{Q}, \Omega^\pm) \\ \Big\downarrow{\scriptstyle \mathscr{F}_{K_2,K_1}} & & \Big\downarrow{\scriptstyle \mathrm{Sh}_{K_2,K_1}} \\ \mathscr{F}_{2d,K_1} & \xrightarrow{j_{K_1}} & \mathrm{Sh}_{K_1}(\mathrm{SO}_\mathbb{Q}, \Omega^\pm) \end{array} \tag{6.5.2}$$

is cartesian. The image of j_K is the complement of a divisor (ibid., 3.10(B)).

6.6 Theorem. *Let Y be a K3 surface over a subfield $k \subset \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Let $H = H^2\big(Y(\mathbb{C}), \mathbb{Z}\big)(1)$, and let $G_\mathrm{B} \subset \mathrm{GL}(H)$ be the Mumford-Tate group. Let $\rho_Y \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(H)(\hat{\mathbb{Z}})$ be the Galois representation on $\hat{H} = H^2\big(Y_{\bar{k}}, \hat{\mathbb{Z}}\big)(1)$, which we identify with $H \otimes \hat{\mathbb{Z}}$ via the comparison isomorphism between singular and étale cohomology. Then the image of ρ_Y has a subgroup of finite index which is an open subgroup of $G_\mathrm{B}(\hat{\mathbb{Z}})$.*

Proof. The proof is very similar to that of Theorem 5.3. The main difference is that for K3 surfaces the Mumford-Tate conjecture is known, due to results of Tankeev [29] and André [1], and that by Proposition 6.2 the Hodge structure on H is always Hodge-maximal.

We retain the notation introduced in 6.4 and 6.5. Choose a primitive polarization λ on Y, say of degree $2d$. Further choose an isometry $i\colon H \xrightarrow{\sim} L_{2d}$. These choices give us a compatible system $\bar{y} = (\bar{y}_K)$ of points $\bar{y}_K \in \mathscr{F}_{2d,K}(\mathbb{C})$, where K runs through the set of open subgroups of $K(3)$. Possibly after replacing k with a finite extension in $\mathbb{C}$, we may assume that $k = k^{\mathrm{conn}}$ and that $\bar{y}_0 = \bar{y}_{K(3)}$ comes from a k-valued point $y_0 \in \mathscr{F}_{2d,K(3)}(k)$ by composing it with the embedding $k \hookrightarrow \mathbb{C}$. Of course, y_0 is just the moduli point of (Y, λ) equipped with a suitable level 3 structure.

Via the chosen isometry i the Hodge structure on $H_\mathbb{Q}$ defines a point $h_0 \in \Omega^\pm$. Let $\bar{t} = (\bar{t}_K)$ be the system of $\mathbb{C}$-valued points $[h_0, eK]$ of $\mathrm{Sh}_K(\mathrm{SO}, \Omega^\pm)$, and abbreviate $\bar{t}_{K(3)}$ to $\bar{t}_0$. The construction of the period map (6.5.1) is such that $j_K(\bar{y}_K) = \bar{t}_K$ for all $K \subset K(3)$.

Let $t_0 = j_{K(3)}(y_0)$, which is a k-valued point of $\mathrm{Sh}_{K(3)}(\mathrm{SO}, \Omega^\pm)$. Let $\mathscr{F}_0 \subset \mathscr{F}_{2d,K(3)} \otimes k$ and $\mathscr{S}_0 \subset \mathrm{Sh}_{K(3)}(\mathrm{SO}, \Omega^\pm) \otimes k$ be the irreducible components containing $\bar{y}_0$ and $\bar{t}_0$, respectively; as they are smooth over k and have a k-rational point, these components are geometrically irreducible. By construction, $j_0 = j_{K(3)}$ restricts to an étale morphism $j_0 \colon \mathscr{F}_0 \to \mathscr{S}_0$ over k.

Consider the homomorphism $\phi_{\bar{t}} \colon \pi_1(\mathscr{S}_0, \bar{t}_0) \to K(3)$ as in 3.1. We also have the homomorphism $\Phi_{\bar{y}} \colon \pi_1(\mathscr{F}_0, \bar{t}_0) \to K(3)$ of (6.4.1). (In both cases we have now extended the base field to k.) The fact that the diagrams (6.5.2) are Cartesian implies that $\Phi_{\bar{y}} = \phi_{\bar{t}} \circ j_{0,*}$.

Let $H_{\mathrm{prim}} \subset H$ be the primitive integral cohomology, and identify the primitive étale cohomology with $\hat{\mathbb{Z}}$-coefficients $\hat{H}_{\mathrm{prim}} \subset \hat{H}$ with $H_{\mathrm{prim}} \otimes \hat{\mathbb{Z}}$. Via the chosen isometry i, the Galois action on $\hat{H}_{\mathrm{prim}}$ is a representation $\rho_{Y,\mathrm{prim}} \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{SO}(\hat{\mathbb{Z}})$. Note that the Galois action ρ_Y on $\hat{H}$ leaves $\hat{H}_{\mathrm{prim}}$ stable and is trivial on the complement; hence the image of ρ_Y is the same as the image of $\rho_{Y,\mathrm{prim}}$. On the other hand, the k-rational point y_0 gives rise to a section σ_{y_0} of the homomorphism $\pi_1(\mathscr{F}_0, \bar{y}_0) \to \mathrm{Gal}(\bar{k}/k)$ induced by the structural morphism $\mathscr{F}_0 \to \mathrm{Spec}(k)$. The composition $\Phi_{\bar{y}} \circ \sigma_{y_0} \colon \mathrm{Gal}(\bar{k}/k) \to K(3) \subset \mathrm{SO}(\hat{\mathbb{Z}})$ is the same as $\rho_{Y,\mathrm{prim}}$. The composition $j_{0,*} \circ \sigma_{y_0} \colon \mathrm{Gal}(\bar{k}/k) \to \pi_1(\mathscr{S}_0, \bar{t}_0)$ is the section σ_{t_0} given by the point $t_0 \in \mathscr{S}_0(k)$. It follows that $\phi_{\bar{t}} \circ \sigma_{t_0} \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{SO}(\hat{\mathbb{Z}})$ is the same as $\rho_{Y,\mathrm{prim}}$.

The rest is the same as in the proof of Theorem 5.3. Let $G_{\mathrm{B}} \subset \mathrm{GSp}_{2g}$ be the Mumford-Tate group, and let $X \subset \Omega^{\pm}$ be the $G_{\mathrm{B}}(\mathbb{R})$-orbit of h_0. The pair (G_{B}, X) is a Shimura datum and we have a morphism $f \colon (G_{\mathrm{B}}, X) \to (\mathrm{SO}, \Omega^{\pm})$. Let $K_0 = f^{-1}\big(K(3)\big)$, let E be the reflex field, and let $\mathrm{Sh}(f) \colon \mathrm{Sh}_{K_0}(G_{\mathrm{B}}, X) \to \mathrm{Sh}_{K(3)}(\mathrm{SO}, \Omega^{\pm}) \otimes E$ be the morphism induced by f. We have a compatible system $\bar{s}$ of points $\bar{s}_K = [h_0, eK] \in \mathrm{Sh}_K(G_{\mathrm{B}}, X)\big(\mathbb{C}\big)$ with $\mathrm{Sh}(f)\big(\bar{s}\big) = \bar{t}$. The point $\bar{s}_0 = \bar{s}_{K_0}$ comes from a k-valued point s_0, and if $S_0 \subset \mathrm{Sh}_{K_0}(G_{\mathrm{B}}, X)_k$ is the irreducible component in which it lies, s_0 gives a section $\sigma_{s_0} \colon \mathrm{Gal}(\bar{k}/k) \to \pi_1(S_0, \bar{s}_0)$. Finally, if $\phi_{\bar{s}} \colon \pi_1(S_0, \bar{s}_0) \to K_0 \subset G_{\mathrm{B}}(\hat{\mathbb{Z}})$ is the representation (3.1.1), it follows from the functoriality explained in Remark 3.2 that $\phi_{\bar{s}} \circ \sigma_{s_0}$ is the same as $\rho_{Y,\mathrm{prim}}$.

By the Mumford-Tate conjecture, s_0 is ℓ-Galois-generic with respect to $\phi_{\bar{s}}$ for every ℓ. By Theorem 4.3 it follows that s_0 is Galois-generic, and by Corollary 3.7(iii), the theorem follows. □

References

[1] Y. André, *On the Shafarevich and Tate conjectures for hyper-Kähler varieties.* Math. Ann. 305 (1996), no. 2, 205–248.

[2] F. Bogomolov, *Sur l'algébricité des représentations ℓ-adiques.* C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 15, A701–A703.

[3] M. Borovoi, *Abelian Galois cohomology of reductive groups.* Mem. Amer. Math. Soc. 132 (1998), no. 626.

[4] N. Bourbaki, *Éléments de mathématique.* Groupes et Algèbres de Lie, Chap. 4, 5 et 6. Masson, 1981.

[5] A. Cadoret, *An adelic open image theorem for Abelian schemes.* Int. Math. Res. Not. 2015, no. 20, 10208–10242.

[6] A. Cadoret, A. Kret, *Galois-generic points on Shimura varieties.* Algebra Number Theory 10 (2016), no. 9, 1893–1934.

[7] P. Deligne, *Travaux de Shimura.* Sém. Bourbaki (1970/71), Exp. No. 389. In: Lecture Notes in Math. 244, Springer, Berlin, 1971; pp. 123–165.

[8] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques.* In: Automorphic forms, representations and L-functions (Proc. Symp. Pure Math. 33, Part 2), pp. 247–289. A.M.S., 1979.

[9] M. Hindry, N. Ratazzi, *Torsion pour les variétés abéliennes de type I et II.* Algebra Number Theory 10 (2016), no. 9, 1845–1891.

[10] C.Y. Hui, *Monodromy of Galois representations and equal-rank subalgebra equivalence.* Math. Res. Lett. **20**, 2013, 1–24.

[11] C.Y. Hui, M. Larsen, *Adelic openness without the Mumford-Tate conjecture.* Preprint, arXiv:1312.3812.

[12] M. Larsen, *Maximality of Galois actions for compatible systems.* Duke Math. J. **80**, 1995, p. 601–630.

[13] M. Larsen, R. Pink, *On ℓ-independence of algebraic monodromy groups in compatible systems of representations.* Invent. Math. 107 (1992), no. 3, 603–636.

[14] M. Larsen, R. Pink, *Abelian varieties, ℓ-adic representations, and ℓ-independence.* Math. Ann. 302 (1995), no. 3, 561–579.

[15] D. Lombardo, *On the ℓ-adic Galois representations attached to nonsimple abelian varieties.* Ann. Inst. Fourier (Grenoble) 66 (2016), no. 3, 1217–1245.

[16] J. Milne, K.-y. Shih, *Conjugates of Shimura varieties.* In: Hodge cycles, motives, and Shimura varieties. Lecture Notes in Math. 900, Springer, 1982; pp. 280–356.

[17] R. Pink, *ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture.* J. reine angew. Math. 495 (1998), 187–237.

[18] V. Platonov, A. Rapinchuk, *Algebraic groups and number theory.* Pure and Applied Mathematics 139. Academic Press, Inc., Boston, MA, 1994.

[19] H. Pohlmann, *Algebraic cycles on abelian varieties of complex multiplication type.* Ann. of Math. (2) 88 (1968), 161–180.

[20] J. Rizov, *Moduli stacks of polarized K3 surfaces in mixed characteristic.* Serdica Math. J. 32 (2006), no. 2–3, 131–178.

[21] J. Rizov, *Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic.* J. reine angew. Math. 648 (2010), 13–67.

[22] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves.* W.A. Benjamin, 1968.

[23] J.-P. Serre, *Représentations ℓ-adiques.* In: Algebraic number theory (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976), pp. 177–193. Japan Soc. Promotion Sci., Tokyo, 1977. [=Œuvres 112.]

[24] J.-P. Serre, *Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981.* In: Œuvres, Volume IV, number 133.

[25] J.-P. Serre, *Résumé des cours de 1985–86.* In: Œuvres, Volume IV, number 136.

[26] J.-P. Serre, *Lettre à Marie-France Vigneras du 10/2/1986.* In: Œuvres, Volume IV, number 137.

[27] J.-P. Serre, *Lettre à Ken Ribet du 7/3/1986.* In: Œuvres, Volume IV, number 138.

[28] J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques.* In: Motives (Proc. Symp. Pure Math. 55, Part 1), pp. 377–400. A.M.S., 1994. [=Œuvres 161.]

[29] S. Tankeev, *Surfaces of K3 type over number fields and the Mumford-Tate conjecture. I, II.* Math. USSR-Izv. 37 (1991), no. 1, 191–208, and Izv. Math. 59 (1995), no. 3, 619–646.

[30] J. Tits, *Classification of algebraic semisimple groups.* In: Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math. 9), pp. 33–62. A.M.S., 1966.

[31] E. Ullmo, A. Yafaev, *Mumford-Tate and generalised Shafarevich conjectures.* Ann. Math. Qué. 37 (2013), no. 2, 255–284.

[32] B. van Geemen, *Real multiplication on K3 surfaces and Kuga-Satake varieties.* Michigan Math. J. 56 (2008), no. 2, 375–399.

[33] J.-P. Wintenberger, *Relèvement selon une isogénie de systèmes ℓ-adiques de représentations galoisiennes associés aux motifs.* Invent. Math. 120 (1995), no. 2, 215–240.

[34] J.-P. Wintenberger, *Une extension de la théorie de la multiplication complexe.* J. reine angew. Math. 552 (2002), 1–14.

[35] J.-P. Wintenberger, *Démonstration d'une conjecture de Lang dans des cas particuliers.* J. reine angew. Math. 553 (2002), 1–16.

[36] Yu. Zarhin, *Hodge groups of K3 surfaces.* J. reine angew. Math. 341 (1983), 193–220.

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