ULTRAPRODUCT COEFFICIENTS IN ÉTALE COHOMOLOGY - A SURVEY

ANNA CADORET

In this paper k always denotes a finite field of characteristic $p>0$. A variety over k means a reduced scheme separated and of finite type over k.

1. Introduction

Etale cohomology and its by-product ℓ-adic cohomology is the solution provided by Grothendieck and his school to the problem of the existence of a Weil cohomology for smooth projective varieties over fields of positive characteristic. The existence of such a cohomology is a central part in Grothendieck's motivic approach to the Weil conjectures; it already gives, via the Grothendieck-Lefschetz trace formula, the rationality and functional equation for the zeta function. Modulo some of the standard conjectures, it should also formally imply the Riemann Hypothesis [Kl]. While the standard conjectures are still widely open, the Riemann hypothesis was proved by Deligne in Weil I [D74], using geometric methods relying on the deepest properties of ℓ-adic cohomology. Deligne later refined these methods in Weil II [D80] to develop a systematic theory of Frobenius weights for constructible ℓ-adic sheaves.

However, another *a priori* natural way to build a Weil cohomology from étale cohomology is to consider ultraproduct of finite fields. Surprisingly, this approach seems to have been almost unexplored till now. Cohomology groups with constant ultraproduct coefficients appear briefly and seemingly for the first time on p. 389 of [S04]. In [T04], Tomasic checked that étale cohomology with constant ultraproduct coefficients indeed gives rise to a Weil cohomology. Actually, the only axioms which do not follow directly from the classical properties of étale cohomology with torsion coefficients are the finiteness of the cohomology groups with constant ultraproduct coefficients and hard Lefschetz. In [T04], this is derived from the similar statements for ℓ-adic cohomology and Gabber's torsion freeness theorem for cohomology groups with $\mathbb{Z}_\ell$-coefficients [G83] - a tricky consequence of the gcd theorem of Weil II [D80, Thm. (4.5.1)]. By devissage using nodal curves in the spirit of [dJ16], Orgogozo gave more recently a direct proof (in the sense that it does use the theory of Frobenius weights for ℓ-adic cohomology) of the finiteness of the cohomology groups with constant ultraproduct coefficients [O19, Thm. 3.1.1, Rem. 3.1.4, §6.2]. Actually Orgogozo's finiteness results are more general and made me realize that one could give a simple definition of étale $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems with a reasonably well-behaved cohomology theory - in particular finite-dimensional cohomology groups - satisfying a theory of Frobenius weights paralleling the one for $\overline{\mathbb{Q}}_\ell$-local systems as developed in Weil II. More precisely, let $\mathcal{L}$ be an infinite set of primes not containing p, $\mathfrak{u}$ a non-principal ultrafilter on $\mathcal{L}$ and $\overline{\mathbb{Q}}_{\mathfrak{u}}$ the quotient of $\overline{\mathbb{F}} := \prod_{\ell\in\mathcal{L}} \overline{\mathbb{F}}_\ell$ by the maximal ideal defined by $\mathfrak{u}$; recall $\overline{\mathbb{Q}}_{\mathfrak{u}}$ is a field isomorphic to $\mathbb{C}$ (which will allow to define weights just as in the $\overline{\mathbb{Q}}_\ell$ setting). One can then define the category of $\overline{\mathbb{Q}}_{\mathfrak{u}}$-étale sheaves as the quotient category of the product of the categories of $\overline{\mathbb{F}}_\ell$-étale sheaves by the full subcategory of those objects $\underline{\mathcal{M}} = \mathcal{M}_\ell$, $\ell\in\mathcal{L}$ for which the set of primes $\ell\in\mathcal{L}$ with $\mathcal{M}_\ell = 0$ is in $\mathfrak{u}$. The category of $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems is the full-subcategory of this quotient category whose objects arise from those $\underline{\mathcal{M}} = \mathcal{M}_\ell$, $\ell\in\mathcal{L}$ with uniformly bounded $\overline{\mathbb{F}}_\ell$-rank and for which there exists an étale cover $X'\to X$ such that the set of primes $\ell\in\mathcal{L}$ with $\mathcal{M}_\ell|_{X'}$ tame is in $\mathfrak{u}$ (See Subsection 2.4 for the definition of tameness for higher-dimensional X). Already when X is a curve and as can be seen from the Grothendieck-Ogg-Shafarevich formula, the almost uniform tameness condition is necessary to ensure the cohomology groups with compact support $(\prod_{\ell\in\mathcal{L}} H_c(X_{\overline{k}}, \mathcal{M}_\ell))\otimes\overline{\mathbb{Q}}_{\mathfrak{u}}$ be finite-dimensional.

The search for a well-behaved category of $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems was originally motivated by extending to $\overline{\mathbb{F}}_\ell$-local systems properties of $\overline{\mathbb{Q}}_\ell$-local systems. For instance, it is an amazingly simple consequence of

Date: The 1st JNT biennal conference - Grand Hotel San Michele, Cetraro, July 22nd - 26th, 2019.

the theory of Frobenius weights that, if $f : Y \to X$ is a smooth proper morphism, the $\mathbb{Q}_\ell$-local system $R^\bullet f_*\mathbb{Q}_\ell|_{X_{\overline{k}}}$ is semisimple [D80, (3.4)]. One may thus ask if the $\mathbb{F}_\ell$-local system $R^\bullet f_*\mathbb{F}_\ell|_{X_{\overline{k}}}$ is semisimple[1] as well for $\ell \gg 0$. Actually, this specific result is proved in [CHT17a]. But the arguments there are rather involved; they rely on a combination of geometric trick, the theory of Frobenius weights for $R^i f_*\overline{\mathbb{Q}}_\ell$ and quite a heavy group-theoretical machinery, including the results of [CT17], a bit of Bruhat-Tits theory, Lie theory *etc.* The rough idea is to compare the $\pi_1(X_{\overline{k}})$-action $R^\bullet f_*\mathbb{Q}_{\ell\,\overline{x}}$ and $R^\bullet f_*\mathbb{F}_{\ell\,\overline{x}}$ by Tannakian methods, using the fact that, by Gabber's theorem mentioned above, $R^\bullet f_*\mathbb{F}_\ell = (R^\bullet f_*\mathbb{Z}_\ell) \otimes \mathbb{F}_\ell$ for $\ell \gg 0$. At the time we were writing [CHT17a], I realized things would become almost straightforward if we would have at disposal a flexible enough theory of Frobenius weights for $\overline{\mathbb{F}}_\ell$-coefficients provided $\ell \gg 0$. Formalizing this using ultraproducts of the $\overline{\mathbb{F}}_\ell$ was, I guess, reminiscent from my reading of Serre's article [S04] and some informal discussions I had with Arno Kret during my stay at the I.A.S. in 2013-14.

The aim of this note is to give an overview of the main results of [C20b] and [CT20]. In the first part of [C20b] we settle the notion of $\overline{\mathbb{Q}}_{\mathrm{u}}$-local system, prove the analogue of the 'fundamental theorem of Weil II for curves' [D80, (3.2.1)] and derive from it some of the classical properties of the theory of Frobenius weights for pure local systems (purity, geometric semisimplicity *etc.*). The second part of [C20b] is devoted to applications to the torsion, unicity and residual semisimplicity / irreducibility properties for integral models in compatible families of $\overline{\mathbb{Q}}_\ell$-local systems. Typical examples of compatible families are the $R^\bullet f_*\overline{\mathbb{Q}}_\ell|_{X_{\overline{k}}}$, $\ell \neq p$ for $f : Y \to X$ a smooth proper morphism but, more generally, every irreducible $\overline{\mathbb{Q}}_\ell$-local system with finite determinant is part of such a compatible family - this is an output of the Langlands correspondance and Deligne's companion conjecture [D80, Conj. (1.2.10)]. In the third part of [C20b], which builds on the results in the second part, we 'complete' the Langlands correspondance and companion conjecture to include $\overline{\mathbb{Q}}_{\mathrm{u}}$-local systems and deduce from this finiteness and lifting results for $\overline{\mathbb{F}}_\ell$-local systems as well as independence results for algebraic monodromy. The paper [CT20] builds further on the companion conjecture for $\overline{\mathbb{Q}}_\ell/\overline{\mathbb{Q}}_{\mathrm{u}}$-local systems to show the Tannakian form of the Cebotarev density theorem 'transfers' from $\overline{\mathbb{Q}}_\ell$- to $\overline{\mathbb{Q}}_{\mathrm{u}}$-local systems. We follow these general guidelines in our exposition. We omitted most of the proofs but, still, kept sketches of some of them in order to give the reader an idea of their flavor and simplicity, especially in the applications.

Conceptually, the fact that $\overline{\mathbb{Q}}_{\mathrm{u}}$-local systems as defined in [C20b] fit in the global picture of the Langlands correspondance and companion conjecture suggests that the definition of [C20b] is the 'right one'. On the other hand, for possible further applications, it would be desirable to embed the category of $\overline{\mathbb{Q}}_{\mathrm{u}}$-local systems into a well-behaved (derived) category of constructible $\overline{\mathbb{Q}}_{\mathrm{u}}$-sheaves stable under the six-operations and satisfying a relative theory of Frobenius weights for arbitrary separated morphisms $f : Y \to X$ paralleling the one developed by Deligne from [D80, (3.3)]. Possibly the most natural attempt would be to enlarge the étale topos and adjust the notion of good stratification introduced by Orgogozo [O19][2]. The first basic statement such a theory of constructible sheaves should imply and that I currently do not know how to prove is the following:

Question. *Let X be a (smooth) variety over k and let $\mathcal{C}$ be ι-pure $\overline{\mathbb{Q}}_{\mathrm{u}}$-local system of weight w. Is $H^i_c(X_{\overline{k}}, \mathcal{C})$ ι-mixed of weights $\leq w+i$?*

Fix an algebraic closure $\overline{k}$ of k; let $\varphi \in \pi_1(k) := Gal(\overline{k}|k)$ denote the geometric Frobenius of k (that is the inverse of the $|k|$th power map). Given a variety X over k and a point $x \in X$ we always denote by $\overline{x}$ a geometric point over x. Let $|X| \subset X$ denote the set of closed points. For $x \in |X|$, write $\varphi_x \in \pi_1(x)$ for the geometric Frobenius at x.

In the following, a sheaf always means a sheaf for the étale topology.

Fix an infinite set of primes $\mathcal{L}$ not containing p and set $\underline{\overline{\mathbb{F}}} := \prod_{\ell \in \mathcal{L}} \overline{\mathbb{F}}_\ell$.

[1] Such properties are required, for instance, to show the growth of the genus and gonality of abstract modular curves attached to such families of $\overline{\mathbb{F}}_\ell$-local systems - See [CT19], [CT16], [C20c].

[2] One technical issue in the notion of good stratification as defined in [O19] stems from the condition that $j_!\mathbb{F}$ be constructible and tame for the topology of alteration in [O19, Thm. 5.1]; this condition is not preserved by $R^i f_*$, which makes delicate, for instance, devissages using Leray spectral sequences as in the proof of [D80, Thm. (3.3.1)].

2. The category of $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems

2.1. Let T be a scheme. For $\ell \in \mathcal{L}$, let $S(T, \overline{\mathbb{F}}_\ell)$ denote the category of $\overline{\mathbb{F}}_\ell$-sheaves and $S(T, \underline{\overline{\mathbb{F}}})$ the 'product category' of the categories $S(T, \overline{\mathbb{F}}_\ell)$, $\ell \in \mathcal{L}$ that is the category whose objects are families $\underline{\mathcal{M}} = \mathcal{M}_\ell$, $\ell \in \mathcal{L}$ with $\mathcal{M}_\ell \in S(T, \overline{\mathbb{F}}_\ell)$, $\ell \in \mathcal{L}$ and whose morphisms $\underline{\mathcal{M}} \to \underline{\mathcal{N}}$ are families $\underline{\phi} = \phi_\ell : \mathcal{M}_\ell \to \mathcal{N}_\ell$, $\ell \in \mathcal{L}$ of morphism $\phi_\ell : \mathcal{M}_\ell \to \mathcal{N}_\ell$ in $S(T, \overline{\mathbb{F}}_\ell)$. Objects (resp. morphisms) in $S(T, \underline{\overline{\mathbb{F}}})$ are denoted by $\underline{\mathcal{M}}$ (resp. $\underline{\phi} : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$) and we write $\mathcal{M}_\ell$ (resp. $\phi_\ell : \mathcal{M}_\ell \to \mathcal{N}_\ell$) for the ℓth component of $\underline{\mathcal{M}}$ (resp. of $\underline{\phi}$). The category $S(T, \underline{\overline{\mathbb{F}}})$ inherits from the $S(T, \overline{\mathbb{F}}_\ell)$, $\ell \in \mathcal{L}$ a structure of rigid abelian monoidal category, namely $\underline{\mathcal{M}} \oplus \underline{\mathcal{N}} = \mathcal{M}_\ell \oplus \mathcal{N}_\ell$, $\ell \in \mathcal{L}$, $\underline{\mathcal{M}} \otimes \underline{\mathcal{N}} = \mathcal{M}_\ell \otimes \mathcal{N}_\ell$, $\ell \in \mathcal{L}$ *etc.*

2.1.1. The cohomology theory we consider on $S(T, \underline{\overline{\mathbb{F}}})$ is the product[3] of the étale cohomology theories on each of the $S(T, \overline{\mathbb{F}}_\ell)$, $\ell \in \mathcal{L}$, namely stalks at geometric points, cohomology groups, cohomology groups with compact support, inverse images, higher direct images *etc.* are defined componentwise. For instance, if $F : S(T, \overline{\mathbb{F}}_\ell) \to Mod_{/\overline{\mathbb{F}}_\ell}$ denotes any of the functors $H^\bullet(T, -)$ (étale cohomology), $H^\bullet_c(T, -)$ (étale cohomology with compact support), $(-)_{\bar{t}}$ (stalk at the geometric point $\bar{t}$) on $S(T, \overline{\mathbb{F}}_\ell)$ then, on $S(T, \underline{\overline{\mathbb{F}}})$, one sets $F : S(T, \underline{\overline{\mathbb{F}}}) \to Mod_{/\underline{\overline{\mathbb{F}}}}$, $\underline{\mathcal{M}} \mapsto \prod_{\ell \in \mathcal{L}} F(\mathcal{M}_\ell)$. Similarly, for a morphism of schemes $f : T' \to T$, one defines the inverse image functor $f^* : S(T, \underline{\overline{\mathbb{F}}}) \to S(T', \underline{\overline{\mathbb{F}}})$, $\underline{\mathcal{M}} \to f^*\underline{\mathcal{M}} = f^*\mathcal{M}_\ell, \ell \in \mathcal{L}$, the higher direct images functor $R^\bullet f_* : S(T', \underline{\overline{\mathbb{F}}}) \to S(T, \underline{\overline{\mathbb{F}}})$, $\underline{\mathcal{M}'} \to R^\bullet f^* \underline{\mathcal{M}'} = R^\bullet f_* \mathcal{M}'_\ell, \ell \in \mathcal{L}$ *etc.*

One says that $\underline{\mathcal{M}} \in S(T, \underline{\overline{\mathbb{F}}})$ is constructible (resp. locally constant constructible - lcc for short) if $\mathcal{M}_\ell$ is, $\ell \in \mathcal{L}$.

2.1.2. Let Π be a topological group. For $\ell \in \mathcal{L}$, let $Rep(\Pi, \overline{\mathbb{F}}_\ell)$ denote the category of finitely generated $\overline{\mathbb{F}}_\ell$-modules M_ℓ equipped with a continuous $\overline{\mathbb{F}}_\ell$-linear action of Π that is such that the image Π_ℓ of Π acting on M_ℓ is finite and the induced morphism $\Pi \twoheadrightarrow \Pi_\ell$ is continuous. When Π is profinite topologically finitely generated ([NS07a], [NS07b]) and, more generally, when every finite index subgroup of Π is open, this latter continuity condition is automatic. Let $Rep(\Pi, \underline{\overline{\mathbb{F}}})$ denote the 'product category' of the $Rep(\Pi, \overline{\mathbb{F}}_\ell)$, $\ell \in \mathcal{L}$ that is the category whose objects are families $\underline{M} = M_\ell$, $\ell \in \mathcal{L}$ with $M_\ell \in Rep(\Pi, \overline{\mathbb{F}}_\ell)$, $\ell \in \mathcal{L}$ and whose morphisms $\underline{M} \to \underline{N}$ are families $\underline{\phi} = \phi_\ell : M_\ell \to N_\ell$, $\ell \in \mathcal{L}$ of morphisms in $Rep(\Pi, \overline{\mathbb{F}}_\ell)$. Again, objects (resp. morphisms) in $Rep(\Pi, \underline{\overline{\mathbb{F}}})$ are denoted by $\underline{M}$ (resp. $\underline{\phi} : \underline{M} \to \underline{N}$) and we write M_ℓ (resp. $\phi_\ell : M_\ell \to N_\ell$) for the ℓth component of $\underline{M}$ (resp. $\underline{\phi} : \underline{M} \to \underline{N}$) *etc.*

2.1.3. If T is connected and $\bar{t}$ is a geometric point on T, the fiber functor $(-)_{\bar{t}} : S(T, \underline{\overline{\mathbb{F}}}) \to Mod_{/\underline{\overline{\mathbb{F}}}}$ induces an equivalence of categories from lcc sheaves in $S(T, \underline{\overline{\mathbb{F}}})$ to $Rep(\pi_1(T; \bar{t}), \underline{\overline{\mathbb{F}}})$. If X is a geometrically connected variety over k and $\overline{x}$ is a geometric point on X, let $W(X, \overline{x}) := \pi_1(X_{\overline{k}}, \overline{x}) \times_{\pi_1(k)} \varphi^{\mathbb{Z}}$ denote the Weil group; it is equipped with the product of the profinite topology on $\pi_1(X_{\overline{k}}, \overline{x})$ and the discrete topology on $\varphi^{\mathbb{Z}}$. Since $\pi_1(X, \overline{x})$ is the profinite completion of $W(X, \overline{x})$, the functor 'restriction to $W(X, \overline{x})$' induces an equivalence of categories $Rep(\pi_1(X, \overline{x}), \underline{\overline{\mathbb{F}}}) \to Rep(W(X, \overline{x}), \underline{\overline{\mathbb{F}}})$; in particular, there is no difference between 'lcc Weil sheaves' and lcc sheaves in our setting.

2.2. A filter on $\mathcal{L}$ is a family of subsets of $\mathcal{L}$ which is stable under finite intersections, supsets and does not contains the empty set. An ultrafilter on $\mathcal{L}$ is a filter which is maximal for $\subset$ among all filters or, equivalently, a filter $\mathfrak{u}$ such that for every $S \subset \mathcal{L}$ either $S \in \mathfrak{u}$ or $\mathcal{L} \setminus S \in \mathfrak{u}$. For every $\underline{n} : \mathcal{L} \to \overline{\mathbb{Z}}_{\geq 1}$, the set of ultrafilters on $\mathcal{L}$ is in bijection with the spectrum of (the 0-dimensional) ring $\underline{\overline{\mathbb{F}}}$:

$$\begin{array}{ccc} \text{Ultrafilters on } \mathcal{L} & \longleftrightarrow & \mathrm{Spec}(\underline{\overline{\mathbb{F}}}) \\ \mathfrak{u} & \longrightarrow & \mathfrak{m}_{\mathfrak{u}} := \langle e_S \mid S \in \mathfrak{u} \rangle \\ \mathfrak{u}_{\mathfrak{m}} := \{ S \subset \mathcal{L} \mid e_S \in \mathfrak{m} \} & \longleftarrow & \mathfrak{m}, \end{array}$$

where $e_S : \mathcal{L} \to \{0, 1\}$ denotes the characteristic function of $\mathcal{L} \setminus S$. For an ultrafilter $\mathfrak{u}$ on $\mathcal{L}$, write $\underline{\overline{\mathbb{F}}} \to \overline{\mathbb{Q}}_{\mathfrak{u}} := \underline{\overline{\mathbb{F}}}/\mathfrak{m}_{\mathfrak{u}}$ for the corresponding ultraproduct. In the above bijection, maximal principal ideals are in bijection with $\mathcal{L}$ and correspond to the so-called principal ultrafilters: $\mathfrak{u}_\ell := \{ S \subset \mathcal{L} \mid \ell \in S \}$.

[3]Not to be confused with the étale cohomological formalism induced by the one on the topos of $\underline{\overline{\mathbb{F}}}$-modules; recall that in general stalks at geometric points, cohomology groups *etc.* do not commute with infinite direct products.

Non principal ultrafilters give rise to characteristic 0 ultraproducts and their intersection is the set of all $S \subset \mathcal{L}$ such that $\mathcal{L} \setminus S$ is finite. In terms of ideals, this means that the intersection of all non-principal maximal ideals of $\underline{\mathbb{F}}$ is $\oplus_{\ell\in\mathcal{L}}\overline{\mathbb{F}}_\ell$.

Let $\mathcal{U}$ denote the set of all *non principal* ultrafilters on $\mathcal{L}$. For $\mathfrak{u} \in \mathcal{U}$ the following holds.
- (2.2.1) $\underline{\overline{\mathbb{F}}} \twoheadrightarrow \overline{\mathbb{Q}}_\mathfrak{u}$ is a flat morphism;
- (2.2.2) $\overline{\mathbb{Q}}_\mathfrak{u}$ is algebraically closed and isomorphic to $\mathbb{C}$.

2.3. Fix $\mathfrak{u} \in \mathcal{U}$ and let $(-)_\mathfrak{u} : S(T, \underline{\overline{\mathbb{F}}}) \to S(T, \overline{\mathbb{Q}}_\mathfrak{u})$, $\underline{\mathcal{M}} \to \mathcal{M}_\mathfrak{u}$ denote the quotient of $S(T, \underline{\overline{\mathbb{F}}})$ by the full subcategory of all $\underline{\mathcal{M}}$ such that $\{\ell \in \mathcal{L} \mid \mathcal{M}_\ell = 0\} \in \mathfrak{u}$. The category $S(T, \overline{\mathbb{Q}}_\mathfrak{u})$ of $\overline{\mathbb{Q}}_\mathfrak{u}$-sheaves inherits from $S(T, \underline{\overline{\mathbb{F}}})$ a structure of rigid abelian monoidal category such that $S(T, \underline{\overline{\mathbb{F}}}) \to S(T, \overline{\mathbb{Q}}_\mathfrak{u})$ becomes an additive tensor functor.

2.3.1. Let $F : S(T, \underline{\overline{\mathbb{F}}}) \to Mod_{/\underline{\overline{\mathbb{F}}}}$ denotes any of the functors $H^\bullet(T, -)$, $H^\bullet_c(T, -)$, $(-)_{\bar{t}}$ on $S(T, \underline{\overline{\mathbb{F}}})$. Then $F : S(T, \underline{\overline{\mathbb{F}}}) \to Mod_{/\overline{\mathbb{Q}}_\mathfrak{u}}$, $\underline{\mathcal{M}} \mapsto F(\underline{\mathcal{M}}) \otimes \overline{\mathbb{Q}}_\mathfrak{u}$ factors through $(-)_\mathfrak{u} : S(T, \underline{\overline{\mathbb{F}}}) \to S(T, \overline{\mathbb{Q}}_\mathfrak{u})$ as $F_\mathfrak{u} : S(T, \overline{\mathbb{Q}}_\mathfrak{u}) \to Mod_{/\overline{\mathbb{Q}}_\mathfrak{u}}$. We simply write $F(\mathcal{M}_\mathfrak{u}) := F_\mathfrak{u}(\mathcal{M}_\mathfrak{u})$ so that $F(\mathcal{M}_\mathfrak{u}) = F(\underline{\mathcal{M}}) \otimes \overline{\mathbb{Q}}_\mathfrak{u}$. Similarly, for a morphism of schemes $f : T' \to T$, the functors $f^* : S(T, \underline{\overline{\mathbb{F}}}) \to S(T', \underline{\overline{\mathbb{F}}}) \to S(T', \overline{\mathbb{Q}}_\mathfrak{u})$, $R^\bullet f_* : S(T', \underline{\overline{\mathbb{F}}}) \to S(T, \underline{\overline{\mathbb{F}}}) \to S(T, \overline{\mathbb{Q}}_\mathfrak{u})$ factor through $(-)_\mathfrak{u} : S(T, \underline{\overline{\mathbb{F}}}) \to S(T, \overline{\mathbb{Q}}_\mathfrak{u})$ as $f^*_\mathfrak{u} : S(T, \overline{\mathbb{Q}}_\mathfrak{u}) \to S(T', \overline{\mathbb{Q}}_\mathfrak{u})$, $R^\bullet f_{*,\mathfrak{u}} : S(T', \overline{\mathbb{Q}}_\mathfrak{u}) \to S(T, \overline{\mathbb{Q}}_\mathfrak{u})$. We simply write $f^*\mathcal{M}_\mathfrak{u} := f^*_\mathfrak{u}\mathcal{M}_\mathfrak{u}$, $R^\bullet f_*\mathcal{M}'_\mathfrak{u} = R^\bullet f_{*,\mathfrak{u}}\mathcal{M}'_\mathfrak{u}$.

2.3.2. **Remark.** Since $S(T, \overline{\mathbb{F}}_\ell) = \mathrm{colim} S(T, \mathbb{F}_{\ell^n})$, one could have alternatively constructed $S(T, \underline{\overline{\mathbb{F}}})$ as follows. For a map $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$, write $\underline{\mathbb{F}_{\underline{n}}} := \prod_{\ell\in\mathcal{L}} \mathbb{F}_{\ell^{n_\ell}}$ and for $\mathfrak{u} \in \mathcal{U}$, let $\underline{\mathbb{F}_{\underline{n}}} \to \mathbb{Q}_{\underline{n},\mathfrak{u}} := \underline{\mathbb{F}_{\underline{n}}}/\mathfrak{m}_\mathfrak{u}$ denote the corresponding ultraproduct. Then $\overline{\mathbb{Q}}_\mathfrak{u} = \mathrm{colim}\mathbb{Q}_{\underline{n},\mathfrak{u}}$. Define as before $S(T, \underline{\mathbb{F}_{\underline{n}}})$ as the 'product category' of the categories $S(T, \mathbb{F}_{\ell^{n_\ell}})$, $\ell \in \mathcal{L}$. Then considering the natural componentwise scalar extension functors $S(T, \underline{\mathbb{F}_{\underline{m}}}) \to S(T, \underline{\mathbb{F}_{\underline{n}}})$, $\underline{m}|\underline{n}$, $S(T, \underline{\overline{\mathbb{F}}}) = \mathrm{colim} S(T, \underline{\mathbb{F}_{\underline{n}}})$. This emphasizes further the parallelism between the construction from torsion coefficients of the cohomology groups with $\overline{\mathbb{Q}}_\ell$- and $\overline{\mathbb{Q}}_\mathfrak{u}$-coefficients as summarized in the table below. Given a prime $\ell(\neq p)$, we always denote by Q_ℓ a finite extension of $\mathbb{Q}_\ell$ and by Z_ℓ, λ_ℓ and F_ℓ the corresponding ring of integers, uniformizer and residue field.

	$\overline{\mathbb{Q}}_\ell$	$F_\mathcal{U}$
torsion coefficients	Z_ℓ/λ_ℓ^n, $n \geq 1$	$\mathbb{F}_{\ell^{n_\ell}}$, $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$
$\varprojlim$ (to char 0 ring)	Z_ℓ	$\underline{\mathbb{F}_{\underline{n}}}$
localization (exact) (to char 0 field)	$Z_\ell \hookrightarrow Q_\ell$	$\underline{\mathbb{F}_{\underline{n}}} \twoheadrightarrow \mathbb{Q}_{\underline{n},\mathcal{U}}$
$\varinjlim$ (to alg. closed char 0 field $\simeq \mathbb{C}$)	$Q_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell$	$\mathbb{Q}_{\underline{n},\mathcal{U}} \hookrightarrow \overline{\mathbb{Q}}_\mathcal{U}$

In both cases, to check cohomology groups with $\overline{\mathbb{Q}}_\ell$- and $\overline{\mathbb{Q}}_\mathfrak{u}$-coefficients behave well, the delicate part of the construction is the projective limit step. In the ℓ-adic setting, difficulties arise from torsion phenomena. In the ultraproduct setting, they arise from wild ramification phenomena. This will lead us to impose a uniform almost tameness assumption in our definition of ultraproduct local systems.

2.3.3. Given $\underline{M} \in Mod_{/\underline{\overline{\mathbb{F}}}}$, the $\mathfrak{u}$-rank of $\underline{M}$ is the $\overline{\mathbb{Q}}_\mathfrak{u}$-dimension of $M_\mathfrak{u} := \underline{M} \otimes \overline{\mathbb{Q}}_\mathfrak{u}$. Given $\underline{\mathcal{M}} \in S(T, \underline{\overline{\mathbb{F}}})$ and a geometric point $\bar{t}$ on T, the $\mathfrak{u}$-rank of $\underline{\mathcal{M}}$ at $\bar{t}$ is the $\mathfrak{u}$-rank of $\underline{\mathcal{M}_{\bar{t}}}$ (equivalently, the $\overline{\mathbb{Q}}_\mathfrak{u}$-dimension of $\mathcal{M}_{\mathfrak{u},\bar{t}}$). One says that $\underline{\mathcal{M}}$ has finite $\mathfrak{u}$-rank if there exists an integer $d \geq 1$ such that the set of all $\ell \in \mathcal{L}$ with $dim(\mathcal{M}_{\ell,\bar{t}}) \leq d$ for every geometric point $\bar{t}$ on T is in $\mathfrak{u}$.

If $\underline{\mathcal{M}} \in S(T, \underline{\overline{\mathbb{F}}})$ is lcc, for every $\ell \in \mathcal{L}$, the $\overline{\mathbb{F}}_\ell$-rank of $\mathcal{M}_{\ell,\bar{t}}$ is independent of $\bar{t}$. In particular the $\mathfrak{u}$-rank of $\underline{\mathcal{M}}$ at $\bar{t}$ is independent of $\bar{t}$; call it the $\mathfrak{u}$-rank of $\underline{\mathcal{M}}$.

If $\underline{M}, \underline{N} \in Mod_{/\underline{\overline{\mathbb{F}}}}$ have finite $\mathfrak{u}$-rank, $(\underline{M} \otimes \underline{N}) \otimes \overline{\mathbb{Q}}_\mathfrak{u} = M_\mathfrak{u} \otimes N_\mathfrak{u}$. In particular, if $\underline{\mathcal{M}}, \underline{\mathcal{N}} \in S(T, \underline{\overline{\mathbb{F}}})$ have finite $\mathfrak{u}$-rank at $\bar{t}$, $(\mathcal{M}_\mathfrak{u} \otimes \mathcal{N}_\mathfrak{u})_{\bar{t}} = \mathcal{M}_{\mathfrak{u},\bar{t}} \otimes \mathcal{N}_{\mathfrak{u},\bar{t}}$.

2.4. Let X be a variety over k, normal and connected (hence integral) and let $X \hookrightarrow \overline{X}$ be a normal compactification. One says that a connected étale cover $X' \to X$ is tamely ramified along $\overline{X} \setminus X$ if every codimension-1 point $\zeta \in \overline{X} \setminus X$ is tamely ramified in the resulting extension $k(X')/k(X)$ of function fields and that a (not necessarily connected) étale cover $X' \to X$ is tamely ramified along $\overline{X}\setminus X$ if each of its connected components is. Fix a geometric point $\overline{x}$ on X. Etale covers which are tamely ramified along $\overline{X}\setminus X$ are classified by a quotient $\pi_1(X,\overline{x})\twoheadrightarrow\pi_1^t(X,\overline{X}\setminus X,\overline{x})$ (whose kernel is generated by the wild inertia groups at all codimension 1 points $\zeta\in\overline{X}\setminus X$); write $K(X,\overline{X}\setminus X,\overline{x}):=\ker(\pi_1(X,\overline{x})\twoheadrightarrow\pi_1^t(X,\overline{X}\setminus X,\overline{x}))$.

Let $X'\to X$ be an étale cover. Following Kersz-Schmidt [KS10], consider the conditions below.
- (2.4.1) (curve-tameness): For every smooth curve C over k and morphism $C\to X$, $X'\times_X C\to C$ is tame;
- (2.4.2) (divisor-tameness): For *every* normal compactification $X\hookrightarrow\overline{X}$, $X'\to X$ is tamely ramified along $\overline{X}\setminus X$.

Divisor-tame covers are classified by a quotient $\pi_1(X,\overline{x})\twoheadrightarrow\pi_1^t(X,x)$ with kernel $K(X,\overline{x}):=\ker(\pi_1(X,\overline{x})\twoheadrightarrow\pi_1^t(X,\overline{x}))$ the (normal) subgroup of $\pi_1(X,\overline{x})$ generated by the $K(X,\overline{X}\setminus X,\overline{x})$ for $X\hookrightarrow\overline{X}$ describing all normal compactifications. Since $\pi_1(X,\overline{x})$, $\pi_1^t(X,\overline{x})$, $\pi_1^t(X,\overline{X}\setminus X,\overline{x})$ are independent of $\overline{x}$ up to inner automorphisms and base points will play no part in the following, we omit them from the notation. If X is smooth over k, [KS10, Thm. 1.1] asserts that (2.4.1), (2.4.2) are equivalent and that, if X admits a smooth compactification $X\hookrightarrow\overline{X}$ such that $\overline{X}\setminus X$ is a normal crossing divisor, (2.4.1), (2.4.2) are also equivalent to the notion of tameness of [SGA1, XIII]. When X is smooth over k, we will say that an étale cover satisfying the equivalent conditions (2.4.1), (2.4.2) is *tamely ramified*. By (2.4.1), the notion of tameness is stable under arbitrary base-changes.

When X is smooth over k, $\pi_1^t(X)$ is topologically finitely generated; this finiteness property will play a crucial part in the following. When X is a curve, this is a consequence of the theory of specialization of the tame étale fundamental group [SGA1, XIII]. In general, it is a consequence of the following Bertini theorem, which is a key technical tool to handle $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems on higher-dimensional smooth varieties.

2.5. **Theorem.** ([C20b, App., Thm. 1.2.1]; Drinfeld, Poonen, Tamagawa ...) *Let X be a normal, geometrically connected variety over $\overline{k}$ and $X'\to X$ a Galois étale cover; the group $K(X'):=\ker(\pi_1(X')\twoheadrightarrow\pi_1^t(X'))$ is normal in $\pi_1(X)$. There exists a smooth, geometrically connected curve C over $\overline{k}$ and a morphism $f:C\to X$ of varieties over k such that the induced morphism $\pi_1(C)\to\pi_1(X)\twoheadrightarrow\pi_1(X)/K(X')$ is surjective and factors through $\pi_1(C)\twoheadrightarrow\pi_1^t(C)$. Furthermore, given a finite set S of closed points contained in a quasi-projective*[4] *open subscheme $U\subset X^{sm}$, one can choose $f:C\to X$ in such a way that it admits a section $g:S\to C$.*

2.6. Let X be a smooth and geometrically connected variety over k. Fix $\mathfrak{u}\in\mathcal{U}$.

2.6.1. For $\ell\in\mathcal{L}$, and a lcc $\mathcal{M}_\ell$ in $S(X,\overline{\mathbb{F}}_\ell)$, one says that $\mathcal{M}_\ell$ is tame if the étale cover $X'\to X$ trivializing $\mathcal{M}_\ell$ is tame. For a lcc $\underline{\mathcal{M}}$ in $S(X,\underline{\overline{\mathbb{F}}})$, one says that $\underline{\mathcal{M}}$ is $\mathfrak{u}$-tame if the set of primes $\ell\in\mathcal{L}$ such that $\mathcal{M}_\ell$ is tame is in $\mathfrak{u}$ and that $\underline{\mathcal{M}}$ is almost $\mathfrak{u}$-tame if there exists an étale cover $X'\to X$ such that $\underline{\mathcal{M}}|_{X'}$ is $\mathfrak{u}$-tame.

2.6.2. Let $S^t_{\mathfrak{u}}(X,\underline{\overline{\mathbb{F}}})\subset S(X,\underline{\overline{\mathbb{F}}})$ denote the full subcategory of almost $\mathfrak{u}$-tame lcc sheaves with finite $\mathfrak{u}$-rank. It is abelian and, as a subcategory of the category of lcc sheaves in $S(X,\underline{\overline{\mathbb{F}}})$, stable under extensions (recall $p\neq\ell$), internal Hom, duals, tensor products, pullback by arbitrary morphisms and push forward by finite étale morphisms. Let $\mathcal{C}(X,\overline{\mathbb{Q}}_{\mathfrak{u}})\subset S(X,\overline{\mathbb{Q}}_{\mathfrak{u}})$ denote the essential image of

$$S^t_{\mathfrak{u}}(X,\underline{\overline{\mathbb{F}}})\subset S(X,\underline{\overline{\mathbb{F}}})\to S(X,\overline{\mathbb{Q}}_{\mathfrak{u}}).$$

2.6.3. Given a (topological) group Π, let $Rep(\Pi,\overline{\mathbb{Q}}_{\mathfrak{u}})$ denote the category of finite-dimensional $\overline{\mathbb{Q}}_{\mathfrak{u}}$-linear representations of Π. Let $Rep^t(\pi_1(X),\overline{\mathbb{Q}}_{\mathfrak{u}})\subset Rep(\pi_1(X),\overline{\mathbb{Q}}_{\mathfrak{u}})$ denote the essential image of the canonical functor

$$S^t_{\mathfrak{u}}(X,\underline{\overline{\mathbb{F}}})\to Rep(\pi_1(X),\underline{\overline{\mathbb{F}}})\to Rep(\pi_1(X),\overline{\mathbb{Q}}_{\mathfrak{u}}).$$

The following (elementary) lemma, which relies on the (non elementary) fact that $\pi_1^t(X)$ is topologically finitely generated provides a useful group-theoretical description of $\mathcal{C}(X,\overline{\mathbb{Q}}_{\mathfrak{u}})$.

[4] Under some mild global assumptions on X^{sm} - see [Stacks, Part 2, 27.9] - every finite set $S\subset X^{sm}$ of closed points is contained in an affine open subscheme $U\subset X^{sm}$.

Lemma. ([C20b, Lem. 3.6.2.3]) *Assume X is smooth over k. Then $Rep^t(\pi_1(X), \overline{\mathbb{Q}}_{\mathfrak{u}})$ is a Tannakian subcategory of $Rep(\pi_1(X), \overline{\mathbb{Q}}_{\mathfrak{u}})$, stable under subobjects. Furthermore the canonical additive tensor functor $S^t_{\mathfrak{u}}(X, \underline{\overline{\mathbb{F}}}) \to Rep^t(\pi_1(X), \overline{\mathbb{Q}}_{\mathfrak{u}})$ factors through an equivalence of categories $\mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}}) \tilde{\to} Rep^t(\pi_1(X), \overline{\mathbb{Q}}_{\mathfrak{u}})$.*

We call $\mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ the category of $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems on X. This terminology is motivated by the above Lemma and, as we will see later, by the fact that it corresponds to the category of $\overline{\mathbb{Q}}_\ell$-local system in the sense of the Langlands correspondance.

From now on and unless some confusion may arise, we simply write $\underline{\mathcal{M}} \to \mathcal{M} := \mathcal{M}_{\mathfrak{u}}$ for the localization functor $S^t_{\mathfrak{u}}(X, \underline{\overline{\mathbb{F}}}) \to \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$.

2.6.4. Since the map $\underline{\overline{\mathbb{F}}}^\times \twoheadrightarrow \overline{\mathbb{Q}}_{\mathfrak{u}}^\times$ is surjective, every $\alpha \in \overline{\mathbb{Q}}_{\mathfrak{u}}$ lifts to some $\underline{\alpha} \in \underline{\overline{\mathbb{F}}}^\times$ hence defines a character

$$\pi_1(X) \twoheadrightarrow \pi_1(k) \overset{\varphi \mapsto \underline{\alpha}}{\to} \underline{\overline{\mathbb{F}}}^\times \twoheadrightarrow \overline{\mathbb{Q}}_{\mathfrak{u}}^\times \in Rep^t_{\mathfrak{u}}(\pi_1(X), \overline{\mathbb{Q}}_{\mathfrak{u}}),$$

which corresponds to a rank-1 object in $\mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$, which we denote by $\overline{\mathbb{Q}}_{\mathfrak{u},X}^{(\alpha)}$. For an arbitrary $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ we write $\mathcal{M}^{(\alpha)} := \mathcal{M} \otimes \overline{\mathbb{Q}}_{\mathfrak{u},X}^{(\alpha)}$ and call it the twist of $\mathcal{M}$ by α. For $\alpha = |k|^{-1}$, we rather write $\mathcal{M}(n) := \mathcal{M}^{(|k|^{-n})}$ and call it the nth Tate twist of $\mathcal{M}$.

Another fundamental consequence of the almost-$\mathfrak{u}$ tameness assumption is the finiteness of the cohomology groups with compact support[5].

2.6.5. **Theorem.** ([C20b, Thm. 3.6.3]) *If $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$, the $\overline{\mathbb{Q}}_{\mathfrak{u}}$-vector space $H^\bullet_c(X_{\overline{k}}, \mathcal{M})$ (hence, by Poincaré duality - recall X is assumed to be smooth over k - $H^\bullet(X_{\overline{k}}, \mathcal{M})$), has finite dimension.*

When X is a curve the finiteness of $H^\bullet_c(X_{\overline{k}}, \mathcal{M})$ directly follows from the Grothendieck-Ogg-Shafarevich formula but when X is higher-dimensional, it requires more elaborated results from ramification theory. The idea is to reduce, by standard arguments using de Jong's alterations [dJ16] and the comparison theorem for curve-tameness and divisor-tameness of Kerz-Schmidt [KS10, Thm. 1.1], to the case where X is the complement of a strict normal crossing divisor in a smooth projective variety, which follows from the uniformity theorem of Orgogozo [O19, Thm. 3.1.1] (plus the already mentioned fact that, in that case, our notion of tameness coincide with the one of [SGA1, XIII]).

3. Weight theory

Let X be a smooth, geometrically connected variety over k. Fix $\mathfrak{u} \in \mathcal{U}$.

3.1. Let $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$.

3.1.1. The finite $\mathfrak{u}$-rank condition allows to define Frobenius weights. For a closed point $x \in |X|$ and a geometric point $\overline{x}$ over x, the geometric Frobenius $\varphi_x \in \pi_1(x)$ at x acts on the finite-dimensional $\overline{\mathbb{Q}}_{\mathfrak{u}}$-vector space $\mathcal{M}_{\overline{x}}$. Given an isomorphism $\iota : \overline{\mathbb{Q}}_{\mathfrak{u}} \tilde{\to} \mathbb{C}$, the ι-weights of $\mathcal{M}$ at x are the $\frac{2\log(|\iota\alpha|)}{\log(|k(x)|)}$ for α describing the set of eigenvalues of φ_x acting on $\mathcal{M}_{\overline{x}}$. If there exists $w \in \mathbb{R}$ such that for every $x \in |X|$ the ι-weights of $\mathcal{M}$ at x are all equal to w, one says that $\mathcal{M}$ is ι-pure of weight w.

For $\alpha \in \overline{\mathbb{Q}}_{\mathfrak{u}}$, $\overline{\mathbb{Q}}_{\mathfrak{u}}^{(\alpha)}$ is ι-pure of weight $\frac{2\log(|\iota\alpha|)}{\log(|k|)}$; in particular $\overline{\mathbb{Q}}_{\mathfrak{u}}(1)$ is ι-pure of weight -2.

3.1.2. The finiteness Theorem 5.5.1 is enough to derive the cohomological interpretation of L-function from the trace formula for the $\mathcal{M}_\ell$, $\ell \in \mathfrak{u}$, namely ([C20b, Thm. 3.5.1]):

$$\prod_{x \in |X|} L_x(\mathcal{M}, T) = \prod_{i \geq 0} \det(1 - T\varphi | \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{M}))^{(-1)^{i+1}}$$

in $\overline{\mathbb{Q}}_{\mathfrak{u}}[[T]]$, where $L_x(\mathcal{M}, T) := \det(1 - T^{\deg(x)}\varphi_x | \mathcal{M}_{\overline{x}})^{-1}$ denotes the local L-factor at $x \in |X|$.

3.1.3. The almost tameness condition also implies the following:

[5]Actually, the assumption that X is smooth is not required for the finiteness of cohomology with compact support

3.1.3.1.([C20b, Thm. 6.1.3]) The radical of the Zariski closure of the image of $\pi_1(X_{\overline{k}})$ acting on $\mathcal{M}_{\overline{x}}$ is unipotent.

This follows by group-theoretic arguments involving Lemma 2.6.3 from the rank-1 case, which is itself a consequence of geometric class field theory:

Fact. (*e.g.* [D80, Thm. (1.3.1)]) *The image of $\pi_1(X_{\overline{k}}) \to \pi_1(X_{\overline{k}})^{ab}$ is the direct product of a pro-p group by a finite group of prime-to-p order.*

The rank-1 case of the above statement can be reformulated by saying that every rank-1 $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ can be written as $\mathcal{M} = \mathcal{E}^{(\alpha)}$ for some $\alpha \in \overline{\mathbb{Q}}_{\mathfrak{u}}$ and with $\mathcal{E}^{\otimes n} \simeq \overline{\mathbb{Q}}_{\mathfrak{u},X}$ for some integer $n \geq 1$. In particular, $\mathcal{M}$ is ι-pure of weight $\frac{2\log(|\iota\alpha|)}{\log(|k|)}$.

3.1.3.2.([C20b, Lem. 5.3.1]) If X is a curve with smooth compactification $j : X \hookrightarrow \overline{X}$ the image of $\pi_1(X_{\overline{k},(\overline{x})})$ acting on $\mathcal{M}_{\overline{\eta}_{\overline{x}}}$ is quasi-unipotent. Here $\overline{\eta}_{\overline{x}}$ denotes a geometric generic point on the strict henselization $X_{\overline{k},(\overline{x})} := X_{\overline{k}} \times_{\overline{X}_{\overline{k}}} spec(\mathcal{O}_{\overline{X}_{\overline{k}},\overline{x}})$.

3.1.4. With these observations in hands, one can adjust Deligne's proof of the fundamental theorem of Weil II for curves [D80, Thm. (3.2.1)] to $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems.

Theorem (Weil II ultraproduct for curves - [C20b, Thm. 1.2.4]). *Assume X is a curve. If $\mathcal{M}$ is ι-pure of weight w then, for every $i \geq 0$, $H^i_c(X_{\overline{k}}, \mathcal{M})$ has ι-weights $\leq w+i$. Equivalently, $H^i(X_{\overline{k}}, \mathcal{M})$ has ι-weights $\geq w+i$.*

Corollary *Assume X is a curve with smooth compactification $j : X \hookrightarrow \overline{X}$. If $\mathcal{M}$ is ι-pure of weight w then, for every $i \geq 0$, $H^i(X_{\overline{k}}, j_*\mathcal{M})$ has ι-weights $w+i$.*

We refer the to [C20b, Sections 5-8] for the proof.

3.2. In Weil II, Deligne uses an elegant devissage to deduce from [D80, Thm. (3.2.1)] what is usually referred to as "the main theorem of Weil II", namely [D80, Thm. (3.3.1)] if $f : X \to Y$ is a morphism of varieties over k and $\mathcal{F}$ a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X, ι-mixed of weight $\leq w$ then $R^i f_! \mathcal{F}$ is ι-mixed of weights $\leq w+i$, $i \geq 0$. With this general statement in hand, the development of a the "yoga of weights" for $\overline{\mathbb{Q}}_\ell$-cohomology [D80, (3.3), (3.4)] is rather straightforward. For $\overline{\mathbb{Q}}_{\mathfrak{u}}$-cohomology however, we do not have yet a good notion of constructible sheaf. However, at the cost of resorting to additional geometric arguments - Lefschetz pencils, elementary fibrations or Bertini like arguments such as theorem 2.5 above, one can go around this issue and establish most of the weight theory for $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems. We summarize below the main results.

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_{\mathfrak{u}} \tilde{\to} \mathbb{C}$.

3.2.1. **(Purity)** Let $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ be ι-pure of weight w. Assume X is proper over k. Then for every $i \geq 0$, $H^i(X, \underline{\mathcal{F}})$ is ι-pure of weights $w+i$.

3.2.2. **(Geometric semisimplicity)** Let $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ be ι-pure of weight w. The following equivalent conditions hold:
(1) $\mathcal{M}|_{X_{\overline{k}}}$ is semisimple in $\mathcal{C}(X_{\overline{k}}, \overline{\mathbb{Q}}_{\mathfrak{u}})$;
(2) For every geometric point $\overline{x}$ on X, $\pi_1(X_{\overline{k}})$ acts semisimply on $\mathcal{M}_{\overline{x}}$;
(3) The set of primes $\ell \in \mathcal{L}$ such that $\mathcal{M}_\ell|_{X_{\overline{k}}}$ is semisimple is in $\mathfrak{u}$.

3.2.3. **(Weight decomposition and filtration for ι-mixed $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems)** One says that $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ is ι-mixed if it admits an increasing filtration $F_\bullet\mathcal{M}$ in $\mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ such that $Gr^F_i(\mathcal{M})$ is ι-pure, $i \in \mathbb{Z}$.

Assume $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ is ι-mixed. Then,
(1) $\mathcal{M}$ admits a decomposition $\mathcal{M} = \oplus_{a \in \mathbb{R}/\mathbb{Z}} \mathcal{M}(a)$ in $\mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ with $\mathcal{M}(a) \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ of ι-weights in $a + \mathbb{Z}$.

(2) If the ι-weights of the $Gr_i^F(\mathcal{M})$, $i \in \mathbb{Z}$ are all in $\mathbb{Z}$ then $\mathcal{M}$ admits an increasing filtration $W_\bullet\mathcal{M}$ in $\mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ with $Gr_i^W\mathcal{M}$ ι-pure of weight i, $i \in \mathbb{Z}$.

3.2.4. **(Weak Cebotarev)** One pathological feature of ultraproducts of finite fields is that their natural topology (the quotient topology of the product of the discrete topologies) is not separated. So it is unclear whether the naïve analogue of the Cebotarev theorem holds. However, the following weak form, which will be enough for our applications, still holds.

Let $\mathcal{M}, \mathcal{M}' \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ be ι-mixed and semisimple. Assume that $L_x(\mathcal{M}, T) = L_x(\mathcal{M}', T)$, $x \in |X|$. Then $\mathcal{M} \simeq \mathcal{M}'$.

We will see later, as a by-product of the Langlands correspondance, that every $\mathcal{M} \in \mathcal{C}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ is ι-mixed (Corollary 5.3.3). We will also deduce from the Langlands correspondance a Tannakian enhancement of the above Weak Cebotarev theorem (Corollary 5.6.1.3).

4. Integral models in compatible families of ℓ-adic local systems

We now come to the applications that were our first motivation to introduce $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems and their theory of Frobenius weights. Let again X be a smooth, geometrically connected variety over k.

For $\ell \in \mathcal{L}$, let $\mathcal{C}(X, \overline{\mathbb{Q}}_\ell)$ denote the category of $\overline{\mathbb{Q}}_\ell$-local systems (that is lcc Weil $\overline{\mathbb{Q}}_\ell$-sheaves) on X. For $\mathcal{F}_\ell \in \mathcal{C}(X, \overline{\mathbb{Q}}_\ell)$ an integral model of $\mathcal{F}_\ell$ is a torsion-free Z_ℓ-local system $\mathcal{H}_\ell$ such that $\mathcal{H}_\ell \otimes \overline{\mathbb{Q}}_\ell = \mathcal{F}_\ell$, where Z_ℓ denotes the ring of integers of a finite field extension Q_ℓ of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Let $\lambda_\ell \in Z_\ell$ be a uniformizer and $F_\ell := Z_\ell/\lambda_\ell$ the residue field.

4.1. Let $\dagger \in \mathcal{L} \cup \mathcal{U}$. One says that a $\overline{\mathbb{Q}}_\dagger$-local system $\mathcal{C}$ on X is algebraic (resp. finite algebraic) if the $\mathbb{Q}$-subextension $Q_{\mathcal{C}} \subset \overline{\mathbb{Q}}_\dagger$ generated by the coefficients of the $\chi_x(\mathcal{C}, T) := \det(1 - T\varphi_x|\mathcal{C}_{\overline{x}})$, $x \in |X|$ is algebraic (resp. a number field). Let $S \subset \mathcal{L} \cup \mathcal{U}$. A compatible family of $\overline{\mathbb{Q}}_\dagger$-local systems on X with index set S is a family $\underline{\mathcal{C}} = \mathcal{C}_\dagger$, $\dagger \in S$ of finite algebraic $\overline{\mathbb{Q}}_\dagger$-local systems such that for every $x \in |X|$, $\chi_x(\underline{\mathcal{C}}, T) := \chi_x(\mathcal{C}_\dagger, T) \in \overline{\mathbb{Q}}[T]$ is independent of S. Write $Q_{\underline{\mathcal{C}}} := Q_{\mathcal{C}_\dagger}$, $\mathcal{L} \cup \mathcal{U}$ and call it the field of coefficients of $\underline{\mathcal{C}}$. Write also $L_x(\underline{\mathcal{C}}, T) := \chi_x(\underline{\mathcal{C}}, T^{\deg(x)})^{-1}$ for the local L-factor of $\underline{\mathcal{C}}$ at x. For $i \geq 0$ and $? = \emptyset, c$, set
$$P_?^i(\mathcal{C}_\dagger, T) := \det(1 - T\varphi|H_?^i(X, \mathcal{C}_\dagger)),\ \dagger \in \mathcal{L} \cup \mathcal{U}.$$
By the trace formula [D80, (1.4.5.1)],
$$L(\underline{\mathcal{C}}, T) := L(\mathcal{C}_\dagger, T) = \prod_{x \in |X|} L_x(\mathcal{C}_\dagger, T) = \prod_{i \geq 0} P_c^i(\mathcal{C}_\dagger, T)^{(-1)^{i+1}} =: \chi_c(\mathcal{C}_\dagger, T) =: \chi_c(\underline{\mathcal{C}}, T)$$
is then in $Q_{\underline{\mathcal{C}}}(T)$ and independent of $\dagger \in S$ as well. One says that $\underline{\mathcal{C}}$ is ι-pure of weight $w \in \mathbb{R}$ if $\mathcal{C}_\dagger$ is $\iota_\dagger$-pure of weight w, $\dagger \in S$ and that $\underline{\mathcal{C}}$ is semisimple if $\mathcal{C}_\dagger$ is, $\dagger \in S$.

The common rank of the $\mathcal{C}_\dagger$, $\dagger \in S$ is called the rank of $\underline{\mathcal{C}}$.

4.2. Let $\underline{\mathcal{F}} := \mathcal{F}_\ell$, $\ell \in \mathcal{L}$ be a compatible family of $\overline{\mathbb{Q}}_\ell$-local systems. For every $\ell \in \mathcal{L}$ and Z_ℓ-model $\mathcal{H}_\ell$ of $\mathcal{F}_\ell$, write $\mathcal{M}_\ell := \mathcal{H}_\ell \otimes_{Z_\ell} \overline{\mathbb{F}}_\ell$ and set $\mathcal{C}_\ell := \mathcal{F}_\ell$ for $\ell \in \mathcal{L}$, $\mathcal{C}_{\mathfrak{u}} := \mathcal{M}_{\mathfrak{u}}$ for $\mathfrak{u} \in \mathcal{U}$. Then,

Lemma / example. *For every $\mathfrak{u} \in \mathcal{U}$, $\underline{\mathcal{M}} \in S_{\mathfrak{u}}^t(X, \underline{\overline{\mathbb{F}}})$ and $\underline{\mathcal{C}} = \mathcal{C}_\dagger$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ is a compatible family of $\overline{\mathbb{Q}}_\dagger$-local systems on X.*

The non-trivial part of the assertion is that $\underline{\mathcal{M}} \in S_{\mathfrak{u}}^t(X, \underline{\overline{\mathbb{F}}})$, $\mathfrak{u} \in \mathcal{U}$. Fix $\ell \in \mathcal{L}$. Up to replacing X by a connected étale cover, one may assume $\mathcal{M}_\ell|_X$ is constant. Let C be a smooth, geometrically connected curve over $\overline{k}$ and $C \to X_{\overline{k}}$ a non-constant morphism. Since $\mathcal{M}_\ell|_{X_{\overline{k}}}$ is constant, $\mathcal{F}_\ell|_C$ is tamely ramified. One has to show that for every $\ell' \in \mathcal{L}$, $\mathcal{F}_{\ell'}|_C$ is tamely ramified as well. By the Grothendieck-Ogg-Shafarevich formula, it is enough to show that the Euler-Poincaré characteristics for compact support cohomology of $\mathcal{F}_\ell|_C$ and $\mathcal{F}_{\ell'}|_C$ coincide. But these are the orders of the pole at

∞ of $\chi(\mathcal{F}_\ell|_C, T)$, $\chi(\mathcal{F}_{\ell'}|_C, T)$ respectively. So the conclusion follows from the compatibility assumption and the trace formula.

4.3. From now on and till the end of Section 5 fix a rank-r compatible family $\underline{\mathcal{F}} := \mathcal{F}_\ell,\ \ell \in \mathcal{L}$ of $\overline{\mathbb{Q}}_\ell$-local systems on X, ι-pure of weight $w \in \mathbb{R}$.

4.3.1. *Uniformity principle.* One nice feature of the formalism of ultraproduct coefficients is its flexibility with respect to the index set $\mathcal{L}$ and the choice of the integral models $\mathcal{H}_\ell$, $\ell \in \mathcal{L}$; this automatically implies that the results obtained are uniform in the choice of the integral models. More precisely, let P_Q (resp. P_Z) be a property of $\overline{\mathbb{Q}}_\ell$-local systems (resp. of $\overline{\mathbb{Z}}_\ell$-local systems) on X. Consider

(1) For every infinite set[6] $\mathcal{L}$ of primes $\neq p$ and compatible family $\underline{\mathcal{F}} = \mathcal{F}_\ell,\ \ell \in \mathcal{L}$ of $\overline{\mathbb{Q}}_\ell$-local systems such that $\mathcal{F}_\ell$ has P_Q, $\ell \in \mathcal{L}$, every family $\underline{\mathcal{H}} = \mathcal{H}_\ell,\ \ell \in \mathcal{L}$ of integral models, $\mathcal{H}_\ell$ has P_Z for $\ell \gg 0$ (depending *a priori* on $\underline{\mathcal{H}}$).

Then, (1) formally implies the following uniform version.

(2) For every infinite set $\mathcal{L}$ of primes $\neq p$ and compatible family $\underline{\mathcal{F}} = \mathcal{F}_\ell,\ \ell \in \mathcal{L}$ of $\overline{\mathbb{Q}}_\ell$-local systems such that $\mathcal{F}_\ell$ has P_Q, $\ell \in \mathcal{L}$, for $\ell \gg 0$ (depending only on $\underline{\mathcal{F}}$) every integral model $\mathcal{H}_\ell$ of $\mathcal{F}_\ell$ has P_Z.

Indeed, otherwise, there would exists an infinite subset $\mathcal{L}' \subset \mathcal{L}$ and for every $\ell \in \mathcal{L}'$ an integral model $\mathcal{H}_\ell$ of $\mathcal{F}_\ell$ such that P_Z fails for $\mathcal{H}_\ell$, contradicting (4.3.1.1) for the compatible family $\mathcal{F}_\ell,\ \ell \in \mathcal{L}'$.

4.3.2. *Integral Models.* The following statement summarizes the torsion, unicity and residual semisimplicity / irreducibility properties for integral models.

4.3.2.1. **Corollary.** *For $\ell \gg 0$ and every choice of Z_ℓ-model $\mathcal{H}_\ell$ of $\mathcal{F}_\ell$ the following holds.*

(1) Assume there exists[7] a smooth, proper morphism $X \to S$ with S a smooth curve over k. Then, for $? = \emptyset, c$, $i \geq 0$ the cohomology groups $H^i_?(X_{\overline{k}}, \mathcal{F}_\ell)$, $H^i_?(X_{\overline{k}}, \mathcal{M}_\ell)$ have the same dimension; in particular $H^i_?(X_{\overline{k}}, \mathcal{H}_\ell)$ is torsion-free and $H^i_?(X_{\overline{k}}, \mathcal{H}_\ell) \otimes F_\ell = H^i_?(X_{\overline{k}}, \mathcal{M}_\ell)$.

(2) (a) $\mathcal{H}_{\ell,\overline{x}}{}^{\pi_1(X_{\overline{k}})} \otimes F_\ell \tilde{\to} \mathcal{M}_{\ell,\overline{x}}{}^{\pi_1(X_{\overline{k}})}$;
(b) if $\underline{\mathcal{F}}$ is semisimple, $\mathcal{H}_{\ell,\overline{x}}{}^{\pi_1(X)} \otimes F_\ell \tilde{\to} \mathcal{M}_{\ell,\overline{x}}{}^{\pi_1(X)}$.

(3) (a) $\mathcal{M}_\ell|_{X_{\overline{k}}}$ is semisimple;
(b) if $\mathcal{F}_\ell$ is semisimple, $\mathcal{M}_\ell$ is semisimple.

(4) (a) if $\mathcal{F}_\ell|_{X_{\overline{k}}}$ is irreducible, $\mathcal{M}_\ell|_{X_{\overline{k}}}$ is irreducible;
(b) if $\mathcal{F}_\ell$ is irreducible, $\mathcal{M}_\ell$ is irreducible.

(5) If $\mathcal{H}'_\ell$ is another Z_ℓ-model of $\mathcal{F}_\ell$ then
(a) $\mathcal{H}_\ell|_{X_{\overline{k}}} \simeq \mathcal{H}'_\ell|_{X_{\overline{k}}}$;
(b) if $\mathcal{F}_\ell$ is semisimple, then $\mathcal{H}_\ell \simeq \mathcal{H}'_\ell$.

(6) (Resp. if $\underline{\mathcal{F}}$ is semisimple) the connected component of the Zariski-closure of the image of the geometric étale fundamental group (resp. of the étale fundamental group) of X acting on the stalks of $\mathcal{H}_\ell$ is a semisimple (resp. a reductive) group scheme over Z_ℓ.

We refer to [C20b, §1.3.2] for the comparison with existing results and approaches. We sketch some of the arguments of the proof of Corollary 4.3.2.1 to emphasize how elementary they are. From 4.3.1, it is enough to prove (1)-(6) for $\ell \gg 0$ depending possibly on $\underline{\mathcal{H}}$. So for every $\ell \in \mathcal{L}$ fix a Z_ℓ-model $\mathcal{H}_\ell$ of $\mathcal{H}_\ell$ and write $\mathcal{M}_\ell := \mathcal{H}_\ell/\lambda_\ell$; set $\mathcal{C}_\ell := \mathcal{F}_\ell$ for $\ell \in \mathcal{L}$, $\mathcal{C}_\mathfrak{u} := \mathcal{M}_\mathfrak{u}$ for $\mathfrak{u} \in \mathcal{U}$.

Proof. (sketch of) (6) essentially follows from (3) but the argument is rather subtle. The key ingredient is the theory of Frobenius tori of Serre and Larsen-Pink (*e.g.* [LarP92]); we refer to [C20b, Proof of 16.2.2] for details. In contrast, (1)-(5) are formal consequences of the theory of Frobenius weights for $\overline{\mathbb{Q}}_\ell$- and $\overline{\mathbb{Q}}_\mathfrak{u}$-local systems and elementary properties of ultraproduct (using the non-elementary fact $\pi_1^t(X)$ is topologically finitely generated!). Let us prove (1) when X is proper over k or a curve (for the general case, see [C20b, Cor. 12.1.5]). The second part of (1) follows from the first part and Lemma 4.3.2.3 below. By Poincaré duality the assertion for $? = \emptyset$ follows from the one for $? = c$ so that one may assume $? = c$. From Lemma 4.3.2.2 below, the $\overline{\mathbb{Q}}_\dagger$-dimension b^i_c of $H^i_c(X_{\overline{k}}, \mathcal{C}_\dagger)$ is

[6] Actually, we may allow finitely many repetitions of each primes in $\mathcal{L}$.

[7] In particular, this includes the case where X is proper over k (we do note impose $X \to S$ to be non-constant) and where $X = S$ is a curve.

independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$. As $b_c^i := dim H_c^i(X_{\overline{k}}, \mathcal{C}_{\mathfrak{u}})$ for every $\mathfrak{u} \in \mathcal{U}$, one also has $b_c^i = H_c^i(X, \mathcal{M}_\ell)$ for $\ell \gg 0$. When X is a curve, (2)(a) is the $? = \emptyset$, $i = 0$ case of (1); the general case of (2)(a) reduces to the case where X is a curve by Theorem 2.5. (3)(a) follows from 3.2.2. (5)(a) follows from (2)(a) and from (3)(a) applied to the compatible family $\mathcal{F}_\ell \otimes \mathcal{F}_\ell^\vee$, $\ell \in \mathcal{L}$ and its family of integral models $\mathcal{H}_\ell \otimes \mathcal{H}'_\ell$, $\ell \in \mathcal{L}$. More precisely, by Cebotarev density theorem, $\mathcal{M}'_\ell{}^{ss} \simeq \mathcal{M}_\ell{}^{ss}$ hence, from (2)(a), $\mathcal{M}'_\ell|_{X_{\overline{k}}} \simeq \mathcal{M}_\ell|_{X_{\overline{k}}}$. Fix an isomorphism $\overline{\phi} : \mathcal{M}'_\ell|_{X_{\overline{k}}} \tilde{\to} \mathcal{M}_\ell|_{X_{\overline{k}}}$. From (3)(a), the reduction-modulo-λ_ℓ map $Hom(\mathcal{H}'_\ell|_{X_{\overline{k}}}, \mathcal{H}_\ell|_{X_{\overline{k}}}) \to Hom(\mathcal{M}'_\ell|_{X_{\overline{k}}}, \mathcal{M}_\ell|_{X_{\overline{k}}})$ is surjective so that $\overline{\phi} : \mathcal{M}'_\ell|_{X_{\overline{k}}} \tilde{\to} \mathcal{M}_\ell|_{X_{\overline{k}}}$ lifts to some $\phi : \mathcal{H}'_\ell|_{X_{\overline{k}}} \to \mathcal{H}_\ell|_{X_{\overline{k}}}$ which, by Nakayama's lemma, is automatically an isomorphism. (5)(b) follows similarly from (2)(b), (3)(b). The proofs of (2)(b), (3)(b), (4) are slightly more involved and we refer to [C20b, Cor. 12.2] for details. □

4.3.2.2. **Lemma.** *Assume X is proper over k or X is a curve. Then for every $i \geq 0$ and $? = \emptyset, c$, $P_?^i(\mathcal{C}_\dagger, T)$ is in $\overline{\mathbb{Q}}[T]$ and independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$.*

Proof. By Poincaré duality, the assertion for $? = \emptyset$; follows from the one for $? = c$. The assertion for $P_c^i(\mathcal{C}_\dagger, T)$ follows from the facts that $\chi(\underline{\mathcal{C}}; T) := \chi(\mathcal{C}_\dagger; T)$ is independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$ (Lemma 4.2), and that

- if X is proper over k, $H_c^i(X_{\overline{k}}, \mathcal{C}_\dagger)$ is $\iota_\dagger$-pure of weight $w + i$ ([D80, Cor. (3.3.6)], 3.2.1)
- if X is an affine curve, $H_c^0(X_{\overline{k}}, \mathcal{C}_\dagger) = 0$, $H_c^1(X_{\overline{k}}, \mathcal{C}_\dagger)$ is of $\iota_\dagger$-weights $< w+2$ ([D80, Cor. (2.2.10)], 3.1.4) and $H_c^1(X_{\overline{k}}, \mathcal{C}_\dagger) = \mathcal{C}_{\dagger, \overline{x}\pi_1(X_{\overline{k}})}(-1)$ is of $\iota_\dagger$-weights $w + 2$.

□

4.3.2.3. **Lemma.** *For $? = \emptyset, c$ and every $i \geq 0$, $H_?^i(X, \mathcal{H}_\ell) \otimes F_\ell = H_?^i(X, \mathcal{M}_\ell) \Leftrightarrow H_?^i(X, \mathcal{H}_\ell)[\lambda_\ell] = 0$ and $H_?^j(X, \mathcal{H}_\ell)[\lambda_\ell] = 0$, $j = i, i+1 \Leftrightarrow dim H_?^i(X, \mathcal{F}_\ell) = dim H_?^i(X, \mathcal{M}_\ell)$.*

Proof. Combine the short exact sequences $0 \to H_?^i(X, \mathcal{H}_\ell) \otimes F_\ell \to H_?^i(X, \mathcal{M}_\ell) \to H_?^{i+1}(X, \mathcal{H}_\ell)[\lambda_\ell] \to 0$ and the equalities $dim(H_?^i(X, \mathcal{H}_\ell) \otimes F_\ell) = dim(H_?^i(X, \mathcal{F}_\ell)) + dim H_?^i(X, \mathcal{H}_\ell)[\lambda_\ell]$, $i \geq 0$. □

5. Langlands correspondance with ultraproduct coefficients and applications

Let X be a smooth, projective, geometrically connected curve over k with generic point η. Let $\mathcal{L}$ denote the set of all primes $\neq p$. For every $\dagger \in \mathcal{L} \cup \mathcal{U}$ weft once for all an isomorphism $\iota_\dagger : \overline{\mathbb{Q}}_\dagger \tilde{\to} \mathbb{C}$.

5.1. For $\dagger \in \mathcal{L} \cup \mathcal{U}$ and every integer $r \geq 1$, let $\mathcal{I}_r(X, \overline{\mathbb{Q}}_\dagger) \subset \mathcal{C}(X, \overline{\mathbb{Q}}_\dagger)$ denote the subset of rank-r irreducible $\overline{\mathbb{Q}}_\dagger$-local systems with finite determinant on X. Write

$$\mathcal{I}_r(\eta, \overline{\mathbb{Q}}_\dagger) := \mathrm{colim} \mathcal{I}_r(U, \overline{\mathbb{Q}}_\dagger) \subset \mathcal{C}(\eta, \overline{\mathbb{Q}}_\dagger) := \mathrm{colim} \mathcal{C}(U, \overline{\mathbb{Q}}_\dagger),$$

where the colimit is over all non-empty open subscheme $U \subset X$.

For $[\mathcal{C}] \in \mathcal{C}(\eta, \overline{\mathbb{Q}}_\dagger)$, there is a largest non-empty open subscheme $j : U_\mathcal{C} \hookrightarrow X$ over which $[\mathcal{C}]$ is unramified that is arises from a (necessarily unique) $\mathcal{C} \in \mathcal{C}(U_\mathcal{C}, \overline{\mathbb{Q}}_\dagger)$. For $x \in |U_\mathcal{C}|$, one thus has the local L-factor $L_x([\mathcal{C}], T) := L_x(\mathcal{C}, T)$ (Subsection 4.1). We simply write $\mathcal{C} := [\mathcal{C}]$ in the following.

5.2. Let $K := k(\eta)$ denote the function field of X and for $x \in |X|$, let K_x denote the completion of K at x, $\widehat{\mathcal{O}}_{X,x}$ its ring of integers and $\mathbb{A} := \mathrm{colim}_S \prod_{x \in S} K_x \prod_{x \in |X| \setminus S} \widehat{\mathcal{O}}_{X,x}$ the ring of adèles of K (where the colimit is over all finite subsets $S \subset |X|$).

Fix a finite character $\pi_1(K)^{ab} \to \mathbb{C}^\times$ and let $\delta : K^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$ denote its composition with the reciprocity morphism $rec : K^\times \setminus \mathbb{A}^\times \hookrightarrow \pi_1(K)^{ab}$ of global class field theory. For every integer $r \geq 1$, $GL_r(\mathbb{A})$ acts by right translation on the $\mathbb{C}$-vector space of locally constant maps $GL_r(K) \setminus GL_r(\mathbb{A}) \to \mathbb{C}$ and this action stabilizes the $\mathbb{C}$-vector space $Cusp_{r,\delta}(\mathbb{A})$ of cuspidal automorphic forms with central character δ that is those locally constant maps $f : GL_r(K) \setminus GL_r(\mathbb{A}) \to \mathbb{C}$ satisfying:
- The $GL_r(\mathcal{O})$-orbit of f generates a finite-dimensional $\mathbb{C}$-vector space ;
- For every $z \in Z(GL_r(\mathbb{A})) = \mathbb{A}^\times$, $f \cdot z = \delta(z) f$;

- (Cuspidality) For every partition $r = r_1 + \cdots + r_s$ with $r_i > 0$ defining a standard parabolic subgroup $P_{\underline{r}} \subset GL_r$ with unipotent radical $U_{\underline{r}}$,

$$\int_{U_{\underline{r}}(K)\backslash U_{\underline{r}}(\mathbb{A})} f(ug)du = 0, \quad g \in GL_r(\mathbb{A}).$$

As a representation of $GL_r(\mathbb{A})$, $Cusp_{r,\delta}(\mathbb{A})$ decomposes as a direct sum of irreducible representations - called automorphic cuspidal, each of them appearing with multiplicity one. These are those which do not arise, by parabolic induction, from lower rank linear groups. Let $\mathcal{A}_r(\eta)$ denote the set of isomorphism classes of complex irreducible cuspidal automorphic representations of $GL_r(\mathbb{A})$ whose central character is of finite order. To every $\pi \in \mathcal{A}_r(\eta)$ is attached a non-empty open subset $j_\pi : U_\pi \hookrightarrow X$ and for every $x \in |X|$ an irreducible $\mathbb{C}$-representation π_x of $GL_r(K_x)$ such that

- π_x is unramified (that is $dim({\pi_x}^{GL_r(\widehat{\mathcal{O}}_{X,x})}) = 1$) if and only if $x \in U_\pi$;
- $\pi = \otimes'_{x\in|X|}\pi_x$ is the restricted tensor product with respect to $|X| \setminus U_\pi$ and the lines ${\pi_x}^{GL_r(\widehat{\mathcal{O}}_{X,x})}$, $x \in |U_\pi|$ (meaning that π is generated by vectors of the form $\otimes_{x\in|X|} f_x$, where for all but finitely many $x \in |U_\pi|$, $f_x \in {\pi_x}^{GL_r(\widehat{\mathcal{O}}_{X,x})}$).

Every r-tuple $\underline{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^{\times\, r}$ defines a character $\chi_{\underline{\lambda}} : B_r(K_x) \to \mathbb{C}^\times$, $\chi_{\underline{\lambda}}(b) = \lambda_1^{v_x(b_{1,1})} \cdots \lambda_n^{v_x(b_{r,r})}$, $b = (b_{i,j}) \in B_r(K_x)$ of the Borel subgroup of upper triangular matrixes in $GL_r(K_x)$. The induced representation from $B_r(K_x)$ to $GL_r(K_x)$ has a unique unramified irreducible subrepresentation $\pi_x(\underline{\lambda})$ and $\pi_x(\underline{\lambda}) \simeq \pi_x(\underline{\mu})$ if and only if the underlying multisets of $\underline{\lambda}$ and $\underline{\mu}$ coincide. For $x \in |U_\pi|$, one can show $\pi_x \simeq \pi_x(\underline{\lambda}_x(\pi))$ for some $\underline{\lambda}_x = (\lambda_{1,x}(\pi), \dots, \lambda_{r,x}(\pi)) \in \mathbb{C}^{\times\, r}$ - the Hecke eigenvalues of π at x. In particular, π_x is uniquely by its local L-factor

$$L_x(\pi, T) := L(\pi_x, T) = \prod_{1\leq i\leq r} \frac{1}{1 - \lambda_{i,x}(\pi)T}.$$

The strong multiplicity one theorem of Piatetski-Shapiro [P79] ensures the local L-factors $L_x(\pi, T)$, $x \in |U_\pi|$ détermines π uniquely; this is the automorphic counterpart of (weak) Cebotarev 3.2.4.

5.3. For $\dagger \in \mathcal{L} \cup \mathcal{U}$, one says that $\pi \in \mathcal{A}_r(\eta)$ and $\mathcal{C} \in \mathcal{C}(\eta, \overline{\mathbb{Q}}_\dagger)$ correspond to each other in the sense of Langlands, and writes $\pi \sim \mathcal{C}$, if $L_x(\pi, T) = L_x(\mathcal{C}, T)$, $x \in U_\pi \cap U_{\mathcal{C}}$.

5.3.1. **Theorem.** (Langlands correspondance (L,r,$\dagger$)) *There exists maps*

$$\mathcal{A}_r(\eta) \underset{\pi_{\dagger,-}}{\overset{\mathcal{I}_{\dagger,-}}{\rightleftarrows}} \mathcal{I}_r(\eta, \overline{\mathbb{Q}}_\dagger)$$

such that $\pi_{\dagger,-} \circ \mathcal{I}_{\dagger,-} = Id$, $\mathcal{I}_{\dagger,-} \circ \pi_{\dagger,-} = Id$ *and* *for every* $\pi \in \mathcal{A}_r(\eta)$, $U_\pi = U_{\mathcal{I}_{\dagger,\pi}}$ *and* $\pi \sim \mathcal{I}_{\dagger,\pi}$;
for every $\mathcal{I} \in \mathcal{I}_r(\eta, \overline{\mathbb{Q}}_\dagger)$, $U_{\mathcal{I}} = U_{\pi_{\dagger,\mathcal{I}}}$ *and* $\pi_{\dagger,\mathcal{I}} \sim \mathcal{I}$.

The compatibility conditions on local L-factors in (L,r,$\dagger$) impose that if $\pi \sim \mathcal{I}$ the central character of π coincides with the determinant of $\mathcal{I}$ *via* $rec : K^\times \setminus \mathbb{A}^\times \hookrightarrow \pi_1(K)^{ab}$.

The proof of (L,r,$\dagger$) for $\dagger \in \mathcal{L}$ was completed by L. Lafforgue [L02, VI.9] building on previous works of Drinfeld, Deligne, Laumon *etc.* As an output of the proof, one obtains that if $\pi \sim \mathcal{C}$ then the local L-factors and ϵ-factors of π and $\mathcal{C}$ coincide at every $x \in |X|$ (see *e.g.* [C20b, 13.1] for the definitions of local L-factors and ϵ-factors at ramified points; we will not use their explicit definitions in the following). L. Lafforgue also proved the Ramanujan-Peterson Conjecture [L02, VI.10 (i)], which implies that the poles of $L_x(\pi, T)$, $x \in |U_\pi|$ are of absolute value 1.

Theorem 5.3.1 and the Ramanujan-Peterson Conjecture immediately imply the curve case of the Companion conjecture of Deligne [D80, Conj. (1.2.10)].

Endow $\mathcal{C}(X, \overline{\mathbb{Q}}_\dagger)$ with the equivalence relation defined by $\mathcal{C} \equiv \mathcal{C}'$ if there exists $\mathcal{I}_j \in \mathcal{I}_{r_j}(X, \overline{\mathbb{Q}}_\dagger)$ and $\alpha_j, \alpha'_j \in \overline{\mathbb{Q}}_\dagger^\times$, $j = 1, \dots, s$ such that

$$\mathcal{C}^{ss} = \oplus_{1\leq j\leq s}\mathcal{I}_j^{(\alpha_j)}, \quad \mathcal{C}'^{\,ss} = \oplus_{1\leq j\leq s}\mathcal{I}_j^{(\alpha'_j)},$$

where $(-)^{ss}$ denotes semisimplification. (It follows from 3.1.3.1 that any semisimple $\mathcal{C}\in\mathcal{C}(X,\overline{\mathbb{Q}}_\dagger)$ can be written in the above form).

5.3.2. **Corollary.** (Companion conjecture) *Let $\dagger\in\mathcal{L}\cup\mathcal{U}$ and $\mathcal{I}_\dagger\in\mathcal{I}_r(X,\overline{\mathbb{Q}}_\dagger)$. Then,*

- *(5.3.2.1) $\mathcal{I}_\dagger$ is pure of weight 0: for every $x\in|X|$ and isomorphism $\iota:\overline{\mathbb{Q}}_\dagger\tilde{\to}\mathbb{C}$ the poles of $\iota L_x(\mathcal{I}_\dagger,T)$ all have absolute value 1 (in particular they are algebraic over $\mathbb{Q}$);*
- *(5.3.2.2) $Q_{\mathcal{I}_\dagger}$ is a finite extension of $\mathbb{Q}$;*
- *(5.3.2.3) For every $\ddagger\in\mathcal{L}\cup\mathcal{U}$ there exists $\mathcal{I}_\ddagger\in\mathcal{I}_r(X,\overline{\mathbb{Q}}_\dagger)$ such that $L_x(\mathcal{I}_\dagger,T)=L_x(\mathcal{I}_\ddagger,T)$, $x\in|X|$.*

For an arbitrary $\dagger\in\mathcal{L}\cup\mathcal{U}$, (5.3.2.1) directly reduces to the curve case by Theorem 2.5. $\dagger,\ddagger\in\mathcal{L}$, (5.3.2.2) , (5.3.2.3) were established in [D12], [Dr12], using geometric arguments to reduce to the curve case.

From (5.3.2.1) applied to the composition factor of a Jordan-Holder filtration and 3.1.3.1 and from 3.2.3, one gets the following fundamental structural result (which shows *a posteriori* that the mixedness condition in 3.2.4 is redundant).

5.3.3. **Corollary.** ($\iota_\dagger$-weight filtration) *For $\dagger\in\mathcal{L}\cup\mathcal{U}$, every $\mathcal{C}\in\mathcal{C}(X,\overline{\mathbb{Q}}_\dagger)$ is $\iota_\dagger$-mixed. More precisely, there exists a filtration - automatically unique and functorial*

$$\mathcal{C}:=W_1\mathcal{C}\supsetneq W_2\mathcal{C}\supsetneq\cdots\supsetneq W_r\mathcal{C}\supsetneq W_{r+1}\mathcal{C}=0$$

such that $Gr_i^W(\mathcal{C})=W_i\mathcal{C}/W_{i+1}\mathcal{C}$ is ι-pure of weight w_i, $i=1,\dots,r$ with $w_1>w_2>\cdots>w_r$.

5.3.4. Corollary (5.3.2.3) easily extends to arbitrary semisimple $\overline{\mathbb{Q}}_\dagger$-local systems as follows. From 3.1.3.1 every semisimple $\mathcal{C}\in\mathcal{C}(X,\overline{\mathbb{Q}}_\dagger)$ can be written as $\mathcal{C}\simeq\oplus_{1\leq j\leq s}\mathcal{I}_j^{(\alpha_j)}$ with $\mathcal{I}_j\in\mathcal{I}_{r_j}(X,\overline{\mathbb{Q}}_\dagger)$ and $\alpha_j\in\overline{\mathbb{Q}}_\dagger$, $j=1,\dots,s$. From (5.3.2.3), for $j=1,\dots,s$ and every $\ddagger\in\mathcal{L}\cup\mathcal{U}$ there exists $\mathcal{I}_{j,\ddagger}\in\mathcal{I}_{r_j}(X,\overline{\mathbb{Q}}_\dagger)$ such that $L_x(\mathcal{I}_j,T)=L_x(\mathcal{I}_{j,\ddagger},T)$, $x\in|X|$. Set also $\alpha_{j,\ddagger}:=\iota_\ddagger^{-1}\circ\iota_\dagger(\alpha_j)$. Then $\mathcal{C}_\ddagger:=\oplus_{1\leq j\leq s}\mathcal{I}_{j,\ddagger}^{(\alpha_{j,\ddagger})}$ satisfied the requested property.

5.4. **About the proofs.** As already mentioned, Theorem 5.3.1 for $\dagger\in\mathcal{L}$ and Corollary 5.3.2 for $\dagger,\ddagger\in\mathcal{L}$) are already settled. In particular, every $\mathcal{I}_\ell\in\mathcal{I}_r(X,\overline{\mathbb{Q}}_\ell)$ gives rise to a compatible family $\underline{\mathcal{I}}=\mathcal{I}_\mathfrak{l}$, $\mathfrak{l}\in\mathcal{L}$ of irreducible $\overline{\mathbb{Q}}_\mathfrak{l}$-local systems which are ι-pure of weight 0. Take any family $\underline{\mathcal{H}}$ of integral models and write $\mathcal{M}_\mathfrak{l}:=\mathcal{H}_\mathfrak{l}\otimes\overline{\mathbb{F}}_\mathfrak{l}$, $\mathfrak{l}\in\mathcal{L}$. Fix $\mathfrak{u}\in\mathcal{U}$. From 4.3.2.1 (4)(b), (5)(b) the map

$$\mathcal{I}_r(X,\overline{\mathbb{Q}}_\ell)\to\mathcal{I}_r(X,\overline{\mathbb{Q}}_\mathfrak{u}),\ \ \mathcal{I}_\ell\longleftrightarrow\underline{\mathcal{I}}\to\mathcal{M}_\mathfrak{u}$$

is well-defined. From 4.2, it preserves the local L-factors at $x\in X$ and from 3.2.4 it is injective. It remains to prove it is surjective; this will both complete the proof of Theorem 5.3.1 (when X is a curve) and Corollary 5.3.2 (when X is of arbitrary dimension).

- Assume X is a curve. The construction of $\mathcal{I}_\dagger:\mathcal{A}_r(\eta)\to\mathcal{I}_r(\eta,\overline{\mathbb{Q}}_\dagger)$ is actually the "difficult" part of Theorem 5.3.1. Once $\mathcal{I}_\dagger:\mathcal{A}_r(\eta)\to\mathcal{I}_r(\eta,\overline{\mathbb{Q}}_\dagger)$ is constructed, the construction of $\pi_\dagger:\mathcal{I}_r(\eta,\overline{\mathbb{Q}}_\dagger)\to\mathcal{A}_r(\eta)$ is by induction on r, through Deligne's "principe de réccurrence". The $r=1$ case is a reformulation of global class field theory. The induction step combines the reciprocity theorem of Piateskii-Shapiro [L02, Thm. B.13] (A purely automorphic result), the product formula for ϵ-factors and weak Cebotarev 3.2.4. For $\dagger\in\mathcal{L}$, the product formula is due to Laumon [Lau87, §3]. For $\dagger\in\mathcal{U}$ it can be derived from the product formula for $\overline{\mathbb{F}}_\ell$-local systems, already established by Deligne [D73, Thm. 7.11]. We refer to [C20b, 14.3] for details.
- Assume now X is of dimension ≥ 2. To prove (5.3.2.2), (5.3.2.3), one can assume $\ddagger=\ell\in\mathcal{L}$. The proof for $\dagger\in\mathcal{U}$ is similar to the one of [D12], [Dr12] for $\dagger\in\mathcal{L}$. Let $Cu(X)$ denote the set of all non-constant morphisms $\phi:C\to X$ from a smooth curve C over k to X. The basic idea is to attach to $\mathcal{I}_\dagger$ a $\overline{\mathbb{Q}}_\ell$-skeleton $\mathcal{S}$ on X that is a collection $\mathcal{S}=\mathcal{I}_{\ell,C}$, $C\in Cu(X)$ of semisimple $\overline{\mathbb{Q}}_\ell$-local systems $\mathcal{I}_{\ell,C}$ on C such that $\mathcal{I}_{\ell,C}$ and $\mathcal{I}_{\ell,C'}$ coincide on $C\times_X C'$ and then shows that $\mathcal{S}$ actually arises from a true $\overline{\mathbb{Q}}_\ell$-local system on X. Namely, one takes for $\mathcal{I}_{\ell,C}$ the (necessarily unique and irreducible) $\overline{\mathbb{Q}}_\ell$-local system compatible with $\mathcal{I}_\dagger|_C$. Since $\mathcal{I}_\dagger$ is a $\overline{\mathbb{Q}}_\mathfrak{u}$-local system on the whole X, the $\mathcal{I}_{\ell,C}$, $C\in Cu(X)$ satisfy the glueing condition on $C\times_X C'$. Also, from (5.3.2.1), the field of definition $Q_{\mathcal{I}_{\ell,C}}$ is algebraic over $\mathbb{Q}$ and, by definition of almost tameness and the compatibility of $\mathcal{I}_{\ell,C}$, $\mathcal{I}_\dagger|_C$, there

exists a common connected étale cover $X' \to X$ such that $\mathcal{I}_{\ell,C}|_{C\times_X X'}$ is tame for every $C \in Cu(X)$. These conditions ensures that the subfield $Q_{\mathcal{S}}$ of $\overline{\mathbb{Q}}$ generated by the $Q_{\mathcal{I}_{\ell,C}}$, $C \in Cu(X)$ is a finite extension of $\mathbb{Q}$ [D12, Thm. 3.1, Rem. 3.10] and that the $\mathcal{I}_{\ell,C}$, $C \in Cu(X)$ all arise as $\mathcal{I}_{\ell,C} = \mathcal{I}_\ell|_C$ from a (necessarily unique and irreducible) $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{I}$ on X [Dr12, Thm. 2.5].

5.5. **First Applications.**

5.5.1. *Finiteness.* Let $X \hookrightarrow \overline{X}$ be a normal compactification, D an effective Cartier divisor on $\overline{X}$ with support in $\overline{X} \setminus X$ and $\alpha : X' \to X$ a Galois cover, $\overline{\alpha} : \overline{X}' \to \overline{X}$ the normalization of $\overline{X}$ in α. For $\dagger \in \mathcal{L} \cup \mathcal{U}$, write $\mathcal{I}_r^{\leq D}(X, \overline{\mathbb{Q}}_\dagger) \subset \mathcal{I}_r(X, \overline{\mathbb{Q}}_\dagger)$ (resp. $\mathcal{I}_r^{\leq \alpha}(X, \overline{\mathbb{Q}}_\dagger) \subset \mathcal{I}_r(X, \overline{\mathbb{Q}}_\dagger)$ for the subset of all $\mathcal{I} \in \mathcal{I}_r(X, \overline{\mathbb{Q}}_\dagger)$ such that for every smooth, separated, connected curve C over $\overline{k}$ and morphism $\phi : C \to \overline{X}_{\overline{k}}$, $Sw(\mathcal{I}|_C) \leq \overline{\phi}^* D$, where $\overline{\phi} : \overline{C} \to \overline{X}_{\overline{k}}$ denotes the extension of $\phi : C \to \overline{X}_{\overline{k}}$ to the smooth compactification $C \hookrightarrow \overline{C}$ of C (resp. $\mathcal{I}|_{X'}$ is tame). Here, $Sw(\mathcal{I}|_C)$ denotes the (global) Swan conductor It is an effective divisor on C, supported on $\overline{C} \setminus C$, whose local degree at $x \in \overline{C} \setminus C$ measures the wild ramification of $\mathcal{I}|_C$ at x (*e.g.* $Sw(\mathcal{I}|_C) = 0$ if and only if $\mathcal{I}|_C$ is tame); it can be recovered from the local ϵ-factors of $\mathcal{I}|_C$. In particular, it follows from the fact that for a smooth curve C on k and $\ell \in \mathcal{L} \cup \mathcal{U}$ the bijection $\mathcal{I}_r(C, \overline{\mathbb{Q}}_\ell) \longleftrightarrow \mathcal{I}_r(C, \overline{\mathbb{Q}}_\dagger)$ from 5.3.2 preserves local ϵ-factors so that the bijection $\mathcal{I}_r(X, \overline{\mathbb{Q}}_\ell) \longleftrightarrow \mathcal{I}_r(X, \overline{\mathbb{Q}}_\dagger)$ restricts to bijections

$$\mathcal{I}_r^{\leq D}(X, \overline{\mathbb{Q}}_\ell) \longleftrightarrow \mathcal{I}_r^{\leq D}(X, \overline{\mathbb{Q}}_\dagger), \ \ \mathcal{I}_r^{\leq \alpha}(X, \overline{\mathbb{Q}}_\ell) \longleftrightarrow \mathcal{I}_r^{\leq \alpha}(X, \overline{\mathbb{Q}}_\dagger).$$

If $D_{\overline{\alpha}} \hookrightarrow \overline{X}$ denotes the discriminant divisor of $\overline{\alpha} : \overline{X}' \to \overline{X}$, one always has $\mathcal{I}_r^{\leq \alpha}(X, \overline{\mathbb{Q}}_\dagger) \subset \mathcal{I}_r^{\leq rD_\alpha}(X, \overline{\mathbb{Q}}_\dagger)$. Hence, it follows from Deligne's finiteness theorem [EKer12, Thm. 2.1], which asserts that $\mathcal{I}_r^{\leq D}(X, \overline{\mathbb{Q}}_\ell)/\equiv$ is finite, that $\mathcal{I}_r^{\leq \alpha}(X, \overline{\mathbb{Q}}_\dagger)/\equiv$ and $\mathcal{I}_r^{\leq D}(X, \overline{\mathbb{Q}}_\dagger)/\equiv$ are finite, of cardinality independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$.

5.5.2. *Lifting.* For $\ell \in \mathcal{L}$, write $\mathcal{I}_r(X, \overline{\mathbb{F}}_\ell)$ for the set of rank-r irreducible $\overline{\mathbb{F}}_\ell$-local systems on X and, with the notation of 5.5.1, $\mathcal{I}_r^{\leq D}(X, \overline{\mathbb{F}}_\ell) \subset \mathcal{I}_r(X, \overline{\mathbb{F}}_\ell)$ (resp. $\mathcal{I}_r^{\leq \alpha}(X, \overline{\mathbb{F}}_\ell) \subset \mathcal{I}_r(X, \overline{\mathbb{F}}_\ell)$) for the subset of all $\mathcal{I} \in \mathcal{I}_r(X, \overline{\mathbb{F}}_\ell)$ such that for every smooth, separated, connected curve C over $\overline{k}$ and morphism $\phi : C \to \overline{X}_{\overline{k}}$, $Sw(\mathcal{I}|_C) \leq \overline{\phi}^* D$ (resp. $\mathcal{I}|_{X'}$ is tame).

Fix a finite character $\chi : \pi_1(X) \to \overline{\mathbb{Q}}^\times$. For $Q = \overline{\mathbb{Q}}_\dagger$ for $\dagger \in \mathcal{L} \cup \mathcal{U}$ or $\overline{\mathbb{F}}_\ell$, $\ell \in \mathcal{L}$, write $\mathcal{I}_r^{\leq \alpha, \chi}(X, Q) \subset \mathcal{I}_r^{\leq \alpha, \chi}(X, Q)$ to be the subset of objects with determinant $\chi \otimes Q : \pi_1(X) \to Q^\times$. Since for every $\ell \in \mathcal{L}$ $|\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell)| < +\infty$ (fixing the determinant imposes that irreducible objects in the same $\equiv$-class only differ by a twist by a root of unity of order dividing the order of the determinant), there exists $\ell_0 := \ell_0(X, \alpha, \chi, r) \in \mathcal{L}$ such that for $\ell \geq \ell_0$, every $\mathcal{I}_\ell \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell)$ has a unique $\overline{\mathbb{Z}}_\ell$-model $\mathcal{H}_\ell$ (Corollary 4.3.2.1 (5)(b)) and $\mathcal{M}_\ell := \mathcal{H}_\ell \otimes \overline{\mathbb{F}}_\ell \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell)$ (Corollary 4.3.2.1 (4)(b)). So that, for $\ell \geq \ell_0$, one has a well-defined 'reduction modulo-ℓ' map

$$\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell) \to \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell), \ \mathcal{I}_\ell \to \mathcal{M}_\ell$$

Corollary. *For $\ell \gg 0$, the reduction modulo-ℓ map $\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell) \to \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell)$ is bijective. In particular, $\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell)$ is finite and every $\mathcal{M}_\ell \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell)$ lifts uniquely to a $\overline{\mathbb{Z}}_\ell$-model $\mathcal{H}_\ell$ of some $\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell)$.*

The injectivity follows from Corollary 4.3.2.1 (2)(b), (4)(b) as in the proof of Corollary 4.3.2.1 (5)(b). If the reduction modulo-ℓ map $\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell) \to \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell)$ were not surjective, there would exist an infinite subset $\mathcal{L}' \subset \mathcal{L}$ such that for every $\ell \in \mathcal{L}'$ there exists a $\mathcal{M}_\ell \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{F}}_\ell)$ which does not lift to a $\overline{\mathbb{Z}}_\ell$-model $\mathcal{H}_\ell$ of some $\mathcal{I}_\ell \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell)$. Since $\mathcal{M}_\ell|_{X'}$ is tame $\ell \in \mathcal{L}'$, for every ultrafilter $\mathfrak{u}$ on $\mathcal{L}'$, $\underline{\mathcal{M}} = \mathcal{M}_\ell, \ell \in \mathcal{L}'$ is in $S_{\mathfrak{u}}^t(X, \overline{\mathbb{F}})$ with $\mathcal{M}_{\mathfrak{u}} \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$. For every $\ell \in \mathcal{L}'$ let $\mathcal{I}_\ell \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell)$ corresponding to $\mathcal{M}_{\mathfrak{u}} \in \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_{\mathfrak{u}})$ under the bijection $\mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\ell) \longleftrightarrow \mathcal{I}_r^{\leq \alpha, \chi}(X, \overline{\mathbb{Q}}_\dagger)$ of 5.5.1. Write $\mathcal{N}_\ell := \mathcal{I}_\ell \otimes \overline{\mathbb{F}}_\ell$, $\ell \in \mathcal{L}'$. By construction, $\mathcal{M}_{\mathfrak{u}} \simeq \mathcal{N}_{\mathfrak{u}}$ and as this holds for every ultrafilter $\mathfrak{u}$ on $\mathcal{L}'$, $\mathcal{M}_\ell \simeq \mathcal{N}_\ell$ for $\ell \in \mathcal{L}'$ large enough. This contradicts the definition of $\mathcal{L}'$.

5.6. **Algebraic monodromy.**

5.6.1. Let $\mathcal{V}(k)$ denote the category of normal varieties over k, $\mathcal{P}(-)\to\to\mathcal{V}(k)$ the fibered (into monoidal categories) category of smooth projective schemes and, for $\dagger\in\mathcal{L}\cup\mathcal{U}$, $\mathcal{C}(-,\overline{Q}_\dagger)\to\to\mathcal{V}(k)$ the fibered (into neutral Tannakian categories over $\overline{Q}_\dagger$) category of $\overline{Q}_\dagger$-local systems. The higher direct image functor give rise to a morphism

$$R^\bullet(-)_*\overline{\mathbb{Q}}_\dagger:\mathcal{P}(-)\to\mathcal{C}(-,\overline{\mathbb{Q}}_\dagger)$$

of categories fibered into monoidal categories over $\mathcal{V}(k)$.

The (conjectural!) output of the philosophy of pure isomotives is the existence of a category $\mathcal{M}ot(-)\to\mathcal{P}(k)$ fibered into semisimple neutral Tannakian categories over $\overline{\mathbb{Q}}$ together with a morphism $R^\bullet(-)_*\overline{\mathbb{Q}}:\mathcal{P}(-)\to\mathcal{M}ot(-)$ of categories fibered into monoidal categories over $\mathcal{P}(k)$ such that the morphisms $R^\bullet(-)_*\overline{\mathbb{Q}}_\dagger:\mathcal{P}(-)\to\mathcal{C}(-,\overline{Q}_\dagger)$ factor through a commutative diagram

$$\begin{tikzcd}
\mathcal{P}(-) \arrow[r, "R^\bullet(-)_*\overline{\mathbb{Q}}"] \arrow[d, "R^\bullet(-)_*\overline{\mathbb{Q}}_\dagger"'] & \mathcal{M}ot(-) \arrow[d, "-\otimes\overline{\mathbb{Q}}_\dagger"] \\
\mathcal{C}(-,\overline{\mathbb{Q}}_\dagger) & \mathcal{M}ot(-)\otimes\overline{\mathbb{Q}}_\dagger \arrow[l, "R_{\overline{\mathbb{Q}}_\dagger}"]
\end{tikzcd}$$

with $R_{\overline{\mathbb{Q}}_\dagger}:\mathcal{M}ot(-)\otimes\overline{\mathbb{Q}}_\dagger\to\mathcal{C}(-,\overline{\mathbb{Q}}_\dagger)$ a fully faithful $\otimes$-functor. (Strictly speaking the construction of the category of pure isomotives is unconditional but its properties are). With this picture in mind, Corollary 5.3.2 reflects the expectation that, for (a smooth) $X\in\mathcal{V}(k)$, objects in $\mathcal{I}_r(X,\overline{\mathbb{Q}}_\dagger)$ are in the essential image of $R_{\overline{\mathbb{Q}}_\dagger}:\mathcal{M}ot(X)\otimes\overline{\mathbb{Q}}_\dagger\to\mathcal{C}(X,\overline{\mathbb{Q}}_\dagger)$. More precisely for every $\mathcal{I}_\dagger\in\mathcal{I}_r(X,\overline{\mathbb{Q}}_\dagger)$ and isomotive $\mathcal{I}\in\mathcal{M}ot(X)$ with $\mathcal{I}_\dagger:=R_{\overline{\mathbb{Q}}_\dagger}(\mathcal{I}\otimes\overline{\mathbb{Q}}_\dagger)$, the field $Q_{\mathcal{I}_\dagger}$ of (5.3.2.1) should be contained in any number field $Q_{\mathcal{I}}$ over which $\mathcal{I}$ is defined and for every $\ddagger\in\mathcal{L}\cup\mathcal{U}$, the $\overline{\mathbb{Q}}_\ddagger$-companion of $\mathcal{I}_\dagger$ is $\mathcal{I}_\ddagger:=R_{\overline{\mathbb{Q}}_\ddagger}(\mathcal{I}\otimes\overline{\mathbb{Q}}_\ddagger)$. In particular, if $G(\mathcal{I})$ denotes the Tannakian group of the full $\otimes$-subcategory $\langle\mathcal{I}\rangle^\otimes\subset\mathcal{M}ot(X)$ generated by $\mathcal{I}$ and $G(\mathcal{I}_\dagger)$ the one of $\langle\mathcal{I}\rangle^\otimes\subset\mathcal{C}(X,\overline{\mathbb{Q}}_\dagger)$ (that is, the Zariski closure of the image of $\pi_1(X)$ acting on $\mathcal{I}_{\dagger,\overline{x}}$), the equivalence of Tannakian categories $R_{\overline{\mathbb{Q}}_\dagger}:\langle\mathcal{I}\rangle^\otimes\otimes\overline{\mathbb{Q}}_\dagger\tilde{\to}\langle\mathcal{I}_\dagger\rangle^\otimes$ corresponds to an isomorphism $(G(\mathcal{I}_\dagger),\mathcal{I}_{\dagger,\overline{x}})\tilde{\to}(G(\mathcal{I}),\mathcal{I})\otimes\overline{\mathbb{Q}}_\dagger$ (well-defined up to conjugation due to the indeterminancy in the choice of the fiber functors). As a consequence of this, one should have:

5.6.1.1. **Conjecture.** *Let $\underline{\mathcal{C}}=\mathcal{C}_\dagger$, $\dagger\in\mathcal{L}\cup\mathcal{U}$ be a compatible family of semisimple $\overline{\mathbb{Q}}_\dagger$-local systems. Then there exist a reductive algebraic group $G(\underline{\mathcal{C}})$ over $\overline{\mathbb{Q}}$ together with a faithful linear finite-dimensional $\overline{\mathbb{Q}}$-representation V such that*

$$(G(\mathcal{C}_\dagger),\mathcal{C}_{\dagger,\overline{x}})\tilde{\to}(G(\underline{\mathcal{C}}),V)\otimes\overline{\mathbb{Q}}_\dagger,\quad \dagger\in\mathcal{L}\cup\mathcal{U}.$$

By Tannakian arguments one can deduce from Corollary 4.3.2.1 and Corollary 5.3.2 the following weak unconditional form of Conjecture 5.6.1.1.

5.6.1.2. **Corollary.** *Let $\underline{\mathcal{C}}=\mathcal{C}_\dagger$, $\dagger\in\mathcal{L}\cup\mathcal{U}$ be a compatible family of semisimple $\overline{\mathbb{Q}}_\dagger$-local systems. Write $G_\dagger:=G(\mathcal{C}_\dagger)$ and $\overline{G}_\dagger:=G(\mathcal{C}_\dagger|_{(X_{\overline{k}}})$.*

- (Connected components) *The morphisms $\pi_1(X_{\overline{k}})\twoheadrightarrow\pi_0(\overline{G}_\dagger)$ (resp. $\pi_1(X)\twoheadrightarrow\pi_0(G_\dagger)$), $\dagger\in\mathcal{L}\cup\mathcal{U}$ are continuous and all have the same kernel. In particular, the groups $\pi_0(\overline{G}_\dagger)$ (resp. $\pi_0(G_\dagger)$), $\dagger\in\mathcal{L}\cup\mathcal{U}$ are all canonically isomorphic.*
- (Neutral component) *There exists a connected reductive algebraic group G over $\overline{\mathbb{Q}}$ together with an irreducible faithful representation V such that*

$$(G_\dagger^\circ,\mathcal{C}_{\dagger,\overline{x}})\simeq(G,V)\otimes_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}_\dagger\ \ (\text{and } (\overline{G}_\dagger^\circ,\mathcal{I}_{\dagger,\overline{x}})\simeq(G^{der},V)\otimes_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}_\dagger),\quad \dagger\in\mathcal{L}\cup\mathcal{U}.$$

For $\ell\in\mathcal{L}$, the continuity of $\pi_1(X)\twoheadrightarrow\pi_0(G_\ell)$ follows from the continuity of the action of $\pi_1(X)$ on $\mathcal{I}_{\ell,\overline{x}}$ but for $\mathfrak{u}\in\mathcal{U}$, this requires the fact that $\pi_1^t(X)$ is topologically finitely generated which ensures that every finite index subgroup of $\pi_1^t(X)$ is open [NS07a], [NS07b].

5.6.1.3. Actually, for $\dagger \in \mathcal{L} \cup \mathcal{U}$, the category $\langle \mathcal{I}_\dagger \rangle^\otimes$ comes with the collection of the $G(\mathcal{I}_\dagger)$-conjugacy classes $\Phi^x_{\mathcal{I}_\dagger}$ of the image $\phi_x \in G(\mathcal{C}_\dagger)$ of the Frobenius φ_x at x, $x \in |X|$ and, similarly, the conjectural category $\langle \mathcal{C} \rangle^\otimes$ comes with the collection of $G(\mathcal{C})$-conjugacy classes $\Phi^x_{\mathcal{I}}$ of the image $\phi_x \in G(\mathcal{I})$ of the Frobenius φ_x at x, $x \in |X|$. Under the isomorphism $(G(\mathcal{I}_\dagger), \mathcal{I}_{\dagger,\overline{x}}) \tilde{\rightarrow} (G(\underline{\mathcal{I}}), \mathcal{I}) \otimes \overline{\mathbb{Q}}_\dagger$ of Conjecture 5.6.1.1, $\Phi^x_{\mathcal{I}_\dagger}$ and $\Phi^x_{\mathcal{I}} \otimes \overline{\mathbb{Q}_\dagger}$ should coincide, $x \in |X|$. In particular, since for $\ell \in \mathcal{L}$ the classical Cebotarev density theorem ensures that the union of the $\Phi^x_{\mathcal{I}_\ell}$, $x \in |X|$ is Zariski-dense in $G(\mathcal{I}_\ell)$, the union of the $\Phi^x_{\mathcal{I}}$, $x \in |X|$ should also be Zariski-dense in $G(\mathcal{I})$ hence for every $\dagger \in \mathcal{L} \cup \mathcal{U}$, the union of the $\Phi^x_{\mathcal{I}_\dagger}$, $x \in |X|$ should Zariski-dense in $G(\mathcal{I}_\dagger)$. Again, using purely tannakian arguments, one can deduce from Corollary 4.3.2.1 and Corollary 5.3.3 the following unconditional statement, which enhances Corollary 3.2.4.

Corollary. (Tannakian Cebotarev - [CT20, Thm. 1.4.1]) *Let $\mathcal{C}$ be an arbitrary $\overline{\mathbb{Q}}_\dagger$-coefficient on X and let $S \subset |X|$ be a subset of closed points. Assume S has upper Dirichlet density $\delta^u(S) > 0$ (resp. $\delta^u(S) = 1$). Then the Zariski-closure of the union of the $\Phi^x_{\mathcal{C}}$, $x \in S$ contains at least one connected component of $G(\mathcal{C})$ (resp. is Zariski-dense in $G(\mathcal{C})$).*

5.6.2. *About the proofs.*

5.6.2.1. For every finite subset $I \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and finite-dimensional vector space V over a field Q of characteristic 0, write $T_I(V) := \oplus_{(m,n)\in I} V^{\otimes m} \otimes V^{\vee \otimes n}$. Let G be a reductive over Q and V a finite-dimensional faithful representation of G over Q. Recall (*e.g.* [D82, Prop. 3.1]) that every finite-dimensional representation of G over Q appears as a sub representation of $T_I(V)$ for some finite subset $I \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and that every (resp. reductive) algebraic subgroup $H \subset G$ is the stabilizer of a line (resp. of a non-zero vector) in some finite-dimensional representation of G over Q.

From these basic observations, one can already deduce from Theorem 3.1.4 the $\dagger$-independency of $\pi_0(\overline{G}_\dagger)$, $\pi_0(G_\dagger)$. Let us explain what happens for $\pi_0(G_\dagger)$ (the proof for $\pi_0(\overline{G}_\dagger)$ is similar). Fix $\dagger \in \mathcal{L} \cup \mathcal{U}$ and replace X with the connected étale cover corresponding to the kernel of $\pi_1(X) \to \pi_0(G_\dagger)$. By symmetry, it is enough to prove that for every other $\ddagger \in \mathcal{L} \cup \mathcal{U}$, $G_\ddagger$ is connected that is, for every finite index subgroup $U \subset \pi_1(X)$ and finite subset $I \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $T_I(\mathcal{C}_{\ddagger,\overline{x}})^U$ and $T_I(\mathcal{C}_{\ddagger,\overline{x}})^{\pi_1(X)}$ have the same dimension. From the continuity of $\pi_1(X) \to \pi_0(G_\dagger)$, one may restrict to open subgroups $U \subset \pi_1(X)$. But open subgroups of $\pi_1(X)$ correspond to connected étale covers of X so it is enough to prove that for every connected étale cover $X' \to X$, and finite subset $I \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $T_I(\mathcal{C}_{\ddagger,\overline{x}})^{\pi_1(X')}$ and $T_I(\mathcal{C}_{\ddagger,\overline{x}})^{\pi_1(X)}$ have the same dimension. Since we already know $T_I(\mathcal{C}_{\dagger,\overline{x}})^{\pi_1(X')}$ and $T_I(\mathcal{C}_{\dagger,\overline{x}})^{\pi_1(X)}$ have the same dimension, it is enough to show that for every connected étale cover $X' \to X$ the dimension of $T_I(\mathcal{C}_{\dagger,\overline{x}})^{\pi_1(X')}$ is independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$. Without loss of generality, we may assume $X' = X$. From Theorem 2.5, one may assume X is a curve. Decomposing $T_I(\mathcal{C}_\dagger) \simeq \oplus_{1 \leq i \leq r} \mathcal{I}^{(\alpha_i)}_{\dagger,i}$ with $\mathcal{I}_{\dagger,i} \in \mathcal{I}_{r_i}(X, \overline{\mathbb{Q}}_\dagger)$ and $\alpha_i \in \overline{\mathbb{Q}}^\times$, $i = 1, \ldots, r$ (recall that, by definition of compatibility, the Frobenius eigenvalues of $\mathcal{C}_\dagger$ are in $\overline{\mathbb{Q}}$) one has $T_I(\mathcal{C}_\ddagger) \simeq \oplus_{1 \leq i \leq r} \mathcal{I}^{(\alpha_i)}_{\ddagger,i}$ with $\mathcal{I}_{\dagger,i} \sim \mathcal{I}_{\ddagger,i}$, $i = 1, \ldots, r$. Since invariants commute with direct sums, it is enough to show that for each $i = 1, \ldots, r$ the dimension of $\mathcal{I}^{(\alpha_i)}_{i,\dagger,\overline{x}}{}^{\pi_1(X')}$ is independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$. From the semisimplicity of geometric monodromy, we have $H^2_c(X_{\overline{k}}, \mathcal{I}^{(\alpha_i)}_{i,\dagger}) \simeq \mathcal{I}^{(\alpha_i)}_{i,\dagger,\overline{x}\ \pi_1(X_{\overline{k}})}(-1) \simeq \mathcal{I}^{(\alpha_i)}_{i,\dagger,\overline{x}}{}^{\pi_1(X_{\overline{k}})}(-1)$. From Lemma 4.3.2.2, $P^2_c(\mathcal{I}^{(\alpha_i)}_{\dagger,i}, T) = P^2_c(\mathcal{I}^{(\alpha_i)}_{\ddagger,i}, T)$ and the conclusion follows from the fact that the dimension of $\mathcal{I}^{(\alpha_i)}_{i,\dagger,\overline{x}}{}^{\pi_1(X_{\overline{k}})}$ is the order of $|k|$ as a root of $P^2_c(\mathcal{I}^{(\alpha_i)}_{\dagger,i}, T)$.

5.6.2.2. The proofs of the $\dagger$-independency of $G^\circ_\dagger$ and of Corollary 5.6.1.3 require a significantly heavier Tannakian machinery and the full strength of Corollary 5.3.2 through the following claim.

Claim. (*e.g.* [C20a, Cor. 8.1]) *For $\dagger, \ddagger \in \mathcal{L} \cup \mathcal{U}$ and semisimple $\mathcal{C}_\dagger \in \mathcal{C}(X, \overline{\mathbb{Q}}_\dagger)$, $\mathcal{C}_\ddagger \in \mathcal{C}(X, \overline{\mathbb{Q}}_\ddagger)$ with $\mathcal{C}_\dagger \sim \mathcal{C}_\ddagger$, the companion correspondence $(\mathcal{O}b(\langle \mathcal{C}_\dagger \rangle^\otimes)/\simeq) \tilde{\rightarrow} (\mathcal{O}b(\langle \mathcal{C}_\ddagger \rangle^\otimes)/\simeq)$ of 5.3.4 induces a canonical semiring isomorphism $\mathbb{C}^+[\iota_\dagger G] \tilde{\rightarrow} \mathbb{C}^+[\iota_\ddagger G_\ell]$, characterized by the fact that it preserves local L-functions and maps irreducible representations to irreducible representations.*

Recall that given an algebraic group G over a field Q of characteristic 0 the semiring $Q^+[G]$ of G is the set of isomorphism classes of finite-dimensional Q-representations of G endowed with the laws $[V_1]+[V_2]:=[V_1\oplus V_2]$ with neutral element $[0]$ and $[V_1]\cdot[V_2]:=[V_1\otimes V_2]$ with neutral element $[Q]$. An element $[V]\in Q^+[G]$ is said to be irreducible if the corresponding representation V is.

With the above claim in hands, the †-independency of $G_\dagger^\circ$ directly follow from the reconstruction theorem [KaLV14, Thm. 1.2].

The proof of Corollary 5.6.1.3 is more subtle. The starting point is that for every semisimple $\mathcal{C}\in\mathcal{C}(X,\overline{\mathbb{Q}}_\dagger)$ there exists $\ell\gg 0$ and an isomorphism $\iota_\ell:\overline{\mathbb{Q}}_\ell\tilde{\to}\mathbb{C}$ such that the $\overline{\mathbb{Q}}_\ell$-compagnon $\mathcal{C}_\ell$ of $\mathcal{C}$ (see 5.3.4) is a $\overline{\mathbb{Q}}_\ell$-local system (not only a lcc Weil $\overline{\mathbb{Q}}_\ell$-sheaf). It then follows from the classical Cebotarev density theorem that Corollary 5.6.1.3 holds for $\mathcal{C}_\ell$. Assume $G:=G(\mathcal{C})$ is connected. Then from the †-independency of $\pi_0(G_\dagger)$, $G_\ell:=G(\mathcal{C}_\ell)$ is connected as well. In particular, $\Phi_{\mathcal{C}}^S:=\cup_{x\in S}\Phi_{\mathcal{C}}^x$ is Zariski-dense in G if and only if $\Phi_{\mathcal{C}_\dagger}^S\cap T$ is Zariski-dense in T for $T\subset G$ a maximal torus. Since the characteristic polynomial map $\chi:G\subset GL(\mathcal{C})\to\mathbb{G}_{m,\overline{\mathbb{Q}}_\dagger}\times\mathbb{A}_{\overline{\mathbb{Q}}_\dagger}^{r-1}$ restricts to a finite morphism $\chi:T\to\chi(G)$, $\Phi_{\mathcal{C}}^S\cap T$ is Zariski-dense in T if and only if $\chi(\Phi_{\mathcal{C}}^S)$ is Zariski-dense in $\chi(T)$. But by the definition of $\mathcal{C}_\ell$, $\iota\chi(\Phi_{\mathcal{C}}^S)=\iota_\ell(\Phi_{\mathcal{C}_\ell}^S)$. And from the above and the classical Cebotarev density theorem, $\iota_\ell(\Phi_{\mathcal{C}_\ell}^S)$ is Zariski-dense in $\iota_\ell(G_\ell)$. On the other hand, by the †-independency of $G_\dagger^\circ=G_\dagger$, $\iota_\ell(G_\ell)$ and $\iota(G)$ have the same dimension. This forces $\iota\chi(\Phi_{\mathcal{C}}^S)$ to be Zariski-dense in $\iota(G)$ and concludes the argument when $\mathcal{C}$ is semisimple and G is connected. When $\mathcal{C}$ is semisimple but G is non-connected, the idea is basically the same but one has to resort to the rather technical formalism of quasi-Cartan developed in [CT20, §3]. Using the weight filtration, one can reduce Corollary 5.6.1.3 for arbitrary $\mathcal{C}$ to Corollary 5.6.1.3 for $\overline{\mathbb{Q}}_\dagger$-local systems $\mathcal{C}$ which are direct sums of pure $\overline{\mathbb{Q}}_\dagger$-local systems. Such a $\mathcal{C}$ is not semisimple in general but $\mathcal{C}|_{X_{\bar{k}}}$ is, which forces $G=\mathbb{G}_{a,\overline{\mathbb{Q}}_\dagger}^\epsilon\times G(\mathcal{C}^{ss})$ with $\epsilon=0,1$ and $\mathcal{C}^{ss}$ the semisimplification of $\mathcal{C}$. Corollary 5.6.1.3 for such a $\mathcal{C}$ then easily follows from Corollary 5.6.1.3 for the semisimplification of $\mathcal{C}$.

References

[BBD82] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100**, SMF, 1982.

[C20a] A. Cadoret, *La conjecture des compagnons [d'après Deligne, Drinfeld, L. Lafforgue, T. Abe...]*, Séminaire Bourbaki no 1156, 71e année, 2018-2019.

[C20b] A. Cadoret, *Weil II ultraproduct for curves and integral models in compatible families of ℓ-adic local systems*, Preprint, 2020.

[C20c] A. Cadoret, *Representations of étale fundamental group and specialization of algebraic cycles*, Contemporary Math., Proceedings volume in honor of Gherard Frey's 75th birthday. To appear.

[CT16] A. Cadoret and A. Tamagawa, *Note on the gonality of abstract modular curves II, with an appendix 'Gonality, isogonality and points of bounded degree on curves'*, Compos. Math. **152**, p. 2405–2442, 2016.

[CT17] A. Cadoret and A. Tamagawa, *On the geometric image of $\mathbb{F}_\ell$-linear representations of étale fundamental groups*, I.M.R.N. **2017**, p. 1–28, 2017.

[CHT17a] A. Cadoret, C.Y. Hui and A. Tamagawa, *Geometric monodromy - semisimplicity and maximality*, Annals of Math.**186**, p. 205–236, 2017.

[CT19] A. Cadoret and A. Tamagawa, *Genus of abstract modular curves with level-ℓ structures*, Journal für die reine und angewandte Mathematik **752**, p. 25–61, 2019

[CT20] A. Cadoret and A. Tamagawa, *Variations on a Tannakian Cebotarev density theorem*, Preprint 2020.

[dJ16] A.J. de Jong, *Smoothness, semi-stability and alterations.* Publ. Math. I.H.E.S. **83**, 51–93, 1996.

[dJ01] A.J. de Jong, *A conjecture on arithmetic fundamental groups.* Israel Journal of Math **121**, p. 61–84, 2001.

[D73] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L.* In Proc. Antwerpen Conference, vol. 2; L.N.M. **349**, Springer-Verlag, p 501–597, 1973.

[D74] P. Deligne, *La conjecture de Weil: I*, Inst. Hautes Études Sci. Publ. Math. **43**, p. 273–307, 1974.

[D80] P. Deligne, *La conjecture de Weil: II*, Inst. Hautes Études Sci. Publ. Math. **52**, p. 137–252, 1980.

[D82] P. Deligne, *Hodge Cycles on Abelian varieties*, in Hoge cycles, motives and Shimura varieties, P. Deligne, J.S. Milne, A. Ogus and K-Y Shih eds, L.N.M. **900**, 1982.

[D12] P. Deligne, *Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius en caractéristique finie*, Moscow Math. J. **12**, p. 497–514, 2012.

[Dr12] V. Drinfeld, *On a conjecture of Deligne*, Mosc. Math. J. **12**, p. 515–542, 2012.

[EKer12] H. Esnault and M. Kerz, *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Math. Viet. **37**, p. 531–562, 2012.

[EK16] H. Esnault and L. Kindler, *Lefschetz theorems for tamely ramified coverings*, Proc. of the A.M.S. **144**, p. 5071–5080, 2016.

[G83] O. Gabber, *Sur la torsion dans la cohomologie ℓ-adique d'une variété*, C.R. Acad. Sci. Paris Ser. I Math. **297**, p. 179–182, 1983.

[KaLV14] D. Kazhdan, M. Larsen and Y. Varshavsky, *The Tannakian formalism and the Langlands conjectures*, Algebra Number Theory **8**, p. 243–256, 2014.

[KS10] M. Kerz and A. Schmidt, *On different notions of tameness in arithmetic geometry*, Math. Annalen **346**, p. 641–668, 2010.

[Kl] , S. L.Kleiman, *Algebraic cycles and the Weil conjectures*, in Dix exposés sur la cohomologie des schémas. Adv. Stud. Pure Math. **3**, North-Holland, p. 359–386, 1968.

[L02] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147**, p.1–241, 2002.

[LarP92] M. Larsen and R. Pink, *On ℓ-independence of algebraic monodromy groups in compatible systems of representations*, Inventiones Math. **107**, p. 603–636, 1992.

[Lau87] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Publ. Math. I.H.E.S **65**, p. 561–579, 1995.

[SGA1] A. Grothendieck et al., *Revêtements étales et groupe fondamental (SGA 1)*. Lecture Notes in Math., **224**, Springer-Verlag, Berlin-New York, 1971.

[SGA4$\frac{1}{2}$] P. Deligne et al, *Cohomologie étale (SGA4 $\frac{1}{2}$)*, Lecture Notes in Math. **569**, Springer-Verlag, Berlin-New York, 1970.

[SGA5] A. Grothendieck et al, *Cohomologie ℓ-adique et fonctions L (SGA5)*, Lecture Notes in Math. **589**, Springer-Verlag, Berlin-New York, 1977.

[SGA7] P. Deligne and N. Katz, *Groupe de monodromie en géométrie algébrique (SGA7) - 2ème partie*, Lecture Notes in Math. **340**, Springer-Verlag, Berlin-New York, 1973.

[NS07a] N. Nikolov and D. Segal, *On finitely generated profinite groups. I. Strong completeness and uniform bounds*, Ann. of Math. **165**, p. 171–238, 2007.

[NS07b] N. Nikolov and D. Segal, *On finitely generated profinite groups. II. Products in quasisimple groups*, Ann. of Math. **165**, p. 239–273, 2007.

[O19] F. Orgogozo, *Constructibilité et modération uniformes en cohomologie étale*, Comp. Math. **755**, p. 711–755, 2019.

[P79] I.I. Piatetski-Shapiro, *Multiplicity one theorems*, in *Automorphic forms, representations and L-functions Part 1*, Proc. Sympos. Pure Math. **33**, p. 209–212, 1979.

[S04] J.P. Serre, *Propriétés conjecturales des groupes de Galois motiviques* in *Motives (Seattle, 1991)*, Proc. of Symposia in Pure Math. **55**, Part I, p. 377–400, 1994.

[Stacks] *The stacks Project* stacks.math.columbia.edu

[T04] I. Tomasic, *A new Weil cohomology theory*, Bulletin of the L.M.S., **36**, p. 663-670, 2004.

anna.cadoret@imj-prg.fr

IMJ-PRG – Sorbonne Université, Paris, FRANCE