RECONSTRUCTING FUNCTION FIELDS FROM MILNOR K-THEORY

ANNA CADORET (JOINT WORK WITH ALENA PIRUTKA)

1. Introduction

This work is motivated by the following general question, for which no counter-example seems to be known.

Question 1. Does the Milnor K-ring $K_*^M(F)$ determine the isomorphism class of the field F?

One may also ask whether or not the above holds in a functorial way, that is, whether or not the Milnor K-ring functor is fully faithfull from the groupoid of fields to the groupoid of $\mathbb{Z}_{\geq 0}$-graded rings.

Since $K_1^M(F) = F^\times$, Question 1 essentially reduces to reconstructing the additive structure of F from the multiplicative group $F^\times$ endowed with additional data that can be detected by the Milnor K-ring. Our main result (Theorem 2.1) asserts that for a finitely generated regular field extension F of transcendence degree ≥ 2 over a perfect field k, the multiplicative group $F^\times/k^\times$ endowed with the equivalence relation induced by algebraic dependence on k determines the isomorphism class of F in a functorial way. We also show that for a finitely generated regular field extension of a field k which is either algebraically closed or finite, the Milnor K-ring detects algebraic dependence. This is a consequence of deep K-theoretic results - the $n = 2$ case of the Bloch-Kato conjecture [MS82] when k is algebraically closed and of the Bass-Tate conjecture [T71] when k is finite. Combined with Theorem 2.1, this enables us show that the Milnor K-ring modulo the ideal of divisible elements (resp. of torsion elements) determines in a functorial way finitely generated regular field extensions of transcendence degree ≥ 2 over algebraically closed fields (resp. over finite fields).

2. Main Result

2.1. Recall that a field extension F/k is *regular* if k is algebraically closed in F. Let F/k be a regular field extension. We say that $\overline{x}, \overline{y} \in F^\times/k^\times$ are *algebraically dependent* and write $\overline{x} \equiv \overline{y}$ if either $\overline{x} = \overline{y} = 1$ or $1 \neq \overline{x}, \overline{y}$ and some (equivalenty, every) lifts $x, y \in F^\times$ of $\overline{x}, \overline{y} \in F^\times/k^\times$ are algebraically dependent over k. The relation $\equiv$ is an equivalence relation on $F^\times/k^\times$.

2.2. Let F/k, F'/k' be regular field extensions. We say that a group morphism $\overline{\psi} : F^\times/k^\times \to F'^\times/k'^\times$ *preserves algebraic dependence* if for every $\overline{x}, \overline{y} \in F^\times/k^\times$ the following holds: $\overline{x} \equiv \overline{y}$ if and only if $\overline{\psi}(\overline{x}) \equiv \overline{\psi}(\overline{y})$.

2.3. Let $\mathrm{Isom}(F, F')$ denote the set of field isomorphisms $F \tilde{\to} F'$ and

$$\mathrm{Isom}(F/k, F'/k') \subset \mathrm{Isom}(F, F')$$

denote the subset of isomorphisms $F \tilde{\to} F'$ inducing field isomorphisms $k \tilde{\to} k'$.

Let $\mathrm{Isom}(F^\times/k^\times, F'^\times/k'^\times)$ denote the set of group isomorphisms $F^\times/k^\times \tilde{\to} F'^\times/k'^\times$ and

$$\mathrm{Isom}^{\equiv}(F^\times/k^\times, F'^\times/k'^\times) \subset \mathrm{Isom}(F^\times/k^\times, F'^\times/k'^\times)$$

the subset of isomorphisms $F^\times/k^\times \tilde{\to} F'^\times/k'^\times$ preserving algebraic dependence. The group $\mathbb{Z}/2$ acts on the set $\mathrm{Isom}^{\equiv}(F^\times/k^\times, F'^\times/k'^\times)$ by $\overline{\psi} \to \overline{\psi}^{-1}$. Write

$$\overline{\mathrm{Isom}}^{\equiv}(F^\times/k^\times, F'^\times/k'^\times)$$

for the resulting quotient.

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Theorem 2.1. *Let k, k' be perfect fields of characteristic $p \geq 0$ and let F/k, F'/k' be finitely generated regular field extensions of transcendence degree ≥ 2. Then the canonical map*

$$\mathrm{Isom}(F/k, F'/k') \to \overline{\mathrm{Isom}}^{\equiv}(F^\times/k^\times, F'^\times/k'^\times)$$

is bijective.

3. Comparison with existing results

Question 1 was considered by Bogomolov and Tschinkel in [BT09], where they prove (a variant of) Theorem 2.1 for finitely generated regular extensions of characteristic 0 fields ([BT09, Thm. 2]) and deduce from it the K-theoretic application for finitely generated field extensions of algebraically closed fields of characteristic 0 ([BT09, Thm. 4]).

Variants of our results were also obtained by Topaz from a smaller amount of K-theoretic information - mod-ℓ Milnor K-rings (for finitely generated field extensions of transcendence degree ≥ 5 over algebraically closed field of characteristic $p \neq \ell$ [To16, Thm. B]) and rational Milnor K-rings (for finitely generated field extensions of transcendence degree ≥ 2 over algebraically closed field of characteristic 0 [To17, Thm. 6.1]) but enriched with the additional data of the so-called "rational quotients" of F/k. See also [To17, Rem. 6.2] for some cases where the additional data of rational quotients can be removed.

Our strategy follows the one of Bogomolov and Tschinkel in [BT09], where the key idea is to parametrize lines in $F^\times/k^\times$ as intersections of multiplicatively shifted (infinite dimensional) projective subspaces of a specific form arising from relatively algebraically closed subextensions of transcendence degree 1. See Subsection 4 for details. The strategy of Topaz is more sophisticated and goes through the reconstruction of the quasi-divisorial valuations of F *via* avatars of the theory of commuting-liftable pairs as developped in the frame of birational anabelian geometry. Though not explicitly stated in the literature, it is likely that for finitely generated field extensions of algebraically closed fields of characteristic $p > 0$ Theorem 2.1 and its K-theoretic application could also be recovered from the technics of birational anabelian geometry as developed by Bogomolov-Tschinkel [BT12], Pop (*e.g.* [P12a], [P12b]) and Topaz.

To our knowledge, Theorem 2.1 for finitely generated regular extensions of arbitrary perfect fields of characteristic $p > 0$ and its K-theoretic application for finitely generated field extensions of finite fields are new.

4. Strategy of proof

For simplicity, write $F^p \subset F$ for the subfield generated by k and the x^p, $x \in F$ and $F^\times/p := F^\times/F^{p\times}$.

According to the fundamental theorem of projective geometry, it would be enough to show that a group isomorphism $\overline{\psi} : F^\times/k^\times \tilde{\to} F'^\times/k'^\times$ preserving algebraic dependence induces a bijection from lines in $F^\times/k^\times$ to lines in $F'^\times/k'^\times$. This would reduce the problem to describing lines in $F^\times/k^\times$ using only $\equiv$ and the multiplicative structure of $F^\times/k^\times$. This classical approach works well if $p = 0$. The key observation of Bogomolov and Tschinkel in [BT09] is that every line can be multiplicatively shifted to a line passing through a "good" pair of points and that those lines can be uniquely parametrized as intersections of multiplicatively shifted (infinite dimensional) projective subspaces of a specific form arising from relatively algebraically closed subextensions of transcendence degree 1 [BT09, Thm. 22]. This is the output of elaborate computations in [BT09]. Later, Rovinsky suggested an alternative argument using differential forms; this is sketched in [BT12, Prop. 9].

When $p > 0$, the situation is more involved. The original computations of [BT09] fail due to inseparability phenomena. Instead, we adjust the notion of "good" for the pair of points in order to refine the argument of Rovinsky. In particular, we use the field-theoretic notion of "regular" element rather than the group-theoretic notion of "primitive" element used in [BT09]. To show that every line can be shifted to a line passing through a "good" pair of points, we use Bertini-like arguments; this is classical when k is infinite but, when k is finite (and F of transcendence degree 2 over k) it seems we cannot avoid the use of the Charles-Poonen Bertini theorem [CP16]. This gives us a parametrization of lines which, when $p > 0$, is much rougher than in [BT09, Thm. 22] - up to prime-to-p powers and certain homographies with F^p-coefficients; this is due to the apparition of constants in F^p when one integrates differentials forms. It is however enough to show that there exists a unique $m \in \mathbb{Z}$ normalized as

$$\begin{array}{ll} |m| = 1 & \text{if } p = 0, 2 \\ 1 \leq |m| \leq \frac{p-1}{2} & \text{if } p > 2; \end{array} \tag{1}$$

such that $\overline{\psi}^m$ induces a bijection from lines in $F^\times/p$ to lines in ${F'}^\times/p$. So that the fundamental theorem of projective geometry gives a unique field isomorphism $\phi : F\tilde{\to}F'$ such that the resulting isomorphism of groups $\phi : F^\times\tilde{\to}{F'}^\times$ coincides with $\overline{\psi}^m$ on $F^\times/p$. This concludes the proof if $p = 0$. But if $p > 0$, the extension F/F^p is much smaller (finite-dimensional!) and one has to perform an additional descent step to show that $m = \pm 1$ and ϕ coincides with $\overline{\psi}^{\pm 1}$ on $F^\times/k^\times$ (not only on $F^\times/p$).

5. Questions

(1) Question 1. For a possible counter-example, consider function fields of curves over algebraically closed or finite fields? In the positive direction, one may ask for the extension of our K-theoretic applications to function fields of transcendence degree ≥ 2 over more general base fields (than algebraically closed or finite fields).

(2) Non birational analogue of Theorem 2.1: One can reformulate Theorem 2.1 (resp. its K-theoretic application) by saying that the birational equivalence class of a normal proper geometrically integral variety X of dimension ≥ 2 over a perfect field k is determined in a functorial way by the inductive systems of the $K^\times$ for K describing the relatively algebraically closed subextensions in the function field of X (resp. (some quotients of) its Milnor K-ring). Find minimal sets of data determining X not only up to birational equivalence but up to isomorphism.

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anna.cadoret@imj-prg.fr

IMJ-PRG – Sorbonne Université, Paris, FRANCE