On uniform boundedness of arithmetico-geometric invariants in one-dimensional families

K3 surfaces and Galois representations
Shepperton, May, 2nd-4th, 2018

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Setting

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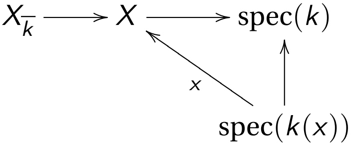

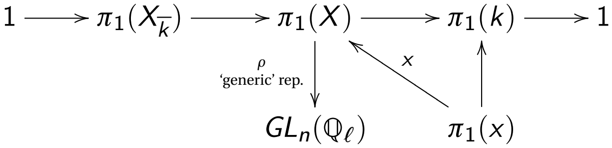

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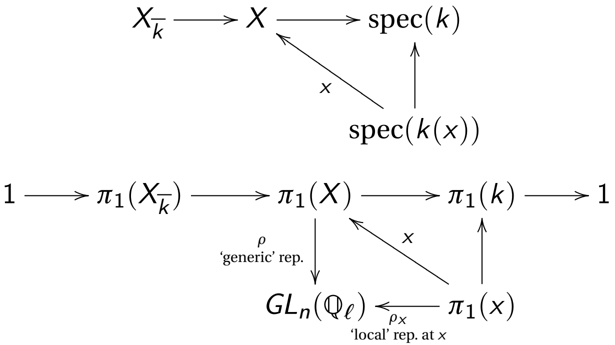

Uniform open image theorem

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{k}}) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(k) & \longrightarrow & 1 \\
 & & & & {\scriptstyle \rho}\big\downarrow & \nwarrow^{x} & \big\uparrow & & \\
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$$Lie(\rho(\pi_1(X_{\overline{k}})))^{ab} = 0$$

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 & & & & GL_n(\mathbb{Q}_\ell) & \underset{\rho_x}{\longleftarrow} & \pi_1(x) & &
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$$X^{?\delta} \quad := \{x \in X \mid codim_{\mathrm{im}\,\rho} \mathrm{im}\,\rho_x ? \delta\} \subset X \quad , ? = \geqslant, <, = etc.$$

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Theorem (UOI)

X curve, $p = 0$ (resp. $p > 0$). For every $d \geqslant 1$ (resp. $d = 1$)

1. ρ GLP $\Rightarrow X^{\geqslant 1}(\leqslant d)$ is finite and

$$B_d := \sup\{[\mathrm{im}(\rho) : \mathrm{im}(\rho_x)] \mid x \in X^{<1}(\leqslant d)\} < +\infty$$

2. $X^{\geqslant 3}(\leqslant d)$ is finite

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Equivalently

$$U_d := \bigcap_{x\in X^{<1}(\leqslant d)} \mathrm{im}\,\rho_x \subset_{op} \mathrm{im}\,\rho$$

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\pi_1(X,\overline{\eta}) & \xrightarrow{\rho} & (R^i f_* \mathbb{Q}_\ell)_{\overline{\eta}} & \xrightarrow{\simeq} & H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell) \\
\uparrow \simeq & & \uparrow \simeq & & \\
\pi_1(X,\overline{x}) & \xrightarrow{\rho} & (R^i f_* \mathbb{Q}_\ell)_{\overline{x}} & \xrightarrow{\simeq} & H^i(Y_{\overline{x}}, \mathbb{Q}_\ell) \\
\uparrow x & \nearrow \rho_x & & & \\
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- Specialization $\rightsquigarrow$ k finite of char $p + \ell$
- $R^i f_* \mathbb{Q}_\ell$ pure of weight i (Weil I)
- $H^1(X_{\overline{k}}, \text{pure of weight } 0)$ of weights $\geqslant 1$ (Weil II)

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(If $p = 0$, comparison Betti $\longleftrightarrow$ ℓ-adic + Hodge II)

Uniform boundedness of ℓ-primary part- two examples

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Conjecture TAV (torsion of abelian varieties)

$k/\mathbb{Q}$ number field, $A \to k$ abelian variety,

$$|A(\overline{k})^{\pi_1(k)}[\ell^\infty] = A(k)[\ell^\infty]| \leqslant C(\dim(A), [k:\mathbb{Q}])$$

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Using moduli spaces, both conjectures amount to bounding uniformly the involved invariants in a specific smooth proper family $f : Y \to X$

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$$\sup\{|Y_x(k(x))[\ell^\infty]| \mid x \in X(\leqslant d)\} < +\infty$$

$X^T(\leqslant d)$: set of $x \in X(\leqslant d)$ satisfying the ℓ-adic Tate conj. for divisors

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- For $d = 1$, both conjectures should follow from the Bombieri-Lang conjecture modulo a general conjecture on abstract modular schemes (C-Tamagawa) - see proof of UOI Thm

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- For $d \geqslant 2$ or $p > 0$, rather questions

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Corollary

If X is a curve and $p = 0$ (resp. $p > 0$), for every $d \geqslant 1$ (resp. $d = 1$)

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- *E.g.* Conj. TAV for $\dim(A)=1$:

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  Manin ~ 69 for k-rational points
  Frey ~ 84 using Mordell-Lang (Faltings $\sim$ 83)

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- First evidence for higher-dimensional A's

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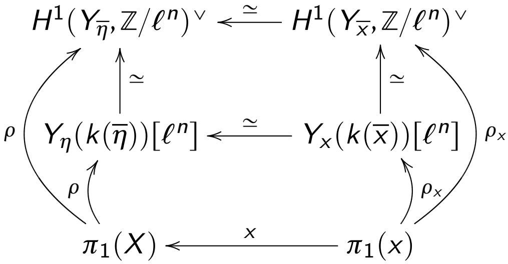

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$$
\begin{array}{ccccc}
H^1(Y_{\overline{\eta}}, \mathbb{Z}/\ell^n)^\vee & \xleftarrow{\simeq} & H^1(Y_{\overline{x}}, \mathbb{Z}/\ell^n)^\vee & := M[\ell^n] \simeq (\mathbb{Z}/\ell^n)^{2d} \\
\uparrow \simeq & & \uparrow \simeq & \\
Y_\eta(k(\overline{\eta}))[\ell^n] & \xleftarrow{\simeq} & Y_x(k(\overline{x}))[\ell^n] & \\
\uparrow \rho & & \uparrow \rho_x & \\
\pi_1(X) & \xleftarrow{x} & \pi_1(x) &
\end{array}
$$

with curved arrows $\rho : \pi_1(X) \to H^1(Y_{\overline{\eta}}, \mathbb{Z}/\ell^n)^\vee$ and $\rho_x : \pi_1(x) \to H^1(Y_{\overline{x}}, \mathbb{Z}/\ell^n)^\vee$.

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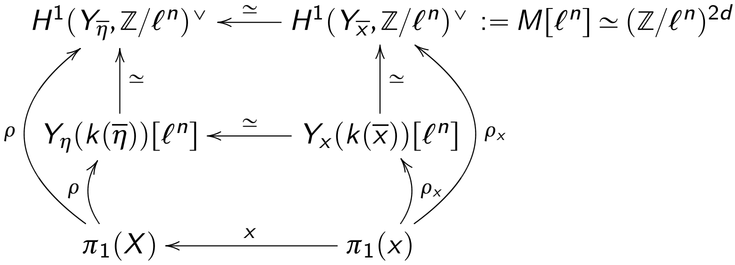

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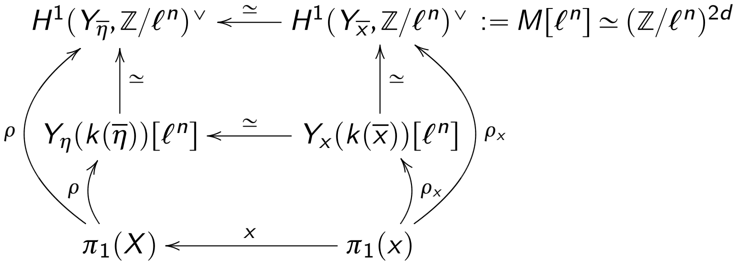

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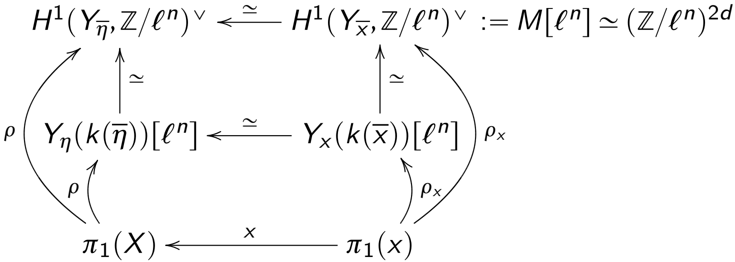

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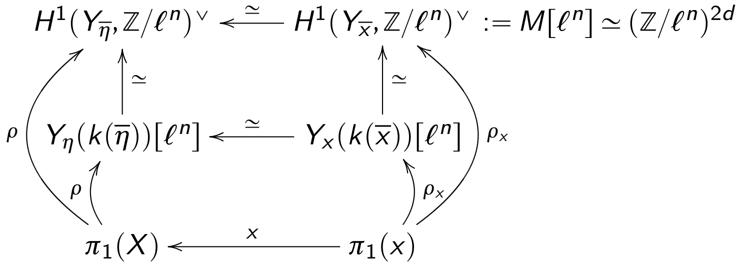

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$$x \in X^{<1}(\leqslant d) \Rightarrow \quad Y_x(k(x))[\ell^\infty] \quad \begin{aligned} &= M^{\pi_1(x)} \\ &\subset M^{\pi_1(X_{U_d})} \\ &= Y_{\eta_{U_d}}(k(\eta_{U_d}))[\ell^\infty] : \text{finite (MWLN)} \end{aligned}$$

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Corollary

If X is a curve and $p = 0$ (resp. $p > 0$), for every $d \geqslant 1$ (resp. $d = 1$)

$$\sup\{Br(Y_{\overline{x}})^{\pi_1(x)}[\ell^\infty] \mid x \in X^T(\leqslant d)\} < +\infty$$

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  Special families of (K3, abelian) surfaces over number fields by reducing to questions about rational points on $X_0(n)$ (Manin, Faltings-Frey, Mazur, Merel)

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- For $f : Y \to X$ abelian scheme or family of K3, $X^T(\leqslant d)=X$
- Can relax the condition on the ℓ-adic Tate conjecture for divisors
- In this talk, $p=0$. For $p>0$ (more delicate !) see Ambrosi's talk

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Lemma (Relation with the Tate conjecture for divisors)

$p \neq \ell$: prime. The following assertions are equivalent

- (1) $c_1 : \mathrm{Pic}(Y_{\overline{k}}) \otimes \mathbb{Q}_\ell \twoheadrightarrow \varinjlim_{U \subset \pi_1(k) \text{ open}} H^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1))^U$;
- (2) $Br(Y_{\overline{k}})^U[\ell^\infty]$ is finite for every open subgroup $U \subset \pi_1(k)$.

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$$0 \to NS(Y_{\overline{k}}) \otimes \mathbb{Z}/\ell^n \to \mathrm{H}^2(Y_{\overline{k}}, \mu_{\ell^n}) \to B[\ell^n] \to 0$$

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$$M^U \subset \mathrm{H}^1(U, T)[\ell^\infty]$$

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U_x varies in a finite set...

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$sp_x : NS(Y_{\overline{\eta}}) \to NS(Y_{\overline{x}})$ is not an isomorphism in general!

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Lemma (Galois generic vs NS-generic)

If $x \in X^{<1}(\leqslant d)$,

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(curved arrow from $\mathrm{H}^2(Y_{\overline{x}}, \mathbb{Q}(1))^{\pi_1(X_{\mathbb{C}})}$ to $\mathrm{H}^2(\overline{Y}_{\mathbb{C}}, \mathbb{Q}(1))$ labelled: semisimplicity $\mathbb{Q}$-PHS; curved arrow from $NS(Y_{\overline{\eta}})$ to $NS(Y_{\overline{x}})$)

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- More generally, Galois-generic $\Rightarrow$ the motivated motivic Galois group does not degenerate (Y.André $\sim$ 96)
- Unconditional formulation is

Corollary (C-Charles, $\sim$ 16, $p=0$; Ambrosi ~ 17, $p>0$)

If X is a curve, $p=0$ (resp. $p>0$) and the Zariski-closure of $\mathrm{im}(\rho)$ is connected, for every $d\geqslant 1$ (resp. $d=1$)

$$\sup\{[Br(Y_{\overline{x}})^{\pi_1(x)}[\ell^\infty] : Br(Y_{\overline{\eta}})^{\pi_1(X)}[\ell^\infty]] \mid x\in X^{<1}(\leqslant d)\} < +\infty$$

(recall $Br(Y_{\overline{\eta}}) \xrightarrow{\simeq} Br(Y_{\overline{x}})$) for $x\in X^{<1}(\leqslant d)$).

UOI Thm - sketch of proof / Strategy

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$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$

Theorem (UOI) (C-Tamagawa, $\sim$ 10, $p=0$; Ambrosi, $\sim$ 17, $p>0$)

X curve, $p=0$ (resp. $p>0$), ρ GLP. For every $d \geqslant 1$ (resp. $d=1$) $X^{\geqslant 1}(\leqslant d)$ is finite and

$$U_d := \bigcap_{x \in X^{<1}(\leqslant d)} \operatorname{im} \rho_x \subset_{op} \operatorname{im} \rho$$

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- Step 1 (works for arbitrary X) Using formalism of Galois categories, attach to ρ a projective system of (non connected) étale covers of X = A(bstract) M(odular) S(schemes)

$$\cdots X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

such that $\mathrm{im}(\varprojlim X_n(\leqslant d) \to X(\leqslant d)) = X^{\geqslant 1}(\leqslant d)$

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  $\Rightarrow |X_n(\leqslant 1)| < +\infty,\ n \gg 0 \Rightarrow |X^{\geqslant 1}(\leqslant 1)| < +\infty$

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  [Uses GLP]

- Step 3 : Show that $\gamma(X_n) \to +\infty$

  + Mordell-Lang Conj.
  $\Rightarrow |X_n(\leqslant d)| < +\infty,\ n \gg 0 \Rightarrow |X^{\geqslant 1}(\leqslant d)| < +\infty$

UOI Thm - sketch of proof / Strategy

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$

- Step 1 (works for arbitrary X) Using formalism of Galois categories, attach to ρ a projective system of (non connected) étale covers of X = A(bstract) M(odular) S(schemes)

$$\cdots X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

  such that $\mathrm{im}(\varprojlim X_n(\leqslant d) \to X(\leqslant d)) = X^{\geqslant 1}(\leqslant d)$

- Step 2 : Show that $g(X_n) \to +\infty$

  + Mordell Conj.

  $\Rightarrow |X_n(\leqslant 1)| < +\infty,\ n \gg 0 \Rightarrow |X^{\geqslant 1}(\leqslant 1)| < +\infty$

  [Uses GLP]

- Step 3 : Show that $\gamma(X_n) \to +\infty$

  + Mordell-Lang Conj.

  $\Rightarrow |X_n(\leqslant d)| < +\infty,\ n \gg 0 \Rightarrow |X^{\geqslant 1}(\leqslant d)| < +\infty$

  [Relies on Step 2]

UOI Thm - sketch of proof / Step 1

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$

UOI Thm - sketch of proof / Step 1

$\rho : \pi_1(X) \rightarrow GL_n(\mathbb{Z}_\ell)$

$$U \subset \pi_1(X) \text{ open subgroup } \leftrightarrow X_U \rightarrow X \text{ étale cover}$$

- $U \cap \pi_1(X_{\overline{k}}) \leftrightarrow X_U \times_k \overline{k} \rightarrow X_{\overline{k}}$;
- $x \in X \;\; \pi_1(x) \rightarrow U \subset \pi_1(X) \Leftrightarrow$

$$\begin{array}{ccc} X_U & & \\ \downarrow & \nwarrow & \\ X & \xleftarrow[x]{} & \mathrm{spec}(k(x)) \end{array}$$

UOI Thm - sketch of proof / Step 1

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$k = \overline{k}$

UOI Thm - sketch of proof / Step 1

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$\Pi := \mathrm{im}(\rho) \subset GL_n(\mathbb{Z}_\ell)$ compact ℓ-adic Lie group
$\Pi(n) := \ker(\Pi \subset \mathrm{GL}(\mathbb{Z}_\ell) \twoheadrightarrow \mathrm{GL}(\mathbb{Z}/\ell^n)),\; n \geqslant 1$

$$\mathcal{V}_n := \{V \subset_{op} \Pi \mid \Phi(\Pi(n-1)) \subset V,\ \Pi(n-1) \not\subset V\}$$

$$X_n := \bigsqcup_{V \in \mathcal{V}_n} X_{\rho^{-1}(V)} \to X$$

$$\begin{array}{ccc} \mathcal{V}_{n+1} & \to & \mathcal{V}_n \\ V & \to & V\Phi(\Pi(n-1)) \end{array} \longleftrightarrow X_{n+1} \to X_n$$

UOI Thm - sketch of proof / Step 1

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- $\Phi(\Pi) \subset_{op} \Pi \Rightarrow |\mathcal{V}_n| < +\infty$
- $K \lhd_{cl} \Pi$, $H \subset_{cl} \Pi$, $H\Phi(K) \supset K \Rightarrow H \supset K$

  $\rightsquigarrow H \subset_{cl} \Pi$, $H \not\subset U_n$, $U_n \in \mathcal{U}_n \Rightarrow \Pi(n-1) \subset H$
- $\Phi(\Pi(n-1)) = \Pi(n)$, $n \gg 0$
- $(V[n]) \in \varprojlim \mathcal{V}_n \Rightarrow \in \bigcap_n V[n] \subset_{cl} \Pi$ not open

UOI Thm - sketch of proof / Step 2

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell), \quad \Pi := \operatorname{im} \rho$

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- $\widehat{X}_n \to X$ Galois closure of $X_n \to X$

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- $\widehat{X}_n \to X$ Galois closure of $X_n \to X$
  $g(\widehat{X}_n) \leqslant 1,\ n \geqslant 0 \Rightarrow \rho$ not GLP [Classif. $Aut(\text{genus} \leqslant 1)$]

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  $g(\widehat{X}_n) \geqslant 2,\ n \gg 0 \Rightarrow g(\widehat{X}_n) \to +\infty$ [R-H formula]

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  $g(\widehat{X}_n) \geqslant 2,\ n \gg 0 \Rightarrow g(\widehat{X}_n) \to +\infty$ [R-H formula]
- Compare $g(\widehat{X}_n)$ and $g(X_n)$: $g(\widehat{X}_n) \to +\infty \Leftrightarrow g(X_n) \to +\infty$

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  Control of ramification terms in R-H formula

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  Control of ramification terms in R-H formula

$$\frac{|I_n \backslash \Pi_n / H_n|}{|\Pi_n / H_n|} \to \frac{1}{|I_H|}$$

Inertia $I \subset_{cl} \Pi_{cl} \supset H := \cap_n V[n]$

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  [Serre-Osterlé's asympotic estimates for cardinality of reduction modulo ℓ^n of ℓ-adic analytic spaces]

UOI Thm - sketch of proof / Step 3

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$, $\Pi := \operatorname{im} \rho$

$$\cdots X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

UOI Thm - sketch of proof / Step 3

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$, $\Pi := \operatorname{im} \rho$

Assume $\gamma(X_n) \leqslant \gamma$ for all n

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UOI Thm - sketch of proof / Step 3

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After extracting [Π compact ℓ-adic Lie group], may assume $X_{n+1} \to X_n$ Galois with group $(\Gamma_n (= \mathbb{Z}/\ell)^{codim_H \Pi})$

UOI Thm - sketch of proof / Step 3

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$, $\Pi := \operatorname{im} \rho$

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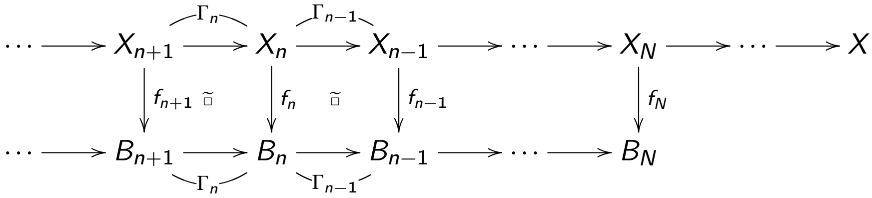

- $B_{n+1} \to B_n$ Galois with group Γ_n
- $g(B_n) = 0$, $deg(f_n) = \gamma$ or $g(B_n) = 1$, $deg(f_n) = \frac{\gamma}{2}$

UOI Thm - sketch of proof / Step 3

$\rho : \pi_1(X) \to GL_n(\mathbb{Z}_\ell)$, $\Pi := \operatorname{im} \rho$

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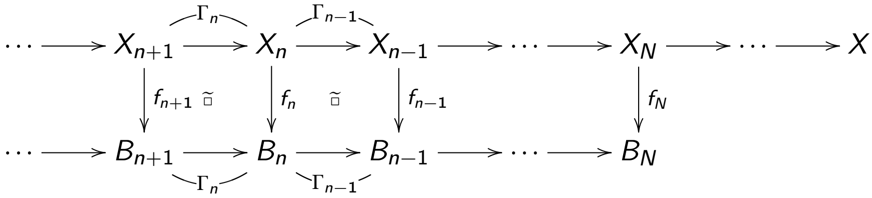

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- $\cdots B_{n+1} \to B_n \to \cdots \to B_N$ 'corresponds to' the induced rep $Ind_{\pi_1(X_N)}^{\pi_1(B_N)}(\rho|_{\pi_1(X_N)})$ which is again GLP

UOI Thm - sketch of proof / Step 3

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$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X_{n+1} & \xrightarrow{\Gamma_n} & X_n & \xrightarrow{\Gamma_{n-1}} & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_N & \longrightarrow \cdots \longrightarrow & X \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & & & \downarrow f_N & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow[\Gamma_n]{} & B_n & \xrightarrow[\Gamma_{n-1}]{} & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_N & &
\end{array}$$

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Contrad. step 2!

UOI Conj - Higher dimensional X

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$\Pi(n):=\ker(\Pi\subset \mathrm{GL}(\mathbb{Z}_\ell)\twoheadrightarrow \mathrm{GL}(\mathbb{Z}/\ell^n))$, $n\geqslant 1$

$$\mathcal{V}_n:=\{V\subset_{op}\Pi \mid \Phi(\Pi(n-1))\subset V,\ \Pi(n-1)\not\subset V\}$$

$$X_n:=\bigsqcup_{V\in\mathcal{V}_n} X_{\rho^{-1}(V)}\to X_{n-1}\to\cdots\to X_1\to X$$

UOI Conj - Higher dimensional X

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$$X_n := \bigsqcup_{V \in \mathcal{V}_n} X_{\rho^{-1}(V)} \to X_{n-1} \to \cdots \to X_1 \to X$$

Conjecture

ρ GLP $(+\,?\,?)$ $\Rightarrow$ X_n of general type for $n \gg 0$

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$$X_n := \bigsqcup_{V \in \mathcal{V}_n} X_{\rho^{-1}(V)} \to X_{n-1} \to \cdots \to X_1 \to X$$

Conjecture

ρ GLP $(+??)$ $\Rightarrow$ X_n of general type for $n \gg 0$

- by noetherian induction and modulo the Lang conj. would imply uniform boundedness conj for torsion of AV, ℓ-primary part of Brauer (for $d = 1$) *etc*
- Need less than $|X_n(k)| < +\infty$, $n \gg 0$... Only

$$|\mathrm{im}(\varprojlim X_n(\leqslant d) \to X(\leqslant d))| < +\infty$$

Thank you !