GENUS OF ABSTRACT MODULAR CURVES WITH LEVEL-ℓ STRUCTURES.

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Abstract. We prove - in arbitrary characteristic - that the genus of abstract modular curves associated to bounded families of continuous geometrically perfect $\mathbb{F}_\ell$-linear representations of étale fundamental groups of curves goes to infinity with ℓ. This applies to the variation of the Galois image on étale cohomology groups with coefficients in $\mathbb{F}_\ell$ in 1-dimensional families of smooth proper schemes or, under certain assumptions, to specialization of first Galois cohomology groups.

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1. Introduction

1.1. **Notation.** Let k be a finitely generated field of characteristic $p \geq 0$ with absolute Galois group $\pi_1(k)$. Let X be a smooth, separated, geometrically connected scheme over k with generic point η and set of closed points $|X|$. Set $\overline{X} := X \times_k \overline{k}$. When X is a curve let $\overline{X}^{cpt}$ denote the smooth compactification of $\overline{X}$, write $\partial\overline{X} := \overline{X}^{cpt} \setminus \overline{X}$ for the divisor at infinity and g_X, γ_X for the genus and gonality of $\overline{X}^{cpt}$respectively. Every $x \in |X|$ produces a quasi-splitting $\sigma_x : \pi_1(x) \hookrightarrow \pi_1(X)$ of the structural projection $\pi_1(X) \to \pi_1(k)$ (recall that $\pi_1(x)$ identifies with the absolute Galois group $\pi_1(k(x))$ of the residue field $k(x)$ at x).

Let $r \in \mathbb{Z}_{\geq 1}$. Let L be an infinite set of primes with $p \notin L$, and for every $\ell \in L$, fix a field F_ℓ of characteristic ℓ and a discrete $F_\ell[\pi_1(X)]$-module H_ℓ with bounded F_ℓ-rank $r_\ell \leq r$ that is, equivalently, a continuous group homomorphism $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell) \simeq \mathrm{GL}_{r_\ell}(F_\ell)$. Set $G_\ell := \mathrm{im}(\rho_\ell)$, $\overline{G}_\ell := \rho_\ell(\pi_1(\overline{X}))$ and $G_{\ell,x} := \mathrm{im}(\rho_\ell \circ \sigma_x)$. Note that $\overline{G}_\ell$ is normal in G_ℓ and $[G_\ell : \overline{G}_\ell G_{\ell,x}] \leq [k(x) : k]$.

1.2. **Leading examples.** From the smooth-proper base change theorem, such families arise from the étale cohomology with coefficients in $\mathbb{F}_\ell$ of the generic fiber of a smooth proper morphism $Y \to X$ (Subsection 5.1)

$$\rho_\ell : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell)), \quad \ell \in L.$$

Other examples arise when considering the specialization of first Galois cohomology groups (Subsection 5.2). More precisely, starting from a bounded family $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$, $\ell \in L$ as above, for every $\ell \in L$ fix an F_ℓ-submodule $V_\ell \subset \mathrm{H}^1(\pi_1(X), H_\ell)$ with bounded F_ℓ-rank. Then the cohomology classes in V_ℓ are classified by an $F_\ell[\pi_1(X)]$-module V_ℓ^{univ}, which is an extension

$$1 \to H_\ell \to V_\ell^{univ} \to V_\ell \to 1,$$

giving rise to an auxiliary bounded family $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell^{univ})$, $\ell \in L$. When $H_\ell = \mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell)$, these are closely related to the specialization of first étale Abel-Jacobi maps modulo ℓ, a typical example of which is the Kummer morphism

$$V_\ell := Y(X)/\ell \hookrightarrow \mathrm{H}^1(\pi_1(X), Y_{\overline{\eta}}[\ell]),$$

for $Y \to X$ an abelian scheme.

1.3. The general problem we want to address is the description of the local Galois images $G_{\ell,x}$ as x varies in $|X|$. In general, given a set $\mathcal{F}_\ell$ (= the 'moduli data') of subgroups of G_ℓ *which does not contain* $\overline{G}_\ell$, one expects that the set $X(\rho_\ell, \mathcal{F}_\ell)$ of all $x \in |X|$ such that $G_{\ell,x}$ is contained in a group of $\mathcal{F}_\ell$ is 'small'. This naturally leads to introducing the *abstract modular scheme* associated with ρ_ℓ, $\mathcal{F}_\ell$

$$X_{\mathcal{F}_\ell}^{\rho_\ell} := \bigsqcup_{U \in \mathcal{F}_\ell} X_U \to X,$$

where $X_U \to X$ denotes the connected étale cover (defined over a finite separable extension k_U of k) corresponding to the open subgroup $\rho_\ell^{-1}(U) \subset \pi_1(X)$ [1]. It follows from the general formalism of Galois categories that
- (Base change) $X_U \times_{k_U} \overline{k} \to \overline{X}$ is the connected étale cover corresponding to $\overline{U} := U \cap \overline{G}_\ell \subset \overline{G}_\ell$.
- (Moduli) $x \in |X|$ lifts to a $k(x)$-rational point on X_U if and only if $G_{\ell,x} \subset U$.

Thus $X(\rho_\ell, \mathcal{F}_\ell)$ is exactly the set of all $x \in |X|$ which lift to a $k(x)$-rational point on $X_{\mathcal{F}_\ell}^{\rho_\ell}$. This gives a diophantine reformulation of our original group representation-theoretic problem and already shows that for every finite extension k' of k the set $X(\rho_\ell, \mathcal{F}_\ell)(k') := X(\rho_\ell, \mathcal{F}_\ell) \cap X(k')$ is thin. But one usually expects stronger sparsity results - for instance that for every integer $d \geq 1$ the set $X(\rho_\ell, \mathcal{F}_\ell)^{\leq d}$ of all $x \in X(\rho_\ell, \mathcal{F}_\ell)$ such that $[k(x) : k] \leq d$ is not Zariski-dense in X (hence finite when X is a curve), that $X(\rho_\ell, \mathcal{F}_\ell)$ is of bounded height, that $X(\rho_\ell, \mathcal{F}_\ell)$ is not Zariski-dense in X or even that $X(\rho_\ell, \mathcal{F}_\ell)$ is empty.

This work focuses on the weakest of these finiteness properties, namely find minimal conditions on the ρ_ℓ, $\mathcal{F}_\ell$, $\ell \in L$ to ensure that $X(\rho_\ell, \mathcal{F}_\ell)(k)$ is not Zariski-dense in X. But even this weakened problem seems out of reach when X has dimension ≥ 2. For curves, the situation is better due to the remarkable fact that the genus controls the finiteness of the set of rational points.

1.3.1. **Fact** (Faltings ($p = 0$, [FW84]), Voloch ($p > 0$, [EElsHKo09, Prop. 3])) *Let k be a finitely generated field. Then there exists an integer $g(k) \geq 2$ such that for every curve C over k with $g_C \geq g(k)$ one has $|C(k)| < +\infty$.*

When X is a curve, which we assume from now and till the end of the introduction, this reduces our original problem to determining under which conditions on the ρ_ℓ, $\mathcal{F}_\ell$, $\ell \in L$ one has

$$g_{X_{\mathcal{F}_\ell}^{\rho_\ell}} := \min\{g_{X_U} \mid U \in \mathcal{F}_\ell\} \to +\infty?$$

1.4. The conditions under which we can prove $g_{X_{\mathcal{F}_\ell}}^{\rho_\ell} \to +\infty$ (Theorem 1.6.1, Corollary 1.6.2) or which appear in the intermediate results (Theorem A, Theorem B, Theorem C) are denoted by (T), (P), (I) in the following. Their precise statements are gathered in Subsection 2.4, to which the reader can refer to.

1.5. If one considers the set $\mathcal{F}_{\ell,tot}$ of all subgroups U of G_ℓ such that $\overline{G}_\ell \not\subset \overline{U}$ (that is $X(\rho_\ell, \mathcal{F}_{\ell,tot})(k)$ is the set of all $x \in X(k)$ such that $G_{\ell,x} \subsetneq G_\ell$), it may happen that there exists an integer $B \geq 1$ and infinitely many ℓ such that $1 < [\overline{G}_\ell : \overline{U}] \leq B$ for some $U \in \mathcal{F}_{\ell,tot}$. This is an obstruction to $g_{X_{\mathcal{F}_{\ell,tot}}^{\rho_\ell}} \to +\infty$. This obstruction disappears if one replaces $\mathcal{F}_{\ell,tot}$ with the set $\mathcal{F}_{\ell,+}$ of all subgroups U of G_ℓ such that $\overline{G}_\ell^+ \not\subset \overline{U}$. Here, given a finite subgroup $G \subset \mathrm{GL}(H_\ell)$, we write $G^+ \subset G$ for the (normal) subgroup generated by the elements of order ℓ. Theorem A below shows that little information is lost when replacing $\mathcal{F}_{\ell,tot}$ with $\mathcal{F}_{\ell,+}$.

[1] This is a slight abuse of notation: more precisely, k_U is the finite separable extension of k corresponding to the image of $\rho_\ell^{-1}(U) \hookrightarrow \pi_1(X) \twoheadrightarrow \pi_1(k)$ and $X_U \to X$ is the connected étale cover corresponding to $\rho_\ell^{-1}(U) \subset \pi_1(X)$ with X_U geometrically connected over k_U.

Theorem A ([CT13b, Thm. 1.1 and Rem. 2.10][2]) *Assume (T). Then there exists an open subgroup* $\Pi \subset \pi_1(\overline{X})$ *such that* $\rho_\ell(\Pi) = \rho_\ell(\Pi)^+$ *for* $\ell \gg 0$.

In particular, $[\overline{G}_\ell : \overline{G}_\ell^+]$ is bounded from above independently of ℓ. This leads to considering the abstract modular curve $X_+^{\rho_\ell} := X_{\mathcal{F}_{\ell,+}}^{\rho_\ell}$.

1.6. The main result of this paper is:

1.6.1. **Theorem** *Assume* $F_\ell = \mathbb{F}_\ell$*, (T) and (P). Then* $g_{X_+^{\rho_\ell}} \to +\infty$.

As one can always construct a family $X_\ell \to X$ of connected étale covers with C_ℓ of genus 0 which are geometrically Galois with group $\mathbb{Z}/\ell$, the above perfectness condition (P) is necessary[3].

Theorem 1.6.1 has the following arithmetic application[4].

1.6.2. **Corollary** *Assume* $F_\ell = \mathbb{F}_\ell$*, (T) and (P). Then for* $\ell \gg 0$ *and all but finitely many* $x \in X(k)$ *one has* $\overline{G}_\ell^+ \subset G_{\ell,x}$*. In particular, there exists an integer* $B \geq 1$ *such that for* $\ell \gg 0$ *and all but finitely many* $x \in X(k)$ *one has* $[G_\ell : G_{\ell,x}] \leq B$*. If* $\overline{G}_\ell = \overline{G}_\ell^+$ *for* $\ell \gg 0$*, one can take* $B = 1$.

To apply Theorem 1.6.1 and Corollary 1.6.2 to families

$$\rho_\ell : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell)), \quad \ell \in L$$

for $Y \to X$ a smooth proper morphism (Subsection 5.1) and (at least in some case) to families arising from first Galois cohomology classes as explained in 1.2 (Subsection 5.2), one has to check that conditions (T) and (P) are satisfied. These follow from alterations and the purity part of the Weil conjectures (Theorem 5.1.1).

As an example, one can apply Corollary 1.6.2 to the particular case of the Kummer morphism to extend the Néron-Silverman specialization theorem to finitely generated fields of arbitrary characteristic (Subsection 5.3.2.2).

1.7. The general problem of the growth of geometric invariants attached to families of abstract modular schemes - especially the geometric genus and gonality of abstract modular curves has been investigated extensively during the past 5 years and essentially settled in the following cases:

- ℓ-adic coefficients, characteristic 0: genus ([CT12b]), gonality ([CT13a]). The results of [CT12b] for the growth of the genus in the case of ℓ-adic coefficients extend to positive characteristic $p > 0$ ($p \neq \ell$) along the guidelines of [CT12a].

- $\mathbb{F}_\ell$-coefficients, characteristic 0: gonality hence genus (this follows essentially from Theorem A and the techniques of [EHKo12] – see the Appendix for details).

The main contribution of this paper is thus to deal in whole generality with the growth of the genus in the case of $\mathbb{F}_\ell$-coefficients in positive characteristic.

[2] More precisely, [CT13b, Thm. 1.1] is Theorem A for bounded families of continuous $\mathbb{F}_\ell$-linear representations of $\pi_1(X)$ but using Larsen-Pink's filtration (see Subsection 3.2) as indicated in [CT13b, Rem. 2.10], the proof of [CT13b, Thm. 1.1] extends as it is to bounded families of continuous F_ℓ-linear representations.

[3] Another way to phrase Theorem 1.6.1 is to say that, if (T) holds then either $g_{X_+^{\rho_\ell}} \to +\infty$ or $\overline{G}_\ell \twoheadrightarrow \mathbb{Z}/\ell$ for infinitely many ℓ.

[4] From Fact 1.3.1 and Theorem 1.6.1, for $\ell \gg 0$ and every $U \in \mathcal{F}_{\ell,+}$, the set $X_U(k)$ is finite hence so is $X(\rho_\ell, \mathcal{F}_{\ell,+})(k) = \bigcup_{U \in \mathcal{F}_\ell^+} \mathrm{im}(X_U(k) \to X(k))$. But, by construction, $\overline{G}_\ell^+ \subset G_{\ell,x}$ for every $x \in X(k) \setminus X(\rho_\ell, \mathcal{F}_{\ell,+})(k)$. This proves the first part of the assertion in Corollary 1.6.2. The second part follows from Theorem A and the fact that for every $x \in X(k)$ one has $G_\ell = \overline{G}_\ell G_{\ell,x}$.

It is natural to ask whether Theorem 1.6.1 could be strengthened with gonality instead of genus. There is *a priori* no hope to extend the complex-analytic arguments of [EHKo12] to the case of positive characteristic (or reduce to a characteristic 0 situation since gonality may decrease under specialization). Actually, gonality is in general much harder to handle than genus, especially in positive characteristic. For a purely algebraic approach and partial results in positive characteristic see [C12] and [CT16], for a graph-theoretical method in the non-archimedean setting and partial results see [CoKKo]. Let us mention that even if one can prove that the gonality of $X_+^{\rho_\ell}$ goes to infinity with ℓ when $p>0$ this would not (at least straightforwardly) imply the above extension of Corollary 1.6.2 for points of bounded degree. This is due to isotriviality phenomena in the positive characteristic form of the Mordell-Lang Conjecture [Hr96] (see [CT16, Appendix] for details).

Another natural problem that does not seem to have been investigated seriously so far is the case of $\overline{\mathbb{F}}_\ell$-coefficients. We discuss briefly this question in the concluding Section 6 and explain how a deep group-theoretical result of Guralnick ([Gu03]) can be used to extend Theorem 1.6.1 to this setting. However, Gurlanick's result relies on delicate satellite theorems of the classification. In contrast, our approach is more geometric and only resort to approximation theory for subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ (r fixed, ℓ varying) as elaborated by Nori ([N87]) and Larsen-Pink ([LaP11]). The main idea is to develop a rough analogue of invariant theory for subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ in order to replacing the curves $X_+^{\rho_\ell}$ by abstract modular curves which are easier to handle. We hope this strategy is interesting in itself and may be used for further development in the study of families of modulo-ℓ representations. In the next subsection, we summarize it.

Eventually, let us point out that Theorem 5.1.1, which asserts in particular that, after possibly replacing X by a connected étale cover, the image of the geometric étale fundamental group on étale cohomology with $\mathbb{F}_\ell$-coefficients is perfect for $\ell \gg 0$, is new in characteristic $p>0$. When $p=0$, this follows from the fact that the geometric étale fundamental group acts semisimply on étale cohomology with $\mathbb{F}_\ell$-coefficients for $\ell \gg 0$ (see the argument in [CT11, §2.2]). When $p>0$ the geometric étale fundamental group is also expected to act semisimply for $\ell \gg 0$ (a proof of this assertion is announced in [CHT16]).

1.8. From now on, assume $F_\ell = \mathbb{F}_\ell$. The strategy of the proof of Theorem 1.6.1 is to construct a 'universal tensor representation' in order to separate groups in $\mathcal{F}_{\ell,+}$ from $\overline{G}_\ell^+$ by lines for $\ell \gg 0$. This allows to construct an auxilliary bounded family $\tilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\tilde{T}_\ell)$, $\ell \in L$ of continuous $\mathbb{F}_\ell$-representations such that every connected component of $X_+^{\rho_\ell}$ dominates a connected component of the abstract modular curve $X_0^{\tilde{\rho}_\ell}$ associated to the family $\mathcal{F}_{\ell,0}$ of all stabilizer of lines in $\tilde{T}_\ell$. This reduces the problem to showing that $g_{X_0^{\tilde{\rho}_\ell}} \to +\infty$ which, due to the specific shape of the moduli problem encoded in $\mathcal{F}_{\ell,0}$, is doable. More precisely, to simplify, assume that $\overline{G}_\ell$ acts semisimply on H_ℓ for $\ell \gg 0$. Then the two main intermediate statements are the following. Let Ω denote the set of all maps $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with finite support (that is such that $f(m,n)=0$ for all but finitely many $(m,n) \in (\mathbb{Z}_{\geq 0})^{\oplus 2}$.

(Special case of) Theorem B: *There exists $f \in \Omega$ such that for $\ell \gg 0$ and every $U \in \mathcal{F}_{\ell,+}$ there exists a line $D \subset T^f(H_\ell) := \bigoplus_{m,n\geq 0}(H_\ell^{\oplus m} \otimes (H_\ell^\vee)^{\oplus n})^{\oplus f(m,n)}$ (depending on U, $\overline{G}_\ell^+$) with the property that $\overline{G}_\ell^+ D \neq D$ but $UD = D$.*

Theorem C: *Assume (T) and (I). Then $g_{X_0^{\rho_\ell}} \to +\infty$.*

To deduce the main Theorem from Theorem B and Theorem C, just set $\tilde{T}_\ell := T_\ell / T_\ell^{\overline{G}_\ell^+}$, where $T_\ell := T^f(H_\ell)$, $\ell \in L$. Then the family $\tilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\tilde{T}_\ell)$, $\ell \in L$ is bounded and satisfies (T) and (I) as soon as the family ρ_ℓ, $\ell \in L$ satisfies (T) and (P). From Theorem B, every connected component of $X_+^{\rho_\ell}$ dominates a connected component of $X_0^{\tilde{\rho}_\ell}$ and, from Theorem C, $g_{X_0^{\tilde{\rho}_\ell}} \to +\infty$.

When one no longer assumes that $\overline{G}_\ell$ acts semisimply on H_ℓ for $\ell \gg 0$, the statement of Theorem B is slightly more involved but the statement of Theorem C remains unchanged (the proof of Theorem C does not use assumptions other than (T) and (I)). See Section 4 for details.

Theorem B is a variant for finite subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ (r fixed, ℓ varying) of the classical Chevalley theorem for algebraic groups and, unsurprisingly, it relies on approximation theory. Approximation theory associates to a subgroup G of $\mathrm{GL}_r(F_\ell)$ a connected algebraic subgroup $\tilde{G} \hookrightarrow \mathrm{GL}_{r,\mathbf{F}_\ell}$ ($\mathbf{F}_\ell \supset F_\ell$) - the algebraic envelope - whose properties reflect those of G and whose rational points approximate well G for $\ell \gg 0$. There are two approaches, one by Nori and Serre, which works only for $F_\ell = \mathbb{F}_\ell$ (with $\mathbf{F}_\ell = \mathbb{F}_\ell$) but is 'functorial' and one by Larsen and Pink, which works for arbitrary fields F_ℓ of characteristic ℓ (with $\mathbf{F}_\ell = \overline{F}_\ell$) but is 'not functorial'. The restriction of our results to $\mathbb{F}_\ell$-coefficients comes from the fact that we resort to the former, where $\tilde{G} \hookrightarrow \mathrm{GL}_{H_\ell}$ is defined as the algebraic subgroup generated by the one-parameter groups $\mathbb{A}^1_{\mathbb{F}_\ell} \to \mathrm{GL}_{H_\ell}$, $t \to \exp(t\log(g))$ for $g \in G$ of order ℓ. By construction $\tilde{G}$ is connected and generated by its unipotent elements and for $\ell \gg 0$ the following properties hold: (i) $\tilde{G}(\mathbb{F}_\ell)^+ = G^+$, (ii) $\tilde{G}(\mathbb{F}_\ell)/\tilde{G}(\mathbb{F}_\ell)^+$ is abelian of order $\leq 2^{r-1}$, (iii) there exists an abelian subgroup of prime-to-ℓ order $A \subset G$ such that G^+A is normal in G with $[G : G^+A] \leq \delta(r)$. To prove Theorem B, one considers a family $\mathrm{GL}_r \times \mathcal{N}_r \supset \mathcal{U}_r \to \mathcal{N}_r$ over R_r parametrizing exponentially generated subgroups of GL_r and, by noetherian induction and the classical Chevalley theorem, one constructs a universal map $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with the property that every exponentially generated subgroup of GL_{r,F_ℓ} ($\ell \geq r$) is the stabilizer of a line in $T^f(F_\ell^{\oplus r})$. By approximation theory (property (i) above), f separates - in the sense of Theorem B - U^+ from $\overline{G}_\ell^+$ for $U \in \mathcal{F}_{\ell,+}$ and $\ell \gg 0$. Then, by *ad-hoc* arguments (including properties (i), (iii) above), one adjusts f so that it satisfies exactly the conclusion of Theorem B.

To prove Theorem C, one proves first that, for the 'Galois closure' $\hat{X}_0^{\rho_\ell}$ of $X_0^{\rho_\ell} \to X$, the ratio $\lambda_{\hat{X}_0^{\rho_\ell}} =$'genus/degree' is bounded from below by an absolute constant $K > 0$. Since the cover $\hat{X}_0^{\rho_\ell} \to X$ is Galois, Stichenoth's bound and the Riemann-Hurwitz formula show that this amounts to proving that $g_{\hat{X}_0^{\rho_\ell}} > 1$ which, in turn, reduces to a combination of group-theoretic arguments involving the classification of finite subgroups of automorphism groups of genus ≤ 1 curves, Theorem A and assumptions (T), (I). One then shows by the Riemann-Hurwitz formula that $\lambda_{\hat{X}_0^{\rho_\ell}} - \lambda_{X_0^{\rho_\ell}} \to 0$. Here, the main difficulty is to control the length of the ramification filtration and the size of the ramification terms. Using assumption (T) and Theorem A, this eventually amounts to a 'non-concentration' estimate (Lemma 4.3.5.1) which, again, is proved using Nori's algebraic envelope.

1.9. The paper is organized as follows. After gathering the notation used throughout the paper in Section 2, we review briefly approximation theory in Section 3. The long Section 4 is devoted to the proof of Theorem 1.6.1 that is, essentially, that of Theorem B (Subsection 4.2) and Theorem C (Subsection 4.3). In Section 5, we give the above mentioned applications of Theorem 1.6.1 and Corollary 1.6.2 to families arising from étale cohomology groups and first Galois cohomology groups. Eventually, in the final Section 6, we discuss briefly the problem of extending Theorem 1.6.1 from $\mathbb{F}_\ell$-coefficients to arbitrary F_ℓ-coefficients.

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2. Notation

Unless explicitly mentioned, k will always denote a finitely generated field of characteristic $p \geq 0$. Fix a separable closure and an algebraic closure $k \hookrightarrow k^{sep} \hookrightarrow \overline{k}$ and write $\pi_1(k)$ for the absolute Galois group of k, which we identify with the étale fundamental group of $\mathrm{spec}(k)$.

2.1. **Notation for X.** Let X be a smooth, separated and geometrically connected scheme over k with generic point η and set of closed points $|X|$. For $x \in |X|$, let $k(x)$ denote its residue field. Set $\overline{X} := X \times_k \overline{k}$. When X is a curve let $\overline{X}^{cpt}$ denote the smooth compactification of $\overline{X}$, write $\partial \overline{X} := \overline{X}^{cpt} \setminus \overline{X}$ for the divisor at infinity and g_X, γ_X for the genus and gonality of $\overline{X}^{cpt}$respectively.
Recall that, by functoriality of étale fundamental group, every $x \in |X|$ gives rise to a commutative diagram of profinite groups with exact row

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\overline{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(k) & \longrightarrow & 1, \\ & & & & \uparrow \sigma_x & \nearrow & & & \\ & & & & \pi_1(x) & & & & \end{array}$$

where $\pi_1(x) \simeq \pi_1(k(x))$. We will always omit base points from the notation for étale fundamental groups (and implicitly work 'up to conjugation').

2.2. **Notation for bounded families of F_ℓ-linear representations of $\pi_1(X)$.** Let L be an infinite set of primes with $p \notin L$, and let $r \in \mathbb{Z}_{\geq 1}$. For every $\ell \in L$, fix a field F_ℓ of characteristic ℓ and a discrete $F_\ell[\pi_1(X)]$-module H_ℓ with F_ℓ-rank $r_\ell \leq r$ that is, equivalently, a continuous group homomorphism $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell) \simeq \mathrm{GL}_{r_\ell}(F_\ell)$.

Write

$$\begin{array}{ll} G_\ell & := \mathrm{im}(\rho_\ell) \subset \mathrm{GL}(H_\ell) \\ \overline{G}_\ell & := \rho_\ell(\pi_1(\overline{X})) \lhd G_\ell \\ G_{\ell,x} & := \mathrm{im}(\rho_\ell \circ \sigma_x) \subset G_\ell,\ x \in |X|. \end{array}$$

Note that for every $x \in |X|$ one has $[G_\ell : \overline{G}_\ell G_{\ell,x}] \leq [k(x) : k]$.

2.2.1. **Remark** By continuity of $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$, the group $G_\ell \subset \mathrm{GL}(H_\ell)$ is finite. In particular, one can always reduce to the case where $F_\ell \subset \overline{\mathbb{F}}_\ell$. Indeed, for every $\ell \in L$ fix an F_ℓ-basis of H_ℓ identifying $\mathrm{GL}(H_\ell) \simeq \mathrm{GL}_{r_\ell}(F_\ell)$. Then, as G_ℓ is finite, there exists a finitely generated subring $R_\ell = \mathbb{F}_\ell[a_1, \cdots, a_s] \subset F_\ell$ such that $G_\ell \subset \mathrm{GL}_{r_\ell}(R_\ell)$. For every $1 \neq g = (g_{i,j})_{1 \leq i,j \leq r_\ell} \in G_\ell$, let $I_g \subset R_\ell$ denote the ideal generated by the elements $g_{i,j}$, $1 \leq i \neq j \leq r_\ell$ and $1 - g_{i,i}$, $i = 1, \ldots, r_\ell$. Then I_g defines a strict closed subscheme $V_g \hookrightarrow \mathrm{spec}(R_\ell) =: V$ and for every closed point $a \in V \smallsetminus \cup_{1 \neq g \in G_\ell} V_g$, the specialization map $G_\ell \hookrightarrow \mathrm{GL}_{r_\ell}(R_\ell) \to \mathrm{GL}_{r_\ell}(\mathbb{F}_\ell(a))$ is injective.

2.3. **Notation for abstract modular curves with level-ℓ structure.** For every $\ell \in L$ and subgroup $U \subset G_\ell$, the continuity of $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$ implies that the inverse image $\rho_\ell^{-1}(U) \subset \pi_1(X)$ is an open subgroup hence corresponds to a connected étale cover $X_U \to X$, which is defined over a finite separable extension k_U of k. The general formalism of Galois categories implies that
- (Base change) $X_U \times_{k_U} \overline{k} \to \overline{X}$ is the connected étale cover corresponding to $\overline{U} := U \cap \overline{G}_\ell \subset \overline{G}_\ell$.
- (Moduli - see [SGA1, Prop. 6.4]) $x \in |X|$ lifts to a $k(x)$-rational point on X_U if and only if $G_{\ell,x} \subset U$.
For every $\ell \in L$, fix a family $\mathcal{F}_\ell$ of subgroups of G_ℓ not containing $\overline{G}_\ell$ and define the *abstract modular scheme associated with* $\mathcal{F}_\ell$ to be the (non-connected) étale cover

$$X_{\mathcal{F}_\ell}^{\rho_\ell} := \bigsqcup_{U \in \mathcal{F}_\ell} X_U \to X.$$

(We will omit the superscript $-^{\rho_\ell}$ when there is no possible confusion). Set

$$d_{X_{\mathcal{F}_\ell}^{\rho_\ell}} := \min\{[\overline{G}_\ell : U \cap \overline{G}_\ell] \mid U \in \mathcal{F}_\ell\}$$

and, when X is a curve,

$$g_{X^{\rho_\ell}_{\mathcal{F}_\ell}} := \min\{g_{X_U} \mid U \in \mathcal{F}_\ell\}, \quad \gamma_{X^{\rho_\ell}_{\mathcal{F}_\ell}} := \min\{\gamma_{X_U} \mid U \in \mathcal{F}_\ell\},$$

which we call respectively the degree, genus and gonality of the abstract modular scheme $X^{\rho_\ell}_{\mathcal{F}_\ell}$ (by convention, if $\mathcal{F}_\ell = \emptyset$, set $d_{X^{\rho_\ell}_{\mathcal{F}_\ell}} = +\infty$, $g_{X^{\rho_\ell}_{\mathcal{F}_\ell}} = +\infty$ and $\gamma_{X^{\rho_\ell}_{\mathcal{F}_\ell}} = +\infty$). Eventually, let $X(\mathcal{F}_\ell) \subset |X|$ denote the set of all $x \in |X|$ which lifts to a $k(x)$-rational point on $X_{\mathcal{F}_\ell}$ and for every $d \in \mathbb{Z}_{\geq 1}$, let $X(\mathcal{F}_\ell)^{\leq d} \subset X(\mathcal{F}_\ell)$ denote the set of all $x \in X(\mathcal{F}_\ell)$ with $[k(x) : k] \leq d$ (when $d = 1$, just write $X(\mathcal{F}_\ell)(k) := X(\mathcal{F}_\ell)^{\leq 1}$).

We will consider more specifically the following families.

2.3.1. **Example** For every $\ell \in L$, let $P : H_\ell \setminus \{0\} \to \mathbb{P}(H_\ell)$ denote the projectivization map and for $M \subset H_\ell$, write $PM := P(M \setminus \{0\})$. Also, for a group G acting on a set A and a subset $B \subset A$, let $\mathrm{Stab}_G(B)$ (resp. $\mathrm{Fix}_G(B)$) denote the subgroup of all $g \in G$ such that $gB = B$ (resp. $gb = b$, $b \in B$); when $B = \{b\}$ is a singleton, we write $\mathrm{Stab}_G(b) = \mathrm{Stab}_G(\{b\})(= \mathrm{Fix}_G(\{b\}))$.

- $X^{\rho_\ell}_+$, $d^{\rho_\ell}_+$, $g^{\rho_\ell}_+$ when $\mathcal{F}_\ell = \mathcal{F}_{\ell,+}$ is the family of all subgroups $U_\ell \subset G_\ell$ such that $\overline{G}^+_\ell \not\subset U_\ell$;
- $X^{\rho_\ell}_0$, $d^{\rho_\ell}_0$, $g^{\rho_\ell}_0$ when $\mathcal{F}_\ell = \mathcal{F}_{\ell,0}$ is the family of all $\mathrm{Stab}_{G_\ell}(Pv)$ for $0 \neq v \in H_\ell$;
- $\widehat{X}^{\rho_\ell}_0, \widehat{d}^{\rho_\ell}_0, \widehat{g}^{\rho_\ell}_0$ when $\mathcal{F}_\ell = \widehat{\mathcal{F}}_{\ell,0}$ is the family of all $\mathrm{Fix}_{G_\ell}(PM)$ for all $F_\ell[\pi_1(X)]$-submodules $0 \neq M \subset H_\ell$;
- $X^{\rho_\ell}_1$, $d^{\rho_\ell}_1$, $g^{\rho_\ell}_1$ when $\mathcal{F}_\ell = \mathcal{F}_{\ell,1}$ is the family of all $\mathrm{Stab}_{G_\ell}(v)$ for $0 \neq v \in H_\ell$.

Here, given a finite subgroup $G \subset \mathrm{GL}(H_\ell)$, we write G^+ for the (characteristic) subgroup generated by the order-ℓ elements in G. For $\ell \geq r$, G^+ is also the (characteristic) subgroup generated by the ℓ-Sylow subgroups in G.

2.4. **Conditions (T), (SS), (P), (I), (F).** For convenience we gather here the statements of conditions (T) (tameness), (P) (perfectness), (I) ((non-)isotriviality), which appear in our main theorems (Theorem 1.6.1, Theorem A and Theorem C) as well as the statement of condition (SS) (semisimplicity), which plays a technical part in our applications (see the proof of Fact 5.1, Remark 5.1.2 and Corollary 5.2.1) and of condition (F) (finiteness), which is an elementary but essential consequence of condition (T) (see Lemma 4.3.1).

(T) For every $x \in \partial\overline{X}$ there exists an open subgroup U_x of the inertia group $I_x \subset \pi_1(\overline{X})$ at x such that $\rho_\ell(U_x)$ is tame for $\ell \gg 0$.

(F) $\pi_1(\overline{X})/(\pi_1(\overline{X}) \cap K)$ is topologically finitely generated, where $K := \bigcap_{\ell \in L} \ker(\rho_\ell)$.

(P) For every open subgroup $\Pi \subset \pi_1(\overline{X})$, there exists an integer $B_\Pi \geq 1$ such that $|\rho_\ell(\Pi)^{ab}| \leq B_\Pi$ for $\ell \gg 0$.

(SS) H_ℓ is a semisimple $\pi_1(\overline{X})$-module for $\ell \gg 0$.

(I) For every open subgroup $\Pi \subset \pi_1(\overline{X})$, one has $H_\ell^\Pi = 0$ for $\ell \gg 0$.

Heuristically, these conditions should be regarded as compatibility conditions for the underlying geometric family $\rho_\ell|_{\pi_1(\overline{X})} : \pi_1(\overline{X}) \to \mathrm{GL}(H_\ell)$, $\ell \in L$.

3. Preliminary: Review of approximation theory for subgroups of $\mathrm{GL}_r(F_\ell)$

Approximation theory for finite subgroups G of $\mathrm{GL}_r(F_\ell)$ (r fixed, ℓ varying) consists in associating to G a connected algebraic subgroup $\mathcal{G} \hookrightarrow \mathrm{GL}_{r,F_\ell}$ whose properties reflect those of G and such that $\mathcal{G}(F_\ell)$ is close to G. The advantage of replacing G with $\mathcal{G}$ is that algebraic groups are more rigid hence easier to handle than arbitrary finite groups.

There are two approaches to this problem: one developed by M. Nori [N87] and J.-P. Serre [S00, 136, 137, 138], which is restricted to the case $F_\ell = \mathbb{F}_\ell$ but is quite explicit and functorial; and one developed by M. Larsen and R. Pink [LaP11], which works for arbitrary fields F_ℓ of coefficients but resorts to (much) heavier machinery from algebraic group theory and is somewhat more difficult to handle. We review briefly the main results of these theories that we will need. Note that Larsen-Pink's approach will only be used when considering arbitrary F_ℓ-coefficients (Theorem A and Subsection 6). The notation introduced here will be used in the rest of the paper.

Given a field F (of characteristic 0 or $\geq r$), we will say that a closed algebraic subgroup $\mathcal{G} \hookrightarrow \mathrm{GL}_{r,F}$ is exponentially generated, if it is generated by one-parameter subgroups of the form

$$\begin{array}{rccc} e_g: & \mathbb{A}^1_F & \to & \mathrm{GL}_{r,F} \\ & t & \to & \exp(t\log(g)) \end{array},$$

with $g \in \mathrm{GL}_r(F)$ unipotent.

3.1. Nori-Serre's approach.

Given a subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$, let $\tilde{G} \hookrightarrow \mathrm{GL}_{r,\mathbb{F}_\ell}$ denote the (exponentially generated) algebraic subgroup generated by the one-parameter subgroups $e_g : \mathbb{A}^1_{\mathbb{F}_\ell} \to \mathrm{GL}_{r,\mathbb{F}_\ell}$, $g \in G$ of order ℓ. The following gathers the main results of [N87].

3.1.1. **Fact** (Approximation Theory I)

(1) $\tilde{G}$ is connected and generated by its unipotent elements;
(2) For $\ell \gg 0$ (depending only on r) one has $G^+ = \tilde{G}(\mathbb{F}_\ell)^+$;
(3) The quotient $\tilde{G}(\mathbb{F}_\ell)/\tilde{G}(\mathbb{F}_\ell)^+$ is abelian of order $\leq 2^{r-1}$;
(4) There exists an integer $d(r) \geq 1$ such that for every prime ℓ and subgroup $G \subset \mathrm{GL}_r(\mathbb{F}_\ell)$ there exists an abelian subgroup $A \subset G$ of prime-to-ℓ order with the properties that G^+A is normal in G and $[G : G^+A] \leq d(r)$;
(5) If G acts semisimply on $\mathbb{F}_\ell^{\oplus r}$ then $\tilde{G}$ is semisimple (and one can choose A in such a way that it commutes with G^+ [CT13b]).

3.2. Larsen-Pink's approach.

[LaP11, Thm. 02 (and its proof)] implies that

3.2.1. **Fact** (Approximation Theory II) *There exists an integer $\delta(r) \geq 1$ such that for every prime ℓ, every field F_ℓ of characteristic ℓ and every finite subgroup $G \subset \mathrm{GL}_{r,F_\ell}$, there exists an algebraic subgroup $\mathcal{G} \hookrightarrow \mathrm{GL}_{r,\overline{F}_\ell}$ such that $[\mathcal{G} : \mathcal{G}^\circ] \leq \delta(r)$, $G \subset \mathcal{G}(\overline{F}_\ell)$, and there exist normal subgroups $R_u(G) \subset R(G) \subset G^\circ \subset G$ with the following properties*

- *$G^\circ = G \cap \mathcal{G}^\circ(\overline{F}_\ell)$ (in particular $[G : G^\circ] \leq \delta(r)$);*
- *$G^\circ/R(G)$ is a direct product of simple groups of Lie type by which one means groups of the form $D(\mathcal{S}^\Phi)$, where $\mathcal{G}^\circ \twoheadrightarrow \mathcal{S}$ is a simple (i.e. absolutely simple and adjoint) quotient and $\Phi : \mathcal{S} \to \mathcal{S}$ a Frobenius map so that the derived subgroup $D(\mathcal{S}^\Phi)$ be simple;*
- *$R(G)/R_u(G)$ embeds into the center of $(\mathcal{G}^\circ/R_u(\mathcal{G}^\circ))(\overline{F}_\ell)$;*
- *$R_u(G) = G \cap R_u(\mathcal{G}^\circ)(\overline{F}_\ell)$.*

One easily shows that for $\ell > \delta(r)$ the group $R_u(G)$ can be defined canonically as the largest normal ℓ-subgroup of G. In particular,

- $R_u(G)$ is characteristic in G for $\ell > \delta(r)$.
- Let G be a subgroup of $\mathrm{GL}_r(F_\ell)$. Let H_ℓ^{ss} denote the G-semisimplification of $F_\ell^{\oplus r} =: H_\ell$. Then the kernel of $G \to \mathrm{GL}(H_\ell^{ss})$ is a normal ℓ-subgroup and coincides with $R_u(G)$ for $\ell > \delta(r)$.

As an application of Larsen-Pink's filtration, one has the following refinement of Theorem A.

3.2.2. **Lemma** *Assume (T) and (SS). Then there exists an open subgroup $\Pi \subset \pi_1(\overline{X})$ such that for every open subgroup $\Pi' \subset \Pi$ one has $\rho_\ell(\Pi') = \rho_\ell(\Pi')^+$ and $\rho_\ell(\Pi')^{ab} = 0$ for $\ell \gg 0$. In particular (P) holds.*

Proof. First, from Theorem A, one may assume that there exists an open subgroup $\Pi \subset \pi_1(\overline{X})$ such that for every open subgroup $\Pi' \subset \Pi$ one has $\rho_\ell(\Pi') = \rho_\ell(\Pi')^+$ for $\ell \gg 0$[5]. Then, as there is no element of order ℓ^2 in $\mathrm{GL}_r(F_\ell)$ for $\ell \geq r$, one has

$$\rho_\ell(\Pi')^{ab} \simeq \mathrm{H}^1(\rho_\ell(\Pi'), \mathbb{F}_\ell).$$

So, it is enough to prove that $\mathrm{H}^1(\rho_\ell(\Pi'), \mathbb{F}_\ell) = 0$. Also, without loss of generality, one may assume that $\rho_\ell(\Pi')$ is normal in $\overline{G}_\ell$ hence acts semisimply on H_ℓ. Indeed, as $\Pi' \subset \pi_1(\overline{X})$ is open, it contains an open subgroup $\Pi'' \subset \Pi'$ which is normal in $\pi_1(\overline{X})$. Assume that one has proved that $\mathrm{H}^1(\rho_\ell(\Pi''), \mathbb{F}_\ell) = 0$, $\ell \gg 0$ for normal subgroups $\Pi'' \subset \Pi$, then the inflation-restriction exact sequence for H^1 gives

$$0 \to \mathrm{H}^1(\rho_\ell(\Pi')/\rho_\ell(\Pi''), \mathbb{F}_\ell) \to \mathrm{H}^1(\rho_\ell(\Pi'), \mathbb{F}_\ell) \to \mathrm{H}^1(\rho_\ell(\Pi''), \mathbb{F}_\ell)$$

and the conclusion follows from the fact that $\mathrm{H}^1(\rho_\ell(\Pi')/\rho_\ell(\Pi''), \mathbb{F}_\ell) = 0$ as soon as $\ell > [\Pi' : \Pi'']$. So, assume that $\rho_\ell(\Pi')$ acts semisimply on H_ℓ. Then $R_u(\rho_\ell(\Pi')) = 1$. The inflation-restriction exact sequence for H^1 again, gives

$$0 \to \mathrm{H}^1(\rho_\ell(\Pi')/\rho_\ell(\Pi')^\circ, \mathbb{F}_\ell) \to \mathrm{H}^1(\rho_\ell(\Pi'), \mathbb{F}_\ell) \to \mathrm{H}^1(\rho_\ell(\Pi')^\circ, \mathbb{F}_\ell)$$

and

$$0 \to \mathrm{H}^1(\rho_\ell(\Pi')^\circ/R(\rho_\ell(\Pi')), \mathbb{F}_\ell) \to \mathrm{H}^1(\rho_\ell(\Pi')^\circ, \mathbb{F}_\ell) \to \mathrm{H}^1(R(\rho_\ell(\Pi')), \mathbb{F}_\ell).$$

As both $\rho_\ell(\Pi')/\rho_\ell(\Pi')^\circ$ and $R(\rho_\ell(\Pi'))$ are of prime-to-ℓ order for $\ell \gg 0$, one is reduced to showing that

$$\mathrm{H}^1(\rho_\ell(\Pi')^\circ/R(\rho_\ell(\Pi')), \mathbb{F}_\ell) = 0.$$

But

$$\mathrm{H}^1(\rho_\ell(\Pi')^\circ/R(\rho_\ell(\Pi')), \mathbb{F}_\ell) = \bigoplus_{i \in I} \mathrm{H}^1(D(\mathcal{S}_i^{\Phi_i}), \mathbb{F}_\ell) \simeq \bigoplus_{i \in I} \mathrm{Hom}(D(\mathcal{S}_i^{\Phi_i})^{ab}, \mathbb{F}_\ell) = 0. \quad \square$$

4. Proof of Theorem 1.6.1

Let $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$, $\ell \in L$ be a bounded family of continuous $\mathbb{F}_\ell$-linear representations satisfying (T) and (P).

4.1. **Theorem B+Theorem C imply Theorem 1.6.1.** The form of Theorem B stated in the introduction was a simplified version of the following more technical statement.

We use the notation Ω from the introduction for the set of all maps $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with finite support. Given a field F, a finite-dimensional F-vector space V and a map $f \in \Omega$, set

$$T^f(V) := \bigoplus_{m,n \geq 0} (V^{\otimes m} \otimes (V^\vee)^{\otimes n})^{\oplus f(m,n)}.$$

(with the convention that $V^{\otimes 0} = F$ and $V^{\oplus 0} = 0$).

Eventually, for a subgroup $U \subset G_\ell$ set $U^{ss} := UR_u(\overline{G}_\ell)/R_u(\overline{G}_\ell) \subset G_\ell^{ss} (:= G_\ell/R_u(\overline{G}_\ell))$.

Theorem B:

(1) *There exists a map $g_r^{ss} \in \Omega$ (depending only on r) satisfying the following property. For every prime $\ell \gg 0$ and subgroups $U_\ell \subset G_\ell \subset \mathrm{GL}(H_\ell)$ such that $G_\ell^+ \not\subset U_\ell$ and that G_ℓ acts semisimply on H_ℓ, there exists a line $D_\ell \subset T^{g_r^{ss}}(H_\ell)$ such that $U_\ell \subset \mathrm{Stab}_{G_\ell}(D_\ell)$ but $G_\ell^+ \not\subset \mathrm{Stab}_{G_\ell}(D_\ell)$.*

[5]More precisely, from Theorem A, there exists open subgroup $\Pi \subset \pi_1(\overline{X})$ and a prime ℓ_0 such that $\rho_\ell(\Pi) = \rho_\ell(\Pi)^+$ for $\ell \geq \ell_0$. Then, for every open subgroup $\Pi' \subset \Pi$ one also has $\rho_\ell(\Pi') = \rho_\ell(\Pi')^+$ for $\ell \geq \max\{\ell_0, [\Pi : \Pi'] + 1\}$.

(2) *For every integer $d \geq 1$ there exists a map $g^u_{r,d} \in \Omega$ (depending only on r, d) satisfying the following property. For every prime $\ell \gg 0$ and subgroups $U_\ell \subset G_\ell \subset \mathrm{GL}(H_\ell)$ such that $G^+_\ell \not\subset U_\ell$, $[G_\ell : G^+_\ell] \leq d$ and $U^{ss,+}_\ell = G^{ss,+}_\ell$ there exists a line $D_\ell \subset T^{g^u_{r,d}}(H_\ell)$ such that $U_\ell \subset \mathrm{Stab}_{G_\ell}(D_\ell)$ but $G^+_\ell \not\subset \mathrm{Stab}_{G_\ell}(D_\ell)$.*

4.1.1. We now explain how to deduce Theorem 1.6.1 from Theorem B and Theorem C. From Theorem A, there exists an integer $d \geq 1$ such that $[\overline{G}_\ell : \overline{G}^+_\ell] \leq d$, $\ell \in L$. For every $\ell \in L$, let H^{ss}_ℓ denote the $\pi_1(\overline{X})$-semisimplification of H_ℓ. Note that H^{ss}_ℓ can be naturally equipped with a structure of $\pi_1(X)$-module. With the notation of Theorem B, set

$$T_\ell := T^{g^{ss}_r}(H^{ss}_\ell) \oplus T^{g^u_{r,d}}(H_\ell).$$

Note that, by construction, the $\mathbb{F}_\ell$-rank of T_ℓ only depends of r, d and $r_\ell \leq r$; in particular, it is uniformly bounded as ℓ varies. Fix $U_\ell \in \mathcal{F}_\ell$ such that $g_{X_{U_\ell}} = g^{\rho_\ell}_+$. From Theorem B, for $\ell \gg 0$ there exists a line $D_\ell \subset T_\ell$ such that $\overline{U}_\ell := U_\ell \cap \overline{G}_\ell \subset \mathrm{Stab}_{G_\ell}(D_\ell)$ but $\overline{G}^+_\ell \not\subset \mathrm{Stab}_{G_\ell}(D_\ell)$. As $\overline{G}^+_\ell$ is characteristic in $\overline{G}_\ell$ hence normal in G_ℓ, $T^+_\ell := T^{\overline{G}^+_\ell}_\ell \subset T_\ell$ is an $\mathbb{F}_\ell[G_\ell]$-submodule and one can consider $\widetilde{T}_\ell := T_\ell / T^+_\ell$, which is an $\mathbb{F}_\ell[G_\ell]$-module as well. Let $\widetilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\widetilde{T}_\ell)$ denote the representation associated with $\widetilde{T}_\ell$ and write $\widetilde{D}_\ell \subset \widetilde{T}_\ell$ for the image of D_ℓ in $\widetilde{T}_\ell$. Then $\widetilde{D}_\ell \neq 0$ and

$$U_\ell \subset \mathrm{Stab}_{G_\ell}(\widetilde{D}_\ell).$$

Thus X_{U_ℓ} is a connected étale cover of one of the connected components of $X^{\widetilde{\rho}_\ell}_0$. By Riemann-Hurwitz, this implies that $g^{\rho_\ell}_+ = g_{X_{U_\ell}} \geq g^{\widetilde{\rho}_\ell}_0$. Theorem 1.6.1 will then follow from Theorem C provided

4.1.2. **Lemma** *The $\widetilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\widetilde{T}_\ell)$, $\ell \in L$ satisfy (T) and (I).*

Proof. Condition (T) is straightforward by construction and the hypothesis (T) on $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$. Condition (I) follows from condition (P). More precisely, for every open subgroup $\Pi \subset \pi_1(\overline{X})$ and $\ell > [\pi_1(\overline{X}) : \Pi]$ one has $\rho_\ell(\Pi)^+ = \overline{G}^+_\ell$. Let $\tau \in T_\ell$ with image $\tilde{\tau}$ in $\tilde{T}_\ell$ fixed by Π that is for every $g \in \rho_\ell(\Pi)$ one has $b(g) : g\tau - \tau \in T^+_\ell$. In particular, $b|_{\overline{G}^+_\ell} : \overline{G}^+_\ell \to T^+_\ell$ is a group morphism (recall that by definition $\overline{G}^+_\ell$ acts trivially on T^+_ℓ hence $H^1(\overline{G}^+_\ell, T^+_\ell) = \mathrm{Hom}(\overline{G}^+_\ell, T^+_\ell)$). If $b|_{\overline{G}^+_\ell} : \overline{G}^+_\ell \to T^+_\ell$ were non-trivial, $\overline{G}^+_\ell$ would have an abelian quotient of order ℓ, contradicting (P). □

We now carry out the proofs of Theorem B and Theorem C, which are independent.

4.2. **Proof of Theorem B - Invariant theory for subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ (r fixed, ℓ varying).** Let G be a finite group and F a field of characteristic $\ell > 0$. Recall the following facts.

- Let M be an $F[G]$-module. Then for every integer $1 \leq n \leq \ell - 1$, symmetrization and antisymmetrization provide well-defined embeddings of $F[G]$-modules

$$\begin{array}{ccc} S^n M & \hookrightarrow & M^{\otimes n} \\ m_1 \cdots m_n & \mapsto & \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)} \end{array}$$

$$\begin{array}{ccc} \bigwedge^n M & \hookrightarrow & M^{\otimes n} \\ m_1 \wedge \cdots \wedge m_n & \mapsto & \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}, \end{array}$$

  where $\epsilon : \mathcal{S}_n \to \{\pm 1\}$ denotes the signature.

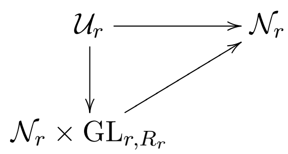
- **(semisimplicity of tensor products -** [S94, Thm. 1]**)** Let $M_1, \ldots, M_s$ be s semisimple $F[G]$-modules of finite rank as F-modules. It is not true in general that $M_1 \otimes_F \cdots \otimes_F M_s$ is also a semisimple $F[G]$-module but this holds if ℓ is large compared with the F-rank of the M_i, $i = 1, \ldots, s$. Precisely, assume $\dim_F(M_1)+\cdots+\dim_F(M_s) < \ell+s$. Then $M_1 \otimes_F \cdots \otimes_F M_s$ is also a semisimple $F[G]$-module.

- Let $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ be a map with finite support, M an $F[G]$-module of finite rank as F-module and $N \subset M$ an $F[G]$-submodule. If M is a semisimple $F[G]$-module then any splitting of $M^\vee \twoheadrightarrow N^\vee$ induces an embedding of $F[G]$-modules $T^f(N) \hookrightarrow T^f(M)$. In the following, we will implicitly assume that a choice for the splitting of $M^\vee \twoheadrightarrow N^\vee$ is given when M is a semisimple $F[G]$-module. If f has support contained in $\mathbb{Z}_{\geq 0} \times \{0\}$ (that is, there is no dual appearing in the tensor space), one always has a canonical embedding of $F[G]$-modules $T^f(N) \hookrightarrow T^f(M)$.

4.2.1. *Separation of exponentially generated subgroups.* Let $N_{r,\mathbb{Z}} \hookrightarrow \mathrm{M}_{r,\mathbb{Z}} \simeq \mathbb{A}^{r^2}_{\mathbb{Z}}$ denote the subscheme of nilpotent matrices, defined by $N_{r,\mathbb{Z}}(R) = \{X \in \mathrm{M}_r(R) \mid X^r = 0\}$ (R: $\mathbb{Z}$-algebra). Write $R_r := \mathbb{Z}[\frac{1}{(r-1)!}]$ and set $\mathcal{N}_r := N^{r^2}_{r,R_r}$. Consider the morphism of R_r-schemes $e : \mathcal{N}_r \times \mathbb{A}^{2r^2}_{R_r} \to \mathrm{GL}_{r,R_r}$ sending $(\underline{n}, \underline{t})$ to

$$e(\underline{n}, \underline{t}) = \exp(t_1 n_1) \cdots \exp(t_{r^2} n_{r^2}) \exp(t_{r^2+1} n_1) \cdots \exp(t_{2r^2} n_{r^2}).$$

Set $\mathcal{U}_r := \mathrm{graph}(e)$ and consider the following commutative diagram of R_r-schemes, where the arrows are the restrictions of the canonical projections:



From [Bo69, I, Prop. 2.2 and its proof], for every prime $\ell \geq r$, field F_ℓ of characteristic ℓ and exponentially generated subgroup $\mathcal{U} \hookrightarrow \mathrm{GL}_{r,F_\ell}$ there exists $\underline{n} \in \mathcal{N}_r(F_\ell)$ such that $\mathcal{U}_{r,\underline{n}} \twoheadrightarrow \mathcal{U} \hookrightarrow \mathrm{GL}_{r,F_\ell}$.

4.2.1.1. **Lemma** *There exists a map $f_r \in \Omega$ (depending only on r) such that for every prime $\ell \gg 0$, field F_ℓ of characteristic ℓ and exponentially generated subgroup $\mathcal{U} \hookrightarrow \mathrm{GL}_{r,F_\ell}$, one has*

$$\mathcal{U} = \mathrm{Fix}_{\mathrm{GL}_{r,F_\ell}}(T^{f_r}(F_\ell^{\oplus r})^{\mathcal{U}}).$$

Proof. This is essentially the same argument as in the proof of [LaP11, Prop. 2.3 (b)]. Let $\mathcal{V}_r \to \mathcal{N}_r$ denote the constant bundle $\mathbb{A}^r_{R_r} \times \mathcal{N}_r \to \mathcal{N}_r$ on which $\mathrm{GL}_{r,R_r} \times \mathcal{N}_r \to \mathcal{N}_r$ acts canonically. We argue by noetherian induction on $\mathcal{N}_r$. Let $\underline{\nu}$ denote the generic point of $\mathcal{N}_r$. Then the image of $\mathcal{U}_{r,\underline{\nu}}$ is a closed algebraic subgroup of $\mathrm{GL}_{r,\underline{\nu}}$ hence, by Chevalley's theorem [Bo69, II, Thm. 5.1], there exists a map $f_{r,0} \in \Omega$ and a subspace $V_1 \subset T^{f_{r,0}}(\mathcal{V}_{r,\underline{\nu}})$ such that the image of $\mathcal{U}_{r,\underline{\nu}}$ is the stabilizer of V_1 in $\mathrm{GL}_{r,\underline{\nu}}$. From [EGAIV.3, Prop. 9.6.1(i)], this property extends to an open neighbourhood $\mathcal{M}_{r,1}$ of $\underline{\nu}$ in $\mathcal{N}_r$. Iterating the argument for each generic point $\underline{\nu}_{1,1}, \ldots, \underline{\nu}_{1,s_1}$ of $\mathcal{N}_{r,1} := \mathcal{N}_r \setminus \mathcal{M}_{r,1}$, one obtains maps $f_{r,1,1}, \ldots, f_{r,1,s_1} \in \Omega$ *etc.* As $\mathcal{N}_r$ is of finite dimension, this process stops and the map with finite support

$$f_r^\circ = \sum_{0 \leq i \leq \dim(\mathcal{N}_r)} \sum_{1 \leq j \leq s_i} f_{r,i,j}$$

(where $s_0 := 1$ and $f_{r,0,1} := f_{r,0}$) has the property that for every exponentially generated subgroup $\mathcal{U} \hookrightarrow \mathrm{GL}_{r,F_\ell}$ there exists an F_ℓ-vector subspace $V_{\mathcal{U}} \subset T^{f_r^\circ}(F_\ell^{\oplus r})$ such that $\mathcal{U}$ is the stabilizer of $V_{\mathcal{U}}$ in $\mathrm{GL}_r(F_\ell)$. Equivalently, $\mathcal{U}$ is the stabilizer of the line $\Lambda^{\dim(V_{\mathcal{U}})} V_{\mathcal{U}}$ in $\mathrm{GL}_r(F_\ell)$ acting on $\Lambda^{\dim(V_{\mathcal{U}})} T^{f_r^\circ}(F_\ell^{\oplus r})$. As $\mathcal{U}$ is an exponentially generated group, it automatically acts trivially on $\Lambda^{\dim(V_{\mathcal{U}})} V_{\mathcal{U}}$. The conclusion then follows by considering the antisymmetrization morphism and replacing f_r° by the map $f_r \in \Omega$ defined by

$$T^{f_r}(F_\ell^{\oplus r}) = \bigoplus_{1 \leq \delta \leq \dim(T^{f_r^\circ}(F_\ell^{\oplus r}))} T^{f_r^\circ}(F_\ell^{\oplus r})^{\otimes \delta}. \ \square$$

4.2.2. *Separation of subgroups of bounded order.*

4.2.2.1. **Lemma** *For every integer $d \geq 1$ there exists a map $f_{r,d} \in \Omega$ (depending only on r, d) with finite support contained in $\mathbb{Z}_{\geq 0} \times \{0\}$ such that for every prime $\ell \gg 0$, field F_ℓ of characteristic ℓ, F_ℓ-module H_ℓ of F_ℓ-rank $r_\ell \leq r$ and finite subgroup $\Gamma \subset \mathrm{GL}(H_\ell)$ of order $\leq d$, there exists a line $D_\Gamma \subset T^{f_{r,d}}(H_\ell)$ such that*

$$\Gamma = \mathrm{Stab}_{\mathrm{GL}(H_\ell)}(D_\Gamma).$$

Proof. For simplicity assume $r_\ell = r$ in the following. More precisely, we are going to show that for every prime $\ell \gg 0$ and subgroup $\Gamma \subset \mathrm{GL}(H_\ell)$ of order $\leq d$ there exists an F_ℓ-vector subspace

$$V_\Gamma \subset \bigoplus_{\delta \leq d} (H_\ell^{\oplus r})^{\otimes \delta}$$

such that $\Gamma = \mathrm{Stab}_{\mathrm{GL}_{H_\ell}}(V_\Gamma)$. Let $\delta(r,d)$ and ν_Γ denote the F_ℓ-rank of $\bigoplus_{\delta \leq d}(H_\ell^{\oplus r})^{\otimes \delta}$ and V_Γ respectively. Then, equivalently, Γ is the stabilizer of the line $\bigwedge^{\nu_\Gamma} V_\Gamma$ in

$$\bigoplus_{\delta' \leq \delta(r,d)} \bigwedge^{\delta'} (\bigoplus_{\delta \leq d} (H_\ell^{\oplus r})^{\otimes \delta}),$$

which, for $\ell > \delta(r,d)$, embeds into

$$\bigoplus_{\delta' \leq \delta(r,d)} (\bigoplus_{\delta \leq d} (H_\ell^{\oplus r})^{\otimes \delta}))^{\otimes \delta'} =: T^{f_{r,d}}(H_\ell)$$

by antisymmetrization.

So, set

$$A_{r,\ell} := F_\ell[\mathrm{GL}_{H_\ell}] = F_\ell[X_{i,j} \mid 1 \leq i,j \leq r, \frac{1}{\det(X_{i,j})}]$$

and let $I_\Gamma \subset A_{r,\ell}$ denote the ideal defining Γ regarded as a closed algebraic subgroup of GL_{H_ℓ}. As a subvariety of GL_{H_ℓ}, Γ is just a set of closed (F_ℓ-rational) points hence

$$I_\Gamma = \bigcap_{\gamma \in \Gamma} I_\gamma = \prod_{\gamma \in \Gamma} I_\gamma,$$

where, writing $\gamma = (\gamma(i,j))_{1 \leq i,j \leq r} \in \mathrm{GL}(H_\ell)$, we write

$$I_\gamma := \langle X_{i,j} - \gamma(i,j) \mid 1 \leq i,j \leq r \rangle$$

for the (maximal) ideal defining $\{\gamma\}$. Then

$$I_\Gamma = \langle \prod_{\gamma \in \Gamma} (X_{i^\gamma, j^\gamma} - \gamma(i^\gamma, j^\gamma)) \mid (i^\gamma)_{\gamma \in \Gamma}, (j^\gamma)_{\gamma \in \Gamma} \in \{1, \ldots, r\}^\Gamma \rangle.$$

In particular, $V_\Gamma := A_{r,\ell,\leq d} \cap I_\Gamma$ contains a set of generators of I_Γ, where we write

$$A_{r,\ell,\leq d} \subset F_\ell[X_{i,j} \mid 1 \leq i,j \leq r] \subset A_{r,\ell}$$

for the vector subspace of degree $\leq d$ polynomials. Now, one easily checks that, for every $g \in \mathrm{GL}(H_\ell)$ the following are equivalent:

(i) $g \in \Gamma$;
(ii) $g\Gamma = \Gamma \subset \mathrm{GL}(H_\ell)$ (where g acts on $\mathrm{GL}(H_\ell)$ *via* translation);
(iii) $gI_\Gamma = I_\Gamma \subset A_{r,\ell}$ (where g acts on $A_{r,\ell}$ *via* $g \cdot P = P(g^{-1}-)$);
(iv) $gV_\Gamma = V_\Gamma \subset A_{r,\ell,\leq d}$.

It remains to show that $A_{r,\ell,\leq d} \subset \bigoplus_{\delta \leq d}(H_\ell^{\oplus r})^{\otimes \delta}$. This follows from the following facts
- The F_ℓ-vector subspace $A_{r,\ell,1} \subset A_{r,\ell,\leq d}$ of degree 1 polynomials is isomorphic to $H_\ell^{\oplus r}$;
- The F_ℓ-vector space $A_{r,\ell,\leq d}$ is isomorphic to $\oplus_{0 \leq \delta \leq d} S^\delta(A_{r,\ell,1})$ as a representation GL_{H_ℓ};
- For $\ell > d \geq \delta$, the symmetrization monomorphisms $S^\delta(A_{r,\ell,1}) \hookrightarrow (A_{r,\ell,1})^{\otimes \delta}$ are well-defined. □

4.2.3. *Separation of the $(-)^+$-part - Proof of Theorem B.* We begin with an elementary reduction. For an integer $n \geq 1$ write $H_{\ell^n} := H_\ell \otimes \mathbb{F}_{\ell^n}$.

4.2.3.1. **Lemma** *It is enough to prove Theorem B (1), (2) with H_ℓ replaced with $H_{\ell^{n(r)}}$ for some integer $n(r)\geq 1$ (depending only on r) and the line D_ℓ replaced by a non-zero submodule of arbitrary rank.*

Proof. Let $g_r\in\Omega$ such that for every prime $\ell\gg 0$ and subgroups $U_\ell\subset G_\ell\subset \mathrm{GL}(H_\ell)$ as in Theorem B (1) or (2), there exists a non-zero submodule $V_\ell\subset T^{g_r}(H_{\ell^{n(r)}})$ such that $U_\ell\subset \mathrm{Stab}_{G_\ell}(V_\ell)$ but $G_\ell^+\not\subset \mathrm{Stab}_{G_\ell}(V_\ell)$. The choice of an isomorphism $\mathbb{F}_{\ell^{n(r)}}\tilde{\rightarrow}\mathbb{F}_\ell^{\oplus n(r)}$ induces an isomorphism of $\mathbb{F}_\ell[G_\ell]$-modules $T^{g_r}(H_{\ell^{n(r)}})\tilde{\rightarrow}T^{g_r^1}(H_\ell)$, where $g_r^1(m,n)=n(r)g_r(m,n)$ and, under this isomorphism, V_ℓ is mapped onto an $\mathbb{F}_\ell$-submodule $W_\ell\subset T^{g_r^1}(H_\ell)$ of rank $\dim_{\mathbb{F}_\ell}(V_\ell)=n(r)\dim_{\mathbb{F}_{\ell^{n(r)}}}(V_\ell)$, which is again U_ℓ-invariant but not G_ℓ^+-invariant. Set $\nu_\ell:=\dim_{\mathbb{F}_\ell}(W_\ell)$ and $\tau(r):=\dim_{\mathbb{F}_\ell}(T^{g_{r,1}}(H_\ell))$. Then

$$D_\ell:=\bigwedge^{\nu_\ell}W_\ell\subset \bigwedge^{\nu_\ell}T^{g_r^1}(H_\ell)\subset \bigoplus_{1\leq\delta\leq\tau(r)}\bigwedge^{\delta}T^{g_r^1}(H_\ell)$$

is a line which is U_ℓ-invariant but not G_ℓ^+-invariant. Furthermore, for $\ell>\tau(r)$,

$$\bigoplus_{1\leq\delta\leq\tau(r)}\bigwedge^{\delta}T^{g_r^1}(H_\ell)\subset \bigoplus_{1\leq\delta\leq\tau(r)}T^{g_r^1}(H_\ell)^{\otimes\delta}=:T^{g_r^2}(H_\ell)$$

by antisymmetrization. □

4.2.3.2. *Proof of Theorem B (1).* By assumption U_ℓ^+ is strictly contained in G_ℓ^+ hence, by Fact 3.1.1(2), $\tilde{U}_\ell$ is strictly contained in $\tilde{G}_\ell$ as well for $\ell\gg 0$. From Lemma 4.2.1.1 there exists a line $D_\ell\subset T_\ell:=T^{f_r}(H_\ell)$ such that

$$\tilde{U}_\ell=\mathrm{Stab}_{G_\ell}(D_\ell).$$

In particular $U_\ell^+D_\ell=D_\ell$ since $U_\ell^+\subset\tilde{U}_\ell(\mathbb{F}_\ell)$ whereas $G_\ell^+D_\ell\neq D_\ell$. Indeed, otherwise, by definition of $\tilde{G}_\ell$, one would also have $\tilde{G}_\ell D_\ell=D_\ell$ hence $\tilde{G}_\ell\subset\tilde{U}_\ell$: a contradiction. Let $0\neq t_\ell\in D_\ell$ and consider the $\mathbb{F}_\ell[U_\ell]$-submodule $M_\ell:=\mathbb{F}_\ell[U_\ell t_\ell]\subset T_\ell$. We distinguish between two cases.

- If M_ℓ is not an $\mathbb{F}_\ell[G_\ell^+]$-submodule then we are done by Lemma 4.2.3.1.

- If M_ℓ is an $\mathbb{F}_\ell[G_\ell^+]$-submodule then the morphism of $\mathbb{F}_\ell[G_\ell^+]$-modules $M_\ell\hookrightarrow \mathrm{Res}^{G_\ell^+}_{G_\ell}T_\ell$ induces by adjunction a morphism of $\mathbb{F}_\ell[G_\ell]$-modules $\mathrm{Ind}^{G_\ell}_{G_\ell^+}M_\ell\rightarrow T_\ell$. Let M_ℓ° denotes its image. Note that $M_\ell\subset M_\ell^\circ$ as an $\mathbb{F}_\ell[G_\ell^+]$-submodule. From Fact 3.1.1(4), U_ℓ contains an abelian subgroup A_ℓ of prime-to-ℓ order such that $U_\ell^+A_\ell$ is normal in U_ℓ and $[U_\ell:U_\ell^+A_\ell]\leq d(r)$. Set

$$m_\ell^\circ:=\dim_{\mathbb{F}_\ell}(M_\ell^\circ)\leq\tau(r):=\dim_{\mathbb{F}_\ell}(T_\ell)$$

and $n(r):=\tau(r)!$. Then there exists an $\mathbb{F}_{\ell^{n(r)}}$-basis $e_1,\dots,e_{m_\ell^\circ}$ of $M^\circ_{\ell^{n(r)}}$ such that $A_\ell e_i\in\mathbb{F}_{\ell^{n(r)}}e_i$, $i=1,\dots,m_\ell^\circ$. Let $e_1^\vee,\dots,e^\vee_{m_\ell^\circ}$ denote the dual basis (that is $e_i^\vee(e_j)=1$ if $i=j$ and $=0$ otherwise). Note that $g\in G_\ell$ fixes $e_i\otimes e_i^\vee\in M^\circ_{\ell^{n(r)}}\otimes M^{\circ\vee}_{\ell^{n(r)}}$ if and only if g stabilizes the line $\mathbb{F}_{\ell^{n(r)}}e_i$ and its complement $\sum_{j\neq i}\mathbb{F}_{\ell^{n(r)}}e_j$. Also, as G_ℓ acts semisimply on T_ℓ for $\ell\gg 0$ and $\mathbb{F}_\ell$ is perfect, G_ℓ acts semisimply on $T_{\ell^{n(r)}}$ hence $M^\circ_{\ell^{n(r)}}\otimes M^{\circ\vee}_{\ell^{n(r)}}$ can be regarded as an $\mathbb{F}_{\ell^{n(r)}}[G_\ell]$-submodule of $T_{\ell^{n(r)}}\otimes T^\vee_{\ell^{n(r)}}$. Set

$$N_{\ell^{n(r)}}:=(M^\circ_{\ell^{n(r)}}\otimes M^{\circ\vee}_{\ell^{n(r)}})^{U_\ell^+A_\ell}\subset T_{\ell^{n(r)}}\otimes T^\vee_{\ell^{n(r)}}.$$

  As $U_\ell^+A_\ell$ is normal in U_ℓ, $N_{\ell^{n(r)}}$ is an $\mathbb{F}_{\ell^{n(r)}}[U_\ell]$-submodule. Again, we distinguish between two cases.

  – If $N_{\ell^{n(r)}}$ is not an $\mathbb{F}_{\ell^{n(r)}}[G_\ell^+]$-submodule. Then we are done by Lemma 4.2.3.1;

  – If $N_{\ell^{n(r)}}$ is an $\mathbb{F}_{\ell^{n(r)}}[G_\ell^+]$-submodule then G_ℓ^+ acts non-trivially on it (since it acts non-trivially on one of the $e_i\otimes e_i^\vee$ for $i=1,\dots,m_\ell$) hence the image $G^+_{N_{\ell^{n(r)}}}$ of G_ℓ^+ acting on $N_{\ell^{n(r)}}$ has order $\geq\ell$ whereas the image $U_{N_{\ell^{n(r)}}}$ of U_ℓ acting on $N_{\ell^{n(r)}}$ has order $\leq d(r)$. In particular, $G^+_{N_{\ell^{n(r)}}}\not\subset U_{N_{\ell^{n(r)}}}$

for $\ell > d(r)$. From Lemma 4.2.2.1, there exists a line $D_\ell \subset T^{f_{\tau(r)^2,d(r)}}(N_{\ell^{n(r)}})$ such that

$$U_{N_{\ell^{n(r)}}} = \mathrm{Stab}_{\mathrm{GL}_{N_{\ell^{n(r)}}}}(D_\ell).$$

Thus $U_\ell D_\ell = D_\ell$ whereas $G_\ell^+ D_\ell \neq D_\ell$. Eventually, note that as $f_{\tau(r)^2,d(r)}$ has support contained in $\mathbb{Z}_{\geq 0} \times \{0\}$, $T^{f_{\tau(r)^2,d(r)}}(N_{\ell^{n(r)}})$ embeds into $T^{f_{\tau(r)^2,d(r)}}(T_{\ell^{n(r)}} \otimes T^\vee_{\ell^{n(r)}})$ as an $\mathbb{F}_{\ell^{n(r)}}[G_\ell^+ U_\ell]$-submodule. Hence we are done by Lemma 4.2.3.1. □

Proof of Theorem B (2). From Fact 3.1.1(2), $G_\ell^+ = \tilde{G}_\ell(\mathbb{F}_\ell)^+$ and $U_\ell^+ = \tilde{U}_\ell(\mathbb{F}_\ell)^+$ for $\ell \gg 0$ so the assumption that U_ℓ^+ is a strict subgroup of G_ℓ^+ implies that $\tilde{U}_\ell$ is a strict subgroup of $\tilde{G}_\ell$ as well for $\ell \gg 0$. From Lemma 4.2.1.1 applied to $\tilde{U}_\ell \hookrightarrow \mathrm{GL}_{H_\ell}$ there exists $0 \neq t_\ell \in T^{f_r}(H_\ell)$ such that $\tilde{U}_\ell$ is the stabilizer of t_ℓ in $\tilde{G}_\ell$. Whence a surjection

$$G_\ell^+/U_\ell^+ \twoheadrightarrow G_\ell^+ t_\ell$$

whose fibers are of cardinality $[\tilde{U}_\ell(\mathbb{F}_\ell) \cap G_\ell^+ : U_\ell^+]$. But one has the following inclusions:

$$\begin{array}{ccccc} & & & \overset{(1)}{\nearrow} & G_\ell^+ \\ U_\ell^+ = \tilde{U}_\ell(\mathbb{F}_\ell)^+ & \hookrightarrow & \tilde{U}_\ell(\mathbb{F}_\ell) \cap G_\ell^+ = \tilde{U}_\ell(\mathbb{F}_\ell) \cap \tilde{G}_\ell(\mathbb{F}_\ell)^+ & & \\ & & & \underset{(2)}{\searrow} & \tilde{U}_\ell(\mathbb{F}_\ell) \end{array}$$

The inclusion (1) shows that $[\tilde{U}_\ell(\mathbb{F}_\ell) \cap G_\ell^+ : U_\ell^+]$ is a power of ℓ by the assumption that $U_\ell^{ss,+} = G_\ell^{ss,+}$, whereas the inclusion (2) shows that $[\tilde{U}_\ell(\mathbb{F}_\ell) \cap G_\ell : U_\ell^+] \leq 2^{r-1}$ by Fact 3.1.1(3). Thus, for $\ell \gg 0$ one has $U_\ell^+ = \tilde{U}_\ell(\mathbb{F}_\ell) \cap G_\ell^+$ hence the above surjection is actually a bijection and $|G_\ell^+ t_\ell|$ is a positive power of ℓ. Again, set $M_\ell := \mathbb{F}_\ell[U_\ell t_\ell]$ and distinguish between two cases.

- If M_ℓ is not an $\mathbb{F}_\ell[G_\ell^+]$-submodule then we are done by Lemma 4.2.3.1.

- If M_ℓ is an $\mathbb{F}_\ell[G_\ell^+]$-submodule then G_ℓ^+ acts non-trivially on M_ℓ since $|G_\ell^+ t_\ell| \geq \ell$. Let $G_{M_\ell}^+$ and U_{M_ℓ} denote the image of G_ℓ^+ and U_ℓ acting on M_ℓ. Then $|U_{M_\ell}| \leq d$ by the assumption that $U_\ell^{ss,+} = G_\ell^{ss,+}$ and $[G_\ell : G_\ell^+] \leq d$, whereas $\ell \,|\, |G_{M_\ell}^+|$. So $G_{M_\ell}^+ \neq U_{M_\ell}$ for $\ell \gg 0$. Set

$$m_\ell := \dim_{\mathbb{F}_\ell}(M_\ell) \leq \tau(r) := \dim_{\mathbb{F}_\ell}(T_\ell).$$

From Lemma 4.2.2.1, there exists a line $D_\ell \subset T^{f_{\tau(r),d}}(M_\ell)$ such that

$$U_{M_\ell} = \mathrm{Stab}_{\mathrm{GL}_{M_\ell}}(D_\ell).$$

Thus $U_\ell D_\ell = D_\ell$ whereas $G_\ell^+ D_\ell \neq D_\ell$. Eventually, as $f_{\tau(r),d}$ has support contained in $\mathbb{Z}_{\geq 0} \times \{0\}$, $T^{f_{\tau(r),d}}(M_\ell)$ embeds into $T^{f_{\tau(r),d}}(T_\ell)$ as an $\mathbb{F}_\ell[G_\ell^+ U_\ell]$-submodule. Hence we are done by Lemma 4.2.3.1.
□

4.3. **Proof of Theorem C.** We will work over the algebraically closed field $\overline{k}$ and for simplicity write X and G_ℓ for $\overline{X}$ and $\overline{G}_\ell$, respectively. We will use freely the notation introduced in Example 2.3.1. We begin with an elementary observation about conditions (T) and (I).

4.3.1. **Lemma** *(T) implies (F).*

Proof. From (T), there exists an open normal subgroup $U \subset \pi_1(X)$ such that the induced family $\rho_\ell|_U : U \to \mathrm{GL}(H_\ell)$ is tame for $\ell \gg 0$. Let $U \twoheadrightarrow U^t$ denote the maximal tame quotient of U. As the action of $\pi_1(X)$ by conjugation on U permutes the inertia groups, the kernel K_U of $U \twoheadrightarrow U^t$ is normal in $\pi_1(X)$. So, for each prime ℓ the representation $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$ factors through $\pi_1(X)/K_U$, thus, $K_U \subset K$. Hence it is enough to prove that $\pi_1(X)/K_U$ is topologically finitely generated. But this follows from the fact that U^t is topologically finitely generated and the short exact sequence of profinite groups

$$1 \to U^t \to \pi_1(X)/K_U \to \pi_1(X)/U \to 1. \ \square$$

Lemma 4.3.1 and Theorem A imply in particular,

4.3.2. **Lemma** *Assume (T) and (I). Then* $d_0^{\rho_\ell} \to +\infty$*.*

Proof. First, from condition (T) and Theorem A, up to replacing X with a connected étale cover, one may assume that $G_\ell = G_\ell^+$ for $\ell \gg 0$. Assume Lemma 4.3.2 does not hold. Then, for some integer $B \geq 1$ and infinitely many primes ℓ, there exists $0 \neq v_\ell \in H_\ell$ such that $|G_\ell P v_\ell| \leq B$. From (F), there are only finitely many possibilities for the connected étale cover $X_{Pv_\ell} \to X$ corresponding to the stabilizers $\mathrm{Stab}_{G_\ell}(Pv_\ell)$ of those $0 \neq v_\ell \in H_\ell$ such that $|G_\ell P v_\ell| \leq B$. So, up to replacing X with a connected étale cover, one may assume that for infinitely many primes ℓ, (a) there exists $0 \neq v_\ell \in H_\ell$ such that $|G_\ell P v_\ell| = 1$ and (b) $G_\ell = G_\ell^+$. But (a) and (b) imply that G_ℓ acts trivially on $\mathbb{F}_\ell v_\ell$, which contradicts (I). $\square$

We introduce a few more notations. Recall that $P : H_\ell \setminus \{0\} \to \mathbb{P}(H_\ell)$ denotes the projectivization map. Correspondingly, write $P : \mathrm{GL}(H_\ell) \to \mathrm{PGL}(H_\ell)$. For every $0 \neq v \in H_\ell$, let $X_v \to X_{Pv} \to X$ denote the connected étale covers corresponding to the inclusions of open subgroups

$$\rho_\ell^{-1}(\mathrm{Stab}_{G_\ell}(v)) \subset \rho_\ell^{-1}(\mathrm{Stab}_{G_\ell}(Pv)) \subset \pi_1(X)$$

and for every $\pi_1(X)$-submodule $0 \neq M \subset H_\ell$, let $X_M \to X_{PM} \to X$ denote the connected étale covers corresponding to the inclusions of open subgroups

$$\rho_\ell^{-1}(\mathrm{Fix}_{G_\ell}(M)) \subset \rho_\ell^{-1}(\mathrm{Fix}_{G_\ell}(PM)) \subset \pi_1(X),$$

where $PM := P(M \setminus \{0\})$. The cover $X_M \to X$ (resp. $X_{PM} \to X$) is Galois with group the image G_M of $G_\ell \to \mathrm{GL}(M)$ (resp. the image $G_{PM} = PG_M$ of $G_\ell \to \mathrm{PGL}(M)$). Write g_v, g_{Pv}, g_M and g_{PM} for the genus of X_v, X_{Pv}, X_M and X_{PM} respectively.

For $0 \neq v \in H_\ell$, let $M(v) := \mathbb{F}_\ell[G_\ell v] \subset H_\ell$ denote the $\pi_1(X)$-submodule generated by v. Then the cover $X_{M(v)} \to X$ (resp. $X_{PM(v)} \to X$) is the Galois closure of $X_v \to X$ (resp. $X_{Pv} \to X$). For every $\pi_1(X)$-submodule $0 \neq M \subset H_\ell$, set

$$\lambda_{PM} := \frac{2g_{PM} - 2}{|G_{PM}|}.$$

The first step of the proof of Theorem C consists in proving that the genus of the 'Galois closure' $\widehat{X}_0^{\rho_\ell}$ of $X_0^{\rho_\ell}$ grows linearly with its degree.

4.3.3. **Lemma** *Assume (T) and (I). Then there exists a constant $K > 0$ such that for every prime $\ell \gg 0$ and $\pi_1(X)$-submodule $0 \neq M \subset H_\ell$, one has*

$$\lambda_{PM} \geq K.$$

The second step consists in comparing the genus of $X_0^{\rho_\ell}$ and the genus of $\widehat{X}_0^{\rho_\ell}$ by the Riemann-Hurwitz formula.

4.3.4. *Proof of Lemma 4.3.3.*

We prove first the following weaker version of Lemma 4.3.3.

4.3.4.1. **Lemma** *Assume (T) and (I). Then* $\widehat{g}_0^{\rho_\ell}\rightarrow +\infty$.

Proof. Observe first that if $\widehat{g}_0^{\rho_\ell}\geq 2$ for $\ell\gg 0$ then $\widehat{g}_0^{\rho_\ell}\rightarrow\infty$. Indeed, there exists a polynomial $P_p(T)\in\mathbb{Z}[T]$ such that the automorphism group of every genus $g\geq 2$ connected curve, smooth and separated over an algebraically closed field of characteristic p is of order $\leq P_p(g)$ [Sti73]. So, if $\widehat{g}_0^{\rho_\ell}\geq 2$ then

$$(d_0^{\rho_\ell}\leq)\widehat{d}_0^{\rho_\ell}\leq \min\{P_p(g_{PM})\}_{0\neq M\subset H_\ell}$$

and the conclusion follows from Lemma 4.3.2.

So, the only cases to rule out are

(i) $g_X=0$ and, for infinitely many primes ℓ, there exists $0\neq M_\ell\subset H_\ell$ such that $g_{PM_\ell}=0$;
(ii) $g_X=0$ and, for infinitely many primes ℓ, there exists $0\neq M_\ell\subset H_\ell$ such that $g_{PM_\ell}=1$;
(iii) $g_X=1$ and, for infinitely many primes ℓ, there exists $0\neq M_\ell\subset H_\ell$ such that $g_{PM_\ell}=1$;

The assumption that $g_{PM}\leq 1$ imposes strong restrictions on the structure of G_M, namely

Fact (Classification) *Let C be a smooth, separated and connected curve over an algebraically closed field k of characteristic $p\geq 0$. Let G be a finite subgroup of the automorphism group of C.*
(1) Assume that $g_C=1$ then G is an extension

$$1\rightarrow A\rightarrow G\rightarrow Q\rightarrow 1$$

with A a finite quotient of $\hat{\mathbb{Z}}^2$ and $|Q|\leq 24$.
(2) ([Su82, Thm. 6.17]) *Assume that $g_C=0$ then G is either (a) a cyclic group; (b) a dihedral group; (c) an alternating group $\mathcal{A}_4$, $\mathcal{A}_5$ or the symmetric group $\mathcal{S}_4$; (d) a split extension*

$$1\rightarrow A\rightarrow G\rightarrow Q\rightarrow 1,$$

where A is an elementary abelian p-group and Q is a cyclic group of prime-to-p order; (e) $\mathrm{PSL}_2(\mathbb{F}_{p^n})$*, $n\geq 1$; or (f)* $\mathrm{PGL}_2(\mathbb{F}_{p^n})$*, $n\geq 1$. The three last cases only occur when $p>0$.*

- Assume first that case (ii) or (iii) holds, then, from Fact (Classification) (1), the group G_{PM_ℓ} contains an abelian subgroup A_{M_ℓ} of index ≤ 24 for infinitely many $\ell\geq 0$. From condition (F) there are only finitely many isomorphism classes of connected étale covers of X of degree ≤ 24 corresponding to the inverse image of A_{M_ℓ} in $\pi_1(X)$. Hence, at least one of them - say $Y\rightarrow X$ - appears infinitely many times. So, up to base-changing by $Y\rightarrow X$, one may assume that G_{PM_ℓ} is abelian for infinitely many ℓ. From Theorem A, one may also assume that $G_{M_\ell}=G_{M_\ell}^+$ hence *a fortiori* that $G_{PM_\ell}=G_{PM_\ell}^+$ that is G_{PM_ℓ} is abelian of order a power of ℓ for infinitely many ℓ. But then, by Schur-Zassenhauss, the short exact sequence

$$1\rightarrow\mathbb{F}_\ell^\times\cap G_{M_\ell}\rightarrow G_{M_\ell}\rightarrow G_{PM_\ell}\rightarrow 1$$

  splits. Write again G_{PM_ℓ} for a complement of $\mathbb{F}_\ell^\times\cap G_{M_\ell}$ in G_{M_ℓ}. As G_{PM_ℓ} is of order a power of ℓ, it fixes a non-zero vector on M_ℓ so

$$N_\ell:=M_\ell^{G_{PM_\ell}}\subset M_\ell$$

  is non-zero. As $\mathbb{F}_\ell^\times\cap G_{M_\ell}$ is central in G_{M_ℓ}, $N_\ell\subset M_\ell$ is a $\pi_1(X)$-submodule such that, by construction, $\mathbb{F}_\ell^\times\cap G_{M_\ell}\twoheadrightarrow G_{N_\ell}$. But as we have reduced to the case where $G_{M_\ell}=G_{M_\ell}^+$, one also has $G_{N_\ell}=G_{N_\ell}^+$ hence $G_{N_\ell}=1$. This contradicts (I).

- The same arguments show that if case (i) holds then G_{PM_ℓ} can only be of type (2) (d), (e) or (f) for $\ell\gg 0$, which is only possible if $p>0$.

  – Assume that case (2) (d) occurs that is, G_{PM_ℓ} is a split extension of a cyclic group Q_{M_ℓ} of prime-to-p order by an elementary abelian p-group A_{M_ℓ}. Letting $\mathrm{PGL}(M_\ell)$ acting by conjugation on $E_\ell:=\mathrm{End}(M_\ell)$, one gets an embedding

$$A_{M_\ell}\hookrightarrow G_{PM_\ell}\hookrightarrow\mathrm{PGL}(M_\ell)\hookrightarrow\mathrm{GL}(E_\ell).$$

As A_{M_ℓ} is abelian of prime-to-ℓ order, it is conjugate in $\mathrm{GL}(E_\ell\otimes\overline{\mathbb{F}}_\ell)$ to a subgroup of the diagonal torus. As a result, the $\mathbb{F}_p$-dimension r_{M_ℓ} of A_{M_ℓ} is $\leq r_\ell^2$. So any complement - which we denote again by Q_{M_ℓ} - of A_{M_ℓ} in G_{PM_ℓ} is an abelian (even cyclic) subgroup of bounded index

$$[G_{PM_\ell}:Q_{M_\ell}]=|A_{M_\ell}|\leq p^{r^2_{M_\ell}}\leq p^{r^2}.$$

And one can conclude as above, using Theorem A .

– Eventually, assume that we are in case (2) (e) or (f) that is, G_{PM_ℓ} is isomorphic to either $\mathrm{PSL}_2(\mathbb{F}_{p^n})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^n})$, hence in particular $\mathrm{PSL}_2(\mathbb{F}_{p^n})\hookrightarrow G_{PM_\ell}$. By considering the upper triangular unipotent subgroup, one has

$$(\mathbb{F}_p)^{\oplus n}\simeq\mathbb{F}_{p^n}\hookrightarrow\mathrm{PSL}_2(\mathbb{F}_{p^n})\hookrightarrow G_{PM_\ell},$$

and one can deduce as above that $n\leq r^2$. Thus, only finitely many $n\geq 1$ occur for cases (2) (e) and (f), which is ruled out by Lemma 4.3.2. □

4.3.4.2. *End of proof of Lemma 4.3.3.* Lemma 4.3.4.1 implies that for $\ell\gg 0$ and every $\pi_1(X)$-submodule $0\neq M\subset H_\ell$ one has $\lambda_{PM}>0$. To deduce Lemma 4.3.3 from this observation, one has to exploit the restrictions imposed by (T) on the ramification filtration of $X_{PM}\to X$ at $x\in\partial X$. More precisely, from (T), for every $x\in\partial X$ there exists a normal open subgroup $U_x\lhd I_x$ such that $\rho_\ell(U_x)$ is tame for $\ell\gg 0$. Set $m_x:=[I_x:U_x]$. Write $I_{x,PM}$ and $U_{x,PM}$ for the images of I_x and U_x in G_{PM} respectively, set $m_{x,PM}:=[I_{x,PM}:U_{x,PM}]$ (which divides m_x) and let

$$(RF_{x,PM})\quad\cdots\lhd I_{x,PM,i+1}\lhd I_{x,PM,i}\lhd\cdots\lhd I_{x,PM,0}=I_{x,PM}$$

denote the ramification filtration (with lower numbering) at x in G_{PM}. Write $\tilde{I}_{x,PM}:=I_{x,PM}/U_{x,PM}$ and $\tilde{I}_{x,PM,i}:=I_{x,PM,i}U_{x,PM}/U_{x,PM}$, $i\geq 0$. Note that, as $U_{x,PM}$ is of order prime-to-p while $I_{x,PM,i}$ is a p-group for $i\geq 1$, one has $I_{x,PM,i}\tilde{\to}\tilde{I}_{x,PM,i}$. Finally, let

$$\cdots\lhd(\tilde{I}_{x,PM})_{i+1}\lhd(\tilde{I}_{x,PM})_i\lhd\cdots\lhd(\tilde{I}_{x,PM})_0=\tilde{I}_{x,PM}$$

denote the ramification filtration (with lower numbering) of $\tilde{I}_{x,PM}$.

Claim 1 *There are only finitely many possibilities for the filtration*

$$\cdots\lhd\tilde{I}_{x,M,i+1}\lhd\tilde{I}_{x,PM,i}\lhd\cdots\lhd\tilde{I}_{x,PM,0}=\tilde{I}_{x,PM}$$

(when ℓ and M varying), and $\tilde{I}_{x,PM,i}=(\tilde{I}_{x,PM})_{j(i)}$, where $u_{x,PM}:=|U_{x,PM}|$ and $j(i)=\lceil\frac{i}{u_{x,PM}}\rceil$.

Proof of Claim 1. As the question is local at x, fixing any place $x_M\in\partial X_{PM}$ above x and completing, the problem amounts to studying the ramification filtration of a Galois extension $K\hookrightarrow K_{PM}$ of fraction fields of complete discrete valuation rings with group $I_{x,PM}$. Then

$$\cdots\lhd(\tilde{I}_{x,PM})_{i+1}\lhd(\tilde{I}_{x,PM})_i\lhd\cdots\lhd(\tilde{I}_{x,PM})_0=\tilde{I}_{x,PM}$$

is the ramification filtration at x associated with the intermediary field extension $K\hookrightarrow K_{PM}^{U_{x,PM}}$. There are only finitely many possibilities for normal open subgroups of I_x containing U_x hence only finitely many possibilities for the intermediary field extension $K\hookrightarrow K_{PM}^{U_{x,PM}}$ so for the possible ramification filtrations

$$\cdots\lhd(\tilde{I}_{x,PM})_{i+1}\lhd(\tilde{I}_{x,PM})_i\lhd\cdots\lhd(\tilde{I}_{x,PM})_0=\tilde{I}_{x,PM}.$$

The epimorphism

$$I_{x,PM}\twoheadrightarrow I_{x,PM}/U_{x,PM}=\tilde{I}_{x,PM,0}$$

induces an isomorphism from the p-Sylow $I_{x,PM,1}$ of $I_{x,PM}$ onto the p-Sylow $\tilde{I}_{x,PM,1}$ of $\tilde{I}_{x,PM}$. But, since $U_{x,PM}$ is of prime-to-p order $u_{x,M}$, it follows from [S68, Chap. IV, Lemma 5] that the group $I_{x,PM,i}$ maps isomorphically onto $(\tilde{I}_{x,PM})_{j(i)}$, where

$$j(i)=\lceil\frac{i}{u_{x,PM}}\rceil.\ \square$$

In particular the length of $(RF_{x,PM})$ $(:= \min\{i \mid I_{x,PM,i} = 1\})$ and the orders of the $I_{x,PM,i}$, $i \geq 1$ can take only finitely many values.

Set $s := |\partial X|$ and $e_{x,PM} := |I_{x,PM}|$. Then the Riemann-Hurwitz formula yields

$$0 < \lambda_{PM} = 2g_X - 2 + \sum_{x\in\partial X}\sum_{i\geq 0}\frac{|I_{x,PM,i}|-1}{e_{x,PM}} = 2g_X - 2 + \sum_{x\in\partial X}(\frac{e_{x,PM}-1}{e_{x,PM}} + \sum_{i>0}\frac{|I_{x,PM,i}|-1}{m_{x,PM}}) = \alpha_{PM} - \beta_{PM}$$

with

$$\alpha_{PM} = 2g_X - 2 + s + \sum_{x\in\partial X}\sum_{i>0}\frac{|I_{x,PM,i}|-1}{m_{x,PM}}, \quad \beta_{PM} = \sum_{x\in\partial X}\frac{1}{e_{x,PM}}.$$

By definition $\beta_{PM} \in \mathcal{E}_s$, where

$$\mathcal{E}_s := \{\sum_{1\leq i\leq s}\frac{1}{e_i} \mid (e_1, \dots, e_s) \in \mathbb{Z}^s_{>0}\}$$

(with the convention $\mathcal{E}_0 := \{0\}$). By Claim 2 below, there exists a rational number $\delta_{PM} > 0$ such that $]\alpha_{PM} - \delta_{PM}, \alpha_{PM}[\ \cap \mathcal{E}_s = \emptyset$. But, from Claim 1, α_{PM} can only take finitely many values so $\delta := \min\{\delta_{PM}\}$ is also > 0 and satisfies $]\alpha_{PM} - \delta, \alpha_{PM}[\ \cap\mathcal{E}_s = \emptyset$ for every $0 \neq M \subset H_\ell$ and prime ℓ. Take any $0 < K \leq \delta$. Then, as $\alpha_{PM} - \lambda_{PM} = \beta_{PM} \in \mathcal{E}_s \cap\ ]0, \alpha_{PM}[$ one has $\alpha_{PM} - \lambda_{PM} \leq \alpha_{PM} - K$.

Claim 2 *For any rational number $\alpha > 0$, there exists a rational number $\beta \in [0; \alpha[$ such that*

$$]\beta; \alpha[\ \cap \mathcal{E}_s = \emptyset.$$

Proof of Claim 2. We proceed by induction on $s \geq 1$. For $s = 1$, write $\alpha = \frac{d}{n}$ with $d, n \in \mathbb{Z}_{>0}$ coprime and take $\beta = \frac{1}{q+1}$, where $q \in \mathbb{Z}_{>0}$ is defined by $n = qd + r$ with $0 \leq r < d$. For $s \geq 2$, pick any $\alpha_0 \in]0, \alpha[\ \cap\mathcal{E}_{s+1}$. Given $\gamma \in \mathcal{E}_{s+1}$, write $\gamma = \frac{1}{e_1} + \cdots + \frac{1}{e_{s+1}}$ with $e_1 \leq \cdots \leq e_{s+1}$. Then $\gamma \leq \frac{s+1}{e_1}$. We distinguish between two cases:

- $\frac{s+1}{e_1} \leq \alpha_0$ hence $\gamma \leq \alpha_0$;
- $\frac{s+1}{e_1} > \alpha_0$ hence $e_1 < \frac{s+1}{\alpha_0}$. This can happen only for finitely many values $1, \dots, \lfloor\frac{s+1}{\alpha_0}\rfloor$ of e_1. By induction hypothesis, for every $i = 1, \dots, \lfloor\frac{s+1}{\alpha_0}\rfloor$ there exists $\beta_i \in [0, \alpha_0 - \frac{1}{i}[$ such that $]\beta_i, \alpha_0 - \frac{1}{i}[\ \cap\mathcal{E}_s = \emptyset$.

Then

$$\beta := \max\{\alpha_0,\ \beta_i + \frac{1}{i},\ i = 1, \dots, \lfloor\frac{s+1}{\alpha_0}\rfloor\}$$

has the requested property. □

4.3.5. *End of proof of Theorem C.* Before turning to the proof of Theorem C itself, we establish Lemma 4.3.5.1 below, which plays a crucial part in the comparison of the genera of $X_0^{\rho_\ell}$ and $\hat{X}_0^{\rho_\ell}$.

For every $\ell \in L$, let H_ℓ be an $\mathbb{F}_\ell$-module of rank $r_\ell \leq r$ and $G_\ell \subset \mathrm{GL}(H_\ell)$ a subgroup. Then,

4.3.5.1. **Lemma** *For every $\ell \in L$, $0 \neq v_\ell \in H_\ell$ and $\mathbb{F}_\ell$-vector subspace $N_\ell \subset H_\ell$ such that $G_\ell^+ v_\ell \not\subset N_\ell$ one has*

$$\frac{|G_\ell^+ v_\ell \cap N_\ell|}{|G_\ell^+ v_\ell|} \leq \frac{r^3}{\ell}.$$

Proof. Since N_ℓ is the intersection of $\mathrm{codim}_{H_\ell}(N_\ell)$ hyperplanes, one can restrict[6] to the case $\mathrm{codim}_{H_\ell}(N_\ell) = 1$ that is $N_\ell = \ker(\lambda_\ell)$ for some non-zero $\mathbb{F}_\ell$-linear form $\lambda_\ell : H_\ell \twoheadrightarrow \mathbb{F}_\ell$. Recall that every $g \in G_\ell$ of order ℓ defines a one-parameter subgroup

$$\begin{array}{rccc} e_g\colon & \mathbb{A}^1_{\mathbb{F}_\ell} & \to & \mathrm{GL}_{H_\ell} \\ & t & \to & \exp(t\log(g)). \end{array}$$

[6]More precisely, if $t = \mathrm{codim}_{H_\ell}(N_\ell)$ and $N_\ell = H_{\ell,1} \cap \cdots \cap H_{\ell,t}$ for hyperplanes $H_{\ell,1}, \cdots, H_{\ell,t}$, the fact that $G_\ell^+ v_\ell \not\subset N_\ell = H_{\ell,1} \cap \cdots \cap H_{\ell,r}$ implies that there exists at least one $1 \leq i \leq t$ such that $G_\ell^+ v_\ell \not\subset H_{\ell,i}$ and it is enough to perform the proof for $N = H_{\ell,i}$.

From [Bo69, Prop. 2.2 and its proof], there exist $g_1, \dots, g_{d_\ell} \in G_\ell$ of order ℓ ($d_\ell := \dim(\tilde{G}_\ell) \leq r^2$) such that the morphism

$$\begin{array}{cccc} f\colon & \mathbb{A}^{d_\ell}_{\mathbb{F}_\ell} & \to & \tilde{G}_\ell \\ & \underline{t} = (t_1, \dots, t_{d_\ell}) & \to & \exp(t_1 \log(g_1)) \cdots \exp(t_{d_\ell} \log(g_{d_\ell})) \end{array}$$

is dominant. Set $\mathcal{E} := \mathbb{A}^{d_\ell}(\mathbb{F}_\ell) \times G^+_\ell$ and introduce the 'homogenizing' map

$$\begin{array}{cccccc} \phi_{v_\ell}\colon & \mathcal{E} & \twoheadrightarrow & G^+_\ell & \twoheadrightarrow & G^+_\ell v_\ell \\ & (\underline{t}, g) & \mapsto & f(\underline{t})g & & \\ & & & g & \mapsto & g v_\ell. \end{array}$$

By construction the fibers of $\phi_{v_\ell} : \mathcal{E} \twoheadrightarrow G^+_\ell v$ all have the same cardinality $\ell^{d_\ell}|\mathrm{Stab}_{G^+_\ell}(v_\ell)|$, hence

$$\frac{|G^+_\ell v_\ell \cap N_\ell|}{|G^+_\ell v_\ell|} = \frac{|(\lambda_\ell \circ \phi_{v_\ell})^{-1}(0)|}{|\mathcal{E}|}.$$

For $g \in G^+_\ell$, set

$$\mathcal{E}_g := \mathbb{A}^{d_\ell}(\mathbb{F}_\ell) \times \{g\}, \quad \mathcal{E}^{\lambda_\ell}_g := \mathcal{E}_g \cap (\lambda_\ell \circ \phi_{v_\ell})^{-1}(0).$$

Then

$$\mathcal{E} = \bigsqcup_{g \in G^+_\ell} \mathcal{E}_g, \quad (\lambda \circ \phi_{v_\ell})^{-1}(0) = \bigsqcup_{g \in G^+_\ell} \mathcal{E}^{\lambda_\ell}_g.$$

Thus, it is enough to show that

$$|\mathcal{E}^{\lambda_\ell}_g| \leq \frac{r^3}{\ell}|\mathcal{E}_g|, \ g \in G^+_\ell.$$

Since g_i is of order ℓ, the element $\log(g_i) \in \mathrm{End}(H_\ell)$ is nilpotent of index $\leq r_\ell$, so $\exp(T_i \log(g_i)) \in \mathrm{End}(H_\ell) \otimes \mathbb{F}_\ell[T_i]$ has degree $< r_\ell \leq r$ and $F^{\lambda_\ell}_{g,v_\ell}(\underline{T}) := \lambda_\ell(f(\underline{T}) g v_\ell) \in \mathbb{F}_\ell[\underline{T}]$ has total degree $< d_\ell r_\ell \leq r^3$. Furthermore, one has $F^{\lambda_\ell}_{g,v_\ell}(\underline{T}) \neq 0$. Indeed, otherwise, we would have

$$G^+_\ell g v_\ell \subset \tilde{G}_\ell(\mathbb{F}_\ell) g v_\ell \subset \overline{f(\mathbb{A}^{d_\ell}_{\mathbb{F}_\ell})}^{zar}(\mathbb{F}_\ell) g v_\ell \subset N_\ell,$$

which would contradict $G^+_\ell v_\ell = G^+_\ell g v_\ell \not\subseteq N_\ell$.

As a result, $|\mathcal{E}^\lambda_g| = |(F^{\lambda_\ell}_{g,v_\ell})^{-1}(0)| \leq r^3 \ell^{d_\ell - 1}$ (see for instance [N87, Lemma 3.3]). □

Projective variant: *There exists a sequence $\epsilon(\ell) \geq 0$, $\ell \in L$ such that $\ln(\ell)\epsilon(\ell) \to 0$ and for For every $\ell \in L$, $0 \neq v_\ell \in H_\ell$ and $\mathbb{F}_\ell$-vector subspace $N_\ell \subset H_\ell$ such that $G^+_\ell v_\ell \not\subset N_\ell$ one has*

$$\frac{|P(G^+_\ell v_\ell \cap N_\ell)|}{|P(G^+_\ell v_\ell)|} \leq \frac{r^3}{\ell}.$$

Proof. Just observe that the fibers of the map

$$G^+_\ell v_\ell \twoheadrightarrow P(G^+_\ell v_\ell)$$

induced by the projectivization all have the same cardinality $[\mathrm{Stab}_{G^+_\ell}(P v_\ell) : \mathrm{Stab}_{G^+_\ell}(v_\ell)]$. Hence

$$\frac{|P(G^+_\ell v_\ell \cap N_\ell)|}{|P(G^+_\ell v_\ell)|} = \frac{|G^+_\ell v_\ell \cap N_\ell|}{|G^+_\ell v_\ell|}$$

and the conclusion follows from Lemma 4.3.5.1. □.

Actually, we will only need the fact (implied by the above projective variant) that the sequence

$$\epsilon(\ell) := \max\{\frac{|P(G^+_\ell v_\ell \cap N_\ell)|}{|P(G^+_\ell v_\ell)|} \mid 0 \neq v_\ell \in H_\ell\}$$

satisfies $\ln(\ell)\epsilon(\ell) \to 0$.

4.3.5.2. We now conclude the proof of Theorem C. We argue by contradiction. Assume that the conclusion of Theorem C does not hold, that is, for some integer $g \geq 0$ and infinitely many primes ℓ there exists $0 \neq v \in H_\ell$ such that $g_{Pv} \leq g$. Write $M := M(v)$ and set $G^+_{M,Pv} := G^+_M \mathrm{Stab}_{G_M}(Pv)$ and let $X^+_{M,Pv} \to X$ denote the connected étale cover corresponding to the inclusion $G^+_{M,Pv} \subset G_M$. The following commutative diagram of connected étale covers sums up the situation

$$\begin{array}{ccc} X_M & \longrightarrow & X_{PM} \\ \downarrow & & \downarrow \\ X_v & \longrightarrow & X_{Pv} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X^+_{M,Pv}. \end{array}$$

By construction, replacing X with the connected étale cover $X^+_{M,Pv} \to X$ does not affect X_{Pv}. From Theorem A and Condition (F), there are only finitely many possibilities for the isomorphism class of $X^+_{M,Pv} \to X$ so, up to replacing X with a finite étale cover, one may assume that for infinitely many primes ℓ there exists $0 \neq v \in H_\ell$ such that $g_{Pv} \leq g$ and $G_M = G^+_{M,Pv}$. Then $G_M Pv = G^+_M Pv$. So, from (the projective variant of) Lemma 4.3.5.1 (see Subsection ?? below), for every $\mathbb{F}_\ell$-vector subspace $N \subset H_\ell$ such that $G_M v \not\subset N$ one has

$$|P(G_M v \cap N)| \leq \epsilon(\ell)|P(G_M v)|$$

For every subgroup $I \subset G_{PM} (= PG_M)$ write

$$\epsilon_I(Pv) := \frac{|I \setminus P(G_M v)|}{|P(G_M v)|} - \frac{1}{|I|}$$

By applying the Riemann-Hurwitz formula to the covers $X_{PM} \to X_{Pv} \to X^+_{M,Pv} = X$, one gets

$$0 \leq \lambda_{PM} - \lambda_{Pv} \leq \sum_{x \in \partial X} (\epsilon_{I_{x,PM}}(Pv) + \sum_{i>0} \frac{|I_{x,PM,i}|}{|I_{x,PM}|} \epsilon_{I_{x,PM,i}}(Pv)),$$

where we write

$$\lambda_{Pv} := \frac{2g_{Pv} - 2}{|P(G_M v)|}.$$

To estimate the $\epsilon_I(Pv)$ that appear in the above formula, introduce

$$P(G_M v)'_I := \bigcup_{J \in \mathcal{M}(I)} P(G_M v)^J,$$

where $\mathcal{M}(I)$ denotes the set of all minimal non-trivial subgroups J of I. Then one has

$$\frac{1}{|I|}(1 - \frac{|P(G_M v)'_I|}{|P(G_M v)|}) \leq \frac{|I \setminus P(G_M v)|}{|P(G_M v)|} \leq \frac{1}{|I|}(1 - \frac{|P(G_M v)'_I|}{|P(G_M v)|}) + \frac{|(P(G_M v)'_I|}{|P(G_M v)|}$$

and

$$\frac{|P(G_M v)'_I|}{|P(G_M v)|} \leq \sum_{J \in \mathcal{M}(I)} \frac{|P(G_M v)^J|}{|P(G_M v)|}.$$

In our case I is a subgroup of $I_{x,PM}$. Let $I^w_{x,PM}$ denote the p-Sylow of $I_{x,PM}$ and $I^t_{x,PM} := I_{x,PM}/I^w_{x,PM}$. Then I can be written as an extension of the cyclic group $I^t := \mathrm{im}(I \to I^t_{x,PM})$ by the p-group $I^w := I \cap I^w_{x,PM}$ which, from (T), is of order $\leq [I_{x,PM} : U_{x,PM}] \leq [I_x : U_x] =: m_x$. In that case, one has

Claim 3 *There exists an absolute constant $C > 0$ such that for any subgroup $I \subset \mathrm{PGL}_n(\mathbb{F}_\ell)$ which is an extension of a cyclic group I_1 by a group I_2 of order $\leq m$, one has*

$$|\mathcal{M}(I)| \leq Cmn\ln(\ell).$$

Proof. First, we shall estimate $|\mathcal{M}(I_1)|$. Let $\tilde{I}$ denote the inverse image of I in $\mathrm{GL}_n(\mathbb{F}_\ell)$; choose an element $\gamma \in \tilde{I}$ whose image in I_1 is a generator of the cyclic group I_1. Set $\tilde{I}_1 = \langle\gamma\rangle$ and denote by I'_1 and

$\tilde{I}'_1$ the maximal prime-to-ℓ subgroups of I_1 and $\tilde{I}_1$, respectively. Set $F := \mathbb{F}_\ell[\tilde{I}'_1] \subset M_n(\mathbb{F}_\ell)$, which is a commutative semisimple subalgebra over $\mathbb{F}_\ell$. As $\mathbb{F}_\ell$ is perfect, $F \otimes \overline{\mathbb{F}}_\ell \subset M_n(\overline{\mathbb{F}}_\ell)$ is again a commutative semisimple subalgebra over $\overline{\mathbb{F}}_\ell$ hence is of the form $\overline{\mathbb{F}}_\ell^r$ with $r \leq n$. This shows

$$\dim_{\mathbb{F}_\ell}(F) = \dim_{\overline{\mathbb{F}}_\ell}(F \otimes \overline{\mathbb{F}}_\ell) \leq n.$$

As a result, $|I'_1| \leq |\tilde{I}'_1| \leq |F| \leq \ell^n$. As I'_1 is cyclic (since I_1 is cyclic), $|\mathcal{M}(I'_1)|$ is the number of prime divisors of $|I'_1|$. In particular, one has

$$2^{|\mathcal{M}(I'_1)|} \leq |I'_1| \leq \ell^n,$$

and

$$|\mathcal{M}(I_1)| \leq |\mathcal{M}(I'_1)| + 1 \leq \frac{n\ln(\ell)}{\ln(2)} + 1.$$

Next, we have a natural map $\pi : \mathcal{M}(I) \to \mathcal{M}(I_1) \cup \{\{1\}\}$ which sends $J \in \mathcal{M}(I)$ to the image of J in I_1. For each $J_1 \in \mathcal{M}(I_1) \cup \{\{1\}\}$, fix a generator γ_{J_1} of J_1. Then, for any $J \in \pi^{-1}(J_1)$, one can choose a generator γ_J of J which maps to $\gamma_{J_1} \in J_1$. Then, since J is determined by γ_J as $J = \langle \gamma_J \rangle$ and γ_J is in the fiber of $I \twoheadrightarrow I_1$ at $\gamma_{J_1} \in I_1$, one concludes $|\pi^{-1}(J_1)| \leq |I_2|$ for any $J_1 \in \mathcal{M}(I_1) \cup \{\{1\}\}$, and

$$|\mathcal{M}(I)| \leq |I_2|(|\mathcal{M}(I_1)| + 1) \leq m\left(\frac{n\ln(\ell)}{\ln(2)} + 2\right),$$

from which the assertion easily follows. (Say, set $C := \frac{3}{\ln(2)}$.) $\square$

From Claim 3, there exists a constant $C > 0$ depending only on r and $\max\{m_x\}_{x\in\partial X}$ such that

$$|\mathcal{M}(I)| \leq C\ln(\ell).$$

For $J \in \mathcal{M}(I)$, let $\gamma_J \in G_M$ denote an element mapping onto a generator of J (which is cyclic by the definition of $\mathcal{M}(I)$). Then

$$P(G_M v)^J = \bigsqcup_{\lambda\in VP(\gamma_J)} P(G_M v \cap \ker(\gamma_J - \lambda Id)),$$

where $VP(\gamma_J)$ denotes the set of $\mathbb{F}_\ell$-rational eigenvalues of γ_J. Note that $|VP(\gamma_J)| \leq r$ and, as J is non-trivial, $G_M v \not\subset \ker(\gamma_J - \lambda Id)$ for every $\lambda \in VP(\gamma_J)$.

As a result, one obtains (with the notation $\epsilon(\ell)$ from 4.3.5.1)

$$\epsilon_I(Pv) \leq \frac{|(P(G_M v)'_I|}{|P(G_M v)|} \leq \sum_{J\in\mathcal{M}(I)} \frac{|P(G_M v)^J|}{|P(G_M v)|} = \sum_{J\in\mathcal{M}(I)} \sum_{\lambda\in VP(\gamma_J)} \frac{|P(G_M v \cap \ker(\gamma_J - \lambda Id))|}{|P(G_M v)|} \leq Cr\ln(\ell)\epsilon(\ell).$$

So, if a denotes the maximal length of the filtrations $(RF_{x,PM})$ (see Claim 1) then

$$0 \leq \lambda_{PM} - \lambda_{Pv} \leq |\partial X|Cr(1+a)\ln(\ell)\epsilon(\ell).$$

As $\ln(\ell)\epsilon(\ell) \to 0$ (Lemma 4.3.5.1), for $\ell \gg 0$ one has

$$g \geq g_{Pv} = \frac{1}{2}|G_{Pv}|(\lambda_{Pv} - \lambda_{PM} + \lambda_{PM}) + 1 \overset{(1)}{\geq} \frac{1}{4}|G_{Pv}|K + 1 \overset{(2)}{\geq} \frac{1}{4}d_0^{\rho_\ell}K + 1,$$

where the inequality (1) follows from Lemma 4.3.3 and the inequality (2) is by definition of $d_0^{\rho_\ell}$. This contradicts Lemma 4.3.2.

5. Applications: Specialization in 1-dimensional families

In this section, L can be (any infinite subset of) the set of all primes but p.

5.1. **Galois image on étale cohomology groups.** Let $Y \to X$ be a smooth proper morphism. By the smooth-proper base change theorem (and modulo the choice of an étale path from $\overline{\eta}$ to $\overline{x}$ and appropriate labeling of base points in étale fundamental groups) for every $i \in \mathbb{Z}_{\geq 0}$, $x \in |X|$ and $\ell \in L$, the representation

$$\rho_\ell^i \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell))$$

identifies with the 'usual' Galois representation

$$\rho_{\ell,x}^i : \Gamma_{k(x)} \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{x}}, \mathbb{F}_\ell))$$

associated to $Y_x \to \mathrm{Spec}(k(x))$. Hence, in this setting, understanding how the $G_{\ell,x}$ vary with $x \in |X|$ amounts to understanding how the images of the $\rho_{\ell,x}$ do. The main result of this subsection is that Theorem 1.6.1 and Corollary 1.6.2 apply to the family $\rho_\ell^i : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell))$, $\ell \in L$. This follows from

5.1.1. **Theorem** *The family*

$$\rho_\ell^i : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell)),\ \ell \in L$$

is bounded and satisfies conditions (T), (P).

Proof. To simplify, write $H^i_{\mathbb{Z}_\ell} := \mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{Z}_\ell)$, $\overline{H}^i_{\mathbb{Z}_\ell} := H^i_{\mathbb{Z}_\ell}/(\mathrm{torsion})$, $H^i_{\mathbb{Q}_\ell} := H^i_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and $H^i_\ell := \mathrm{H}^i(Y_{\overline{\eta}}, \mathbb{F}_\ell)$.

(1) **$H^i_{\mathbb{Z}_\ell}$ is torsion free for $\ell \gg 0$.** When $p = 0$, this follows from the comparison isomorphism between Betti and étale cohomology with finite coefficients and the fact that Betti cohomology with coefficients in $\mathbb{Z}$ is finitely generated. When $p > 0$, see [G83] (projective case) and [Or13, Rem. 3.1.5]. As $H^{i+1}_{\mathbb{Z}_\ell}$ is also torsion free for $\ell \gg 0$, by considering the long exact sequence associated to the short exact sequence

$$0 \to \mathbb{Z}_\ell \xrightarrow{\ell} \mathbb{Z}_\ell \to \mathbb{F}_\ell \to 0$$

of constant sheaves on $X_{\text{ét}}$, one sees that the canonical map $H^i_{\mathbb{Z}_\ell}/\ell \to H^i_\ell$ is an isomorphism for $\ell \gg 0$.

(2) **Boundedness.** This is [Or13, Prop. 3.1.3].

(3) **Condition (T).** Condition (T) for $H^i_{\mathbb{Q}_\ell}$ follows from de Jong's alteration theorem [B96, Prop. 6.3.2] that is, for any $x \in \partial\overline{X}$ there exists an open subgroup U_x of the inertia group $I_x \subset \pi_1(\overline{X})$ at x such that the image of U_x in $\mathrm{GL}(H^i_{\mathbb{Q}_\ell})$ is unipotent for $\ell \neq p$. As $H^i_{\mathbb{Z}_\ell} \simeq \overline{H}^i_{\mathbb{Z}_\ell} \subset H^i_{\mathbb{Q}_\ell}$ for $\ell \gg 0$, the image of U_x in $\mathrm{GL}(\overline{H}^i_{\mathbb{Z}_\ell})$ is unipotent for $\ell \gg 0$ and as $H^i_{\mathbb{Z}_\ell}/\ell \simeq \overline{H}^i_{\mathbb{Z}_\ell}/\ell \simeq H^i_\ell$ for $\ell \gg 0$, the image of U_x in $\mathrm{GL}(H^i_\ell)$ is also unipotent for $\ell \gg 0$.

(4) **Condition (P).** One may freely replace X with a connected étale cover. In particular, one may assume that $\Pi = \pi_1(\overline{X})$ and that $\overline{G}_\ell = \overline{G}_\ell^+$ by Theorem A. Let $H^{i,ss}_\ell$ denote the $\pi_1(\overline{X})$-semisimplification of H^i_ℓ. Then $R_u(\overline{G}_\ell)$ is the kernel of $\overline{G}_\ell \to \mathrm{GL}(H^{i,ss}_\ell)$ for $\ell \gg 0$ by the comments following Fact 3.2.1. Set $\overline{G}^{ss}_\ell := \overline{G}_\ell/R_u(\overline{G}_\ell)$. From the right-exactness of the abelianization functor, one has the exact sequence (of $\mathbb{F}_\ell$-vector spaces for $\ell \geq r$)

$$R_u(\overline{G}_\ell)^{ab} \to \overline{G}^{ab}_\ell \to \overline{G}^{ss,ab}_\ell \to 0$$

and from Lemma 3.2.2, one may assume that $\overline{G}^{ss,ab}_\ell = 0$ for $\ell \gg 0$. So, it is enough to prove that the image of $R_u(\overline{G}_\ell)^{ab}$ in $\overline{G}^{ab}_\ell$ is trivial for $\ell \gg 0$. This will follow from a Weil-weight argument by reduction modulo ℓ of $H^i_{\mathbb{Z}_\ell}$, which is torsion free with $H^i_{\mathbb{Z}_\ell}/\ell \simeq H^i_\ell$ for $\ell \gg 0$. More precisely, by the standard specialization argument of tame étale fundamental group, one may assume that k is finite. Let φ denote the Frobenius of k. From the Weil conjectures [D80], the characteristic polynomials $P_{\varphi,1}$ and $P_{\varphi,2}$ of φ acting respectively on $\pi_1(\overline{X})^{(\ell),ab}$ and $H^i_{\mathbb{Z}_\ell}$ both lie in $\mathbb{Z}[T]$ and are independent of ℓ. Furthermore, the zeroes of $P_{\varphi,1}$ are Weil numbers of weight 1 or 2 and those of

$P_{\varphi,2}$ are Weil numbers of weight i. As a result, the characteristic polynomial $P_{\varphi,3}$ of φ acting (*via* the adjoint representation) on $\mathrm{End}_{\mathbb{Q}_\ell}(H^i_{\mathbb{Q}_\ell})$ lies in $\mathbb{Q}[T]$, is independent of ℓ and its zeroes are Weil numbers of weight 0. Fix an integer $a \geq 1$ such that $P^\circ_{\varphi,3} := aP_{\varphi,3}$ lies in $\mathbb{Z}[T]$. (In fact, one can take $a \in |k|^{\mathbb{Z}_{\geq 0}}$. But this fact is not used later.) As $P_{\varphi,1}$ and $P^\circ_{\varphi,3}$ have no common zero, there exists $U, V \in \mathbb{Z}[T]$ and $0 \neq b \in \mathbb{Z}$ such that

$$UP_{\varphi,1} + VP^\circ_{\varphi,3} = b.$$

In particular, for $\ell \gg 0$, the reductions $\overline{P_{\varphi,1}}^\ell$ and $\overline{P^\circ_{\varphi,3}}^\ell$ of $P_{\varphi,1}$ and $P^\circ_{\varphi,3}$ modulo ℓ are coprime. On the one hand, as $\pi_1(\overline{X})^{(\ell),ab}/\ell \twoheadrightarrow \overline{G}^{ab}_\ell$, the characteristic polynomial $P_{\varphi,1,\ell}$ of φ acting on $\overline{G}^{ab}_\ell$ divides $\overline{P_{\varphi,1}}^\ell$. On the other hand, let $r_\ell := \dim_{\mathbb{F}_\ell}(H^i_\ell) \leq r$ and write $N_\ell \subset \mathrm{End}(H^i_\ell)$ for the subset of nilpotent matrices and $U_\ell \subset \mathrm{GL}(H^i_\ell)$ for the subset of unipotent matrices. Then the logarithm and exponential $\log : U_\ell \to N_\ell$, $\exp : N_\ell \to U_\ell$ defined by

$$\log(u) = -\sum_{1 \leq i \leq \ell-1} \frac{(1-u)^i}{i}, \quad \exp(n) = \sum_{0 \leq i \leq \ell-1} \frac{n^i}{i!}$$

define bijections which are inverse to each other. Furthermore, it follows from the Campbell-Haussdorff formula that $\log(DR_u(\overline{G}_\ell))$ and $\log(R_u(\overline{G}_\ell))$ are $\mathbb{F}_\ell$-Lie-algebras (see [N87, Remark 1.8]) and from [N87, Lemma 1.5] (applied to $W_1 = W_2 = \log(DR_u(\overline{G}_\ell))$ and $S = \log(R_u(\overline{G}_\ell))$) plus the fact that $DR_u(\overline{G}_\ell)$ is normal in $R_u(\overline{G}_\ell)$ that $\log(DR_u(\overline{G}_\ell))$ is actually a Lie-ideal in $\log(R_u(\overline{G}_\ell))$ for $\ell \gg 0$. Thus, one obtains, for $\ell \gg 0$, a well-defined commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & DR_u(\overline{G}_\ell) & \longrightarrow & R_u(\overline{G}_\ell) & \longrightarrow & R_u(\overline{G}_\ell)^{ab} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \log} & & \downarrow{\scriptstyle \log} & & \downarrow{\scriptstyle \overline{\log}} & & \\
0 & \longrightarrow & \log(DR_u(\overline{G}_\ell)) & \longrightarrow & \log(R_u(\overline{G}_\ell)) & \longrightarrow & L_\ell & \longrightarrow & 0
\end{array}$$

with the property that the upper row is a short exact sequence of finite groups, the lower one is a short exact sequence of Lie algebras and the two left vertical arrows are bijections. The right vertical arrow is then automatically surjective by commutativity of the diagram hence bijective by cardinality. Furthermore, one has

$$(*) \quad \overline{\log}(\overline{g}\overline{g}') = \overline{\log}(\overline{g}) + \overline{\log}(\overline{g}'),\ \overline{g}, \overline{g}' \in R_u(\overline{G}_\ell)^{ab}.$$

As the logarithm

$$\log : R_u(\overline{G}_\ell) \tilde{\to} \log(R_u(\overline{G}_\ell)) \subset \mathrm{End}_{\mathbb{F}_\ell}(H^i_\ell),$$

is φ-equivariant, one sees that the characteristic polynomial $P_{\varphi,3,\ell}$ of φ acting[7] on L_ℓ divides $\overline{P^\circ_{\varphi,3}}^\ell$. Write $P_{\varphi,3,\ell} = \sum_{\nu \geq 0} a_\nu T^\nu$. The bijectivity of $\overline{\log}$ and the relation (*) then show that

$$\prod_{\nu \geq 0} \varphi^\nu \overline{g}^{a_\nu} \varphi^{-\nu} = 1$$

or equivalently, with additive notation, $P_{\varphi,3,\ell}(\varphi)(\overline{g}) = 0$, $g \in R_u(\overline{G}_\ell)^{ab}$. In other words, the minimal polynomial of φ acting on $R_u(\overline{G}_\ell)^{ab}$ divides $P_{\varphi,3,\ell}$ hence $\overline{P^\circ_{\varphi,3}}^\ell$: this is only possible if the image of $R_u(\overline{G}_\ell)^{ab}$ in $\overline{G}^{ab}_\ell$ is trivial, as requested. □

5.1.2. **Remark** Note that Condition (P) in characteristic $p > 0$ does not seem to have been noticed before. Condition (SS) holds if $p = 0$ (more precisely, condition (SS) for the Betti cohomology with coefficients in $\mathbb{Q}$ follows from Deligne's semisimplicity theorem [D71, Cor. 4.2.9]; to deduce the result for étale cohomology with coefficients in $\mathbb{F}_\ell$, use [CT11, Lemma 2.5] and the comparison between Betti and étale cohomologies). Condition (SS) also holds if $p > 0$, when $Y \to X$ is an abelian scheme ([Ta66], [Z77], [Sz85]). It is expected to hold in general. (Note that the ℓ-adic version of it holds [D80, Cor. 3.4.13]).

[7] Note that as $R_u(\overline{G}_\ell)$ is characteristic in $\overline{G}_\ell$ and $DR_u(\overline{G}_\ell)$ is characteristic in $R_u(\overline{G}_\ell)$, $DR_u(\overline{G}_\ell)$ is characteristic in $\overline{G}_\ell$ hence the action of φ on $\overline{G}_\ell$ restricts to an action on $R_u(\overline{G}_\ell)$ and $DR_u(\overline{G}_\ell)$.

5.2. **First Galois cohomology groups.** Another situation where Corollary 1.6.2 applies is the following. Let $\pi_1(X)\rightarrow \mathrm{GL}(V_\ell)$, $\ell\in L$ be a bounded family of continuous $\mathbb{F}_\ell$-linear representations of $\pi_1(X)$. Fix an integer $r\geq 1$ and for every $\ell\in L$ an $\mathbb{F}_\ell$-submodule

$$H_\ell\subset \mathrm{H}^1(\pi_1(X),V_\ell)=\mathrm{Ext}^1_{\mathbb{F}_\ell[\pi_1(X)]}(\mathbb{F}_\ell,V_\ell)$$

of $\mathbb{F}_\ell$-rank $r_\ell\leq r$. Then,

5.2.1. **Corollary** *Assume that the family $\pi_1(X)\rightarrow \mathrm{GL}(V_\ell)$, $\ell\in L$ satisfies (T), (SS) and (I). Then for $\ell\gg 0$ and all but finitely many $x\in X(k)$ the specialization map*

$$sp_x: H_\ell\hookrightarrow \mathrm{H}^1(\pi_1(X),V_\ell)\rightarrow \mathrm{H}^1(k(x),V_\ell)$$

is injective.

Proof. We refer to [C16, §1.1] for more details. First, recall that there exists an $\mathbb{F}_\ell[\pi_1(X)]$-module

$$0\rightarrow V_\ell\rightarrow H_\ell^{univ}\rightarrow H_\ell\rightarrow 0$$

with the property that the pull-back morphism

$$-^*H_\ell^{univ}:\mathrm{Hom}(\mathbb{F}_\ell,H_\ell)\rightarrow \mathrm{Ext}^1_{\mathbb{F}_\ell[\pi_1(X)]}(\mathbb{F}_\ell,V_\ell)$$

induces an isomorphism onto $H_\ell\subset \mathrm{Ext}^1_{\mathbb{F}_\ell[\pi_1(X)]}(\mathbb{F}_\ell,V_\ell)$. Assuming that the family $\pi_1(X)\rightarrow \mathrm{GL}(V_\ell)$, $\ell\in L$ satisfies (T), (SS) and (I) one easily shows that the family $\rho_\ell:\pi_1(X)\rightarrow \mathrm{GL}(H_\ell^{univ})$, $\ell\in L$ satisfies (T) and (P) (this is a special case of [C16, Lemma 2.5 (2)]). In particular, from Corollary 1.6.2, there exists an integer $K\geq 1$ such that for $\ell\gg 0$ and all but finitely many $x\in X(k)$ one has $[G_\ell:G_{\ell,x}]\leq K$. Then, from [C16, Lemma 2.4 (2)], the specialization map $sp_x: H_\ell\hookrightarrow \mathrm{H}^1(\pi_1(X),V_\ell)\rightarrow \mathrm{H}^1(k(x),V_\ell)$ is injective as soon as $\ell>K$. □

5.3. **Abelian varieties: Néron-Silverman specialization theorem for arbitrary finitely generated fields of characteristic $p\geq 0$.** As an application of Corollary 5.2.1, one can extend the Néron-Silverman specialization theorem [Si83, Thm. C] to arbitrary finitely generated fields of characteristic $p\geq 0$. Before doing this, we need a preliminary result on the 'almost' uniform boundedness of ℓ-torsion in 1-dimensional family of abelian varieties, which is of interest in itself.

5.3.1. *'Almost' uniform boundedness of ℓ-torsion in 1-dimensional family of abelian varieties.* Let $Y\rightarrow X$ be an abelian scheme. The following statement extends [EHKo12, Thm.7] to finitely generated fields k of characteristic $p>0$ (see also [CT11, Cor 1.4 and §3.3]). For a comparison of our results and technics with those of [EHKo12] - and, in particular, the limitation to k-rational points versus points of bounded degree - see Subsection 1.7.

5.3.1.1. **Corollary** *The set of all $x\in X(k)$ such that $Y_x[\ell](k)\neq 0$ is finite for $\ell\gg 0$.*

Proof. Considering the family

$$\rho_\ell:\pi_1(X)\rightarrow \mathrm{GL}(Y_\eta[\ell]),\ \ell\in L,$$

the statement amounts to showing that $X_1^{\rho_\ell}(k)$ is finite for $\ell\gg 0$. For this, let $(Y_\eta)_0\subset Y_\eta$ denote the largest abelian subvariety isogenous to a k-isotrivial abelian variety; set $Y_\eta^0:=Y_\eta/(Y_\eta)_0$ and, for every $v\in Y_\eta$, let v^0 denote the image of v in Y_η^0 (see [CT12a, §1.1] for more details). Then one can write

$$X_1^{\rho_\ell}=X_\ell^0\sqcup X_{\ell,0},$$

where

$$X_{\ell,0}:=\bigsqcup_{0\neq v\in (Y_\eta)_0[\ell]} X_v.$$

From [CT11, Prop. 3.18], one has $X_{\ell,0}(k)=\emptyset$ for $\ell\gg 0$ thus it is enough to show that $X_\ell^0(k)$ is finite for $\ell\gg 0$. But, by functoriality of the étale fundamental group, for any $0\neq v\in Y_\eta[\ell]$, one has a finite étale cover $X_v\rightarrow X_{v^0}$ hence it is enough to make the proof for Y_η^0. In other words, we may assume that $(Y_\eta)_0 = 0$. Under this assumption, it follows from Theorem C that the minimum of the genera of the connected components of $X_1^{\rho_\ell}$ goes to $+\infty$ with ℓ. So the conclusion follows from Fact 1.3.1. □

5.3.2. *Néron-Silverman specialization theorem for arbitrary finitely generated fields of characteristic $p \geq 0$.* Let again $Y \to X$ be an abelian scheme and assume furthermore that $Y_{\overline{\eta}}$ contains no non-trivial isotrivial abelian subvariety. For every $\ell \in L$, fix an $\mathbb{F}_\ell$-submodule $H_\ell \subset \mathrm{H}^1(\pi_1(X), Y_{\overline{\eta}}[\ell])$ of finite $\mathbb{F}_\ell$-rank $r_\ell \leq r$. Then

5.3.2.1. **Corollary** *For $\ell \gg 0$ and all but finitely many $x \in X(k)$ the specialization map*

$$sp_x : H_\ell \hookrightarrow \mathrm{H}^1(\pi_1(X), Y_{\overline{\eta}}[\ell]) \to \mathrm{H}^1(k, Y_{\overline{x}}[\ell])$$

is injective. In particular, for $\ell \gg 0$ and all but finitely many $x \in X(k)$ the specialization map

$$Y_\eta(k(\eta))/\ell \to Y_x(k)/\ell$$

is injective.

Proof. By the geometric Lang-Néron theorem [LN59], the assumption that $Y_{\overline{\eta}}$ contains no non-trivial isotrivial abelian subvariety ensures that the $Y_{\overline{\eta}}[\ell]$, $\ell \in L$ satisfy (I). They also satisfy (T) and (SS) (See Remark 5.1.2 above). So, the first part of Corollary 5.3.2.1 follows from Corollary 5.2.1. The second part is a special case of the first part. More precisely, the long exact sequence of Galois cohomology groups associated to the Kummer short exact sequence ($\ell \neq p$) yields the following commutative diagram, where the horizontal arrows are injective and the upper left vertical arrow is an isomorphism (extension property of Néron models).

$$\begin{array}{ccc}
Y_\eta(k(\eta))/\ell & \hookrightarrow & \mathrm{H}^1(k(\eta), Y_\eta[\ell]) \\
\| & & \uparrow{\scriptstyle inf} \\
Y(X)/\ell & \hookrightarrow & \mathrm{H}^1(X, Y[\ell]) \simeq \mathrm{H}^1(\pi_1(X), Y_\eta[\ell]) \\
\downarrow & & \downarrow{\scriptstyle res} \\
Y_x(k(x))/\ell & \hookrightarrow & \mathrm{H}^1(k(x), Y_x[\ell])
\end{array}$$

To apply the first part of Corollary 5.3.2.1, one only needs to show that $H_\ell = Y_\eta(k(\eta))/\ell$ has finite $\mathbb{F}_\ell$-rank bounded from above independently of ℓ. But this follows from the arithmetic Lang-Néron's theorem [LN59], which asserts that $Y_\eta(k(\eta))$ is a finitely generated abelian group. □

5.3.2.2. **Corollary** (Néron-Silverman's specialization for arbitrary finitely generated fields of characteristic $p \geq 0$) *For $\ell \gg 0$ and all but finitely many $x \in X(k)$ the specialization map $sp_x : Y(k(\eta)) \to Y_x(k)$ is injective.*

Proof. We apply the Criterion of [S89, §11.1, p. 152], which asserts that the specialization map $sp_x : Y(k(\eta)) \to Y_x(k)$ is injective as soon as

(i) $Y(k(\eta))$ is a finitely generated abelian group;
(ii) the specialization map $Y_\eta(k(\eta))/\ell \to Y_x(k)/\ell$ is injective;
(iii) the specialization map $Y_\eta(k(\eta))_{tors} \to Y_x(k)_{tors}$ is injective;
(iv) the specialization map $Y_\eta(k(\eta))[\ell] \tilde{\to} Y_x(k)[\ell]$ is an isomorphism.

Condition (i) is the arithmetic Lang-Néron's theorem. Condition (ii) for $\ell \gg 0$ and all but finitely many $x \in X(k)$ is Corollary 5.3.2.1. Condition (iii) for $\ell \gg 0$ and all but finitely many $x \in X(k)$ comes from the following facts:

- If $p = 0$, the specialization map is always injective on torsion;
- If $p > 0$, the specialization map is always injective on p'-torsion and p-torsion is locally constructible. More precisely, the scheme-theoretic closure Z of $Y_\eta(k(\eta))[p]$ in Y is finite over X and the generic fiber of $Z \to X$ is (finite) étale. Thus $Z \to X$ is generically étale that is there exists a non-empty open subscheme $U \subset X$ such that $Z \times_X U \to U$ is finite étale. Then, for all $x \in U(k)$

(hence for all but finitely many $x \in X(k)$) the specialization map $sp_x : Z_\eta(k(\eta)) \to Z_x(k(x))$ is injective.

Condition (iv) for $\ell \gg 0$ and all but finitely many $x \in X(k)$ is Condition (iii) and Corollary 5.3.1.1 (which ensure that $Y_\eta(k(\eta))[\ell] = Y_x(k)[\ell] = 0$). $\square$

5.3.2.3. **Remark** (Comparison with [Si83]) As pointed out by Brian Conrad, one could also try and extend directly the proof of [Si83, Thm. C]. The setting in [Si83] is the following: k is a global field, C is a smooth, *projective*, geometrically connected curve over k and A is a *smooth*, projective, irreducible variety equipped with a flat morphism $A \to C$ whose generic fiber is an abelian variety. Let $C^\circ \subset C$ denote a (non-empty) open subscheme such that $A^\circ := A \times_C C^\circ \to C^\circ$ is an abelian scheme. Then, the main result of *loc. cit.* ([Si83, Thm. B]) is the following limit formula relating the height h_C on C and the Neron-Tate height pairings $\langle -, -\rangle_{A_\eta}$ on A_η and $\langle -, -\rangle_{A_x}$ on A_x for $x \in C^\circ$: for every $P, Q \in A_\eta(k(\eta))$

$$\lim_{x \in |C^\circ|,\, h_C(x) \to +\infty} \frac{\langle P_x, Q_x\rangle_{A_x}}{h_C(x)} = \langle P_\eta, Q_\eta\rangle_{A_\eta}.$$

Combining this limit formula with the non-degeneracy of the Neron-Tate height pairing on the generic fiber shows that the set of all $x \in |C^\circ|$ where the specialization map $A(k(\eta)) \to A_x(k(x))$ is non-injective is of bounded height. To apply Silverman's argument to our setting, one should first extend $Y \to X$ to a family $Y^{cpt} \to X^{cpt}$ over the[8] smooth compactification X^{cpt} of X and then use resolution of singularities (in characteristic 0) to ensure the smoothness of Y^{cpt} over k. However, as mentioned to us by Philipp Habegger, it does not seem that the smoothness of A is really required in Silverman's argument. More precisely, the only height-theoretic statement among those listed in [Si83, Section 2] that really requires the smoothness is (d). However, Silverman only applies (d) to the base curve C. Desingularizing curves is not a problem in any characteristic. Thus it seems that (without resorting to resolution of singularities or alterations for higher dimensional varieties) Silverman's argument shows that the set of all $x \in |X|$ where the specialization map $Y_\eta(k(\eta)) \to Y_x(k(x))$ is non-injective is of bounded height. However, it does not seem that, over finitely generated fields of transcendence degree ≥ 1 in characteristic 0 and ≥ 2 in characteristic $p > 0$ the boundedness of heights implies the finiteness of rational points. In characteristic 0, this might be achieved using arithmetic height functions as constructed in [M00], using Arakelov theory and resolution of singularities but such constructions do not seem to be available in characteristic $p > 0$.

Compared with Silverman (possibly generalized) proof, let us mention that our approach shows that the Néron-Silverman specialization theorem is only a special case of a more general statement about specialization of first cohomology classes (Corollary 5.3.2.1), which seems to be currently out of reach of the elementary heights techniques used in [Si83]. In characteristic 0, we still require deep arguments (even when k is a number field): to obtain the finiteness of points $x \in X(k)$ (resp. of points $x \in |X|$ with $[k(x) : k] \leq d$) where the specialization map is non-injective, we need the Mordell conjecture (resp. the Mordell-Lang conjecture for subvarieties of abelian varieties plus the strengthened form of Theorem 1.6.1 for gonality discussed in Subsection 1.7). But, in contrast, our proof in characteristic $p > 0$ is fairly elementary. Note that checking the assumptions of Corollary 5.2.1 is easier when $Y \to X$ is an abelian scheme than a general smooth, proper morphism and that the proof of Fact 1.3.1 in characteristic $p > 0$ is much easier than in characteristic 0).

6. Arbitrary F_ℓ-coefficients

For most of the applications, considering $\mathbb{F}_\ell$-coefficients is enough. However, one may ask whether Theorem 1.6.1 extends to F_ℓ-coefficients, where F_ℓ is an arbitrary subfield of $\overline{\mathbb{F}}_\ell$ (see Remark 2.2.1). This seems to be doable at the cost of invoking much deeper group-theoretical results.

So, let $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell)$, $\ell \in L$ be a bounded family of continuous F_ℓ-linear representations. We begin with the following observation. For every $\ell \in L$, with the notation of Fact 3.2.1, write

$$\overline{G}_\ell^\circ / R(\overline{G}_\ell) = \prod_{i \in I_\ell} D_{\ell,i}^{n_{\ell,i}}$$

[8]Possibly after replacing k by a finite extension.

with $D_{\ell,i} := D(\mathcal{S}_{\ell,i}^{\Phi_{\ell,i}})$ and $D_{\ell,i} \not\simeq D_{\ell,j}$, $i \neq j \in I_\ell$, where $\Phi_{\ell,i} = \varphi_{\ell,i} Fr_{\ell,i}$ for some standard Frobenius map $Fr_{\ell,i} : \mathcal{S}_{\ell,i} \to \mathcal{S}_{\ell,i}$ and some automorphism $\varphi_{\ell,i} : \mathcal{S}_{\ell,i} \tilde{\to} \mathcal{S}_{\ell,i}$ induced by an automorphism of the root system of $\mathcal{S}_{\ell,i}$. Given $U_\ell \in \mathcal{F}_{\ell,+}$, the geometric monodromy group of $X_{U_\ell} \to X$ is

$$M(U_\ell) := \overline{G}_\ell / K_{\overline{G}_\ell}(\overline{U}_\ell),$$

where $K_{\overline{G}_\ell}(\overline{U}_\ell) := \bigcap_{g \in \overline{G}_\ell} g \overline{U}_\ell g^{-1}$ is the largest normal subgroup of $\overline{G}_\ell$ contained in $\overline{U}_\ell$.

Lemma 6.1. *Assume (T) and (P). Then for every $U_\ell \in \mathcal{F}_{\ell,+}$ there exists $i \in I$ such that $D(\mathcal{S}_i^{\Phi_i})$ is a subquotient of $M(U_\ell)$.*

Proof. From Lemma 4.3.1, Theorem A and Condition (P), up to replacing X with a connected étale cover, one may assume that $\overline{G}_\ell^+ = \overline{G}_\ell = \overline{G}_\ell^\circ$ and that $\overline{G}_\ell^{ab} = 0$ for every $\ell \in L$. As $K_{\overline{G}_\ell}(\overline{U}_\ell) \subset \overline{U}_\ell \subsetneq \overline{G}_\ell$, the group $M(U_\ell)$ is non-trivial hence, being a quotient of $\overline{G}_\ell = \overline{G}_\ell^+$, has order divisible by ℓ. Then one has a commutative diagram of the following form, where the vertical arrows are surjective and the horizontal rows are short exact sequences.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R(\overline{G}_\ell) & \longrightarrow & \overline{G}_\ell & \longrightarrow & \overline{G}_\ell/R(\overline{G}_\ell) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_\ell & \longrightarrow & M(U_\ell) & \longrightarrow & Q_\ell & \longrightarrow & 1.
\end{array}$$

If $Q_\ell = 1$ then $M(U_\ell)$ is a quotient of $R(\overline{G}_\ell)$ generated by its order-ℓ elements that is a quotient of $R(\overline{G}_\ell)^+$. But $R(G_\ell)^+ = R_u(\overline{G}_\ell)$. (Indeed, observe that $R(\overline{G}_\ell)/R_u(\overline{G}_\ell)$ is of prime-to-ℓ order.) Thus $M(U_\ell)$ would be of order a power of ℓ hence have a non-trivial quotient isomorphic to $\mathbb{Z}/\ell$. As $M(U_\ell)$ is also a quotient of $\overline{G}_\ell$, this would contradict Condition (P). This shows that $Q_\ell \neq 1$, which implies that Q_ℓ (hence also $M(U_\ell)$) admits a quotient isomorphic to $D(\mathcal{S}_i^{\Phi_i})$ for some $i \in I$, as desired. □

Theorem 1.6.1 for arbitrary F_ℓ-coefficients can now be deduced from Lemma 6.1 and the results of [Gu03], which rely on delicate satellite group-theoretical results of the classification (such as the Aschbacher-O'Nan-Scott theorem and Aschbacher's theorem on subgroups of classical groups over finite fields). More precisely, [Gu03, Thm. 1.5] asserts that[9] for every pair of integers $r \geq 1$, $g \geq 0$ there are only finitely many simple groups of the form $D(\mathcal{S}_\ell^{\Phi_\ell})$, with $\mathcal{S}_\ell$ a simple adjoint group over $\overline{\mathbb{F}}_\ell$ of rank $\leq r$, which occur as a composition factor of the monodromy group of a connected étale cover $X_\ell \to B_\ell$ with X_ℓ of genus $\leq g$. From this and Lemma 6.1, one immediately deduces

Theorem 6.2. *Assume (P) and (T). Then $g_+^{\rho_\ell} \to +\infty$.*

It is probable (though the authors did not work out all the details) and it would be in any case desirable that one could give a proof of Theorem 6.2 along the same guidelines as the proof of Theorem 1.6.1 thus resorting to 'more elementary' material than [Gu03].

7. Appendix: the gonality variant of Theorem 1.6.1 in characteristic 0

We sketch the proof of the following result. We refer to [EHKo12, §2 and §3] for details and references. With the notation of Subsection 2.3, we write $\gamma_+^{\rho_\ell} := \gamma_{X_+^{\rho_\ell}}$.

7.1. **Theorem** *Assume $p = 0$ and (P). Then $\gamma_+^{\rho_\ell} \to +\infty$.*

Proof (sketch). Recall that for a finite morphism of curves $C \xrightarrow{f} B$, one has $\gamma_B \leq \gamma_C \leq \deg(f)\gamma_B$ so, to prove that $\gamma_+^{\rho_\ell} \to +\infty$, one can freely replace X by a connected étale cover hence, by Theorem A, assume that (*) $\overline{G}_\ell = \overline{G}_\ell^+$ and $\overline{G}_\ell^{ab} = 0$ for $\ell \gg 0$. Eventually, after fixing an embedding $k \hookrightarrow \mathbb{C}$, one may reduce to the case where $k = \mathbb{C}$. Fix a finite 'symmetric' subset S of generators of the topological

[9] Note that in [Gu03], the terminology 'Chevalley groups' includes arbitrary finite simple groups of Lie type in the sense of Larsen-Pink (contrary to the somewhat standard conventions).

fundamental group $\pi_1^{top}(X(\mathbb{C}))$. Write $s := |S|$ and let S_ℓ denote the image of S in G_ℓ. For every $\ell \in L$, fix $U_\ell \in \mathcal{F}_{\ell,+}$ such that $\gamma_{X_{U_\ell}} = \gamma_+^{\rho_\ell}$. Then, from a deep group-theoretical result of Pyber and Szabo, properties (*) ensure that the family of Cayley-Schreier graphs $C(G_\ell/\overline{U}_\ell, S_\ell)$, $\ell \in L$ is an 'esperantist' family that is the possible decreasing of the first eigenvalue of the combinatorial Laplacian of the graphs is controlled by the order of the graphs in an explicit way. Assuming this, it is easy to show that $g_{X_{U_\ell}} \geq 2$ for $\ell \gg 0$ and, then, to relate the first eigenvalue of the combinatorial Laplacian to the first eigenvalue of the Laplacian acting on the hyperbolic L^2-functions on $X_{U_\ell} \simeq \Gamma_\ell \setminus \mathbb{H}$ (here $\mathbb{H}$ is the hyperbolic plane and $\Gamma_\ell \subset \mathrm{PSL}_2(\mathbb{Z})$ a discrete subgroup). The latter, in turn, is related to $\gamma_{X_{U_\ell}}$ by the Li-Yau inequality. Combining these facts and the esperantist property, one gets the announced result. □

It has the following arithmetic consequence ([F94], [Fr94]).

7.2. **Corollary** *Assume $p = 0$ and (P). Then for every integer $d \geq 1$, for $\ell \gg 0$ and all but finitely many $x \in X$ with $[k(x) : k] \leq d$ one has $\overline{G}_\ell^+ \subset G_{\ell,x}$. In particular, there exists a constant $K \geq 1$ such that for $\ell \gg 0$ and all but finitely many $x \in X$ with $[k(x) : k] \leq d$ one has $[G_\ell : G_{\ell,x}] \leq K$.*

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