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The ℓ-primary torsion conjecture for abelian surfaces with real multiplication

By

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Abstract

We prove that the Bombieri-Lang conjecture implies the ℓ-primary torsion conjecture for abelian surfaces with real multiplication.

§ 1. Introduction

Let $\mathfrak{O}$ be the ring of integers of a quadratic real field extension of $\mathbb{Q}$ with discriminant D and let $X(n)$ denote the coarse moduli scheme for the stack of polarized abelian surfaces with real multiplication by $\mathfrak{O}$ and μ_n-level structure (See Subsection 2.2 for precise definitions). The irreducible components of the $X(n)$ are normal, separated, geometrically connected surfaces (defined over a number field depending on the involved data) and it is known that the smallest integer $n(D)$ such that $X(n(D))$ is of general type is 1 except for finitely many cases, where it is 2 or 3 (Theorem 2.2).

The Bombieri-Lang conjecture (Conjecture 2.1) predicts that if X is a surface of general type over a number field k then the set of k-rational points $X(k)$ is not Zariski-dense in X. Thus, conjecturally, polarized abelian surfaces with real multiplication by $\mathfrak{O}$ and $\mu_{n(D)}$-level structure defined over k are parametrized by a closed subscheme $S \hookrightarrow X(n(D))$ whose irreducible components have dimension ≤ 1.

For such a scheme S and any abelian scheme $A \to S$, works of A. Tamagawa and the author show that, for any integer $d \geq 1$ and prime ℓ the ℓ-primary rational torsion is uniformly bounded in the fibres at closed points $s \in S$ whose residue degree $[k(s) : k]$ is $\leq d$ (Theorem 2.4). So, assuming the Bombieri-Lang conjecture, one can expect that for any number field k and prime ℓ the k-rational ℓ-primary torsion of polarized

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abelian surfaces with real multiplication by $\mathfrak{O}$ and $\mu_{n(D)}$-level structure defined over k is uniformly bounded.

The only difficulty stems from the fact that, for $n(D) \leq 2$, the coarse moduli scheme $X(n)$ is not fine hence there is no universal abelian surface with real multiplication by $\mathfrak{O}$ and $\mu_{n(D)}$-level structure over it, to which one could apply directly the uniform boundedness result of A. Tamagawa and the author. However, this can be overcome by elementary cohomological and rigidification technics.

The main result we obtain in this short note is the following.

Theorem 1.1. *Assume that the Bombieri-Lang conjecture holds. Then for any number field k and prime ℓ there exists an integer $N := N(\mathfrak{O}, k, \ell)$ such that, for any polarized abelian surface A with real multiplication by $\mathfrak{O}$ and $\mu_{n(D)}$-level structure defined over k:*

$$|A(k)[\ell^\infty]| \leq \ell^N.$$

In Section 2 we gather the several ingredients involved in the proof of Theorem 1.1. The proof itself is carried out in Section 3.

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§ 2. Preliminaries

Given a scheme S over a field k and an integer $d \geq 1$, we will write

$$S^{\leq d} := \{s \in S \mid [k(s) : k] \leq d\},$$

where $k(s)$ denotes the residue field of S at s.

§ 2.1. Bombieri-Lang conjecture

Given a field k of characteristic 0, let $\mathcal{P}(k)$ (resp. $\mathcal{B}(k)$) denote the category of all schemes smooth, projective and geometrically connected over k (resp. of all normal schemes separated, of finite type and geometrically connected over k). Also, let $\sim$ denote the birational equivalence on $\mathcal{P}(k)$ (resp. $\mathcal{B}(k)$), it follows from Nagata's compactification theorem and Hironaka's desingularization theorem that the canonical map

$$\mathcal{P}(k)/\sim \to \mathcal{B}(k)/\sim$$

is bijective (*e.g.* [1, §2.1]). Thus any birational invariant defined on $\mathcal{P}(k)$ naturally extends to $\mathcal{B}(k)$; this is in particular the case for Kodaira dimension and we say that $S \in \mathcal{B}(k)$ is of general type if its Kodaira dimension $\kappa(S)$ is equal to its dimension $\dim(S)$. With this convention, we can formulate the so-called Bombieri-Lang conjecture

Conjecture 2.1. *Let k be a number field and let $S \in \mathcal{B}(k)$ be a surface of general type. Then the set of k-rational points $S(k)$ is not Zariski-dense in S.*

Conjecture 2.1 was stated by E. Bombieri for surfaces and generalized by S. Lang for schemes $S \in \mathcal{B}(k)$ of arbitrary dimension under the following more precise form. Let k be a number field and let $S \in \mathcal{B}(k)$ be of general type. Then there existe a closed subscheme $Z \hookrightarrow S$, $Z \neq S$ such that for any finite field extension $k \hookrightarrow k'$ the set $S(k') \setminus Z(k')$ is finite.

The Lang conjecture (and even the Bombieri-Lang conjecture) is widely open. The most striking result is that it holds for subvarieties of abelian varieties [7].

§ 2.2. The stack of abelian surfaces with real multiplication

Let $\mathbb{Q} \hookrightarrow K$ be a degree g totally real field extension and let $\mathfrak{O}$ denote the ring of integers of K. Let $\mathfrak{C}$ and $\mathfrak{C}^+$ denote the class group and strict class group of $\mathfrak{O}$, respectively. Fix a system $\mathcal{J}_1, \dots, \mathcal{J}_{h^+}$ of fractional ideals of $\mathfrak{O}$, which, endowed with their natural notion of positivity, form a complete system of representatives of $\mathfrak{C}^+$. For $\mathcal{J}$ one of the $\mathcal{J}_1, \dots, \mathcal{J}_{h^+}$, a g-dimensional $\mathcal{J}$-polarized abelian scheme with real multiplication by $\mathfrak{O}$ is a triple (A, ι, λ), where:
- $A \to S$ is an abelian scheme of relative dimension g;
- $\iota : \mathfrak{O} \hookrightarrow \mathrm{End}_S(A)$ is a ring homomorphism;
- $\lambda : (\mathcal{M}_A, \mathcal{M}_A^+) \tilde{\to} (\mathcal{J}, \mathcal{J}^+)$ is a polarization that is, an $\mathfrak{O}$-linear isomorphism of étale sheaves between the invertible $\mathfrak{O}$-module $\mathcal{M}_A$ of all symmetric $\mathfrak{O}$-linear homomorphisms from A to $A^\vee$ and $\mathcal{J}$, identifying the positive cone of polarizations $\mathcal{M}_A^+$ with the totally real elements $\mathcal{J}^+$ in $\mathcal{J}$.

Let $\mathbb{Q}(\mathcal{J})$ denote the field of definition of $\mathcal{J}$ and $\mathcal{S}_{\mathfrak{O},g,\mathcal{J}} \to \mathrm{spec}(\mathbb{Q}(\mathcal{J}))$ the étale stack of g-dimensional $\mathcal{J}$-polarized abelian schemes with real multiplication by $\mathfrak{O}$. Set

$$\mathcal{S}_{\mathfrak{O},g} := \bigsqcup_{1 \leq i \leq h^+} \mathcal{S}_{\mathfrak{O},g,\mathcal{J}_i}.$$

One can furthermore endow $\mathcal{J}$-polarized abelian schemes with real multiplication by $\mathfrak{O}$ with a μ_n level-structure, that is an injective $\mathfrak{O}$-linear homomorphism of étale sheaves:

$$\epsilon : \mu_n \otimes_{\mathbb{Z}} \mathfrak{D}^{-1} \hookrightarrow A,$$

where $\mathfrak{D}$ is the different of $\mathbb{Q} \hookrightarrow K$. Let $\mathcal{S}_{\mathfrak{O},g,\mathcal{J}}(n)$ and $\mathcal{S}_{\mathfrak{O},g}(n)$ denote the corresponding stacks.

The stacks $\mathcal{S}_{\mathfrak{O},g}(n)$, $n \geq 0$ are Deligne-Mumford stacks locally of finite presentation and with finite inertia; let $c : \mathcal{S}_{\mathfrak{O},g,\mathcal{J}}(n) \to S_{\mathfrak{O},g,\mathcal{J}}(n)$ denote their coarse moduli schemes. For $n \geq 3$, $c : \mathcal{S}_{\mathfrak{O},g,\mathcal{J}}(n) \to S_{\mathfrak{O},g,\mathcal{J}}(n)$ is a fine moduli scheme. See [12], [14, Chap. X], [6] for more details.

For $g = 2$, the coarse moduli schemes $S_{\mathfrak{O},2,\mathcal{J}}(n)$ are 2-dimensional normal schemes, separated and geometrically connected over $\mathbb{Q}(\mathcal{J})$. The analytic space associated with $S_{\mathfrak{O},2,\mathcal{J}}(n) \times_{\mathbb{Q}(\mathcal{J})} \mathbb{C}$ can be identified with $\mathrm{PSL}_2^n(\mathfrak{O} \oplus \mathcal{J}) \setminus \mathfrak{H}^2$. Here

$$\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$$

and $\mathrm{PSL}_2^n(\mathfrak{O} \oplus \mathcal{J}) = \mathrm{SL}_2^n(\mathfrak{O} \oplus \mathcal{J})/\{\pm 1\}$, where $\mathrm{SL}_2^n(\mathfrak{O} \oplus \mathcal{J})$ is the subgroup of $\mathrm{SL}_2(k)$ of matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a \equiv d \bmod \mathcal{J}$, $c \in n\mathcal{J}$ and $b \in n\mathcal{J}^{-1}$. Using this analytic description, one can show the following [9], [10], [14, Thm. VII.3.3 and VII.3.4].

Theorem 2.2. *For any real quadratic field extension $\mathbb{Q} \hookrightarrow K$ with ring of integers $\mathfrak{O}$ and discriminant D one has*

- $S_{\mathfrak{O},2,\mathcal{J}}$ *is of general type except if* $D = 5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44, 53, 56, 57, 60, 61, 65, 69, 73, 77, 85, 88, 92, 93, 105, 120, 140, 165$ *(we will then say that D is exceptional).*

- $S_{\mathfrak{O},2,\mathcal{J}}(2)$ *is of general type except if $D = 5,\ 8,\ 12$, in which case $S_{\mathfrak{O},2,\mathcal{J}}(3)$ is of general type.*

We set $n(D) = 1$ if D is non-exceptional, $n(D) = 2$ if D is exceptional $\neq 5,\ 8,\ 12$ and $n(D) = 3$ if $D = 5,\ 8,\ 12$.

§ 2.3. Strong uniform boundedness of ℓ-primary torsion in 1-dimensional families of abelian varieties

The Mordell-Weil theorem asserts that for any number field k and abelian variety A over k the (abelian) group $A(k)$ of k-rational points is finitely generated. In particular its torsion subgroup $A(k)_{tors}$ is finite. Given a prime ℓ, the ℓ-primary torsion conjecture is a uniform conjectural form of Mordell-Weil theorem for the ℓ-Sylow subgroups $A(k)[\ell^\infty]$ of $A(k)_{tors}$.

Conjecture 2.3 (ℓ-primary torsion conjecture). *For any integer $\delta \geq 1$*

- **Weak form:** *For any number field k and prime ℓ there exists an integer $N := N(\delta, \ell, k)$ depending only on δ, ℓ and k such that for any δ-dimensional abelian variety A over k one has*

$$|A(k)[\ell^\infty]| \leq \ell^N.$$

- **Strong form:** *For any integer $d \geq 1$ and prime ℓ there exists an integer $N := N(\delta, \ell, d)$ depending only on δ, ℓ and d such that for any number field k with $[k : \mathbb{Q}] \leq d$ and δ-dimensional abelian variety A over k one has*

$$|A(k)[\ell^\infty]| \leq \ell^N.$$

The $\delta = 1$ case of the strong ℓ-primary torsion conjecture essentially follows from the fact that for any prime ℓ the gonality of the modular curves $Y_1(\ell^n)$ (classifying elliptic curves with a torsion point of order exactly ℓ^n) goes to $+\infty$ with n. Indeed, combined with Faltings-Lang-Frey [8], this implies that for any integer $d \geq 1$ the sets $Y_1(\ell^n)^{\leq d}$ are finite for n large enough (depending on d). But if $Y_1(\ell^n)^{\leq d}$ is non-empty for all $n \geq 0$, it follows from the usual compactness argument that

$$\varprojlim Y_1(\ell^n)^{\leq d}$$

is non-empty as well, which contradicts Mordell-Weil theorem.

For $\delta \geq 2$, the ℓ-primary torsion conjecture is widely open. However, the following higher dimensional relative variant of the strong ℓ-primary torsion conjecture holds [2], [3], [4].

Theorem 2.4. *Let k be a number field, S a normal curve, separated, of finite type and geometrically connected over k and $A \to S$ an abelian scheme. Then, for any integer $d \geq 1$ and prime ℓ there exists an integer $N := N(A, \ell, d)$ depending only on A, ℓ and d such that for any $s \in S^{\leq d}$ on has*

$$|A_s(k(s))[\ell^\infty]| \leq \ell^N.$$

When applied to the 'universal elliptic scheme'

$$E \equiv y^2 + xy - x^3 + \frac{36}{j - 1728}x + \frac{1}{j - 1728} \to \mathbb{P}^1_j \setminus \{0, 1728, +\infty\}$$

theorem 2.4 yields the 1-dimensional case of the ℓ-primary torsion conjecture.

§ 3. Proof of theorem 1.1

We now proceed to the proof of Theorem 1.1 by combining the ingredients gathered in Section 2.

Fix a finite extension k of $\mathbb{Q}(\mathcal{J})$ and, for simplicity, write n for $n(D)$ and $c : \mathcal{X}(n) \to X(n)$ for the pullback of $c : \mathcal{S}_{\mathfrak{O},2,\mathcal{J}}(n) \to S_{\mathfrak{O},2,\mathcal{J}}(n)$ *via* $\mathrm{spec}(k) \to \mathrm{spec}(\mathbb{Q}(\mathcal{J}))$. In any case, the canonical morphism of stacks $F : \mathcal{X}(6) \to \mathcal{X}(n)$ induces a commutative diagram

$$\begin{array}{ccc}\mathcal{X}(6) & \xrightarrow{F} & \mathcal{X}(n)\\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c}\\ X(6) & \xrightarrow[f]{} & X(n),\end{array}$$

where the bottom arrow $f : X(6) \to X(n)$ is a finite surjective morphism of surfaces. By Theorem 2.2 (and assuming the Bombieri-Lang conjecture), the irreducible components of the Zariski closure $\mathfrak{X}(n)$ of $X(n)(k)$ in $X(n)$ have dimension ≤ 1. Since $\mathfrak{X}(n)$ has finitely many irreducible components (being of finite type over k), it is enough to bound uniformly the ℓ-primary torsion of the abelian surfaces associated with points in $\mathcal{X}(n)(k)$ lying over $\mathfrak{I}$ for each irreducible component $\mathfrak{I}$ of $\mathfrak{X}(n)$. So, without loss of generality, we may assume that $\mathfrak{X}(n)$ is irreducible.

§ 3.1. $\dim(\mathfrak{X}(n)) = 0$ (*i.e.* $\mathfrak{X}(n)$ is a singleton)

Since $c : \mathcal{X}(n) \to X(n)$ is a coarse moduli scheme, it induces a bijection

$$\mathcal{X}(n)(\overline{k})/\sim \tilde{\to} X(n)(\overline{k}).$$

In particular, any two $A_i (= (A_i, \iota_i, \lambda_i, \epsilon_i)) \in \mathcal{X}(n)(k)$, $i = 1,\ 2$ become isomorphic in $\mathcal{X}(n)(\overline{k})$. The conclusion then follows from Lemma 3.1 below applied to the forgetful functor $\mathcal{X}(n) \to \mathcal{A}_2$ and the fact that for any abelian variety A over a number field k and integer $d \geq 1$ the set

$$A^{\leq d} \cap A(\overline{k})_{tors}$$

is finite [5, Lemma 3.19].

Lemma 3.1. *Let S be any scheme, let $\mathcal{A}_\delta \to S$ denote the étale stack of δ-dimensional abelian varieties over S and let $\mathcal{X} \to S$ be an algebraic stack with quasi-finite inertia*[1]*. Then for any morphism $A : \mathcal{X} \to \mathcal{A}_\delta$ of fibered categories, there exists an integer $\Delta := \Delta(\delta) \geq 1$ such that for any S-field k and $x,\ ;y \in \mathcal{X}(k)$ such that $x|_{\overline{k}}$ and $y|_{\overline{k}}$ are isomorphic in $\mathcal{X}(\overline{k})$, there exists a finite extension $k \hookrightarrow k_{x,y}$ with degree $[k_{x,y} : k] \leq \Delta$ such that $A(x)|_{k_{x,y}}$ and $A(y)|_{k_{x,y}}$ become isomorphic in $\mathcal{A}_\delta(k_{x,y})$.*

Here, given a morphisme of S-schemes $V \to U$ and an object $x \in \mathcal{X}(U)$, we write $x|_V$ for the image of x *via* the canonical pull-back functor $\mathcal{X}(U) \to \mathcal{X}(V)$. For instance, with this notation, $A(x)|_{k_{x,y}}$ is nothing but $A(x) \times_k k_{x,y}$.

[1] Recall that the inertia stack of $\mathcal{X}$ is defined to be $\mathcal{I}_S(\mathcal{X}) := \mathcal{X} \times_{\mathcal{X}\times_S\mathcal{X}} \mathcal{X} \to \mathcal{X}$. Hence, in particular, it follows from the definition of the fibre product in the 2-category of S-groupoids that for any geometric point $x \in \mathcal{X}(\Omega)$ the fibre $\mathcal{I}_S(\mathcal{X})_x$ can be identified with $\mathrm{Aut}(x)$. The assumption that $\mathcal{X}$ has finite inertia is only there to ensure that the automorphism group of geometric points is finite.

Proof. Recall first that there exists an integer $B(\delta) \geq 1$ such that for any field k and δ-dimensional abelian variety A over k, any finite subgroup G of $\mathrm{Aut}(A)$ has order $\leq B(\delta)$. Indeed, let ℓ be a prime different from the characteristic of k. Then, since $\mathrm{End}(A)$ acts faithfully on $T_\ell(A)$ [11, Lemma 12.2], G is a finite subgroup of $\mathrm{GL}_{2\delta}(\mathbb{Z}_\ell)$. But such subgroups have order bounded by a constant depending only on δ and ℓ [13, LG 4.27, Thm. 5]. The conclusion thus follows from the fact that we can take $\ell = 2$ or 3.

Given $x \in \mathcal{X}(k)$, write $\mathrm{Tw}(x/k)$ for the set of isomorphism classes of twists of x over k that is pairs (x', ϕ), where $x' \in \mathcal{X}(k)$ and $\phi : x|_{\overline{k}} \tilde{\to} x'|_{\overline{k}}$ is an isomorphism in $\mathcal{X}(\overline{k})$. One has a commutative diagram of pointed sets:

$$\begin{tikzcd}
\mathrm{Tw}(A(x)/k) \arrow[rr, "\beta"] & & \mathrm{H}^1(\Gamma_k, \mathrm{Aut}(A(x)|_{\overline{k}})) & \\
& & & \mathrm{H}^1(\Gamma_k, A(\mathrm{Aut}(x|_{\overline{k}}))) \arrow[ul] \\
\mathrm{Tw}(x/k) \arrow[uu, "\alpha"] \arrow[rr, "a"'] & & \mathrm{H}^1(\Gamma_k, \mathrm{Aut}(x|_{\overline{k}})), \arrow[ur, "b"'] &
\end{tikzcd}$$

where β is an isomorphism of pointed sets. By the quasi-finite inertia assumption, the group $\mathrm{Aut}(x|_{\overline{k}})$ is finite so $A(\mathrm{Aut}(x|_{\overline{k}}))$ is a finite subgroup of $\mathrm{Aut}(A(x|_{\overline{k}}))$ and, by the observation above, has order $\leq B(\delta)$. Set $k_x := \overline{k}^{\ker(\varphi)}$, where $\varphi : \Gamma_k \to \mathfrak{S}(A(\mathrm{Aut}(x|_{\overline{k}})))$ is the permutation representation associated with the Γ_k-set $A(\mathrm{Aut}(x|_{\overline{k}}))$. Then $[k_x : k] \leq B(\delta)!$ and $A(\mathrm{Aut}(x|_{\overline{k}}))$ is a trivial Γ_{k_x}-module so $\mathrm{H}^1(k_x, A(\mathrm{Aut}(x|_{\overline{k}})))$ can be identified with the set of orbits of

$$\mathrm{Hom}_{Group}(\Gamma_{k_x}, A(\mathrm{Aut}(x|_{\overline{k}})))$$

under the right action by inner automorphisms of $A(\mathrm{Aut}(x|_{\overline{k}}))$. Now, given $y \in \mathrm{Tw}(x/k)$, set

$$k_{x,y} := k_x^{\ker(ba(y)|_{k_x})}$$

then $[k_{x,y} : k] \leq B(\delta)!B(\delta)$ and $ba(y)|_{k_{x,y}}$ is trivial. Since β is an isomorphism of pointed sets and the above diagram commutes, this implies that $\beta\alpha(y)|_{k_{x,y}}$ is trivial as well. So $\Delta(\delta) = B(\delta)!B(\delta)$ works. $\square$

§ 3.2. $\dim(\mathfrak{X}(n)) = 1$

Let $\mathfrak{X}(6)$ denote the pullback of $\mathfrak{X}(n) \hookrightarrow X(n)$ *via* $f : X(6) \to X(n)$. Up to considering separately the (finitely many) irreducible components of $\mathfrak{X}(6)$, we may assume that $\mathfrak{X}(6)$ is irreducible. From Subsection 3.1, we may freely remove finitely many points from $\mathfrak{X}(n)$ hence also assume that $\mathfrak{X}(6)$ is a curve smooth (separated and of finite type) over k. Eventually, we may freely replace k by any finite extension $k \hookrightarrow k'$ hence assume that $\mathfrak{X}(6)$ is geometrically connected over k.

As $c : \mathcal{X}(6) \to X(6)$ is a fine moduli scheme, there is a section

$$\begin{tikzcd} \mathcal{X}(6) \arrow[r, "c"] & X(6) \arrow[l, bend left] \end{tikzcd}$$

corresponding to a $\mathcal{J}$-polarized abelian scheme with real multiplication by $\mathfrak{O}$ and μ_6-level structure $(A, \iota, \lambda, \epsilon) \to X(6)$. For simplicity, write again

$$A := A \times_{X(6)} \mathfrak{X}(6) \to \mathfrak{X}(6)$$

for the pullback of $A \to X(6)$ *via* $\mathfrak{X}(6) \hookrightarrow X(6)$. Then, from Theorem 2.4, for any prime ℓ and integer $d \geq 1$ there exists an integer $N := N(A, \ell, d)$ such that for any $x \in \mathfrak{X}(6)^{\leq d}$ one has

$$|A_x(k(x))[\ell^\infty]| \leq \ell^N.$$

This is true, in particular, for

$$d = e\Delta(2),$$

where e denotes the degree of $f : \mathfrak{X}(6) \to \mathfrak{X}(n)$. But for any $A' = (A', \iota, \lambda, \epsilon) \in \mathcal{X}(n)(k)$ above some $x \in \mathfrak{X}(n)(k)$ and any $x' \in f^{-1}(x)$, $A' \times_k \overline{k}$ and $F(A_{x'} \times_{k(x')} \overline{k})$ become isomorphic in $\mathcal{X}(n)(\overline{k})$. So the conclusion, again, follows from Lemma 3.1 applied to the forgetful functor $\mathcal{X}(n) \to \mathcal{A}_2$.

Remark 3.2.

1. The proof given here applies as it is to other similar situations. For instance if $\mathcal{A}_{\delta,\gamma}$ denotes the stack of δ-dimensional abelian varieties endowed with a degree γ^2 polarization and $c : \mathcal{A}_{\delta,\gamma} \to A_{\delta,\gamma}$ its coarse moduli scheme then, for any number field k and 2-dimensional subscheme $S \hookrightarrow A_{\delta,\gamma}$ such that $S \in \mathcal{B}(k)$ is of general type, the ℓ-primary torsion of the abelian varieties associated with points in $\mathcal{A}_{\delta,\gamma}(k)$ lying over S is uniformly bounded.

2. Even in the fine moduli situation, that is $A \to S$ is an abelian scheme with $S \in \mathcal{B}(k)$ of general type, our proof *does not* show that the Bombieri-Lang conjecture implies that for integer $d \geq 2$ and prime ℓ the ℓ-primary $k(s)$-rational torsion in the fibres A_s, $s \in S^{\leq d}$ is uniformly bounded. The obstruction comes from the fact that the set $S^{\leq d}$ might be Zariski dense in S even if for each finite extension $k \hookrightarrow k'$ with degree $[k' : k] \leq d$ the set $S(k')$ is not.

Remark 3.3 (Adding structures). One can show [1, Lemma 4.11] that the ℓ-primary torsion conjecture is equivalent to

Conjecture 3.4. *Let k be a number field, $S \in \mathcal{B}(k)$, $A \to S$ an abelian scheme and ℓ a prime, then there exists an integer $N := N(A, \ell)$ such that $A_s[\ell^\infty](k) \subset A_s[\ell^N]$ for all $s \in S(k)$.*

When S is a surface, combining the technics of [1], [4, Lemma 3.5] and assuming the Lang conjecture for surfaces, one can show that conjecture 3.4 holds *provided*

- for all $v \in T_\ell(A_\eta) \smallsetminus \ell T_\ell(A_\eta)$ the dimension of $\pi_1(S)v$ as a ℓ-adic analytic space is ≥ 3;
- $A_\eta[\ell^{n_\ell}](k(\eta)) = A_\eta[\ell^{n_\ell}]$ (in other words, $A \to S$ is endowed with a full-level-ℓ^{n_ℓ}-structure defined over S) with

$$n_\ell = \begin{array}{l} 1 \text{ if } \ell \geq 5; \\ 2 \text{ if } \ell = 3; \\ 3 \text{ if } \ell = 2. \end{array}$$

This shows that adding structures (especially full-level structures) should force the ℓ-primary torsion conjecture to hold; a main difficulty is thus to remove the hypothesis that small full-level structures are defined over the base field.

In Theorem 1.1, we add two kind of structures: real multiplication, so that the parameter space be a surface (recall that $A_{g,1}$ is three-dimensional) and $\mu_{n(D)}$-level structures so that the parameter space be of general type. In view of the above considerations and though it makes the proof more complicated (for $n(D) \geq 3$, the difficulty stemming from the fact that $c : \mathcal{S}_{\mathfrak{O},2,\mathcal{J}}(n) \to S_{\mathfrak{O},2,\mathcal{J}}(n)$ is not a fine moduli scheme disappears), it is important that $n(D)$ be as small as possible. The fact that we can choose $n(D) = 1$ for almost all D and in any case $n(D) \leq 3$ (so, in particular, $n(D)$ independent of ℓ) relies on the highly non-trivial classification theorem 2.2.

References

[1] Cadoret, A., Kodaira dimension of abstract modular surfaces, preprint (2011).

[2] Cadoret, A. and Tamagawa, A., A uniform boundedness of p-primary torsion on abelian schemes, preprint (2008).

[3] A. CADORET and A. TAMAGAWA, *A uniform open image theorem for ℓ-adic representations I*, preprint 2009.

[4] Cadoret, A. and Tamagawa, A., A uniform open image theorem for ℓ-adic representations II, preprint (2009).

[5] Cadoret, A. and Tamagawa, A., On a weak variant of the geometric torsion conjecture, preprint (2010).

[6] Deligne, P. and Pappas, G., Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant, *Compositio Math.*, **90** (1994), 59–79.

[7] Faltings, G., The general case of Lang's conjecture, *Barsotti's symposium in algebraic geometry*, Academic Press, (1994), 175–182.

[8] Frey, G., Curves with infinitely many points of finite degree, *Israel Journal of Math.* **85**, (1994), 79–83.

[9] Hirzebruch, F. and Van de Ven, A., Hilbert modular surfaces and the classification of algebraic surfaces, *Inventiones Math.*, **23** No 1, (1974), 1–29.

[10] Hirzebruch, F. and Zagier, D., Classification of Hilbert modular surfaces, *Complex analysis and algebraic geometry*, Iwanami Shoten and Cambridge Univ. Press, (1977), 43–77.

[11] Milne, J.S., Abelian varieties, *Arithmetic geometry* (Cornell/Silverman, eds.), Springer-Verlag, (1986), 103–150.

[12] Rapoport, M., Compactifications de l'espace de modules de Hilbert-Blumenthal, *Compositio Math.* **36**, (1978), 255–335.

[13] Serre, J.-P., Lie algebras and Lie groups, W.A. Benjamin, (1965).

[14] Van der Geer, G., Hilbert Modular Surfaces, *E.M.G.* **16**, Springer, (1987).