

RIGIDITY PROPERTIES OF RAMIFICATION AND APPLICATIONS

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Notation / conventions.

For a scheme S , write $|S|$ for the set of closed points of S .

For a connected scheme S and a geometric point \bar{s} on S , write $\pi_1(S, \bar{s})$ for the étale fundamental group of S ; we almost always omit the base point \bar{s} from the notation and, if $S = \text{spec}(A)$ is affine, we often abbreviate $\pi_1(A) := \pi_1(\text{spec}(A))$. Recall that if $A = k$ is a field, $\pi_1(k)$ is the absolute Galois group of k .

A variety over a field k is a scheme separated and of finite type over k . If S is an integral scheme with generic point η and X is a smooth, geometrically connected variety over $k := k(\eta)$, a model of X over S is a surjective morphism $\mathcal{X} \rightarrow S$ such that X fits into a Cartesian diagram

$$(1) \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \\ \text{spec}(k) & \xrightarrow{\eta} & S. \end{array}$$

One says that X admits a *smooth model over S* if it fits into Cartesian diagram (1) with \mathcal{X} integral and $\mathcal{X} \rightarrow S$ surjective, smooth, separated and of finite type, and one says that X admits a *smooth model with relative normal crossing compactification (a smooth NCC model for short) over S* if it admits a smooth model $\mathcal{X} \rightarrow S$ over S which fits into a diagram

$$(2) \quad \begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathcal{X}^{\text{cpt}} \\ & \searrow & \downarrow \\ & & S \end{array}$$

with $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}}$ an open immersion, \mathcal{X}^{cpt} integral, $\mathcal{X}^{\text{cpt}} \rightarrow S$ smooth, proper, and $\mathcal{D} := \mathcal{X}^{\text{cpt}} \setminus \mathcal{X} \hookrightarrow \mathcal{X} \rightarrow S$ a relative normal crossing divisor. If $S = \{\eta, s\}$ is the spectrum of a discrete valuation ring with (generic point η and) closed point s , one says that X admits a *semistable model over S* if it fits into a Cartesian diagram (1) with $\mathcal{X} \rightarrow S$ semistable (see Paragraph 1.1.3), and one says that X admits a *semistable model with relative normal crossing compactification (a semistable NCC model for short) over S* if it admits a semistable model $\mathcal{X} \rightarrow S$ over S which fits into a diagram (2) with $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}}$ an open immersion, $\mathcal{X}^{\text{cpt}} \rightarrow S$ proper, semistable and $\mathcal{D} := \mathcal{X}^{\text{cpt}} \setminus \mathcal{X} \hookrightarrow \mathcal{X} \rightarrow S$ a relative normal crossing divisor.

1. INTRODUCTION

Let k be a field and X a smooth, geometrically connected variety over k . The general topics of this article is the investigation of rigidity phenomena for ramification properties of \mathbb{Q}_p -local systems on X or, more generally, continuous representations $\rho : \pi_1(X) \rightarrow \Pi$ of the étale fundamental group $\pi_1(X)$ of X with target a profinite group Π . By rigidity phenomena we roughly mean the following. Let P be a property (which may involve extra data such as suitable models of X) of such representations. For a given representation $\rho : \pi_1(X) \rightarrow \Pi$, consider the set of closed points

$$|X|_\rho^P := \{x \in |X| \mid x^* \rho : \pi_1(x) \xrightarrow{x} \pi_1(X) \xrightarrow{\rho} \Pi \text{ satisfies } P\} \subset |X|.$$

Then one may ask the relation between the following properties:

- (i) $\rho : \pi_1(X) \rightarrow \Pi$ satisfies P ;
- (ii) $|X|_\rho^P$ is "huge" in a precise sense;
- (iii) $|X|_\rho^P \neq \emptyset$.

The implication (ii) \Rightarrow (iii) is tautological and, if P is functorial enough, the implication (i) \Rightarrow (ii) should also be automatic. In general the implications (iii) \Rightarrow (ii) \Rightarrow (i) are more delicate.

These questions were originally prompted by several applications, in particular to Conjecture (T), a consequence of the unramified Fontaine-Mazur conjecture - see Subsection 1.2.3 and Appendix B.

1.1. Ramification with respect to an open embedding. Let \mathcal{X} be a connected normal scheme and let $\emptyset \neq X \hookrightarrow \mathcal{X}$ an open immersion. For every $x \in \mathcal{X}$, set

$$\mathcal{X}_{(\bar{x})} := \text{spec}(\mathcal{O}_{\mathcal{X}, \bar{x}}), \quad X_{(\bar{x})} := X \times_{\mathcal{X}} \mathcal{X}_{(\bar{x})}$$

and write

$$I_{X,x} := \text{im}(\pi_1(X_{(\bar{x})}) \rightarrow \pi_1(X)) \subset \pi_1(X)$$

for the inertia group of x in $\pi_1(X)$ with respect to $X \hookrightarrow \mathcal{X}$ and $I_{X,x} \twoheadrightarrow I_{X,x}^t$ for its tame quotient, that is $I_{X,x}^t = I_{X,x}$ if x has residue characteristic 0 and $I_{X,x}^t = I_{X,x}^{(p')}$ if x has residue characteristic $p > 0$. It follows from the definition that if x_1 specializes to x_0 in \mathcal{X} then $I_{X,x_1} \subset I_{X,x_0}$.

1.1.1. One says that a morphism of profinite groups $\pi_1(X) \rightarrow \Pi$ is *unramified¹ with respect to $X \hookrightarrow \mathcal{X}$* if for every $x \in \mathcal{X}$ the resulting morphism $I_{X,x} \hookrightarrow \pi_1(X) \rightarrow \Pi$ is trivial. As \mathcal{X} is normal, the following properties are equivalent:

- (i) $\pi_1(X) \rightarrow \Pi$ is unramified with respect to $X \hookrightarrow \mathcal{X}$;
- (ii) For every $x \in |\mathcal{X} \setminus X|$, the resulting morphism $I_{X,x} \hookrightarrow \pi_1(X) \rightarrow \Pi$ is trivial;
- (iii) $\pi_1(X) \rightarrow \Pi$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X})$,

and if \mathcal{X} is regular, by Zariski-Nagata purity theorem [SGA1, Exp. X, Thm. 3.1], Properties (i), (ii), (iii) are also equivalent to

- (iv) For every $x \in \mathcal{X} \setminus X$ of codimension 1 in \mathcal{X} , the resulting morphism $I_{X,x} \hookrightarrow \pi_1(X) \rightarrow \Pi$ is trivial.

1.1.2. One says that a morphism of profinite groups $\pi_1(X) \rightarrow \Pi$ is *tamely ramified¹ with respect to $X \hookrightarrow \mathcal{X}$* if for every $x \in \mathcal{X}$ the resulting morphism $I_{X,x} \hookrightarrow \pi_1(X) \rightarrow \Pi$ factors through $I_{X,x} \twoheadrightarrow I_{X,x}^t$. Let

$$\langle\langle I_{X,x}^w \mid x \in \mathcal{X} \setminus X \rangle\rangle \subset \pi_1(X)$$

denote the subgroup normally topologically generated by the subgroups $I_{X,x}^w := \ker(I_{X,x} \twoheadrightarrow I_{X,x}^t) \subset \pi_1(X)$, $x \in \mathcal{X} \setminus X$ and set

$$\pi_1(X) \twoheadrightarrow \pi_1^\dagger(\mathcal{X}; \mathcal{X} \setminus X) := \text{coker}(\langle\langle I_{X,x}^w \mid x \in \mathcal{X} \setminus X \rangle\rangle \rightarrow \pi_1(X))$$

for the profinite quotient classifying étale covers of X tamely ramified with respect to $X \hookrightarrow \mathcal{X}$. Again, the following properties are equivalent:

- (i) $\pi_1(X) \rightarrow \Pi$ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$;
- (ii) For every $x \in |\mathcal{X} \setminus X|$, the resulting morphism $I_{X,x} \hookrightarrow \pi_1(X) \rightarrow \Pi$ factors through $I_{X,x} \twoheadrightarrow I_{X,x}^t$;
- (iii) $\pi_1(X) \rightarrow \Pi$ factors through $\pi_1(X) \twoheadrightarrow \pi_1^\dagger(\mathcal{X}; \mathcal{X} \setminus X)$,

¹ The normality of \mathcal{X} is used to ensure that $\mathcal{X}_{(\bar{x})}$ - hence $X_{(\bar{x})}$ - is integral so that it makes sense to consider the étale fundamental group $\pi_1(X_{(\bar{x})})$ of $X_{(\bar{x})}$. One could extend the definitions of being unramified (resp. tamely ramified) with respect to $X \hookrightarrow \mathcal{X}$ to arbitrary schemes \mathcal{X} by requiring that for every $u \in X_{(\bar{x})}$ the image of $\pi_1(u) \rightarrow \pi_1(X) \rightarrow \Pi$ be trivial (resp. factors through $\pi_1(u) \twoheadrightarrow \pi_1(u)^{(p')}$, where p denote the residue characteristic of x).

and if \mathcal{X} is regular and $\mathcal{D} := \mathcal{X} \setminus X \hookrightarrow \mathcal{X}$ is a normal crossing divisor, Properties (i), (ii), (iii) are also equivalent to [SGA1, Exp. XIII]

- (iv) For every $x \in \mathcal{X} \setminus X$ of codimension 1 in \mathcal{X} , the resulting morphism $I_{X,x} \hookrightarrow \pi_1(X) \rightarrow \Pi$ factors through $I_{X,x} \twoheadrightarrow I_{X,x}^t$.

1.1.3. *Schemes over dvr in mixed characteristic.* Let \mathcal{O}_k be a complete discrete valuation ring with fraction field k , of characteristic 0, maximal ideal \mathfrak{m} , and perfect residue field $\kappa := \mathcal{O}_k/\mathfrak{m}$ of characteristic $p > 0$. Write $S := \text{spec}(\mathcal{O}_k) = \{s, \eta\}$, where s denotes the closed point and η the generic point.

Assume now $\mathcal{X} \rightarrow S$ is surjective, separated and of finite type and the open immersion $X \hookrightarrow \mathcal{X}$ is the inclusion of the generic fiber; For every $x \in \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|)$, let $S_x := \text{spec}(\mathcal{O}_{k(x)}) \rightarrow \mathcal{X}$ denote the normalization of the reduced subscheme $\overline{\{x\}}^{\text{zar}}$ underlying the Zariski closure of x in \mathcal{X} . The technical core of this article is to investigate the relation between the (tame) ramification of a continuous morphism of profinite groups $\rho : \pi_1(X) \rightarrow \Pi$ with respect to $X \hookrightarrow \mathcal{X}$ and the (tame) ramification of $x^*\rho : \pi_1(x) \rightarrow \Pi$ with respect to $\{x\} \hookrightarrow S_x$, $x \in \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|)$ when $\mathcal{X} \rightarrow S$ is semistable (which is enough for most applications) that is $\mathcal{X} \rightarrow S$ is geometrically irreducible, separated, flat, of finite type, X is integral, smooth over k , and for every $x \in |\mathcal{X}_s|$, $\mathcal{O}_{\mathcal{X},\bar{x}}$ is isomorphic to the strict henselization of the localization at $\langle T_1, \dots, T_n \rangle$ of

$$\mathcal{O}_k^{\text{ur}}[T_1, \dots, T_n] / \langle T_1 \cdots T_m - \pi^r \rangle$$

for some integers $r \geq 1$, $n \geq m \geq 1$. Here k^{ur} denotes the maximal unramified extension of k and π a uniformizer of k .

Our main result is the following. Let X be a smooth, geometrically connected variety over k . For a profinite group Π and a continuous morphism $\rho : \pi_1(X) \rightarrow \Pi$, let

$$|X|_{\rho}^{\text{ur}} \subset |X|$$

denote the subset of all $x \in |X|$ such that $x^*\rho : \pi_1(x) \rightarrow \Pi$ is unramified with respect to $\{x\} \hookrightarrow S_x$ that is factors through $\pi_1(x) \twoheadrightarrow \pi_1(S_x)$. Similarly, let

$$|X|_{\rho}^{\text{tr}} \subset |X|$$

denote the subset of all $x \in |X|$ such that $x^*\rho : \pi_1(x) \rightarrow \Pi$ is tamely ramified with respect to $\{x\} \hookrightarrow S_x$ that is factors through $\pi_1(x) \twoheadrightarrow \pi_1^t(S_x; s_x)$.

Theorem 1. *Let Π be a profinite group and let $\rho : \pi_1(X) \rightarrow \Pi$ be a continuous morphism.*

(1) *For every semistable model $\mathcal{X} \rightarrow S$ of X over S one has*

$$\rho : \pi_1(X) \rightarrow \Pi \text{ is unramified with respect to } X \hookrightarrow \mathcal{X} \Leftrightarrow |X|_{\rho}^{\text{ur}} \supset \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|).$$

(2) *For every semistable NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ of X over S , one has*

$$\rho : \pi_1(X) \rightarrow \Pi \text{ is unramified with respect to } X \hookrightarrow \mathcal{X}^{\text{cpt}} \Leftrightarrow |X|_{\rho}^{\text{ur}} = |X|.$$

(3) *For every semistable model $\mathcal{X} \rightarrow S$ of X over S , one has*

$$\rho : \pi_1(X) \rightarrow \Pi \text{ is tamely ramified with respect to } X \hookrightarrow \mathcal{X} \Leftrightarrow |X|_{\rho}^{\text{tr}} \supset \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|).$$

When $\rho : \pi_1(X) \rightarrow \Pi$ arises from a \mathbb{Q}_{ℓ} -local system \mathcal{V}_{ℓ} on X (possibly $\ell = p$), write $|X|_{\mathcal{V}_{\ell}}^{\text{ur}}$ and $|X|_{\mathcal{V}_{\ell}}^{\text{tr}}$ instead of $|X|_{\rho}^{\text{ur}}$ and $|X|_{\rho}^{\text{tr}}$. Define also

$$|X|_{\mathcal{V}_{\ell}}^{\text{sst}} \subset |X|$$

to be the subset of all $x \in |X|$ such that $x^*\mathcal{V}_{\ell}$ is semistable *viz* the corresponding inertia group $I_x := I_{S_x, s_x}$ acts unipotently on $V_{\ell} := \mathcal{V}_{\ell, \bar{x}}$.

If $\ell \neq p$ and $\mathcal{V}_{\ell} = R^i f_* \mathbb{Q}_{\ell}$ for some smooth proper morphism $f : Y \rightarrow X$ and integer $i \geq 0$, the locus $|X|_{\mathcal{V}_{\ell}}^{\text{ur}}$ controls to some extent (at least contains) the subset $|X|_f^{\text{gdred}} \subset |X|$ of all $x \in |X|$ where $Y_x \rightarrow \text{spec}(k(x))$ has good reduction. Similarly, the locus $|X|_{\mathcal{V}_{\ell}}^{\text{sst}}$ controls to some extent (at least contains) the subset $|X|_f^{\text{sstred}} \subset |X|$ of all $x \in |X|$ where $Y_x \rightarrow \text{spec}(k(x))$ has semistable reduction. In view of geometric applications, this prompts the following variant of Theorem 1 (1), which is, actually, a corollary of Theorem 1 (3).

Assume X admits a semistable model $\mathcal{X} \rightarrow S$ over S . Let $\ell \neq p$ a prime. One says that a \mathbb{Q}_{ℓ} -local system \mathcal{V}_{ℓ} on X is *semistable with respect to $X \hookrightarrow \mathcal{X}$* if for every² $x \in \mathcal{X}$, the corresponding inertia group $I_{X,x}$ acts unipotently on $V_{\ell} := \mathcal{V}_{\ell, \bar{x}}$.

²If $\mathcal{X} \rightarrow S$ is smooth (resp. in general), this is actually equivalent to requiring that $I_{X,x}$ acts unipotently on V_{ℓ} for every generic point in \mathcal{X}_s (resp. for every generic point and for every non smooth point in \mathcal{X}_s).

Corollary 2. *Let $\ell \neq p$ a prime and let \mathcal{V}_ℓ be a \mathbb{Q}_ℓ -local system on X . Then, for every semistable model $\mathcal{X} \rightarrow S$ of X over S , one has*

$$\mathcal{V}_\ell \text{ is semistable with respect to } X \hookrightarrow \mathcal{X} \iff |X|_{\mathcal{V}_\ell}^{\text{sst}} \supset \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|).$$

Remark 3.

- (1) The right-hand side conditions in Theorem 1 and Corollary 2 can be weakened - See Remark 16 for the precise statement.
- (2) If $\rho : \pi_1(X) \rightarrow \Pi$ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$, then Theorem 1 (1) and Corollary 2 can be significantly strengthened. For instance, if X admits a smooth model $\mathcal{X} \rightarrow S$ and $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,m}$ denote the connected components of \mathcal{X}_s , then

$\rho : \pi_1(X) \rightarrow \Pi$ is unramified (resp. \mathcal{V}_ℓ is semistable) with respect to $X \hookrightarrow \mathcal{X}$

\iff for each $i = 1, \dots, m$ there exists $x_i \in \text{im}(\mathcal{X}(\mathcal{O}_{k^u}) \rightarrow |X|)$ specializing to some $u_i \in |\mathcal{X}_{s,i}|$ and such that $x_i^* \rho$ is unramified (resp. $x^* \mathcal{V}_\ell$ is semistable);

$\stackrel{(*)}{\iff} |X|_\rho^{\text{tr}} \supset \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|)$ (resp. $|X|_{\mathcal{V}_\ell}^{\text{tr}} \supset \text{im}(\mathcal{X}(\mathcal{O}_{\bar{k}}) \rightarrow |X|)$),
and for each $i = 1, \dots, m$ there exists $x_i \in \text{im}(\mathcal{X}(\mathcal{O}_{k^u}) \rightarrow |X|)$ specializing to some $u_i \in |\mathcal{X}_{s,i}|$ and such that $x_i^* \rho$ is unramified (resp. $x^* \mathcal{V}_\ell$ is semistable).

where the equivalence (*) is Theorem 1 (3). Here $k \subset k^u \subset \bar{\mathbb{Q}}_p$ denotes the maximal unramified extension of k in $\bar{\mathbb{Q}}_p$. See Remark 24 for the general case. Actually, when X admits a smooth model $\mathcal{X} \rightarrow S$ and $\rho : \pi_1(X) \rightarrow \Pi$ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$, Theorem 1 (1) and Corollary 2 are classical corollaries of Abhyankar's lemma and are well-known. The main difficulty and novelty is to handle the case of non-tame representation and of semistable models.

- (3) Theorem 1 (1), (2) and Corollary 2 completed by Remark 16 when $\rho : \pi_1(X) \rightarrow \Pi$ arises from a \mathbb{Q}_ℓ -local system on X for a prime $\ell \neq p$ provide scheme-theoretic ℓ -adic counterparts to recent results in variational p -adic Hodge theory by Guo-Yang and Diao-Yao. Here is tentative dictionary.

Assumption on (formal) model $\mathcal{X} \rightarrow S$	ℓ -adic over a scheme X	p -adic over formal scheme X
smooth	Theorem 1 (1)+Remarks 16, 24	[GuY24, Thm. 7.2]
smooth	Theorem 1 (2)+Remark 16	[DiY25, Thm. 1.6]
semistable	Theorem 1 (1)+Remarks 16, 24	[DiY25, Thm. 3.11]
semistable	Corollary 2 + Remark 16	[GuY24, Thm. 7.2]

1.2. Applications.

1.2.1. *Geometric translations and enhancements.* We retain the notation and assumptions of Subsection 1.1.3 for k, S etc. Let X be a smooth, geometrically connected variety over k . As already mentioned, if $\ell \neq p$ and $\mathcal{V}_\ell = R^i f_* \mathbb{Q}_\ell$ for some smooth proper morphism $f : Y \rightarrow X$ and integer $i \geq 0$, one always has

$$|X|_{\mathcal{V}_\ell}^{\text{ur}} \supset |X|_f^{\text{gdred}}, \quad |X|_{\mathcal{V}_\ell}^{\text{sst}} \supset |X|_f^{\text{sstred}}.$$

When equality holds - e.g. for abelian schemes or for curves, Theorem 1 (1), (3) plus Remarks 16, 24 easily translate as follows. (Compare with [DiY25, Cor. 1.2]).

Corollary 4. *Let X be a smooth, geometrically connected variety over k admitting a smooth model $\mathcal{X} \rightarrow S$. Let $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,m}$ denote the irreducible components of \mathcal{X}_s . Let $Y \rightarrow X$ be an abelian scheme or a smooth, proper, geometrically connected curve of genus $g \geq 2$. Assume for every $x \in \text{im}(\mathcal{X}(\mathcal{O}_{k^t}) \rightarrow |X|)$, Y_x has semistable reduction and for each $i = 1, \dots, m$ there exists $x_i \in \text{im}(\mathcal{X}(\mathcal{O}_{k^u}) \rightarrow |X|)$ specializing to some $u_i \in |\mathcal{X}_{s,i}|$ such that Y_{x_i} has good reduction. Then, for every $x \in |X|$, Y_x has good reduction.*

Here $k \subset k^u \subset k^t \subset \bar{\mathbb{Q}}_p$ denote respectively the maximal unramified and maximal tamely ramified extension of k in $\bar{\mathbb{Q}}_p$.

Remark 5. In Corollary 4, if one only assumes X admits a semistable model $\mathcal{X} \rightarrow S$, the same assertions hold with the condition that for each $i = 1, \dots, m$ there exists $x_i \in \text{im}(\mathcal{X}(\mathcal{O}_{k^u}) \rightarrow |X|)$ specializing to some $u_i \in |\mathcal{X}_{s,i}|$ such that Y_{x_i} has good reduction replaced with the condition that for every $x \in \text{im}(\mathcal{X}(\mathcal{O}_{k^t}) \rightarrow |X|)$ specializing to some $u \in |\mathcal{X}_s^{\text{n-sm}}|$, Y_x has good reduction.

If one imposes additional restrictions on $Y \rightarrow X$ (e.g. adding level structures), the conclusion of Corollary 4 can be strengthened to the extent that $Y \rightarrow X$ extends to an abelian scheme (resp. a smooth proper curve) over \mathcal{X} . We give the proof of Corollary 4 and discuss samples of such enhancements of it in Subsection 4.3.

1.2.2. *Arithmetic combinatorial fundamental group.* Again, we retain the notation and assumptions of Subsection 1.1.3 for k, S etc. and let X be a smooth, geometrically connected variety over k . A second application of Theorem 1 is an intrinsic characterization of the combinatorial part of the arithmetic fundamental group of semistable NCC models $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ of X under the mild assumption that

(3) The irreducible components $\mathcal{X}_{s,1}^{\text{cpt}}, \dots, \mathcal{X}_{s,t}^{\text{cpt}}$ of $\mathcal{X}_s^{\text{cpt}}$ are smooth, geometrically irreducible over κ .

Remark 6. As least when X is a curve with semistable NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$, after possibly enlarging k and blowing up the finitely many singular points of the irreducible components of $\mathcal{X}_s^{\text{cpt}}$, one may always assume that Assumption (3) holds.

More precisely, write $\mathcal{D} := \mathcal{X}^{\text{cpt}} \setminus \mathcal{X}$ and recall the following canonical specialization diagram of tame fundamental groups

$$(4) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{\bar{k}}) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(k) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1^t(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) & \longrightarrow & \pi_1^t(\mathcal{X}^{\text{cpt}}; \mathcal{D}) & \longrightarrow & \pi_1(S) \longrightarrow 1 \\ & & \parallel & & \simeq \uparrow & & \simeq \uparrow \\ 1 & \longrightarrow & \pi_1^t(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) & \longrightarrow & \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) & \longrightarrow & \pi_1(\kappa) \longrightarrow 1, \end{array}$$

Set $\mathcal{X}_{s,i} := \mathcal{X}_{s,i}^{\text{cpt}} \cap \mathcal{X}_s$, $i = 1, \dots, t$ for the irreducible components of \mathcal{X}_s , and $\mathcal{D}_{s,i} := \mathcal{D}_s \cap \mathcal{X}_{s,i}^{\text{cpt}} = \mathcal{X}_{s,i}^{\text{cpt}} \setminus \mathcal{X}_{s,i}$, $\mathcal{D}_{\bar{s},i} := \mathcal{D}_{\bar{s}} \cap \mathcal{X}_{\bar{s},i}^{\text{cpt}} = \mathcal{X}_{\bar{s},i}^{\text{cpt}} \setminus \mathcal{X}_{\bar{s},i}$, $i = 1, \dots, t$. Write

$$K(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) \subset \pi_1^t(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}})$$

for the subgroup topologically normally generated by the images of the canonical morphisms $\pi_1^t(\mathcal{X}_{\bar{s},i}^{\text{cpt}}; \mathcal{D}_{\bar{s},i}) \rightarrow \pi_1^t(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}})$, $i = 1, \dots, t$ and, similarly, write

$$K(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \subset \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)$$

for the subgroup topologically normally generated by the images of the canonical morphisms $\pi_1^t(\mathcal{X}_{s,i}^{\text{cpt}}; \mathcal{D}_{s,i}) \rightarrow \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)$, $i = 1, \dots, t$. As $\bigsqcup_{1 \leq i \leq t} \mathcal{X}_{s,i}^{\text{cpt}} \rightarrow \mathcal{X}_s^{\text{cpt}}$ is a morphism of effective descent for finite covers of $\mathcal{X}_s^{\text{cpt}}$, étale over \mathcal{X}_s and tamely ramified along \mathcal{D}_s , the third line of (4) - in particular, the *combinatorial parts*

$$\pi_1^{\text{comb}}(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) := \text{coker}(K(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) \rightarrow \pi_1^t(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}})),$$

$$\pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) := \text{coker}(K(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \rightarrow \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s))$$

of $\pi_1^t(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}})$ and $\pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)$ respectively, admits a simple combinatorial group-theoretic description in terms of the intersection complex of the $\mathcal{X}_{s,1}^{\text{cpt}}, \dots, \mathcal{X}_{s,t}^{\text{cpt}}$ - see [SGA1, IX] or [St06]. For instance, when $\mathcal{X}^{\text{cpt}} \rightarrow S$ is a relative curve, one gets

$$\pi_1^{\text{comb}}(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) \xrightarrow{\sim} \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \simeq \pi_1(\Gamma(\mathcal{X}_s))^{\wedge},$$

where $\pi_1(\Gamma(\mathcal{X}_s)) \rightarrow \pi_1(\Gamma(\mathcal{X}_s))^{\wedge}$ is the profinite completion of the topological fundamental group of the dual graph $\Gamma(\mathcal{X}_s)$ of \mathcal{X}_s . Under assumption (3), we give the following intrinsic description of the morphism

$$\pi_1(X) \twoheadrightarrow \pi_1^t(\mathcal{X}; \mathcal{D}) \simeq \pi_1^t(\mathcal{X}_s; \mathcal{D}_s) \twoheadrightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s).$$

Let $K(X) \subset \pi_1(X)$ denote the subgroup topologically normally generated by the images of the canonical morphisms $\pi_1(x) \rightarrow \pi_1(X)$, $x \in |X|$ and set

$$\pi_1^{\text{comb}}(X) := \text{coker}(K(X) \rightarrow \pi_1(X)).$$

Corollary 7. *One has canonical isomorphisms*

$$\begin{array}{ccc} \pi_1^{\text{comb}}(X) & \xrightarrow{\simeq} & \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \\ \simeq \downarrow & & \downarrow \simeq \\ \pi_1^{\text{comb}}(\mathcal{X}^{\text{cpt}}) & \xrightarrow{\simeq} & \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \emptyset). \end{array}$$

In particular, $\pi_1(X) \twoheadrightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)$ is independent of the semistable model satisfying (3).

1.2.3. *Trivial locus of \mathbb{Q}_p -local systems.* Our original motivation for Theorem 1 was to attack Conjecture (T) below. We refer to Appendix B for a more detailed discussion about Conjecture (T) and its variants. Let k be a field and let X be a smooth, geometrically connected variety over k . Let p be a prime and let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . For every $x \in X$ and geometric point \bar{x} over x , set $V_p := \mathcal{V}_{p,\bar{x}}$ and let $\bar{G}_p, G_{p,x} \subset G_p \subset \mathrm{GL}_{V_p}$ denote the Zariski closures of the images $\bar{\Pi}_p, \Pi_{p,x}$ and Π_p of $\pi_1(X_{\bar{k}}), \pi_1(x)$ and $\pi_1(X)$ acting on V_p respectively. Define the *trivial locus* of \mathcal{V}_p as

$$|X|_{\mathcal{V}_p}^{\mathrm{triv}} := \{x \in |X| \mid G_{p,x}^\circ = 1\},$$

(where $(-)^{\circ}$ denotes the neutral component). The following conjecture predicts that, when k is a number field, the trivial locus should be empty unless $\mathcal{V}_p|_{X_{\bar{k}}}$ is finite.

Conjecture (T). *Assume k is a number field. For a \mathbb{Q}_p -local system \mathcal{V}_p on X , one has*

$$|X|_{\mathcal{V}_p}^{\mathrm{triv}} \neq \emptyset \Leftrightarrow |X|_{\mathcal{V}_p}^{\mathrm{triv}} = |X|.$$

By Hilbert's irreducibility theorem, the conditions $|X|_{\mathcal{V}_p}^{\mathrm{triv}} = |X|$ is equivalent to $G_p^\circ = 1$. See Subsection B.1.1 for equivalent formulations of Conjecture (T) and variants about the *unipotent locus*

$$|X|_{\mathcal{V}_p}^{\mathrm{uni}} := \{x \in |X| \mid G_{p,x}^\circ \text{ is unipotent}\}$$

and centralizing locus. In this introductory section, for simplicity, we mostly focus on the trivial locus. Also, we formulated Conjecture (T) for *smooth* geometrically connected varieties over k but, actually, this implies (hence is equivalent to) Conjecture (T) for geometrically irreducible varieties over k which are only assumed to be normal - see Proposition 56.

On top of the reformulations given in Subsection B.1.1, Conjecture (T) is motivated by its relations with classical conjectures - see Subsection B.1.2, in particular with the unramified Fontaine-Mazur conjecture

Conjecture (U F-M). (Unramified Fontaine-Mazur [FoM95, Conj. 5.1a]) *Let k be a number field. Let p be a prime and let $U \subset \mathrm{spec}(\mathcal{O}_k)$ be an open subset containing all finite places $v|p$ of k . Then every \mathbb{Q}_p -local system on U is finite.*

and its higher-dimensional variant

Conjecture (VU F-M). (Variational unramified Fontaine-Mazur) *Let $U \subset \mathrm{spec}(\mathcal{O}_k)$ be a non-empty open subset and let $\mathcal{X} \rightarrow U$ be surjective, flat, of finite type with \mathcal{X} normal. Let p be a prime such that U contains all places $v|p$ of k . Then every \mathbb{Q}_p -local system on \mathcal{X} is finite.*

More precisely, we prove the following.

Corollary 8.

- (1) *Conjecture (U F-M) \Rightarrow Conjecture (VU F-M)*
(2) *Conjecture (VU F-M) for $\mathcal{X} \rightarrow U$ of relative dimension d \Rightarrow Conjecture (T) for X of dimension d .*

If k is a number field we prove that Conjecture (T) holds for \mathcal{V}_p part of a \mathbb{Q} -compatible family $\mathcal{V} = (\mathcal{V}_\ell)_\ell$ of \mathbb{Q}_ℓ -local system on X - see Corollary 39. This can be proved directly or deduced from the following p -adic variant of Conjecture (T) - see Corollary 34 and Remark 35 (3).

Corollary 9. *Let k be a p -adic field and let X be a smooth, geometrically connected variety over k . Let \mathcal{V}_p be a \mathbb{Q} -rational \mathbb{Q}_p -local system on X . Then*

$$|X|_{\mathcal{V}_p}^{\mathrm{triv}} \neq \emptyset \Leftrightarrow |X|_{\mathcal{V}_p}^{\mathrm{triv}} = |X|.$$

Similarly, $|X|_{\mathcal{V}_p}^{\mathrm{uni}} \neq \emptyset \Leftrightarrow |X|_{\mathcal{V}_p}^{\mathrm{uni}} = |X|$.

Remark 10. In general, one cannot expect the equivalent conditions in Corollary 9 to be also equivalent to $G_p^\circ = 1$ (or G_p° is unipotent) as predicted by Conjecture (T) when k is a number field. Indeed, already if X is a curve, from Corollary 7, these conditions are also equivalent to the following: after possibly replacing k by a finite field extension so that X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \rightarrow S$ satisfying Condition (3), \mathcal{V}_p is combinatorial *viz* the corresponding representation $\pi_1(X) \rightarrow \mathrm{GL}(\mathcal{V}_{p,\bar{x}})$ factors through

$$\pi_1(X) \twoheadrightarrow \pi_1^{\mathrm{comb}}(\mathcal{X}_s^{\mathrm{cpt}}, \mathcal{D}_s) \simeq \pi_1(\Gamma(\mathcal{X}_s^{\mathrm{cpt}}))^\vee.$$

If one removes the \mathbb{Q} -rationality assumption, the arguments in the proof of Corollary 9 still yield information about the distribution of trivial points. For instance, when X is a curve, one gets the following weak form of Conjecture (T), where condition (iii) is weakened to the condition (iii)' asserting that $|X|_{\mathcal{V}_p}^{\mathrm{triv}}$ has to be " v -adically small" for all finite places $v|p$ of k .

Let k be a number field and let X be a smooth, geometrically connected curve over k admitting a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \rightarrow S$ over some non-empty open subscheme $S \subset \mathrm{spec}(\mathcal{O}_k)$.

Corollary 11. *For $p \in \text{im}(|S| \rightarrow |\text{spec}(\mathbb{Z})|)$ the following holds. Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X with \mathbb{Q}_p -rank $< p - 1$ and such that \overline{G}_p° is not unipotent. Then, for every $v \in S_p$, there exists a 0-dimensional Zariski-closed subset $\mathcal{Z}_v \subset \mathcal{X}_v^{\text{cpt}}$ such that*

$$(iii)' \quad |X|_{\mathcal{V}_p}^{\text{uni}} \subset \bigcap_{v \in S_p} sp_v^{-1}(\mathcal{Z}_v),$$

where $sp_v : |\mathcal{X}^{\text{cpt}}| \rightarrow |\mathcal{X}_v^{\text{cpt}}|$ denotes the specialization map.

Remark 12. The 0-dimensional Zariski-closed subsets $\mathcal{Z}_v \subset \mathcal{X}_v^{\text{cpt}}$ appearing in the statement of Corollary 11 are rather explicit. For instance, if \mathcal{V}_p is a summand of a \mathbb{Q}_p -local system of the form $R^{2i}f_*\mathbb{Q}_p(i)$ for some smooth proper morphism $f : Y \rightarrow X$, up to shrinking S , one may also assume $f : Y \rightarrow X$ admits a smooth proper model $f : \mathcal{Y} \rightarrow \mathcal{X}$ over S . Then, for $p \gg 0$ and for every $v \in S_p$, $\mathcal{X}_v \setminus \mathcal{Z}_v$ can be taken to be the largest open subset over which the convergent F-isocrystal $R^{2i}f_{\text{cris}}^\dagger \mathcal{O}_{\mathcal{Y}_v|\mathcal{X}_v}^\dagger$ admits a slope filtration.

After introducing some technical level conditions in Section 2, we carry out the proof of Theorem 1 and Corollary 2 in Section 3, which is the technical core of the article. We give first applications of Theorem 1 in Section 4. In Section 5 we prove Corollary 7 and give applications of Theorem 1 to Conjecture (T) and to its variants over p -adic fields and for the unipotent and centralizing loci; in particular, we prove there Corollary 8 and Corollary 9. Appendix A provides a very brief overview of the results from (variational) p -adic Hodge theory used in Section 5 and Appendix B while, in Appendix B, we discuss in more details Conjecture (T).

Acknowledgements: In a first version of this work, mostly aimed at attacking Conjecture (T) via methods from p -adic Hodge theory, we only considered Theorem 1 under the assumption that X admits a smooth model over S . We are very grateful to Sasha Petrov for asking whether this could be extended to the case where X admits a semistable model over S , which yields a definitively wider range of applications. The reduction to the case where \mathcal{V}_ℓ is semisimple in Lemma 33 is also due to him. We would also like to thank Ziquan Yang for exchanges around a consistent notion of ℓ -adic semistable local system, which lead us to Corollary 2, and for sketching the argument in Remark 17. More generally, we enjoyed constructive discussions with Haoyang Guo and Ziquan Yang.

2. LEVEL

Some of the proofs and statements involve level assumptions that we list here for the convenience of the reader.

Let S be a connected scheme. Fix a prime ℓ and let \mathcal{V}_ℓ be a \mathbb{Q}_ℓ -local system on S . Fix a geometric point \bar{s} on S and set $V_\ell := \mathcal{V}_{\ell, \bar{s}}$; let $\Pi_\ell \subset \text{GL}(V_\ell)$ denote the image of $\pi_1(S, \bar{s})$ acting on V_ℓ . Consider the following "level conditions" on \mathcal{V}_ℓ :

Lev₁(\mathcal{V}_ℓ) There exists a Π_ℓ -stable \mathbb{Z}_ℓ -lattice $V_\ell^\circ \subset V_\ell$ such that $\Pi_\ell \subset \text{Id} + \tilde{\ell}\text{End}_{\mathbb{Z}_\ell}(V_\ell^\circ)$, where $\tilde{\ell} = 4$ if $\ell = 2$ and $\tilde{\ell} = \ell$ otherwise.

Lev₂(\mathcal{V}_ℓ) Π_ℓ is torsion free.

Lev₃(\mathcal{V}_ℓ) Π_ℓ is pro- ℓ .

Lev₄(\mathcal{V}_ℓ) The torsion elements in Π_ℓ are of prime-to- ℓ order.

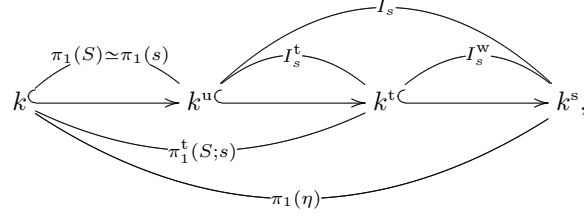
One easily checks the following implications.

$$\text{Lev}_1(\mathcal{V}_\ell) \implies \text{Lev}_2(\mathcal{V}_\ell) \iff (\text{Lev}_3(\mathcal{V}_\ell) + \text{Lev}_4(\mathcal{V}_\ell)) \implies \text{Lev}_4(\mathcal{V}_\ell) \iff \ell > \dim(V_\ell) + 1.$$

In practice, Lev₁(\mathcal{V}_ℓ) can always be achieved after replacing S by a connected étale cover.

3. POINTWISE VERSUS GLOBAL RAMIFICATION PROPERTIES

Let \mathcal{O}_k be a complete discrete valuation ring with fraction field k , of characteristic 0, maximal ideal \mathfrak{m} , and perfect residue field κ , of characteristic $p > 0$. Set $S := \text{spec}(\mathcal{O}_k) = \{\eta, s\}$, where η is the generic point and s the closed point of S . Fix a separable(= algebraic) closure $k \hookrightarrow k^s = \bar{k}$. Considering the normal crossing divisor $s \hookrightarrow S$, and writing $I_s := I_{\eta, s}$ for simplicity (see Subsection 1.1), one gets



where $I_s^w := \ker(I_s \rightarrow I_s^t)$ denotes the wild inertia group. For $\# := u, t, s$ let $\mathcal{O}_{k^\#}$ denote the valuation ring of $k^\#$ and $S^\# := \text{spec}(\mathcal{O}_{k^\#})$.

For a variety X over k and a closed point $x \in |X|$ with residue field $k(x)$, write $\mathcal{O}_{k(x)}$ for the valuation ring of $k(x)$, $S_x := \text{spec}(\mathcal{O}_{k(x)}) = \{x, s_x\}$, with $(x$ the generic point and) s_x the closed point of S_x .

Let $\mathcal{X} \rightarrow S$ be a semistable morphism. Set $X := \mathcal{X}_\eta$; up to replacing S by its normalization in $\mathcal{X} \rightarrow S$, we may and will assume that X is geometrically connected over k .

Let $\psi : Y \rightarrow X$ be a Galois (in particular finite, étale and connected) cover. For a subset $\Sigma \subset |X|$, say that $\psi : Y \rightarrow X$ is Σ -pointwise unramified (resp. tame) if for every $x \in \Sigma$ and for one (equivalently every) $y \in Y_x$ the resulting cover $S_y \rightarrow S_x$ is étale (resp. tamely ramified along s_x). We will apply this terminology to subsets of the following form. For $\# = u, t, s$ and $u \in |\mathcal{X}_s|$,

$$\Sigma =]u[^\# := \text{im}(\mathcal{X}(\mathcal{O}_{k^\#})_u \hookrightarrow X(k^\#) \rightarrow |X|) \subset |X|,$$

where $\mathcal{X}(\mathcal{O}_{k^\#})_u$ denotes the fiber of $\mathcal{X}(\mathcal{O}_{k^\#}) \rightarrow \mathcal{X}(\bar{\kappa}) \rightarrow |\mathcal{X}_s|$ over u ; equivalently, $]u[^\#$ is the subset of all $x \in |X|$ such that

$$\begin{array}{ccccc} \text{spec}(k(x)) & \xrightarrow{x} & X & & \\ \downarrow & & \downarrow & & \\ \text{spec}(\mathcal{O}_{k^\#}) & \longrightarrow & \text{spec}(\mathcal{O}_{k(x)}) & \xrightarrow{\exists! \bar{x}} & \mathcal{X} \\ \uparrow & \square & \uparrow & \square & \uparrow \\ \text{spec}(\bar{\kappa}) & \longrightarrow & \text{spec}(\kappa(s_x)) & \longrightarrow & \text{spec}(\kappa(u)) \xrightarrow{u} \mathcal{X}_s \end{array}$$

More generally, for a subset $F \subset |\mathcal{X}_s|$, we will consider

$$\Sigma =]F[^\# := \bigcup_{u \in F}]u[^\# \subset |X|$$

and, when $F = |\mathcal{X}_s|$, we will simply write

$$|\mathcal{X}_s[^\# :=]|\mathcal{X}_s[^\# (= \text{im}(\mathcal{X}(\mathcal{O}_{k^\#}) \hookrightarrow X(k^\#) \rightarrow |X|)).$$

We are now ready to state the core technical result of this article - Theorem 13 below. Let $\mathcal{X} \rightarrow S$ be a semistable morphism with $X := \mathcal{X}_\eta$ geometrically connected over k . Let $\psi : Y \rightarrow X$ be a Galois cover. Let $\psi_\mathcal{X} : \mathcal{Y} \rightarrow \mathcal{X}$ denote the normalization of \mathcal{X} in $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}$.

Theorem 13. *Let $\mathcal{X}_s^{\text{sm}} = \mathcal{X}_{s,1}^{\text{sm}} \sqcup \dots \sqcup \mathcal{X}_{s,m}^{\text{sm}}$ denote the decomposition of the regular (viz smooth) locus $\mathcal{X}_s^{\text{sm}} \hookrightarrow \mathcal{X}_s$ into irreducible (viz connected) components.*

(U) *There exists a non-empty open subset $\mathcal{X}_{s,i}^o \subset \mathcal{X}_{s,i}^{\text{sm}}$, $i = 1, \dots, m$ (depending on $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}^{\text{sm}}$) such that, for every $u_i \in \mathcal{X}_{s,i}^o$, $i = 1, \dots, m$, the following conditions are equivalent.*

- (U-1) $\ker(\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X})) \subset \pi_1(Y)$;
- (U-2) $\psi_\mathcal{X} : \mathcal{Y} \rightarrow \mathcal{X}$ is (finite) étale;
- (U-3) $\psi : Y \rightarrow X$ is $]\mathcal{X}_s[^\#$ -pointwise unramified;
- (U-4) $\psi : Y \rightarrow X$ is $\cup_{1 \leq i \leq m}]u_i[^\# \cup \mathcal{X}_s^{\text{n-sm}}[^\#$ -pointwise unramified.

(T) *There exists a non-empty open subset $\mathcal{X}_{s,i}^o \subset \mathcal{X}_{s,i}^{\text{sm}}$, $i = 1, \dots, m$ (depending on $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}^{\text{sm}}$) such that, for every $u_i \in \mathcal{X}_{s,i}^o$, $i = 1, \dots, m$, the following conditions are equivalent.*

- (T-1) $\ker(\pi_1(X) \twoheadrightarrow \pi_1^t(\mathcal{X}; \mathcal{X}_s)) \subset \pi_1(Y)$;
- (T-2) $\psi_\mathcal{X} : \mathcal{Y} \rightarrow \mathcal{X}$ is (finite) tamely ramified along \mathcal{X}_s ;
- (T-3) $\psi : Y \rightarrow X$ is $]\mathcal{X}_s[^\#$ -pointwise tame;

(T-4) $\psi : Y \rightarrow X$ is $\cup_{1 \leq i \leq m} u_i [{}^t \cup] \mathcal{X}_s^{n\text{-sm}} [{}^t \text{-pointwise tame}$.

Remark 14. If $\mathcal{X} \rightarrow S$ is smooth and \mathcal{Y} is regular, one can take $\mathcal{X}_{s,i}^\circ = \mathcal{X}_{s,i}$, $i = 1, \dots, m$ in Theorem 13 (U). If, furthermore, for every intermediate cover $Y \rightarrow Y' \rightarrow X$ the normalization $\mathcal{Y}' \rightarrow \mathcal{X}$ of \mathcal{X} in $Y' \rightarrow X \hookrightarrow \mathcal{X}$ is smooth over S with \mathcal{Y}' regular, then one can also take $\mathcal{X}_{s,i}^\circ = \mathcal{X}_{s,i}$, $i = 1, \dots, m$ in Theorem 13 (T).

We will also use the following Corollary / variant of Theorem 13 (U). Let $(\mathcal{X}^{\text{cpt}}, \mathcal{D}) \rightarrow S$ be a semistable marked pair with smooth generic fiber (X^{cpt}, D) , that is $\mathcal{X}^{\text{cpt}} \rightarrow S$ is a proper semistable scheme, $\mathcal{D} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ is a relative normal crossing divisor on $\mathcal{X}^{\text{cpt}} \rightarrow S$ and X^{cpt} is smooth over k . Write again $\mathcal{X}_s^{\text{cpt,sm}} = \sqcup_{1 \leq i \leq m} \mathcal{X}_{s,i}^{\text{cpt,sm}}$ for the decomposition of $\mathcal{X}_s^{\text{cpt,sm}}$ into irreducible (*viz* connected) components. Set $\mathcal{X} := \mathcal{X}^{\text{cpt}} \setminus \mathcal{D}$ and $X := X^{\text{cpt}} \setminus D$. Let $\psi : Y \rightarrow X$ be a Galois cover and let $\psi_{\mathcal{X}^{\text{cpt}}} : \mathcal{Y}^{\text{cpt}} \rightarrow \mathcal{X}^{\text{cpt}}$ denote the normalization of \mathcal{X}^{cpt} in $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}^{\text{cpt}}$.

Corollary 15. *There exists a non-empty open subset $\mathcal{X}_{s,i}^{\text{cpt,sm},\circ} \subset \mathcal{X}_{s,i}^{\text{cpt,sm}}$, $i = 1, \dots, m$ (depending on $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}^{\text{sm}}$) such that for every $u_i \in \mathcal{X}_{s,i}^{\text{cpt,sm},\circ}$, $i = 1, \dots, m$, the following conditions are equivalent.*

- (U-1) $\ker(\pi_1(X) \rightarrow \pi_1(\mathcal{X}^{\text{cpt}})) \subset \pi_1(Y)$;
- (U-2) $\psi_{\mathcal{X}^{\text{cpt}}} : \mathcal{Y}^{\text{cpt}} \rightarrow \mathcal{X}^{\text{cpt}}$ is (finite) étale;
- (U-3) $\psi : Y \rightarrow X$ is $|X|$ -pointwise unramified;
- (U-4) $\psi : Y \rightarrow X$ is $\cup_{1 \leq i \leq m} u_i [{}^t \cup] \mathcal{X}_s^{\text{cpt,n-sm}} [{}^t \text{-pointwise unramified}$.

Before we embark in the proof of Theorem 13 and Corollary 15, which will occupy the remaining part of Section 3, we explain how to deduce from them the proof of Theorem 1.

Proof of Theorem 1: The implications \Rightarrow are tautological. We prove the converse implications \Leftarrow . Consider Assertion (1). By definition, if $|X|_\rho^{\text{ur}} \supset]\mathcal{X}_s[{}^s$ then, for every normal open subgroup $U \subset \Pi$ the corresponding Galois cover $X_U \rightarrow X$ is $]\mathcal{X}_s[{}^s$ -pointwise unramified. By Theorem 13 (U-3) \Rightarrow (U-1), the morphism $\pi_1(X) \twoheadrightarrow \Pi/U$ then factors through $\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X})$. One concludes by passing to the limit on U . This proves (1). The proofs of (2) and (3) are exactly similar using Corollary 15 (U-3) \Rightarrow (U-1) and Theorem 13 (T-3) \Rightarrow (T-1) respectively.

Remark 16.

- (1) Label by (i) the left hand side conditions and by (ii)' the right-hand side conditions of the equivalence in Assertions (1), (2), (3) of Theorem 1. The proof of Theorem 1 shows more precisely that (i) \Rightarrow (ii)' \Rightarrow (ii)'' \Rightarrow (i), where, for (1), (3),

$$(ii)'' \quad |X|_\rho \supset]\mathcal{X}_s[{}^\#,$$

with

- in (1), $\# = u$ if $\mathcal{X} \rightarrow S$ is smooth, and $\# = t$ in general,
- in (3), $\# = t$.

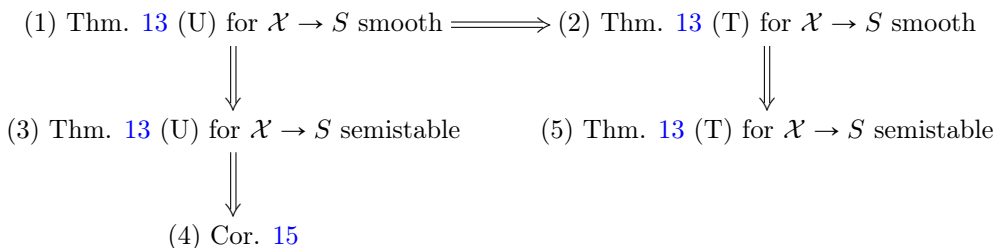
while, for (2),

$$(ii)'' \quad |X|_\rho \supset]\mathcal{X}_s^{\text{cpt}}[{}^t \cap |X|.$$

- (2) The construction of the open subsets $\mathcal{X}_{s,i}^\circ \subset \mathcal{X}_{s,i}$, $i = 1, \dots, m$ is effective. For instance, in Theorem 13 (U), if Y is geometrically connected over k and $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,m}$ are geometrically connected over κ , $\mathcal{X}_{s,i}^\circ = \mathcal{X}_{s,i} \setminus (\psi_{\mathcal{X}}(\mathcal{Y}^{\text{n-reg}}) \cap \mathcal{X}_{s,i})$, $i = 1, \dots, m$.

Remark 17. Guo-Yang proved Theorem 1 (1) in the setting of smooth p -adic formal schemes over \mathcal{O}_k - see [GuY24, Thm. 6.31]; as they mention, this can be regarded as the (scheme-theoretic in our setting) ℓ -adic analogue of Fact 43 (3) / Crystalline part of [GuY24, Thm. 7.2]. Ziquan Yang pointed out to us that one can actually deduce Theorem 1 (1) when X admits a smooth model $\mathcal{X} \rightarrow S$ from its formal analogue. Here is a brief sketch of the argument. (To complete)

3.1. **Strategy of proofs.** Here is the general architecture of the proofs.



The delicate implications are (U-4) \Rightarrow (U-2) and (T-4) \Rightarrow (T-2). Roughly, one proceeds as follows.

- (1) After some reduction, in particular shrinking \mathcal{X}_s , one may assume \mathcal{Y} is smooth. After this reduction, we prove one can take $\mathcal{X}_{s,i}^\circ = \mathcal{X}_{s,i}$, $i = 1, \dots, m$. More precisely, if $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ were not étale, it would be non-étale along one of the $\mathcal{X}_{s,i}$, which would imply that for every $u \in \mathcal{X}_{s,i}$ there would exist $v \in \mathcal{Y}_u$ such that the resulting morphism at the level of cotangent spaces $\psi_{\mathcal{X}}^\# : \mathfrak{m}_u/\mathfrak{m}_u^2 \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2$ is not injective. On the other hand, using the local structure of $\widehat{\mathcal{O}}_{\mathcal{X},u}$, one can always construct a section $\tilde{x} : S \rightarrow \mathcal{X}$ mapping the closed point $s \in S$ to u and such that the resulting morphism at the level of cotangent spaces $\tilde{x}^\# : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2$ does not vanish at a . This contradicts the fact that, by (U-4), $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is $]u[^\mathfrak{u}$ -pointwise unramified.
- (2) After some reduction, in particular shrinking \mathcal{X}_s , one may assume \mathcal{Y} is smooth and that for every intermediate cover $Y \rightarrow Y' \rightarrow X$ the normalization $\mathcal{Y}' \rightarrow \mathcal{X}$ of \mathcal{X} in $Y' \rightarrow X \hookrightarrow \mathcal{X}$ is smooth over S . After this reduction, we prove one can take $\mathcal{X}_{s,i}^\circ = \mathcal{X}_{s,i}$, $i = 1, \dots, m$. More precisely, one can consider the intermediate cover $Y \rightarrow Y_{I^w} \rightarrow X$ corresponding to the wild inertia group $I^w \subset G := \text{Aut}(\psi)$ and the normalization $\mathcal{Y}_{I^w} \rightarrow \mathcal{X}$ of \mathcal{X} in $Y_{I^w} \rightarrow X \hookrightarrow \mathcal{X}$. Then (T-4) forces $\mathcal{Y} \rightarrow \mathcal{Y}_{I^w}$ to be $] \mathcal{Y}_{I^w,s} [^\mathfrak{u}$ -unramified which, applying Theorem 13 (U-4) \Rightarrow (U-2) to $\mathcal{Y} \rightarrow \mathcal{Y}_{I^w}$, in turn imposes that $\mathcal{Y} \rightarrow \mathcal{Y}_{I^w}$ is trivial.

Once Theorem 13 is established for $\mathcal{X}^{\text{sm}} \rightarrow S$, the issue is to control what happens locally around the finitely many non-smooth points $u \in \mathcal{X}^{\text{n-sm}} := \mathcal{X} \setminus \mathcal{X}^{\text{sm}}$. To do so, we first describe in Subsection 3.3.1 the étale fundamental group of the smooth locus of $\mathcal{X}_{(\bar{u})} = \text{spec}(\mathcal{O}_{\mathcal{X},\bar{u}}) \rightarrow \text{spec}(\mathcal{O}_{k^{\text{ur}}})$ (Lemma 20) and the prime-to- p completion of the étale fundamental group of $X_{(\bar{u})} = \mathcal{X}_{(\bar{u})} \times_{\mathcal{X}} X$ (Lemma 21). Then,

- (3), (4) The common strategy to prove Theorem 13 (U) and Corollary 15 when $\mathcal{X} \rightarrow S$ is semistable the following. Assume there exists $u \in \mathcal{X}_s$ and $v \in \mathcal{Y}_u$ (resp. $u \in \mathcal{D}_s$ and $v \in (\mathcal{Y}^{\text{cpt}})_u$) which violate (U-2) - that is such that $\mathcal{Y}_{(\bar{v})} \rightarrow \mathcal{X}_{(\bar{u})}$ (resp. $\mathcal{Y}_{(\bar{v})}^{\text{cpt}} \rightarrow \mathcal{X}_{(\bar{u})}^{\text{cpt}}$) is finite not étale. Then one constructs explicitly a k^t -point $x' \in X$ violating (U-4) using the description of étale covers of $\mathcal{X}_{(\bar{u})}^{\text{sm}}$ (resp. of $\mathcal{X}_{(\bar{u})}$ tamely ramified with respect to $\mathcal{X}_{(\bar{u})} \hookrightarrow \mathcal{X}_{(\bar{u})}^{\text{cpt}}$) given in Lemma 20 (resp. by Abhyankar's lemma).
- (5) From Theorem 13 (T) for $\mathcal{X}^{\text{sm}} \rightarrow S$, one already knows that $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is tamely ramified along the generic points of \mathcal{X}_s . The proof of Theorem 13 (T) is achieved exploiting this and the structure of $\pi_1(X_{(\bar{u})})^{(p')}$ given in Lemma 21.

3.2. Proof of Theorem 13 when $\mathcal{X} \rightarrow S$ is smooth. In this Subsection, assume $\mathcal{X} \rightarrow S$ is smooth.

3.2.1. Elementary remarks. Let $k \hookrightarrow k'$ be a finite field extension and let $s' \in S'$ denote the closed point of $S' := \text{spec}(\mathcal{O}_{k'})$.

- (1) Assume $k' \subset k^{\text{ur}}$ (resp. $k' \subset k^t$). Then for every $u \in \mathcal{X}_s$, the square

$$\begin{array}{ccc} \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathfrak{u}} \longrightarrow]u[^{\mathfrak{u}} & (\text{resp. the square } \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathfrak{t}} \longrightarrow]u[^{\mathfrak{t}}) \\ \downarrow & \square & \downarrow \\ |X \times_k k'| \longrightarrow]X| & & |X \times_k k'| \longrightarrow]X| \end{array}$$

is (well-defined and) Cartesian.

Indeed, let $x \in]X|$ and $x' \in (X \times_k k')_x$. We are to prove that $x \in]u[^{\mathfrak{u}}$ if and only if $x' \in \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathfrak{u}}$ (resp. $x \in]u[^{\mathfrak{t}}$ if and only if $x' \in \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathfrak{t}}$). The if part of the assertion, which ensures that the upper horizontal arrow is well defined, follows from the definition of $] - [^{\mathfrak{u}}$ (resp. $] - [^{\mathfrak{t}}$) and the fact that being unramified (resp. tamely ramified) is stable by composition and subextensions. Let us prove the only if part. Assume $x \in]u[^{\mathfrak{u}}$ (resp. $x \in]u[^{\mathfrak{t}}$). The fact that $x' : \text{spec}(k(x')) \rightarrow X \times_k k'$ extends (uniquely) to $\tilde{x}' : S_{x'} \rightarrow \mathcal{X} \times_S S'$ follows from the fact that $x : \text{spec}(k(x)) \rightarrow X$ extends to $\tilde{x} : S_x \rightarrow \mathcal{X}$ and the properness of $\mathcal{X} \times_S S' \rightarrow \mathcal{X}$ while the fact that $\mathcal{O}_{k(x')} \subset \mathcal{O}_{k^{\text{ur}}}$ (resp. $\mathcal{O}_{k(x')} \subset \mathcal{O}_{k^t}$) follows from the fact that $\mathcal{X} \times_S S' \rightarrow S$ is étale (resp. at most tamely ramified along \mathcal{X}_s).

- (2) Let $k \hookrightarrow k'$ be a finite field extension and let $S' := \text{spec}(\mathcal{O}_{k'}) \rightarrow S$ denote the normalization of S in $\text{spec}(k') \rightarrow \text{spec}(k) \rightarrow S$. Let $\mathcal{Y}'_1 \rightarrow \mathcal{X}$ denote the normalization of $\mathcal{X} \times_S S'$ in $Y \times_k k' \rightarrow X \times_k k' \hookrightarrow \mathcal{X} \times_S S'$ and let $\mathcal{Y}'_2 \rightarrow \mathcal{X} \times_S S'$ denote the normalization of $\mathcal{X} \times_S S'$ in $\mathcal{Y} \times_S S' \rightarrow \mathcal{X} \times_S S'$. By the universal property of normalization, for $i = 1, 2$, the morphism $\mathcal{Y}'_i \rightarrow \mathcal{X} \times_S S'$ factors canonically as $\mathcal{Y}'_i \rightarrow \mathcal{Y} \rightarrow \mathcal{X}$. Furthermore the canonical morphism $\mathcal{Y}'_1 \rightarrow \mathcal{Y}'_2$ is an isomorphism and if $S' \rightarrow S$ is étale, $\mathcal{Y} \times_S S' \rightarrow \mathcal{Y}'_2$ is an isomorphism [Stacks, Tag 03GV].

3.2.2. *Proof of Theorem 13 (U).* The implications (U-1) \Rightarrow (U-2) \Rightarrow (U-3) \Rightarrow (U-4) and (U-2) \Rightarrow (U-1) are (almost) tautological so we are left to prove (U-4) \Rightarrow (U-2). If $\mathcal{X}_s = \emptyset$, there is nothing to prove, so we may and will assume that $\mathcal{X}_s \neq \emptyset$.

Let us first observe that it is enough to prove (U-4) \Rightarrow (U-2) after:

- (1) Replacing k by the algebraic closure k_Y of k in the function field of Y ;
- (2) Base-change along $\text{spec}(k') \rightarrow \text{spec}(k)$ for some finite field extension $k \subset k' \subset k^{\text{ur}}$.

We explain this observation; the arguments are very similar for (1) and (2).

- In case (1), let $S_Y \rightarrow S$ denote the normalization of S in $\text{spec}(k_Y) \rightarrow \text{spec}(k) \rightarrow S$. As $\mathcal{X}_s \neq \emptyset$, by condition (U-4) $S_Y \rightarrow S$ is dominated by an étale cover of S hence is étale. As $\mathcal{X} \times_S S_Y$ is normal, one has a canonical factorization $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X} \times_S S_Y \rightarrow \mathcal{X}$ and $\mathcal{Y} \rightarrow \mathcal{X} \times_S S_Y$ is the normalization of $\mathcal{X} \times_S S_Y$ in $Y \rightarrow X \times_k k_Y \rightarrow \mathcal{X} \times_S S_Y$.
- In case (2), let $S' \rightarrow S$ denote the normalization of S in $\text{spec}(k') \rightarrow \text{spec}(k) \rightarrow S$. By assumption $S' \rightarrow S$ is étale hence $\mathcal{Y} \times_S S' \rightarrow \mathcal{X} \times_S S'$ is the normalization of $\mathcal{X} \times_S S'$ in $Y \times_k k' \rightarrow X \times_k k' \rightarrow \mathcal{X} \times_S S'$.

To unify the end of the argument, in case (1), write $k' := k_Y$ and

$$Y' \rightarrow X' := Y \rightarrow X \times_k k_Y, \quad \mathcal{Y}' \rightarrow \mathcal{X}' \rightarrow S' := \mathcal{Y} \rightarrow \mathcal{X} \times_S S_Y \rightarrow S_Y,$$

while, in case (2), write

$$Y' \rightarrow X' := Y \times_k k' \rightarrow X \times_k k', \quad \mathcal{Y}' \rightarrow \mathcal{X}' \rightarrow S' := \mathcal{Y} \times_S S' \rightarrow \mathcal{X} \times_S S' \rightarrow S'.$$

Let $s' \in S'$ denote the closed point of S' . Let $\mathcal{X}_{s,i} \times_s s' = \sqcup_{1 \leq j \leq m_i} \mathcal{X}_{s',i,j}$ be the decomposition into irreducible (*viz* connected) components of $\mathcal{X}_{s',i} \times_s s'$, $i = 1, \dots, m$. For every non-empty open subset $\mathcal{X}_{s',i,j}^\circ \subset \mathcal{X}_{s',i,j}$, the image $\mathcal{X}_{s,i}^\circ \subset \mathcal{X}_s$ of $\sqcup_{1 \leq j \leq m_i} \mathcal{X}_{s',i,j}^\circ$ via $\mathcal{X}_{s'} \rightarrow \mathcal{X}_s$ is again a non-empty open subset of $\mathcal{X}_{s,i}$. Now, assume (U-4) \Rightarrow (U-2) holds for $Y \times_k k' \rightarrow X \times_k k'$ and the open subsets $\mathcal{X}_{s',i,j}^\circ \subset \mathcal{X}_{s',i,j}$, $j = 1, \dots, m_i$, $i = 1, \dots, m$. Then (U-4) \Rightarrow (U-2) holds for $Y \rightarrow X$ and the open subsets $\mathcal{X}_{s,i}^\circ \subset \mathcal{X}_{s,i}$, $i = 1, \dots, m$. Indeed, assume for every $u_i \in \mathcal{X}_{s,i}^\circ$, $\psi : Y \rightarrow X$ is $]u_i[^\text{u}$ -pointwise unramified. Let $x' \in]u'_{i,j}[^\text{u}$ with image $x \in]u_i[^\text{u}$ (See (1) in Subsection 3.2.1) and for every $y \in Y_{x'}$, consider the factorization $S_y \rightarrow S_{x'} \rightarrow S_x$. As $\psi : Y \rightarrow X$ is $]u_i[^\text{u}$ -pointwise unramified, $S_y \rightarrow S_x$ is étale hence $S_y \rightarrow S_{x'}$ is étale as well. This shows $Y' \rightarrow X'$ is $]u'_{i,j}[^\text{u}$ -pointwise unramified. By (U-4) \Rightarrow (U-2) for $Y' \rightarrow X'$ and the open subsets $\mathcal{X}_{s',i,j}^\circ \subset \mathcal{X}_{s',i,j}$, $j = 1, \dots, m_i$, $i = 1, \dots, m$, $\mathcal{Y}' \rightarrow \mathcal{X}'$ is étale. As $\mathcal{X}' \rightarrow \mathcal{X}$ is étale, this implies $\mathcal{Y}' \rightarrow \mathcal{X}$ is étale hence that $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is étale.

So, after possibly replacing k by k_Y , we may and will assume Y is geometrically integral over k , and after possibly replacing k by a finite field extension $k \subset k' \subset k^{\text{ur}}$, we may and will assume $\mathcal{X}_{s,i}$ is geometrically irreducible (*viz* connected) over κ , $i = 1, \dots, m$.

Let $\xi_i \in \mathcal{X}_{s,i}$ denote the generic point of $\mathcal{X}_{s,i}$, $i = 1, \dots, m$. As \mathcal{X} is regular, it follows from Zariski-Nagata purity theorem that $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is étale if (and only if) it is étale at ξ_i , $i = 1, \dots, m$. As \mathcal{Y} is normal, it is regular in codimension 1 hence the non-regular locus $\mathcal{Y}^{\text{n-reg}} \subset \mathcal{Y}$ is a closed subset of codimension ≥ 2 in \mathcal{Y} . As $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is finite, $\mathcal{Z} := \psi_{\mathcal{X}}(\mathcal{Y}^{\text{n-reg}}) \subset \mathcal{X}$ is also closed of codimension ≥ 2 in \mathcal{X} . On the other hand, for every $i = 1, \dots, m$, as $\mathcal{X} \rightarrow S$ is smooth, $\mathcal{X}_{s,i} \subset \mathcal{X}$ is closed of codimension 1 in \mathcal{X} hence $\mathcal{Z}_{s,i} := \mathcal{Z} \cap \mathcal{X}_{s,i} \subset \mathcal{X}_{s,i}$ is closed of codimension ≥ 1 and $\mathcal{X}_{s,i}^\circ := \mathcal{X}_{s,i} \setminus \mathcal{Z}_{s,i} \subset \mathcal{X}_{s,i}$ is a non-empty open subscheme; in particular, $\xi_i \in \mathcal{X}_{s,i}^\circ$. So, setting $\mathcal{X}^\circ := \mathcal{X} \setminus \mathcal{Z}$, it is enough to prove that $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^\circ \rightarrow \mathcal{X}^\circ$ is étale at ξ_i , $i = 1, \dots, m$. So, up to replacing $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ with $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^\circ \rightarrow \mathcal{X}^\circ$, we may and will assume \mathcal{Y} is also regular. Fix $u_i \in \mathcal{X}_{s,i}$, $i = 1, \dots, m$. Up to replacing further $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ by its base-change along $\text{spec}(\mathcal{O}_{k'}) \rightarrow S$ for some finite field subextension $k \subset k' \subset k^{\text{ur}}$, we may and will assume that $\kappa(v) = \kappa$ for every $v \in \mathcal{Y}_{u_i}$ (in particular, $\kappa(u_i) = \kappa$). We argue by contradiction. Assume the subset $\mathcal{Y}^{\text{n-ét}} \subset \mathcal{Y}$ of all $y \in \mathcal{Y}$ such that $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is non-étale at y is non-empty. Then, again by Zariski-Nagata purity theorem, $\mathcal{Y}^{\text{n-ét}} \subset \mathcal{Y}$ is closed and pure of codimension 1 in \mathcal{Y} . Fix a generic point $\xi \in \mathcal{Y}^{\text{n-ét}}$. As $\psi : Y \rightarrow X = \mathcal{X}_\eta$ is étale, ξ necessarily lies over one of the ξ_i , $i = 1, \dots, m$ - say ξ_i . But as $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is finite, $\psi_{\mathcal{X}}(\mathcal{Y}^{\text{n-ét}}) \subset \mathcal{X}$ is closed in \mathcal{X} hence contains $\mathcal{X}_{s,i}$.

For simplicity, write $u := u_i \in \mathcal{X}_{s,i}$ and fix $v \in (\mathcal{Y}^{\text{n-ét}})_u$. By Lemma 18 (v) \Rightarrow (ii) applied to $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ (here, we use the regularity of \mathcal{Y}), the canonical morphism $\psi_{\mathcal{X}}^\# : \mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2$ induced by $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ at the level of cotangent spaces is not injective. Recall that, from our preliminary reduction $\kappa = \kappa(u) = \kappa(v)$. Fix $0 \neq a \in \ker(\mathfrak{m}_u/\mathfrak{m}_u^2 \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2)$. The idea is to construct a $\tilde{x} \in \mathcal{X}(\mathcal{O}_k)_u$ such that the resulting morphism of cotangent spaces $\tilde{x}^\# : \mathfrak{m}_u/\mathfrak{m}_u^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$ satisfies $\tilde{x}^\#(a) \neq 0$. Assume such a $\tilde{x} \in \mathcal{X}(\mathcal{O}_k)_u$ exists and let $x \in]u[^\text{u}$ denote its image in $|X|$. By (U-4), the normalization $\tilde{Y}_x \rightarrow S$ of S in $Y_x \rightarrow \text{spec}(k) \rightarrow S$ is étale. Actually, $Y_x = \text{spec}(k_{x,1} \times \dots \times k_{x,t})$

with $k \hookrightarrow k_{x,j}$ a finite unramified extension, $j = 1, \dots, t$ and $\tilde{Y}_x = \text{spec}(\mathcal{O}_{k_{x,1}} \times \dots \times \mathcal{O}_{k_{x,t}})$. For $j = 1, \dots, t$, write $\tilde{Y}_{x,j} := \text{spec}(\mathcal{O}_{k_{x,j}}) = \{y_j, v_j\}$, where y_j is the generic point and v_j the closed point of $\tilde{Y}_{x,j}$. As $\tilde{Y}_x \rightarrow S \xrightarrow{\tilde{x}} \mathcal{X}$ also coincides with the normalization of \mathcal{X} in $Y_x \rightarrow \text{spec}(k) \rightarrow S \xrightarrow{\tilde{x}} \mathcal{X}$, by the universal property of $\tilde{Y}_x \rightarrow S \xrightarrow{\tilde{x}} \mathcal{X}$, one gets a unique factorization

$$(5) \quad \begin{array}{ccccc} Y_x & \longrightarrow & \tilde{Y}_x & \cdots \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow \psi_{\mathcal{X}} \\ \text{spec}(k) & \longrightarrow & S & \xrightarrow{\tilde{x}} & \mathcal{X} \\ & \searrow & \xrightarrow{x} & & \end{array}$$

As $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is integral, by the Going down theorem [Stacks, Tag 00H8], there exists $y \in \mathcal{Y}$ specializing to v and mapping to x . By construction $y \in Y_x$ hence coincides with the generic point of one of the irreducible components - say $\tilde{Y}_{x,j}$ - of \tilde{Y}_x . As $\tilde{Y}_x \rightarrow S$ is finite, v_j maps to v . The commutative square

$$\begin{array}{ccc} \tilde{Y}_{x,j} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \psi_{\mathcal{X}} \\ S & \xrightarrow{\tilde{x}} & \mathcal{X} \end{array}$$

induces a commutative square at the level of cotangent spaces

$$\begin{array}{ccc} \mathfrak{m}_{v_j}/\mathfrak{m}_{v_j}^2 & \longleftarrow & \mathfrak{m}_v/\mathfrak{m}_v^2 \\ \uparrow & & \uparrow \psi_{\mathcal{X}}^{\#} \\ \mathfrak{m}/\mathfrak{m}^2 & \longleftarrow_{\tilde{x}^{\#}} & \mathfrak{m}_u/\mathfrak{m}_u^2 \end{array}$$

But as $\tilde{Y}_x \rightarrow S$ is étale, it follows from Lemma 18 (ii) \Rightarrow (v) that the morphism $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_{v_j}/\mathfrak{m}_{v_j}^2$ is injective; this contradicts the fact that $\tilde{x}^{\#}(a) \neq 0$ while $\psi_{\mathcal{X}}^{\#}(a) = 0$.

It remains to construct $\tilde{x} \in \mathcal{X}(\mathcal{O}_k)_u$ such that the resulting morphism of cotangent spaces $\tilde{x}^{\#} : \mathfrak{m}_u/\mathfrak{m}_u^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$ satisfies $\tilde{x}^{\#}(a) \neq 0$. For this, as \mathcal{X} is smooth at u and the residue field of u is κ , $\mathcal{O}_k[[T_1, \dots, T_n]] \xrightarrow{\sim} \hat{\mathcal{O}}_{\mathcal{X},u}$ and, modulo this isomorphism, $\hat{\mathfrak{m}}_u \subset \hat{\mathcal{O}}_{\mathcal{X},u}$ identifies with the ideal $\langle \pi, T_1, \dots, T_n \rangle$, where $\pi \in \mathfrak{m}$ is a uniformizer, $\mathfrak{m}_u/\mathfrak{m}_u^2 \xrightarrow{\sim} \kappa\bar{\pi} \oplus \kappa\bar{T}_1 \oplus \dots \oplus \kappa\bar{T}_n$ and $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \kappa\bar{\pi}$. Fix any κ -linear morphism $\bar{f}_a^{\#} : \mathfrak{m}_u/\mathfrak{m}_u^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$ such that $\bar{f}_a^{\#}(a) \neq 0$ and $\bar{f}_a^{\#}(\bar{\pi}) = \bar{\pi}$ (if $p_i : \mathfrak{m}_u/\mathfrak{m}_u^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$ denotes the projection onto the $\bar{\pi}$ th component for $i = 0$ and the \bar{T}_i th component for $i = 1, \dots, n$, and $a = a_0\bar{\pi} + \sum_{1 \leq i \leq n} a_i\bar{T}_i$ then, one can take $\bar{f}_a^{\#} = p_0$ if $a_0 \neq 0$ and $\bar{f}_a^{\#} = p_0 + p_i$ for some $1 \leq i \leq n$ such that $a_i \neq 0$ if $a_0 = 0$). Let $f_a^{\#}(T_1), \dots, f_a^{\#}(T_n) \in \mathfrak{m}$ lifting $\bar{f}_a^{\#}(\bar{T}_1), \dots, \bar{f}_a^{\#}(\bar{T}_n) \in \mathfrak{m}/\mathfrak{m}^2$; these define a unique morphism $f_a^{\#} : \mathcal{O}_k[[T_1, \dots, T_n]] \rightarrow \mathcal{O}_k$ of \mathcal{O}_k -algebras, and the resulting \mathcal{O}_k -point

$$\tilde{x} : \text{spec}(\mathcal{O}_k) \xrightarrow{f_a^{\#}} \text{spec}(\mathcal{O}_k[[T_1, \dots, T_n]]) \simeq \text{spec}(\hat{\mathcal{O}}_{\mathcal{X},u}) \rightarrow \text{spec}(\mathcal{O}_{\mathcal{X},u}) \rightarrow \mathcal{X}$$

has the expected property.

Lemma 18. *Let $\pi : V \rightarrow U$ be a finite surjective morphism between integral normal noetherian schemes. Let $v \in V$ and set $u := \pi(v)$. Let $\kappa(u) \leftarrow \mathcal{O}_{U,u} =: \mathcal{O}_u \supset \mathfrak{m}_u$ (resp. $\kappa(v) \leftarrow \mathcal{O}_{V,v} =: \mathcal{O}_v \supset \mathfrak{m}_v$) denote the residue field, local ring and maximal ideal of U at u (resp. of V at v).*

(1) *The following conditions are equivalent*

- (i) $\pi : V \rightarrow U$ is étale at v ;
- (ii) $\pi : V \rightarrow U$ is unramified at v ;
- (iii) the canonical morphism $\mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2$ is an epimorphism.

(2) *Assume furthermore U is regular at u , then the equivalent conditions in (1) are also equivalent to:*

- (i) the canonical morphism $\mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2$ is an isomorphism,

(and imply V is regular at v).

(4) *Assume furthermore V is regular at v , then the equivalent conditions in (1) are also equivalent to:*

- (i) the canonical morphism $\mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \rightarrow \mathfrak{m}_v/\mathfrak{m}_v^2$ is a monomorphism,

(and imply U is regular at u).

Proof. The implication (i) \Rightarrow (ii) is tautological while the implication (ii) \Rightarrow (i) is [SGA1, Exp. I, Thm. 9.5 (ii)]. Condition (iii) is equivalent to $\mathfrak{m}_u \mathcal{O}_v + \mathfrak{m}_v^2 = \mathfrak{m}_v$ so the implication (ii) \Rightarrow (iii) is tautological while the implication (iii) \Rightarrow (ii) follows from Nakayama's lemma since \mathfrak{m}_v is a finitely generated \mathcal{O}_v -module [Stacks, Tag 07RC (4)]. The implication (iv) \Rightarrow (iii) and (iv) \Rightarrow (v) are tautological. By Nakayama's lemma and Krull's principal ideal, one always has $\dim_{\kappa(u)} \mathfrak{m}_u / \mathfrak{m}_u^2 \geq \dim \mathcal{O}_u$ with equality if and only if U is regular at u and similarly for V at v . This shows the implication (iii) \Rightarrow (iv) assuming U is regular at u and the implication (v) \Rightarrow (iv) assuming V is regular at v . \square

3.2.3. *Proof of Theorem 13 (T).* Again, the implications (T-1) \Rightarrow (T-2) \Rightarrow (T-3) \Rightarrow (T-4) and (T-2) \Rightarrow (T-1) are (almost) tautological so we are left to prove (T-4) \Rightarrow (T-2).

Let $\mathcal{X}^o \subset \mathcal{X}$ be an open subscheme such that $\mathcal{X}^o \cap \mathcal{X}_{s,i} \neq \emptyset$, $i = 1, \dots, m$. As \mathcal{X} is regular, $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is tamely ramified along \mathcal{X}_s if (and only) if $\psi_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{X}^o : \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^o \rightarrow \mathcal{X}^o$ is tamely ramified along \mathcal{X}_s^o so that, in proving (T-4) \Rightarrow (T-2), one may freely replace $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ by its base-change along such an open subscheme $\mathcal{X}^o \hookrightarrow \mathcal{X}$.

For $i = 1, \dots, m$, consider the Cartesian diagram

$$\begin{array}{ccccc} & & \mathcal{Y}_s^{\text{red}} & \longleftarrow & \mathcal{Y}_{s,i} \\ & & \downarrow & & \downarrow \\ \mathcal{Y} & \longleftarrow & \mathcal{Y}_s & \square & \\ \psi_{\mathcal{X}} \downarrow & & \downarrow \psi_{\mathcal{X},s} & & \downarrow \\ \mathcal{X} & \longleftarrow & \mathcal{X}_s & \longleftarrow & \mathcal{X}_{s,i} \end{array}$$

(where $\mathcal{Y}_s^{\text{red}} \hookrightarrow \mathcal{Y}_s$ denotes the reduced closed subscheme) and write

$$\mathcal{Y}_{s,i} = \bigcup_{1 \leq j \leq m_i} \mathcal{Y}_{s,i,j}$$

for the decomposition of $\mathcal{Y}_{s,i}$ into irreducible components. The non-regular locus $\mathcal{Y}_{s,i}^{\text{n-reg}} \subsetneq \mathcal{Y}_{s,i}$ is a strict closed subscheme. As $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is finite, $\psi_{\mathcal{X}}(\mathcal{Y}_{s,i}^{\text{n-reg}}) \subsetneq \mathcal{X}_{s,i}$ is again a strict closed subscheme. So that, replacing $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ by its base-change along

$$\mathcal{X}^o := \mathcal{X} \setminus \bigsqcup_{1 \leq i \leq m} \psi_{\mathcal{X}}(\mathcal{Y}_{s,i}^{\text{n-reg}}) \hookrightarrow \mathcal{X},$$

we may and will assume that $\mathcal{Y}_{s,i}$ is regular (*viz* smooth over κ as κ is perfect), $i = 1, \dots, m$. Actually, later in the argument we will have to ensure this property holds not only for the normalization $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ of \mathcal{X} in $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}$ but also for the normalization $\psi'_{\mathcal{X}} : \mathcal{Y}' \rightarrow \mathcal{X}$ of \mathcal{X} in $Y' \xrightarrow{\psi'} X \hookrightarrow \mathcal{X}$ for some intermediate covers $Y \rightarrow Y' \xrightarrow{\psi'} X$. But as there are only finitely such intermediate covers, we can do so by shrinking \mathcal{X} further (namely removing not only $\bigsqcup_{1 \leq i \leq m} \psi_{\mathcal{X}}(\mathcal{Y}_{s,i}^{\text{n-reg}})$ but the union of all $\bigsqcup_{1 \leq i \leq m'} \psi_{\mathcal{X}}(\mathcal{Y}'_{s,i}{}^{\text{n-reg}})$ for $Y \rightarrow Y' \xrightarrow{\psi'} X$ describing the finitely many intermediate covers of $\psi : Y \rightarrow X$).

For a subgroup $H \subset G := \text{Aut}(\psi)$, write $Y_H \rightarrow X$ for the corresponding connected étale cover and $\mathcal{Y}_H \rightarrow \mathcal{X}$ for the normalization of \mathcal{X} in $Y_H \rightarrow X \hookrightarrow \mathcal{X}$. Fix $i = 1, \dots, m$, let $\xi := \xi_i \in \mathcal{X}_{s,i}$ denote the generic point of $\mathcal{X}_{s,i}$, $\zeta := \zeta_{i,j} \in \mathcal{Y}_{s,i,j}$ the generic point of $\mathcal{Y}_{s,i,j}$ and write

$$G \supset D := D_{\zeta/\xi} \supset I := I_{\zeta/\xi} \supset I^w := I_{\zeta/\xi}^w \subset 1$$

for the decomposition, inertia and wild inertia groups of ζ/ξ respectively. These yield a commutative diagram

$$\begin{array}{ccccccccc} Y & \longrightarrow & Y_{I^w} & \longrightarrow & Y_I & \longrightarrow & Y_D & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}_{I^w} & \longrightarrow & \mathcal{Y}_I & \longrightarrow & \mathcal{Y}_D & \longrightarrow & \mathcal{X}. \end{array}$$

Let ζ_{I^w} , ζ_I and ζ_D denote the image of ζ in \mathcal{Y}_{I^w} , \mathcal{Y}_I and \mathcal{Y}_D respectively. By construction, $\text{spec}(\mathcal{O}_{\mathcal{Y}_D, \zeta_D}) \rightarrow \text{spec}(\mathcal{O}_{\mathcal{X}, \xi})$ and $\text{spec}(\mathcal{O}_{\mathcal{Y}_I, \zeta_I}) \rightarrow \text{spec}(\mathcal{O}_{\mathcal{Y}_D, \zeta_D})$ are unramified, $\text{spec}(\mathcal{O}_{\mathcal{Y}_{I^w}, \zeta_{I^w}}) \rightarrow \text{spec}(\mathcal{O}_{\mathcal{Y}_I, \zeta_I})$ is tamely ramified and $\text{spec}(\mathcal{O}_{\mathcal{Y}, \zeta}) \rightarrow \text{spec}(\mathcal{O}_{\mathcal{Y}_{I^w}, \zeta_{I^w}})$ is wildly ramified.

Also, just as in the proof of Theorem 13 (U), it is enough to prove (T-4) \Rightarrow (T-2) after:

- Replacing k by the algebraic closure k_Y of k in the function field of Y . The argument is exactly similar to the one of the proof of Theorem 13 (U), replacing "étale" with "tamely ramified".

In particular, after possibly replacing k by k_Y , we may and will assume Y is geometrically integral over k .

- Base-change along $\text{spec}(k') \rightarrow \text{spec}(k)$ for some finite field extension $k \subset k' \subset k^t$. Again, the argument is exactly similar to the one of the proof of Theorem 13 (U), replacing "étale" with "tamely ramified" and $\mathcal{Y} \times_S S'$ with the normalization of $\mathcal{X} \times_S S'$ in $Y \times_k k' \rightarrow X \times_k k' \rightarrow \mathcal{X} \times_S S'$.

So, after possibly replacing k by a finite field extension $k \subset k' \subset k^t$, we may and will assume that $\mathcal{X}_{s,i}$ is geometrically irreducible (*viz* connected) over κ , $i = 1, \dots, m$ and, by Abhyankar's lemma ([Stacks, Tag 0BRM]), that $\text{spec}(\mathcal{O}_{\mathcal{Y}_{I^w}, \zeta_{I^w}}) \rightarrow \text{spec}(\mathcal{O}_{\mathcal{X}, \xi_i})$ is unramified (in other words, $I_{i,j} = I_{i,j}^w$), $j = 1, \dots, m_i$, $i = 1, \dots, m$.

We now define $\mathcal{X}_{s,i}^\circ \subset \mathcal{X}_{s,i}$ as in Theorem 13 (U) (namely $\mathcal{X}_{s,i}^\circ = \mathcal{X}_{s,i} \setminus \mathcal{Z}_{s,i}$, where $\mathcal{Z} := \psi_{\mathcal{X}}(\mathcal{Y}^{\text{n-reg}})$ and $\mathcal{Z}_{s,i} = \mathcal{Z} \cap \mathcal{X}_{s,i}$), $i = 1, \dots, m$.

Fix $1 \leq i \leq m$, $u_i \in |\mathcal{X}_{s,i}^\circ|$, $x \in]u_i[{}^t$ and $y \in Y_x$. By definition, $x : \text{spec}(k(x)) \rightarrow X$ extends (uniquely as $\mathcal{X} \rightarrow S$ is separated) to $\tilde{x} : S_x \rightarrow \mathcal{X}$ and, as $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is finite hence proper, $y : \text{spec}(k(y)) \rightarrow Y$ extends uniquely as

$$\begin{array}{ccc} \text{spec}(k(y)) & \xrightarrow{y} & \mathcal{Y} \\ \downarrow & \searrow \exists! \tilde{y} & \downarrow \psi_{\mathcal{X}} \\ S_y & \xrightarrow{\tilde{x}} & \mathcal{X} \end{array}$$

Then there exists a unique $1 \leq i \leq m$ such that $\tilde{x}_s \in \mathcal{X}_{s,i}$ and a unique (recall that \mathcal{Y}_s is regular) $1 \leq j \leq m_i$ such that $\tilde{y}_s \in \mathcal{Y}_{s,i,j}$. Write $I^w := I_{\zeta_{i,j}/\xi_i}^w (= I_{\zeta_{i,j}/\xi_i})$ and $D := D_{\zeta_{i,j}/\xi_i}$. For $H = D, I^w$ and 1, let $\mathcal{Y}_{H,s}^\circ \subset \mathcal{Y}_{H,s}$ denote the irreducible component of ζ_H and set $\mathcal{Y}_H^\circ := \mathcal{Y}_H \setminus (\mathcal{Y}_{H,s} \setminus \mathcal{Y}_{H,s}^\circ)$. As ζ_D is inert in $\mathcal{Y}_{I^w} \rightarrow \mathcal{Y}_D$ and ζ_{I^w} is totally wildly ramified in $\mathcal{Y} \rightarrow \mathcal{Y}_{I^w}$, one actually has a partly Cartesian diagram

$$\begin{array}{ccccccc} \mathcal{Y}^\circ & \longrightarrow & \mathcal{Y}_{I^w}^\circ & \longrightarrow & \mathcal{Y}_D^\circ & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \parallel \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}_{I^w} & \longrightarrow & \mathcal{Y}_D & \longrightarrow & \mathcal{X}. \end{array}$$

In particular, $\mathcal{Y}_{I^w}^\circ \rightarrow \mathcal{Y}_D^\circ$ is an étale cover and $\mathcal{Y}_D^\circ \rightarrow \mathcal{X}$ is an étale morphism (but *a priori* not finite). Let y^w denote the image of y in Y_{I^w} and $\tilde{y}^w : S_{y^w} = \text{spec}(\mathcal{O}_{y^w}) \rightarrow \mathcal{Y}_{I^w}^\circ$ the normalization of $\mathcal{Y}_{I^w}^\circ$ in $\text{spec}(k(y^w)) \xrightarrow{y^w} Y_{I^w} \rightarrow \mathcal{Y}_{I^w}^\circ$ so that one has a commutative diagram

$$\begin{array}{ccccccc} S_y & \longrightarrow & S_{y^w} & \longrightarrow & S_x & & \\ \downarrow \tilde{y} & & \downarrow \tilde{y}^w & & \downarrow \tilde{x} & & \\ \mathcal{Y}^\circ & \longrightarrow & \mathcal{Y}_{I^w}^\circ & \longrightarrow & \mathcal{Y}_D^\circ & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \parallel \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}_{I^w} & \longrightarrow & \mathcal{Y}_D & \longrightarrow & \mathcal{X}. \end{array}$$

As $\mathcal{Y}_{I^w}^\circ \rightarrow \mathcal{X}$ is étale, $S_{y^w} \rightarrow S_x$ is unramified while as $Y \rightarrow Y_{I^w}$ is Galois with group I^w of order a power of p , the ramification index of $S_y \rightarrow S_{y^w}$ is a power of p . On the other hand, as $x \in]u_i[{}^t$, by (T-4), $S_y \rightarrow S_x$ (hence *a fortiori* $S_y \rightarrow S_{y^w}$) is at most tamely ramified. This forces $S_y \rightarrow S_x$ to be unramified and proves that $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is $]u_i[{}^t$ -pointwise unramified (hence *a fortiori* $]u_i[{}^u$ -pointwise unramified). By Theorem 13 (U-4) \Rightarrow (U-2), $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is an étale cover, which imposes that $\mathcal{Y} \rightarrow \mathcal{Y}_{I^w}$ is trivial.

3.3. Semistable base schemes. We define the *smooth locus* $\mathcal{X}^{\text{sm}/S} \subset \mathcal{X}$ of a morphism of schemes $\mathcal{X} \rightarrow S$ to be the set of all $x \in \mathcal{X}$ such that $\mathcal{X} \rightarrow S$ is smooth at x [Stacks, Tag 01V5]. When $S = \{\eta, s\}$ is the spectrum of a discrete valuation ring with (generic point η and) closed point s , $\mathcal{X}^{\text{sm}/S} \subset \mathcal{X}$ is thus the set of all $x \in \mathcal{X}$ such that there exists a Zariski-open neighbourhood $\mathcal{U} \hookrightarrow \mathcal{X}$ of x such that $\mathcal{U} \hookrightarrow \mathcal{X} \rightarrow S$ is a smooth morphism. If one furthermore assumes $\mathcal{X} \rightarrow S$ is flat and of finite type, then for every $x \in \mathcal{X}_s$, $x \in \mathcal{X}^{\text{sm}/S}$ if and only if $x \in \mathcal{X}_s^{\text{sm}/s}$ (and for every $x \in X$, $x \in X^{\text{sm}/\eta}$ if and only if $x \in \mathcal{X}^{\text{sm}/S}$) so that, in that case, we will omit the base scheme S from the notation.

Let now $S = \text{spec}(\mathcal{O}_k)$ denote, again, the spectrum of a discrete valuation ring as in Theorem 13 and let $\mathcal{X} \rightarrow S$ be semistable. As already mentioned, now that we proved Theorem 13 for the smooth locus $\mathcal{X}^{\text{sm}} \rightarrow S$ of $\mathcal{X} \rightarrow S$, the

remaining issue is to control what happens locally around the finitely many non-smooth points $u \in \mathcal{X}^{\text{n-sm}} := \mathcal{X} \setminus \mathcal{X}^{\text{sm}}$. To do so, we first describe in Subsection 3.3.1 the étale fundamental group of the smooth locus of $\mathcal{X}_{(\bar{u})} = \text{spec}(\mathcal{O}_{\mathcal{X}, \bar{u}}) \rightarrow \text{spec}(\mathcal{O}_{k^{\text{ur}}})$ and the prime-to- p completion of the étale fundamental group of $X_{(\bar{u})} = \mathcal{X}_{(\bar{u})} \times_{\mathcal{X}} X$.

3.3.1. *Preliminary computations.* For a morphism of rings $f : \mathcal{O} \rightarrow \mathcal{O}'$, we again denote by $f : \text{spec}(\mathcal{O}') \rightarrow \text{spec}(\mathcal{O})$ the corresponding morphism of affine schemes. To simplify the notation, we also sometimes write \mathcal{O} instead of $\text{spec}(\mathcal{O})$.

For every integers $r \geq 1$, $n \geq m \geq 1$ set

$$\mathcal{O}_r := \mathcal{O}_{k^{\text{ur}}}[T_1, \dots, T_n] / \langle T_1 \cdots T_m - \pi^r \rangle, \quad \mathfrak{m}_r := \langle \pi, T_1, \dots, T_n \rangle,$$

and let

$$c_{r,0 \rightarrow \emptyset} : \mathcal{O}_r \rightarrow \mathcal{O}_{r,0} = (\mathcal{O}_r)_{\mathfrak{m}_r}, \quad \mathfrak{m}_{r,0} := \mathfrak{m}_r \mathcal{O}_{r,0}$$

denote the localization of \mathcal{O}_r at \mathfrak{m}_r and

$$c_{r,\bar{0} \rightarrow 0} : \mathcal{O}_{r,0} \rightarrow \mathcal{O}_{r,\bar{0}} := \mathcal{O}_{r,0}^{\text{sh}}, \quad \mathfrak{m}_{r,\bar{0}} := \mathfrak{m}_{r,0} \mathcal{O}_{r,\bar{0}}$$

the strict henselization of $\mathcal{O}_{r,0}$ at $\mathfrak{m}_{r,0}$. For $\# = \emptyset, 0, \bar{0}$, write also $\Theta_{r,\#} := \text{spec}(\mathcal{O}_{r,\#})$, and $\theta_{r,\#} \in \Theta_{r,\#}$ for the closed point corresponding to the maximal ideal $\mathfrak{m}_{r,\#}$. If we need to keep track of the data (n, m) , we will replace the subscript " $(-)_r$ " with the subscript " $(-)_{n,m,r}$ ". Note that the rings $\mathcal{O}_{r,\#}$ are all normal and integral. The morphism $\Theta_{r,\bar{0}} \rightarrow \Theta_r$ can also be described as the limit

$$(6) \quad \Theta_{r,\bar{0}} = \lim_{U \rightarrow \Theta_r} U$$

indexed over all étale-open neighbourhoods U of θ_r in Θ_r viz commutative diagrams

$$(7) \quad \begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ \text{spec}(\bar{\kappa}) & \xrightarrow{\theta_r} & \Theta_r \end{array}$$

where $U \rightarrow \Theta_r$ is an étale morphism.

Let $t_i := t_{i,\#} \in \mathcal{O}_{r,\#}$ denote the image of T_i , $i = 1, \dots, r$. For $r' | r$, the canonical morphism of $\mathcal{O}_{k^{\text{ur}}}$ -algebras

$$\mathcal{O}_{k^{\text{ur}}}[T_1, \dots, T_n] \rightarrow \mathcal{O}_{r'}, \quad (T_1, \dots, T_n) \mapsto (t_1^{r'}, \dots, t_m^{r'}, t_{m+1}, \dots, t_n)$$

factors canonically through a finite morphism $\alpha_{r' \rightarrow r} : \mathcal{O}_r \rightarrow \mathcal{O}_{r'}$ and induces, by localization at \mathfrak{m}_r , $\mathfrak{m}_{r'}$ and strict henselization at $\mathfrak{m}_{r,0}$, $\mathfrak{m}_{r',0}$ respectively, canonical finite morphisms of $\mathcal{O}_{k^{\text{ur}}}$ -algebras

$$\alpha_{r' \rightarrow r,0} : \mathcal{O}_{r,0} \rightarrow \mathcal{O}_{r',0}, \quad \alpha_{r' \rightarrow r,\bar{0}} : \mathcal{O}_{r,\bar{0}} \rightarrow \mathcal{O}_{r',\bar{0}}.$$

Note that $\alpha_{r' \rightarrow r,\bar{0}} : \Theta_{r',\bar{0}} \rightarrow \Theta_{r,\bar{0}}$ decomposes as $\Theta_{r',\bar{0}} \hookrightarrow \Theta_{r,\bar{0}} \times_{\Theta_r} \Theta_{r'} \rightarrow \Theta_{r,\bar{0}}$, where $\Theta_{r',\bar{0}} \hookrightarrow \Theta_{r,\bar{0}} \times_{\Theta_r} \Theta_{r'}$ is the inclusion of one of the finitely many connected components of $\Theta_{r,\bar{0}} \times_{\Theta_r} \Theta_{r'}$ [Stacks, Tag 05WP]. In other words, one has a diagram cartesian up to connected component, which we denote by

$$(8) \quad \begin{array}{ccc} \Theta_{r',\bar{0}} & \xrightarrow{\alpha_{r' \rightarrow r,\bar{0}}} & \Theta_{r,\bar{0}} \\ c_{r',\bar{0} \rightarrow \emptyset} \downarrow & \wedge \square & \downarrow c_{r,\bar{0} \rightarrow \emptyset} \\ \Theta_{r'} & \xrightarrow{\alpha_{r' \rightarrow r}} & \Theta_r \end{array}$$

3.3.1.1. *Smooth locus.* We define the smooth locus $\Theta_{r,\bar{0}}^{\text{sm}} \subset \Theta_{r,\bar{0}}$ to be the inverse image of $\Theta_r^{\text{sm}} \subset \Theta_r$ via $c_{r,\bar{0} \rightarrow \emptyset} : \Theta_{r,\bar{0}} \rightarrow \Theta_r$. As for every diagram (7), $U^{\text{sm}} \subset U$ is the inverse image of $\Theta_r^{\text{sm}} \subset \Theta_r$ via $U \rightarrow \Theta_r$ [Stacks, Tag 02K5], $\Theta_r^{\text{sm}} \subset \Theta_r$ is also the inverse image of $U^{\text{sm}} \subset U$ via $\Theta_{r,\bar{0}} \rightarrow U$.

Lemma 19. *Assume $m \geq 2$. Then, $\Theta_{r,\bar{0}}^{\text{n-sm}} (:= \Theta_{r,\bar{0}} \setminus \Theta_{r,\bar{0}}^{\text{sm}}) = \bigcup_{1 \leq i \neq j \leq m} V(t_i, t_j) \subset \Theta_{r,\bar{0}}$. In particular, $\Theta_{r,\bar{0}}^{\text{n-sm}} \subset \Theta_{r,\bar{0}}$ has codimension 2.*

Proof. By our definition of $\Theta_{r,\bar{0}}^{\text{n-sm}}$, it is enough to prove that $\Theta_{r,\bar{0}}^{\text{n-sm}} = \bigcup_{1 \leq i \neq j \leq m} V(t_i, t_j) \subset \Theta_{r,\bar{0}}$. As $\Theta_r \rightarrow S^{\text{ur}}$ is flat, of finite presentation, and of relative dimension $n - 1$, one can describe $\Theta_r^{\text{sm}} \subset \Theta_r$ as the set of all $x \in \Theta_r$ such that there exists a surjective morphism $\mathcal{O}_{\Theta_r, x}^{\oplus(n-1)} \rightarrow \Omega_{\Theta_r | S^{\text{ur}}, x}^1$ of $\mathcal{O}_{\Theta_r, x}$ -modules. On the other hand, one can compute $\Omega_{\Theta_r | S^{\text{ur}}}^1 = \Omega_{\mathcal{O}_r | \mathcal{O}_{k^{\text{ur}}}}^1$ explicitly via the exact sequence of \mathcal{O}_r -modules:

$$\mathcal{O}_r \xrightarrow{(\hat{t}_1, \dots, \hat{t}_m, 0, \dots, 0)} \mathcal{O}_r^{\oplus n} \rightarrow \Omega_{\mathcal{O}_r | \mathcal{O}_{k^{\text{ur}}}}^1 \rightarrow 0,$$

where $\hat{t}_i := \frac{t_1 \cdots t_m}{t_i}$, $i = 1, \dots, m$. Thus

$$\Theta_r^{\text{sm}} = \{x \in \Theta_r \mid \hat{t}_i \in \mathcal{O}_{\Theta_r, x}^\times \text{ for some } 1 \leq i \leq m\}.$$

Equivalently, $\Theta_r^{\text{n-sm}} := \{x \in \Theta_r \mid \hat{t}_i \in \mathfrak{m}_{\Theta_r, x}, i = 1, \dots, m\}$ \square

$$= \bigcap_{1 \leq i \leq m} V(\hat{t}_i) = \bigcap_{1 \leq i \leq m} \bigcup_{1 \leq i+j \leq m} V(t_j) = \bigcup_{1 \leq i+j \leq m} V(t_i, t_j) \subset \Theta_r.$$

Write $r = r'p^\nu$ with $p \nmid r'$.

Lemma 20. *Assume $m > 1$. One has $\pi_1(\Theta_{r,0}^{\text{sm}}) \simeq (\mathbb{Z}/r')^{m-1}$. More precisely, the universal cover $\tilde{\Theta}_{r,0}^{\text{sm}} \rightarrow \Theta_{r,0}^{\text{sm}}$ of $\Theta_{r,0}^{\text{sm}}$ is the base change*

$$\begin{array}{ccc} \tilde{\Theta}_{r,0}^{\text{sm}} & \hookrightarrow & \Theta_{p^\nu, \bar{0}} \\ \downarrow & \square & \downarrow \alpha_{p^\nu \rightarrow r, \bar{0}} \\ \Theta_{r,0}^{\text{sm}} & \hookrightarrow & \Theta_{r, \bar{0}}. \end{array}$$

Proof. By construction, for $i = 1, \dots, m$, the diagram

$$\begin{array}{ccccc} \Theta_{r', \bar{0}}[\frac{1}{\hat{t}_i}] & \longrightarrow & \Theta_{r, \bar{0}}[\frac{1}{\hat{t}_i}] & \longrightarrow & \Theta_r[\frac{1}{\hat{t}_i}] \\ \downarrow & & \downarrow & & \downarrow \\ \Theta_{r', \bar{0}} & \longrightarrow & \Theta_{r, \bar{0}} & \longrightarrow & \Theta_r \end{array}$$

is cartesian.

Claim 1. *One has $\pi_1(\Theta_{1,0}^{\text{sm}}) = 1$.*

Proof of Claim 1. As Θ_1 is a Noetherian scheme of dimension n , regular at t_1 , $\mathcal{O}_{1, \bar{0}}$ is a Noetherian, regular local ring of dimension n [Stacks, Tag 06LN], [Stacks, Tag 06LK], hence $\Theta_{1, \bar{0}}$ is a Noetherian, regular [M87, Thm. 19.3] scheme of dimension n . From Lemma 19 and Zariski-Nagata purity [SGA1, X, Cor. 3.3], one thus gets $\pi_1(\Theta_{1,0}^{\text{sm}}) \xrightarrow{\sim} \pi_1(\Theta_{1, \bar{0}}) = 1$. \square

Claim 2. *For every integer $\nu \geq 1$, one has $\pi_1(\Theta_{p^\nu, \bar{0}}^{\text{sm}}) = 1$.*

Proof of Claim 2. From Claim 1, $\pi_1(\Theta_{1,0}^{\text{sm}}) = 1$ so that, by the general formalism of Galois categories, the universal cover $\tilde{\Theta}_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{p^\nu, \bar{0}}^{\text{sm}}$ of $\Theta_{p^\nu, \bar{0}}^{\text{sm}}$ is finite and the finite cover $\Theta_{1,0}^{\text{sm}} \rightarrow \Theta_{p^\nu, \bar{0}}^{\text{sm}}$ factors as $\Theta_{1,0}^{\text{sm}} \rightarrow \tilde{\Theta}_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{p^\nu, \bar{0}}^{\text{sm}}$ (as $\tilde{\Theta}_{p^\nu, \bar{0}}^{\text{sm}}$ is normal, $\tilde{\Theta}_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{p^\nu, \bar{0}}^{\text{sm}}$ can also be described, in more down-to-earth terms, as the normalization of $\Theta_{p^\nu, \bar{0}}^{\text{sm}}$ in the maximal subextension of $k(\Theta_{1,0}^{\text{sm}})/k(\Theta_{p^\nu, \bar{0}}^{\text{sm}})$ unramified over $\Theta_{p^\nu, \bar{0}}^{\text{sm}}$). Consider the notation in the following base change diagram

$$\begin{array}{ccccccc} V_0 & \longrightarrow & \tilde{U}_0 & \longrightarrow & U_0 & \longrightarrow & \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Theta_{1, \bar{0}}[\frac{1}{\hat{t}_1}] & \longrightarrow & \tilde{\Theta}_{p^\nu, \bar{0}} & \longrightarrow & \Theta_{p^\nu, \bar{0}}[\frac{1}{\hat{t}_1}] & \longrightarrow & \mathcal{O}_{k^{\text{ur}}} \\ \downarrow & & \downarrow & & \downarrow & & \parallel \\ \Theta_{1,0}^{\text{sm}} & \longrightarrow & \tilde{\Theta}_{p^\nu, \bar{0}}^{\text{sm}} & \longrightarrow & \Theta_{p^\nu, \bar{0}}^{\text{sm}} & \longrightarrow & \mathcal{O}_{k^{\text{ur}}}. \end{array}$$

The cover $V_0 \rightarrow U_0$ also identifies with a connected component of the base-change

$$\begin{array}{ccccc} V_0 & \longrightarrow & U_0 & \hookrightarrow & \Theta_{p^\nu, \bar{0}} \times_{\mathcal{O}_{k^{\text{ur}}}} \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} \\ \downarrow & & \downarrow & & \downarrow \\ \underbrace{\Theta_1[\frac{1}{\hat{t}_1}] \times_{\mathcal{O}_{k^{\text{ur}}}} \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}_{:=V} & \longrightarrow & \underbrace{\Theta_{p^\nu}[\frac{1}{\hat{t}_1}] \times_{\mathcal{O}_{k^{\text{ur}}}} \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}_{:=U} & \hookrightarrow & \Theta_{p^\nu} \times_{\mathcal{O}_{k^{\text{ur}}}} \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} \end{array}$$

Subclaim. *The finite cover $V \rightarrow U$ has no non-trivial étale intermediate subcovers.*

Proof of the Subclaim. The cover $V \rightarrow U$ is given by

$$\begin{array}{ccc} \mathcal{O}_{p^\nu} \left[\frac{1}{t_1} \right] \otimes_{\mathcal{O}_{k^{\text{ur}}}} \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} & \xrightarrow{t \mapsto (t_1^{p^\nu}, \dots, t_m^{p^\nu}, t_{m+1}, \dots, t_n)} & \mathcal{O}_1 \left[\frac{1}{t_1} \right] \otimes_{\mathcal{O}_{k^{\text{ur}}}} \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} \\ \simeq \downarrow & & \downarrow \simeq \\ \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [T_2, T_2^{-1}, \dots, T_m, T_m^{-1}, T_{m+1}, \dots, T_n] & \xrightarrow{(T_2, \dots, T_n) \mapsto (T_2^{p^\nu}, \dots, T_m^{p^\nu}, T_{m+1}, \dots, T_n)} & \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [T_2, T_2^{-1}, \dots, T_m, T_m^{-1}, T_{m+1}, \dots, T_n], \end{array}$$

hence is a direct product of $m - 1$ -copies of the Kummer cover

$$\mathbb{G}_{m, \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}} \rightarrow \mathbb{G}_{m, \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}, \quad t \mapsto t^{p^\nu},$$

and $n - m$ copies of the trivial cover

$$\mathbb{A}_{\mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}^1 \rightarrow \mathbb{A}_{\mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}^1, \quad t \mapsto t.$$

In particular, $V \rightarrow U$ has automorphism group $(\mathbb{Z}/p^\nu)^{m-1}$ and its only non-trivial étale subcover is the trivial one. Equivalently, the Kummer cover

$$\varphi : D := \mathbb{G}_{m, \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}^{m-1} \rightarrow D, \quad (t_2, \dots, t_m) \mapsto (t_2^{p^\nu}, \dots, t_m^{p^\nu})$$

has no degree- p étale subcover. Indeed, as $\varphi : D \rightarrow D$ is an isogeny of split tori, the map

$$\begin{array}{ccc} D & \xrightarrow{\varphi_+} & D \mapsto \ker(\varphi_+) \\ \varphi \downarrow & \swarrow \varphi_- & \\ D & & \end{array}$$

induces a bijection between index p subgroups of $\ker(\varphi) \simeq (\mu_{p^\nu, \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}})^{\oplus m-1} \simeq (\mathbb{Z}/p^s \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})})^{\oplus m-1}$ and degree- p subcovers $\varphi_- : D \rightarrow D$. But for every index- p subgroup $M \subset (\mathbb{Z}/p^s)^{\oplus m-1}$ lifting to an index- p subgroup $\tilde{M} \subset \mathbb{Z}^{\oplus m-1}$ there exists an automorphism $\tilde{\alpha} \in \text{GL}_{m-1}(\mathbb{Z})$ such that $\tilde{\alpha}(\tilde{M}) = p^{s-1}\mathbb{Z} \oplus \mathbb{Z}^{\oplus m-2} \subset \mathbb{Z}^{\oplus m-1}$ hence for every degree- p subcover $\varphi_- : D \rightarrow D$ there exists an automorphism $\tilde{\alpha} \in \text{Aut}_{\mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}}(D)$ such that $\varphi_- = \tilde{\alpha}^{-1} \phi_- \tilde{\alpha} : D \rightarrow D$, with

$$\phi_- : D \rightarrow D, \quad t \mapsto (t_2^p, t_3, \dots, t_m).$$

But $\phi_- : D \rightarrow D$ is not étale, otherwise the extension of $\mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}$ -algebras

$$\mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [X, X^{-1}] \hookrightarrow \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [Y, Y^{-1}], \quad X \mapsto Y^p$$

would be étale. But from the isomorphism of $\mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})}$ -algebras

$$\begin{array}{ccc} A := \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [X, X^{-1}] & \xrightarrow{X \mapsto Y^p} & \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [Y, Y^{-1}], \\ \downarrow X \mapsto X & \nearrow \cong & \\ B := \mathcal{O}_{k^{\text{ur}}(\mu_{p^\nu})} [X, X^{-1}] \mathcal{X}_s [{}^s T] / \langle T^p - X \rangle & & \end{array}$$

this would force $\Omega_{B|A}^1 = 0$, while

$$\Omega_{B|A}^1 \simeq B/p t^{p-1} \simeq B/p \neq 0 : \quad \text{a contradiction.}$$

This concludes the proof of the Subclaim. \square

It remains to prove that the Subclaim implies

$$(\deg(\tilde{\Theta}_{p^\nu, \tilde{0}}^{\text{sm}} \rightarrow \Theta_{p^\nu, \tilde{0}}^{\text{sm}}) =) \deg(\tilde{U}_{\tilde{0}} \rightarrow U_{\tilde{0}}) = 1$$

(hence $\pi_1(U) = 1$). We proceed in two steps.

- a) For every intermediate cover $V \rightarrow W \rightarrow U$, $W \times_U U_{\tilde{0}} \rightarrow U_{\tilde{0}}$ is étale if and only if $W \rightarrow U$ is étale *viz* trivial by the Subclaim. This essentially follows from the fact that the canonical morphism $U_{\tilde{0}} \rightarrow U$ is pro-étale. More precisely let $V \rightarrow W \rightarrow U$ be a non-trivial intermediate cover such that $W \times_U U_{\tilde{0}} \rightarrow U_{\tilde{0}}$ is étale; without loss of generality, one may assume $W \rightarrow U$ has degree p and is given by

$$A := \mathcal{O}_{p^\nu} \left[\frac{1}{t_1} \right] \otimes_{k^{\text{ur}}} k^{\text{ur}}(\mu_{p^\nu}) \hookrightarrow A \left[\frac{1}{t_2} \right] =: B$$

Consider the notation in the base-change diagram

$$\begin{array}{ccc} \mathrm{spec}(B') := W \times_U U_{\bar{0}} & \longrightarrow & U_{\bar{0}} =: \mathrm{spec}(B) \\ \downarrow & \square & \downarrow \\ \mathrm{spec}(B) := W & \longrightarrow & U := \mathrm{spec}(A). \end{array}$$

As $U_{\bar{0}} \rightarrow U$ is pro-étale, so is $W \times_U U_{\bar{0}} \rightarrow W$ hence, as W is normal, so is $W \times_U U_{\bar{0}}$. On the other hand, as $W \rightarrow U$ is finite, so is $W \times_U U_{\bar{0}} \rightarrow U_{\bar{0}}$. Hence $W \times_U U_{\bar{0}} \rightarrow U_{\bar{0}}$ is the normalization of $U_{\bar{0}}$ in $\mathrm{spec}(k(W \times_U U_{\bar{0}})) \rightarrow \mathrm{spec}(k(U_{\bar{0}})) \rightarrow U_{\bar{0}}$, where $k(W \times_U U_{\bar{0}})$ stands for the total fraction ring of B' . In particular, $W \times_U U_{\bar{0}} \rightarrow U_{\bar{0}}$ is étale if and only if it is unramified. But, one has

$$\Omega_{B'|A}^1 \simeq \Omega_{B|A}^1 \otimes_B B' \simeq B/p \otimes_B B' \simeq B'/p \neq 0 : \text{ a contradiction.}$$

b) Step a) implies that the canonical morphism

$$\mathrm{Gal}(k(V_{\bar{0}})|k(U_{\bar{0}})) \rightarrow \mathrm{Gal}(k(V)|k(U))$$

is surjective - hence an isomorphism. Indeed, otherwise, there would exist a degree- p intermediate cover $V \rightarrow U' \rightarrow U$ such that $U' \times_U U_{\bar{0}} \rightarrow U_{\bar{0}}$ is totally split (recall $k(V)/k(U)$ is Galois with group $(\mathbb{Z}/p^\nu)^{m-1}$ hence all its subcovers are again Galois covers) - hence étale: a contradiction. Thus $\tilde{U}_{\bar{0}} \rightarrow U_{\bar{0}}$ arises as the base change $\tilde{U} \times_U U_{\bar{0}} \rightarrow U$ of an étale intermediate cover $V \rightarrow \tilde{U} \rightarrow U$, which is necessarily trivial by the Subclaim. \square

End of the proof of Lemma 20. From Claim 2, it is enough to show that the cover $\Theta_{p^\nu, \bar{0}}^{\mathrm{sm}} \rightarrow \Theta_{r, \bar{0}}^{\mathrm{sm}}$ is étale or, equivalently, that for $i = 1, \dots, m$, the cover $\Theta_{p^\nu, \bar{0}}[\frac{1}{t_i}] \rightarrow \Theta_{r, \bar{0}}[\frac{1}{t_i}]$ is étale. But, up to a connected component, $\Theta_{p^\nu, \bar{0}}[\frac{1}{t_i}] \rightarrow \Theta_{r, \bar{0}}[\frac{1}{t_i}]$ is the base change along $\Theta_{r, \bar{0}}[\frac{1}{t_i}] \rightarrow \Theta_r$ of the finite cover $\alpha_{p^\nu \rightarrow r} : \Theta_{p^\nu} \rightarrow \Theta_r$ which, as explained in the proof of Claim 2, writing $D := \mathbb{G}_{m, \mathcal{O}_{k, \mathrm{ur}}}^{m-1}$, $A := \mathbb{A}_{\mathcal{O}_{k, \mathrm{ur}}}^{n-m}$, is isomorphic to

$$\varphi \times \mathrm{Id}_A : D \times A \rightarrow D \times A,$$

with $\varphi : D \rightarrow D$, $t \mapsto (t_1^{r'}, \dots, t_m^{r'})$. But by assumption $r' \in \mathcal{O}(D)^\times$ hence the same computation as in the proof of Claim 2 shows that $\Omega_\varphi^1 = 0$ that is $\varphi \times \mathrm{Id}_A : D \times A \rightarrow D \times A$ is étale. \square

As $\Theta_{r, \bar{0}}^{\mathrm{sm}}$ is normal, Lemma 20 can be reformulated by saying that the function field functor

$$\Theta \rightarrow \Theta_{r, \bar{0}}^{\mathrm{sm}} \mapsto k(\Theta)/k(\Theta_{r, \bar{0}})$$

is an equivalence of categories between finite connected étale covers of $\Theta_{r, \bar{0}}^{\mathrm{sm}}$ and subextensions of the Kummer extension

$$k(\Theta_{p^\nu, \bar{0}}) \simeq k(\Theta_{r, \bar{0}})(t_1^{\frac{1}{r'}}, \dots, t_{m-1}^{\frac{1}{r'}})/k(\Theta_{r, \bar{0}})$$

(note that, as $p \nmid r'$, k^{ur} contains all r' -roots of unity), with quasi-inverse the functor which sends a Kummer subextension $K/k(\Theta_{r, \bar{0}})$ of $k(\Theta_{p^\nu, \bar{0}})/k(\Theta_{r, \bar{0}})$ to the normalization $\Theta_K \rightarrow \Theta_{r, \bar{0}}^{\mathrm{sm}}$ of $\Theta_{r, \bar{0}}^{\mathrm{sm}}$ in $K/k(\Theta_{r, \bar{0}})$. Instead of the normalization $\Theta_K \rightarrow \Theta_{r, \bar{0}}^{\mathrm{sm}}$ of $\Theta_{r, \bar{0}}^{\mathrm{sm}}$ in $K/k(\Theta_{r, \bar{0}})$ one can consider the normalization $\Theta_{K,+} \rightarrow \Theta_{r, \bar{0}}$ of $\Theta_{r, \bar{0}}$ in $K/k(\Theta_{r, \bar{0}})$; this induces an equivalence of categories between Kummer subextensions of $k(\Theta_{p^\nu, \bar{0}})/k(\Theta_{r, \bar{0}})$ and finite connected covers $\Theta_+ \rightarrow \Theta_{r, \bar{0}}$ with Θ_+ normal and $\Theta_+ \times_{\Theta_{r, \bar{0}}} \Theta_{r, \bar{0}}^{\mathrm{sm}} \rightarrow \Theta_{r, \bar{0}}^{\mathrm{sm}}$ étale.

3.3.1.2. *Inertia.* For simplicity, write

$$I_{r, \bar{0}} := \pi_1(\Theta_{r, \bar{0}}[\frac{1}{\pi}]),$$

and let $I_{r, \bar{0}} \rightarrow I_{r, \bar{0}}^t := I_{r, \bar{0}}^{(p')}$ denote its maximal prime-to- p quotient.

Lemma 21. *The group $I_{r, \bar{0}}^t$ is abelian and fits into the following exact functorial diagram of abelian profinite groups*

$$\begin{array}{ccccccc} 1 & \longrightarrow & I_{1, \bar{0}}^t & \longrightarrow & I_{r', \bar{0}}^t & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 1 & \longrightarrow & I_{p^\nu, \bar{0}}^t & \longrightarrow & I_{r, \bar{0}}^t & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1, \end{array}$$

with $I_{1, \bar{0}}^t \simeq (\widehat{\mathbb{Z}}^{(p')})^m$.

Proof. The assertion when $r = 1$ immediately follows from Abhyankar's lemma as $\mathcal{O}_{1,\bar{0}}$ is regular. More precisely, for every $N \geq 1$, $p \nmid N$, let

$$\Theta_{1,\bar{0}}[\frac{1}{\pi}]_N \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$$

denote the Galois cover corresponding to the normalization of $\Theta_{1,\bar{0}}[\frac{1}{\pi}]$ in the extension

$$k(\Theta_{1,\bar{0}})(\pi^{\frac{1}{N}}, t_1^{\frac{1}{N}}, \dots, t_{m-1}^{\frac{1}{N}})/k(\Theta_{1,\bar{0}}).$$

Then the covers $\Theta_{1,\bar{0}}[\frac{1}{\pi}]_N \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$, $p \nmid N$ are cofinal among all connected étale covers of $\Theta_{1,\bar{0}}[\frac{1}{\pi}]$ tamely ramified at $\theta_{1,\bar{0}}$. The commutative square

$$\begin{array}{ccc} \Theta_{1,\bar{0}}[\frac{1}{\pi}] & \xrightarrow{\alpha_{1 \rightarrow r',\bar{0}}} & \Theta_{r',\bar{0}}[\frac{1}{\pi}] \\ \alpha_{1 \rightarrow p^\nu,\bar{0}} \downarrow & & \downarrow \alpha_{r' \rightarrow r,\bar{0}} \\ \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}] & \xrightarrow{\alpha_{p^\nu \rightarrow r,\bar{0}}} & \Theta_{r,\bar{0}}[\frac{1}{\pi}] \end{array}$$

and the fact that $\alpha_{1 \rightarrow r',\bar{0}} : \Theta_{1,\bar{0}}[\frac{1}{\pi}] \rightarrow \Theta_{r',\bar{0}}[\frac{1}{\pi}]$ and $\alpha_{p^\nu \rightarrow r,\bar{0}} : \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}] \rightarrow \Theta_{r,\bar{0}}[\frac{1}{\pi}]$ are Galois cover obtained by pulling back the universal covers $\alpha_{1 \rightarrow r',\bar{0}} : \Theta_{1,\bar{0}}^{\text{sm}} \rightarrow \Theta_{r',\bar{0}}^{\text{sm}}$ and $\alpha_{p^\nu \rightarrow r,\bar{0}} : \Theta_{p^\nu,\bar{0}}^{\text{sm}} \rightarrow \Theta_{r,\bar{0}}^{\text{sm}}$ respectively, yields the commutative diagram with exact lines

$$\begin{array}{ccccccc} 1 & \longrightarrow & I_{1,\bar{0}} & \longrightarrow & I_{r',\bar{0}} & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \simeq \\ 1 & \longrightarrow & I_{p^\nu,\bar{0}} & \longrightarrow & I_{r,\bar{0}} & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1. \end{array}$$

which, from Lemma 22 below, yields the commutative diagram with exact lines

$$\begin{array}{ccccccc} 1 & \longrightarrow & I_{1,\bar{0}}^t & \longrightarrow & I_{r',\bar{0}}^t & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \simeq \\ 1 & \longrightarrow & I_{p^\nu,\bar{0}}^t & \longrightarrow & I_{r,\bar{0}}^t & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1. \end{array}$$

It thus remains to prove that:

- i) $I_{r',\bar{0}}^t$ is abelian that is, equivalently, every Galois cover $\Theta \rightarrow \Theta_{r',\bar{0}}[\frac{1}{\pi}]$ totally tamely ramified at $\theta_{r',\bar{0}}$ is abelian. To check this consider the base-change diagram

$$\begin{array}{ccc} \square & \longrightarrow & \Theta \\ \downarrow & \square & \downarrow \\ \Theta_{1,\bar{0}}[\frac{1}{\pi}] & \longrightarrow & \Theta_{r',\bar{0}}[\frac{1}{\pi}]. \end{array}$$

As $\square \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$ is tamely ramified, there exists $p \nmid N$ such that $\Theta_{1,\bar{0}}[\frac{1}{\pi}]_N \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$ factors as

$$\begin{array}{ccc} \Theta_{1,\bar{0}}[\frac{1}{\pi}]_N & \longrightarrow & \square \\ \downarrow & \swarrow & \\ \Theta_{1,\bar{0}}[\frac{1}{\pi}] & & \end{array}$$

By construction, the resulting cover

$$\Theta_{1,\bar{0}}[\frac{1}{\pi}]_N \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}] \rightarrow \Theta_{r',\bar{0}}[\frac{1}{\pi}]$$

is the normalization of $\Theta_{r',\bar{0}}[\frac{1}{\pi}]$ in the extension

$$k(\Theta_{r',\bar{0}})(\pi^{\frac{1}{N}}, t_1^{\frac{1}{r'N}}, \dots, t_{m-1}^{\frac{1}{r'N}})/k(\Theta_{r',\bar{0}})$$

hence is abelian with group $(\mathbb{Z}/Nr')^m$ (as k^{ur} contains all $r'N$ th root of unity).

- ii) $I_{1,\bar{0}}^t \rightarrow I_{p^\nu,\bar{0}}^t$ is an isomorphism that is,

- $I_{1,\bar{0}}^t \rightarrow I_{p^\nu,\bar{0}}^t$ is surjective. Equivalently, for every connected étale cover $\Theta \rightarrow \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}]$ totally tamely ramified at $\theta_{p^\nu,\bar{0}}$, the pull-back

$$\begin{array}{ccc} \square & \longrightarrow & \Theta \\ \downarrow & \square & \downarrow \\ \Theta_{1,\bar{0}}[\frac{1}{\pi}] & \longrightarrow & \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}] \end{array}$$

is again connected. But this follows from the fact that $\Theta_{1,\bar{0}}[\frac{1}{\pi}] \rightarrow \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}]$ is of degree p^ν while $\Theta \rightarrow \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}]$ is totally tamely ramified at $\theta_{p^\nu,\bar{0}}$ so that $\square \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$ is again totally tamely ramified at $\theta_{1,\bar{0}}$ hence, in particular, connected.

- $I_{1,\bar{0}}^t \rightarrow I_{p^\nu,\bar{0}}^t$ is injective. Equivalently, for every connected étale cover $\Theta \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$ totally tamely ramified at $\theta_{1,\bar{0}}$ there exists a connected étale cover $\Theta_- \rightarrow \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}]$ totally tamely ramified at $\theta_{p^\nu,\bar{0}}$ such that

$$\begin{array}{ccc} & \Theta & \\ & \swarrow & \searrow \\ \square & \longrightarrow & \Theta_{1,\bar{0}}[\frac{1}{\pi}] \\ \downarrow & \square & \downarrow \\ \Theta_- & \longrightarrow & \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}] \end{array}$$

It is enough to prove this for $\Theta = \Theta_{1,\bar{0}}[\frac{1}{\pi}]_N \rightarrow \Theta_{1,\bar{0}}[\frac{1}{\pi}]$, $p \nmid N$. But, then, $\Theta \rightarrow \Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}]$ is obtained explicitly as the normalization of $\Theta_{p^\nu,\bar{0}}[\frac{1}{\pi}]$ in

$$k(\Theta_{p^\nu,\bar{0}})(t_1^{\frac{1}{p^\nu}}, \dots, t_{m-1}^{\frac{1}{p^\nu}})(\pi^{\frac{1}{N}}, t_1^{\frac{1}{Np^\nu}}, \dots, t_{m-1}^{\frac{1}{Np^\nu}})/k(\Theta_{p^\nu,\bar{0}}).$$

As p and N are coprime, writing $ap^\nu + bN = 1$ for some $a, b \in \mathbb{Z}$, one gets

$$\begin{aligned} k(\Theta_{p^\nu,\bar{0}})(t_1^{\frac{1}{p^\nu}}, \dots, t_{m-1}^{\frac{1}{p^\nu}})(\pi^{\frac{1}{N}}, t_1^{\frac{1}{Np^\nu}}, \dots, t_{m-1}^{\frac{1}{Np^\nu}}) &= k(\Theta_{p^\nu,\bar{0}})(t_1^{\frac{1}{p^\nu}}, \dots, t_{m-1}^{\frac{1}{p^\nu}})(\pi^{\frac{1}{N}}, t_1^{\frac{a}{N}}, \dots, t_{m-1}^{\frac{a}{N}}) \\ &= k(\Theta_{p^\nu,\bar{0}})(\pi^{\frac{1}{N}}, t_1^{\frac{a}{N}}, \dots, t_{m-1}^{\frac{a}{N}})k(\Theta_{1,\bar{0}})/k(\Theta_{p^\nu,\bar{0}}). \end{aligned}$$

□

Lemma 22. Fix a prime p . Let $1 \rightarrow \Pi' \rightarrow \Pi \rightarrow \Pi'' \rightarrow 1$ be a short exact sequence of profinite groups with Π'' of prime-to- p order. Then the canonical morphism $\Pi'^{(p')} \rightarrow \Pi^{(p')}$ is injective.

Proof. Fix a p -Sylow subgroup $P' \subset \Pi'$. As Π'' is of prime-to- p order, P' maps isomorphically to a p -Sylow $P' \xrightarrow{\sim} P \subset \Pi$; let us identify $P' := P$. We are to show that the subgroup

$$\ker(\Pi' \rightarrow \Pi'^{(p')}) = \langle\langle P' \rangle\rangle_{\Pi'} \subset \Pi'$$

topologically normally generated by P' in Π' maps isomorphically to

$$\ker(\Pi \rightarrow \Pi^{(p')}) = \langle\langle P' \rangle\rangle_{\Pi} \subset \Pi.$$

But, as Π' is normal in Π and $P' \subset \Pi'$, for every $\pi \in \Pi$, $\pi P' \pi^{-1} \subset \Pi'$; in particular, $\pi P' \pi^{-1} \subset \Pi'$ is also a p -Sylow of Π' hence there exists $\pi' \in \Pi'$ such that $\pi P' \pi^{-1} = \pi' P' \pi'^{-1}$. □

As $\Theta_{r,\bar{0}}$ is normal, Lemma 21 can be reformulated by saying that the function field functor

$$\Theta \rightarrow \Theta_{r,\bar{0}} \mapsto k(\Theta)/k(\Theta_{r,\bar{0}})$$

is an equivalence of categories between finite connected covers of $\Theta_{r,\bar{0}}$ totally tamely ramified above $\theta_{r,\bar{0}}$ and subextensions of the Kummer extension

$$k(\Theta_{r,\bar{0}})^t := \operatorname{colim}_{p \nmid N} k(\Theta_{r,\bar{0}})(\pi^{\frac{1}{N}}, t_1^{\frac{1}{N}}, \dots, t_{m-1}^{\frac{1}{N}})/k(\Theta_{r,\bar{0}})$$

(note that, as $p \nmid r'$, k^{ur} contains all prime-to- p -roots of unity), with quasi-inverse the functor which sends a Kummer subextension $K/k(\Theta_{r,\bar{0}})$ of $k(\Theta_{r,\bar{0}})^t/k(\Theta_{r,\bar{0}})$ to the normalization $\Theta_K \rightarrow \Theta_{r,\bar{0}}$ of $\Theta_{r,\bar{0}}$ in $K/k(\Theta_{r,\bar{0}})$.

From now and until the end of Subsection 3.3, let $\mathcal{X} \rightarrow S$ be a semistable morphism; recall that this means $\mathcal{X} \rightarrow S$ is separated, flat, of finite type, X is integral, smooth over k , and for every $u \in |\mathcal{X}_s|$,

$$\mathcal{O}_{\mathcal{X},\bar{u}} \simeq \mathcal{O}_{n,m,r,\bar{0}}$$

for some integers $r \geq 1$, $n \geq m \geq 1$. In particular, $u \in \mathcal{X}^{\text{sm}}$ if and only if $m = 1$ and $u \in \mathcal{X}^{\text{reg}}$ if and only if $m = 1$ or $r = 1$. As both $\mathcal{X} \rightarrow S$ and $\Theta_{n,m,r} \rightarrow S$ are of finite presentation, the condition that $\mathcal{O}_{\mathcal{X},\bar{u}} \simeq \mathcal{O}_{n,m,r,\bar{0}}$ is equivalent to the existence of a commutative diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\simeq} & \mathcal{U} \\ \downarrow & \swarrow & \searrow \\ \mathcal{X} & \xleftarrow{\bar{u}} \text{spec}(\bar{k}) \xrightarrow{\bar{0}} & \Theta_{n,m,r} \end{array}$$

with $U \rightarrow \Theta_{n,m,r}$, $U \rightarrow \mathcal{X}$ étale morphisms. In particular, $\Theta_{n,m,r,\bar{0}}^{\text{sm}} \subset \Theta_{n,m,r,\bar{0}}$ also identifies with the inverse image of $\mathcal{X}^{\text{sm}} \subset \mathcal{X}$ via $\Theta_{n,m,r,\bar{0}} \simeq \text{spec}(\mathcal{O}_{\mathcal{X},\bar{u}}) \rightarrow \mathcal{X}$.

3.3.2. Proof of Theorem 13 (U) and Corollary 15. For both statements, again, the implications (U-1) \Rightarrow (U-2) \Rightarrow (U-3) \Rightarrow (U-4) and (U-2) \Rightarrow (U-1) are (almost) tautological. We prove (U-4) \Rightarrow (U-2) applying the strategy described in Subsection 3.1.

3.3.2.1. Proof of Theorem 13 (U). From Subsection 3.2.2, we may and will assume that $\mathcal{X}^{\text{n-sm}} (= \mathcal{X}_s^{\text{n-sm}}) \neq \emptyset$. From *loc. cit.*, $\psi_{\mathcal{X}^{\text{sm}}} : \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^{\text{sm}} \rightarrow \mathcal{X}^{\text{sm}}$ is (finite) étale. Assume (U-2) does not hold so that there exists $u \in \mathcal{X}_s^{\text{n-sm}}$ and $v \in \mathcal{Y}_u$ such that $\Theta_+ := \mathcal{Y}_{(\bar{v})} \rightarrow \mathcal{X}_{(\bar{u})} \simeq \Theta_{n,m,r,\bar{0}} =: \Theta_{r,\bar{0}}$ is finite not étale [Stacks, Tag 02GU]. Write $r = p^\nu r'$ with $p \nmid r'$. From Lemma 20, and as $\Theta := \Theta_+ \times_{\Theta_{r,\bar{0}}} \Theta_{r,\bar{0}}^{\text{sm}} \rightarrow \Theta_{r,\bar{0}}^{\text{sm}}$ is étale, the extension of function fields $k(\Theta)/k(\Theta_{r,\bar{0}})$ is a non-trivial subextension of the Kummer extension

$$k(\Theta_{p^\nu,\bar{0}}) \simeq k(\Theta_{r,\bar{0}})(t_1^{\frac{1}{r}}, \dots, t_{m-1}^{\frac{1}{r}})/k(\Theta_{r,\bar{0}}).$$

Fix a subextension $k(\Theta_{r,\bar{0}}) \subset K \subset k(\Theta)$ with $K/k(\Theta_{r,\bar{0}})$ of prime degree ℓ . By abelian Kummer theory (as $p \nmid r'$, k^{ur} contains all r' -roots of unity), one has $K = k(\Theta_{r,\bar{0}})(f^{\frac{1}{\ell}})$ for some $f = t_1^{e_1} \cdots t_{m-1}^{e_{m-1}}$ with $0 \leq e_i \leq \ell - 1$, $i = 1, \dots, m-1$ and $e_i \geq 1$ for at least one of $i = 1, \dots, m$. From these data, we construct a non trivial extension k'/k^{ur} contained in k^t and a morphism of $\mathcal{O}_{k^{\text{ur}}}$ -algebras

$$\mathcal{O}_{r,\bar{0}} \rightarrow \mathcal{O}_{k'}$$

as follows: let $k_1, \dots, k_m \geq 1$ be integers and $\ell_0 \neq \ell$ a prime such that, setting,

$$N := \sum_{1 \leq i \leq m-1} k_i e_i$$

one has

$$(i) \quad \sum_{1 \leq i \leq m} \frac{k_i}{\ell_0} = r \quad \text{and} \quad (ii) \quad \ell, \ell_0 \nmid N.$$

For instance, if $e_i \geq 1$, one can take $k_i = 1$, $k_j = \ell$ for $1 \leq i \neq j \leq m-1$,

$$\ell_0 \geq \max\left\{\frac{1 + \ell(m-2)}{r}, N + 1\right\},$$

and $k_m = \ell_0 r - (1 + \ell(m-2))$. Set $k' := k^{\text{ur}}(\pi^{\frac{1}{\ell_0}})$. From (i), the morphism

$$\mathcal{O}_{k^{\text{ur}}}[T_1, \dots, T_n] \rightarrow \mathcal{O}_{k'}, \quad \underline{T} \mapsto (\pi^{\frac{k_1}{\ell_0}}, \dots, \pi^{\frac{k_m}{\ell_0}}, 0, \dots, 0)$$

factors canonically through

$$\begin{array}{ccc} \mathcal{O}_{k^{\text{ur}}}[T_1, \dots, T_n] & \longrightarrow & \mathcal{O}_{k'} \\ \downarrow & \nearrow & \\ \mathcal{O}_{r,\bar{0}} & & \end{array}$$

The morphism of $\mathcal{O}_{k^{\text{ur}}}$ -algebras $\mathcal{O}_{r,\bar{0}} \rightarrow \mathcal{O}_{k'}$ gives rise to an $\mathcal{O}_{k'}$ -point - hence a k' -point:

$$\begin{array}{ccccc} & & \Theta_{r,\bar{0}} & \longleftarrow & \Theta_{r,\bar{0}}^{\text{sm}} & \longleftarrow & \Theta_{r,\bar{0}}[\frac{1}{\pi}] \\ & \nearrow & \uparrow \hat{x}' & & & & \nearrow x' \\ \text{spec}(\bar{k}) & \longleftarrow & \text{spec}(\mathcal{O}_{k'}) & \longleftarrow & \text{spec}(k') & & \end{array}$$

whence

$$\begin{array}{ccccccc}
& & \text{spec}(k') & \longleftarrow & \Theta_{K,x'} & \longleftarrow & \Theta_{x'} \\
& & \downarrow x' & & \downarrow & & \downarrow \\
& & \square & & \square & & \square \\
\text{spec}(k^{\text{ur}}) & \longleftarrow & \Theta_{r,0}[\frac{1}{\pi}] & \longleftarrow & \Theta_K[\frac{1}{\pi}] & \longleftarrow & \Theta[\frac{1}{\pi}] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{spec}(\mathcal{O}_{k^{\text{ur}}}) & \longleftarrow & \Theta_{r,0}^{\text{sm}} & \longleftarrow & \Theta_K & \longleftarrow & \Theta
\end{array}$$

But one can compute explicitly $\Theta_{K,x'}$. Indeed, as $\mathcal{O}_{r,0}[\frac{1}{\pi}][T]/\langle T^\ell - f \rangle$ is normal with fraction field K one has an identification $\Theta_K[\frac{1}{\pi}] \simeq \text{spec}(\mathcal{O}_{r,0}[\frac{1}{\pi}][T]/\langle T^\ell - f \rangle) \rightarrow \Theta_{r,0}[\frac{1}{\pi}]$, whence

$$(\mathcal{O}_{r,0}[\frac{1}{\pi}][T]/\langle T^\ell - f \rangle) \otimes_{\mathcal{O}_{r,0}[\frac{1}{\pi}],x'} k' \simeq k'[T]/\langle T^\ell - \pi^{\frac{N}{\ell_0}} \rangle \simeq k^{\text{ur}}((\pi^{\frac{1}{\ell_0}})^{\frac{1}{\ell}}),$$

where the last isomorphism follows from condition (ii). This contradicts (U-4).

3.3.2.2. *Proof of Corollary 15.* From Theorem 13 (U-4) \Rightarrow (U-2), one already knows that $\psi_{\mathcal{X}} : \mathcal{Y}^{\text{cpt}} \times_{\mathcal{X}^{\text{cpt}}} \mathcal{X} \rightarrow \mathcal{X}$ is finite étale. Assume (U-2) does not hold so that there exists $u \in \mathcal{D}_s$ and $v \in \mathcal{Y}_u^{\text{cpt}}$ such that $\mathcal{Y}_{(v)}^{\text{cpt}} \rightarrow \mathcal{X}_{(x)}^{\text{cpt}}$ is finite, tamely ramified but not étale over $\mathcal{D}_{(\bar{u})} := \mathcal{D} \times_{\mathcal{X}^{\text{cpt}}} \mathcal{X}_{(\bar{u})}^{\text{cpt}}$. Write

$$\mathcal{O}_{\mathcal{X}^{\text{cpt}},\bar{u}} \simeq \mathcal{O}_{k^{\text{ur}}}[X_1, \dots, X_n, T_1, \dots, T_m]_{\langle \pi, X_1, \dots, X_n, T_1, \dots, T_m \rangle}^h$$

with $t_1 = \dots = t_m = 0$ corresponding to $\mathcal{D}_{(\bar{u})}$. From Abhyankar's lemma, the extension of function fields $k(\mathcal{Y}_{(v)}^{\text{cpt}})/k(\mathcal{X}_{(\bar{u})}^{\text{cpt}})$ is a subextension of a Kummer extension

$$k(\mathcal{X}_{(\bar{u})}^{\text{cpt}})(t_1^{\frac{1}{N}}, \dots, t_m^{\frac{1}{N}})/k(\mathcal{X}_{(\bar{u})}^{\text{cpt}})$$

for some integer $N \geq 1$ with $p \nmid N$. As in Subsection 3.3.2.1, fix a subextension

$$k(\mathcal{X}_{(\bar{u})}^{\text{cpt}}) \subset K = k(\mathcal{X}_{(\bar{u})}^{\text{cpt}})(f^{\frac{1}{\ell}}) \subset k(\mathcal{Y}_{(v)}^{\text{cpt}})$$

of prime degree ℓ with $f = t_1^{e_1} \dots t_m^{e_m}$ and $0 \leq e_i \leq \ell - 1$, $i = 1, \dots, m$ and $e_i \geq 1$ for at least one $i = 1, \dots, m$. Choose $k_1, \dots, k_r \geq 1$, $\ell_0 \neq \ell$ satisfying (i), (ii) of the proof of Subsection 3.3.2.1; set $k' := k^{\text{ur}}(\pi^{\frac{1}{\ell_0}})$. The morphism of $\mathcal{O}_{k^{\text{ur}}}$ -algebras

$$\begin{array}{ccc}
\mathcal{O}_{k^{\text{ur}}}[X_1, \dots, X_n, T_1, \dots, T_m] & \xrightarrow{(X_1, \dots, X_n, T_1, \dots, T_m) \mapsto (0, \dots, 0, \pi^{\frac{k_1}{\ell_0}}, \dots, \pi^{\frac{k_m}{\ell_0}})} & \mathcal{O}_{k'} \\
\downarrow & \nearrow & \\
\mathcal{O}_{\mathcal{X}^{\text{cpt}},\bar{u}} & &
\end{array}$$

gives rise to an $\mathcal{O}_{k'}$ -point $\tilde{x}' \in \mathcal{X}^{\text{cpt}}(\mathcal{O}_{k'})$ with associated k' -point $x' \in X(k')$

$$\begin{array}{ccccc}
& & \mathcal{X}^{\text{cpt}} & \longleftarrow & X \\
& & \uparrow & & \uparrow \\
& & \mathcal{X}_{(\bar{u})}^{\text{cpt}} & & x' \\
& \nearrow & \uparrow & & \\
\text{spec}(\bar{k}) & \hookrightarrow & \text{spec}(\mathcal{O}_{k'}) & \longleftarrow & \text{spec}(k')
\end{array}$$

such that

$$Y_{x'} = (\mathcal{Y}_{(v)}^{\text{cpt}})_{x'} \twoheadrightarrow \text{spec}(\mathcal{O}_{\mathcal{X}^{\text{cpt}},\bar{u}}[t_1^{-1}, \dots, t_m^{-1}][f^{\frac{1}{\ell}}]) \otimes_{\mathcal{O}_{\mathcal{X}^{\text{cpt}},\bar{u}}[t_1^{-1}, \dots, t_m^{-1}],x'} k' \simeq \text{spec}(k^{\text{ur}}((\pi^{\frac{1}{\ell_0}})^{\frac{1}{\ell}})).$$

This contradicts (U-4).

3.3.3. *Proof of Theorem 13 (T).* Again the implications (T-1) \Rightarrow (T-2) \Rightarrow (T-3) \Rightarrow (T-4) and (T-2) \Rightarrow (T-1) are (almost) tautological. We prove (T-4) \Rightarrow (T-2). From Subsection 3.2.3, we may and will assume that $\mathcal{X}^{\text{n-sm}} (= \mathcal{X}_s^{\text{n-sm}}) \neq \emptyset$. From *loc. cit.*, $\psi_{\mathcal{X}^{\text{sm}}} : \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^{\text{sm}} \rightarrow \mathcal{X}^{\text{sm}}$ is finite, tamely ramified along $\mathcal{X}_s^{\text{sm}}$; in particular, $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is tamely ramified over the generic points of \mathcal{X}_s (which all lie in $\mathcal{X}_s^{\text{sm}}$). We are to prove that $\psi_{\mathcal{X}} : \mathcal{Y} \rightarrow \mathcal{X}$ is also tamely ramified at every $u \in \mathcal{X}_s^{\text{n-sm}}$ viz that the corresponding representation of profinite group

$$\pi_1(X_{(\bar{u})}) \rightarrow \pi_1(X) \rightarrow \text{Aut}(Y/X) =: \Pi$$

factors through the tame - viz prime-to- p completion

$$\pi_1(X_{(\bar{u})}) \twoheadrightarrow \pi_1^t(X_{(\bar{u})}) := \pi_1(X_{(\bar{u})})^{(p')}.$$

Recall that $\mathcal{O}_{\mathcal{X}, \bar{u}} \simeq \mathcal{O}_{m, n, r, \bar{0}} =: \mathcal{O}_{r, \bar{0}}$ so that $\pi_1(X_{(\bar{u})}) \simeq I_{r, \bar{0}}$ and that, writing $r = p^\nu r'$ with $p \nmid r'$, one has a canonical short exact sequence

$$1 \rightarrow I_{p^\nu, \bar{0}} \rightarrow I_{r, \bar{0}} \rightarrow (\mathbb{Z}/r')^{m-1} \rightarrow 1$$

obtained by pulling-back the universal cover $\Theta_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{r, \bar{0}}^{\text{sm}}$ along $\Theta_{r, \bar{0}}[\frac{1}{\pi}] \rightarrow \Theta_{r, \bar{0}}^{\text{sm}}$ (See proof of Lemma 21). For $i = 1, \dots, m$, let $\zeta_{r, i}$ denote the generic point of the irreducible component of $\Theta_{r, \bar{0}}$ defined by $t_i = 0$ and write

$$I_{r, i} := \text{im}(\pi_1(\Theta_{r, \bar{0}}(\zeta_{r, i}) \times_{\Theta_{r, \bar{0}}} \Theta_{r, \bar{0}}[\frac{1}{\pi}]) \rightarrow \pi_1(\Theta_{r, \bar{0}}[\frac{1}{\pi}])) \subset \pi_1(\Theta_{r, \bar{0}}[\frac{1}{\pi}]) = I_{r, \bar{0}}$$

for the inertia group of $\zeta_{r, i}$ in $I_{r, \bar{0}}$. As $\zeta_{r, i} \in \Theta_{r, \bar{0}}^{\text{sm}}$ and $\Theta_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{r, \bar{0}}^{\text{sm}}$ is étale, one has

$$\begin{array}{ccccccc} 1 & \longrightarrow & I_{p^\nu, \bar{0}} & \longrightarrow & I_{r, \bar{0}} & \longrightarrow & (\mathbb{Z}/r')^{m-1} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \\ & & I_{p^\nu, i} & \xrightarrow{\simeq} & I_{r, i} & & \end{array}$$

- If $\nu = 0$, $\Theta_{1, \bar{0}}$ is regular and $\Theta_{1, \bar{0}, s} \hookrightarrow \Theta_{1, \bar{0}}$ is a normal crossing divisor in $\Theta_{1, \bar{0}}$ so that, as the pullback $\mathcal{Y} \times_{\mathcal{X}} \Theta_{1, \bar{0}} \rightarrow \Theta_{1, \bar{0}}$ is tamely ramified at $\zeta_{1, i}$, $i = 1, \dots, m$, it follows from Abhyankar's lemma that $\mathcal{Y} \times_{\mathcal{X}} \Theta_{1, \bar{0}} \rightarrow \Theta_{1, \bar{0}}$ is tamely ramified along $\Theta_{1, \bar{0}, s}$ hence, in particular, at $\bar{0}$. This proves that the morphism of profinite groups $I_{1, \bar{0}} \hookrightarrow I_{r, \bar{0}} \rightarrow \Pi$ factors through $I_{1, \bar{0}} \twoheadrightarrow I_{1, \bar{0}}^t$. The assertion then follows from the fact that

$$\ker(I_{1, \bar{0}} \twoheadrightarrow I_{1, \bar{0}}^t) = \ker(I_{r, \bar{0}} \twoheadrightarrow I_{r, \bar{0}}^t)$$

(See Lemma 22).

- If $\nu > 0$,

(a) Assume k contains all p^ν th roots of unity. The sequence of Galois covers $\Theta_{1, \bar{0}}^{\text{sm}} \rightarrow \Theta_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{r, \bar{0}}^{\text{sm}}$ induces, at the level of inertia groups, the following canonical diagram of inclusions

$$\begin{array}{ccccc} I_{1, \bar{0}} & \xrightarrow{(\mathbb{Z}/p^\nu)^{m-1}} & I_{p^\nu, \bar{0}} & \xrightarrow{(\mathbb{Z}/r')^{m-1}} & I_{r, \bar{0}} \\ \uparrow & & \uparrow & & \uparrow \\ I_{1, i} & \xrightarrow{(\mathbb{Z}/p^\nu)^{m-1}} & I_{p^\nu, i} & \xrightarrow{\simeq} & I_{r, i} \end{array}$$

where the isomorphism $I_{p^\nu, i} \xrightarrow{\simeq} I_{r, i}$ arises from the fact $\Theta_{p^\nu, \bar{0}}^{\text{sm}} \rightarrow \Theta_{r, \bar{0}}^{\text{sm}}$ is étale and $\zeta_{r, i} \in \Theta_{r, \bar{0}}^{\text{sm}}$ while the isomorphism

$$I_{p^\nu, i}/I_{1, i} \xrightarrow{\simeq} I_{p^\nu, \bar{0}}/I_{1, \bar{0}} \simeq (\mathbb{Z}/p^\nu)^{m-1}$$

follows from the fact $\Theta_{1, \bar{0}}^{\text{sm}} \rightarrow \Theta_{p^\nu, \bar{0}}^{\text{sm}}$ is totally wildly ramified along $\Theta_{p^\nu, \bar{0}, s}$ (See Subclaim in proof of Lemma 20). For a subgroup $I \subset I_{r, \bar{0}}$, write

$$I_\Pi := \text{im}(I \hookrightarrow I_{r, \bar{0}} \twoheadrightarrow \Pi) \subset (I_{r, \bar{0}})_\Pi = \Pi.$$

So, with this notation, one has

$$(9) \quad \begin{array}{ccccccc} 1 & \longrightarrow & (I_{1, \bar{0}})_\Pi & \longrightarrow & (I_{p^\nu, \bar{0}})_\Pi & \longrightarrow & (I_{p^\nu, \bar{0}})_\Pi / (I_{1, \bar{0}})_\Pi \longrightarrow 1 \\ & & & & & & \uparrow \\ & & & & & & (I_{p^\nu, i})_\Pi \end{array}$$

On the other hand, as the morphism of profinite groups $I_{r, i} \rightarrow \Pi$ factors through $I_{r, i} \twoheadrightarrow I_{r, i}^t$, $i = 1, \dots, m$, one has, *a fortiori*, that

- the morphism $I_{p^\nu, i} \hookrightarrow I_{r, i} \rightarrow \Pi$ factors through $I_{p^\nu, i} \twoheadrightarrow I_{p^\nu, i}^t$, $i = 1, \dots, m$ - hence $(I_{p^\nu, i})_\Pi$ is of prime-to- p order;
- the morphism $I_{1, i} \hookrightarrow I_{r, i} \rightarrow \Pi$ factors through $I_{1, i} \twoheadrightarrow I_{1, i}^t$, $i = 1, \dots, m$ which, again from the fact that $\Theta_{1, \bar{0}}$ is regular and Abhyankar's lemma, implies that $I_{1, \bar{0}} \hookrightarrow I_{r, \bar{0}} \rightarrow \Pi$ also factors through $I_{1, \bar{0}} \twoheadrightarrow I_{1, \bar{0}}^t$ - hence $(I_{1, \bar{0}})_\Pi$ is of prime-to- p order.

Combined with (9), this proves $(I_{p^\nu, \bar{0}})_\Pi$ has prime-to- p order which, as $(I_{r, \bar{0}})_\Pi / (I_{p^\nu, \bar{0}})_\Pi$ is also of prime-to- p order, ensures that $I_{r, \bar{0}} \twoheadrightarrow \Pi$ factors through $I_{r, \bar{0}} \twoheadrightarrow I_{r, \bar{0}}^t$.

- (b) If we no longer assume that k contains all p^ν th roots of unity, the base-change diagram

$$\begin{array}{ccc} \Theta_{1, \bar{0}}[\frac{1}{\pi}] \times_{k^{\text{ur}}} k^{\text{ur}}(\mu_{p^\nu}) & \longrightarrow & \Theta_{p^\nu, \bar{0}}[\frac{1}{\pi}] \times_{k^{\text{ur}}} k^{\text{ur}}(\mu_{p^\nu}) \\ \downarrow & \square & \downarrow \\ \Theta_{1, \bar{0}}[\frac{1}{\pi}] & \longrightarrow & \Theta_{p^\nu, \bar{0}}[\frac{1}{\pi}] \end{array}$$

yields a canonical diagram

$$(10) \quad \begin{array}{ccccc} & & I_{p^\nu, i, \mu_{p^\nu}} & & \\ & & \downarrow & \searrow & \\ 1 & \longrightarrow & I_{1, \bar{0}, \mu_{p^\nu}} \hookrightarrow I_{p^\nu, \bar{0}, \mu_{p^\nu}} & \longrightarrow & I_{p^\nu, \bar{0}, \mu_{p^\nu}} / I_{1, \bar{0}, \mu_{p^\nu}} \\ & & \downarrow \triangleleft & \square & \downarrow \triangleleft \\ & & I_{1, \bar{0}} \hookrightarrow I_{p^\nu, \bar{0}} & \xrightarrow{(*)} & I_{p^\nu, \bar{0}} / I_{1, \bar{0}} \\ & & \downarrow & \nearrow & \downarrow \simeq \\ & & I_{p^\nu, i} & \xrightarrow{(*)} & I_{p^\nu, \bar{0}} / I_{1, \bar{0}} \end{array}$$

where, for $\# = 1, p^\nu$, we write $I_{\#, \bar{0}, \mu_{p^\nu}} := \pi_1(\Theta_{\#, \bar{0}}[\frac{1}{\pi}] \times_{k^{\text{ur}}} k^{\text{ur}}(\mu_{p^\nu}))$ and $I_{\#, i, \mu_{p^\nu}} \subset I_{\#, \bar{0}, \mu_{p^\nu}}$ for the corresponding inertia group along $t_i = 0$, $i = 1, \dots, m$. Be aware that the maps $(*)$ are only set-theoretic maps as $I_{1, \bar{0}}$ is no longer normal in $I_{p^\nu, \bar{0}}$. Still, what we care about is that the map $I_{p^\nu, i} \twoheadrightarrow I_{p^\nu, \bar{0}} / I_{1, \bar{0}}$ be surjective. Mapping Diagram (10) to Π , one gets that $(I_{1, \bar{0}})_\Pi$ becomes normal in $(I_{p^\nu, \bar{0}})_\Pi$ (as $(I_{1, \bar{0}, \mu_{p^\nu}})_\Pi = (I_{p^\nu, \bar{0}, \mu_{p^\nu}})_\Pi$ is normal both in $(I_{1, \bar{0}})_\Pi$ and $(I_{p^\nu, \bar{0}})_\Pi$). As $(I_{p^\nu, i})_\Pi \twoheadrightarrow (I_{p^\nu, \bar{0}})_\Pi / (I_{1, \bar{0}})_\Pi$ remains of course surjective, we are in the setting of Diagram (9) and we can conclude as in Case (a).

3.4. Proof of Corollary 2. This subsection is devoted to the proof of Corollary 2. We retain the notation of Subsection 1.1.3.

3.4.1. Tame inertia. Let $u \in |\mathcal{X}_s|$ and let $x \in |u|^\#$. Write $\overline{\{x\}}^{\text{zar}} \hookrightarrow \mathcal{X}$ for the Zariski closure of x in \mathcal{X} and $\tilde{x} : S_x \rightarrow \mathcal{X}$ for the (unique) extension of $x : \text{spec}(k(x)) \rightarrow \mathcal{X}$. Then the canonical morphism $S_x \rightarrow \overline{\{x\}}^{\text{zar}}$ identifies with the normalization of the integral scheme $\overline{\{x\}}^{\text{zar}}$ and one gets a canonical commutative diagram

$$(11) \quad \begin{array}{ccc} & & \mathcal{X}_{(u)} := \text{spec}(\mathcal{O}_{\mathcal{X}, u}) \\ & & \downarrow \\ S_x & \longrightarrow & \overline{\{x\}}^{\text{zar}} \hookrightarrow \mathcal{X} \end{array}$$

with the closed point $s_x \in S_x$ mapping to the closed point $u \in \mathcal{X}_{(u)}$. Indeed, without loss of generality one may assume $\mathcal{X} = \text{spec}(R)$ is affine. Let $\mathfrak{p} \subset R$ denote the prime ideal corresponding to $\overline{\{x\}}^{\text{zar}} \hookrightarrow \mathcal{X}$ viz. the kernel of the ring map $\tilde{x}^\# : R \rightarrow \mathcal{O}_{k(x)}$ and let $\mathfrak{m} (= (\tilde{x}^\#)^{-1}(\mathfrak{m}_x)) \subset R$ denote the maximal ideal corresponding to u . With these notation, one has a canonical commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_k & \longrightarrow & R & \xrightarrow{\tilde{x}^\#} & \mathcal{O}_{k(x)} \hookrightarrow k(x) \\ & \searrow & \downarrow & \nearrow & \\ & & R/\mathfrak{p} & & \end{array}$$

so that R/\mathfrak{p} is local with maximal ideal \mathfrak{m} and $\mathcal{O}_{k(x)}$ is the integral closure of R/\mathfrak{p} in $k(x)(= \text{Frac}(R/\mathfrak{p}))$. In particular, one has

$$R/\mathfrak{p} \simeq (R/\mathfrak{p})_{\mathfrak{m}} \simeq R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}} \leftarrow R_{\mathfrak{m}}.$$

Fix a geometric point \bar{x} over $x \in S_x$ with residue field $k(\bar{x}) \simeq k(x)^s = k^s$ and let $k(x)^{\text{ur}} \subset k(\bar{x})$ denote the maximal extension of $k(x)$ in $k(\bar{x})$ unramified over s_x so that the closed point of S_x^{ur} defines a geometric point \bar{s}_x over $s_x \in S_x$ hence a geometric point \bar{u} over $u \in \mathcal{X}_{(u)}$. These choices induce a canonical morphism $S_x^{\text{ur}} = S_{x,(\bar{s}_x)} \rightarrow \mathcal{X}_{(\bar{u})}$ hence a canonical morphism

$$\text{spec}(k(x)^{\text{ur}}) \rightarrow X_{(\bar{u})} \rightarrow X$$

yielding

$$\begin{array}{ccccccc} \pi_1(k(x)^{\text{ur}}) & \longrightarrow & \pi_1(X_{(\bar{u})}) & \twoheadrightarrow & I_{X,u} \hookrightarrow & \pi_1(X) & \\ \downarrow & & \downarrow & & \downarrow & \downarrow & \\ \pi_1(k(x)^{\text{ur}})^{(p')} & \longrightarrow & \pi_1(X_{(\bar{u})})^{(p')} & \twoheadrightarrow & I_{X,u}^{(p')} & \twoheadrightarrow & \pi_1^t(\mathcal{X}; \mathcal{X}_s) \\ \parallel & & \parallel & & \text{not. } \parallel & & \\ (I_x^t =) \pi_1^t(S_{x,(\bar{s}_x)}; \bar{s}_x) & \longrightarrow & \pi_1^t(\mathcal{X}_{(\bar{u})}; \bar{u}) & \twoheadrightarrow & I_{X,u}^t & & \end{array}$$

Write

$$I_{x,u}^t := \text{im}(I_x^t \rightarrow I_{X,u}^t) \subset I_{X,u}^t.$$

Note that, as $I_{X,u}^t$ is abelian (Lemma 21), the subgroup $I_{x,u}^t \subset I_{X,u}^t$ depends only on x and not on the choice of the geometric \bar{x} over it (whence our omitting \bar{x} in the notation). Recall also that, by definition, if $x_1 \in \mathcal{X}$ specializes to $x_0 \in \mathcal{X}$ then $I_{X,x_1} \subset I_{X,x_0}$; in particular, if $u \in \mathcal{X}_s^\circ$ for some irreducible component $\mathcal{X}_s^\circ \hookrightarrow \mathcal{X}_s$ with generic point ζ , then $I_{X,\zeta} \subset I_{X,u}$. To sum it up, one always has

$$(12) \quad I_{X,\zeta}^t \rightarrow I_{X,u}^t \supset I_{x,u}^t.$$

Lemma 23.

- (1) Assume $u \in \mathcal{X}_s^{\text{sm}}$ and set $\mathcal{X}^\circ := X \sqcup \mathcal{X}_s^{\text{sm},\circ}$, where $\mathcal{X}_s^{\text{sm},\circ} \hookrightarrow \mathcal{X}_s^{\text{sm}}$ denote the irreducible (viz. connected) component of $\mathcal{X}_s^{\text{sm}}$ containing u . Write $\bar{I}_{X,\zeta}^t \subset \bar{I}_{X,u}^t \supset \bar{I}_{x,u}^t$ for the images in $\pi_1^t(\mathcal{X}^\circ; \mathcal{X}_s^\circ)$ of the groups in (12). In general, $\bar{I}_{x,u}^t \subset \bar{I}_{\zeta}^t = \bar{I}_{X,u}^t$ and, if $x \in]u[^u$, $\bar{I}_{x,u}^t = \bar{I}_{X,\zeta}^t = \bar{I}_{X,u}^t$.
- (2) Assume $u \in \mathcal{X}_s^{\text{n-sm}}$. Then the group $I_{X,u}^t$ is generated by the $I_{x,u}^t$, $x \in]u[^s$.

Proof. (1) Replacing $\mathcal{X} \rightarrow S$ with $\mathcal{X}^\circ \rightarrow S$, we may assume $\mathcal{X} \rightarrow S$ is smooth. For a connected étale cover $X' \rightarrow X$ tamely ramified along \mathcal{X}_s , let $\mathcal{X}' \rightarrow \mathcal{X}$ denote the normalization of \mathcal{X} in $X' \rightarrow X \rightarrow \mathcal{X}$ and consider the following conditions

- (i) $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_{(\bar{u})} \rightarrow \mathcal{X}_{(\bar{u})}$ admits a section;
- (ii) $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_{(\bar{\zeta})} \rightarrow \mathcal{X}_{(\bar{\zeta})}$ admits a section;
- (iii) $\mathcal{X}' \times_{\mathcal{X}} S_{x,(\bar{s}_x)} \rightarrow S_{x,(\bar{s}_x)}$ admits a section.

We are to show that (i) \Leftrightarrow (ii) \Rightarrow (iii) and that, if $x \in]u[^u$, one also has (iii) \Rightarrow (ii). From Abhyankar's lemma [Stacks, Tag 0BSE], for every connected étale cover $X' \rightarrow X$ tamely ramified along \mathcal{X}_s one gets a Cartesian diagram

$$\begin{array}{ccc} \bigsqcup_{1 \leq i \leq r} \text{spec}(A[T]/\langle T^{e_i} - \pi \rangle) & \longrightarrow & \mathcal{X}' \\ \downarrow & \square & \downarrow \\ \text{spec}(A) & \longrightarrow & \mathcal{X}_{(u)} \longrightarrow \mathcal{X} \end{array}$$

with $\text{spec}(A) \rightarrow \mathcal{X}_{(u)}$ a finite étale cover such that $\zeta = \pi \mathcal{O}_{\mathcal{X},u} \in \text{spec}(\mathcal{O}_{\mathcal{X},u})$ lifts uniquely to $\zeta_A \in \text{spec}(A)$ with $\zeta_A = \pi A$ and $e_1, \dots, e_r \geq 1$ some integers prime to p . As $\text{spec}(A) \rightarrow \mathcal{X}_{(u)}$ is a finite étale cover, the canonical morphisms $S_{x,(\bar{s}_x)} \rightarrow \mathcal{X}_{(u)}$, $\mathcal{X}_{(\bar{\zeta})} \rightarrow \mathcal{X}_{(u)}$ both factor through

$$\begin{array}{ccccccc} & & \text{spec}(A) & & & & \\ & \swarrow & \downarrow & \nwarrow & & & \\ S_{x,(\bar{s}_x)} & \longrightarrow & S_x & \longrightarrow & \mathcal{X}_{(u)} & \longleftarrow & \mathcal{X}_{(\bar{\zeta})} \longleftarrow \mathcal{X}_{(\bar{\zeta})} \end{array}$$

so that

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_{(\bar{u})} \simeq \bigsqcup_{1 \leq i \leq r} \operatorname{spec}(\mathcal{O}_{\mathcal{X}, \bar{u}}[T]/\langle T^{e_i} - \pi \rangle) \rightarrow \mathcal{X}_{(\bar{u})};$$

$$\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_{(\bar{c})} \simeq \bigsqcup_{1 \leq i \leq r} \operatorname{spec}(\mathcal{O}_{\mathcal{X}, \bar{c}}[T]/\langle T^{e_i} - \pi \rangle) \rightarrow \mathcal{X}_{(\bar{c})};$$

$$\mathcal{X}' \times_{\mathcal{X}} S_{x, (\bar{s}_x)} \simeq \bigsqcup_{1 \leq i \leq r} \operatorname{spec}(\mathcal{O}_{S_x, \bar{s}_x}[T]/\langle T^{e_i} - \pi \rangle) \rightarrow S_{x, (\bar{s}_x)}.$$

As π has not root in $\mathcal{O}_{\mathcal{X}, \bar{u}}$ nor $\mathcal{O}_{\mathcal{X}, \bar{c}}$, one has (i) \Leftrightarrow (ii) $\Leftrightarrow e_i = 1$ for some $i = 1, \dots, r$. On the other hand, if $e_x \geq 1$ denotes the ramification index of $k(x)/k$ that is $\pi = u_x \pi_x^{e_x}$ for some uniformizer $\pi_x \in \mathfrak{m}_x$ and $u_x \in \mathcal{O}_x^\times$, then $\mathcal{X}' \times_{\mathcal{X}} S_{x, (\bar{s}_x)} \rightarrow S_{x, (\bar{s}_x)}$ admits a section if and only if $e_i | e_x$ for some $i = 1, \dots, r$. As a result, one always has (i) \Leftrightarrow (ii) \Rightarrow (iii) and, if $x \in]u[$ viz $e_x = 1$ then (iii) \Rightarrow (i) \Leftrightarrow (ii).

- (2) Recall that $\mathcal{O}_{\mathcal{X}, \bar{u}} \simeq \mathcal{O}_{r, \bar{0}} := \mathcal{O}_{n, m, r, \bar{0}}$, and that $I_u^t = (\widehat{\mathbb{Z}}(p'))^m \simeq \operatorname{Gal}(k(\Theta_{r, \bar{0}})^t | k(\Theta_{r, \bar{0}}))$. Let $I_u^- \subset I_u^t$ denote the subgroup of I_u^t generated by the $I_{x, u}^t$, $x \in]u[$ and assume $I_u^- \subsetneq I_u^t$ so that there exists a subgroup $I_u^- \subset I_u^{-+} \subset I_u^t$ of prime index $[I_u^t : I_u^{-+}] = \ell \neq p$. Let $K = k(\Theta_{r, \bar{0}})(f^{\frac{1}{\ell}})/k(\Theta_{r, \bar{0}})$ denote the corresponding extension, where $f = \pi^{e_\pi} t_1^{e_1} \dots t_{m-1}^{e_{m-1}}$ for some integers $0 \leq e_\pi, e_1, \dots, e_{m-1} \leq \ell - 1$ and at least one of $e_\pi, e_1, \dots, e_{m-1} \geq 1$, and let $\Theta_K \rightarrow \Theta_{r, \bar{0}}$ denote the normalization of $\Theta_{r, \bar{0}}$ in $\operatorname{spec}(K) \rightarrow \operatorname{spec}(k(\Theta_{r, \bar{0}})) \rightarrow \Theta_{r, \bar{0}}$.

Again, we apply the strategy of Subsection 3.3.2.1. Define integers $k_1, \dots, k_m \geq 1$ as follows.

- (a) If $e_\pi \geq 1$: fix a prime $\ell_0 > \ell$ such that $r\ell_0 - \ell(m-1) > 0$ and set

$$(k_1, \dots, k_m) = (\ell, \dots, \ell, r\ell_0 - \ell(m-1)).$$

- (b) If $e_\pi = 0$: then, there exists $1 \leq i \leq m-1$ such that $e_i \geq 1$ - say $i = 1$; fix a prime $\ell_0 > \ell$ such that $r\ell_0 - \ell(m-2) > 0$ and set

$$(k_1, \dots, k_m) = (1, \ell, \dots, \ell, r\ell_0 - (1 + \ell(m-2))).$$

In both cases, set $k' := k^{\operatorname{ur}}(\pi^{\frac{1}{\ell_0}})$. The morphism of $\mathcal{O}_{k^{\operatorname{ur}}}$ -algebras

$$\begin{array}{ccc} \mathcal{O}_{k^{\operatorname{ur}}}[T_1, \dots, T_n] & \xrightarrow{(T_1, \dots, T_m, T_{m+1}, \dots, T_n) \mapsto (\pi^{\frac{k_1}{\ell_0}}, \dots, \pi^{\frac{k_m}{\ell_0}}, 0, \dots, 0)} & \mathcal{O}_{k'} \\ \downarrow & \nearrow & \\ \mathcal{O}_{\mathcal{X}, \bar{u}} \simeq \mathcal{O}_{r, \bar{0}} & & \end{array}$$

gives rise to a $\mathcal{O}_{k'}$ -point $\tilde{x}' \in \mathcal{X}(\mathcal{O}_{k'})$ hence to a k' -point $x' \in X(k')$

$$\begin{array}{ccccc} \Theta_K & \longrightarrow & \operatorname{spec}(\mathcal{O}_{\mathcal{X}, \bar{u}}[T]/\langle T^\ell - f \rangle) & \longrightarrow & \mathcal{X}_{(\bar{u})} \longleftarrow \tilde{x}' \operatorname{spec}(\mathcal{O}_{k'}) \\ & & \square & & \uparrow \\ & & & & X_{(\bar{u})} \\ & & & & \uparrow \\ \Theta_{K, x'} & \longrightarrow & \operatorname{spec}((\mathcal{O}_{\mathcal{X}, \bar{u}}[T]/\langle T^\ell - f \rangle) \otimes_{\mathcal{O}_{\mathcal{X}, \bar{u}}} k') & \longrightarrow & \operatorname{spec}(k'), \\ & & & & \uparrow \\ & & & & x' \end{array}$$

with

- In case (a):

$$(\mathcal{O}_{\mathcal{X}, \bar{u}}[T]/\langle T^\ell - f \rangle) \otimes_{\mathcal{O}_{\mathcal{X}, \bar{u}}} k' \simeq k'[T]/\langle T^\ell - \pi^{\frac{e_1 + \dots + e_{m-1}}{\ell_0}} \pi^{e_\pi} \rangle \simeq k'(\pi^{\frac{e_\pi}{\ell}}) \simeq k'(\pi^{\frac{1}{\ell}}) \not\subset k'^{\operatorname{ur}};$$

- In case (b):

$$(\mathcal{O}_{\mathcal{X}, \bar{u}}[T]/\langle T^\ell - f \rangle) \otimes_{\mathcal{O}_{\mathcal{X}, \bar{u}}} k' \simeq k'[T]/\langle T^\ell - \pi^{\frac{e_2 + \dots + e_{m-1}}{\ell_0}} \pi^{\frac{e_1}{\ell_0}} \rangle \simeq k'(\pi^{\frac{e_1}{\ell_0 \ell}}) \simeq k'(\pi^{\frac{1}{\ell}}) \not\subset k'^{\operatorname{ur}},$$

so that, in both cases, we get a contradiction. \square

3.4.2. *Proof of Corollary 2.* Assume first \mathcal{V}_ℓ is semistable with respect to $X \hookrightarrow \mathcal{X}$. As $\ell \neq p$, \mathcal{V}_ℓ is automatically tamely ramified with respect to $X \hookrightarrow \mathcal{X}$. The implication \Rightarrow then immediately follows from Lemma 23 (as a subgroup of a unipotent group is unipotent). Assume now that $|X|_{\mathcal{V}_\ell}^{\text{sst}} \supseteq |\mathcal{X}_s|^\text{t}$. As $\ell \neq p$, $|X|_{\mathcal{V}_\ell}^{\text{sst}} \subset |X|_{\mathcal{V}_\ell}^\text{t}$ hence from Theorem 1 (3) \Leftarrow and Remark 16, \mathcal{V}_ℓ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$. As for every $x_1 \in \mathcal{X}$ specializing to $x_0 \in X$, one has $I_{X,x_1} \subset I_{X,x_0}$, it is enough to prove that $\bar{I}_{X,u}^k$ acts unipotently on $V_\ell := \mathcal{V}_{\ell,\bar{\eta}}$ for every $u \in |\mathcal{X}_s|$. If $u \in \mathcal{X}_s^{\text{sm}}$, this follows from Lemma 23 (1) and the fact that, by formal smoothness of $\mathcal{X}^{\text{sm}} \rightarrow S$, one has $\mathcal{X}^{\text{sm}}(\mathcal{O}_{k^{\text{ur}}})_u \neq \emptyset$ hence, *a fortiori*, $]u]^\text{u} \neq \emptyset$. If $u \in \mathcal{X}_s^{\text{n-sm}}$, the assertion follows from Lemma 21 and Lemma 23 (2), using that an *abelian* subgroup of $\text{GL}(V_\ell)$ generated by unipotent elements is again unipotent.

Remark 24.

(1) Again, label by (i) the left hand side condition and by (ii)' the right-hand side condition of the equivalence in Corollary 2. The proof of Corollary 2 shows more precisely that

$$(i) \Rightarrow (ii)' \Rightarrow (ii)'' \Rightarrow (i),$$

where (ii)'' denotes the condition $|X|_{\mathcal{V}_\ell}^{\text{sst}} \supseteq |\mathcal{X}_s|^\text{u}$ if $\mathcal{X} \rightarrow S$ is smooth, and $|X|_{\mathcal{V}_\ell}^{\text{sst}} \supseteq |\mathcal{X}_s|^\text{t}$ in general.

(2) Lemma 23 also shows that, if $\rho : \pi_1(X) \rightarrow \Pi$ (resp. \mathcal{V}_ℓ) is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$, Theorem 1 (1) and Corollary 2 can be significantly strengthened. Namely,

(2.1) If X admits a smooth model $\mathcal{X} \rightarrow S$ and $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,m}$ denote the connected components of \mathcal{X}_s ,

$\rho : \pi_1(X) \rightarrow \Pi$ is unramified (resp. \mathcal{V}_ℓ is semistable) with respect to $X \hookrightarrow \mathcal{X}$

\Leftrightarrow for each $i = 1, \dots, m$ there exists $u_i \in |\mathcal{X}_{s,i}|$ such that $|X|_\rho^{\text{ur}} \cap]u_i]^\text{u} \neq \emptyset$ (resp. $|X|_{\mathcal{V}_\ell}^{\text{sst}} \cap]u_i]^\text{u} \neq \emptyset$).

(2.2) If X admits a semistable model $\mathcal{X} \rightarrow S$,

$\rho : \pi_1(X) \rightarrow \Pi$ is unramified (resp. \mathcal{V}_ℓ is semistable) with respect to $X \hookrightarrow \mathcal{X}$

\Leftrightarrow for each $u \in |\mathcal{X}_s^{\text{n-sm}}|$, $|X|_\rho^{\text{ur}} \supseteq]u]^\text{t}$ (resp. $|X|_{\mathcal{V}_\ell}^{\text{sst}} \supseteq]u]^\text{t}$).

From Theorem 1 (3) and Remark 16 (1), the condition that $\rho : \pi_1(X) \rightarrow \Pi$ (resp. \mathcal{V}_ℓ) is tamely ramified can be replaced by the pointwise condition that $|X|_\rho^{\text{ur}} \supseteq |\mathcal{X}_s|^\text{t}$ (resp. $|X|_{\mathcal{V}_\ell}^{\text{t}} \supseteq |\mathcal{X}_s|^\text{t}$).

4. FIRST APPLICATIONS OF THEOREM 1

4.1. **Automatic extension to compactifications.** Let k be a discrete valuation field as in Section 3 and let X be a smooth, geometrically connected variety over k .

Corollary 25. *Let Π be a profinite group and let $\rho : \pi_1(X) \rightarrow \Pi$ be a continuous morphism. Assume $|X|_\rho^{\text{ur}} = |X|$. Then for every smooth compactification $X_{k^s} \hookrightarrow X_{k^s}^{\text{cpt}}$, the morphism $\rho : \pi_1(X_{k^s}) \rightarrow \Pi$ factors through $\pi_1(X_{k^s}) \twoheadrightarrow \pi_1(X_{k^s}^{\text{cpt}})$.*

Proof. Note that $|X|_\rho^{\text{ur}} = |X|$ implies $|X'|_\rho^{\text{ur}} = |X'|$ for every surjective morphism $X' \rightarrow X$. Assume first X is a curve. After possibly replacing k by a finite field extension, one may assume X admits a semistable model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow \text{spec}(\mathcal{O}_k)$ in the sense that $\mathcal{X}^{\text{cpt}} \rightarrow \text{spec}(\mathcal{O}_k)$ is a semistable curve and $\mathcal{D} := \mathcal{X}^{\text{cpt}} \setminus \mathcal{X}$ is the union of $\text{deg}(X^{\text{cpt}} \setminus X)$ disjoint sections $\text{spec}(\mathcal{O}_k) \rightarrow \mathcal{X}^{\text{cpt}}$ [DM69], [Kn83]³ so that, from Theorem 1 (2) \Leftarrow , the morphism $\rho : \pi_1(X) \rightarrow \Pi$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X}^{\text{cpt}})$ hence, *a fortiori*, the morphism $\rho : \pi_1(X_{k^s}) \rightarrow \Pi$ factors through $\pi_1(X_{k^s}) \twoheadrightarrow \pi_1(X_{k^s}^{\text{cpt}})$. The general case reduces to the case of curves as follows. As all schemes are of finite type, up to replacing k by a finite field extension, we may and will assume the smooth compactification $X_{k^s} \hookrightarrow X_{k^s}^{\text{cpt}}$ over k^s descends to a smooth compactification $X \hookrightarrow X^{\text{cpt}}$ over k . From⁴ [Dr12, Cor. 5.2], if $\rho : \pi_1(X_{k^s}) \rightarrow \Pi$ does not factor through $\pi_1(X_{k^s}) \twoheadrightarrow \pi_1(X_{k^s}^{\text{cpt}})$, there exists $x \in |X_{k^s}^{\text{cpt}} \setminus X_{k^s}|$ and a line $\Delta \subset T_x X_{k^s}^{\text{cpt}}$ such that for every smooth connected curve C over k^s , closed point $c \in C$ and morphism $\alpha : C \rightarrow X_{k^s}^{\text{cpt}}$ satisfying

$$(*) \quad \alpha^{-1}(X_{k^s}) \neq \emptyset, \alpha(c) = x, \text{ and } T_c \alpha : T_c C \rightarrow T_x X_{k^s}^{\text{cpt}} \text{ induces an isomorphism } T_c \alpha : T_c C \xrightarrow{\sim} \Delta \subset T_x X^{\text{cpt}},$$

³Strictly speaking the references [DM69], [Kn83] only deal with the case of hyperbolic curves but, after possibly replacing k by a finite field extension, the case of non hyperbolic curves reduce to the case of \mathbb{P}_k^1 or of an elliptic curve, which are well-known.

⁴[Dr12, Cor. 5.2] is stated for regular schemes of finite type over $\mathbb{Z}[\frac{1}{p}]$ but the proof works as it is - and is even simpler - for smooth schemes over an algebraically closed field of characteristic 0.

$\alpha^* \rho : \pi_1(\alpha^{-1}(X_{k^s})) \xrightarrow{\alpha} \pi_1(X_{k^s}) \xrightarrow{\rho} \Pi$ does not factor through $\pi_1(\alpha^{-1}(X_{k^s})) \rightarrow \pi_1(\alpha^{-1}(X_{k^s}) \cup \{c\})$, where the notation are as in the following Cartesian square

$$\begin{array}{ccc} \alpha^{-1}(X_{k^s}) & \xrightarrow{\alpha} & X_{k^s} \\ \downarrow & \square & \downarrow \\ C & \xrightarrow{\alpha} & X_{k^s}^{\text{cpt}}. \end{array}$$

On the other hand, by Lemma 26 below, one can always construct a smooth projective connected curve C over k^s , a closed point $c \in C$ and a morphism $\alpha : C \rightarrow X_{k^s}^{\text{cpt}}$ satisfying (*). As all schemes are of finite type, up to replacing k by a finite field extension, one may assume $\alpha : C \rightarrow X_{k^s}^{\text{cpt}}$ descends to a morphism $\alpha_0 : C_0 \rightarrow X^{\text{cpt}}$ over k . From the case of curve, and as $|\alpha_0^{-1}(X)|_{\rho}^{\text{ur}} = |\alpha_0^{-1}(X)|$, $\alpha^* \rho : \pi_1(\alpha^{-1}(X_{k^s})) \rightarrow \Pi$ factors through $\pi_1(\alpha^{-1}(X_{k^s})) \rightarrow \pi_1(C^{\text{cpt}})$: a contradiction. \square

Lemma 26. *Let $k = k^s$ be an algebraically closed field, let X be a smooth variety over k , irreducible and of dimension $d \geq 2$, let $\emptyset \neq U \subsetneq X$ a non-empty open subscheme, $x \in X \setminus U$ and $\Delta \subset T_x X$ a line. Then there exists a smooth connected curve C over k , a closed point $c \in C$ and a morphism $\alpha : C \rightarrow X$ satisfying*

$$\alpha^{-1}(U) \neq \emptyset, \alpha(c) = x, \text{ and } T_c \alpha : T_c C \rightarrow T_x X \text{ induces an isomorphism } T_c \alpha : T_c C \xrightarrow{\sim} \Delta \subset T_x X,$$

Furthermore, Zariski-locally around x , $\alpha : C \rightarrow X$ is a closed embedding.

Proof. We begin with the following result of commutative algebra.

Claim: *Let R be a regular local ring with maximal ideal \mathfrak{m} , residue field $k = R/\mathfrak{m}$ and Krull dimension d . Fix a $d-1$ -dimension k -vector subspace $V \subset \mathfrak{m}/\mathfrak{m}^2$. Fix also an ideal $0 \subsetneq I \subset \mathfrak{m}$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ denote the minimal prime ideals of I . Then there exists $f_1, \dots, f_{d-1} \in \mathfrak{m}$ such that*

- (1) f_1, \dots, f_{d-1} are part of a regular system of parameters;
- (2) $\bar{f}_1, \dots, \bar{f}_{d-1} \in \mathfrak{m}/\mathfrak{m}^2$ generate the k -vector subspace V ;
- (3) $\sqrt{\mathfrak{p}_i, f_1, \dots, f_{d-1}} = \mathfrak{m}$, $i = 1, \dots, r$.

Proof of the claim We argue by induction on d (observe that, as $0 \neq I$, $d \geq 1$). If $d = 1$, R is a dvr and there is nothing to prove. Assume the claim holds for R of Krull dimension $d-1 \geq 1$. Fix $v_1, \dots, v_{d-1} \in \mathfrak{m}$ such that $\bar{v}_1, \dots, \bar{v}_{d-1} \in \mathfrak{m}/\mathfrak{m}^2$ is a k -basis of V . If $\sqrt{I} = \mathfrak{m}$, again, there is nothing to prove as, then, \mathfrak{m} is the only prime ideal of I . Assume $\sqrt{I} \subsetneq \mathfrak{m}$ that is $\mathfrak{p}_i \subsetneq \mathfrak{m}$, $i = 1, \dots, r$. In particular, $v_1 R + \mathfrak{m}^2 \not\subset \mathfrak{p}_i$, $i = 1, \dots, r$ and, by assumption, $v_1 R + \mathfrak{m}^2 \not\subset \mathfrak{m}^2$ so that, by [M80, I, B]

$$v_1 R + \mathfrak{m}^2 \not\subset \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r.$$

Pick $f_1 \in v_1 R + \mathfrak{m}^2$, $f_1 \notin \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$. As $\bar{f}_1 = \bar{v}_1 \neq 0$ in $\mathfrak{m}/\mathfrak{m}^2$, $R_1 := R/f_1$ is again a regular local ring with maximal ideal $\mathfrak{m}_1 := \mathfrak{m}/f_1$, residue field $R_1/\mathfrak{m}_1 = k$ and Krull dimension $d-1$. Also, $I_1 := (I + Rf_1)/f_1 \neq 0$. Indeed, otherwise, in $\text{spec}(R)$, one would have

$$\bigcup_{1 \leq i \leq r} V(\mathfrak{p}_i) = V(I) = V(f_1) = V(I) \cap V(f_1) = \bigcup_{1 \leq i \leq r} V(\mathfrak{p}_i, f_1).$$

On the other hand, for $i = 1, \dots, r$, by construction, $f_1 \notin \mathfrak{p}_i$ so that $V(\mathfrak{p}_i, f_1) \subsetneq V(\mathfrak{p}_i)$, hence, as $V(\mathfrak{p}_i)$ is irreducible, $\dim(V(\mathfrak{p}_i, f_1)) < \dim(V(\mathfrak{p}_i))$: a contradiction. One can thus apply the induction hypothesis to R_1 , I_1 and

$$V_1 := \text{im}(V \subset \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}/(\mathfrak{m}^2 + f_1) = \mathfrak{m}_1/\mathfrak{m}_1^2)$$

with k -basis $\bar{v}_{2,1}, \dots, \bar{v}_{d-1,1}$, where $v_{i,1}$ denotes the image of v_i via $\mathfrak{m} \rightarrow \mathfrak{m}_1/\mathfrak{m}_1^2$, $i = 2, \dots, d-1$, to get $f_2, \dots, f_{d-1} \in \mathfrak{m}$ whose images $\bar{f}_{2,1}, \dots, \bar{f}_{d-1,1} \in \mathfrak{m}_1$ satisfy condition (1), (2), (3) of the Claim for R_1 , I_1 , V_1 . The family f_1, f_2, \dots, f_{d-1} constructed in this way satisfies (1), (2), (3) of the Claim for R , I , V . \square

We derive Lemma 26 from the claim as follows. Set $Z := X \setminus U$, $R := \mathcal{O}_{X,x}$, $I := \ker(R \rightarrow \mathcal{O}_{Z,x}) \subset R$. Note that $I \neq 0$ as, otherwise, there would be a Zariski-open neighbourhood U_x of x in X such that $Z \cap U_x = U_x$ whence, taking Zariski-closure and using that X is irreducible, $Z = X$: a contradiction. By duality, the line $\Delta = k\delta \subset T_x X = \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ defines a k -vector subspace $V := \ker(\delta) \subset \mathfrak{m}/\mathfrak{m}^2$ of dimension $d-1$. Pick $f_1, \dots, f_{d-1} \in R$ as in the claim and an affine open neighbourhood $U_x = \text{spec}(A)$ of x in X such that $f_1, \dots, f_{d-1} \in R$ arise from elements $f_1, \dots, f_{d-1} \in A$. Set

$$C := \text{spec}(A/\langle f_1, \dots, f_{d-1} \rangle) \hookrightarrow U_x.$$

Up to shrinking U_x further, one may also assume:

$$- C \cap Z = \{x\};$$

- C is regular (hence smooth over k as $k = k^s$);
- C is irreducible.

Also, by construction,

$$\mathfrak{m} \supset \langle f_1, \dots, f_{d-1} \rangle = \ker(R \rightarrow \mathcal{O}_{C,x})$$

hence,

$$V = \ker(\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_{C,x}/\mathfrak{m}_{C,x}^2 = \mathfrak{m}/(\mathfrak{m}^2 + I))$$

and, by duality, $\Delta = \text{im}(T_x C \rightarrow T_x X)$. □

Example 27. Say that a smooth, geometrically connected variety X over k is *quasi-compactly geometrically simply connected over k* if there exists a morphism $X' \rightarrow X$ such that $\pi_1(X'_{k^s}) \rightarrow \pi_1(X_{k^s})$ is open and for some (equivalently, every) smooth compactification $X'_{k^s} \hookrightarrow X'_{k^s}{}^{\text{cpt}}$, $\pi_1(X'_{k^s}{}^{\text{cpt}})$ is finite. Typically, a connected open subvariety of a smooth, proper, connected and geometrically simply connected variety over k is quasi-compactly geometrically simply connected over k . Then, if X is a smooth, quasi-compactly geometrically simply connected over k , for every \mathbb{Q}_p -local system \mathcal{V}_p on X with $|X|_{\mathcal{V}_p}^{\text{ur}} = |X|$ (e.g. $|X|_{\mathcal{V}_p}^{\text{pot-ur}} \neq \emptyset$ and $\text{Lev}_2(\mathcal{V}_p)$ holds - see Subsection 5.2.3.2), $\mathcal{V}_p|_{X_{k^s}}$ is finite.

Remark 28.

- (1) Let k be a number field, let X be a smooth, geometrically connected variety over k and $\rho : \pi_1(X) \rightarrow \Pi$ a continuous morphism of profinite groups such that for one finite place v of k one has $|X_{k_v}|_{\rho}^{\text{ur}} = |X_{k_v}|$. Then, by invariance of étale fundamental group under extension of algebraically closed fields of characteristic 0 and Corollary 25, for every smooth compactification $X_{k^s} \hookrightarrow X_{k^s}{}^{\text{cpt}}$, $\mathcal{V}_p|_{X_{k^s}}$ extends to $X_{k^s}{}^{\text{cpt}}$.
- (2) Let k be a field of characteristic 0 and let X be a smooth, geometrically connected variety over k . For every continuous morphism $\rho : \pi_1(X) \rightarrow \Pi$ of profinite groups and smooth compactification $X \hookrightarrow X^{\text{cpt}}$ of X over k , it formally follows from the morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{k^s}) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & \pi_1(X_{k^s}{}^{\text{cpt}}) & \longrightarrow & \pi_1(X^{\text{cpt}}) & \longrightarrow & \pi_1(k) & \longrightarrow & 1 \end{array}$$

that if $\rho : \pi_1(X_{k^s}) \rightarrow \Pi$ factors through $\pi_1(X_{k^s}) \rightarrow \pi_1(X_{k^s}{}^{\text{cpt}})$ then $\rho : \pi_1(X) \rightarrow \Pi$ automatically factors through $\pi_1(X) \rightarrow \pi_1(X^{\text{cpt}})$.

- (3) Combining (1), (2) and Corollary 44, one gets the following evidence for Conjecture (T). Let k be a number field, X a smooth, geometrically connected variety over k and \mathcal{V}_p a \mathbb{Q}_p -local system on X . Assume $\text{Lev}_2(\mathcal{V}_p)$ holds and $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$. Then \mathcal{V}_p extends to every smooth compactification $X \hookrightarrow X^{\text{cpt}}$ of X over k .

4.2. Base-changing along étale covers. In the applications of Theorem 1, one usually is in a situation where one can only ensure $|X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X|$. If \mathcal{V}_p is crystalline, $|X|_{\mathcal{V}_p}^{\text{ur}} = |X|_{\mathcal{V}_p}^{\text{pot-ur}}$ but, in general, the inclusion $|X|_{\mathcal{V}_p}^{\text{ur}} \subset |X|_{\mathcal{V}_p}^{\text{pot-ur}}$ may be strict so that one has to replace \mathcal{V}_p on X with $\mathcal{V}_p|_{X'}$ on X' for some connected étale cover $X' \rightarrow X$ to ensure $|X'|_{\mathcal{V}_p}^{\text{ur}} = |X'|_{\mathcal{V}_p}^{\text{pot-ur}}$; however, replacing X with X' may affect some properties of X , e.g. that X admits a smooth NCC model over S . We discuss briefly a way to go around this difficulty and refer to Corollary 37 for an application.

Assume X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$; write $\mathcal{D} := \mathcal{X}^{\text{cpt}} \setminus \mathcal{X}$. Then, by Abhyankar's lemma

$$\pi_1^t(\mathcal{X}; \mathcal{X}_s) \simeq \pi_1^t(\mathcal{X}^{\text{cpt}}; \mathcal{D}) \times_{\pi_1(s)} \pi_1^t(S; s) \simeq \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \times_{\pi_1(s)} \pi_1^t(S; s)$$

while

$$\pi_1(\mathcal{X}) \simeq \pi_1^t(\mathcal{X}^{\text{cpt}}; \mathcal{D}) \simeq \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s).$$

- a) By Theorem 1 (3) \Leftarrow , if $\text{Lev}_3(\mathcal{V}_p)$ holds and $|X|_{\mathcal{V}_p}^{\text{tr}} \supset]\mathcal{X}_s[^s$, then the action of $\pi_1(X)$ on $\mathcal{V}_{p,\bar{x}}$ factors through $\pi_1(X) \rightarrow \pi_1(\mathcal{X}^{\text{cpt}})$.
- b) Say that a connected étale cover $X' \rightarrow X$ is *good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$* if the following holds. Let k' be the algebraic closure of k in the function field of X' and $\mathcal{X}' \rightarrow \mathcal{X}$, $\mathcal{X}'^{\text{cpt}} \rightarrow \mathcal{X}^{\text{cpt}}$ the normalization of \mathcal{X} , \mathcal{X}^{cpt} in $X' \rightarrow X \rightarrow \mathcal{X}$, $X' \rightarrow X \rightarrow \mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}}$ respectively. Then the resulting canonical sequence of morphisms $\mathcal{X}' \rightarrow \mathcal{X}'^{\text{cpt}} \rightarrow S' := \text{spec}(\mathcal{O}_{k'})$ is again a smooth NCC over S' . Say that an open subgroup $U \subset \pi_1(X)$ is good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ if the corresponding connected étale cover $X_U \rightarrow X$ is. The open subgroups

$$U_1 \times_{\pi_1(s)} U_2 \subset \pi_1^t(\mathcal{X}^{\text{cpt}}; \mathcal{D}) \times_{\pi_1(s)} \pi_1^t(S; s)$$

with $U_1 \subset \pi_1^t(\mathcal{X}^{\text{cpt}}; \mathcal{D})$, $U_2 \subset \pi_1^t(S; s)$ open subgroups form a cofinal family of open subgroups of

$$\pi_1^t(\mathcal{X}; \mathcal{X}_s) \simeq \pi_1^t(\mathcal{X}^{\text{cpt}}; \mathcal{D}) \times_{\pi_1(s)} \pi_1^t(S; s)$$

and, if X is a curve, the inverse images of these groups in $\pi_1(X)$ are good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$.

These observations also yield the following variant / strengthening of Theorem 1 (1) when X is a curve admitting a smooth NCC model over S .

Corollary 29. *Let X be a curve. Assume X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$. Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X such that $\text{Lev}_4(\mathcal{V}_p)$ holds. Assume $|X|_{\mathcal{V}_p}^{\text{pot-ur}} \neq \emptyset$. Then there exists a connected étale cover $X' \rightarrow X$, which is good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ and such that $\text{Lev}_3(\mathcal{V}_p|_{X'})$ holds. In particular, $\mathcal{V}_p|_{X'}$ extends to a \mathbb{Q}_p -local system on $\mathcal{X}'^{\text{cpt}}$, where $\mathcal{X}'^{\text{cpt}} \rightarrow \mathcal{X}^{\text{cpt}}$ denotes the normalization of \mathcal{X}^{cpt} in $X' \rightarrow X \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}}$.*

Proof. One has

$$\begin{aligned} |X|_{\mathcal{V}_p}^{\text{pot-ur}} \neq \emptyset &\Rightarrow |X| \stackrel{(1)}{=} |X|_{\mathcal{V}_p}^{\text{pot-ur}} \stackrel{(2)}{=} |X|_{\mathcal{V}_p}^{\text{tr}} \\ &\stackrel{(3)}{\Rightarrow} \text{the action of } \pi_1(X) \text{ on } \mathcal{V}_{p, \bar{x}} \text{ corresponding to } \mathcal{V}_p \text{ factors through } \pi_1(X) \twoheadrightarrow \pi_1^t(\mathcal{X}; \mathcal{X}_s). \end{aligned}$$

where (1) follows from Corollary 44, (2) from the assumption that $\text{Lev}_4(\mathcal{V}_p)$ holds and (3) from Theorem 1 (3) \Leftarrow . From Observation b) above, there exists a connected étale cover $X' \rightarrow X$ which is good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ and such that $\text{Lev}_1(\mathcal{V}_p|_{X'})$ - hence $\text{Lev}_3(\mathcal{V}_p|_{X'})$ hold. In particular, $|X'|_{\mathcal{V}_p}^{\text{ur}} = |X'|_{\mathcal{V}_p}^{\text{tr}} = |X'|$ so that the last part of the assertion follows from Observation a) above (or directly from Theorem 1 (2) \Leftarrow). \square

4.3. Geometric translations and enhancements - samples. Let $S = \text{spec}(\mathcal{O}_k)$ be the spectrum of a discrete valuation ring with closed point s as in Section 3. For a profinite group Π and a set \mathbb{L} of prime numbers, write $\Pi \twoheadrightarrow \Pi^{\mathbb{L}}$ for the pro- \mathbb{L} completion of Π , *viz* the maximal profinite quotient of Π with order divisible only by primes in \mathbb{L} .

4.3.1. Proof of Corollary 4. We retain the notation of Subsection 1.2.1.

- Let $f : Y \rightarrow X$ be an abelian scheme. Fix a prime $\ell \neq p$ and consider the \mathbb{Q}_ℓ -local system $\mathcal{V}_\ell := R^1 f_* \mathbb{Q}_\ell$. The assumption that Y_x has semistable reduction for every $x \in \text{im}(\mathcal{X}(\mathcal{O}_{k^t}) \rightarrow |X|)$ ensures that

$$|X|_{\mathcal{V}_\ell}^{\text{tr}} \supset |X|_{\mathcal{V}_\ell}^{\text{sst}} \supset \mathcal{X}_s[\ell^t]$$

hence, by Theorem 1 (3) plus Remark 16, that \mathcal{V}_ℓ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$. But, then, we are in the setting of Remark 24 (2), and hence \mathcal{V}_ℓ is actually unramified with respect to $X \hookrightarrow \mathcal{X}$. This implies that for every $x \in |X|$, $x^* \mathcal{V}_\ell$ is unramified and the conclusion follows from the Serre-Tate criterion [SeT68].

- Let $f : Y \rightarrow X$ be a smooth proper and geometrically connected curve of genus ≥ 2 . Fix a prime $\ell \neq p$ and consider

- the \mathbb{Q}_ℓ -local system $\mathcal{V}_\ell := R^1 f_* \mathbb{Q}_\ell$;
- the outer pro- ℓ representation $\rho_\ell : \pi_1(X) \rightarrow \Pi := \text{Out}(\pi_1(X_{\bar{\eta}})^{\{\ell\}})$.

Again, the assumption that Y_x has semistable reduction for every $x \in \text{im}(\mathcal{X}(\mathcal{O}_{k^t}) \rightarrow |X|)$ ensures that

$$|X|_{\mathcal{V}_\ell}^{\text{tr}} \supset |X|_{\mathcal{V}_\ell}^{\text{sst}} \supset \mathcal{X}_s[\ell^t]$$

hence, by Theorem 1 (3) plus Remark 16, that \mathcal{V}_ℓ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$. As Π fits into a $\pi_1(X)$ -equivariant extension of profinite groups

$$1 \rightarrow K \rightarrow \Pi \rightarrow \text{GL}(\mathcal{V}_{\ell, \bar{\eta}}) \rightarrow 1$$

with K a pro- ℓ group [As95], the fact that \mathcal{V}_ℓ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$ in turn ensures that $\rho_\ell : \pi_1(X) \rightarrow \Pi$ is tamely ramified with respect to $X \hookrightarrow \mathcal{X}$. By Remark 24 (2), $\rho_\ell : \pi_1(X) \rightarrow \Pi$ is thus unramified with respect to $X \hookrightarrow \mathcal{X}$ and the conclusion follows from Oda-Tamagawa criterion [O90], [O95], [T97, Prop. (0.8)].

4.3.2. Proof of Corollary 4. For every profinite group Π , consider the functor

$$H_{\Pi}^{\text{rep}} := \text{Hom}_{\pi_1(S)}(\pi_1(-), \Pi) : \text{Sch}/S \rightarrow \text{Sets}.$$

Let $\mathcal{X} \rightarrow S$ be flat, separated, and of finite type, with generic fiber X . Another way to phrase Theorem 1 (1) and Remark 16 is the following. Assume $\mathcal{X} \rightarrow S$ is semistable (resp. smooth), then the canonical set-theoretic square

$$(13) \quad \begin{array}{ccc} H_{\Pi}^{\text{rep}}(X) & \longrightarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\Pi}^{\text{rep}}(x) \\ \uparrow & \square & \uparrow \\ H_{\Pi}^{\text{rep}}(\mathcal{X}) & \longrightarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\Pi}^{\text{rep}}(S_x) \end{array}$$

is Cartesian for $\# = \text{t}$ (resp. $\# = \text{ur}$)⁵. One may ask if the corresponding canonical square

$$(14) \quad \begin{array}{ccc} H_{\mathcal{M}}(X) & \longrightarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\mathcal{M}}(x) \\ \uparrow & \square & \uparrow \\ H_{\mathcal{M}}(\mathcal{X}) & \longrightarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\mathcal{M}}(S_x) \end{array}$$

obtained by replacing $H_{\Pi}^{\text{rep}}(-) : \text{Sch}/S \rightarrow \text{Sets}$ by a geometric functor

$$H_{\mathcal{M}} := \text{Hom}_S(-, \mathcal{M}) : \text{Sch}/S \rightarrow \text{Sets}$$

for $\mathcal{M} \rightarrow S$ a sufficiently nice algebraic stack, is Cartesian as well. If the functor $H_{\mathcal{M}} : \text{Sch}/S \rightarrow \text{Sets}$ is "anabelian enough", one may hope that this is indeed the case. The condition of being "anabelian enough" may depend on the stack \mathcal{M} but, typically, one could ask for the existence of a morphism of profinite groups $\rho : \pi_1(\mathcal{M}) \rightarrow \Pi$ such that \mathcal{M} endowed with $\rho : \pi_1(\mathcal{M}) \rightarrow \Pi$ is smooth- (resp. semistable-) admissible⁶ in the sense that for every smooth (resp. semistable) $\mathcal{X} \rightarrow S$ the square (1) in Diagram (15) below is Cartesian.

$$(15) \quad \begin{array}{ccccccc} H_{\mathcal{M}}(X) & \xrightarrow{\quad} & H_{\Pi}^{\text{rep}}(X) & \longrightarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\Pi}^{\text{rep}}(x) & \longleftarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\mathcal{M}}(x) \\ \uparrow & & \uparrow & \square & \uparrow & & \uparrow \\ H_{\mathcal{M}}(\mathcal{X}) & \xrightarrow{\quad} & H_{\Pi}^{\text{rep}}(\mathcal{X}) & \longrightarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\Pi}^{\text{rep}}(S_x) & \longleftarrow & \prod_{x \in]\mathcal{X}_s[^\#} H_{\mathcal{M}}(S_x), \end{array}$$

(The horizontal arrows in (1) and (2) are those arising from the morphism of functors

$$H_{\mathcal{M}} \xrightarrow{\pi_1(-)} \text{Hom}_{\pi_1(S)}(\pi_1(-), \pi_1(\mathcal{M})) \xrightarrow{\rho} H_{\Pi}^{\text{rep}}.)$$

Clearly, if \mathcal{M} is endowed with a morphism of profinite groups $\rho : \pi_1(\mathcal{M}) \rightarrow \Pi$ such that it becomes smooth (resp. semistable)-admissible, then the fact that (13) is Cartesian for $\# = \text{u}$ (resp. $\# = \text{t}$) immediately implies that for every smooth (resp. semistable) $\mathcal{X} \rightarrow S$ the square (14) is Cartesian for $\# = \text{u}$ (resp. $\# = \text{t}$).

Examples:

- (1) Pointed smooth proper curves: Let $\mathcal{M} = \mathcal{M}_{g,n}/S$ denote the stack of smooth, proper, geometrically connected curves of genus g with n disjoint ordered sections and $2 - 2g - n < 0$, and $\rho : \pi_1(\mathcal{M}) \rightarrow \Pi$ be the universal outer pro- \mathbb{L} representation

$$\rho_{\mathbb{L}} : \pi_1(\mathcal{M}_{g,n}) \rightarrow \Pi := \text{Out}(\pi_1(\mathcal{C}_{\bar{\eta}} \setminus \{s_{1,\bar{\eta}}, \dots, s_{n,\bar{\eta}}\})^{\mathbb{L}}),$$

where $\mathbb{L} := \{\ell_1, \ell_2\}$ for two distinct primes $\ell_1 \neq \ell_2$ and $(\mathcal{C}, s_1, \dots, s_n)/\mathcal{M}_{g,n}$ is the universal pointed curve. Then $\mathcal{M}_{g,n} \rightarrow S$ endowed with $\rho_{\mathbb{L}} : \pi_1(\mathcal{M}_{g,n}) \rightarrow \text{Out}(\pi_1(\mathcal{C}_{\bar{\eta}} \setminus \{s_{1,\bar{\eta}}, \dots, s_{n,\bar{\eta}}\})^{\mathbb{L}})$ is semistable-admissible [St05, Thm. 2.10. B]. In particular, for every smooth (resp. semistable) $\mathcal{X} \rightarrow S$ and smooth, proper, geometrically connected curve $C \rightarrow X$ of genus $g \geq 2$, the following holds. Assume for every $x \in]\mathcal{X}_s[^\text{u}$ (resp. every $x \in]\mathcal{X}_s[^\text{t}$), C_x has good reduction. Then $C \rightarrow X$ extends to a smooth, proper, geometrically connected curve of genus g over \mathcal{X} .

- (2) Canonical integral models: Let (G, \mathfrak{X}) be a Shimura datum and $K \subset G(\mathbb{A}_f)$ be a neat compact open subgroup; write $M := \text{Sh}_K(G, \mathfrak{X})$ for the corresponding Shimura variety, defined over the reflex field $E = E(G, \mathfrak{X})$. Then every connected component M° of M comes equipped with a canonical (up to choice of base points) continuous morphism of profinite groups $\rho : \pi_1(M^\circ) \rightarrow K$; for every prime ℓ , let $\rho_\ell : \pi_1(M^\circ) \xrightarrow{\rho} K \rightarrow K_\ell$ denote its

⁵If one restricts to the functor $H_{\Pi}^{\text{rep}, \text{t}} := \text{Hom}_{\pi_1(S)}(\pi_1^\dagger(-; (-)_s), \Pi) : \text{Sch}/S \rightarrow \text{Sets}$ of representations which are tamely ramified with respect to $X \hookrightarrow \mathcal{X}$ then, by Remark 24 (2) one can replace $] \mathcal{X}_s [^\#$ with $] \mathcal{X}_s^{\text{n-sm}} [^\text{t}$ (resp. with $\{x_1, \dots, x_m\}$ for some $x_i \in]u_i[^\text{u}$ and $u_i \in \mathcal{X}_{s,i}$, $i = 1, \dots, m$).

⁶We warn the reader that our terminology diverges from the one of [BST24].

ℓ th component. For $p \neq \ell$ and a place $v|p$ of E , write $k := E_v$. An integral canonical model of M_k over $S := \text{spec}(\mathcal{O}_k)$ can be defined as a smooth model $M \hookrightarrow \mathcal{M} \rightarrow S$ such that for every connected component M° of M the representation $\rho_\ell : \pi_1(M^\circ) \rightarrow K_\ell$ factors through $\pi(M^\circ) \twoheadrightarrow \pi_1(\mathcal{M}^\circ)$ and $\mathcal{M}^\circ \rightarrow S$ endowed with $\rho_\ell : \pi_1(\mathcal{M}^\circ) \rightarrow K_\ell$ is smooth-admissible. Such models always exist for $p \gg 0$ [BST24] and it is expected that they exist as soon as $K_p \subset G(\mathbb{Q}_p)$ is hyperspecial; this is known if (G, \mathfrak{X}) is of Abelian type [Kot92], [Ki10]. For instance, the moduli space $\mathcal{A}_{g,n}$ of g -dimensional principally polarized abelian varieties with full level n -structure with $n \geq 3$ over $\mathbb{Z}[\frac{1}{n}]$ yields a canonical integral model $\mathcal{M} := \mathcal{A}_{g,n,\mathbb{Z}_p} \rightarrow \text{spec}(\mathbb{Z}_p)$ of $M := \mathcal{A}_{g,n,\mathbb{Q}_p}$ for every $p \nmid n$ hence the following holds. Let k/\mathbb{Q}_p be a p -adic field with $p \nmid n$, $\mathcal{X} \rightarrow S$ a smooth morphism with geometrically connected generic fiber X and $A \rightarrow X$ a principally polarized abelian scheme with full level- n structure. Let $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,m}$ denote the connected components of \mathcal{X}_s . Assume for every $i = 1, \dots, m$ there exists $u_i \in \mathcal{X}_{s,i}$ and $x_i \in]u_i[^\mathfrak{u}$ such that A_x has good reduction *viz* extends to an abelian scheme over S_x (note that, if so, the principal polarization and full level- n structure automatically extend over S_x as well). Then $A \rightarrow X$ extends to an abelian scheme over \mathcal{X} .

- (3) Abelian varieties: If one does not impose conditions on the level structure, Example (2) still holds at least provided k has absolute ramification index $e \leq p-1$. Indeed, let again $\mathcal{X} \rightarrow S$ be a smooth morphism with geometrically connected generic fiber X and $f : A \rightarrow X$ an abelian scheme. Let $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,m}$ denote the connected components of \mathcal{X}_s and let ζ_1, \dots, ζ_m their generic points. If for every $x \in]\mathcal{X}_s[^\mathfrak{u}$, A_x has good reduction then $R^1 f_* \mathbb{Q}_\ell$ extends to a \mathbb{Q}_ℓ -local system on \mathcal{X} hence, $\eta^* R^1 f_* \mathbb{Q}_\ell \simeq R^1 f_{\eta*} \mathbb{Q}_\ell$ extends to a \mathbb{Q}_ℓ -local system on $\mathcal{O}_{\mathcal{X},\zeta_i}$, $i = 1, \dots, r$. By the Serre-Tate criterion [SeT68] and spreading out, for each $i = 1, \dots, r$ there exists a Zariski open neighbourhood $\mathcal{U}_i \subset \mathcal{X}$ of ζ_i in \mathcal{X} such that $f : A \rightarrow X$ extends to an abelian scheme $\mathcal{A}_i \rightarrow \mathcal{U}_i$. By the uniqueness in the Neron model property $\mathcal{A}_i \rightarrow \mathcal{U}_i$, $i = 1, \dots, r$ and $A \rightarrow X$ glue to an abelian scheme $\mathcal{A} \rightarrow \mathcal{U}$ on

$$\mathcal{U} := X \cup \bigcup_{1 \leq i \leq r} \mathcal{U}_i.$$

If $e \leq p-1$, by [VZ10, Cor. 5], \mathcal{X} is healthy regular in the sense of [VZ10, Def. 1] hence $\mathcal{A} \rightarrow \mathcal{U}$ extends further to an abelian scheme over \mathcal{X} . The restriction on the absolute ramification index $e \leq p-1$ is optimal, as shown by the counter-examples in [VZ10, Thm. 28 (i)]. More precisely, consider the following properties:

- (i) for every $x \in]\mathcal{X}_s[^\mathfrak{s}$, A_x extends to an abelian scheme over S_x ;
- (ii) for every $x \in]\mathcal{X}_s[^\mathfrak{u}$, A_x extends to an abelian scheme over S_x ;
- (iii) there exists an open subscheme $X \subset \mathcal{U} \subset \mathcal{X}$ containing all the generic points of \mathcal{X}_s such that $A \rightarrow X$ extends to an abelian scheme over \mathcal{U} ;
- (iv) $A \rightarrow X$ extends to an abelian scheme over \mathcal{X} .

Then our argument shows (i) \Rightarrow (ii) \Rightarrow (iii) $\xrightarrow{e \leq p-1}$ (iv) \Rightarrow (i). But, as \mathcal{X} is regular, it follows from Zariski-Nagata purity that $\pi_1(\mathcal{X} \setminus \mathcal{U}) \xrightarrow{\sim} \pi_1(\mathcal{X})$ so that, actually, one also has (iii) \Rightarrow (i). In other words, one cannot expect a monodromy criterion like [St05, Thm. 2.10. B] to hold for abelian schemes in general, even when $\mathcal{X} \rightarrow S$ is smooth.

- (4) Stable proper curves: The same argument as in Example (3) but invoking Corollary 2 (and using the Grothendieck-Deligne-Mumford criterion [DM69, Thm. (2.4)], [SGA7] in place of the Serre-Tate criterion and [DM69, Thm. 1.11] in place of the uniqueness in the Neron model property) yields, for instance: for every smooth $\mathcal{X} \rightarrow S$ and stable, proper, geometrically connected curve $C \rightarrow X$ of genus $g \geq 2$, the following holds. Assume for every $x \in]\mathcal{X}_s[^\mathfrak{u}$ C_x has semistable reduction *viz* extends to a stable curve over S_x . Then there exists an open subscheme $X \subset \mathcal{U} \subset \mathcal{X}$ containing all the generic points of \mathcal{X}_s such that $C \rightarrow X$ extends to a stable curve over \mathcal{U} .

5. COMBINATORIAL π_1 AND p -ADIC VARIANTS OF CONJECTURE (T)

In this section, we discuss some results around the analogue of Conjecture (T) and Conjecture (U) over p -adic fields and give (rather straightforward) applications to Conjecture (T). As recalled in the introduction and Appendix B, all the implications of Conjecture (T) and Conjecture (U) but (iii) \Rightarrow (ii) are known. When one replaces the number field k with a p -adic field, the implication (ii) \Rightarrow (i) fails. In Subsection 5.1, we explain that this failure is precisely measured by the combinatorial part of the étale fundamental group of the special fiber of a (any) semistable NCC model $X \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ (when it exists).

5.1. Intrinsic description of arithmetic combinatorial π_1 - proof of Corollary 7. We retain the notation and assumptions of Subsection 1.2.3 and perform the proof of Corollary 7, which is a consequence of Theorem 1 (2) \Leftarrow .

The existence and surjectivity of the morphism $\pi_1^{\text{comb}}(X) \rightarrow \pi_1^{\text{comb}}(X^{\text{cpt}})$ follows from the tautological inclusion $|X| \subset |X^{\text{cpt}}|$ and the existence and surjectivity of the morphism $\pi_1^{\text{comb}}(X^{\text{cpt}}) \rightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \emptyset)$ follows from

$$|X^{\text{cpt}}| \leftarrow \mathcal{X}^{\text{cpt}}(\mathcal{O}_{k^s}) \rightarrow \bigcup_{1 \leq i \leq t} |\mathcal{X}_{s,i}^{\text{cpt}}|.$$

The existence and surjectivity of the morphism $\pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \rightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \emptyset)$ follows from the canonical commutative functorial diagrams

$$\begin{array}{ccc} \pi_1^t(\mathcal{X}_{s,i}^{\text{cpt}}; \mathcal{D}_{s,i}) & \longrightarrow & \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s), \quad i = 1, \dots, t. \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{X}_{s,i}^{\text{cpt}}) & \longrightarrow & \pi_1(\mathcal{X}_s^{\text{cpt}}) \end{array}$$

Eventually, by the valuative criterion of properness, every $x \in |X|$ induces a unique $\tilde{x} : S_x \rightarrow \mathcal{X}^{\text{cpt}}$; let $I_x \subset \{1, \dots, r\}$ denote the subset of those i such that $\tilde{x} : S_x \rightarrow \mathcal{X}^{\text{cpt}}$ specializes to some closed point on $\mathcal{X}_{i,s}^{\text{cpt}}$. The existence and surjectivity of the morphism $\pi_1^{\text{comb}}(X) \rightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)$ then follows from the following canonical commutative functorial diagrams

$$\begin{array}{ccc} \pi_1(x) & \longrightarrow & \pi_1^t(S_x; \tilde{x}^* \mathcal{D}) \quad , \quad i \in I_x, x \in |X|. \\ \downarrow & & \downarrow \\ \pi_1(X) & \longrightarrow & \pi_1^t(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) \longleftarrow \pi_1^t(\mathcal{X}_{s,i}^{\text{cpt}}; \mathcal{D}_{s,i}) \end{array}$$

It remains to prove that $\pi_1^{\text{comb}}(X) \rightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \emptyset)$ is an isomorphism. For this, it is enough to prove that the canonical morphism $\pi_1(X) \rightarrow \pi_1^{\text{comb}}(X)$ factors as

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1^{\text{comb}}(X) . \\ \downarrow & \nearrow & \\ \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \emptyset) & & \end{array}$$

From Theorem 1 (2) \Leftarrow applied to $\pi_1(X) \rightarrow \pi_1^{\text{comb}}(X)$, one gets a canonical factorization

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1^{\text{comb}}(X) \\ \downarrow & \nearrow & \\ \pi_1(\mathcal{X}_s^{\text{cpt}}) & & \end{array}$$

For every $i = 1, \dots, r$, and $u \in \mathcal{X}_{s,i} \cap \mathcal{X}_s^{\text{sm}}$ with residue field $\kappa(u)$, let $k \subset k_u \subset k^{\text{ur}}$ denote the unique intermediate field extension corresponding to $\kappa \subset \kappa(u) \subset \bar{\kappa}$ via $\text{Gal}(k^{\text{ur}}|k) \simeq \pi_1(S) \leftarrow \pi_1(\kappa)$. By formal smoothness, u lifts to some $\tilde{x} : \text{spec}(\mathcal{O}_{k_u}) \rightarrow \mathcal{X}^{\text{sm}}$ giving rise to a closed point $x \in |X|$ which specializes to u on $\mathcal{X}_{s,i}$ hence, by Cebotarev, the image of

$$\begin{array}{ccccc} \pi_1(\mathcal{X}_{s,i} \cap \mathcal{X}_s^{\text{sm}}) & \longrightarrow & \pi_1(\mathcal{X}_s^{\text{cpt}}) & \longrightarrow & \pi_1^{\text{comb}}(X) \\ & \searrow & \uparrow & & \\ & & \pi_1(\mathcal{X}_{s,i}^{\text{cpt}}) & & \end{array}$$

is trivial. This concludes the proof for Corollary 7.

Remark 30. From Remark 16, the proof of Corollary 7 also implies that $K(X) \subset \pi_1(X)$ is the subgroup topologically normally generated by the images of the canonical morphisms $\pi_1(x) \rightarrow \pi_1(X)$ for $x \in]\mathcal{X}_s^{\text{cpt}}[{}^t \cap |X|$.

5.2. p -adic variant of Conjecture (T). Let $S = \text{spec}(\mathcal{O}_k)$ be the spectrum of a the ring of integers of a p -adic field k and let s denote its closed. Let X be a smooth, geometrically connected variety over k . Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . One has the following implications

$$|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset \xrightarrow{\text{Fact 43 (1)}} |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset \text{ and } \mathcal{V}_p \text{ is Hodge-Tate} \xrightarrow{\text{Par. A.1.2}} |X|_{\mathcal{V}_p}^{\text{pot-ur}} \neq \emptyset \xrightarrow{\text{Fact 43 (1)}} |X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X|.$$

In particular, if one of the following holds

$$(C-1) \quad |X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset \text{ and } \text{Lev}_2(\mathcal{V}_p) \text{ holds;}$$

$$(C-2) \quad |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset, \mathcal{V}_p \text{ is Hodge-Tate and } \text{Lev}_2(\mathcal{V}_p) \text{ holds;}$$

$$(C-3) \quad |X|_{\mathcal{V}_p^{\text{uni}}} \neq \emptyset \text{ and } |X|_{\mathcal{V}_p^{\text{cris}}} = |X|,$$

then one automatically has $|X|_{\mathcal{V}_p^{\text{ur}}} = |X|$.

Assume X admits a semistable NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ over S . If $|X|_{\mathcal{V}_p^{\text{ur}}} = |X|$, it follows from Theorem 1 (2) \Leftrightarrow that \mathcal{V}_p extends to a \mathbb{Q}_p -local system $\tilde{\mathcal{V}}_p$ on \mathcal{X}^{cpt} so that, one can consider its restriction $\tilde{\mathcal{V}}_p|_{\mathcal{X}_s^{\text{cpt}}}$ and try and exploit the theory of companions. To do so, we recall first the relation between \mathbb{Q}_p -local systems and (over)convergent F -isocrystals on $\mathcal{X}_s^{\text{cpt}}$.

5.2.1. *\mathbb{Q}_p -local systems versus (over)convergent F -isocrystals.* Let κ be a finite field of characteristic $p > 0$ and let X be a smooth, separated and geometrically connected curve over κ . Let $X \hookrightarrow X^{\text{cpt}}$ denote its smooth compactification. Let $\text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)$, $\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)$ denote respectively the categories of convergent and overconvergent F -isocrystals on $X/W(\kappa)$ with scalar extended from $W(\kappa)[\frac{1}{p}]$ to $\overline{\mathbb{Q}}_p$ [A18, 1.4, 2.14 et seq.]. From [Ke04, Thm. 1.1], there is a fully faithful⁷ exact \otimes -functor

$$\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p) \rightarrow \text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p).$$

Let

$$\text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)^0 \subset \text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)$$

denote the full subcategory of unit-root (*viz* isoclinic of slope 0) convergent F -isocrystals on $X/\overline{\mathbb{Q}}_p$ and

$$\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)^0 \subset \text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)$$

the full subcategory of unit-root overconvergent ones *viz* of those objects in $\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)$ whose image in $\text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)$ lies in $\text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)^0$. From [K73, Prop. 4.1.1], [Cr87, 2.2, Thm.] there is a canonical equivalence of Tannakian categories

$$\text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)^0 \xrightarrow{\sim} \text{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1(X))$$

which restricts to an equivalence of Tannakian categories ([Ts98, Thm. 7.2.3], [Sh11, Prop. 4.2])

$$\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)^0 \xrightarrow{\sim} \text{Rep}_{\overline{\mathbb{Q}}_p}^\dagger(\pi_1(X))$$

onto the full subcategory $\text{Rep}_{\overline{\mathbb{Q}}_p}^\dagger(\pi_1(X)) \subset \text{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1(X))$ of potentially unramified representations. These equivalences preserve characteristic polynomials of Frobenii on both sides.

5.2.2. *Q -rationality.* Let Q be a field; for a polynomial $P \in Q[T]$ with roots $\alpha_1, \dots, \alpha_r \in \overline{Q}$, and every integer m , set

$$P^{(m)}(T) := \prod_{1 \leq i \leq r} (T - \alpha_i^m) \in Q[T].$$

For a potentially unramified \mathbb{Q}_p -local system \mathcal{V}_p on $x = \text{spec}(k)$, and every lifts $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \pi_1(x)$ of the Frobenius of $\pi_1(s_x) \simeq \text{Gal}(k(x)^{\text{ur}}|k(x))$ the roots of the characteristic polynomials $\tilde{\chi}_1, \tilde{\chi}_2 \in \mathbb{Q}_p[T]$ of $\tilde{\varphi}_1, \tilde{\varphi}_2$ acting on $\mathcal{V}_{p, \bar{x}}$ differ by roots of unity of order bounded by the order of

$$\text{im}(\pi_1(x) \rightarrow \text{GL}(\mathcal{V}_{p, \bar{x}}^\circ) \rightarrow \text{GL}(\mathcal{V}_{p, \bar{x}}^\circ/p)),$$

where \mathcal{V}_p° is any \mathbb{Z}_p -model of \mathcal{V}_p . If $N(\mathcal{V}_p)$ denotes the order of the subgroup of $\mathbb{G}_m(\overline{\mathbb{Q}}_p)_{\text{tors}}$ generated by all these roots of unity (for all pairs of lifts $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \pi_1(x)$ of the Frobenius of $\pi_1(s_x)$) then, for every lift $\tilde{\varphi}_x \in \pi_1(x)$ of the Frobenius of $\pi_1(s_x)$ with characteristic polynomial

$$\tilde{\chi} = \prod_{1 \leq i \leq r} (T - \alpha_i) \in \mathbb{Q}_p[T],$$

the polynomial $\tilde{\chi}^{(N)} \in \mathbb{Q}_p[T]$ depends only on \mathcal{V}_p and not on the lift $\tilde{\varphi}_x$. By a slight abuse of notation, we write, again $\chi_{\mathcal{V}_p} := \tilde{\chi}^{(N)}$ for the "characteristic polynomial of Frobenius attached to \mathcal{V}_p ". When \mathcal{V}_p is unramified, $N = 1$ hence one recovers the usual definition.

Let now X be a smooth, geometrically connected variety over k and let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . For a subextension $\mathbb{Q} \subset Q \subset \mathbb{Q}_p$, and an integer $w \in \mathbb{Z}$, write

$$|X|_{\mathcal{V}_p^{\text{pot-ur}}} \supset |X|_{\mathcal{V}_p^Q} \supset |X|_{\mathcal{V}_p^{Q, w}}$$

⁷Actually, we will only apply these facts when $X = X^{\text{cpt}}$, in which case $\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p) \rightarrow \text{Isoc}^\varphi(X/\overline{\mathbb{Q}}_p)$ is an equivalence. But, to emphasize the relevance of the fact that \mathbb{Q}_p -local systems at stake extend to the smooth compactification we do not make these assumptions here.

for the subset of all $x \in |X|_{\mathcal{V}_p}^{\text{pot-ur}}$ such that the corresponding characteristic polynomial of Frobenius χ_x lies in $Q[T]$, and such that the corresponding characteristic polynomial of Frobenius χ_x lies in $Q[T]$ and is pure of weight w respectively. One says that \mathcal{V}_p is Q -rational (resp. Q -rational and pure of weight w) if $|X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X|_{\mathcal{V}_p}^Q$ (resp. $|X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X|_{\mathcal{V}_p}^{Q,w}$).

The following observation shows that the existence of a trivial point forces $\overline{\mathbb{Q}}$ -rationality and purity.

Corollary 31. *One has $|X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}}} \neq \emptyset$ if and only if $|X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}}} = |X|$ and, for every integer $w \in \mathbb{Z}$, $|X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}},w} \neq \emptyset$ if and only if $|X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}},w} = |X|$.*

Proof. To prove Corollary 31, one may freely replace X by a connected étale cover; in particular one may assume $\text{Lev}_2(\mathcal{V}_p)$ holds so that $|X'|_{\mathcal{V}_p}^{\text{ur}} = |X'|_{\mathcal{V}_p}^{\text{pot-ur}}$ for every connected étale cover $X' \rightarrow X$. Assume first X is a curve. After possibly replacing k by a finite field extension, one may assume X admits a semistable model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ as in the proof of Corollary 25 so that, from Theorem 1 (2) \Leftarrow , \mathcal{V}_p extends to a \mathbb{Q}_p -local system $\tilde{\mathcal{V}}_p$ on \mathcal{X}^{cpt} . Let $\mathcal{X}_{s,1}^{\text{cpt}}, \dots, \mathcal{X}_{s,t}^{\text{cpt}}$ denote the irreducible components of $\mathcal{X}_s^{\text{cpt}}$. We are to prove that $\tilde{\mathcal{V}}_p|_{\mathcal{X}_s}$ is $\overline{\mathbb{Q}}$ -rational or, equivalently, that $\mathcal{V}_{p,i} := \tilde{\mathcal{V}}_p|_{\mathcal{X}_{s,i}}$ is $\overline{\mathbb{Q}}$ -rational, $i = 1, \dots, t$. Let $x \in |X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}}}$ (resp. $x \in |X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}},w}$); by the valuative criterion of properness, it extends to $\tilde{x} : S_x \rightarrow \mathcal{X}^{\text{cpt}}$ and specializes to a closed point $x_s \in |\mathcal{X}_s^{\text{cpt}}|$. Fix $1 \leq i \leq t$ be such that $x_s \in \mathcal{X}_{s,i}^{\text{cpt}}$. To prove that $\mathcal{V}_{p,i}$ is $\overline{\mathbb{Q}}$ -rational one may freely replace it by its semisimplification hence assume $\mathcal{V}_{p,i}$ is semisimple so that it decomposes as a direct sum

$$\mathcal{V}_{p,i} = \bigoplus_{1 \leq j \leq r_i} \mathcal{S}_{p,i,j}$$

of simple $\overline{\mathbb{Q}}_p$ -local systems, each of them being a twist $\mathcal{S}_{p,i,j} = \mathcal{S}_{p,i,j}^{\circ(\alpha_{i,j})}$ of a simple $\overline{\mathbb{Q}}_p$ -local system $\mathcal{S}_{p,i,j}^{\circ}$ with finite determinant. Here, we use the projectivity of $\mathcal{X}_{s,i}$ to ensure that the image of $\pi_1(\mathcal{X}_{s,i}) \rightarrow \pi_1(\mathcal{X}_{s,i})^{\text{ab}}$ is finite (e.g. [D80, Proof of (1.3.1)]). Let $\mathfrak{S}_{p,i,j}$ denote the (automatically simple) unitroot overconvergent F-isocrystal corresponding to $\mathcal{S}_{p,i,j}^{\circ}$ via the \otimes -equivalence

$$\text{Isoc}^{\varphi,\dagger}(X/\overline{\mathbb{Q}}_p)^0 \xrightarrow{\sim} \text{Rep}_{\overline{\mathbb{Q}}_p}^{\dagger}(\pi_1(X)).$$

From [A18], $\mathfrak{S}_{p,i,j}$ is $Q_{i,j}$ -rational for some number field $Q_{i,j}$ and pure of weight 0, hence so is $\mathcal{S}_{p,i,j}^{\circ}$. On the other hand, the existence of $x_s \in \mathcal{X}_{s,i}$ ensures that $\alpha_{i,j} \in \overline{\mathbb{Q}}$ (resp. and is pure of weight w). This shows $\mathcal{S}_{p,i,j}$ is $Q_{i,j}(\alpha_{i,j}) \subset \overline{\mathbb{Q}}$ -rational (resp. and pure of weight w), $j = 1, \dots, r_i$. Hence $\mathcal{V}_{p,i}$ is $\overline{\mathbb{Q}}$ -rational (resp. and pure of weight w). But then, for every $1 \leq i \neq j \leq t$ such that $\mathcal{X}_{s,j}^{\text{cpt}} \cap \mathcal{X}_{s,i}^{\text{cpt}} \neq \emptyset$, one has $|\mathcal{X}_{s,j}^{\text{cpt}}|_{\mathcal{V}_{p,j}}^{\overline{\mathbb{Q}}} \neq \emptyset$ (resp. $|\mathcal{X}_{s,j}^{\text{cpt}}|_{\mathcal{V}_{p,j}}^{\overline{\mathbb{Q}},w} \neq \emptyset$) and one can apply the same argument to $\mathcal{V}_{p,j}$ on $\mathcal{X}_{s,j}^{\text{cpt}}$. The conclusion follows from the fact that \mathcal{X}_s is connected. The general case follows from the curve case, using that, by Bertini, for every $x_0 \in |X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}}}$ (resp. $x_0 \in |X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}},w}$) and closed point $x \in |X|$, and up to possibly replacing further k by a finite field extension, one can always find a smooth geometrically connected curve C over k and a morphism $\alpha : C \rightarrow X$ such that $x_0, x \in \alpha(C)$. \square

Remark 32. It is unclear whether one can expect the conclusion of Lemma 31 to hold with $|X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}}} \neq \emptyset$ replaced by $|X|_{\mathcal{V}_p}^Q \neq \emptyset$ for Q a number field unless X admits a semistable NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ (in which case, one can apply directly the above argument to \mathcal{V}_p on X resorting to [AE19, Comment after Thm. 0.3] instead of [A18]).

In particular, if \mathcal{V}_p on X satisfies one of the assumption (C-1), (C-2), (C-3), then $|X| = |X|_{\mathcal{V}_p}^{\overline{\mathbb{Q}},w}$.

5.2.3. \mathbb{Q} -rational \mathbb{Q}_p -local systems. If one furthermore imposes that the \mathbb{Q}_p -local system \mathcal{V}_p on X is \mathbb{Q} -rational, then the strongest possible local variants of Conjectures (T), (U) hold. This is the content of Corollary 34 and Corollary 36 below.

5.2.3.1. A key lemma. Let κ be a finite field of characteristic $p > 0$ and let X be a smooth, separated and geometrically connected curve over κ . Let $X \hookrightarrow X^{\text{cpt}}$ denote its smooth compactification.

Lemma 33. *Let \mathcal{V}_ℓ be a \mathbb{Q}_ℓ -local system on X such that $|X|_{\mathcal{V}_\ell}^{\text{uni}} \neq \emptyset$. If $\ell = p$, assume furthermore \mathcal{V}_p is potentially unramified. Then*

- (1) either $|X|_{\mathcal{V}_\ell}^{\text{uni}} = |X|$ and $\mathcal{V}_\ell|_{X_{\bar{\kappa}}}$ is finite;
- (2) or $|X|_{\mathcal{V}_\ell}^{\text{uni}}$ is finite.

If $\ell = p$, and \mathcal{V}_p is \mathbb{Q} -rational, then we are in case (1).

Proof. First, if the claim holds if \mathcal{V}_ℓ is semisimple then it holds in general. Indeed consider a Jordan-Holder filtration

$$\mathcal{V}_{\ell,0} = 0 \subsetneq \mathcal{V}_{\ell,1} \subsetneq \cdots \subsetneq \mathcal{V}_{\ell,r-1} \subsetneq \mathcal{V}_{\ell,r} = \mathcal{V}_\ell$$

and set $\mathcal{S}_{\ell,i} := \mathcal{V}_{\ell,i}/\mathcal{V}_{\ell,i-1}$, $i = 1, \dots, r$ for its simple graded pieces so that

$$\mathcal{V}_\ell^{\text{ss}} \simeq \bigoplus_{1 \leq i \leq r} \mathcal{S}_{\ell,i}$$

By construction

$$|X|_{\mathcal{V}_\ell^{\text{ss}}}^{\text{uni}} = |X|_{\mathcal{V}_\ell}^{\text{uni}} \neq \emptyset$$

and $\mathcal{V}_\ell^{\text{ss}}$ is \mathbb{Q} -rational if \mathcal{V}_ℓ is. In particular, $|X|_{\mathcal{V}_\ell^{\text{ss}}}^{\text{uni}}$ is finite if and only if $|X|_{\mathcal{V}_\ell}^{\text{uni}}$ is finite and, if $\ell = p$ and \mathcal{V}_p is \mathbb{Q} -rational, $\mathcal{V}_\ell^{\text{ss}}|_{X_{\bar{\kappa}}}$ is finite. It remains to prove that $\mathcal{V}_\ell^{\text{ss}}|_{X_{\bar{\kappa}}}$ is finite (if and) only if $\mathcal{V}_\ell|_{X_{\bar{\kappa}}}$ is finite. Up to replacing X by a connected étale cover, one may assume $\mathcal{V}_\ell^{\text{ss}}|_{X_{\bar{\kappa}}}$ is trivial *viz* $\mathcal{S}_{\ell,i}|_{X_{\bar{\kappa}}}$ is trivial, $i = 1, \dots, r$. It is then enough to prove that if

$$0 \rightarrow \mathcal{F}'_\ell \rightarrow \mathcal{F}_\ell \rightarrow \mathcal{F}''_\ell \rightarrow 0$$

is an extension of \mathbb{Q}_ℓ -local systems on X with (*) $\mathcal{F}'_\ell|_{X_{\bar{\kappa}}}$, $\mathcal{F}''_\ell|_{X_{\bar{\kappa}}}$ trivial and \mathcal{F}'_ℓ , \mathcal{F}''_ℓ unipotent, then

$$0 \rightarrow \mathcal{F}'_\ell|_{X_{\bar{\kappa}}} \rightarrow \mathcal{F}_\ell|_{X_{\bar{\kappa}}} \rightarrow \mathcal{F}''_\ell|_{X_{\bar{\kappa}}} \rightarrow 0$$

splits. But the class of such an extension is an element of

$$\mathrm{H}^1(X_{\bar{\kappa}}, \mathcal{F}''_\ell \vee \otimes \mathcal{F}'_\ell)^{\varphi_\kappa} \xleftarrow{\simeq} (\mathrm{H}^1(X_{\bar{\kappa}}, \mathbb{Q}_\ell) \otimes \mathrm{H}^0(X_{\bar{\kappa}}, \mathcal{F}''_\ell \vee \otimes \mathcal{F}'_\ell))^{\varphi_\kappa} \simeq 0,$$

where the second isomorphism follows the fact $\mathrm{H}^0(X_{\bar{\kappa}}, \mathcal{F}''_\ell \vee \otimes \mathcal{F}'_\ell)$ is pure of weight 0 by (*) while $\mathrm{H}^1(X_{\bar{\kappa}}, \mathbb{Q}_\ell)$ is pure of weight 1. By a straightforward induction, this shows $\mathcal{V}_\ell|_{X_{\bar{\kappa}}}$ is trivial.

So, assume \mathcal{V}_ℓ is semisimple.

- If $\ell = p$, let \mathfrak{V}_p denote the (automatically semisimple) unitroot overconvergent⁸ F -isocrystal corresponding to $\mathcal{V}_p, \overline{\mathbb{Q}}_p$ via the \otimes -equivalence

$$\mathrm{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)^0 \xrightarrow{\sim} \mathrm{Rep}_{\overline{\mathbb{Q}}_p}^\dagger(\pi_1(X)).$$

- If $\ell \neq p$, fix an isomorphism $\tau_{\ell,p} : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ and let $\mathfrak{V}_p := \tau_{\ell,p} \mathcal{V}_\ell$ denote the unique semisimple overconvergent F -isocrystal which is the $\tau_{\ell,p}$ -companion of \mathcal{V}_ℓ [L02], [A18].

Write

$$\mathfrak{V}_p = \bigoplus_{1 \leq i \leq r} \mathfrak{S}_{p,i}$$

for its direct sum decomposition into simple summands in $\mathrm{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)$. As

$$|X|_{\mathfrak{S}_{p,i}}^{\text{uni}} \supset |X|_{\mathfrak{V}_p}^{\text{uni}} = |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset,$$

$\mathfrak{S}_{p,i}$ has finite determinant (to see this, if $\ell \neq p$, use that the image of $\pi_1(X_{\bar{\kappa}}) \rightarrow \pi_1(X)^{\text{ab}}$ is the product of a pro- p group by a finite group and, if $\ell = p$, use that, after possibly replacing X by a connected étale cover, one may assume \mathcal{V}_p extends to X^{cpt} and use that the image of $\pi_1(X_{\bar{\kappa}}^{\text{cpt}}) \rightarrow \pi_1(X^{\text{cpt}})^{\text{ab}}$ is finite - see the proof of Corollary 31), $i = 1, \dots, r$.

The companion correspondance induces an action with finite orbits of $\mathrm{Aut}(\overline{\mathbb{Q}}_p)$ on the set of isomorphism classes of simple objects with finite determinant in $\mathrm{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)$. Let $\mathfrak{V}_{p,1} := \mathfrak{V}_p, \dots, \mathfrak{V}_{p,s}$ denote the finitely many (up to isomorphism) semisimple companions of \mathfrak{V}_p . By construction, the overconvergent F -isocrystal

$$\mathfrak{F}_p := \mathfrak{V}_{p,1} \oplus \cdots \oplus \mathfrak{V}_{p,s}$$

is semisimple, \mathbb{Q} -rational, each of its simple summand has finite determinant and one has

$$|X|_{\mathfrak{V}_{p,1}}^{\text{uni}} = \cdots = |X|_{\mathfrak{V}_{p,s}}^{\text{uni}}.$$

In particular, if for every $x \in |X|$, $\chi_x \in \mathbb{Q}[T]$ denotes the characteristic polynomial of Frobenius attached to $x^* \mathfrak{F}_p$, then ii) $\chi_x \in \mathbb{Q}[T]$ and, as every simple summand of \mathfrak{F}_p has finite determinant, i) χ_x is pure of weight 0 [A18].

Let $U \subset X$ denote the largest (non-empty) open subscheme over which \mathfrak{F}_p admits a slope filtration (in the category of convergent F -isocrystals) [K79, Thm. 2.3.1, 2.4.2]

$$0 = S_0(\mathfrak{F}_p|_U) \subsetneq S_1(\mathfrak{F}_p|_U) \subsetneq \cdots \subsetneq S_s(\mathfrak{F}_p|_U) = \mathfrak{F}_p|_U,$$

with

$$\mathrm{Gr}_i^S(\mathfrak{F}_p|_U) := S_i(\mathfrak{F}_p|_U)/S_{i-1}(\mathfrak{F}_p|_U)$$

of slope q_i and $q_1 < \cdots < q_s$. We distinguish two cases:

⁸This is to ensure the overconvergence of \mathfrak{V}_p that we assume \mathcal{V}_p is potentially unramified.

- At least one of the q_i is $\neq 0$, which forces $|X|_{\mathcal{V}_\ell}^{\text{uni}} (= |X|_{\mathfrak{F}_p}^{\text{uni}} \subset |X|_{\mathfrak{F}_p}^{\text{uni}}) \subset X \setminus U$;
- $\mathfrak{F}_p|_U$ is unit-root. By semicontinuity of the slope filtration [K79, Thm. 2.3.1], this imposes $U = X$ and \mathfrak{F}_p is unit-root. In particular, for every $x \in |X|$, iv) $\chi_x \in \overline{\mathbb{Z}}_p[T]$. Eventually, the fact that every simple summand of \mathfrak{F}_p has finite determinant implies that for every prime $\ell \neq p$ and field isomorphism $\tau : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, the unique semisimple τ -companion ${}^\tau \mathfrak{F}_p$ of \mathfrak{F}_p is étale; in particular, for every $x \in |X|$, iii) $\chi_x \in \overline{\mathbb{Z}}_\ell[T]$.
 - If $\ell = p$, let \mathcal{F}_p denote the potentially unramified $\overline{\mathbb{Q}}_p$ -local system corresponding to \mathfrak{F}_p via

$$\text{Isoc}^{\varphi, \dagger}(X/\overline{\mathbb{Q}}_p)^0 \xrightarrow{\sim} \text{Rep}_{\overline{\mathbb{Q}}_p}^{\dagger}(\pi_1(X)).$$

- If $\ell \neq p$, set $\mathcal{F}_\ell := \tau_{\ell, p}^{-1} \mathfrak{F}_p$.

We have just shown that for every $x \in |X|$, the characteristic polynomial χ_x of the Frobenius $\varphi_{x, \ell} : \mathcal{F}_{\ell, \bar{x}} \xrightarrow{\sim} \mathcal{F}_{\ell, \bar{x}}$ satisfies

$$\text{i) } \chi_x \text{ is pure of weight } w = 0; \quad \text{ii) } \chi_x \in \mathbb{Q}[T]; \quad \text{iii) } \chi_x \in \overline{\mathbb{Z}}_{\ell'}[T], \ell' \neq p; \quad \text{iv) } \chi_x \in \overline{\mathbb{Z}}_p[T],$$

hence is a product of cyclotomic polynomials thus, up to replacing X by a connected étale cover, one may assume $\chi_x = (T-1)^r$, $x \in |X|$. By Cebotarev, this implies \mathcal{F}_ℓ - hence *a fortiori* $\mathcal{V}_{\ell, \overline{\mathbb{Q}}_\ell}$, is quasi-unipotent - hence finite (since $\mathcal{F}_\ell, \mathcal{V}_{\ell, \overline{\mathbb{Q}}_\ell}$ are semisimple)⁹.

If $\ell = p$ and \mathcal{V}_p is \mathbb{Q} -rational, and if $\chi_x \in \mathbb{Z}_p[T]$ denotes the characteristic polynomial of Frobenius attached to $x^* \mathcal{V}_p$ then ii), iv) are automatic, and i), iii) follows from the companion correspondance as above. \square

5.2.3.2. \mathbb{Q} -rational Hodge-Tate \mathbb{Q}_p -local systems. Let X be a smooth hyperbolic curve over k . One says that a morphism of profinite groups $\pi_1(X) \rightarrow \Pi$ is *combinatorial* (resp. *geometrically combinatorial*) if X admits a semistable NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ satisfying (3) and such that the morphism $\pi_1(X) \rightarrow \Pi$ is combinatorial (resp. geometrically combinatorial) with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ in the sense that it factors through $\pi_1(X) \twoheadrightarrow \pi_1^{\text{comb}}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)$ (resp. that its restriction $\pi_1(X_{\bar{k}}) \hookrightarrow \pi_1(X) \rightarrow \Pi$ factors through $\pi_1(X_{\bar{k}}) \twoheadrightarrow \pi_1^{\text{comb}}(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}})$). One says that a morphism of profinite groups $\pi_1(X) \rightarrow \Pi$ is *quasi-combinatorial* (resp. *quasi-geometrically combinatorial*), if it becomes combinatorial (resp. geometrically combinatorial) after possibly replacing X by a connected étale cover. In particular, one says that a \mathbb{Q}_p local system \mathcal{V}_p on X is combinatorial, quasi-combinatorial *etc.* if the corresponding representation of profinite groups $\pi_1(X) \twoheadrightarrow \Pi_p \subset \text{GL}(\mathcal{V}_{p, \bar{x}})$ is.

Corollary 34. *Let X be a smooth hyperbolic curve over k and let \mathcal{V}_p be a \mathbb{Q} -rational \mathbb{Q}_p -local system on X . Then,*

$$\begin{array}{ll} \text{(T)} \quad \text{(i)'} & \mathcal{V}_p \text{ is quasi-geometrically combinatorial and } |X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset; \\ & \Leftrightarrow \text{(ii) } |X|_{\mathcal{V}_p}^{\text{triv}} = |X|; \\ & \Leftrightarrow \text{(iii) } |X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset. \end{array} \quad \begin{array}{ll} \text{(U)} \quad \text{(i)'} & \mathcal{V}_p \text{ is quasi-geometrically combinatorial and } |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset; \\ & \Leftrightarrow \text{(ii) } |X|_{\mathcal{V}_p}^{\text{uni}} = |X|; \\ & \Leftrightarrow \text{(iii) } |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset. \end{array}$$

Proof. We treat the assertion for (T); the assertion for (U) is exactly similar after observing that $|X|_{\mathcal{V}_p}^{\text{uni}} = |X|_{\mathcal{V}_p^{\text{ss}}}^{\text{uni}}$ and that $|X|_{\mathcal{V}_p^{\text{ss}}}^{\text{uni}} \neq \emptyset$ imposes that $\mathcal{V}_p^{\text{ss}}$ is Hodge-Tate. The implication (ii) \Rightarrow (iii) is straightforward. The implication (i)' \Rightarrow (ii) is by definition of "geometrically combinatorial". Indeed if (i)' holds, after possibly replacing X by a connected étale cover, the representation $\pi_1(X) \twoheadrightarrow \Pi_p \subset \text{GL}(\mathcal{V}_{p, \bar{x}})$ corresponding to \mathcal{V}_p factors through

$$\begin{array}{ccc} \pi_1(X) & \twoheadrightarrow & \Pi_p \\ \downarrow & & \uparrow \\ \pi_1^{\dagger}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s) & \twoheadrightarrow & \pi_1^{\dagger}(\mathcal{X}_s^{\text{cpt}}; \mathcal{D}_s)/K(\mathcal{X}_{\bar{s}}^{\text{cpt}}; \mathcal{D}_{\bar{s}}) \simeq \pi_1(\Gamma(\mathcal{X}_s))^{\wedge} \times \pi_1(\kappa) \end{array}$$

so that, in particular, for every $x \in |X|$, the representation $\pi_1(x) \rightarrow \pi_1(X) \twoheadrightarrow \Pi_p$ corresponding to $x^* \mathcal{V}_p$ identifies with

$$\pi_1(x) \subset \pi_1(k) \twoheadrightarrow \pi_1(\kappa) \subset \pi_1(\Gamma(\mathcal{X}_s))^{\wedge} \times \pi_1(\kappa) \twoheadrightarrow \Pi_p$$

hence $G_{p, x}^{\circ}$ is independent of x . We prove the implication (iii) \Rightarrow (i)'. Replacing X by a connected étale cover one may assume, $\text{Lev}_2(\mathcal{V}_p)$ holds so that $|X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X|_{\mathcal{V}_p}^{\text{ur}} = |X|_{\mathcal{V}_p}^{\mathbb{Q}}$, and condition (C-1) of Subsection holds so that $|X|_{\mathcal{V}_p}^{\text{ur}} = |X|$. Replacing k by a finite field extension, one may assume X admits a semistable NCC compactification $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ over S such that the irreducible components $\mathcal{X}_{s,1}, \dots, \mathcal{X}_{s,t}$ of \mathcal{X}_s are geometrically connected over κ

⁹For this part of the argument, see also [Kos17, Prop. 1.1].

and the points in $\mathcal{X}_s^{\text{n-sm}}$ are all κ -rational. By Theorem 1 (2) \Leftarrow , \mathcal{V}_p extends to a \mathbb{Q}_p -local system $\tilde{\mathcal{V}}_p$ on \mathcal{X}^{cpt} . Let $x \in]\mathcal{X}_s[^{\text{triv}}$; assume x specializes to $x_s \in \mathcal{X}_{s,1}^{\text{cpt}}$ so that

$$|\mathcal{X}_{s,1}^{\text{cpt}}|_{\tilde{\mathcal{V}}_p}^{\text{triv}} \neq \emptyset.$$

From Lemma 33, $\tilde{\mathcal{V}}_p|_{\mathcal{X}_{s,1}^{\text{cpt}}}$ is then finite - hence trivial as Π_p is torsion-free, and

$$|\mathcal{X}_{s,1}^{\text{cpt}}|_{\tilde{\mathcal{V}}_p}^{\text{triv}} = |\mathcal{X}_{s,1}^{\text{cpt}}|.$$

In particular, for every $2 \leq i \leq t$ such that $\mathcal{X}_{s,i}^{\text{cpt}}$ intersects $\mathcal{X}_{s,1}^{\text{cpt}}$, one has $|\mathcal{X}_{s,i}^{\text{cpt}}|_{\tilde{\mathcal{V}}_p}^{\text{uni}} \neq \emptyset$ hence, iterating the argument and using that $\mathcal{X}_s^{\text{cpt}}$ is connected, one gets that $\tilde{\mathcal{V}}_p|_{\mathcal{X}_{s,i}^{\text{cpt}}}$ is trivial, $i = 1, \dots, t$. \square

Remark 35.

- (1) The equivalent condition (i)', (ii), (iii) in (T) are also equivalent to \mathcal{V}_p is quasi-combinatorial.
- (2) Under additional assumptions - e.g. \mathcal{V}_p is crystalline and X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ over S with $\Gamma(\mathcal{X}_s) = 1$, one can replace the condition (i)' by the original conditions (i) (namely, $G_p^\circ = 1$ in case (T) and G_p° is unipotent in case (U)) in the statement of Corollary 34 - see Corollary 37 for details.

If X is an arbitrary connected variety over k , combining Chow's lemma and Bertini's theorem, for every $x, x' \in |X|$ one can always find a sequence of morphisms $\alpha_i : C_i \rightarrow X$, $i = 1, \dots, r$ with C_i a smooth irreducible curve over k , $x \in \alpha_1(C_1)$, $x' \in \alpha_r(C_r)$ and $C_i \times_X C_{i+1} \neq \emptyset$, $i = 1, \dots, r-1$. Hence Corollary 34 implies:

Corollary 36. *Let X be a smooth connected variety over k and let \mathcal{V}_p be a \mathbb{Q} -rational \mathbb{Q}_p -local system on X . Then,*

$$(T) \quad |X|_{\mathcal{V}_p}^{\text{triv}} = |X| \Leftrightarrow |X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset, \quad (U) \quad |X|_{\mathcal{V}_p}^{\text{uni}} = |X| \Leftrightarrow |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset.$$

5.2.4. *A weak local variant of Conjecture (T).* Using the same arguments as in the proof of Corollary 34 but without assuming \mathbb{Q} -rationality, one can still prove a weak local variant of Conjecture (T), at least when X is a curve admitting a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$, and under mild assumptions on \mathcal{V}_p .

Corollary 37. *Let X be a curve admitting a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ and let \mathcal{V}_p be a \mathbb{Q}_p -local system on X satisfying one of the following:*

- a) $\mathcal{X} = \mathcal{X}^{\text{cpt}}$ and \mathcal{V}_p is crystalline;
- b) \mathcal{V}_p is Hodge-Tate and $\text{Lev}_4(\mathcal{V}_p)$ holds (e.g. $p > \text{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) + 1$).

Let

$$sp : |\mathcal{X}^{\text{cpt}}| \rightarrow |\mathcal{X}_s^{\text{cpt}}|$$

denote the specialization map. Then there exists a 0-dimensional Zariski-closed subset $\mathcal{Z}_s \subset \mathcal{X}_s^{\text{cpt}}$ such that

$$\begin{aligned} & (i) \quad G_p^\circ \text{ is unipotent;} \\ & \Leftrightarrow (ii) \quad |X|_{\mathcal{V}_p}^{\text{uni}} = |X|; \\ & \Leftrightarrow (iii)' \quad |X|_{\mathcal{V}_p}^{\text{uni}} \not\subset sp^{-1}(|\mathcal{Z}_s|). \end{aligned}$$

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii)' are straightforward. We prove the implication (iii)' \Rightarrow (i).

- In case a), (C-3) holds so that $|X| = |X|_{\mathcal{V}_p}^{\text{ur}}$ and from Theorem 1 (1) \Leftarrow , \mathcal{V}_p extends to a \mathbb{Q}_p -local system $\tilde{\mathcal{V}}_p$ on $\mathcal{X} = \mathcal{X}^{\text{cpt}}$.
- In case b), \mathcal{V}_p is Hodge-Tate and $|X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$, hence $|X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X|$. As $\text{Lev}_4(\mathcal{V}_p)$ holds, one can fix a connected étale cover $X' \rightarrow X$ as in Corollary 29 so that $\mathcal{V}_p|_{X'}$ extends to a \mathbb{Q}_p -local system $\tilde{\mathcal{V}}'_p$ on $\mathcal{X}'^{\text{cpt}}$. From the canonical chain of morphisms arising from specialization [SGA1, X]

$$(16) \quad \begin{array}{ccccccc} \pi_1(\mathcal{X}'^{\text{cpt}}) & \xrightarrow{\cong} & \pi_1(\mathcal{X}'^{\text{cpt}}) & \longleftarrow & \pi_1(X'^{\text{cpt}}) & \longleftarrow & \pi_1(X') \xrightarrow[\triangleleft]{\text{open}} \pi_1(X) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \pi_1(\mathcal{X}_s'^{\text{cpt}}) & \longleftarrow & \pi_1(X_{k_s}'^{\text{cpt}}) & \longleftarrow & \pi_1(X_{k_s}') & \xrightarrow[\triangleleft]{\text{open}} & \pi_1(X_{k_s}) \end{array}$$

one gets that

- \mathcal{V}_p is unipotent if and only if $\tilde{\mathcal{V}}'_p|_{\mathcal{X}_s'^{\text{cpt}}}$ is unipotent;

Also, one can freely replace \mathcal{Z}_s by a larger 0-dimensional closed subscheme of \mathcal{X}_s . In particular, for every 0-dimensional closed subscheme $\mathcal{Z}'_s \subset \mathcal{X}_s'^{\text{cpt}}$ with image $\mathcal{Z}_s \subset \mathcal{X}_s^{\text{cpt}}$ via $\mathcal{X}_s'^{\text{cpt}} \rightarrow \mathcal{X}_s^{\text{cpt}}$, up to replacing $\mathcal{Z}'_s \subset \mathcal{X}_s'^{\text{cpt}}$ with the inverse image $\mathcal{Z}''_s (\supset \mathcal{Z}'_s)$ of \mathcal{Z}_s in $\mathcal{X}_s'^{\text{cpt}}$, one has:

- $|X|_{\mathcal{V}_p}^{\text{uni}} \not\subset sp^{-1}(|\mathcal{Z}_s|)$ if and only if $|X'|_{\mathcal{V}_p}^{\text{uni}} \not\subset sp^{-1}(|\mathcal{Z}'_s|)$.

So, in both cases, and without loss of generality, one may assume $\mathcal{X} = \mathcal{X}^{\text{cpt}}$ and \mathcal{V}_p extends to a \mathbb{Q}_p -local system $\tilde{\mathcal{V}}_p$ on \mathcal{X} with $|\mathcal{X}_s|_{\tilde{\mathcal{V}}_p}^{\text{uni}} \neq \emptyset$ and we are to show that $\tilde{\mathcal{V}}_p|_{\mathcal{X}_s}$ is finite. From Lemma 33, only two cases can occur:

- $|\mathcal{X}_s|_{\tilde{\mathcal{V}}_p}^{\text{uni}} = |\mathcal{X}_s|$ and $\tilde{\mathcal{V}}_p|_{\mathcal{X}_s}$ is finite;
- $|\mathcal{X}_s|_{\tilde{\mathcal{V}}_p}^{\text{uni}}$ is finite.

In the former case, one can take $\mathcal{Z}_s = \emptyset$. In the latter case, one takes $\mathcal{Z}_s = |\mathcal{X}_s|_{\tilde{\mathcal{V}}_p}^{\text{uni}}$. \square

5.3. Applications to Conjecture (T) and Conjecture (U) over number fields. In this Section, unless otherwise mentioned, k is a number field. For a finite place v of k , let $k \hookrightarrow k_v$ denote the completion of k at v . If X is a variety over k and $x \in X(k)$ a k -point, let $x_v : \text{spec}(k_v) \rightarrow \text{spec}(k) \xrightarrow{x} X$ denote the induced k_v -point on X .

5.3.1. Conjecture (T) for \mathbb{Q} -compatible families. We first recall the definition of \mathbb{Q} -compatibility.

For a \mathbb{Q}_ℓ -local system \mathcal{V}_ℓ on $x = \text{spec}(k)$, let $U_{\mathcal{V}_\ell} \subset |\text{spec}(\mathcal{O}_k)|$ be the set of all finite places v of k such that, writing $p := p_v$ for the residue characteristic of v , the following holds:

- If $\ell \neq p$, $x_v^* \mathcal{V}_\ell$ is unramified *viz* extends to a \mathbb{Q}_ℓ -local system over $\text{spec}(\mathcal{O}_{k_v})$;
- If $\ell = p$, $x_v^* \mathcal{V}_p$ is crystalline.

For $v \in U_{\mathcal{V}_\ell}$ with residue characteristic p and $\ell \neq p$, let $\chi_{x_v^* \mathcal{V}_\ell} \in \mathbb{Q}_\ell[T]$ denote the characteristic polynomial of the geometric Frobenius

$$\varphi_{x_v^* \mathcal{V}_\ell} : \mathcal{V}_{\ell, \bar{x}} \rightarrow \mathcal{V}_{\ell, \bar{x}}$$

and for $\ell = p$, let $\chi_{x_v^* \mathcal{V}_p} \in k_{v,0}[T]$ denote the characteristic polynomial of the linearized crystalline Frobenius¹⁰

$$\varphi_{x_v^* \mathcal{V}_p, \text{cris}} : D_{\text{cris}}(x_v^* \mathcal{V}_p) \rightarrow D_{\text{cris}}(x_v^* \mathcal{V}_p).$$

See Subsection A.1 for a very brief review of basic definitions from p -adic Hodge theory, in particular the one of Fontaine's Riemann-Hilbert functor $D_{\text{cris}} : \text{Rep}_{\mathbb{Q}_p}(\pi_1(k_v)) \rightarrow M_{k_{v,0}}^\varphi$.

One says that \mathcal{V}_ℓ is *almost everywhere unramified* (AEU for short) if $U_{\mathcal{V}_\ell} \subset |\text{spec}(\mathcal{O}_k)|$ is a non-empty open subset and that it is \mathbb{Q} -rational (resp. and pure of weight $w \in \mathbb{R}$) if there exists a non-empty open subset $U'_\ell \subset U_{\mathcal{V}_\ell}$ such that for every $v \in |U'_\ell|$ the polynomial $\chi_{x_v^* \mathcal{V}_\ell}$ is in $\mathbb{Q}[T]$ (resp. the polynomial $\chi_{x_v^* \mathcal{V}_\ell}$ is in $\mathbb{Q}[T]$ and is pure of weight w , that is for every root α of $\chi_{x_v^* \mathcal{V}_\ell}$ and infinite place $\mathbb{Q}(\alpha) \xrightarrow{\varphi} \mathbb{C}$, $|\alpha|_\infty = |\kappa_v|^{\frac{w}{2}}$).

Let $\mathcal{V} := (\mathcal{V}_\ell)_\ell$ be a family of \mathbb{Q}_ℓ -local systems on $x = \text{spec}(k)$ (indexed by the set $|\text{spec}(\mathbb{Z})|$ of all rational primes). Write

$$U_{\mathcal{V}} := \bigcap_{\ell} U_{\mathcal{V}_\ell} \subset |\text{spec}(\mathcal{O}_k)|.$$

One says that \mathcal{V} is \mathbb{Q} -compatible (resp. and pure of weight $w \in \mathbb{R}$) if $U_{\mathcal{V}} \subset |\text{spec}(\mathcal{O}_k)|$ is a non-empty open subset and there exists a non-empty open subset $U'_\mathcal{V} \subset U_{\mathcal{V}}$ such that for every $v \in U'_\mathcal{V}$ the polynomial $\chi_{x_v^* \mathcal{V}} := \chi_{x_v^* \mathcal{V}_\ell}$ is in $\mathbb{Q}[T]$ (resp., pure of weight $w \in \mathbb{R}$), and independent of the prime ℓ .

Let X be a variety over k . One says that a family of \mathbb{Q}_ℓ -local systems $\mathcal{V} := (\mathcal{V}_\ell)_\ell$ on X is \mathbb{Q} -compatible (resp. and pure of weight $w \in \mathbb{R}$) if $x^* \mathcal{V}$ is, $x \in |X|$. The purity assumption ensures that \overline{G}_ℓ is semisimple [D80, Thm. (1.3.8), (1.11), Cor. (3.4.12)].

Classical examples of \mathbb{Q} -compatible families \mathcal{V} of \mathbb{Q}_ℓ -local systems on X are the geometric ones, namely those of the form $\mathcal{V}_\ell = R^i f_* \mathbb{Q}_\ell(j)$ for some smooth proper morphism $f : Y \rightarrow X$ and $i \geq 0$, j integers; these are pure of weight $w = i - 2j$ [D80], [KM74]. More generally, any family of \mathbb{Q}_ℓ -local systems on X cut out by a family of algebraic correspondances on a geometric one are \mathbb{Q} -compatible (see *e.g.* [CT25, Lem. 10]).

¹⁰More precisely, if $\phi_{x_v^* \mathcal{V}_p, \text{cris}} : D_{\text{cris}}(x_v^* \mathcal{V}_p) \rightarrow D_{\text{cris}}(x_v^* \mathcal{V}_p)$ denotes the (σ -semilinear) crystalline Frobenius then $\varphi_{x_v^* \mathcal{V}_p, \text{cris}} := \phi_{x_v^* \mathcal{V}_p, \text{cris}}^{m_v}$, where $m_v := [k_{v,0} : \mathbb{Q}_p]$.

Remark 38. Note that it tautologically follows from the definition of a \mathbb{Q} -compatible family $\mathcal{V} := (\mathcal{V}_\ell)_\ell$ of \mathbb{Q}_ℓ -local systems on $x = \text{spec}(k)$ that for $\ell \gg 0$ and every place $v|\ell$ of k , $x_v^* \mathcal{V}_\ell$ is crystalline - hence Hodge-Tate. This implies the following. Let X be a smooth, connected variety over k and $\mathcal{V} := (\mathcal{V}_\ell)_\ell$ a \mathbb{Q} -compatible family of \mathbb{Q}_ℓ -local systems on X . For every $x \in |X|$, write $\ell_{\mathcal{V},x}$ for the smallest prime such that for every prime $\ell \geq \ell_{\mathcal{V},x}$ and finite place $v|\ell$ of $k(x)$, $x_v^* \mathcal{V}_\ell$ is Hodge Tate. Set

$$\ell_{\mathcal{V}} := \min\{\ell_{\mathcal{V},x} \mid x \in |X|\}$$

From Fact 43 (1), for every $\ell \geq \ell_{\mathcal{V}}$ and every place $v|\ell$ of k , $\mathcal{V}_\ell|_{X_{k_v}}$ is Hodge-Tate. But then, in particular, for every $x \in |X|$ and finite place $v|\ell$ of $k(x)$, $x_v^* \mathcal{V}_\ell$ is Hodge-Tate as well, so that $\ell_{\mathcal{V},x} \leq \ell_{\mathcal{V}}$. This shows $\ell_{\mathcal{V},x} = \ell_{\mathcal{V}}$ is independent of $x \in |X|$.

We now prove that Conjecture (T) and Conjecture (U) hold when \mathcal{V}_ℓ is part of a \mathbb{Q} -compatible family \mathcal{V} of \mathbb{Q}_ℓ -local systems on X . Note that we do not assume purity *a priori*.

Corollary 39. *Let $\mathcal{V} = (\mathcal{V}_\ell)_\ell$ be a \mathbb{Q} -compatible family of \mathbb{Q}_ℓ -local systems on X . Then,*

- (1) *The set $|X|_{\mathcal{V}_\ell}^{\text{uni}} := |X|_{\mathcal{V}_\ell}^{\text{uni}}$ is independent of the prime ℓ and $|X|_{\mathcal{V}_\ell}^{\text{uni}} = |X|_{\mathcal{V}_\ell}^{\text{triv}}$ for $\ell \gg 0$;*
- (2) *If $|X|_{\mathcal{V}_\ell}^{\text{uni}} \neq \emptyset$ then G_ℓ° is unipotent (hence $\overline{G}_\ell^\circ = 1$ as G_ℓ° is unipotent implies that \mathcal{V} is pure of weight 0) for every prime ℓ and $G_\ell^\circ = 1$ for $\ell \gg 0$.*

Proof. The first part of Corollary 39 (1) follows from the fact that one can also describe $|X|_{\mathcal{V}_\ell}^{\text{uni}}$ as

$$|X|_{\mathcal{V}_\ell}^{\text{uni}} := \{x \in |X| \mid \text{rank}(G_{\ell,x}^\circ) = 0\},$$

and that $\text{rank}(G_{\ell,x}^\circ)$ is independent of $\ell \in |\text{spec}(\mathbb{Z})|$ [Se81, §3]. The second part of Theorem 39 (1), follows from Corollary 50 and Remark 38; more precisely, one has $|X|_{\mathcal{V}_\ell}^{\text{uni}} = |X|_{\mathcal{V}_\ell}^{\text{triv}}$ as soon as $\ell \geq \ell_{\mathcal{V}}$.

For Theorem 39 (2), it is enough to prove that $|X|_{\mathcal{V}_\ell}^{\text{uni}} = |X|$. Indeed, from Theorem 39 (1) this implies $|X|_{\mathcal{V}_\ell}^{\text{triv}} = |X|$ for primes $\ell \gg 0$. By Fact 46, the condition $|X|_{\mathcal{V}_\ell}^{\text{uni}} (= |X|_{\mathcal{V}_\ell}^{\text{uni}}) = |X|$ implies G_ℓ° is unipotent while the condition $|X|_{\mathcal{V}_\ell}^{\text{triv}} = |X|$ implies $G_\ell^\circ = 1$. As the assumptions of Theorem 39 and the property $|X|_{\mathcal{V}_\ell}^{\text{uni}} = |X|$ are invariant under base-change, one may freely replace X by a connected étale cover hence assume that $\text{Lev}_1(\mathcal{V}_{\ell_0})$ holds for at least one prime ℓ_0 , which implies the following.

- (*) For every $x \in |X|$, and finite place v of $U'_{x^* \mathcal{V}}$ above a prime $\ell \neq \ell_0$, the subgroup $\Xi_{x_v} \subset \overline{\mathbb{Q}}^\times$ generated by the roots of $\chi_{x_v^* \mathcal{V}} (= \chi_{x_v^* \mathcal{V}_\ell} = \chi_{x_v^* \mathcal{V}_{\ell_0}})$ is torsion-free.

By the Chebotarev density theorem, this implies in particular that for every $x \in |X|$, $G_{\ell,x}^\circ = G_{\ell,x}$. Hence

- $$x \in |X|_{\mathcal{V}_\ell}^{\text{uni}} \begin{array}{l} \stackrel{(1)}{\iff} \text{the set of places } v \text{ of } U'_{x^* \mathcal{V}} \text{ such that } \chi_{x_v} = (T-1)^r \text{ has density 1} \\ \stackrel{(2)}{\iff} \text{for } p \gg 0, \text{ and every place } v|p \text{ of } U'_{x^* \mathcal{V}}, \chi_{x_v^* \mathcal{V}} = (T-1)^r \\ \stackrel{(3)}{\iff} \text{for } p \gg 0, \text{ and every places } v|p \text{ of } k \text{ and } v_x|v \text{ of } k(x), x_{v_x} \in |X_{k_v}|_{\mathcal{V}_p}^{\text{ur}} \cap |X_{k_v}|_{\mathcal{V}_p}^{\text{uni}} \\ \stackrel{(4)}{\iff} \text{for } p \gg 0, \text{ and every place } v|p \text{ of } k, |X_{k_v}| = |X_{k_v}|_{\mathcal{V}_p}^{\text{uni}}, \text{ and for every place } v_x|v \text{ of } k(x), x_{v_x}^* \mathcal{V}_p \\ \text{is crystalline} \\ \stackrel{(5)}{\iff} \text{for } p \gg 0, \text{ and every place } v|p \text{ of } k, \mathcal{V}_p|_{X_{k_v}} \text{ is Hodge-Tate, } \mathbb{Q}\text{-rational, with } |X_{k_v}|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset, \text{ and} \\ \text{for every place } v_x|v \text{ of } k(x), x_{v_x}^* \mathcal{V}_p \text{ is crystalline.} \end{array}$$

Equivalence (1) follows from (*), the Chebotarev density theorem and the fact that a connected algebraic group is unipotent if and only if it contains a Zariski-dense set of unipotent elements. Implication (2) is tautological. Implication (3) follows from (*) and Lemma 42. Implication (4) follows from the fact that if $x_v^* \mathcal{V}_p$ is both crystalline and unipotent then it is \mathbb{C}_p -admissible (*viz* potentially unramified) and crystalline hence unramified - see Subsection A.1.2. Implication (5) is Corollary 36. As the latest of the assertion in this chain of implications holds by assumption (see Remark 38), this concludes the proof of Theorem 39. \square

In general, Corollary 39 does not imply Conjecture (C) for \mathcal{V}_ℓ part of a \mathbb{Q} -compatible family $\mathcal{V} = (\mathcal{V}_\ell)_\ell$ of \mathbb{Q}_ℓ -local systems on X unless one could prove *e.g.* that, for every $x \in |X|$, the family of \mathbb{Q}_ℓ -local system $A_x(\mathcal{V}) := (A_x(\mathcal{V}_\ell))_\ell$ introduced in Subsection B.2.2 is also \mathbb{Q} -compatible. However, Theorem 39 does imply Conjecture (C) under the following easy "large geometric monodromy" assumptions.

Corollary 40. *Let $\mathcal{V} = (\mathcal{V}_\ell)_\ell$ be a \mathbb{Q} -compatible family of \mathbb{Q}_ℓ -local systems on X . Assume one of the following conditions hold: (1) \overline{G}_ℓ° is a Levi subgroup of G_ℓ° , or (2) \overline{G}_ℓ° and the homotheties torus $\mathbb{G}_m(V_\ell) \simeq \mathbb{G}_{m,\mathbb{Q}_\ell} \subset \text{GL}(V_\ell)$*

generate a Levi subgroup of G_ℓ° . Then, for every prime ℓ , Conjecture (C) holds for \mathcal{V}_ℓ , namely $|X|_{\mathcal{V}_\ell}^{\text{cent}} = \emptyset$ unless $\overline{G}_\ell^\circ = 1$ and $|X|_{\mathcal{V}_\ell}^{\text{uni}} = \emptyset$ unless G_ℓ° is unipotent.

Proof. The assumptions impose that \overline{G}_ℓ° is reductive - hence semisimple (e.g. [KL81, Thm. 6]); in particular \overline{G}_ℓ° remains unchanged if one replaces \mathcal{V}_ℓ by its semisimplification $\mathcal{V}_\ell^{\text{ss}}$. As one also has $|X|_{\mathcal{V}_\ell}^{\text{cent}} \subset |X|_{\mathcal{V}_\ell^{\text{ss}}}^{\text{cent}}$, one may thus assume that \mathcal{V}_ℓ is semisimple for every ℓ . In that case, Condition (1) becomes simply $\overline{G}_\ell^\circ = G_\ell^\circ$ and Condition (2) that \overline{G}_ℓ° and $\mathbb{G}_m(V_\ell)$ generate G_ℓ° . By construction the family of \mathbb{Q}_ℓ -local systems $\mathcal{E} = (\mathcal{E}_\ell := \mathcal{V}_\ell \otimes \mathcal{V}_\ell^\vee)_\ell$ is \mathbb{Q} -compatible. Fix a prime ℓ and let $x \in |X|_{\mathcal{V}_\ell}^{\text{cent}}$. As \overline{G}_ℓ° is semisimple, the assumptions impose that $G_{\ell,x}^\circ \subset \mathbb{G}_m(V_\ell)$ hence $x \in |X|_{\mathcal{E}_\ell}^{\text{triv}}$ and, by Theorem 39, $\overline{G}_\ell^{\text{ad}\circ} = 1$ hence $\overline{G}_\ell^\circ = 1$ since \overline{G}_ℓ° is semisimple. \square

5.3.2. *Proof of Corollary 11.* Let X be a smooth, geometrically connected curve over k admitting a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\text{cpt}} \rightarrow S$ over some non-empty open subscheme $S \subset \text{spec}(\mathcal{O}_k)$. For every $p \in \text{im}(|S| \rightarrow |\text{spec}(\mathbb{Z})|)$, let \mathcal{V}_p be a \mathbb{Q}_p -local system on X with $\text{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) < p - 1$ and \overline{G}_p° is not unipotent. If $|X|_{\mathcal{V}_p}^{\text{uni}} = \emptyset$, there is nothing to prove. So assume $|X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$. As $|X|_{\mathcal{V}_p}^{\text{uni}} = |X|_{\mathcal{V}_p^{\text{ss}}}^{\text{uni}}$, replacing \mathcal{V}_p by its semisimplification $\mathcal{V}_p^{\text{ss}}$ one may assume \mathcal{V}_p is semisimple; note that replacing \mathcal{V}_p by $\mathcal{V}_p^{\text{ss}}$ does not affect the assumption that \overline{G}_p° is not unipotent. Then, from Corollary 50 (and its proof), for every place $v|p$ of k , $\mathcal{V}_p|_{X_{k_v}}$ is automatically Hodge-Tate and $|X|_{\mathcal{V}_p}^{\text{triv}} = |X|_{\mathcal{V}_p}^{\text{uni}}$. Corollary 11 now immediately follows from Corollary 37.

5.3.3. *Relation with the unramified Fontaine-Mazur conjecture.*

5.3.3.1. *AEU \mathbb{Q}_p -local systems.* Generalizing the definition of an AEU \mathbb{Q}_p -local system on $X = x = \text{spec}(k)$, say that a \mathbb{Q}_p -local \mathcal{V}_p on X is AEU if there exists a smooth model $\mathcal{X} \rightarrow U$ of X over a non-empty open subscheme $U \subset \text{spec}(\mathcal{O}_k)$ such that \mathcal{V}_p extends to a \mathbb{Q}_p -local system on \mathcal{X} . Note that, if $\mathcal{X}_i \rightarrow U_i$, $i = 1, 2$ are two smooth models of X then there exists a non-empty open subscheme $U \subset U_1 \cap U_2$ such that $\mathcal{X}_1 \times_{U_1} U \xrightarrow{\sim} \mathcal{X}_2 \times_{U_2} U$ as U -schemes. In particular, \mathcal{V}_p is AEU if and only if for every smooth model $\mathcal{X} \rightarrow U$ of X over a non-empty open subscheme $U \subset \text{spec}(\mathcal{O}_k)$, there exists a non-empty open subscheme $U' \subset U$ such that \mathcal{V}_p extends to a \mathbb{Q}_p -local system on $\mathcal{X} \times_U U'$.

The property of being pointwise AEU is also rigid.

Fact 41. ([LiZh17, Prop. 4.1], [P23, Prop. 6.1]) *Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . Consider the following properties*

- (i) \mathcal{V}_p is AEU;
- (ii) for every $x \in |X|$, $x^*\mathcal{V}_p$ is AEU;
- (iii) there exists $x \in |X|$ such that $x^*\mathcal{V}_p$ is AEU.

Then (i) \Rightarrow (ii) \Leftrightarrow (iii) and, if \mathcal{V}_p is semisimple, then (iii) \Rightarrow (i).

5.3.3.2. *Proof of Corollary 8.* We prove the following implications.

- (a) Conjecture (VU F-M) for $\mathcal{X} \rightarrow U$ of relative dimension 1 \Rightarrow Conjecture (T);
- (b) Conjecture (U F-M) \Rightarrow Conjecture (T);
- (c) Conjecture (U F-M) \Rightarrow For every $\mathcal{X} \rightarrow U$ and \mathbb{Q}_p -local system \mathcal{V}_p on \mathcal{X} as in Conjecture (VU F-M) one has $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$.

In particular, as \mathcal{X} is normal (hence the canonical morphism $\pi_1(X) \rightarrow \pi_1(\mathcal{X})$ is surjective), from implication (c) and a second application of Conjecture (U F-M) via implication (b), Conjecture (U F-M) \Rightarrow Conjecture (VU F-M).

- Proof of (a): Let k be a number field and let X be a smooth, geometrically connected variety over k . Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X and assume $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$. We are to prove that $|X|_{\mathcal{V}_p}^{\text{triv}} = |X|$. Without loss of generality, one may assume \mathcal{V}_p is semisimple (see Proposition 47). By Chow's lemma and Bertini's theorem, one may assume X is a curve (see Corollary 36). After possibly replacing X by a connected étale cover, one may assume X has genus ≥ 2 and $\text{Lev}_2(\mathcal{V}_p)$ holds. In particular, for every finite place $v|p$ of k , $\mathcal{V}_p|_{X_{k_v}}$ satisfies condition (C-1) of Subsection 5.2.3.2 so that $|X_{k_v}|_{\mathcal{V}_p}^{\text{triv}} = |X_{k_v}|$. From Remark 28 (1) and (2), \mathcal{V}_p extends to a \mathbb{Q}_p -local system on the smooth compactification $X \hookrightarrow X^{\text{cpt}}$ of X . So, without loss of generality, one may assume X is a smooth projective curve of genus ≥ 2 . By the semistable reduction theorem for curves, after possibly replacing further k by a finite field extension, one may assume X admits a regular semistable model $\mathcal{X} \rightarrow \text{spec}(\mathcal{O}_k)$. From Fact 41¹¹, there exists a non-empty open subscheme $U \subset \text{spec}(\mathcal{O}_k)$ such that \mathcal{V}_p extends to a \mathbb{Q}_p -local system on $\mathcal{X}_U := \mathcal{X} \times_{\text{spec}(\mathcal{O}_k)} U$. Actually, in Fact 41, in the process of shrinking $\text{spec}(\mathcal{O}_k)$ to U to ensure \mathcal{V}_p extends to a \mathbb{Q}_p -local system on \mathcal{X}_U , one typically throws away the places v of $\text{spec}(\mathcal{O}_k)$ dividing p (even if $\mathcal{X}_{\mathcal{O}_{k_v}} := \mathcal{X} \times_{\mathcal{O}_k} \mathcal{O}_{k_v}$ is smooth over $\text{spec}(\mathcal{O}_{k_v})$).

¹¹As X is smooth over k , there exists a non-empty open subscheme $U \subset \text{spec}(\mathcal{O}_k)$ such that $\mathcal{X}_U := \mathcal{X} \times_{\mathcal{O}_k} U \rightarrow U$ is smooth and one can apply Fact 41 and the comments preceding it to $\mathcal{X}_U \rightarrow U$.

Still, as for every finite place $v|p$ of k , $\mathcal{V}_p|_{X_{k_v}}$ satisfies condition (C-1), again $|X_{k_v}|_{\mathcal{V}_p}^{\text{ur}} = |X_{k_v}|$; this ensures we can assume U contains v . Indeed, write $U_+ = U \cup \text{spec}(\mathcal{O}_k)_p$ and let $v \in U_+ \setminus U$. From Theorem 1 (1) \Rightarrow , $\mathcal{V}_p|_{X_{k_v}}$ extends to a \mathbb{Q}_p -local system on $\mathcal{X}_{\mathcal{O}_{k_v}}$, namely the corresponding representation of $\pi_1(X_{k_v})$ on $V_p := \mathcal{V}_{p,\bar{x}}$ factors through $\pi_1(X_{k_v}) \twoheadrightarrow \pi_1(\mathcal{X}_{\mathcal{O}_{k_v}}^{\text{cpt}})$. In particular, the inertia group of the generic point of each connected component of $\mathcal{X}_{\mathcal{O}_{k_v}}$ acts trivially on V_p . By Zariski-Nagata purity (recall \mathcal{X} is regular), this implies \mathcal{V}_p extends to a \mathbb{Q}_p -local system on $\mathcal{X}_{U_+} \times_{U_+} (U \cup \{v\})$. We have thus proved that \mathcal{V}_p extends to a \mathbb{Q}_p -local system on \mathcal{X}_U with $U \hookrightarrow \text{spec}(\mathcal{O}_k)$ a non-empty open subscheme containing all finite places $v|p$ of k . From Conjecture (VU F-M) \mathcal{V}_p is finite.

- Proof of (b): Let k be a number field and let X be a smooth, geometrically connected variety over k . Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X and assume $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$. Again, we are to prove that $|X|_{\mathcal{V}_p}^{\text{triv}} = |X|$. Up to replacing X by a connected étale cover and k by a finite field extension, one may assume $\text{Lev}_2(\mathcal{V}_p)$ holds. From Fact 41 and from Corollary 44, for every $x \in |X|$, $x^*\mathcal{V}_p$ is AEU and for every place $v|p$ of $k(x)$, $x_v^*\mathcal{V}_p$ is potentially unramified - hence unramified by $\text{Lev}_2(\mathcal{V}_p)$. But then, by Conjecture (U F-M), $x \in |X|_{\mathcal{V}_p}^{\text{triv}}$.

- Proof of (c): Let $U \subset \text{spec}(\mathcal{O}_k)$ be a non-empty open subscheme containing all finite places $v|p$ of k , let $\mathcal{X} \rightarrow U$ be a morphism surjective, flat of finite type with \mathcal{X} normal; write X for the generic fiber of $\mathcal{X} \rightarrow U$. Let \mathcal{V}_p be a \mathbb{Q}_p -local system on \mathcal{X} . One may freely replace \mathcal{V}_p on \mathcal{X} by its restriction to any non-empty open subscheme of \mathcal{X} . By Epp's theorem [Stacks, Tag 09IL], there exists a finite field extension K/k such that, if $U^{\nu,K} \rightarrow U$ denotes the normalization of U in $\text{spec}(K) \rightarrow \text{spec}(k) \hookrightarrow U$, replacing $\mathcal{X} \rightarrow U$ with the normalized base changed

$$(\mathcal{X} \times_U \tilde{U})^\nu \rightarrow \mathcal{X} \times_U U^{\nu,K} \rightarrow U^{\nu,K},$$

one may assume for every $u \in U$, and every generic point ξ_u of \mathcal{X}_u , $\mathcal{X} \rightarrow U$ is smooth at ξ_u (note that $U^{\nu,K}$ also contains all places $v|p$ of K). In particular, replacing \mathcal{X} by a non-empty open subscheme one may assume $\mathcal{X} \rightarrow U$ is smooth. Replacing one more time $\mathcal{X} \rightarrow U$ by its base change along the normalization $U^{\nu,K} \rightarrow U$ of U in $\text{spec}(K) \rightarrow \text{spec}(k) \hookrightarrow U$ for some finite field extension K of k , one may also assume that for every place $v|p$ of k , $\mathcal{X}(\mathcal{O}_{k_v}) \neq \emptyset$. But then, as $\mathcal{X} \times_U \text{spec}(\mathcal{O}_{k_v}) \rightarrow \text{spec}(\mathcal{O}_{k_v})$ is smooth, $\mathcal{X}(\mathcal{O}_{k_v}) \subset X(k_v) (= X^{\text{sm}}(k_v))$ is a non-empty, automatically *open* subset, so that it follows from a classical Corollary of [MB89, Thm. 1.3] - see e.g. [Co06, Cor. 1.5], that there exists a finite field extension K/k , with U_p totally split in K , and $x \in X(K)$ such that for every place $v|p$ of k and finite place $w|v$ of K , $x_w \in \mathcal{X}(\mathcal{O}_{k_v})$. In other words, there exists a non-empty open subscheme $U_- \subset U \times_{\mathcal{O}_k} \mathcal{O}_K$ containing all the finite places of K above p such that $x \in \mathcal{X}(U_-)$. But then, Conjecture (U F-M) imposes that $x^*\mathcal{V}_p$ is finite that it $x \in |X|_{\mathcal{V}_p}^{\text{triv}}$.

APPENDIX A. BRIEF REVIEW OF (VARIATIONAL) p -ADIC HODGE THEORY

Let k be a p -adic field with ring of integers \mathcal{O}_k , maximal ideal \mathfrak{m} and residue field $\kappa := \mathcal{O}_k/\mathfrak{m}$; let s denote the closed point and η the generic point of $S := \text{spec}(\mathcal{O}_k)$. Let $\mathbb{Q}_p \subset k_0 \subset k$ be the maximal unramified extension of \mathbb{Q}_p contained in k and $\sigma : k_0 \xrightarrow{\sim} k_0$ its arithmetic Frobenius.

A.1. (Very) brief recollection of classical p -adic Hodge theory.

A.1.1. Let $B_{\text{cris}} \subset B_{\text{dR}} =: B_{\text{dR}}(k) = B_{\text{dR}}(\bar{k})$ and $B_{\text{HT}} = \text{Gr}(B_{\text{dR}})$ denote Fontaine's crystalline, de Rham and Hodge-Tate period rings respectively, and consider the associated "Riemann-Hilbert" \otimes -functors

$$\begin{aligned} D_{\text{cris}} &: \text{Rep}_{\mathbb{Q}_p}(\pi_1(\eta)) \rightarrow \text{M}_{k_0}^\sigma, V \mapsto (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{\pi_1(k)} \\ D_{\text{dR}} &: \text{Rep}_{\mathbb{Q}_p}(\pi_1(\eta)) \rightarrow F\text{-M}_k, V \mapsto (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{\pi_1(k)}, \\ D_{\text{HT}} &: \text{Rep}_{\mathbb{Q}_p}(\pi_1(\eta)) \rightarrow \text{Gr}_k, V \mapsto (B_{\text{HT}} \otimes_{\mathbb{Q}_p} V)^{\pi_1(k)}, \end{aligned}$$

Here $\text{M}_{k_0}^\sigma$ (resp. $F\text{-M}_k$, resp. Gr_k) denotes the category of k_0 -modules of finite rank D equipped with a σ -semilinear endomorphism $\phi : D \rightarrow D$ (resp. of k -modules of finite rank D equipped with a descending separated exhaustive filtration F^\bullet by k -submodules, resp. of k -modules of finite rank D equipped with a direct sum decomposition D^\bullet by k -submodules). Let

$$\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\pi_1(k)) \subset \text{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\pi_1(\eta)) \subset \text{Rep}_{\mathbb{Q}_p}^{\text{HT}}(\pi_1(\eta)) \subset \text{Rep}_{\mathbb{Q}_p}(\pi_1(\eta))$$

denote the full subcategories of crystalline (*viz* such that $\text{rank}_{\mathbb{Q}_p}(V) = \text{rank}_{k_0}(D_{\text{cris}}(V))$), de Rham (*viz* such that $\text{rank}_{\mathbb{Q}_p}(V) = \text{rank}_k(D_{\text{dR}}(V))$) and Hodge-Tate (*viz* such that $\text{rank}_{\mathbb{Q}_p}(V) = \text{rank}_k(D_{\text{HT}}(V))$) representations. The functors $D_{\text{dR}} : \text{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\pi_1(k)) \rightarrow F\text{-M}_k$ and $D_{\text{HT}} : \text{Rep}_{\mathbb{Q}_p}^{\text{HT}}(\pi_1(k)) \rightarrow \text{Gr}_k$ are faithful exact \otimes -functors

A.1.2. The following implications are classical. Note that being Hodge-Tate with single Hodge-Tate weight 0 is the same thing as being \mathbb{C}_p -admissible. In particular, being Hodge-Tate and unipotent - hence a successive extension of the trivial representation \mathbb{Q}_p , implies being \mathbb{C}_p -admissible.

$$\begin{array}{ccccccc} \mathbb{C}_p\text{-admissible} & \xleftrightarrow{(1)} & \text{potentially unramified} & \implies & \text{potentially crystalline} & \implies & \text{de Rham} \implies \text{Hodge-Tate} \\ & & \uparrow & & \square (2) & & \uparrow \\ & & \text{unramified} & \implies & \text{crystalline} & & \end{array}$$

The implication $\xrightarrow{(1)}$ is a theorem of Sen [S80, Cor. to Thm. 11]. For the fact that (2) is "Cartesian", namely that

$$(2) \quad \text{crystalline} + \text{potentially unramified} \implies \text{unramified},$$

see e.g. [Ca19, Prop. 4.3.2]. Lemma 42 is certainly well-known to expert but we could not find a reference; its proof was explained to us by Benjamin Schraen.

Lemma 42. *Let $V_p \in \text{Rep}_{\mathbb{Q}_p}(\pi_1(S))$. Then the elementary divisors of*

- the image $\varphi_p : V_p \xrightarrow{\sim} V_p$ of the geometric Frobenius $\varphi \in \pi_1(s) \simeq \pi_1(S)$;
- the linearized crystalline Frobenius $\varphi : D_{\text{cris}}(V_p) \xrightarrow{\sim} D_{\text{cris}}(V_p)$,

coincide. In particular, the characteristic polynomial of $\varphi : D_{\text{cris}}(V_p) \xrightarrow{\sim} D_{\text{cris}}(V_p)$ is in $\mathbb{Z}_p[T]$ and its roots are v -adic units.

Proof. Write again $I_s := \ker(\pi_1(\eta) \rightarrow \pi_1(S) \simeq \pi_1(s))$ for the inertia group of k , let $k_0 \subset k_0^{\text{ur}} \subset \bar{k}$ denote the maximal unramified extension of k_0 and let \hat{k}_0^{ur} denote its completion. By definition $\varphi = \phi^m : D_{\text{cris}}(V_p) \rightarrow D_{\text{cris}}(V_p)$, where $m := [k_0 : \mathbb{Q}_p]$ and $\phi : D_{\text{cris}}(V_p) \rightarrow D_{\text{cris}}(V_p)$ is the crystalline Frobenius. As V_p is crystalline, and using that $(B_{\text{cris}})^{I_s} = \hat{k}_0^{\text{ur}}$ (e.g. [Fo94, Prop. 5.1.2]),

$$D_{\text{cris}}(V_p) = (V_p \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\pi_1(\eta)} = (V_p \otimes_{\mathbb{Q}_p} (B_{\text{cris}})^{I_s})^{\pi_1(S)} = (V_p \otimes_{\mathbb{Q}_p} \hat{k}_0^{\text{ur}})^{\pi_1(S)} =: D_{\hat{k}_0^{\text{ur}}}(V_p)$$

has k_0 -dimension $\dim_{\mathbb{Q}_p}(V_p)$. In other words, V_p is \hat{k}_0^{ur} -admissible, hence the canonical \hat{k}_0^{ur} -linear injective morphism

$$\alpha : D_{\hat{k}_0^{\text{ur}}}(V_p) \otimes_{k_0} \hat{k}_0^{\text{ur}} \rightarrow V_p \otimes_{\mathbb{Q}_p} \hat{k}_0^{\text{ur}}$$

is an isomorphism, which is equivariant with the following structures:

- the $\pi_1(\eta)$ -action (with $D_{\hat{k}_0^{\text{ur}}}(V_p)$ viewed as a trivial $\pi_1(\eta)$ -representation);
- the crystalline Frobenii (with the crystalline Frobenius on V_p being the identity and the one on \hat{k}_0^{ur} the lift $\sigma : \hat{k}_0^{\text{ur}} \xrightarrow{\sim} \hat{k}_0^{\text{ur}}$ of the arithmetic Frobenius on the residue field).

In particular, $\alpha : D_{\hat{k}_0^{\text{ur}}}(V_p) \otimes_{k_0} \hat{k}_0^{\text{ur}} \xrightarrow{\sim} V_p \otimes_{\mathbb{Q}_p} \hat{k}_0^{\text{ur}}$ exchanges

$$Id \otimes_{k_0} \sigma^m \longleftrightarrow \varphi_p^{-1} \otimes_{\mathbb{Q}_p} \sigma^m, \quad \phi \otimes_{k_0} \sigma \longleftrightarrow Id \otimes_{\mathbb{Q}_p} \sigma.$$

As a result,

$$\alpha \circ (\phi^m \otimes_{k_0} Id) \circ \alpha^{-1} = \alpha \circ (\phi \otimes_{k_0} \sigma)^m (Id \otimes_{k_0} \sigma^m)^{-1} \circ \alpha^{-1} = (Id \otimes_{\mathbb{Q}_p} \sigma)^m (\varphi_p^{-1} \otimes_{\mathbb{Q}_p} \sigma^m)^{-1} = \varphi_p \otimes_{\mathbb{Q}_p} Id.$$

This shows the two k_0 -linear morphisms $\varphi_p \otimes_{\mathbb{Q}_p} Id_{k_0} : V_p \otimes_{\mathbb{Q}_p} k_0 \xrightarrow{\sim} V_p \otimes_{\mathbb{Q}_p} k_0$ and $\phi^m : D_{\hat{k}_0^{\text{ur}}}(V_p) \xrightarrow{\sim} D_{\hat{k}_0^{\text{ur}}}(V_p)$ have the same elementary divisors hence, in particular, the same characteristic polynomial. \square

A.2. Pointwise versus global properties. Let X be a smooth variety over k . Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X , write

$$|X|_{\mathcal{V}_p}^{\text{ur}} \subset |X|_{\mathcal{V}_p}^{\text{cris}} \subset |X|_{\mathcal{V}_p}^{\text{dR}} \subset |X|_{\mathcal{V}_p}^{\text{HT}} \subset |X|$$

for the subsets of all $x \in |X|$ such that $x^* \mathcal{V}_p$ is unramified, crystalline, de Rham and Hodge-Tate respectively; let also

$$(|X|_{\mathcal{V}_p}^{\text{ur}} \subset) |X|_{\mathcal{V}_p}^{\text{pot-ur}} (\subset |X|_{\mathcal{V}_p}^{\text{dR}})$$

denote the subset of all $x \in |X|$ such that $x^* \mathcal{V}_p$ is potentially unramified. Say that \mathcal{V}_p is *pointwise unramified* if $|X|_{\mathcal{V}_p}^{\text{ur}} = |X|$; define similarly the notion of being *pointwise crystalline*, *pointwise de Rham*, *pointwise Hodge-Tate* and *pointwise potentially unramified*.

A.2.1. *Hodge-Tate, de Rham and crystalline local systems.* There are also global notions of Hodge-Tate, de Rham and crystalline \mathbb{Q}_p -local systems on X defined using geometric versions of Fontaine's Riemann-Hilbert functors. More precisely, let $X^{\text{an}} \rightarrow X$ denote the rigid-analytification of X . The natural morphism of sites $X_{\text{et}}^{\text{an}} \rightarrow X_{\text{et}}$ induces a faithful exact \otimes -functor

$$(-)^{\text{an}} : \text{Loc}_{\mathbb{Z}_p}(X_{\text{et}}) \rightarrow \text{Loc}_{\mathbb{Z}_p}(X_{\text{et}}^{\text{an}})$$

from the category $\text{Loc}_{\mathbb{Z}_p}(X_{\text{et}})$ of \mathbb{Z}_p -local systems on X_{et} to the category $\text{Loc}_{\mathbb{Z}_p}(X_{\text{et}}^{\text{an}})$ of \mathbb{Z}_p -local systems on $X_{\text{et}}^{\text{an}}$ hence, passing to the isogeny category, a faithful exact \otimes -functor

$$(-)^{\text{an}} : \text{Loc}_{\mathbb{Q}_p}(X_{\text{et}}) \rightarrow \text{Loc}_{\mathbb{Q}_p}(X_{\text{et}}^{\text{an}}).$$

- Let $\text{Higgs}(X^{\text{an}})$ denote the category of vector bundles with a nilpotent Higgs field on X^{an} . If

$$D_{\text{HT}} : \text{Loc}_{\mathbb{Q}_p}(X_{\text{et}}^{\text{an}}) \rightarrow \text{Higgs}(X^{\text{an}})$$

denotes the natural Hodge-Tate Riemann-Hilbert functor constructed in [LiZh17, §2.1], one says that a \mathbb{Q}_p -local system \mathcal{V}_p on $X_{\text{et}}^{\text{an}}$ is *Hodge-Tate* if

$$\text{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) = \text{rank}(D_{\text{HT}}(\mathcal{V}_p)),$$

and that a \mathbb{Q}_p -local system \mathcal{V}_p on X_{et} is Hodge-Tate if $\mathcal{V}_p^{\text{an}}$ is.

- Let $F\text{-Vect}^\nabla(X^{\text{an}})$ denote the category of filtered vector bundles on X^{an} with a flat connection satisfying Griffith's transversality. If

$$D_{\text{dR}} : \text{Loc}_{\mathbb{Q}_p}(X_{\text{et}}^{\text{an}}) \rightarrow F\text{-Vect}^\nabla(X^{\text{an}})$$

denotes the natural de Rham Riemann-Hilbert functor constructed in [LiZh17, §3.2], one says that a \mathbb{Q}_p -local system \mathcal{V}_p on $X_{\text{et}}^{\text{an}}$ is *de Rham* if

$$\text{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) = \text{rank}(D_{\text{dR}}(\mathcal{V}_p)),$$

and that a \mathbb{Q}_p -local system \mathcal{V}_p on X_{et} is de Rham if $\mathcal{V}_p^{\text{an}}$ is.

Assume furthermore $X \rightarrow \text{spec}(k)$ admits a model $\mathcal{X} \rightarrow S$, smooth, separated and of finite type. Write $\widehat{\mathcal{X}}$ for the formal completion of \mathcal{X} along the closed fiber \mathcal{X}_v . Let $\widehat{\mathcal{X}}_\eta$ denote the rigid-analytic fiber of $\widehat{\mathcal{X}}$ so that one gets an open immersion $\widehat{\mathcal{X}}_\eta \hookrightarrow X^{\text{an}}$ of rigid analytic spaces.

- Let $F\text{-wIsoc}(\mathcal{X}_v/\mathcal{O}_{k_0})$ denote the category of weak F -isocrystals on $\mathcal{X}_v/\mathcal{O}_{k_0}$ [GuY24, Def. 5.10]. If

$$D_{\text{cris}}^{\text{an}} : \text{Loc}_{\mathbb{Q}_p}(\widehat{\mathcal{X}}_{\eta, \text{et}}) \rightarrow F\text{-wIsoc}(\mathcal{X}_v/\mathcal{O}_{k_0})$$

denotes the natural crystalline Riemann-Hilbert functor constructed in [GuY24, Thm. 1.10] and one defines

$$D_{\text{cris}} : \text{Loc}_{\mathbb{Z}_p}(X_{\text{et}}^{\text{an}}) \xrightarrow{|\widehat{\mathcal{X}}_\eta} \text{Loc}_{\mathbb{Q}_p}(\widehat{\mathcal{X}}_{\eta, \text{et}}) \xrightarrow{D_{\text{cris}}^{\text{an}}} F\text{-wIsoc}(\mathcal{X}_v/\mathcal{O}_{k_0}),$$

one says that a \mathbb{Q}_p -local system \mathcal{V}_p on $X_{\text{et}}^{\text{an}}$ is *crystalline* if $D_{\text{cris}}(\mathcal{V}_p)$ has constant rank $\text{rank}(D_{\text{cris}}(\mathcal{V}_p))$ and

$$\text{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) = \text{rank}(D_{\text{cris}}(\mathcal{V}_p)).$$

One says that a \mathbb{Q}_p -local system \mathcal{V}_p on X_{et} is *crystalline* (with respect to $\widehat{\mathcal{X}}_\eta$) if $\mathcal{V}_p^{\text{an}}$ is.

The following summarizes the relation between the pointwise and global Hodge-Tate, de Rham and crystalline properties.

Fact 43. *Let X be a smooth, geometrically connected variety over k . Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . Then,*

(1) ([P23, §7]; see also [Shim18]) *One has*

$$(i) \mathcal{V}_p \text{ is Hodge-Tate} \Leftrightarrow (ii) |X|_{\mathcal{V}_p}^{\text{HT}} = |X| \Leftrightarrow (iii) |X|_{\mathcal{V}_p}^{\text{HT}} \neq \emptyset;$$

Furthermore, if \mathcal{V}_p is Hodge-Tate, the multiset $HT(\mathcal{V}_p) := HT(x^\mathcal{V}_p)$ of Hodge-Tate weights of $x^*\mathcal{V}_p$ is independent of $x \in |X|$.*

(2) ([LiZh17, Thm. 1.1, Thm. 1.3]) *One has*

$$(i) \mathcal{V}_p \text{ is de Rham} \Leftrightarrow (ii) |X|_{\mathcal{V}_p}^{\text{dR}} = |X| \Leftrightarrow (iii) |X|_{\mathcal{V}_p}^{\text{dR}} \neq \emptyset.$$

(3) ([GuY24, Thm. 7.2]) *Assume furthermore X admits a smooth model $\mathcal{X} \rightarrow S$. One has*

$$(i) \mathcal{V}_p \text{ is crystalline (with respect to } \widehat{\mathcal{X}}_\eta) \Leftrightarrow (ii)' |X|_{\mathcal{V}_p}^{\text{cris}} \supset \text{im}(\mathcal{X}(\mathcal{O}_{k_s}) \rightarrow |X|)$$

Note that, in particular, the property of being a Hodge-Tate, de Rham or crystalline \mathbb{Q}_p -local system is preserved by passing to subquotients.

Corollary 44. *Let X be a smooth, geometrically connected variety over k and let \mathcal{V}_p a \mathbb{Q}_p -local system on X . Then, one has*

$$(ii) |X|_{\mathcal{V}_p}^{\text{pot-ur}} = |X| \Leftrightarrow (iii) |X|_{\mathcal{V}_p}^{\text{pot-ur}} \neq \emptyset,$$

and, if $|X|_{\mathcal{V}_p}^{\text{pot-ur}} \neq \emptyset$, then $|X|_{\mathcal{V}_p}^{\text{cris}} = |X|_{\mathcal{V}_p}^{\text{ur}}$ and for every $x \in |X|_{\mathcal{V}_p}^{\text{cris}}$ the characteristic polynomials of

- the geometric Frobenius $\varphi_{x,p} : \mathcal{V}_{p,\bar{x}} \xrightarrow{\sim} \mathcal{V}_{p,\bar{x}}$;
- the linearized crystalline Frobenius $\varphi_{x,\text{cris}} : D_{\text{cris}}(x^*\mathcal{V}_p) \xrightarrow{\sim} D_{\text{cris}}(x^*\mathcal{V}_p)$,

coincide. In particular, the characteristic polynomial of $\varphi_{x,\text{cris}} : D_{\text{cris}}(x^*\mathcal{V}_p) \xrightarrow{\sim} D_{\text{cris}}(x^*\mathcal{V}_p)$ is in $\mathbb{Z}_p[T]$ and its roots are v -adic units.

Proof. According to the equivalence (1) of Subsection A.1.2, the equivalence (ii) \Leftrightarrow (iii) is a special case of Fact 43 (1). The equality $|X|_{\mathcal{V}_p}^{\text{ur}} = |X|_{\mathcal{V}_p}^{\text{cris}}$ follows from the implication (2) in Subsection A.1.2. The last part of the assertion then follows from Lemma 42. \square

Eventually, we record the following consequence of [P23, Thm. 4.1].

Corollary 45. *Let X be a smooth, geometrically connected variety over k and let \mathcal{V}_p a \mathbb{Q}_p -local system on X . Assume \mathcal{V}_p is semisimple and $|X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$. Then \mathcal{V}_p is Hodge-Tate.*

Proof. Let $x \in |X|_{\mathcal{V}_p}^{\text{uni}}$. After possibly replacing X by a connected étale cover, one may assume $\overline{G}_p^\circ = \overline{G}_p$, $G_p^\circ = G_p$ and $G_{p,x}^\circ = G_{p,x}$. Since G_p is reductive and $G_{p,x} \hookrightarrow G_p \twoheadrightarrow G_p/\overline{G}_p$ is surjective, one has $\overline{G}_p = G_p$. Decompose $\mathcal{V}_{p,\mathbb{Q}_p}$ as a direct sum of simple \mathbb{Q}_p -local systems

$$\mathcal{V}_{p,\mathbb{Q}_p} = \bigoplus_{1 \leq i \leq r} \mathcal{V}_{p,i}.$$

Fix $1 \leq i \leq r$. The equality $\overline{G}_p = G_p$ imposes that $\mathcal{V}_{p,i}|_{X_{\bar{k}}}$ is simple hence, from [P23, Thm. 4.1], that $\mathcal{V}_{p,i}(\chi_{i,v})$ is Hodge-Tate for some character $\chi_{i,v} : \pi_1(k_v) \rightarrow \overline{\mathbb{Q}}_p^\times$. In particular, $x_v^*\mathcal{V}_{p,i}$ is Hodge-Tate and so is any of its quotient. But as $x_v^*\mathcal{V}_{p,i}$ is unipotent, this proves $\chi_{i,v} : \pi_1(k_v) \rightarrow \overline{\mathbb{Q}}_p^\times$ is Hodge-Tate hence $\mathcal{V}_{p,i}$ is also Hodge-Tate. \square

APPENDIX B. AROUND CONJECTURE (T)

In this appendix, we gather observations about Conjecture (T) and its variants - Conjecture (U) and Conjecture (C) below - which, we feel, are of independent interest.

Notation / conventions for algebraic groups.

For an algebraic group G , let $G^\circ \subset G$ denote the neutral component of G and, for a closed subgroup $H \subset G$, let $Z_G(H) \subset N_G(H) \subset G$ denote the centralizer and normalizer of H in G respectively; set $Z(G) := Z_G(G)$ for the center of G . For a profinite group Π and a topological field Q , let $\text{Rep}_Q(\Pi)$ denote the category of continuous, finite-dimensional Q -representations of Π .

B.1. Trivial, unipotent and centralizing loci. Let k be a field and let X be a smooth, geometrically connected variety over k . Let p be a prime and let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . For every $x \in X$ and geometric point \bar{x} over x , set $V_p := \mathcal{V}_{p,\bar{x}}$ and let $\overline{G}_p, G_{p,x} \subset G_p \subset \text{GL}_{V_p}$ denote the Zariski closures of the images $\overline{\Pi}_p, \Pi_{p,x}$ and Π_p of $\pi_1(X_{\bar{k}}, \bar{x})$, $\pi_1(x)$ and $\pi_1(X)$ acting on V_p respectively (so that $\Pi_{p,\eta} = \Pi_p$, $G_{p,\eta} = G_p$). Define the *degeneracy locus* or *Tate locus* (restricted to closed points) of \mathcal{V}_p as

$$|X|_{\mathcal{V}_p} := \{x \in |X| \mid G_{p,x}^\circ \subsetneq G_p^\circ\}.$$

Informally, $|X|_{\mathcal{V}_p}$ is the set of all $x \in |X|$ where $x^*\mathcal{V}_p$ degenerates. When k is a number field, under mild assumptions on $\overline{\Pi}_p$, one expects that $X(k) \cap |X|_{\mathcal{V}_p}$ is not Zariski-dense in X - see [C23]. Here, we focus on the most degenerate stratum of $|X|_{\mathcal{V}_p}$, namely the *trivial locus*

$$|X|_{\mathcal{V}_p}^{\text{triv}} := \{x \in |X| \mid G_{p,x}^\circ = 1\}.$$

The following conjecture predicts that the trivial locus should be empty unless $\mathcal{V}_p|_{X_{k^s}}$ is finite.

Conjecture (T). *Assume k is a number field. For a \mathbb{Q}_p -local system \mathcal{V}_p on X , the following implications hold.*

$$(i)' \overline{G}_p^\circ = 1 \Leftrightarrow (i) G_p^\circ = 1 \Leftrightarrow (ii) |X|_{\mathcal{V}_p}^{\text{triv}} = |X| \Leftrightarrow (iii) |X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset.$$

The implications (i)' \Leftrightarrow (i) \Rightarrow (ii) \Rightarrow (iii) are tautological. The implication (ii) \Rightarrow (i) follows from Hilbert's irreducibility theorem.

Fact 46. (Hilbert's irreducibility - [Se89, §9.6, 10.6, Thm.]) *Let k be a number field and let X be a smooth, geometrically connected variety over k . For every \mathbb{Q}_p -local system \mathcal{V}_p on X there exists infinitely many $x \in |X|$ such that $\Pi_{p,x} = \Pi_p$ - hence such that $G_{p,x} = G_p$.*

To summarize, one has:

$$(i)' \longleftarrow (i) \iff (ii) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} (iii)$$

Conjecture (T) is motivated by the following reformulations and relations with classical conjectures.

B.1.1. Reformulation of Conjecture (T).

- (1) An equivalent formulation of Conjecture (T) in terms of the degeneracy locus $|X|_{\mathcal{V}_p}$ is the following - see Lemma 48. Assume k is a number field and Conjecture (T) holds (for every \mathbb{Q}_p -local system on X). Then, for every \mathbb{Q}_p -local system \mathcal{V}_p on X the following holds. For every $x \in |X|$, $G_{\ell,x}^\circ$ normally generates G_p° . Equivalently,

$$|X|_{\mathcal{V}_p} = \{x \in |X| \mid G_{\ell,x}^\circ \not\subseteq \text{Nor}_{G_p^\circ}(G_{\ell,x}^\circ)\}.$$

- (2) As a compact p -adic Lie group is a closed subgroup of $\text{GL}_m(\mathbb{Z}_p)$ for some integer $m \geq 0$ [Lu88, Prop. 4], one has the following diophantine reformulation of Conjecture (T): Assume k is a number field and let

$$\cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$$

be a projective system of finite étale covers with $X_n \rightarrow X$ Galois of group Π_n , $n \geq 0$. Assume $\Pi := \lim \Pi_n$ is a p -adic Lie group of dimension > 0 for some prime ℓ . Then,

$$\lim X_n(k) = \emptyset.$$

B.1.2. Relation to classical conjectures.

- (1) Assume $k = \mathbb{C}$ and let \mathcal{V} be a polarizable \mathbb{Z} -variation of pure Hodge structures on the complex-analytification X^{an} of X . One can define similarly the degeneracy locus or Hodge locus $|X|_{\mathcal{V}}$, the trivial locus $|X|_{\mathcal{V}}^{\text{triv}}$ and the centralizing locus $|X|_{\mathcal{V}}^{\text{cent}}$ of \mathcal{V} using¹² the Mumford-Tate group G_x of $x^*\mathcal{V}$, and the generic Mumford-Tate group G of \mathcal{V} in place of the ℓ -adic algebraic monodromy groups $G_{\ell,x}$, G_p . In that setting, the statements corresponding to Conjecture (T) (and Conjecture (C) that we will state later) easily follows *e.g.* from the constancy of Hodge numbers¹³ and the fact that the neutral component \overline{G}° of the Zariski-closure \overline{G} of the image of $\pi_1(X^{\text{an}})$ acting on \mathcal{V}_x is contained in G [An92, Thm. 1]. In particular, for \mathbb{Q}_p -local systems arising from motives, Conjecture (T) (and Conjecture (C)) should follow from classical motivic realization conjectures (Hodge [Ho52], Tate [Ta94]; see also [DiLaLiZh23, Conj. 1.4]).

- (2) In whole generality, Conjecture (T) follows from the unramified Fontaine-Mazur conjecture (Conjecture (U F-M)) - see Corollary 8.

The trivial locus $|X|_{\mathcal{V}_p}^{\text{triv}}$ is closely related to the *unipotent locus*

$$|X|_{\mathcal{V}_p}^{\text{uni}} := \{x \in |X| \mid G_{p,x}^\circ \text{ is unipotent}\},$$

and *centralizing locus*

$$|X|_{\mathcal{V}_p}^{\text{cent}} := \{x \in |X| \mid G_{p,x}^\circ \subset Z_{G_p}(\overline{G}_p^\circ)\}.$$

The following are the variants of Conjecture (T) for the unipotent and centralizing loci respectively.

Conjecture (U). *Assume k is a number field. For a \mathbb{Q}_p -local system \mathcal{V}_p on X , the following implications hold.*

$$(i)' \overline{G}_p^\circ = 1 \iff (i) G_p^\circ \text{ is unipotent} \iff (ii) |X|_{\mathcal{V}_p}^{\text{uni}} = |X| \iff (iii) |X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset.$$

Conjecture (C). *Assume k is a number field. For a \mathbb{Q}_p -local system \mathcal{V}_p on X , the following implications hold.*

$$(i)' \overline{G}_p^\circ = 1 \iff (i) \overline{G}_p^\circ \subset Z(G_p^\circ) \iff (ii) |X|_{\mathcal{V}_p}^{\text{cent}} = |X| \iff (iii) |X|_{\mathcal{V}_p}^{\text{cent}} \neq \emptyset.$$

Before reviewing the various implications involved in Conjectures (U), (C) and the relations between Conjectures (T), (U), (C), we record some preliminary technical observations.

Convention for number fields: For a number field k and a place $v|p$ of k , let k_v denote the completion of k at v and $\mathcal{O}_{k_v} \twoheadrightarrow \kappa_v$ its ring of integers and residue field respectively; let also $\mathbb{Q}_p \subset k_{v,0} \subset k_v$ denote the maximal unramified

¹²Recall that these are connected and reductive.

¹³If $|X^{\text{an}}|_{\mathcal{V}}^{\text{triv}} \neq \emptyset$, the only non-zero Hodge number is $h^{0,0}$; equivalently, $G_x = 1$, $x \in X^{\text{an}}$ - hence $G = 1$.

extension of \mathbb{Q}_p contained in k_v , $m_v := [k_{v,0} : \mathbb{Q}_p]$ its degree and $\sigma : k_{v,0} \xrightarrow{\sim} k_{v,0}$ its Frobenius. If X is a variety over k and $x \in |X|$ a closed point, for a place $v|p$ of k we write $x_v : \text{spec}(k(x)_v) \rightarrow \text{spec}(k(x)) \xrightarrow{x} X$ for the resulting $k(x)_v (= k(x_v))$ -point, which we also regard as a closed point $x_v \in |X_{k_v}|$.

B.2. Preliminaries.

B.2.1. *A consequence of local class field theory and Sen's theorem.*

Proposition 47. *Let k be a number field. Let p be a prime and \mathcal{V}_p a \mathbb{Q}_p -local system on $x = X = \text{spec}(k)$. Assume $x_v^* \mathcal{V}_p$ is Hodge-Tate for every place $v|p$ of k . Then $(G_p^\circ)^{\text{ab}}$ is reductive.*

Proof. Fix a geometric point \bar{x} over x . Write $V_p := \mathcal{V}_{p,\bar{x}}$ and let $\rho_p : \pi_1(x) = \pi_1(k) \rightarrow G_p(\mathbb{Q}_p) \subset \text{GL}(V_p)$ denote the continuous representation corresponding to \mathcal{V}_p . After possibly replacing k by a finite field extension one may assume $G_p = G_p^\circ$. Fix a Levi subgroup $L_p \subset G_p$ and let $N_p \subset G_p$ denote the smallest normal algebraic subgroup of G_p containing L_p . If G_p^{ab} is not reductive, then $N_p \subsetneq G_p$. By Lemma 48 below, there exists a G_p -subrepresentation $W_p \subset T(V_p)$ such that $N_p = \ker(G_p \rightarrow \text{GL}_{W_p})$. By construction, the non-trivial unipotent group G_p/N_p acts faithfully on W_p . Let \mathcal{W}_p denote the \mathbb{Q}_p -local system on x corresponding to W_p viewed as a $\pi_1(k)$ -representation via $\pi_1(k) \rightarrow G_p(\mathbb{Q}_p) \rightarrow (G_p/N_p)(\mathbb{Q}_p)$. Then, as \mathcal{W}_p lies in the Tannakian category generated by \mathcal{V}_p , $x_v^* \mathcal{W}_p$ is Hodge-Tate for every finite place $v|p$ of k . As a result, it is enough to prove that if G_p is unipotent, then it is trivial. If G_p is unipotent non-trivial then there exists a surjective morphism $p : G_p \rightarrow \mathbb{G}_{a,\mathbb{Q}_p}$ and a factorization

$$\begin{array}{ccc} \pi_1(x) = \pi_1(k) & \xrightarrow{\rho_p} & G_p(\mathbb{Q}_p) \xrightarrow{p} \mathbb{G}_{a,\mathbb{Q}_p}(\mathbb{Q}_p) \simeq \mathbb{Q}_p \\ \downarrow & \nearrow \rho_p^{\text{ab}} & \\ \pi_1(k)^{\text{ab}} & & \end{array}$$

such that $\text{im}(\rho_p^{\text{ab}}) \simeq \mathbb{Z}_p$. As \mathbb{Z}_p is torsion-free and as $\mathcal{O}_v^\times \simeq \mathbb{Z}_\ell^{\oplus r} \times (\mathcal{O}_v^\times)_{\text{tor}}$ for every prime $\ell \neq p$ and place $v|p$ of k , it follows from local class field theory that $\rho_p^{\text{ab}} : \pi_1(k)^{\text{ab}} \rightarrow \mathbb{Z}_p$ factors through $\pi_1(\mathcal{O}_k[\frac{1}{p}])^{\text{ab}} \rightarrow \mathbb{Z}_p$. On the other hand, for every place $v|p$ of k , $x_v^* \mathcal{V}_p$ is unipotent, so that it has a single Hodge-Tate weight, which is 0; equivalently (see equivalence (1) in Paragraph A.1.2), it is potentially unramified. In particular, for every finite place $v|p$ of k , $\rho_p^{\text{ab}}|_{\pi_1(k_v)^{\text{ab}}} : \pi_1(k_v)^{\text{ab}} \rightarrow \mathbb{Z}_p$ is potentially unramified - hence unramified since \mathbb{Z}_p is torsion-free. This proves that $\rho_p^{\text{ab}} : \pi_1(k)^{\text{ab}} \rightarrow \mathbb{Z}_p$ actually factors through $\pi_1(\mathcal{O}_k)^{\text{ab}} \rightarrow \mathbb{Z}_p$, which contradicts the finiteness of $\pi_1(\mathcal{O}_k)^{\text{ab}}$. \square

Lemma 48. *Let Q be a field of characteristic 0, V a finite dimensional Q -vector space and $N \subset G \subset \text{GL}_V$ algebraic subgroups with N normal in G . Then there exists a G -subrepresentation*

$$W_N \subset T(V) := \bigoplus_{m,n} V^{\otimes m} \otimes V^{\vee \otimes n}$$

such that $N = \ker(G \rightarrow \text{GL}_{W_N})$.

Proof. By [D82, Prop. 3.1 (a), (b)], there exists a GL_V -subrepresentation $V_1 \subset T(V)$ such that N is the stabilizer of a line $L_1 \subset V_1$; let $L_1 \subset V_2 \subset V_1$ denote the smallest G -subrepresentation containing L_1 and let $N_2 \subset G_2 \subset \text{GL}_{V_2}$ denote the image of N and G acting on V_2 respectively. By construction N_2 is contained in a split torus of GL_{V_2} - hence is reductive. By [D82, Prop. 3.1 (a), (c)], there exists a GL_{V_2} -subrepresentation $V_3 \subset T(V_2) \subset T(V_1) \subset T(V)$ and a finite subset $A \subset V_3$ such that N_2 is the algebraic subgroup of GL_{V_2} fixing the elements in A . Let $A \subset V_4 \subset V_3$ denote the smallest G_2 -subrepresentation containing A . By construction, $N_2 = \ker(G_2 \rightarrow \text{GL}_{V_4})$ hence one can take $W_N := V_4$. \square

Remark 49. Using Lemma 48, one immediately sees that Conjecture (T) is equivalent to the characterization of the degeneracy locus of \mathcal{V}_p given in Subsection B.1.1 (1).

Corollary 50. *Let k be a number field and X a smooth, geometrically connected variety over k . Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . Assume for every place $v|p$ of k , $\mathcal{V}_p|_{X_{k_v}}$ is Hodge-Tate (e.g. $x^* \mathcal{V}_p|_{X_{k_v}}$ is Hodge-Tate for some $x \in |X_{k_v}|$). Then $|X|_{\mathcal{V}_p}^{\text{triv}} = |X|_{\mathcal{V}_p}^{\text{uni}}$. In particular, for every \mathbb{Q}_p -local system on X , one has*

$$|X|_{\mathcal{V}_p}^{\text{uni}} = |X|_{\mathcal{V}_p^{\text{ss}}}^{\text{uni}} = |X|_{\mathcal{V}_p^{\text{ss}}}^{\text{triv}}.$$

Proof. The first part of the assertion follows from Fact 43 (1) and Proposition 47. For the second part of the assertion, if $|X|_{\mathcal{V}_p}^{\text{uni}} = \emptyset$, there is nothing to prove. So assume $|X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$. The equality $|X|_{\mathcal{V}_p}^{\text{uni}} = |X|_{\mathcal{V}_p^{\text{ss}}}^{\text{uni}}$ is straightforward. From the first part of the assertion, it is thus enough to prove that if \mathcal{V}_p is semisimple with $|X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$ then for every place $v|p$ of k , $\mathcal{V}_p|_{X_{k_v}}$ is automatically Hodge-Tate. Let $x \in |X|_{\mathcal{V}_p}^{\text{uni}}$. After replacing X by a connected étale cover, one may assume $\overline{G}_p = \overline{G}_p$, $G_p^\circ = G_p$ and $G_{p,x}^\circ = G_{p,x}$. Since G_p is reductive and $G_{p,x} \hookrightarrow G_p \twoheadrightarrow G_p/\overline{G}_p$ is surjective, one has

$\overline{G}_p = G_p$. In particular, for every place $v|p$ of k , $\mathcal{V}_p|_{X_{k_v}}$ remains semisimple hence, from Corollary 45, is automatically Hodge-Tate. \square

B.2.2. A construction of Beilinson-Petrov. Let k be a field and let X be a smooth, geometrically connected variety over k . The following construction, which is introduced in the proof of [P23, Thm. 8.1], where it is attributed to Beilinson.

Construction 51. Assume $X(k) \neq \emptyset$ and fix $x \in X(k)$, which we regard as a section of the structural morphism $s_X : X \rightarrow \text{spec}(k)$. Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X and let

$$A_x(\mathcal{V}_p) \subset E_x(\mathcal{V}_p) := \mathcal{V}_p^\vee \otimes s_X^*(x^*\mathcal{V}_p)$$

denote the minimal sub-local system $\mathcal{S}_p \subset E_x(\mathcal{V}_p)$ such that $\mathcal{S}_{p,\bar{x}}$ contains $Id_{\mathcal{V}_{p,\bar{x}}}$; explicitly, it corresponds to the $\pi_1(X, \bar{x})$ -subrepresentation

$$A_x(\mathcal{V}_p)_{\bar{x}} = \mathbb{Q}_p[\overline{\Pi}_p] \subset E_x(\mathcal{V}_p)_{\bar{x}} = \text{End}_{\mathbb{Q}_p}(\mathcal{V}_{p,\bar{x}}).$$

Note that, by definition, $E_x(\mathcal{V}_p)_{\bar{x}}$ is $E_x(\mathcal{V}_p)_{\bar{x}} \simeq \text{End}_{\mathbb{Q}_p}(\mathcal{V}_{p,\bar{x}})$ equipped with the action

$$\pi \cdot f = (x s_X)(\pi) \cdot f \cdot \pi^{-1}, \quad \pi \in \pi_1(X, \bar{x}), \quad f \in E_x(\mathcal{V}_p)_{\bar{x}}.$$

In particular, one has

- a canonical quotient morphism $A_x(\mathcal{V}_p) \otimes s_X^*(x^*\mathcal{V}_p)^\vee \twoheadrightarrow \mathcal{V}_p^\vee$ (sending $g \otimes \phi$ to the linear form $a \mapsto \phi(g(a))$);
- If $\overline{G}_p^\circ = \overline{G}_p$, then $x \in |X|_{\mathcal{V}_p}^{\text{cent}}$ if and only if $x \in |X|_{A_x(\mathcal{V}_p)}^{\text{triv}}$.

Fact 52. ([P23, Prop. 8.2]) *If k be a p -adic field the \mathbb{Q}_p -local system $A_x(\mathcal{V}_p)$ is de Rham.*

* * *

In the remaining part of this section, unless otherwise mentioned, k is a number field and X is a smooth, geometrically connected variety over k .

B.3. Comparing $|X|_{\mathcal{V}_p}^{\text{triv}}$, $|X|_{\mathcal{V}_p}^{\text{uni}}$, $|X|_{\mathcal{V}_p}^{\text{cent}}$. Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . Tautologically $|X|_{\mathcal{V}_p}^{\text{triv}} \subset |X|_{\mathcal{V}_p}^{\text{uni}}$ and if $x^*\mathcal{V}_p$ is semisimple for every $x \in |X|$, then $|X|_{\mathcal{V}_p}^{\text{triv}} = |X|_{\mathcal{V}_p}^{\text{uni}}$. If k is a number field, one has the following less obvious result.

Proposition 53. *Assume k is a number field. Then,*

- (1) *One has $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset \Rightarrow |X|_{\mathcal{V}_p}^{\text{triv}} = |X|_{\mathcal{V}_p}^{\text{uni}}$;*
- (2) *In general, one always has*

$$|X|_{\mathcal{V}_p}^{\text{triv}} \subset |X|_{\mathcal{V}_p}^{\text{uni}} \subset |X|_{\mathcal{V}_p}^{\text{cent}}.$$

Proof. Assertion (1) is a special case of Corollary 50 as every $x \in |X|_{\mathcal{V}_p}^{\text{triv}}$ satisfies the assumption of Corollary 50. For Assertion (2), if $|X|_{\mathcal{V}_p}^{\text{uni}} = \emptyset$ there is nothing to prove. Otherwise, one may replace k by a finite field extension hence assume $|X|_{\mathcal{V}_p}^{\text{uni}} \cap X(k) \neq \emptyset$ and X by a connected étale cover hence assume $\overline{G}_p^\circ = \overline{G}_p$. Let $x \in |X|_{\mathcal{V}_p}^{\text{uni}} \cap X(k)$. With the notation of Construction 51, $x \in |X|_{E_x(\mathcal{V}_p)}^{\text{uni}} \subset |X|_{A_x(\mathcal{V}_p)}^{\text{uni}}$. But then, from Fact 52 and Corollary 50,

$$|X|_{A_x(\mathcal{V}_p)}^{\text{triv}} = |X|_{A_x(\mathcal{V}_p)}^{\text{uni}},$$

so that the conclusion follows from the fact that $x \in |X|_{\mathcal{V}_p}^{\text{cent}}$ if and only if $x \in |X|_{A_x(\mathcal{V}_p)}^{\text{triv}}$.

B.4. "Easy" implications in Conjectures (U), (C). Again, in Conjecture (U), the implications (i) \Rightarrow (ii) \Rightarrow (iii), and, in Conjecture (C), the implications (i) \Leftarrow (i)' \Rightarrow (ii) \Rightarrow (iii) are tautological while, both in Conjecture (U) and Conjecture (C), the implications (ii) \Rightarrow (i) follow from Fact 46. In Conjecture (C), the implication (i) \Rightarrow (ii) follows from the fact G_p° is generated by $\overline{G}_p^\circ, G_{p,x}^\circ$. Eventually, the implication (i) \Rightarrow (i)' both in Conjecture (U) and Conjecture (C) follows from Lemma 54, which is a consequence of geometric class field theory. More precisely, for Conjecture (C), this is straightforward as (i) implies $\overline{G}_p^\circ \xrightarrow{\sim} (\overline{G}_p^{\circ, \text{ab}})_{G_p^\circ/\overline{G}_p^\circ}$ while, for Conjecture (U), as (i) implies that both \overline{G}_p° and $G_p^\circ/\overline{G}_p^\circ$ are unipotent, if $\overline{G}_p^\circ \neq 1$ one would have $\overline{G}_p^\circ \simeq \mathbb{G}_{a, \mathbb{Q}_p}^{\oplus r}$ for some integer $r \geq 1$ hence $(\overline{G}_p^{\circ, \text{ab}})_{G_p^\circ/\overline{G}_p^\circ} \neq 0$, contradicting Lemma 54.

Lemma 54. *One has $(\overline{G}_p^{\circ, \text{ab}})_{G_p^\circ/\overline{G}_p^\circ} = 1$.*

Proof. Replacing \mathcal{V}_p on X with $\alpha^*\mathcal{V}_p$ on X' for some connected étale cover $X' \rightarrow X$, one may assume $G_p^\circ = G_p$, $\overline{G}_p^\circ = \overline{G}_p$ and $X(k) \neq \emptyset$. Let $x \in X(k)$ and fix a geometric point \bar{x} over x , which gives rise to a morphism of split short exact sequences of profinite groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{k^s}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & \pi_1(k) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & (\pi_1(X_{k^s}, \bar{x})^{\text{ab}})_{\pi_1(k)} & \longrightarrow & \pi_1(X, \bar{x})^{\text{ab}} & \longrightarrow & \pi_1(k) \longrightarrow 1 \end{array}$$

\xleftarrow{x} (top row) \xrightarrow{x} (bottom row)

From [KL81, Thm. 1], $(\pi_1(X_{k^s}, \bar{x})^{\text{ab}})_{\pi_1(k)}$ is finite. The assertion thus follows from the fact that, by construction / definition, the image of the canonical map $(\pi_1(X_{k^s}, \bar{x})^{\text{ab}})_{\pi_1(k)} \rightarrow (\overline{G}_p^{\circ, \text{ab}})_{G_p^\circ/\overline{G}_p^\circ}(\mathbb{Q}_p)$ is Zariski-dense in $(\overline{G}_p^{\circ, \text{ab}})_{G_p^\circ/\overline{G}_p^\circ}$. \square

To summarize, one has:

$$\begin{array}{ccccc} \text{(iii)} & \begin{array}{c} \xrightarrow{??} \\ \xleftarrow{\quad} \end{array} & \text{(ii)} & \longleftrightarrow & \text{(i)} & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{(C)} \end{array} & \text{(i)'} \end{array}$$

B.5. Relations between Conjectures (T), (U), (C). From Proposition 53 (2), for every \mathbb{Q}_p -local system on X , one has

$$\begin{aligned} \text{[(iii)} \Rightarrow \text{(i) in Conjecture (C) for } \mathcal{V}_p] & \implies \text{[(iii)} \Rightarrow \text{(i) in Conjecture (U) for } \mathcal{V}_p] \\ & \implies \text{[(iii)} \Rightarrow \text{(i) in Conjecture (T) for } \mathcal{V}_p]. \end{aligned}$$

Construction 51 can also be used to prove the following.

Corollary 55. *For every \mathbb{Q}_p -local system on X , one has*

$$\text{[(iii)} \Rightarrow \text{(i) in Conjecture (T) for } A_x(\mathcal{V}_p) \text{ and some } x \in |X|_{\mathcal{V}_p}^{\text{cent}}] \implies \text{[(iii)} \Rightarrow \text{(i) in Conjecture (C) for } \mathcal{V}_p].$$

Proof. Let \mathcal{V}_p be a \mathbb{Q}_p -local system on X such that $|X|_{\mathcal{V}_p}^{\text{cent}} \neq \emptyset$. One may replace k by a finite field extension hence assume $|X|_{\mathcal{V}_p}^{\text{cent}} \cap X(k) \neq \emptyset$ and X by a connected étale cover hence assume $\overline{G}_p^\circ = \overline{G}_p$. Let $x \in |X|_{\mathcal{V}_p}^{\text{cent}} \cap X(k)$. Then $x \in |X|_{A_x(\mathcal{V}_p)}^{\text{triv}}$. In particular, $|X|_{A_x(\mathcal{V}_p)}^{\text{triv}} \neq \emptyset$ so that by Conjecture (T) (T) for $A_x(\mathcal{V}_p)$, the étale fundamental group $\pi_1(X)$ - hence *a fortiori* $\pi_1(X_{k^s})$, acts on $A_x(\mathcal{V}_p)_{\bar{x}} = \mathbb{Q}_p[\overline{\Pi}_p]$ through a finite quotient. But this means in particular that the orbit $\overline{\Pi}_p \simeq \overline{\Pi}_p \cdot Id$ is finite. \square

B.6. Relaxing the smoothness assumption on X . We formulated Conjecture (T) (resp. (U), (C)) for *smooth* geometrically connected varieties over k but, actually, this implies (hence is equivalent) to Conjecture (T) (resp. (U), (C)) for geometrically irreducible varieties over k which are only assumed to be normal. More precisely, let (P) be a property of \mathbb{Q}_p -local systems preserved by base-change along morphisms $Y \rightarrow X$ of varieties over k .

Proposition 56. *Assume Conjecture (T) (resp. (U), (C)) holds for \mathbb{Q}_p -local systems satisfying (P) on smooth, geometrically connected curves X over k . Let X be a geometrically irreducible variety over k and let \mathcal{V}_p be a \mathbb{Q}_p -local system on X . Assume that X is normal and that $\mathcal{V}_p|_{X^{\text{sm}}}$ satisfies (P), where $X^{\text{sm}} \hookrightarrow X$ denotes the non-empty open smooth (viz regular, as k has characteristic 0) locus. Then Conjecture (T) (resp. (U), (C)) for $\mathcal{V}_p|_{X^{\text{sm}}}$ implies Conjecture (T) (resp. (U), (C)) for \mathcal{V}_p .*

Proof. As X is geometrically irreducible over k and $X^{\text{sm}} \hookrightarrow X$ is an open immersion, X^{sm} is geometrically irreducible over k . As X is normal and k is perfect, $X_{\bar{k}}$ is again normal and $(X^{\text{sm}})_{\bar{k}} = X_{\bar{k}}^{\text{sm}}$. In particular, the canonical morphisms $\pi_1(X^{\text{sm}}) \rightarrow \pi_1(X)$ and $\pi_1(X_{\bar{k}}^{\text{sm}}) \rightarrow \pi_1(X_{\bar{k}})$ are both surjective. So, in view of statement (i) in Conjecture (T) (resp. (U), (C)), it is enough to prove that $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$ (resp. $|X|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$, resp. $|X|_{\mathcal{V}_p}^{\text{cent}} \neq \emptyset$) implies $|X^{\text{sm}}|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$ (resp. $|X^{\text{sm}}|_{\mathcal{V}_p}^{\text{uni}} \neq \emptyset$, resp. $|X^{\text{sm}}|_{\mathcal{V}_p}^{\text{cent}} \neq \emptyset$). We treat the case of Conjecture (T); the cases of Conjectures (U) and (C) are exactly similar. Assume $|X|_{\mathcal{V}_p}^{\text{triv}} \neq \emptyset$ and let $x_0 \in |X|_{\mathcal{V}_p}^{\text{triv}}$. Fix any $x \in X^{\text{sm}}$. From *e.g.* [Mu70, Lem. p.56], there exists a finite field extension K of k , a smooth, geometrically connected curve C over K and a morphism $\alpha : C \rightarrow X$ such that $x_0, x \in \alpha(C)$. In particular, $|C|_{\alpha^*\mathcal{V}_p}^{\text{triv}} \supset \alpha^{-1}(x_0) \neq \emptyset$ hence, Conjecture (T) applied to $\alpha^*\mathcal{V}_p$ on C implies $|C|_{\alpha^*\mathcal{V}_p}^{\text{triv}} = |C|$ which, in turn, yields $x \in |X|_{\mathcal{V}_p}^{\text{triv}}$. \square

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