SPECIALIZATION OF MODULO ℓ GALOIS GROUPS IN 1-DIMENSIONAL FAMILIES

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Let k be a finitely generated field of characteristic $p \geq 0$ with absolute Galois group $\Gamma_k := \mathrm{Gal}(k^{sep}|k)$. Let X be a smooth, separated, geometrically connected scheme over k with generic point η and set of closed points $|X|$. Set $\overline{X} := X \times_k \overline{k}$. When X is a curve let $\overline{X}^{cpt}$ denote the smooth compactification of $\overline{X}$, write $\partial\overline{X} := \overline{X}^{cpt} \setminus \overline{X}$ for the divisor at infinity and g_X, γ_X for the genus and gonality of $\overline{X}^{cpt}$respectively. Recall that any $x \in |X|$ produces a quasi-splitting $\sigma_x : \Gamma_{k(x)} \hookrightarrow \pi_1(X)$ of the structural projection $\pi_1(X) \to \Gamma_k$ (here $k(x)$ denotes the residue field at x).
Let $r \in \mathbb{Z}_{\geq 1}$. Let L be an infinite set of primes, $p \notin L$ and for every $\ell \in L$, fix a field F_ℓ of characteristic ℓ and a discrete $F_\ell[\pi_1(X)]$-module H_ℓ with F_ℓ-rank $r_\ell \leq r$ that is, equivalently, a continuous group morphism $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell) \simeq \mathrm{GL}_{r_\ell}(F_\ell)$. Set $G_\ell := \mathrm{im}(\rho_\ell)$, $\overline{G}_\ell := \rho_\ell(\pi_1(\overline{X}))$ and $G_{\ell,x} := \rho_\ell \circ \sigma_x(\Gamma_{k(x)})$. Note that $\overline{G}_\ell \lhd G_\ell$ and $[G_\ell : \overline{G}_\ell G_{\ell,x}] \leq [k(x) : k]$.

The problem we want to address is the description of the local Galois images $G_{\ell,x}$ as x varies in $|X|$. In general, given a family $\mathcal{F}_\ell$ (= the'moduli data') of subgroups of G_ℓ *which does not contain* G_ℓ, one expects that the set $X(\mathcal{F}_\ell)$ of all $x \in |X|$ such that $G_{\ell,x}$ is contained in a groups of $\mathcal{F}_\ell$ is 'small'. This naturally yields to introduce the *abstract modular curve* (AMC for short) associated with ρ_ℓ, $\mathcal{F}_\ell$

$$X_{\mathcal{F}_\ell}^{(\rho_\ell)} := \bigsqcup_{U \in \mathcal{F}_\ell} X_U \to X,$$

where $X_U \to X$ denotes the connected étale cover (defined over the finite extension k_U of k) corresponding to the open subgroup $\rho_\ell^{-1}(U) \subset \pi_1(X)$. It follows from the general formalism of Galois categories that
- $X_U \times_{k_U} \overline{k} \to \overline{X}$ is the connected étale cover corresponding to $\overline{U} := U \cap \overline{G}_\ell \subset \overline{G}_\ell$.
- (Moduli) $x \in |X|$ lifts to a $k(x)$-rational point on X_U if and only if $G_{\ell,x} \subset U$.

Thus $X(\mathcal{F}_\ell)$ is exactly the set of all $x \in |X|$ which lift to a $k(x)$-rational point on $X_{\mathcal{F}_\ell}$. This gives a diophantine reformulation of our original group representation-theoretic problem and already shows that for every finite extension k' of k the set $X(\mathcal{F}_\ell)(k') := X(\mathcal{F}_\ell) \cap X(k')$ is thin. But, of course, one expects much more to hold. For instance, depending on the situation, that for every integer $d \geq 1$ the set $X(\mathcal{F}_\ell)^{\leq d}$ of all $x \in X(\mathcal{F}_\ell)$ such that $[k(x) : k] \leq d$ is not Zariski-dense (hence finite when X is a curve), that $X(\mathcal{F}_\ell)$ is of bounded height, that $X(\mathcal{F}_\ell)$ is not Zariski-dense or even that $X(\mathcal{F}_\ell)$ is empty. In this work, we focus on the weakest of these finiteness properties, namely, we would like to find minimal conditions on the ρ_ℓ, $\mathcal{F}_\ell$, $\ell \in L$ which ensure that $X(\mathcal{F}_\ell)(k)$ is not Zariski-dense. But even this weakened problem seems out of reach in whole generality. However, for curves, one has the remarkable fact that the genus controls the finiteness of its set of rational points. More precisely, recall

Fact (Faltings ($p = 0$), Voloch ($p > 0$)): *Let k be a finitely generated field. Then there exists an integer $g(k) \geq 2$ such that for every curve C over k with $g_C \geq g(k)$ one has $|C(k)| < +\infty$.*

This reduces our original problem to determining under which conditions on the ρ_ℓ, $\mathcal{F}_\ell$, $\ell \in L$ one has

$$g_{X_{\mathcal{F}_\ell}} := \min\{g_{X_U} \mid U \in \mathcal{F}_\ell\} \to +\infty?$$

If one consider the family $\mathcal{F}_{\ell,tot}$ of all subgroups U of G_ℓ such that $\overline{G}_\ell \not\subset \overline{U}$ (that is $X(\mathcal{F}_\ell)(k)$ is the set of all $x \in X(k)$ such that $G_{\ell,x} \subsetneq G_\ell$), it may happen that there exists an integer $B \geq 1$ and infinitely many ℓ such that $1 < [\overline{G}_\ell : \overline{U}] \leq B$ for some $U \in \mathcal{F}_{\ell,tot}$. This is an obstruction to $g_{X_{\mathcal{F}_{\ell,tot}}} \to +\infty$. This obstruction disappears if one replaces $\mathcal{F}_{\ell,tot}$ with the set $\mathcal{F}_{\ell,+}$ of all subgroups U of G_ℓ such that $\overline{G}_\ell^+ \not\subset \overline{U}$. Here, given a subgroup $G \subset \mathrm{GL}(H_\ell)$, we write $G^+ \subset G$ for the subgroup generated by its ℓ-Sylow. Surprisingly, almost no information is lost when replacing $\mathcal{F}_{\ell,tot}$ with $\mathcal{F}_{\ell,+}$; this is a general property of bounded families of continuous F_ℓ-representation of $\pi_1(X)$ for X a curve over a *finitely generated field* k.

Theorem A: *Assume (T): For very $x \in \partial\overline{X}$ there exists an open subgroup U_x of the inertia group at x such that $p \nmid \rho_\ell(U_x))$, $\ell \in L$. Then there exists an open subgroup $\Pi \subset \pi_1(X)$ such that $\rho_\ell(\Pi) = \rho_\ell(\Pi)^+$, $\ell \in L$.*

Date: Oberwolfach: Langlands Correspondence and Constructive Galois Theory (February 2nd-7th, 2014).

In particular, $[\overline{G}_\ell : \overline{G}_\ell^+]$ is bounded from above independently of ℓ. This yields to consider the AMC $X_{\ell,+} := X_{\mathcal{F}_{\ell,+}}$. Also, as one can always construct family $C_\ell \to C$ of connected étale covers with group $\mathbb{Z}/\ell$ and C_ℓ of genus 0, the following perfectness condition is necessary

(P): For every open subgroup $\Pi \subset \pi_1(\overline{X})$ there exists an integer $B_\Pi \geq 1$ such that $|\rho_\ell(\Pi)^{ab}| \leq B_\Pi$, $\ell \in L$.

When $p = 0$, $F_\ell = \mathbb{F}_\ell$ and assuming (P), one already knows (see below) that $g_{X_{\ell,+}} := \min\{\gamma_{X_U} \mid U \in \mathcal{F}_\ell\} \to +\infty$. Thus, *when* $F_\ell = \mathbb{F}_\ell$, which we assume from now on unless otherwise mentionned, the following seems the best possible result.

(Main) Theorem: *Assume (T) and (P). Then* $g_{X_{\ell,+}} \to +\infty$.

Corollary: *Assume (T) and (P). Then there exists an integer $B \geq 1$ such that for $\ell \gg 0$ and all but finitely many $x \in X(k)$ one has $[G_\ell : G_{\ell,x}] \leq B$ (and if $\overline{G}_\ell = \overline{G}_\ell^+$ for $\ell \gg 0$, one can even take $B = 1$).*

The main Theorem and its Corollary apply to families of the form

$$\rho_\ell : \pi_1(X) \to \mathrm{GL}(\mathrm{H}(Y_{\overline{\eta}}, \mathbb{F}_\ell)),\ \ \ell \in L$$

for $Y \to X$ a smooth proper morphism. In that case, (T) and the boundedness condition follow from de Jong's alterations and the fact - due to Gabber - that $\mathrm{H}(Y_{\overline{\eta}}, \mathbb{Z}_\ell)$ is torsion-free for $\ell \gg 0$. After several reductions (including Theorem A, Nori-Serre's approximation theory and specialization of tame fundamental group), (P) essentially reduces to the Weil conjectures.
They also apply to specialization of first cohomology groups and a consequence of them is the following. For $x \in |X|$, consider the restriction map

$$res_x : V_\ell \hookrightarrow \mathrm{H}^1(\pi_1(X), H_\ell) \overset{res_x}{\to} \mathrm{H}^1(k(x), H_\ell),$$

where V_ℓ is a $\mathbb{F}_\ell$-subvector space with $\mathbb{F}_\ell$-rank $s_\ell \leq s$. Assume that the family $\pi_1(X) \to \mathrm{GL}(H_\ell)$, $\ell \in L$ is bounded, satisfies (T), (SS): H_ℓ is a semi-simple $\pi_1(\overline{X})$-module for $\ell \gg 0$ and (I): for every open subgroup $\Pi \subset \pi_1(\overline{X})$, one has $H_\ell^\Pi = 0$ for $\ell \gg 0$. Then, for $\ell \gg 0$ and all but finitely many $x \in X(k)$, the restriction map $res_x : V_\ell \to \mathrm{H}^1(k, H_\ell)$ is injective. In particular, if $A \to X$ is an abelian scheme such that $A_{\overline{\eta}}$ contains no non-trivial isotrivial abelian subvariety, for $\ell \gg 0$ and all but finitely many $x \in X(k)$, the restriction map $A(X)/\ell \overset{Kummer}{\hookrightarrow} \mathrm{H}^1(\pi_1(X), H_\ell) \overset{res_x}{\to} \mathrm{H}^1(k, H_\ell)$ is injective which, as observed by Serre, implies that $A(X) \hookrightarrow A_x(k)$ is injective as well. This is an extension to arbitrary characteristic $p \geq 0$ of the Néron-Silverman specialization theorem. More generally, one can apply this kind of argument to specialization of the reduction modulo ℓ of the first higher ℓ-adic Abel-Jacobi maps.

The strategy of the proof of the main Theorem is to construct a 'universal tensor representation' in order to separate by lines groups in $\mathcal{F}_{\ell,+}$ from $\overline{G}_\ell^+$ for $\ell \gg 0$. This allows to construct an auxilliary bounded family $\tilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\tilde{T}_\ell)$, $\ell \in L$ of continuous $\mathbb{F}_\ell$-representations such that every connected component of $X_{\ell,+}$ dominates a connected component of the AMC $X_{\ell,0}^{\tilde{\rho}_\ell}$ associated to the family $\mathcal{F}_{\ell,0}$ of all stabilizer of lines in $\tilde{T}_\ell$. This reduces the problem to showing that $g_{X_{\ell,0}^{\tilde{\rho}_\ell}} \to +\infty$ which, due to the specific shape of the moduli problem encoded in $\mathcal{F}_{\ell,0}$, is doable. More precisely, the two main intermediate statements are the following.

Theorem B: *There exists a map $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with finite support such that for $\ell \gg 0$ and every $U \in \mathcal{F}_{\ell,+}$ there exists a line $D \subset T^f(H_\ell) := \bigoplus_{m,n \geq 0} (H_\ell^{\oplus m} \otimes (H_\ell^\vee)^{\oplus n})^{\oplus f(m,n)}$ (depending on U, $\overline{G}_\ell^+$) with the property that $\overline{G}_\ell^+ D \neq D$ but $UD = D$.*

Theorem C: *Assume (T) and (I). Then* $g_{X_{\ell,0}} \to +\infty$.

To deduce the main Theorem from Theorem B and Theorem C, just set $\tilde{T}_\ell := T_\ell / T_\ell^{\overline{G}_\ell^+}$, where $T_\ell := T^f(H_\ell)$, $\ell \in L$. Then the family $\tilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\tilde{T}_\ell)$, $\ell \in L$ is bounded and satisfies (T) and (I) as soon as the family ρ_ℓ, $\ell \in L$ satisfies (T) and (P). From Theorem B, every connected component of $X_{\ell,+}$ dominates a connected component of $X_{\ell,0}^{\tilde{\rho}_\ell}$ and, from Theorem C, $g_{X_{\ell,0}^{\tilde{\rho}_\ell}} \to +\infty$.

Theorem B is a variant for finite subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ (r fixed, ℓ varying) of the classical Chevalley theorem for algebraic groups and, unsurprisingly, it relies on approximation theory. Approximation theory[1] associates to a subgroup G of $\mathrm{GL}_r(F_\ell)$ a connected algebraic subgroup $\tilde{G} \hookrightarrow \mathrm{GL}_{r,\mathbf{F}_\ell}$ ($\mathbf{F}_\ell \subset F_\ell$) - the algebraic enveloppe

[1]Here, we consider, again, an arbitrary field F_ℓ of characteristic ℓ

- whose properties reflect those of G and whose rational points approximate well G for $\ell \gg 0$. There are two approaches, one by Nori and Serre, which works only for $F_\ell = \mathbb{F}_\ell$ but is 'functorial' and one by Larsen and Pink, which works for arbitrary fields F_ℓ of characteristic ℓ but is 'not functorial'. The restriction of our results to $\mathbb{F}_\ell$-coefficients comes from the fact that we resort to the former[2], where $\tilde{G} \hookrightarrow \mathrm{GL}_{H_\ell}$ is defined as the algebraic subgroup generated by the one-parameter groups $\mathbb{A}^1_{\mathbb{F}_\ell} \to \mathrm{GL}_{H_\ell}$, $t \to \exp(t\log(g))$ for $g \in G$ of order ℓ. By construction $\tilde{G}$ is connected and generated by its unipotent elements and for $\ell \gg 0$ the following properties hold: (i) $\tilde{G}(\mathbb{F}_\ell)^+ = G^+$, (ii) $\tilde{G}(\mathbb{F}_\ell)/\tilde{G}(\mathbb{F}_\ell)^+$ is abelian of order $\leq 2^{r-1}$, (iii) there exists an abelian subgroup of prime-to-ℓ order $A \subset G$ such that G^+A is normal in G with $[G : G^+A] \leq \delta(r)$. To prove Theorem B, one considers a family $\mathrm{GL}_r \times \mathcal{N}_r \supset \mathcal{U}_r \to \mathcal{N}_r$ over $\mathbb{Z}[\frac{1}{r!}]$ parametrizing exponentially generated subgroups of GL_r and, by induction on dimension and the classical Chevalley theorem, one constructs a universal map $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with the property that every exponentially generated subgroup of GL_{r,F_ℓ} ($\ell > r$) is the stabilizer of a line in $T^f(F_\ell^{\oplus r})$. By approximation theory (property (i) above), f separates - in the sense of Theorem B - U^+ from $\overline{G}_\ell^+$ for $U \in \mathcal{F}_{\ell,+}$ and $\ell \gg 0$. Then, by *ad-hoc* arguments (including properties (i), (iii) above), one adjusts f so that it satisfies exactly the conclusion of Theorem B.

To prove Theorem C, one proves first that, for the 'Galois closure' $\hat{X}_{\ell,0}$ of $X_{\ell,0} \to X$, the ratio $\lambda_{\hat{X}_{\ell,0}}$ ='genus/degree' is bounded from below by an absolute constant $K > 0$. Since the cover $\hat{X}_{\ell,0} \to X$ is Galois, Stichenoth's bound and the Riemann-Hurwitz formula show that this amounts to prove that $g_{\hat{X}_{\ell,0}} \leq 1$ which, in turn, reduces to a combination of group-theoretic arguments involving the classification of finite subgroups of automorphism groups of genus ≤ 1 curves, Theorem A and assumptions (T), (I). One then shows by Riemann-Hurwitz formula that $(\lambda_{\hat{X}_{\ell,0}} - \lambda_{X_{\ell,0}}) \to 0$. Here, the main difficulty is to control the length of the ramification filtration and the size of the ramification terms. Using assumption (T) and Theorem A, this eventually amounts to the following 'non-concentration' estimate: *there exists a sequence* $\epsilon(\ell)$, $\ell \in L$ *such that* $\epsilon(\ell)\ln(\ell) \to 0$ *and for every* $\mathbb{F}_\ell$*-vector subspace* $N_\ell \subset H_\ell$ *and* $0 \neq v_\ell \in H_\ell$, *if* $\overline{G}_\ell^+ v_\ell \not\subset N_\ell$ *then* $\frac{|\overline{G}_\ell^+ v_\ell \cap N_\ell|}{|\overline{G}_\ell^+ v_\ell|} \leq \epsilon(\ell)$, which, again, is proved using Nori's algebraic enveloppe.

To conclude, let us mention two further possible directions, still in the case where the base scheme X is a curve.

Arbitrary F_ℓ-coefficients: By an easy specialization argument, one can always assume that $F_\ell \subset \overline{\mathbb{F}}_\ell$. Using Theorem A and Larsen-Pink's approximation theory, one can reduces our main theorem for arbitrary F_ℓ-coefficients to a deep[3] group-theoretical result by Guralnick. However, we know no counter-example to the following conjectural statement:

Conjecture: *Let* $\rho_\ell : \pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$, $\ell \in L$ *be a bounded family of continuous representations satisfying (T), (P). Then there exists an integer* $s \geq 1$ *such that* $\rho_\ell|_{\pi_1(\overline{X})} : \pi_1(\overline{X}) \to \mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$ *is* $\mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$*-conjugate to a representation with coefficients in* $\mathbb{F}_{\ell^s}$, $\ell \in L$.

This conjecture is in the spirit of the ℓ-independence conjectures/statements for families of automorphic representations but the compatibility condition (P) and the arithmetic input that the representations are not only representations of $\pi_1(\overline{X})$ but also of $\pi_1(X)$ are weaker than the standard ℓ-independency and purity assumptions about the characteristic polynomials of Frobenii. One could try and tackle first this conjecture when k is finite and for $\overline{\mathbb{Q}}_\ell$-coefficients or replacing (P) by the assumption that for every $x \in |X|$ the characteristic polynomial of the Frobenius at x is the reduction modulo ℓ of a polynomial independent of ℓ, with coefficients in the completion of a finite extension of $\mathbb{Q}$ independent of x and pure.

Gonality: When $p = 0$ and $F_\ell = \mathbb{F}_\ell$, it was shown by Ellenberg, Hall and Kovalski that Theorem A combined with Cayley-Schreier graphs and complex-analytic technics implies that, under (P), $\gamma_{X_{\ell,+}} \to +\infty$. The generalization of this result to arbitrary F_ℓ-coefficients seems to be conditioned by the extension of the Cayley-Schreier graphs part of the proof (due to Pyber and Szabo). One can also ask for similar results when $p > 0$. Akio Tamagawa and I have obtained some partial positive results in this direction when $F_\ell = \mathbb{F}_\ell$ by purely algebraic methods[4].

[2]Though it is possible that, resorting to much more elaborate group-theoretic arguments, our approach extends to arbitrary F_ℓ-coefficients *via* Larsen-Pink's approximation theory.

[3]involving satellite theorems of the classification like Aschbasher's theorem for maximal subgroups of finite classical groups.

[4]Gonality may decrease under specialization and there is *a priori* no hope to reduce the study of gonality when $p > 0$ to a characteristic 0 setting.