MOTIVATED CYCLES UNDER SPECIALIZATION

par

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Résumé. — This paper is essentially a survey of André's theory of pure motivated motives with an emphasis on specialization theory in characteristic zero.

We review first the classical construction of pure motives and then turn to pure motivated motives whose construction is modeled upon the one of pure homological motives, replacing homological cycles by motivated cycles. Basically, motivated cycles are obtained from homological cycles by adjoining formally the Lefschetz involution so that the so-called standard conjectures become true in the category of pure motivated motives; in particular, this category is a semisimple Tannakian category naturally equipped with fibre functors coming from Weil cohomologies.

The last section is devoted to the ℓ-adic version of André's specialization theorem for motivated cycles, which asserts that, given a family of motivated motives M over a scheme S of finite type over a finitely generated field k of characteristic 0, the locus of all $s \in S(k)$ where the motivated motivic Galois group associated with $M_{\overline{s}}$ degenerates is thin in $S(k)$. When S is a curve, we improve André's statement by resorting to a uniform open image theorem for ℓ-adic cohomology proved by A. Tamagawa and the author. We conclude by some applications of this specialization theorem.

Cet article est une introduction à la théorie des motifs motivés purs développée par André. Nous nous intéressons plus particulièrement au problème de la spécialisation de ces motifs en caractéristique 0.

Nous commençons par rappeler la construction classique des motifs purs puis nous présentons la construction des motifs purs motivés comme une variante de la construction des motifs purs homologiques où les cycles homologiques sont remplacés par les cycles motivés. En gros, les cycles motivés sont obtenus en adjoignant formellement l'involution de Lefschetz aux cycles homologiques de sorte que les conjectures dites standard deviennent vraies dans la catégorie des motifs purs motivés; en particulier, cette catégorie est une catégorie tannakienne semisimple naturellement munie de foncteurs fibres provenant des cohomologies de Weil considérées.

La dernière partie de cet article est consacrée à la version ℓ-adique du théorème d'André sur la spécialisation des cyces motivés. Celui-ci peut s'énoncer comme suit. Soit k un corps de type fini et de caractéristique nulle, S un schéma de type fini sur k et M une famille de motifs motivés sur S. Alors l'ensemble des points $s \in S(k)$ où le groupe de Galois motivé motivique associé à $M_{\overline{s}}$ dégénère est mince dans $S(k)$. Lorsque S est une courbe, nous améliorons le résultat d'André en invoquant un théorème d'image ouverte uniforme du à A. Tamagawa et l'auteur. Nous concluons en donnant quelques applications de ce théorème de spécialisation.

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1. Introduction

Classically, (pure) motives can be presented either as an attempt to construct a universal cohomology or as an attempt to "embed" the category of smooth projective varieties into a neutral semisimple Tannakian category over a field E. Both points of view are intrinsically connected but we will rather adopt the second one, which is more adapted to André's theory of motivated motives.

Recall that a neutral Tannakian category over a field E is a rigid abelian tensor category which admits a faithful tensor functor with value in the category of finite dimensional E-modules. The main theorem of Tannakian formalism asserts that a neutral Tannakian category is equivalent to the category of finite dimensional E-rational representations of a pro-algebraic group over E (pro-reductive if the category is furthermore assumed to be semisimple).

Fix a field k of characteristic 0 and let $\mathcal{P}(k)$ denote the category of smooth, projective schemes over k and $\mathcal{P}(k)^{op}$ its opposite category. These are tensor categories. The first step of the construction of pure motives is to "embed" $\mathcal{P}(k)^{op}$ into an additive tensor category - this is the category of homological correspondences. Once one has an additive tensor category, one can, in turn, "embed" it into its Karoubian enveloppe, which is a pseudoabelian tensor category - this is the category of effective motives. The third step consists in inverting formally the so-called Lefschetz motive to obtain the category of pure motives, which is a rigid pseudoabelian tensor category. The category of pure motives, however, is not Tannakian yet and, unfortunately, the remaining part of the construction is only conjectural, based on the so-called standard conjectures. These conjectures are all implied by the so-called Lefschetz type conjecture, which predicts that the Lefschetz involution is a morphism in the category of pure motives.

The key idea of André's construction of motivated cycles is to adjoin formally the Lefschetz involutions to the set of homological correspondances in order to force Lefschetz conjecture to hold and construct a category of motives which is a semisimple neutral Tannakian category - the category of pure motivated motives. In particular, to any $X \in \mathcal{P}(k)$ one can associate the tensor subcategory $\langle X\rangle^{\otimes}$ generated by X in the category of pure motivated motives; this is again a semisimple neutral Tannakian category and its Galois group $G_{mot}(X)$ is a reductive algebraic group.

Now, given a scheme S, smooth, separated and geometrically connected over k with generic point η and a smooth projective morphism $f : X \to S$ with geometrically connected fibres, one can ask how the categories $\langle X_{\overline{s}}\rangle^{\otimes}$ vary with $s \in S$ or, equivalently, how the $G_{mot}(X_{\overline{s}})$ do. This problem is dealt with in [**A96**, §5].

First, one has to find a way to compare $\langle X_{\overline{s}}\rangle^{\otimes}$ and $\langle X_{\overline{\eta}}\rangle^{\otimes}$, $s \in S$. Using the semisimplicity of the category of pure motivated motives and Deligne's fixed part theorem, one can show that the specialization isomorphism for ℓ-adic cohomology

$$sp_s : \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{\eta}}, \mathbb{Q}_\ell)(i) \tilde{\to} \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{s}}, \mathbb{Q}_\ell)(i)$$

maps motivated cycles to motivated cycles (corollary 4.7). So, as motivated motivic Galois groups are reductive, one can identify $G_{mot}(X_{\overline{s}})$ with a subgroup of $G_{mot}(X_{\overline{\eta}})$ and equality holds if and only if the specialization morphism for ℓ-adic cohomology induces an isomorphism onto motivated cycles for all fibre power $X \times_k X \times_k \cdots \times_k X$.

The next natural question is to understand the structure of the set of all $s \in S$ such that $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$; André's specialization theorem for motivated cycles answers it, at least partially.

Theorem 1.1. — ([**A96**, Thm. 5.2]) *For any finite field extension k'/k the set of all $s \in S(k')$ such that*

$$G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$$

is thin in $S(k')$.

The proof is along the following guidelines. First, one observes that $G_{mot}(X_{\overline{s}})$ contains an open subgroup of the image G_s of the ℓ-adic Galois representation

$$\rho_{f,s} : \Gamma_{k(s)} \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{s}}, \mathbb{Q}_\ell)).$$

As a result, the degeneration of $G_{mot}(X_{\overline{s}})$ forces the degeneration of G_s. Similarly, $G_{mot}(X_{\overline{\eta}})$ contains an open subgroup of the image G of the generic ℓ-adic Galois representation

$$\rho_{f,\eta} : \Gamma_{k(\eta)} \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell))$$

and identifying $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Q}_\ell)$ *via* $sp_s : \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{\eta}}, \mathbb{Q}_\ell)(i) \tilde{\to} \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{s}}, \mathbb{Q}_\ell)(i)$, one can regard G_s as a closed subgroup of G. So, the set where $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$ is contained in the set where G_s is not open in G. The problem thus amounts to studying this second set.

To control this second set, André resorts to a profinite variant of Serre's irreducibility theorem [**Se89**, p.148]. When S is a curve, this can be replaced by a uniform open image theorem for ℓ-adic represenions of étale fundamental groups proved by A. Tamagawa and the author ([**CT09b**, Thm. 1.1] - see theorem 5.3) to obtain the following. Given an integer $d \geq 1$, let $S^{\leq d}$ denote the set of all closed points $s \in S$ such that $[k(s) : k] \leq d$.

Theorem 1.2. — *Assume that S is a curve and that k is a finitely generated field of characteristic 0. Then, for any integer $d \geq 1$, the set of all $s \in S^{\leq d}$ such that $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$ is finite.*

The paper is organized as follows. In section 2, we review the construction of the category of pure motives (after some preliminaries - gathered in subsection 2.1 - about algebraic cycles and Weil cohomologies) and, in section 3, we discuss the formalism of the standard conjectures. In section 4, we give the main features of André's theory of motivated cycles and explain how to specialize them. Section 5 is devoted to the statement and proof of the specialization theorem for motivated motivic Galois groups (theorem 5.1, which gathers theorem 1.1 and 1.2). We conclude this last section by discussing related topics such as jumping of the Neron-Severi rank or Tate conjectures.

It goes without saying that I am very much indebted to the reading of André's works [**A96**] and [**A04**] for the writing of this paper. I am also grateful to the referee for his or her constructive remarks.

2. The category of pure motives

2.1. Algebraic cycles and Weil cohomologies. — The aim of this preliminary section is to review the formalism of algebraic cycles and Weil cohomologies required to introduce the category of algebraic correspondences, which is the starting point of the construction of the category of pure motives. The content here is very standard and can be skipped by any reader familiar with these notions. We assume basic knowledge about intersection theory [**F84**], usual Weil cohomologies (say

Betti and ℓ-adic) and Tannakian formalism [**DM82**].

Given a field K, we write $\mathrm{Mod}_{/K}$ for the category of K-modules, $\mathrm{Mod}_{/K}^{\mathbb{Z}_{\geq 0}}$ for the category of $\mathbb{Z}_{\geq 0}$-graded K-modules, $\mathrm{Alg}_{/K}^{\mathbb{Z}_{\geq 0}}$ for the category of $\mathbb{Z}_{\geq 0}$-graded K-algebras and $\mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}}$ for the category of anticommutative $\mathbb{Z}_{\geq 0}$-graded K-algebras regarded as a $\otimes$-category whose commutativity constraint is given by Koszul rule that is, for any two $\mathbb{Z}_{\geq 0}$-graded algebras $M = \oplus_{i\geq 0} M_i$, $N = \oplus_{i\geq 0} N_i$, the commutativity constraint

$$c_{M,N} : M \otimes_K N \tilde{\to} N \otimes_K M$$

can be written as

$$c_{M,N} = \bigoplus_{i,j\geq 0} c_{i,j},$$

where

$$c_{i,j} : \begin{array}{lcl} M_i \otimes_K N_j & \tilde{\to} & N_j \otimes_K M_i \\ m_i \otimes n_j & \mapsto & (-1)^{ij} n_j \otimes m_i \end{array} \quad , \ i, \ j \geq 0.$$

Fix a field k, of characteristic 0.

Given a connected $X \in \mathcal{P}(k)$, we will write d_X for its dimension. Some statements involving d_X below only make sense if X is equidimensional. We will not necessarily recall this hypothesis mainly because the functors at stake commute with coproducts. So, the reader can think of X as being connected.

2.1.1. Algebraic cycles. — Let E be a field of characteristic 0 and $\sim$ an adequate relation on $\mathcal{P}(k)$ (for instance, rational equivalence *rat*, algebraic equivalence *alg*, homological equivalence H or numerical equivalence *num*). For any $X \in \mathcal{P}(k)$, write

$$\mathrm{Z}^*_\sim(X)_E := \mathrm{Z}^*(X) \otimes_{\mathbb{Z}} E/\sim$$

for the $\mathbb{Z}_{\geq 0}$-graded algebra of algebraic cycles with coefficients in E modulo $\sim$. Recall that this defines covariant functors

$$(\mathrm{Z}^*_\sim(-)_E, (-)^*) : \mathcal{P}(k)^{op} \to \mathrm{Alg}_{/E}^{\mathbb{Z}_{\geq 0}}$$

and

$$(\mathrm{Z}^*_\sim(-)_E, (-)_*) : \mathcal{P}(k) \to \mathrm{Mod}_{/E}.$$

More precisely, given a morphism $f : X \to Y$ in $\mathcal{P}(k)$, the morphism $f_* : \mathrm{Z}^*_\sim(X)_E \to \mathrm{Z}^*_\sim(Y)_E$ is defined as follows. For any $x \in X$, set $f_*(\overline{\{x\}}) = [k(x) : k(f(x))]\overline{\{f(x)\}}$ if $\dim(\overline{\{x\}}) = \dim(\overline{\{f(x)\}})$ and $f_*(\overline{\{x\}}) = 0$ else. Then, extend f_* by E-linearity and check that it factors *via* $\sim$. Equivalently, one has

$$f_*(\alpha) = p^{XY}_{Y*}(\Gamma_f \cdot p^{XY*}_X(\alpha)) \quad , \alpha \in \mathrm{Z}^*_\sim(X)_E, \tag{1}$$

where $\Gamma_f \in Z^{d_Y}_\sim(X \times_k Y)_E$ denotes the graph of $f : X \to Y$ and $p^{XY}_X : Y \times_k X \to X$ (resp. $p^{XY}_Y : Y \times_k X \to Y$) the first (resp. second) projection. The morphism $f^* : \mathrm{Z}^*_\sim(Y)_E \to \mathrm{Z}^*_\sim(X)_E$ is defined by

$$f^*(\beta) = p^{YX}_{X*}({}^t\Gamma_f \cdot p^{YX*}_Y(\beta)) \quad , \beta \in \mathrm{Z}^*_\sim(Y)_E. \tag{2}$$

Note that $f_* : \mathrm{Z}^*_\sim(X)_E \to \mathrm{Z}^{*-\delta}_\sim(Y)_E$ shifts the degree by $-\delta$, where δ denotes the dimension of the generic fibre of $f : X \to Y$. Also, $f_* : \mathrm{Z}^*_\sim(X)_E \to \mathrm{Z}^{*-\delta}_\sim(Y)_E$ is not compatible with the intersection product but, however, the following relations hold

$$f_*(\alpha \cdot f^*\beta) = (f_*\alpha) \cdot \beta \quad , \alpha \in \mathrm{Z}^*_\sim(X)_E, \ \beta \in \mathrm{Z}^*_\sim(Y)_E \text{ (projection)}. \tag{3}$$

2.1.2. Weil cohomology. — Let $E \hookrightarrow K$ be an extension of fields of characteristic 0 and fix a Weil cohomology $(\mathrm{H}^*, (-)^*) : \mathcal{P}(k)^{op} \to \mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}}$ that is a covariant functor of tensor categories [(1)] satisfying the finiteness properties (1), (2) below and endowed with traces and a cycle map satisfying properties (3), (4), (5) below.

1. For any $X \in \mathcal{P}(k)$,
   (a) $\dim_K(\mathrm{H}^i(X)) < +\infty$ for $0 \leq i \leq 2\dim(X)$;
   (b) $\mathrm{H}^i(X) = 0$ for $i \geq 2\dim(X) + 1$.

2. $\dim_K(\mathrm{H}^2(\mathbb{P}^1_k)) = 1$ and $\mathrm{H}^1(\mathbb{P}^1_k) = 0$.

For any $n \in \mathbb{Z}$, write

$$-(n) := - \otimes_K \mathrm{H}^2(\mathbb{P}^1_k)^{\otimes_K(-n)} : \mathrm{Mod}_{/K}^{\mathbb{Z}_{\geq 0}} \to \mathrm{Mod}_{/K}^{\mathbb{Z}_{\geq 0}}$$

for the *nth Tate twist operator* (which shifts the degree by $-2n$). Also, given $X \in \mathcal{P}(k)$, write

$$\mathbf{H}^*(X) := \bigoplus_{r \geq 0} \mathrm{H}^{2r}(X)(r).$$

This gives raise to a twisted $\mathbb{Z}_{\geq 0}$-graded covariant functor

$$(\mathbf{H}^*(-), (-)^*) : \mathcal{P}(k)^{op} \to \mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}}.$$

The *trace* of $X \in \mathcal{P}(k)$ is a morphism

$$Tr_X : \mathrm{H}^{2d_X}(X)(d_X) \to K$$

in $\mathrm{Mod}_{/K}$ and the *cycle map* is a morphism of $\mathbb{Z}_{\geq 0}$-graded functors

$$\gamma_H^*(-) : (Z^*_{rat}(-)_E, (-)^*) \to (\mathbf{H}^*(-), (-)^*).$$

3. <u>Compatibility with the tensor structures</u>: The following two diagrams commute (here we use "=" for "isomorphic")

$$\begin{tikzcd}
\mathrm{H}^{2d_X}(X)(d_X) \otimes_K \mathrm{H}^{2d_Y}(Y)(d_Y) \arrow[r, "Tr_X \otimes_K Tr_Y"] \arrow[d, equal] & K \\
\mathrm{H}^{2(d_X+d_Y)}(X \times_k Y)(d_X + d_Y); \arrow[ur, "Tr_{X \times_k Y}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
Z^*_{alg}(X)_E \otimes_E Z^*_{alg}(Y)_E \arrow[r, "\gamma_H^*(X) \otimes \gamma_H^*(Y)"] \arrow[d, equal] & \mathbf{H}^*(X) \otimes_K \mathbf{H}^*(Y) \arrow[d, equal] \\
Z^*_{alg}(X \times_k Y)_E \arrow[r, "\gamma_H^*(X \times_k Y)"'] & \mathbf{H}^*(X \times_k Y);
\end{tikzcd}$$

4. <u>Normalization of the trace</u>: The following diagram commutes (X connected)

$$\begin{tikzcd}
Z^{d_X}_{alg}(X)_E \otimes_E K \arrow[r, "\gamma_H^{d_X}(X) \otimes_E Id_K"] \arrow[d] & \mathrm{H}^{2d_X}(X)(d_X) \arrow[d, "Tr_X"] \\
E \otimes_E K \arrow[r, equal] & K,
\end{tikzcd}$$

   where the first vertical arrow is induced from the canonical degree morphism. If X is geometrically connected then $\gamma_H^{d_X}(X)$ and Tr_X are both isomorphisms.

[(1)]In particular, for any $X \in \mathcal{P}(k)$, the diagonal morphism $\Delta_{X|k} : X \to X \times_k X$ induces the cup product $\cup$ on H^* and the structural morphism $X \to \mathrm{spec}(k)$ induces the k-algebra structure on H^*.

5. Poincaré duality: The pairing

$$\langle -,- \rangle : H^*(X) \otimes_K H^{2d_X-*}(X)(d_X) \overset{\cup}{\to} H^{2d_X}(X)(d_X) \overset{Tr_X}{\to} K$$

is non-degenerate (X connected).

In particular, for any $f : X \to Y$, one can define

$$f_* : \mathrm{H}^*(X) \to \mathrm{H}^{*-2\delta}(Y)(-\delta)$$

as the composite

$$\begin{array}{rcl} H^i(X) & \overset{P.D.}{\tilde{\to}} & \mathrm{Hom}_{\mathrm{Mod}(K)}(H^{2d_X-i}(X)(d_X), K) \\ & \overset{{}^t f^*}{\to} & \mathrm{Hom}_{\mathrm{Mod}(K)}(H^{2d_X-i}(Y)(d_X), K) \\ & \overset{P.D.}{\tilde{\to}} & H^{i-2\delta}(Y)(-\delta). \end{array}$$

This yields a covariant functor $(\mathrm{H}^*, (-)_*) : \mathcal{P}(k) \to \mathrm{Mod}_{/K}$ such that $\gamma_H(-)$ becomes a morphism of functors

$$\gamma_H(-) : (Z_{alg}(-)_E, (-)_*) \to (\mathbf{H}^*(-), (-)_*)$$

and one has the projection formula

$$f_*(\alpha \cup f^*\beta) = (f_*\alpha) \cup \beta,\ \alpha \in \mathrm{H}^*(X),\ \beta \in \mathrm{H}^*(Y).$$

If $f : X \to \mathrm{spec}(k)$ is the structural morphism, then $f_* = Tr_X$.

2.1.3. Algebraic cycles modulo homological equivalence. — By definition of homological equivalence, the cycle map $\gamma_H^*(-) : \mathrm{Z}^*_{alg}(-)_E \to \mathbf{H}^*(-)$ factors through

$$\begin{tikzcd} \mathrm{Z}^*_{rat}(-)_E \arrow[rr, "\gamma_H^*(-)"] \arrow[dr] & & \mathbf{H}^*(-) \\ & \mathrm{Z}^*_H(-)_E, \arrow[ur, hook] & \end{tikzcd}$$

where $\mathrm{Z}^*_H(-)_E : \mathcal{P}(k)^{op} \to \mathrm{Alg}_{/E}^{\mathbb{Z}_{\geq 0}\text{-grad}}$ denotes the functor of cycles with coefficients in E modulo homological equivalence.

From now on, we assume that H^* is a classical Weil cohomology that is, one of the following

- If $k \subset \mathbb{C}$, Betti cohomology with coefficients in $\mathbb{Q}$ (not.: $\mathrm{H}^*_B(X^{an}, \mathbb{Q}) =: \mathrm{H}^*_B(-)$);
- ℓ-adic cohomology with coefficients in $\mathbb{Q}_\ell$ (not.: $\mathrm{H}^*_{\text{ét}}(- \times_k \overline{k}, \mathbb{Q}_\ell) =: \mathrm{H}^*_\ell(- \times_k \overline{k})$);
- De Rham cohomology (not.: $\mathbb{H}^*_{\mathrm{zar}}(-, \Omega^*_{-|k}) =: \mathrm{H}^*_{DR}(-)$).

Since the comparison isomorphisms commute with the cycle map ($k \subset \mathbb{C}$, $E = \mathbb{Q}$)

$$\begin{tikzcd} \mathbf{H}^*_B(-) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \arrow[rr, equal, "c_{B,\ell}"] & & \mathbf{H}^*_\ell(- \times_k \overline{k}) \\ & \mathrm{Z}^*_{alg}(-) \arrow[ul, "\gamma_B^*(-)"'] \arrow[ur, "\gamma_\ell^*(-)"] \arrow[dl, "\gamma_B^*(-)"'] \arrow[dr, "\gamma_{DR}^*(-)"] & \\ \mathbf{H}^*_B(-) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[rr, equal, "c_{B,DR}"'] & & \mathbf{H}^*_{DR}(-) \otimes_k \mathbb{C} \end{tikzcd}$$

the definition of $Z^*_H(-)_E$ is independent of the choice of the classical Weil cohomology.

2.1.4. Algebraic correspondences. — We now come to the definition of the category of algebraic correspondences. For any $X,\ Y\in\mathcal{P}(k)$, applying Kunneth formula and Poincaré duality, one gets the following identifications

$$\begin{aligned}\mathrm{H}^*(X\times_k Y) &\overset{Kunneth}{=} \mathrm{H}^*(X)\otimes_K \mathrm{H}^*(Y)\\ &\overset{P.D.}{=} \mathrm{Hom}_{\mathrm{Mod}_{/K}}(\mathrm{H}^*(X),K)\otimes_K \mathrm{H}^*(Y)(-d_X)\\ &= \mathrm{Hom}_{\mathrm{Mod}_{/K}}(\mathrm{H}^*(X),\mathrm{H}^*(Y))(-d_X).\end{aligned}$$

Explicitly, one has

$$u(\alpha)=p^{XY}_{Y*}(u\cup p^{XY*}_X(\alpha)) \qquad ,\ u\in \mathrm{H}^*(X\times_k Y),\ \alpha\in \mathrm{H}^*(X);$$

$$v\circ u=p^{XYZ}_{XZ*}(p^{XYZ*}_{XY}u\cup p^{XYZ*}_{YZ}v) \quad ,\ u\in \mathrm{H}^*(X\times_k Y),\ v\in \mathrm{H}^*(Y\times_k Z).$$

And, *via* these identifications, the following two diagrams commute

(4)
$$\begin{tikzcd} \mathrm{H}^{2d_X}(X\times_k Y)(d_X) \arrow[r, equal] & \bigoplus_{i=0}^{2d_X}\mathrm{Hom}_{\mathrm{Mod}_{/K}}(\mathrm{H}^i(X),\mathrm{H}^i(Y)) \\ \mathrm{Hom}_{\mathcal{P}(k)^{op}}(X,Y) \arrow[u, "\gamma_H^{d_X}(X\times_k Y)({}^t\Gamma_-)"] \arrow[ur, "(-)^*"'] & \end{tikzcd}$$

(5)
$$\begin{tikzcd} \mathrm{H}^{2d_Y}(X\times_k Y)(d_Y) \arrow[r, equal] & \bigoplus_{i=2\delta}^{2d_X}\mathrm{Hom}_{\mathrm{Mod}_{/K}}(\mathrm{H}^i(X),\mathrm{H}^{i-2\delta}(Y))(-\delta) \\ \mathrm{Hom}_{\mathcal{P}(k)}(X,Y) \arrow[u, "\gamma_H^{d_Y}(X\times_k Y)(\Gamma_-)"] \arrow[ur, "(-)_*"'] & \end{tikzcd}$$

This motivates the introduction of the category $\mathcal{C}_\sim(k)_E$ of *(degree 0) correspondences with coefficients in E modulo* $\sim$ defined as follows. The objects of $\mathcal{C}_\sim(k)_E$ are those of $\mathcal{P}(k)$ and, given $X,\ Y$ (connected) in $\mathcal{P}(k)$, the set of morphisms in $\mathcal{C}_\sim(k)_E$ from X to Y is

$$C_\sim(X,Y)_E:=Z_H^{d_X}(X\times_k Y)_E.$$

Composition in $\mathcal{C}_\sim(k)_E$ is defined by the rule

$$\begin{array}{rccl}\circ: & \mathcal{C}_\sim(Y,Z)_E\times\mathcal{C}_H(X,Y)_E & \to & \mathcal{C}_\sim(X,Z)_E\\ & (\beta,\alpha) & \mapsto & \beta\circ\alpha:=(p^{XYZ}_{XZ})_*((p^{XYZ}_{XY})^*(\alpha)\cdot(p^{XYZ}_{YZ})^*(\beta))\end{array}$$

Remark 2.1. — More generally, one could define the category $\mathcal{C}^*_\sim(k)_E$ of $\mathbb{Z}$-*graded correspondences with coefficients in E modulo homological equivalence* whose objects are those of $\mathcal{P}(k)$ and, given $X,\ Y$ (connected) in $\mathcal{P}(k)$, the set of morphisms in $\mathcal{C}^*_\sim(k)_E$ from X to Y is

$$C^*_\sim(X,Y)_E:=Z_H^{d_X+*}(X\times_k Y)_E.$$

Composition in $\mathcal{C}^*_\sim(k)_E$ is well-defined by the rule

$$\begin{array}{rccl}\circ: & \mathcal{C}^s_\sim(Y,Z)_E\times\mathcal{C}^r_\sim(X,Y)_E & \to & \mathcal{C}^{r+s}_\sim(X,Z)_E\\ & (\beta,\alpha) & \mapsto & \beta\circ\alpha:=(p^{XYZ}_{XZ})_*((p^{XYZ}_{XY})^*(\alpha)\cdot(p^{XYZ}_{YZ})^*(\beta)).\end{array}$$

The transpose of the graph gives raise to a natural covariant essentially surjective functor of tensor categories (for the obvious tensor structure on $\mathcal{C}_\sim(k)_E$)

$$\mathcal{P}(k)^{op}\to\mathcal{C}_\sim(k)_E.$$

Furthermore, the covariant functors of tensor categories

$$(Z^*_\sim(-),(-)^*):\mathcal{P}(k)^{op}\to\mathrm{Alg}^{\mathbb{Z}_{\geq 0}}_{/E},\quad (Z^*_\sim(-),(-)_*):\mathcal{P}(k)\to\mathrm{Mod}_{/E}$$

extends to $\mathcal{C}_\sim(k)_E$ and $\mathcal{C}_\sim(k)_E^{op}$ respectively by the rules of (1), (2), (3). Then, if homological equivalence is coarser than $\sim$, the diagrams (4) and (5) show how to extend the functors

$$(\mathrm{H}^*(-),(-)^*):\mathcal{P}(k)^{op}\to \mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}},\ \ (\mathrm{H}^*,(-)_*):\mathcal{P}(k)\to \mathrm{Mod}_{/K}$$

to $\mathcal{C}_\sim(k)_E$ so that the cycle map remains a morphism of functors of tensor categories from $(Z^*_\sim(-),(-)^*)$ to $(\ \mathbf{H}^*(-),(-)^*)$ and from $(Z^*_\sim(-),(-)_*)$ to $(\ \mathbf{H}^*(-),(-)_*)$ respectively.

By construction, $\mathcal{C}_\sim(k)_E$ is an additive tensor category. We are now ready to proceed to the next step in the construction of pure motives.

2.2. End of the construction of the category of pure motives. —

2.2.1. Pure effective motives. —

2.2.1.1. Karoubian enveloppe of additive categories. —

An additive category $\mathcal{A}$ is said to be *pseudoabelian* if any idempotent in $\mathcal{A}$ admits a kernel in $\mathcal{A}$. Given an additive category $\mathcal{A}$, there exists a pseudoabelian category $\mathcal{A}^\#$ and a covariant additive functor

$$\kappa:\mathcal{A}\to\mathcal{A}^\#$$

which is universal for additive functors from $\mathcal{A}$ to pseudoabelian categories. Thus $\kappa:\mathcal{A}\to\mathcal{A}^\#$ is unique up to equivalence of categories and called the *pseudoabelian or Karoubian enveloppe of* $\mathcal{A}$. It can be easily described as follows. Objects of $\mathcal{A}^\#$ are pairs (A,e), where A is an object in $\mathcal{A}$ and $e:A\to A$ is an idempotent in $\mathcal{A}$ and given (A,e), (A',e') in $\mathcal{A}^\#$, the set of morphisms from (A,e) to (A',e') in $\mathcal{A}^\#$ is

$$e'\circ \mathrm{Hom}_{\mathcal{A}}(A,A')\circ e\subset \mathrm{Hom}_{\mathcal{A}}(A,A')$$

and composition is induced by the composition in $\mathcal{A}$. Eventually, the functor $\kappa:\mathcal{A}\to\mathcal{A}^\#$ sends A to (A,Id_A) and $\phi:A\to A'$ to $\phi:(A,Id_A)\to(A',Id_{A'})$; in particular, $\kappa:\mathcal{A}\to\mathcal{A}^\#$ is fully faithful.

By construction, if $e_i:A\to A$, $i=1,\dots,r$ is a family of orthogonal idempotents in $\mathcal{A}$ then

$$A=\bigsqcup_{1\leq i\leq r}(A,e_i).$$

(Here, we write A for (A,Id_A)).

Also, let $f:A\to B$ be a morphism in $\mathcal{A}$. Then any section $g:B\to A$ of $f:A\to B$ in $\mathcal{A}$ defines an idempotent $e_g:=g\circ f:A\to A$ in $\mathcal{A}$ hence, from the above, a decomposition

$$A=(A,e_g)\sqcup(A,Id_A-e_g).$$

One then has the following elementary categorical lemma.

Lemma 2.2. — *With the above notation*

- *The morphism* $f\circ e_g=f:(A,e_g)\tilde{\to}B$ *is an isomorphism (with inverse* $e_g\circ g=e_g$*);*
- *For any two sections* $g,\ g':B\to A$ *in* $\mathcal{A}$*, the morphism*

$$(Id_A-e_{g'})\circ(Id_A-e_g)=Id_A-e_g:(A,Id_A-e_g)\tilde{\to}(A,Id_A-e_{g'})$$

is an isomorphism (with inverse $(Id_A-e_g)\circ(Id_A-e_{g'})=Id_A-e_{g'}$*).*

In other words, any morphism $f:A\to B$ admitting a section in $\mathcal{A}$ induces a decompostion

$$A=B\sqcup\mathbb{L}_f,$$

which is independent of the section.

2.2.1.2. Pure effective motives. —

The Karoubian enveloppe $\kappa : \mathcal{C}_\sim(k)_E \to \mathcal{C}_\sim(k)_E^{\#} =: \mathrm{M}_\sim^{eff}(k)_E$ of $\mathcal{C}_\sim(k)_E$ is called the *category of pure effective motives with coefficient in E modulo $\sim$.*

Let $\mathbb{I}$ denote the image of $\mathrm{spec}(k)$ in $\mathrm{M}_\sim(k)_E$. From lemma 2.2, any $X \in \mathcal{P}(k)$ such that $X(k) \neq \emptyset$ admits a decomposition

$$X = \mathbb{I} \sqcup \mathbb{L}_X.$$

When $X = \mathbb{P}_k^1$, the motive $\mathbb{L}_{\mathbb{P}_k^1} =: \mathbb{L}$ is simply called the *Lefschetz motive.* The following requires a bit more work.

Lemma 2.3. — *There is a unique tensor structure on $\mathrm{M}_\sim^{eff}(k)_E$ such that $\mathcal{C}_\sim(k)_E \to \mathrm{M}_\sim^{eff}(k)_E$ becomes a functor of tensor categories. Furthermore, the functor:*

$$- \otimes_E \mathbb{L} : \mathrm{M}_\sim^{eff}(k)_E \to \mathrm{M}_\sim^{eff}(k)_E$$

is fully faithful (in other words, $\mathbb{L}$ is quasi-invertible).

By the universal property of pseudoabelian enveloppe, if homological equivalence is coarser than $\sim$, the covariant functor of tensor categories

$$(\mathrm{H}^*(-), (-)^*) : \mathcal{C}_\sim(k)_E \to \mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}}$$

extends to $\mathrm{M}_\sim^{eff}(k)_E$.

2.2.2. Pure Motives. —

2.2.2.1. Inverting the Lefschetz motive. —

Given a tensor category $\mathcal{A}$ and a pseudo-invertible object $\mathbb{L}$ in $\mathcal{A}$ such that[(2)] the permutation $(1,2,3)$ acts as the identity on $\mathbb{L}^{\otimes 3}$, there exists a tensor category $\mathcal{A}[\mathbb{L}^{-1}]$ and a functor of tensor categories

$$\iota : \mathcal{A} \to \mathcal{A}[\mathbb{L}^{-1}]$$

which is universal for functors of tensor categories from $\mathcal{A}$ sending $\mathbb{L}$ to an invertible object. Again, $\iota : \mathcal{A} \to \mathcal{A}[\mathbb{L}^{-1}]$ can be easily described as follows. Objects of $\mathcal{A}[\mathbb{L}^{-1}]$ are pairs (A, m), where A is an object in $\mathcal{A}$ and $m \in \mathbb{Z}$ and given (A, m), (A', m') in $[\mathbb{L}^{-1}]$, the set of morphisms from (A, m) to (A', m') in $\mathcal{A}[\mathbb{L}^{-1}]$ is

$$\varinjlim_{k \geq -m, -m'} \mathrm{Hom}_{\mathcal{A}}(A \otimes \mathbb{L}^{k+m}, A' \otimes \mathbb{L}^{k+m'})$$

(note that, since $\mathbb{L}$ is pseudo-invertible, the transition morphisms are all bijective) and composition is induced by the composition in $\mathcal{A}$. Eventually, the functor $\iota : \mathcal{A} \to \mathcal{A}[\mathbb{L}^{-1}]$ sends A to $(A, 0)$ and $\phi : A \to A'$ to $\phi : (A, 0) \to (A', 0)$; in particular, $\iota : \mathcal{A} \to \mathcal{A}[\mathbb{L}^{-1}]$ is fully faithful.

2.2.2.2. Pure motives. —

The category of pure motives with coefficient in E modulo $\sim$ is the category

$$\iota : \mathrm{M}_\sim^{eff}(k)_E \to \mathrm{M}_\sim^{eff}(k)_E[\mathbb{L}^{-1}] =: \mathrm{M}_\sim(k)_E.$$

Note that $Id_{\mathbb{L}}$ identifies $\mathbb{L} := \iota(\mathbb{L})$ with $(\mathbb{I}, 1)$ in $\mathrm{M}_\sim(k)_E$; we will also write $\mathbb{I} := \iota(\mathbb{I})$. Also, one can check that morphisms from (X, e, m) to (X', e', m') in $\mathrm{M}_\sim(k)_E$ can be identified with

$$e' \mathcal{C}_\sim^{m'-m}(X, X') e.$$

(2) This condition is an if and only if condition to ensure that the category $\mathcal{A}[\mathbb{L}^{-1}]$ constructed below is indeed a tensor category. In our situation, one can show that transpositions already act as the identity on $\mathbb{L}^{\otimes 2}$.

The category $\mathrm{M}_\sim(k)_E$ is a pseudoabelian tensor category which is now rigid. The dual of (X,e,m) is $(X,\ {}^te,d_X-m)$. In particular, for any M in $\mathrm{M}_\sim(k)_E$ one has an evaluation morphism $\epsilon_M : M\otimes M^\vee\to\mathbb{I}$ (corresponding to $Id_{M^\vee}$) and a coevaluation morphism $\eta_M:\mathbb{I}\to M^\vee\otimes M$ (corresponding to Id_M) hence a trace morphism

$$Tr_M:\mathrm{End}_{\mathrm{M}_\sim(k)_E}(M)\to\mathrm{End}_{\mathrm{M}_\sim(k)_E}(\mathbb{I})=E$$

sending $f:M\to M$ to the composite

$$\mathbb{I}\overset{\eta_M}{\to}M^\vee\otimes M\overset{Id_{M^\vee}\otimes f}{\to}M^\vee\otimes M\overset{c_{M^\vee,M}}{\to}M\otimes M^\vee\overset{\epsilon_M}{\to}\mathbb{I}$$

and a rank defined to be

$$\mathrm{rank}(M)=Tr_M(Id_M)\in E.$$

Define the *motivic cohomology functor* to be resulting covariant functor of tensor categories

$$\mathfrak{h}_\sim:\mathcal{P}(k)^{op}\to\mathrm{M}_\sim(k)_E$$

(sending $X\in\mathcal{P}(k)$ to $(X,Id_X,0)$ and $f:X\to Y$ to ${}^t\Gamma_f$). If homological equivalence is coarser than $\sim$ and, since $\mathrm{H}^*(\mathbb{L})=(Id-x\circ\pi)^*\mathrm{H}^*(\mathbb{P}^1_k)=\mathrm{H}^2(\mathbb{P}^1_k)$ is invertible in $\mathrm{AAlg}^{\mathbb{Z}_{\geq 0}}_{/K}$, the covariant functor of tensor categories

$$(\mathrm{H}^*(-),(-)^*):\ \mathrm{M}^{eff}_\sim(k)_E\to\mathrm{AAlg}^{\mathbb{Z}_{\geq 0}}_{/K}$$

extends to $\mathrm{M}_\sim(k)_E$ that is, one has a commutative diagram of functors of tensor categories

$$\begin{tikzcd}
\mathcal{P}(k)^{op} \arrow[r, "\mathrm{H}^*"] \arrow[d, "\mathfrak{h}_\sim"'] & \mathrm{Alg}^{\mathbb{Z}_{\geq 0}}_{/K} \\
\mathrm{M}_\sim(k)_E. \arrow[ur, "\mathrm{H}^*"'] &
\end{tikzcd}$$

Classically, one writes $(X,e,m):=e\mathfrak{h}_\sim(X)(m)$ hence $\mathrm{H}^*(e\mathfrak{h}_\sim(X)(m))=e^*\mathrm{H}^*(X)(m)$. By construction, the trace and dual in $\mathrm{M}_\sim(k)_E$ are compatible with the trace and Poincaré duality of the given Weil cohomogy.

3. Kunneth type and Lefschetz type conjectures

We have now constructed pseudoabelian rigid tensor categories $\mathrm{M}_\sim(k)_E$ and motivic cohomology functors $\mathfrak{h}_\sim:\mathcal{P}(k)^{op}\to\mathrm{M}_\sim(k)_E$. When homological equivalence is coarser than $\sim$, classical Weil cohomologies factor *via* motivic cohomology and give rise to functors of tensor categories

$$(\mathrm{H}^*(-),(-)^*):\ \mathrm{M}_\sim(k)_E\to\mathrm{AAlg}^{\mathbb{Z}_{\geq 0}}_{/K},$$

which are faithful and exact when $\sim$ is the homological equivalence.

However, to make the (neutral) Tannakian formalism work, two questions remain to be solved, namely:

1. Is $\mathrm{M}_\sim(k)_E$ an abelian category?
2. If $\mathrm{M}_\sim(k)_E$ is an abelian category, does it admit fibre functors

$$F:\mathrm{M}_\sim(k)_E\to\mathrm{Mod}_{/E}?$$

The first question is answered by the theorem of Jannsen, which gives an if and only if condition for $\mathrm{M}_\sim(k)_E$ to be abelian semisimple.

Theorem 3.1. — (Jannsen [**J92**]) *If $\sim$ is an adequate equivalence relation, the following three properties are equivalent. (i) $\mathrm{M}_\sim(k)_E$ is an abelian semisimple category;*
(ii) $\mathcal{C}_\sim(X,X)_E$ is a semisimple finite dimensional E-algebra for all $X\in\mathcal{P}(k)$;
(iii) $\mathrm{M}_\sim(k)_E=\mathrm{M}_{\mathrm{num}}(k)_E$.

In particular, $\mathrm{M}_{\mathrm{num}}(k)_E$ is a semisimple abelian rigid tensor category over E hence close to be Tannakian. However, Deligne's criterion, gives an obstruction for $\mathrm{M}_{\mathrm{num}}(k)_E$ to be Tannakian.

Theorem 3.2. — (Deligne's criterion [**D90**]) *Let E be a field of characteristic 0 and $\mathcal{T}$ an abelian rigid tensor category over E with $\mathrm{End}_{\mathcal{T}}(\mathbb{I}) = E$. Then $\mathcal{T}$ is Tannakian if and only if for any M in $\mathcal{T}$ one has*

$$\mathrm{rank}(M) \in \mathbb{Z}_{\geq 0}.$$

Indeed, since numerical equivalence is coarser than other adequate equivalence relations on $\mathcal{P}(k)$, one has a canonical functor of tensor categories $R_{\sim} : \mathrm{M}_{\sim}(k)_E \to \mathrm{M}_{\mathrm{num}}(k)_E$, which is essentially surjective and, given any M in $\mathrm{M}_H(k)_E$, one has

$$\mathrm{rank}(R_H(M)) \quad = \sum_{i\geq 0}(-1)^i \mathrm{dim}_K(\mathrm{H}^i(M)),$$

Thus, before going further, one has to remedy the default of positivity of the rank. This is conjecturally done by Kunneth type conjecture.

3.1. Kunneth type conjecture. — For any X in $\mathcal{P}(k)$, the *Kunneth projectors*

$$\pi^i_{X,H} : \mathrm{H}^*(X) \twoheadrightarrow \mathrm{H}^i(X) \hookrightarrow \mathrm{H}^*(X),\ i \geq 0$$

form a complete system of orthogonal central idempotents in $\mathrm{End}_{\mathrm{Mod}^{\mathbb{Z}_{\geq 0}}_{/K}}(\mathrm{H}^*(X))$. Kunneth type conjecture claims that they should be homological correspondences that is, in motivic terms:

Conjecture 3.3. — (Kunneth type) *For any $X \in \mathcal{P}(k)$, the Kunneth projectors $\pi^i_{X,H} : \mathrm{H}^*(X) \to \mathrm{H}^*(X)$, $i \geq 0$ are realizations of morphisms in $M_H(k)_E$.*

Assuming Kunneth type conjecture, for any $X \in \mathcal{P}(k)$, one can set

$$\mathfrak{h}^i_H(X) := \pi^i_{X,H}(\mathfrak{h}_H(X)) \in M_H(k)_E,\ i \geq 0.$$

and decompose

$$\mathfrak{h}_H(X) = \bigoplus_{i\geq 0} \mathfrak{h}^i_H(X)$$

in $M_H(k)_E$. Also, since $\pi^i_{X,H}(\mathbb{L}) = 0$ except for $i = 2$, any $M = e\mathfrak{h}(X)(m)$ in $M_H(k)_E$ can be canonically graded as $M = \oplus_{i\geq 0} M_i$, where

$$M_i = e\mathfrak{h}^{i+2m}_H(X)(m),\ i \geq 0.$$

By definition of the Kunneth projectors, this graduation endows $M_H(k)_E$ with a structure of $\mathbb{Z}_{\geq 0}$-graded tensor category for which $\mathrm{H}^* : M_H(k)_E \to \mathrm{AAlg}^{\mathbb{Z}_{\geq 0}}_{/K}$ becomes a $\mathbb{Z}_{\geq 0}$-graded functor of tensor categories. Similarly, setting

$$\mathfrak{h}^i_{\mathrm{num}}(X) := R_H(\pi^i_{X,H})(\mathfrak{h}_{\mathrm{num}}(X)),\ i \geq 0,$$

one can endow $\mathrm{M}_{\mathrm{num}}(k)_E$ with a structure of $\mathbb{Z}_{\geq 0}$-graded tensor category.

Note that by construction, for any M, N in $M_{\sim}(k)_E$, the commutativity constraint

$$c_{M,N} : M \otimes_K N \tilde{\to} N \otimes_K M$$

in $M_H(k)_E$ is given by

$$c_{M,N} = \bigoplus_{i,j\geq 0} c_{M_i,N_j},$$

where

$$\begin{array}{rccc} c_{M_i,N_j} : & M_i \otimes N_j & \tilde{\to} & N_j \otimes M_i \\ & m_i \otimes n_j & \mapsto & n_j \otimes m_i \end{array},\ i,\ j \geq 0.$$

We can use the graduation provided by Kunneth type conjecture to modify this commutativity constraint according to Koszul rule; write $\dot{\mathrm{M}}_H(k)_E$, $\dot{\mathrm{M}}_{\mathrm{num}}(k)_E$ for the resulting $\mathbb{Z}_{\geq 0}$-graded tensor categories. Then, for any M in $\dot{\mathrm{M}}_{\mathrm{num}}(k)_E$, one has

$$\mathrm{rank}(M) = \sum_{i\geq 0} \dim_K(\mathrm{H}^i(M)) \in \mathbb{Z}_{\geq 0}.$$

Hence, from Deligne's criterion, one gets

Corollary 3.4. — *Assuming Kunneth type conjecture, the category* $\dot{\mathrm{M}}_{\mathrm{num}}(k)_E$ *is a semisimple Tannakian category.*

However, there is no natural way to construct fibre functors on $\dot{\mathrm{M}}_{\mathrm{num}}(k)_E$ whereas modified classical Weil cohomology functors

$$\dot{\mathrm{H}}^* : \dot{M}_H(k)_E \xrightarrow{\mathrm{H}^*} \mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}} \xrightarrow{For} \mathrm{Mod}_{/K}$$

(where For denotes the forgetful functor) provide natural candidates for fibre functors on $\dot{M}_H(k)_E$. This motivates the following conjecture.

Conjecture 3.5. — (num=H) *For any* $X \in \mathcal{P}(k)$ *numerical equivalence and homological equivalence coincide on* $Z^*(X)$*.*

Corollary 3.6. — *Assuming Kunneth type conjecture and the 'num=H' conjecture, the category* $\dot{\mathrm{M}}_{\mathrm{num}}(k)_E$ *is a semisimple Tannakian category with fibre functors*

$$\dot{\mathrm{H}}^* : \dot{M}_H(k)_E \to \mathrm{Mod}_{/K}.$$

In characteristic 0, both the Kunneth type conjecture and the 'num=H' conjecture follow from Lefschetz type conjecture.

3.2. Lefschetz type conjecture. — For any X in $\mathcal{P}(k)$, a *polarization* η *on* X is the image

$$\eta = \gamma_H^1(X)([D]) \in \mathrm{H}^2(X)(1)$$

of the class $[D] \in Z_H^1(X)$ of an ample divisor on X. Let $Pol(X)$ denote the set of polarizations on X. The strong Lefschetz theorem asserts that for any polarization η on X, the associated *Lefschetz operator* $L_\eta := - \cup \eta$ induces isomorphisms

$$L_\eta^{d_X-i} : \mathrm{H}^i(X)(r) \tilde{\to} \mathrm{H}^{2d_X-i}(X)(d_X - i + r),\ i \leq d_X,\ r \in \mathbb{Z}.$$

In particular, one can consider the Lefschetz decompositions

$$\mathrm{H}^j(X)(r) = \bigoplus_{\max\{0, j-d_X\} \leq k \leq j/2} L_\eta^k P^{j-2k}(X)(r),$$

where we set

$$P^i(X)(r) := \mathrm{H}^i(X)(r) \cap \ker(L_\eta^{d_X-i+1}),\ i \leq d_X$$

and define

- The *Lefschetz involution*

$$*_{L,\eta} : \bigoplus_{i\geq 0,\, r\in\mathbb{Z}} \mathrm{H}^i(X)(r) \tilde{\to} \bigoplus_{i\geq 0,\, r\in\mathbb{Z}} \mathrm{H}^i(X)(r)$$

  by

$$\begin{array}{ll} *_{L,\eta}|_{\mathrm{H}^i(X)(r)} = L_\eta^{d_X-i} : \mathrm{H}^i(X)(r) \tilde{\to} \mathrm{H}^{2d_X-i}(X)(d-i+r) & \text{if } i \leq d_X; \\ *_{L,\eta}|_{\mathrm{H}^i(X)(r)} = (L_\eta^{i-d_X})^{-1} : \mathrm{H}^i(X)(r) \tilde{\to} \mathrm{H}^{2d-i}(X)(d_X - i + r) & \text{if } i > d_X. \end{array}$$

- The *Hodge involution*

$$*_{H,\eta}: \bigoplus_{i\geq 0,\ r\in\mathbb{Z}} \mathrm{H}^i(X)(r) \tilde{\to} \bigoplus_{i\geq 0,\ r\in\mathbb{Z}} \mathrm{H}^i(X)(r)$$

by multiplying $*_{L,\eta}$ by a factor

$$(-1)^{\frac{(j-2k)(j-2k+1)}{2}} \frac{k!}{(d_X - j + k)!}$$

on $L_\eta^k P^{j-2k}(X)(r)$.

Lefschetz type conjecture claims that the Lefschetz involution should be an algebraic correspondance that is, in motivic terms:

Conjecture 3.7. — (Lefschetz type) *For any $X \in \mathcal{P}(k)$ and polarization η on X, the Lefschetz involution*

$$*_{L,\eta}: \bigoplus_{i\geq 0,\ r\in\mathbb{Z}} \mathrm{H}^i(X)(r) \tilde{\to} \bigoplus_{i\geq 0,\ r\in\mathbb{Z}} \mathrm{H}^i(X)(r)$$

is the realization of a morphism in $M_H(k)$.

Also, note that for any $X \in \mathcal{P}(k)$, one always has

Proposition 3.8. — (Kleiman)

$$\mathbb{Q}[\pi^i_{X,H},\ i\geq 0] \subset \mathbb{Q}[L_\eta, *_{L,\eta}] = \mathbb{Q}[L_\eta, *_{H,\eta}].$$

(as $\mathbb{Q}$-subalgebras of the endomorphism ring of $\bigoplus_{i\geq 0,\ r\in\mathbb{Z}} \mathrm{H}^i(X)(r)$). In particular,

- *one can replace $*_{L,\eta}$ by $*_{H,\eta}$ in the statement of Lefschetz type conjecture;*
- *Lefschetz type conjecture implies Kunneth type conjecture;*
- *if Lefschetz type conjecture holds for $X \in \mathcal{P}(k)$ and a fixed polarization η on X then it holds for all polarization η on X.*

Also, combined with the Hodge index theorem, Lefschetz type conjecture implies the 'num=H' conjecture.

Theorem 3.9. — (Hodge index theorem) *For any $X \in \mathcal{P}(k)$ and polarization η on X, the pairing:*

$$\begin{array}{ccc} Z^r_H(X) \times Z^r_H(X) & \to & \mathbb{Q} \\ (\alpha,\beta) & \mapsto & \langle \alpha, *_{H,\eta}\beta\rangle \end{array}$$

takes its value in $\mathbb{Q}$ and is positive definite.

Indeed, since Lefschetz type conjecture implies that the Hodge involution $*_{H,X}$ is the realization of an automorphism in $M_H(k)$, theorem 3.9 shows that the pairing

$$\begin{array}{ccc} Z^r_H(X) \times Z^{d_X-r}_H(X) & \to & \mathbb{Q} \\ (\alpha,\beta) & \mapsto & \langle \alpha, \beta\rangle \end{array}$$

is non-degenerate as well. But then, for any $\alpha \in Z^*_H(X)$, if $R_H(\alpha) = 0$, by definition of numerical equivalence, for any $\beta \in Z^*_H(X)$ one has $\langle\alpha,\beta\rangle = 0$ hence $\alpha = 0$.

Corollary 3.10. — *Assuming Lefschetz type conjecture, $\dot{\mathrm{M}}_H(k)_E$ is a semisimple Tannakian category with fibre functors*

$$\dot{\mathrm{H}}^*: \dot{M}_H(k)_E \to \mathrm{Mod}_{/K}.$$

4. André's theory of motivated cycles

The key idea of André's theory of motivated cycles [**A96**] is to adjoin formally the Lefschetz involutions to the set of morphisms in $M_H(k)$ in order to force Lefschetz conjecture to hold and construct a category of motives which is a semisimple neutral Tannakian category - the category $M_{mot,H}(k)_E$ of pure motivated motives. In subsection 4.1, we give the technical definition of motivated cycles and explain that they behave well-enough (basically like algebraic cycles) to play the part of algebraic cycles in the construction of a category of pure motives. In subsection 4.2, we explain how to specialize motivated cycles and motivated motivic Galois groups by means of Deligne's fixed part theorem.

4.1. Construction. — For any $X \in \mathcal{P}(k)$, the set of *motivated cycles on X with coefficients in E* is the subset $Z_{mot,H}(X)_E \subset \mathbf{H}^*(X)$ defined as

$$Z_{mot,H}(X)_E := \left\{ (p_X^{XY})_*(\alpha \cup *_{L,\eta}\beta) \mid \alpha,\ \beta \in Z_H^*(X \times_k Y)_E,\ \begin{array}{l} Y \in \mathcal{P}(k), \\ \eta_X \in Pol(X),\ \eta_Y \in Pol(Y), \\ \eta = (p_X^{XY})^*\eta_X + (p_Y^{XY})^*\eta_Y. \end{array} \right\}$$

André's original definition is more elaborate; one fixes a full subcategory $\mathcal{V}$ of $\mathcal{P}(k)$ (stable by products, direct sums and connected components) - called the category of base pieces and only allows the Y to vary in $\mathcal{V}$. But for simplicity, here, we will only consider the case where $\mathcal{V} = \mathcal{P}(k)$.

Also, set

$$Z^r_{mot,H}(X)_E := Z_{mot,H}(X)_E \cap \mathrm{H}^{2r}(X)(r),\ r \geq 0.$$

The fact that $Z_{mot,H}(X)_E$ is stable by sum, coproduct direct and inverse images by projections requires a few computations.

Lemma 4.1. — [**A96**, Prop. 2.1]

1. *The structure of $\mathbb{Z}_{\geq 0}$-graded K-algebra of $\mathbf{H}^*(X)$ induces a structure of $\mathbb{Z}_{\geq 0}$-graded E-algebra on $Z_{mot,H}(X)_E$ and the natural inclusions*

$$Z^*_H(X)_E \subset Z^*_{mot,H}(X)_E \subset \mathbf{H}^*(X)$$

are morphisms of $\mathbb{Z}_{\geq 0}$-graded E-algebras.

2. *For any $X,\ Y \in \mathcal{P}(k)$, one has*

$$(p_X^{XY})^* Z^*_{mot,H}(X)_E \subset Z_{mot,H}(X \times_k Y)_E$$

and

$$(p_X^{XY})_* Z^*_{mot,H}(X \times_k Y)_E \subset Z_{mot,H}(X)_E.$$

In particular, since algebraic cycles are motivated, the rules (1), (2), (3) of subsection 2.1.1 give rise to two covariant functors

$$(Z^*_{mot,H}(-)_E, (-)^*) : \mathcal{P}(k)^{op} \to \mathrm{Alg}^{\mathbb{Z}_{\geq 0}}_{/E}$$

and

$$(Z^*_{mot,H}(-)_E, (-)_*) : \mathcal{P}(k) \to \mathrm{Mod}_{/E}$$

such that the following diagrams of morphisms of functors commute.

$$\begin{tikzcd} (Z^*_H(-)_E, (-)^*) \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathbb{Z}}_{/E} \\ (Z^*_{mot,H}(-)_E, (-)^*) \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} (Z^*_H(-)_E, (-)_*) \arrow[r] \arrow[d] & \mathrm{Mod}_{/E} \\ (Z^*_{mot,H}(-)_E, (-)_*) \arrow[ur] & \end{tikzcd}$$

Then, one can transpose the formal construction of $M_\sim(k)_E$ with $Z^*_{mot,H}(X)_E$ replacing $Z^*_\sim(X)_E$ to obtain the category $M_{mot,H}(k)_E$ of *pure motivated motives with coefficient in E*. Note also that, from the above and the comparison isomorphisms, up to canonical isomorphism of $\mathbb{Z}_{\geq 0}$-graded $\mathbb{Q}$-algebras, the definition of $Z^*_{mot,H}(X)_\mathbb{Q}$ is independent of the classical Weil cohomology.

In the setting of pure motivated motives Lefschetz conjecture becomes an easy lemma.

Lemma 4.2. — [**A96**, Prop. 2.2, 2.3] *For any $X \in \mathcal{P}(k)$ and polarization η on X, the Lefschetz involution $*_{L,\eta}$ is a morphisms in $M_{mot,H}(k)_E$ (hence $*_{H,\eta}$ and $\pi^i_{X,H}$, $i \geq 0$ are endomorphisms in $M_{mot,H}(k)_E$ as well).*

One can also define the analogue of numerical equivalence on motivated cycles. Namely, a motivated cycle $\alpha \in Z^*_{mot,H}(X)_E$ is said to be numerically equivalent to 0 if $\langle \alpha, \beta \rangle = 0$ for all $\beta \in Z^*_{mot,H}(X)_E$. After observing that the Hodge index theorem extends to motivated cycles (which are of type (p,p)), one can follow the guidelines of the standard conjectures as exposed above in section 3 (see [**A96**, section 3]) that is, prove the analogue of Jansenn theorem for motivated numerical motives and, combining it with lemma 4.2 and Hodge index theorem, obtain

Theorem 4.3. — ([**A96**, Thm.0.4]) *The category $\dot{\mathrm{M}}_{mot,H}(k)_E$ of pure motivated motives is a semisimple Tannakian category with fibre functors*

$$\dot{\mathrm{H}}^* : \dot{M}_{mot,H}(k)_E \to \mathrm{Mod}_{/K}.$$

In particular, for $E = K$, the category $\dot{M}_{mot,H}(k)_K$ is a neutral semisimple Tannakian category over K. It then follows from the general theory of Tannakian categories [**DM82**], [**D90**] that the *motivic Galois group*

$$G_{mot,H} := \mathrm{Aut}^{\otimes}(\dot{\mathrm{H}}^*)$$

is a pro-reductive algebraic group over K and that $\dot{\mathrm{H}}^* : \dot{M}_{mot,H}(k)_K \to \mathrm{Mod}_{/K}$ factors through

$$\begin{tikzcd} \dot{M}_{mot,H}(k)_K \arrow[r, "\dot{\mathrm{H}}^*"] \arrow[d, "\dot{\mathrm{H}}^*"'] & \mathrm{Mod}_{/K} \\ \mathrm{Rep}_K(G_{mot,H}), \arrow[ur, "For"'] & \end{tikzcd}$$

where $\mathrm{Rep}_K(G_{mot,H})$ denotes the category of finite dimensional algebraic representations of $G_{mot,H}$ over K and $\dot{\mathrm{H}}^* : \dot{M}_{mot,H}(k)_K \to \mathrm{Rep}_K(G_{mot,H})$ is an equivalence of categories.

For any M in $M_{mot,H}(k)_K$ one can also consider the Tannakian subcategory $\langle M \rangle^{\otimes} \subset \dot{M}_{mot,H}(k)$ and the associated motivic Galois group $G_{mot,H}(M) := \mathrm{Aut}^{\otimes}(\dot{\mathrm{H}}^*|_{\langle M \rangle^{\otimes}}) \subset \mathrm{GL}(\mathrm{H}^*(M))$, which is a reductive algebraic group over K. Then

$$G_{mot,H} = \varprojlim G_{mot,H}(M),$$

where the index set is the set of pure motivated motives partially ordered by the inclusion relation. Hence, the transition morphisms are the faithfully flat morphisms of algebraic groups $G_{mot,H}(N) \twoheadrightarrow G_{mot,H}(M)$ corresponding to the inclusion of Tannakian categories $\langle M \rangle^{\otimes} \subset \langle N \rangle^{\otimes}$.

For $M = \mathfrak{h}_{mot,H}(X)$, we will write $G_{mot,H}(X)$ for $G_{mot,H}(M)$.

Remark 4.4. — It follows from the definition of motivated cycles that

1. If Lefschetz conjecture holds then motivated cycles and algebraic cycles modulo homological equivalence coincide.
2. If $k = \overline{k}$ is algebraically closed then the collections of De Rham and ℓ-adic realizations (for all field embeddings $k \hookrightarrow \mathbb{C}$) is an absolute Hodge cycle in the sense of [**D82**, §2]. As a result, if the Hodge conjecture holds, algebraic cycles, motivated cycles and absolute Hodge cycles coincide. For codimension 1 cycles, the Hodge conjecture is known as the Lefschetz theorem on $(1,1)$-classes. In particular
$$Z^1_{mot,H}(X)_{\mathbb{Q}} = Z^1_H(X)_{\mathbb{Q}}$$
Also, algebraic and numerical (hence, *a fortiori* homological) equivalence coincide modulo torsion, whence
$$Z^1_H(X)_{\mathbb{Q}} = NS(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$
where $NS(X) := Z^1_{alg}(X)$ denotes the Néron-Severi group of X.

4.2. Comparison and specialization of motivated cycles. — Recall that the definition of motivated cycles is independent of the classical Weil cohomology $\mathrm{H}^*(-)$ so, here, we will use ℓ-adic cohomology but, for simplicity, we keep the imprecise notation 'H' and 'K' for H_ℓ and $\mathbb{Q}_\ell$ respectively. Also, to simplify the notation, we will omit the subscript '$-_E$'.

4.2.1. Invariance under algebraically closed extension and Galois descent. — Assume that k is a finitely generated field of characteristic 0, let $\overline{k}$ be a fixed algebraic closure of k and write Γ_k for the automorphism group of $\overline{k}/k$. Classical arguments [**D82**, Prop. 2.9], [**A96**, Scolie 2.5 a)] show that

Lemma 4.5. — *Let $\Omega/\overline{k}$ be an extension of algebraically closed fields. Then, for any $X \in \mathcal{P}(k)$, the specialization isomorphism for ℓ-adic cohomology $\mathbf{H}_\ell^*(X_\Omega)\tilde{\to}\mathbf{H}_\ell^*(X_{\overline{k}})$ restricts to an isomorphism*

$$Z_{mot,H}(X_\Omega)\tilde{\to}Z_{mot,H}(X_{\overline{k}}).$$

Furthermore, $Z_{mot,H}(X_{\overline{k}}) \subset \mathbf{H}_\ell^*(X_{\overline{k}})$ is a sub-Γ_k-module such that the induced representation

$$\Gamma_k \to \mathrm{GL}(Z_{mot,H}(X_{\overline{k}}))$$

has finite image and

$$Z_{mot,H}(X_{\overline{k}})^{\Gamma_k} = Z_{mot,H}(X).$$

4.2.2. Specialization of motivated cycles. —

4.2.2.1. Setting. — Let S be a scheme of finite type over k and $X \to S$ a smooth, projective morphism. Given any point $s \in S$, let $\overline{s} : \mathrm{spec}(k(\overline{s})) \to S$ denote an associated geometric point. From [**SGA6**, X, App. §7], for any $s, t \in S$ such that $s \in \overline{\{t\}}$, there exists a specialization monomorphism

$$sp_{t,s} : Z_H^*(X_{\overline{t}}) \hookrightarrow Z_H^*(X_{\overline{s}})$$

of $\mathbb{Z}_{\geq 0}$-graded E-algebras such that the following diagram commutes.

$$\begin{tikzcd} Z_H^*(X_{\overline{t}}) \arrow[r, hook, "\gamma_\ell^*(X_{\overline{t}})"] \arrow[d, hook, "sp_{t,s}"'] & \mathbf{H}_\ell^*(X_{\overline{t}}) \arrow[d, equal, "sp_{t,s}"] \\ Z_H^*(X_{\overline{s}}) \arrow[r, hook, "\gamma_\ell^*(X_{\overline{s}})"'] & \mathbf{H}_\ell^*(X_{\overline{s}}), \end{tikzcd}$$

where the right vertical arrow is the specialization isomorphism for ℓ-adic cohomology.

It is thus natural to ask whether $sp_{t,s}(Z_{mot,H}^*(X_{\overline{t}})) \subset Z_{mot,H}^*(X_{\overline{s}})$. For codimension 1 motivated cycles, this is true since they are algebraic. For higher codimensional motivated cycles, this is not clear from their definition. Indeed, for instance, there is no reason why the auxilliary $Y \in \mathcal{P}(k(\overline{t}))$ involved in the definition of a motivated cycle in $Z_{mot,H}^*(X_{\overline{t}})$ should have good reduction at s.

A way to prove that $sp_{t,s}(Z_{mot,H}^*(X_{\overline{t}})) \subset Z_{mot,H}^*(X_{\overline{s}})$ is to use Grothendieck's parallel transport for motivated cycles.

4.2.2.2. Grothendieck's parallel transport.— Let S be a scheme geometrically connected, smooth and separated over k and $f : X \to S$ a smooth proper morphism with geometrically connected fibres. By the general theory of étale fundamental group, the constructible sheaf $R^n f_*\mathbb{Q}_\ell(m)$ is described by any of the representations

$$\rho_{\overline{s}} : \pi_1(S; \overline{s}) \to \mathrm{GL}((R^n f_*\mathbb{Q}_\ell(m)_{\overline{s}}),\ s \in S.$$

In particular,

$$\mathrm{H}^0(S, R^n f_*\mathbb{Q}_\ell)(m) = (R^n f_*\mathbb{Q}_\ell(m)_{\overline{s}})^{\pi_1(S;\overline{s})}$$

and

$$\mathrm{H}^0(S_{\overline{k}}, R^n f_*\mathbb{Q}_\ell)(m) = (R^n f_*\mathbb{Q}_\ell(m)_{\overline{s}})^{\pi_1(S_{\overline{k}};\overline{s})}.$$

Also, the specialization isomorphism for ℓ-adic cohomology is not canonically defined but depends on the choice of an étale path. More precisely, given two points $s,\ t\in S$ any étale path $\alpha\in\pi_1(S;\overline{t},\overline{s})$ induces an isomorphism

$$(sp_{t,s}=)\pi_{t,s}^{\alpha,m,n}: R^nf_*\mathbb{Q}_\ell(m)_{\overline{t}}\tilde{\to}R^nf_*\mathbb{Q}_\ell(m)_{\overline{s}}.$$

As $\pi_1(S_{\overline{k}};\overline{t},\overline{s})$ is a $\pi_1(S_{\overline{k}};\overline{t})$-torsor, this isomorphism restricts to an isomorphism

$$\pi_{t,s}^{m,n}:(R^nf_*\mathbb{Q}_\ell(m)_{\overline{t}})^{\pi_1(S_{\overline{k}};\overline{t})}\tilde{\to}(R^nf_*\mathbb{Q}_\ell(m)_{\overline{s}})^{\pi_1(S_{\overline{k}};\overline{s})},$$

which is independent of α and is called the *parallel transport*.

The key point is that the parallel transport is a morphism in $M_{mot,H}(k)_E$. This follows, essentially, from Deligne's fixed part theorem and the fact that $M_{mot,H}(k)_E$ is a semisimple abelian category.

Lemma 4.6. — *There exists submotives $\Delta_{\overline{t}}^{m,n}\hookrightarrow\mathfrak{h}^n_{mot,H}(X_{\overline{t}})(m)$ and $\Delta_{\overline{s}}^{m,n}\hookrightarrow\mathfrak{h}^n_{mot,H}(X_{\overline{s}})(m)$ and a morphism $\pi_{t,s}^{m,n,0}:\Delta_{\overline{t}}^{m,n}\to\Delta_{\overline{s}}^{m,n}$ in $M_{mot,H}(k)$ such that $\mathrm{H}^*_\ell(\pi_{t,s}^{m,n,0})=\pi_{t,s}^{m,n}$.*

Proof. Let $X\hookrightarrow\overline{X}$ be a smooth compactification of X (which, since X is smooth and separated over k, exists by combining Nagata's compactification theorem and Hironaka's desingularization theorem) and consider the following sequence of canonical morphisms

$$\begin{tikzcd}
\mathrm{H}^n_\ell(\overline{X}_{\overline{k}})(m) \arrow[r] \arrow[rr, bend right, "u^{m,n}"'] & \mathrm{H}^n_\ell(X_{\overline{k}})(m) \arrow[r] & \mathrm{H}^0(S_{\overline{k}},R^nf_*\mathbb{Q}_\ell)(m) \arrow[r, hook, "c_{\overline{t}}^{m,n}"] & \mathrm{H}^n_\ell(X_{\overline{t}})(m).
\end{tikzcd}$$

Deligne's fixed part theorem [**D71**, Thm. 4.1.1] and the comparison isomorphism between Betti and ℓ-adic cohomologies imply that the morphism $u^{m,n}$ is surjective. Write

$$i_{\overline{s}}:X_{\overline{s}}\hookrightarrow X\hookrightarrow\overline{X}\ \text{ and }\ i_{\overline{t}}:X_{\overline{t}}\hookrightarrow X\hookrightarrow\overline{X}.$$

This induces a commutative diagram

$$\begin{tikzcd}
 & & \mathrm{H}^n_\ell(\overline{X}_{\overline{k}})(m) \arrow[d, "u^{m,n}"] \arrow[dll, "i_{\overline{s}}^{m,n}"'] \arrow[drr, "i_{\overline{t}}^{m,n}"] & & \\
 & & \mathrm{H}^0(S_{\overline{k}},R^nf_*\mathbb{Q}_\ell)(m) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & \\
\mathrm{H}^n_\ell(X_{\overline{s}})(m) & \mathrm{H}^n_\ell(X_{\overline{s}})(m)^{\pi_1(S_{\overline{k}};\overline{s})} \arrow[l, hook', "c_{\overline{s}}^{m,n}"'] & & \mathrm{H}^n_\ell(X_{\overline{t}})(m)^{\pi_1(S_{\overline{k}};\overline{t})} \arrow[ll, dotted, "\pi_{t,s}"] \arrow[r, hook, "c_{\overline{t}}^{m,n}"] & \mathrm{H}^n_\ell(X_{\overline{t}})(m),
\end{tikzcd}$$

where $i_{\overline{s}}^{m,n}$ and $i_{\overline{t}}^{m,n}$ are realizations of morphisms in $M_{mot,H}(k)$ (actually, they are algebraic correspondances). In particular, since $M_{mot,H}(k)$ is abelian, $K:=\ker(u^{m,n})=\ker(i_{\overline{s}}^{m,n})=\ker(i_{\overline{t}}^{m,n})$ is a pure motivated motive (independent of $\overline{s},\ \overline{t}$) and, by semi-simplicity of $M_{mot,H}(k)$, one can decompose $\mathfrak{h}^n_{mot,H}(\overline{X}_{\overline{k}})(m)=K\oplus N$ in $M_{mot,H}(k)$. Let $i_N:N\hookrightarrow\mathfrak{h}^n_{mot,H}(\overline{X})(m)$ denote the canonical monomorphism. Again, as $M_{mot,H}(k)$ is abelian, $\Delta_{\overline{s}}^{m,n}:=\mathrm{im}(i_{\overline{s}}^{m,n}\circ i_N)$ and $\Delta_{\overline{t}}^{m,n}:=\mathrm{im}(i_{\overline{t}}^{m,n}\circ i_N)$ are pure motivated motives. It follows from the exactness of $\mathrm{H}^*_\ell(-)$ that $\mathrm{H}^*_\ell(\Delta_{\overline{s}}^{m,n})=\mathrm{H}^*_\ell(X_{\overline{s}})^{\pi_1(S_{\overline{k}};\overline{s})}$ and $\mathrm{H}^*_\ell(\Delta_{\overline{t}})=\mathrm{H}^*_\ell(X_{\overline{t}})^{\pi_1(S_{\overline{k}};\overline{t})}$. As $i_{\overline{s}}^{m,n}\circ i_N:N\tilde{\to}\Delta_{\overline{s}}$ and $i_{\overline{t}}^{m,n}\circ i_N:N\tilde{\to}\Delta_{\overline{t}}$ are isomorphism in $M_{mot,H}(k)$ one can set $\pi_{t,s}^{m,n,0}=(i_{\overline{s}}^{m,n}\circ i_N)\circ(i_{\overline{t}}^{m,n}\circ i_N)^{-1}$, which is an isomorphism in $M_{mot,H}(k)$ and, by construction, satisfies $\mathrm{H}^*_\ell(\pi_{t,s}^{m,n,0})=\pi_{t,s}^{m,n}$. □

In particular, $\pi_{t,s}:\mathbf{H}^*_\ell(X_{\overline{t}})^{\pi_1(S_{\overline{k}};\overline{t})}\tilde{\to}\mathbf{H}^*_\ell(X_{\overline{t}})^{\pi_1(S_{\overline{k}};\overline{s})}$ maps motivated cycles to motivated cycles.

Corollary 4.7. — (Specialization of motivated cycles) *Let S be a connected scheme smooth and separated over a field k of characteristic 0 and $f : X \to S$ a smooth proper morphism with geometrically connected fibres. For any $s, t \in S$ such that $s \in \overline{\{t\}}$, one has: $sp_{t,s}(Z^*_{mot,H}(X_{\overline{t}})) \subset Z^*_{mot,H}(X_{\overline{s}})$.*

Proof. Assume first that $S = \mathrm{spec}(R)$ with R a local ring, that t is the generic point and s is the closed point of S. From lemma 4.5, the action of $\pi_1(S; \overline{t})$ on $Z^*_{mot,H}(X_{\overline{t}})$ has finite image. So let $S' \to S$ be the connected étale cover corresponding to the open subgroup

$$\ker(\pi_1(S; \overline{t}) \to \mathrm{GL}(Z^*_{mot,H}(X_{\overline{t}}))) \subset \pi_1(S; \overline{t})$$

and consider the base change

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r] & S. \end{tikzcd}$$

Then, for any $t' \in S'$ above t, and associated geometric point $\overline{t}' : \mathrm{spec}(\Omega) \to S'$ with image $\overline{s} : \mathrm{spec}(\Omega) \xrightarrow{\overline{s}'} S' \to S$, it follows from the universal property of fibre product that $X_{\overline{t}}$ and $X'_{\overline{t}'}$ are isomorphic as Ω-schemes hence $Z_{mot,H}(X_{\overline{t}}) = Z_{mot,H}(X'_{\overline{t}'})$. So, without loss of generality, one may replace S with S' and assume that $\pi_1(S; \overline{t})$ acts trivially on $Z^*_{mot,H}(X_{\overline{t}})$. Hence, in particular, $Z^*_{mot,H}(X_{\overline{t}}) \subset \mathrm{H}_\ell(X_{\overline{t}})^{\pi_1(S_{\overline{k}}; \overline{t})}$. But then, it follows from the above that

$$sp_{t,s}(Z^*_{mot,H}(X_{\overline{t}})) \subset Z^*_{mot,H}(X_{\overline{s}}).$$

For the general case, just observe that, without loss of generality one can replace S by $\mathrm{spec}(\mathcal{O}_{\overline{\{t\}},s}) \to S$. $\square$

Remark 4.8. — More generally, the above shows that, if for $s \in S$ one writes $Z^{ess*}_{mot,H}(X_{\overline{s}}) \subset Z^*_{mot,H}(X_{\overline{s}})$ for the subset of *essentially invariant motivated cycles* that is the subset of all $v \in Z^*_{mot,H}(X_{\overline{s}})$ such that $|\pi_1(S_{\overline{k}}; \overline{s})v| < +\infty$, one always has

$$sp_{t,s}(Z^{ess*}_{mot,H}(X_{\overline{t}})) \subset Z^{ess*}_{mot,H}(X_{\overline{s}})$$

hence, by symmetry

$$sp_{t,s}(Z^{ess*}_{mot,H}(X_{\overline{t}})) = Z^{ess*}_{mot,H}(X_{\overline{s}}).$$

In particular, if η denote the generic point of S and s any point of S, it follows from $Z^*_{mot,H}(X_{\overline{\eta}}) = Z^{ess*}_{mot,H}(X_{\overline{\eta}})$ that one always has

$$sp_{\eta,s}(Z^*_{mot,H}(X_{\overline{\eta}})) = Z^{ess*}_{mot,H}(X_{\overline{s}}) \subset Z^*_{mot,H}(X_{\overline{s}}).$$

4.2.3. Cospecialization of motivated motivic Galois groups. — For any abelian rigid tensor category $\mathcal{T}$ and M in $\mathcal{T}$, define the mixed tensors of bidegree (m, n) on M to be

$$T^{m,n}(M) := M^{\otimes m} \otimes (M^\vee)^{\otimes n} \in \mathcal{T}$$

and for any $I \subset \mathbb{Z}^2_{\geq 0}$, $|I| < +\infty$, set

$$T^I(M) := \bigoplus_{(m,n) \in I} T^{m,n}(M).$$

4.2.3.1. Chevalley's theorem for reductive groups.— Let G be a *reductive* algebraic group over a field K of characteristic 0 and let V be a finite dimensional faithful representation of G over K.

Lemma 4.9. — (Chevalley's theorem [**D82**, Prop. 3.1]) *There exists $I \subset \mathbb{Z}^2_{\geq 0}$, $|I| < +\infty$ and $t \in T^I(V) \setminus \{0\}$ such that*

$$G = \mathrm{Stab}_{\mathrm{GL}(V)}(t).$$

In particular, if

$$Fix(G) := \bigcup_{m,n\geq 0} T^{m,n}(V)^G$$

denotes the set of all mixed tensors on V fixed by G then

$$G = \mathrm{Stab}_{\mathrm{GL}(V)}(Fix(G))$$

and, actually, there exists a finite subset $F \subset Fix(G)$ such that

$$G = \mathrm{Stab}_{\mathrm{GL}(V)}(F).$$

Now, recall that for $M \in M_{mot,H}(k)_K$, the group $G_{mot,H}(M)$ is a reductive algebraic group over K. So, the above considerations can be applied to $G_{mot,H}(M)$, with $V := \mathrm{H}^*(M)$. By definition of $G_{mot,H}(M)$, for any m, $n \geq 0$ and $v \in T^{m,n}(V)^{G_{mot,H}(M)}$, there exists a submotive $S_v^{m,n} \hookrightarrow T^{m,n}(M)$ and a morphism of motives $i_v : \mathbb{I} \to S_v^{m,n}$ such that $\mathrm{H}^*(S_v^{m,n}) = Kv$ and $\mathrm{H}^*(i_v) : \mathrm{H}^*(\mathbb{I}) \to \mathrm{H}^*(S_v^{m,n})$ is the morphism of $G_{mot,H}(M)$-representations $K \to Kv$ sending 1 to v (here, K stands for the trivial representation of $G_{mot,H}(M)$). Hence

$$Fix(G_{mot,H}(M)) = \bigcup_{m,n\geq 0} \mathrm{H}^*(\mathrm{Hom}_{M_{mot,H}(k)_K}(\mathbb{I}, T^{m,n}(M)))$$

so, for $X \in \mathcal{P}(k)$ and $M = \mathfrak{h}_{mot,H}(X)$, one has

$$Fix(G_{mot,H}(X)) = \bigcup_{m,n\geq 0} T^{m,n}(\mathrm{H}^*(X)) \cap Z^*_{mot,H}(X^{m+n}).$$

Hence, from the reductivity of $G_{mot,H}(X)$, one has

$$G_{mot,H}(X) = \mathrm{Stab}_{\mathrm{GL}(\mathrm{H}^*(X))}\left(\bigcup_{m,n\geq 0} T^{m,n}(\mathrm{H}^*(X)) \cap Z^*_{mot,H}(X^{m+n})\right).$$

4.2.3.2. Application to cospecialization of motivated motivic Galois groups.— We now come back to the setting of subsection 4.2.2 that is, S is a scheme geometrically connected, smooth and separated over k with geometric point η and $f : X \to S$ is a smooth proper morphism with geometrically connected fibres. For any s, $t \in S$ such that $s \in \overline{\{t\}}$, we have the following commutative diagram:

$$\begin{tikzcd} Z^{ess*}_{mot,H}(X_{\overline{t}}) \arrow[r, hook] \arrow[d, equal, "sp_{t,s}"'] & Z^*_{mot,H}(X_{\overline{t}}) \arrow[r, hook] & \mathbf{H}^*_\ell(X_{\overline{t}}) \arrow[d, equal, "sp_{t,s}"] \\ Z^{ess*}_{mot,H}(X_{\overline{s}}) \arrow[r, hook] & Z^*_{mot,H}(X_{\overline{s}}) \arrow[r, hook] & \mathbf{H}^*_\ell(X_{\overline{s}}). \end{tikzcd}$$

So, when $Z^{ess*}_{mot,H}(X_{\overline{t}}) = Z^*_{mot,H}(X_{\overline{t}})$ (for instance, when $t = \eta$ is the generic point of S), identifying $\mathrm{H}^*_\ell(X_{\overline{t}})$ and $\mathrm{H}^*_\ell(X_{\overline{s}})$ by means of the specialization isomorphism for ℓ-adic cohomology, we obtain an inclusion of algebraic groups

$$G_{mot,H}(X_{\overline{s}}) \hookrightarrow G_{mot,H}(X_{\overline{t}}).$$

In the next section, we study when this inclusion is an isomorphism.

5. Variation of motivated motivic Galois groups

5.1. Statement and proof of the main theorem. — We now come to the statement and proof of (the ℓ-adic version of) André's deformation theorem for motivated motivic Galois groups, improved by resorting to the finiteness theorem [**CT09b**, Thm. 1.1]. We retain the notation and conventions of paragraph 4.2.3.2. Given a point $s \in S$, since the specialization isomorphism for ℓ-adic cohomology

$$sp_{\eta,s}(= \pi^\alpha_{\eta,s}) : \mathrm{H}^*_\ell(X_{\overline{\eta}}) \tilde{\to} \mathrm{H}^*_\ell(X_{\overline{s}})$$

is compatible with

$$\alpha - \alpha^{-1} : \pi_1(S; \overline{\eta}) \tilde{\to} \pi_1(S; \overline{s}),$$

we will identify below $\mathrm{H}^* := \mathrm{H}^*_\ell(X_{\overline{\eta}}) = \mathrm{H}^*_\ell(X_{\overline{s}})$, $\Pi := \pi_1(S;\overline{\eta}) = \pi_1(S;\overline{s})$ and $\Pi^{geo} := \pi_1(S_{\overline{k}};\overline{\eta}) = \pi_1(S_{\overline{k}};\overline{s})$. Consequently, we will regard $G_{mot,H}(X_{\overline{s}})$ as an algebraic subgroup of $G_{mot,H}(X_{\overline{\eta}})$. If $s\in S$ is a closed point, it induces a quasi-splitting of the fondamental short exact sequence for $\Pi(=\pi_1(S;\overline{s}))$

$$\begin{tikzcd} 1 \arrow[r] & \Pi^{geo} \arrow[r] & \Pi \arrow[r] & \Gamma_k \arrow[r] & 1 \\ & & & \Gamma_{k(s)} \arrow[ul, hook, "\sigma_s"] \arrow[u, hook] & \end{tikzcd}$$

We write Π_s for the image of $\Gamma_{k(s)} \overset{\sigma_s}{\hookrightarrow} \Pi$. Eventually, let G, G^{geo} and G_s denote the images of Π, Π^{geo} and Π_s acting on H^* respectively. Note that these are ℓ-adic Lie groups and that, by definition, the morphism

$$G_s \to G/G^{geo}$$

has open image.

Theorem 5.1. —

1. *$G_{mot,\ell}(X_{\overline{s}}) \subset G_{mot,\ell}(X_{\overline{\eta}})$, $s\in S$ and the following three properties are equivalent.*
   (a) *$G_{mot,\ell}(X_{\overline{s}}) = G_{mot,\ell}(X_{\overline{\eta}})$;*
   (b) *$G_{mot,\ell}(X_{\overline{s}})$ contains an open subgroup of G^{geo};*
   (c) *The specialization morphisms*
   $$sp_{\eta,s} : Z^*_{mot,H}(X^n_{\overline{\eta}}) \hookrightarrow Z^*_{mot,H}(X^n_{\overline{s}}),\ n\geq 1$$
   are isomorphisms.
2. *Assume that k is finitely generated over $\mathbb{Q}$. Let $S_f\subset S$ denote the set of all $s\in S$ such that $G_{mot,\ell}(X_{\overline{s}}) \neq G_{mot,\ell}(X_{\overline{\eta}})$.*
   (a) *For any finite field extension k'/k the set $S_f\cap S(k')$ is thin in $S(k')$.*
   (b) *Assume that S is a **curve**. Then, for any integer $d\geq 1$, the set $S_f^{\leq d} := S_f\cap S^{\leq d}$ is finite.*

Remark 5.2. — Part (1) and (2)(a) of Theorem 5.1 are essentially contained in [**A96**] whereas part (2)(b) of Theorem 5.1 follows from [**CT09b**].

Proof. We first prove assertion (1). For any m, $n\in\mathbb{Z}_{\geq 0}$, write

$$T^{m,n} := T^{m,n}(\mathrm{H}^*);$$
$$(T^{m,n})^{ess} := \{v\in T^{m,n} \mid |\Pi^{geo}v| < +\infty\};$$

and, for any $t\in S$

$$T^{m,n}_{mot,t} := T^{m,n}\cap Z^*_{mot,H}(X^{m+n}_{\overline{t}});$$
$$(T^{m,n}_{mot,t})^{ess} := (T^{m,n})^{ess}\cap T^{m,n}_{mot,t}.$$

Recall that

$$T^{m,n}_{mot,t} = (T^{m,n})^{G_{mot,\ell}(X_{\overline{t}})}$$

and

$$G_{mot,\ell}(X_{\overline{t}}) = \mathrm{Stab}_{\mathrm{GL}(\mathrm{H}^*)}\left(\bigcup_{m,n\geq 0} T^{m,n}_{mot,t}\right),$$

which already shows (a) $\Leftrightarrow$ (c).

As already mentioned in remark 4.8, the set $(T^{m,n}_{mot,t})^{ess}$ is actually independent of t and equal to $T^{m,n}_{mot,\eta}$; so we simply denote it by $(T^{m,n}_{mot})^{ess}$. For simplicity, write

$$\Gamma := G_{mot,\ell}(X_{\overline{\eta}}) = \mathrm{Stab}_{\mathrm{GL}(\mathrm{H}^*)}\left(\bigcup_{m,n\geq 0} (T^{m,n}_{mot})^{ess}\right).$$

Also, as one can always find a finite subset

$$F \subset \bigcup_{m,n\geq 0} (T^{m,n}_{mot})^{ess}$$

such that

$$\Gamma = \mathrm{Stab}_{\mathrm{GL}(\mathrm{H}^*)}(F),$$

the group Γ contains an open subgroup of G^{geo}, whence (a) $\Rightarrow$ (b).

Now, assume that $G_{mot,\ell}(X_{\overline{s}})$ contains an open subgroup of G^{geo} then

$$T^{m,n}_{mot,s} \subset (T^{m,n}_{mot})^{ess} = T^{m,n}_{mot,\eta} \subset T^{m,n}_{mot,s},$$

whence $T^{m,n}_{mot,s} = T^{m,n}_{mot,\eta}$. This shows (b) $\Rightarrow$ (c).

We now prove assertion (2). For any closed point $s \in S$ there exists a finite subset

$$F_s \subset \bigcup_{m,n\geq 0} T^{m,n}_{mot,s}$$

such that $G_{mot,\ell}(X_{\overline{s}}) = \mathrm{Stab}_{\mathrm{GL}(\mathrm{H}^*)}(F_s)$. So, from lemma 4.5, the group $G_{mot,\ell}(X_{\overline{s}})$ contains an open subgroup U_s of G_s. Assume that $s \in S_f$ that is, $G_{mot,\ell}(X_{\overline{s}})$ contains no open subgroup of G^{geo}. Then, in particular, $U_s \cap G^{geo}$ is not open in G^{geo}, which is equivalent to the fact that U_s is not open in G. So, the set where $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$ is contained in the set where G_s is not open in G. The problem thus amounts to studying this second set.

To prove (a), observe that giving $\rho_{f,\eta} : \Gamma_{k(\eta)} \to \mathrm{GL}(\mathrm{H}^*)$ is equivalent to giving a Galois extension $K/k(\eta)$ with group $G := \mathrm{im}(\rho_{f,\eta})$ and unramified over S. The group G_s can then be identified with the decomposition group of s in $K/k(\eta)$. In this setting, a profinite variant of Hilbert irreducibility theorem, due to Serre [**Se89**, Thm. p.149] (and [**Se92**, Prop. 3.3.1]), asserts that there exists a thin subset $\Omega \subset S(k)$ such that for all $s \in S(k) \smallsetminus \Omega$ one has $G_s = G$.

As for (b), observe that the generic ℓ-adic Galois representation $\rho_{f,\eta} : \Gamma_{k(\eta)} \to \mathrm{GL}(\mathrm{H}^*)$ factors through the natural ℓ-adic representation

$$\rho_f : \pi_1(S; \overline{\eta}) \to \mathrm{GL}(\mathrm{H}^*).$$

In this setting and when S is a curve, one can replace the above profinite variant of Hilbert's irreducibility theorem by the following stronger result.

Theorem 5.3. — ([**CT09b**, Thm. 1.1]) *Let k be a finitely generated field of characteristic* 0 *and S a smooth, separated and geometrically connected curve over k. Let $\rho : \pi_1(S) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a ℓ-adic representation such that*

$$(\dagger)\ \ Lie(\rho(\pi_1(S_{\overline{k}})))^{ab} = 0.$$

Then, for any integer $d \geq 1$, the set $S_{\rho,d}$ of all $s \in S^{\leq d}$ such that G_s is not open in G is finite. Furthermore, there exists an integer $B_{\rho,d} \geq 1$ such that $[G : G_s] \leq B_{\rho,d}$ for any closed point $s \in S \smallsetminus S_{\rho,d}$ such that $[k(s) : k] \leq d$.

As the representations $\rho_f : \pi_1(S; \overline{\eta}) \to \mathrm{GL}(\mathrm{H}^*)$ satisfy the condition ($\dagger$) [**CT09a**, Thm. 5.7], one gets (b). (Note that, to prove theorem 1.2, one only uses the first part of theorem 5.3). □

5.2. Applications. — In this section, we assume that k is finitely generated over $\mathbb{Q}$ and that S is a smooth, separated, geometrically connected *curve* over k.

5.2.1. Néron-Severi groups. — For codimension 1 motivated cycles, theorem 5.1 (3) and remark 4.4 (2) imply in particular

Corollary 5.4. — *For any integer $d \geq 1$ the set of all $s \in S^{\leq d}$ such that*

$$\mathrm{rank}(NS(X_{\overline{\eta}})) < \mathrm{rank}(NS(X_{\overline{s}}))$$

is finite.

(See also [**MaP10**] for a p-adic approach of the study of the jumping locus of the Néron-Severi rank). More generally, under the Leftschetz type conjecture, homological and motivated cycles coincide hence for any integer $d \geq 1$ the set of all $s \in S^{\leq d}$ such that

$$\mathrm{rank}(Z_H(X_{\overline{\eta}})) < \mathrm{rank}(Z_H(X_{\overline{s}}))$$

is conjecturally finite.

As observed in [**MaP10**, Prop. 1.13], this also implies that if $A \to S$ is an abelian scheme over S then the set of all $s \in S^{\leq d}$ such that

$$\mathrm{End}(A_{\overline{\eta}}) \hookrightarrow \mathrm{End}(A_{\overline{s}})$$

is not an isomorphism is finite. When k is a number field, this is related to a result of Masser, which shows the following. Assume here that S is affine of dimension ≥ 1 and fix an affine embedding $\phi : S \hookrightarrow \mathbb{A}^n_k$; let h_ϕ denote the associated Weil's logarithmic height on S and write $S^{\leq d,h}$ for the set of all $s \in S^{\leq d}$ such that $h_\phi(s) \leq h$. It is known that $S^{\leq d,h}$ is always finite. Let $\omega(k,d,h)$ (resp. $\omega(A,d,h)$) denote the minimal degree of a polynomial in $k[T_1,\ldots,T_n]$ vanishing on $S^{\leq d,h}$ (resp. on the set of all $s \in S^{\leq d,h}$ such that $\mathrm{End}(A_{\overline{\eta}}) \hookrightarrow \mathrm{End}(A_{\overline{s}})$ is not an isomorphism) but not on S. Then ([**M96**, Thm. p. 459]) there exists constants $C = C(A)$ and $\lambda = \lambda(\dim(A_\eta))$ such that

$$\omega(A,d,h) \leq C\max\{d,h\}^\lambda.$$

As one always has

$$\omega(k,d,h) \geq \exp(ch),$$

for some constant $c = c(k,d)$ and $d \geq 2\dim(S)$, this shows that the set of all $s \in S^{\leq d,h}$ such that $\mathrm{End}(A_{\overline{\eta}}) \hookrightarrow \mathrm{End}(A_{\overline{s}})$ is not an isomorphism is 'sparse' with respect to $S^{d,h}$ when d is fixed and h goes to ∞. When S is a curve, our result improves the one of Masser. However, Masser's result is effective and, if we let $h(d)$ denote the maximal height of an exceptional $s \in S^{\leq d}$, it implies that the number of exceptional $s \in S^{\leq d}$ is bounded from above by the constant $Ch(d)^\lambda$. This yields the question of trying to estimate $h(d)$.

5.2.2. Motivated Tate conjecture. —

5.2.2.1. Statement. —

Let $\mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ denote the category of continuous finite-dimensional $\mathbb{Q}_\ell$-linear representations of Γ_k. Then one gets a factorization

$$\begin{tikzcd} \dot{M}_{mot,H}(k) \arrow[r, "\dot{H}^*"] \arrow[d, "\dot{H}^*"'] & \mathrm{Mod}_{/\mathbb{Q}_\ell} \\ \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k) \arrow[ur, "For"'] & \end{tikzcd}$$

For $X \in \mathcal{P}(k)$, it is natural to formulate the motivated variant of Tate conjecture, namely

Conjecture 5.5. — (Motivated Tate conjecture for X) *The induced functor $\dot{H}^* : \langle \mathfrak{h}_{mot,H}(X)\rangle^\otimes \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ is full.*

This is equivalent to

$$\mathbf{H}^*(X^n_{\overline{k}})^{\Gamma_k} = Z^*_{mot,H}(X^n),\ n \geq 1.$$

But, on the other hand, one has

$$\mathbf{H}^*(X^n_{\overline{k}})^{G_{mot,H}(X)} = Z^*_{mot,H}(X^n),\ n \geq 1.$$

So, the motivated Tate conjecture for X implies that $G_{mot,H}(X)$ contains the Zariski-closure G^z of the image G of $\Gamma_k \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{k}}))$ and, if $\Gamma_k \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{k}}))$ is semisimple, that $G_{mot,H}(X) = G^z$. This motivates

Conjecture 5.6. — (Generalized Motivated Tate conjecture for X) *The two following equivalent conditions hold*

1. *X satisfies the motivated Tate conjecture and* $\Gamma_k \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{k}}))$ *is semisimple;*
2. $G_{mot,H}(X) = G^z$.

In particular, as the dimension of $G_{mot,H}(X)$ is independent of ℓ, the generalized motivated Tate conjecture for X implies that the dimension of G as ℓ-adic Lie group is independent of ℓ as well.

One can of course extend these conjectures from

$$\dot{H}^* : \langle \mathfrak{h}_{mot,H}(X)\rangle^{\otimes} \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$$

to

$$\dot{H}^* : \dot{M}_{mot,H}(k) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$$

to get

Conjecture 5.7. —
- (Motivated Tate conjecture) *The functor* $\dot{H}^* : \dot{M}_{mot,H}(k) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ *is full.*
- (Generalized Motivated Tate conjecture) *The functor* $\dot{H}^* : \dot{M}_{mot,H}(k) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ *is full and for each* $X \in \mathcal{P}(k)$, *the representation* $\Gamma_k \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{k}}))$ *is semisimple.*

Remark 5.8. — The classical generalized Tate conjecture is for homological cycles. It implies the Lefschetz type conjecture hence, in particular, the motivated generalized Tate conjecture [**A04**, Prop. 7.3.2.1.].

5.2.2.2. Dependency on ℓ of the special locus of theorem 5.3. —

Let $f : X \to S$ be a smooth, projective morphism with geometrically connected fibres and write G_ℓ (resp. $G_{\ell,s}$) for the image of $\rho_{f,\ell} : \pi_1(S) \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell))$ (resp. $\rho_{f,\ell,s} : \Gamma_{k(s)} \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{s}}, \mathbb{Q}_\ell))$). Then, from theorem 5.3, for any integer $d \geq 1$ the set $S_{d,\ell}$ of all $s \in S^{\leq d}$ such that $G_{\ell,s}$ is not open in G_ℓ is finite. Also, from theorem 5.1, it contains the set D_d (independent of ℓ) of all $s \in S^{\leq d}$ such that $G_{mot,H}(X_{\overline{s}}) \subsetneq G_{mot,H}(X_{\overline{\eta}})$.

If, in addition, X satisfies the generalized Tate conjecture then $D_d = S_{d,\ell}$ for all prime ℓ. This yields us to formulate the following conjecture on ℓ-independency of the exceptional loci.

Conjecture 5.9. — *With the notation above, for any integer $d \geq 1$, the exceptional locus $S_{d,\ell}$ is independent of ℓ.*

(See also [**CT09a**, Conj. 5.5]).

5.2.2.3. 'Specialization' of the generalized motivated Tate conjecture. —

We retain the notation of the preceding paragraph. Almost nothing is known concerning generalized Tate conjectures (see [**A04**, §7.3] for a brief survey). However, one can make the following observation. Assume that X_η satisfies the generalized motivated Tate conjecture, that is $G_\ell^z = G_{mot,H}(X_\eta)$ and that $G_{mot,H}(X_\eta)$ is connected. Then, for every $s \in S^{\leq d} \setminus S_{\ell,d}$, it follows from theorem 5.3 (and the connectedness of G_ℓ^z) that $G_{\ell,s}^z = G_\ell^z$ and from theorem 5.1 that $Z^*_{mot,H}(X^n_{\overline{\eta}}) = Z^*_{mot,H}(X^n_{\overline{s}})$, $n \geq 1$ hence that

$$Z^*_{mot,H}(X^n_\eta) = Z^*_{mot,H}(X^n_{\overline{\eta}})^{G_\ell} = Z^*_{mot,H}(X^n_{\overline{\eta}})^{G_\ell^z} = Z^*_{mot,H}(X^n_{\overline{s}})^{G_{\ell,s}^z} = Z^*_{mot,H}(X^n_{\overline{s}})^{G_{\ell,s}} = Z^*_{mot,H}(X^n_s),\ n \geq 1$$

which is equivalent to $G_{mot,H}(X_\eta) = G_{mot,H}(X_s)$. In other words,

Proposition 5.10. — *Assume that X_η satisfies the generalized motivated Tate conjecture and that $G_{mot,H}(X_\eta)$ is connected then, for any integer $d \geq 1$, the set of all $s \in S^{\leq d}$ such that X_s does not satisfy the generalized motivated Tate conjecture is finite.*

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