UNIFORM BOUNDEDNESS OF p-PRIMARY TORSION OF ABELIAN SCHEMES (*Preliminary version*)

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ABSTRACT. Let k be a field finitely generated over $\mathbb{Q}$ and denote by Γ_k its absolute Galois group. Let A be an abelian variety over k. Given a prime p and a character $\chi : \Gamma_k \to \mathbb{Z}_p^*$, define the *twisted p-primary torsion* associated with χ to be the Γ_k-module:

$$A[p^\infty](\chi) := \{T \in A[p^\infty] \mid {}^\sigma T = \chi(\sigma)T,\ \sigma \in \Gamma_k\},$$

where $A[p^\infty]$ denotes the p-primary torsion of $A(\overline{k})$. Say that χ is *non-Tate* if it does not appear as a subrepresentation of the p-adic representation associated with an abelian variety over k. (This holds, for instance, if χ is the trivial or the p-adic cyclotomic character.) If χ is non-Tate, $A[p^\infty](\chi)$ is always finite. Our main result is about the uniform boundedness of $A[p^\infty](\chi)$ when A varies in a 1-dimensional family. More precisely, if S is a smooth, geometrically connected curve over k and A is an abelian scheme over S, then there exists an integer $N := N(A, S, k, p, \chi)$, such that $A_s[p^\infty](\chi) \subset A_s[p^N]$ holds for any $s \in S(k)$.

This result is obtained as a corollary of the following geometric statement for the p-primary torsion of abelian varieties over function fields of curves. Let K be the function field of a smooth, geometrically connected curve over an algebraically closed field of characteristic 0 and let A be an abelian variety over K. Assume for simplicity that A contains no nontrivial isotrivial subvariety. Then, for any $c \geq 0$, there exists an integer $N := N(c, A, S, k, p) \geq 0$ such that $A[p^\infty](K') \subset A[p^N]$ for all finite extension K'/K with K' of genus $\leq c$.

Our result about the uniform boundedness of twisted p-primary torsion for the p-adic cyclotomic character together with certain descent methods yields a proof of the 1-dimensional case of the modular tower conjecture, which was, actually, the original motivation for this work. The modular tower conjecture is a conjecture from regular inverse Galois theory posed by M. Fried in the early 1990s, (a generalized variant of) which roughly states that if $(H_{n+1} \to H_n)$ is a projective system of moduli spaces classifying G-covers of genus g curves with group G_n with r ramification points, then, for any field k finitely generated over $\mathbb{Q}$, $H_n(k) = \emptyset$ for n large enough, provided the projective limit of $(G_{n+1} \to G_n)$ admits an open subgroup which is free pro-p of finite positive rank.

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1. INTRODUCTION

Fix a prime p and let k be a field of characteristic $q \neq p$ and let $S \to k$ be a smooth, geometrically connected curve over k of type (g, r) (that is such that there exists a proper, smooth, geometrically connected, genus g curve $\overline{S} \to k$ over k and a degree r etale divisor $D \subset \overline{S}$ such that $S = \overline{S} \smallsetminus D$). Denote by $\eta : k(S) \to S$ the generic point of S.

Given a geometric generic point $s : \Omega \to S$ denote by $\Pi := \pi_1(S, s)$ the etale fundamental group of S. Recall that, in our situation, it is simply the Galois group of the maximal algebraic extension of $k(S)$ in Ω which is unramified on S; in particular, it is a quotient of the absolute Galois group $\Gamma_\eta := \mathrm{Gal}(k(S)^{sep}/k(S))$, where $k(S)^{sep}$ stands for the separable closure of $k(S)$ in Ω.

Now, fix an abelian scheme $A \to S$ with generic fiber $A_\eta \to \eta$ of dimension d. We refer to section 2.1 for the notation used for abelian varieties. Since $A \to S$ is an abelian scheme and $q \neq p$, the natural action of Γ_η on the Tate module $T_p(A_\eta)$ factors through Π. This, in turn, defines an action of Π over $A_\eta[p^n]$, $n \geq 0$ hence over $A_\eta[p^\infty]$. Finally, for any $v \in A_\eta[p^\infty]$, write Π_v for the stabilizer of v under Π; it is a closed subgroup of Π of index $\leq |\langle v\rangle|^{2d}$, hence it is also open in Π, and, by Galois theory, corresponds to a connected finite etale cover $S_v \to S$ (defined over a finite extension k_v/k). Similarly, write $\Pi_{\langle v\rangle}$ for the stabilizer of $\langle v\rangle$ under Π and $S_{\langle v\rangle} \to S$ for the resulting connected finite etale cover (defined over a finite extension $k_{\langle v\rangle}/k$). The inclusion of open subgroups $\Pi_v \subset \Pi_{\langle v\rangle} \subset \Pi$

yields a commutative diagram of finite etale covers:

$$\begin{array}{ccc} S_v & \longrightarrow & S_{\langle v\rangle} \\ \downarrow & \swarrow & \\ S & & \end{array}$$

We will write g_v and $g_{\langle v\rangle}$ for the genus of the smooth compactifications of $S_v \times_{k_v} \overline{k}$ and $S_{\langle v\rangle} \times_{k_{\langle v\rangle}} \overline{k}$, respectively.

Eventually, denote by $(A_\eta)_0$ the largest abelian subvariety of A_η which is isogenous to an isotrivial abelian variety (see section 2).

We can now state the main geometric result of this paper:

Theorem 1.1. *Assume that k is algebraically closed. Then, for any $c \geq 0$, there exists an integer $N := N(c, A, S, k, p) \geq 0$ such that for all $v \in A_\eta[p^\infty]$ either $g_v \geq c$ or $p^N v \in (A_\eta)_0$. The same statement holds if g_v is replaced by $g_{\langle v\rangle}$.*

Say that a character $\chi : \Gamma_k \to \mathbb{Z}_p^*$ is *non-Tate* if it does not appear as a subrepresentation of the p-adic representation associated with an abelian variety over k. This holds, for instance, if χ is the trivial or the p-adic cyclotomic character, provided k is finitely generated over the prime field. When χ is non-Tate, $A[p^\infty](\chi)$ is always finite. Theorem 1.1 yields the following corollary about the uniform boundedness of $A[p^\infty](\chi)$ when A varies in a 1-dimensional family.

Corollary 1.2. *Assume that k is finitely generated over $\mathbb{Q}$ and let $\chi : \Gamma_k \to \mathbb{Z}_p^*$ be a non-Tate character. Then there exists an integer $N := N(A, S, k, p, \chi)$, such that, for any $s \in S(k)$, the Γ_k-module $A_s[p^\infty](\chi) := \{T \in A_s[p^\infty] \mid {}^\sigma T = \chi(\sigma)T,\ \sigma \in \Gamma_k\}$ is contained in $A_s[p^N]$.*

The strategy of the proof of theorem 1.1 is as follows. First, we show that it is enough to prove the following statement (Theorem 4.1): *Assume that A_η contains no nontrivial abelian subvariety isogenous to an isotrivial abelian variety, and, for any $v \in T_p(A_\eta)^*$, set $v_n := v \bmod p^n T_p(A_\eta) \in A_\eta[p^n]^*$, $n \geq 0$. Then $g_{v_n} \to \infty$ and $g_{\langle v_n\rangle} \to \infty$.*

Then, to any $v \in T_p(A_\eta)^*$ associate the Π-module $M := \mathbb{Q}_p[\Pi]v \cap T_p(A_\eta)$ and denote by $\Pi(n)$ the kernel of the reduction morphism $\mathrm{GL}(M) \twoheadrightarrow \mathrm{GL}(M/p^n)$. The inclusion of open subgroups $\Pi(n) \subset \Pi_{v_n} \subset \Pi$ corresponds to a sequence of finite etale covers $S(n) \to S_{v_n} \to S$, with $S(n) \to S$ Galois, which makes it easier to handle. We first prove that the genus $g(n)$ of $S(n)$ goes to infinity with n. This is achieved by using the fact that the image of Π in $\mathrm{GL}(M)$ cannot be almost-abelian (section 2.3). Then, using the Riemann-Hurwitz formula, we compare $g(n)$ and g_{v_n} and show that the "increasing rate" of g_{v_n} is "close enough" to that of $g(n)$ for $n \gg 0$. This relies on the asymptotic bound given by J. Oesterlé for the number of points on reduction modulo p^n of p-adic analytic subspaces of $\mathbb{Z}_p^m$. The proof of the variant for $g_{\langle v\rangle}$ is along the same lines but working with the projectivization of the representation of Π over M.

To deduce corollary 1.2 from theorem 1.1, we introduce a sequence of (disconnected) finite etale covers $\mathcal{S}_{n+1,\chi} \to \mathcal{S}_{n,\chi}$ of S with the property that k-rational points $s_n : k \to \mathcal{S}_{n,\chi}$ lying above $s : k \to S$ correspond to elements of order exactly p^n in $A_s[p^\infty](\chi)$. It is thus enough to prove that $\mathcal{S}_{n,\chi}(k) = \emptyset$ for $n \gg 0$. According to theorem 1.1, $\mathcal{S}_{n,\chi}$ can be written as a disjoint union $\mathcal{S}_{n,\chi} = \mathcal{S}_{n,\chi}^{(1)} \coprod \mathcal{S}_{n,\chi}^{(2)}$, where $\mathcal{S}_{n,\chi}^{(1)}$ corresponds to the "isotrivial part" of $A_\eta[p^n]$ and $\mathcal{S}_{n,\chi}^{(2)}$ to the "non-isotrivial" one. On one hand, we show that $\mathcal{S}_{n,\chi}^{(1)}(k) = \emptyset$ for $n \gg 0$, and, on the other hand, from theorem 1.1, the genus of each component of $\mathcal{S}_{n,\chi}^{(2)}$ is larger than 2 for $n \gg 0$. Applying the Mordell conjecture (Faltings' finiteness theorem) and the definition of non-Tate p-adic character then yields the result.

Our original motivation for corollary 1.2 was to prove the 1-dimensional case of the modular tower conjecture. The modular tower conjecture is a conjecture from regular inverse Galois theory posed by M. Fried in the early 1990s (cf. [Fr95]), (a generalized variant of) which roughly states that if

$(H_{n+1} \to H_n)$ is a projective system of moduli spaces classifying G-covers of genus g curves with group G_n and ramification indices $e_{n,1}, \dots, e_{n,r}$, then, for any field k finitely generated over $\mathbb{Q}$, $H_n(k) = \emptyset$ for n large enough, provided the projective limit of $(G_{n+1} \to G_n)$ admits an open subgroup which is free pro-p of finite positive rank. As the H_n have dimension $3g-3+r$ (under the hyperbolicity condition $2-2g-r<0$), the 1-dimensional case corresponds to $(g,r) = (0,4),\ (1,1)$. (The 0-dimensional case corresponds to $(g,r) = (0,3)$ and is easy to treat. See lemma 5.11.) In that case, we thus have (theorem 5.12): *The modular tower conjecture holds for* $(g,r) = (0,4),\ (1,1)$. A typical example of the modular tower conjecture is when one takes for G_n the dihedral group D_{2p^n} of order $2p^n$, $(g,r) = (0,2s)$, and $e_{n,1} = \cdots = e_{n,2s} = 2$. Then, roughly speaking, H_n classifies jacobians of genus $s-1$ hyperelliptic curve with a torsion point of order excatly p^n. In particular, for $s=2$, $(H_{n+1} \to H_n)$ is just the projective system of modular curves $(Y_1(p^{n+1}) \to Y_1(p^n))$ and, in that special case, the modular tower conjecture can be deduced from the Mordell conjecture (Faltings' finiteness theorem) and the fact that $\varprojlim Y_1(p^n)(k) = \emptyset$. This is the basic idea (essentially due to Manin [Ma69]) hidden behind our proof of corollary 1.2 and its application to the modular tower conjecture. In the general case, one has to distinguish between the fine and coarse moduli situations. If H_n is a fine moduli scheme for $n \geq 0$, then one can apply directly corollary 1.2 to the p-adic cyclotomic character and the jacobian $\mathrm{Pic}^0_{\mathcal{C}_0|H_0} \to H_0$ of the universal curve $\mathcal{C}_0 \to H_0$. When H_n is a mere coarse moduli scheme for $n \geq 0$, the idea is, roughly, to use descent techniques to construct an auxilliary tower $(\tilde{H}_{n+1} \to \tilde{H}_n)$ of fine moduli schemes with the property that $\tilde{H}_n(k) \neq \emptyset$ as soon as $H_n(k) \neq \emptyset$.

The paper is organized as follows. In section 2, we list the notation used for abelian varieties (section 2.1) and collect technical preliminaries about non-Tate characters (section 2.2) and almost-abelian p-adic representations (section 2.3). In section 3, we prove a result (theorem 3.1) concerning reduction modulo p^n of certain p-adic analytic spaces, which is the technical core of our proofs of the main results. In section 4, the proofs of theorem 1.1 corollary 1.2 are carried out in sections 4.1 and 4.2, respectively. Eventually, in section 5, we convey the proof of the 1-dimensonal modular tower conjecture. After recalling the basic notion about G-curves and their stacks (section 5.1), we describe carefully our descent techniques (section 5.2) and apply them together with corollary 1.2 (section 5.3).

Remark 1.3. In [Fr06], Fried outlined a proof, which is different from ours, of (the original version of) the modular tower conjecture for $(g,r) = (0,4)$.

2. Preliminaries on abelian schemes

2.1. **Notation for abelian varieties.** Fix a prime p and let k be a field of characteristic $q \neq p$. Given an abelian variety $A \to k$, we will use the following (classical) notation:

$A[p^n] := ker([p^n] : A(\overline{k}) \to A(\overline{k}))$ for the kernel of the multiplication-by-p^n endomorphism;
$A[p^\infty] := \bigcup_{n \geq 0} A[p^n]$ for the p-primary torsion of $A(\overline{k})$.
$T_p(A) := \varprojlim A[p^n]$ for the p-adic Tate module of A;
$A[p^n]^* := A[p^n] \smallsetminus pA[p^n]$;
$T_p(A)^* := T_p(A) \smallsetminus pT_p(A) = \varprojlim A[p^n]^*$;
$V_p(A) := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$;

Now, if $A \to \eta$ is defined over the function field η of a k-curve $S \to k$, we will write A_0 for the largest abelian subvariety of A which is isogeneous to an isotrivial abelian variety. (Herem we say that an abelian variety $B \to \eta$ is isotrivial, if $B \times_\eta \overline{\eta}$ descends to an abelian variety over $\overline{k}$.) Equivalently, A_0 is the largest abelian subvariety of A which is a homomorphic image of an isotrivial abelian variety. Indeed, if $A_1 \to A_2$ is an epimorphism of abelian varieties, then, by Poincaré's complete reducibility theorem, A_2 is isogeneous to an abelian subvariety of A_1. But as any abelian subvariety of an isotrivial abelian variety is again isotrivial, if A_1 is isotrivial then A_2 is isogeneous to an isotrivial abelian subvariety of A_1. In particular, if $A_i \subset A$ is isogeneous to an isotrivial abelian variety, $i = 1, 2$, then so is $A_1 + A_2 \subset A$ (as a homomorphic image of $A_1 \times A_2$); thus A_0 is well-defined. (When k has characteristic 0, any abelian variety isogeneous to an isotrivial abelian variety is again isotrivial, so A_0 is simply the largest isotrivial subvariety of A.) We define A^0 to be the quotient abelian variety A/A_0. Then (again by Poincaré's complete reducibility theorem) we have $(A^0)_0 = 0$.

2.2. **Non-Tate characters.** Fix a prime p and let k be a field of characteristic $q \neq p$. Let $\chi : \Gamma_k \to \mathbb{Z}_p^*$ be a character. Say that χ is *non-Tate* if it does not appear as a subrepresentation of the p-adic representation associated with an abelian variety over k. When k is finitely generated over its prime field, the trivial character and the p-adic cyclotomic character are typical examples of non-Tate characters.

Lemma 2.1. *For any finitely generated extension k' of k, $\chi : \Gamma_k \to \mathbb{Z}_p^*$ is non-Tate if and only if $\chi|_{\Gamma_{k'}} : \Gamma_{k'} \to \mathbb{Z}_p^*$ is non-Tate.*

Proof. First, the 'if' part is trivial. Indeed, if χ appears in $T_p(A)$ for some abelian variety A over k, $\chi|_{\Gamma_{k'}}$ appears in $T_p(A \times_k k')$. Next, to show the 'only if' part, suppose that $\chi|_{\Gamma_{k'}}$ appears in $T_p(A')$ for some abelian variety A' over k'.

Assume first that k' is a finite extension of k. In this case, one can easily reduce the problem to the following two cases: k'/k is separable; and k'/k is purely inseparable (for $q > 0$). In the former case, define A to be the Weil restriction $Res_{k'/k}(A')$, which is an abelian variety over k with dimension $[k' : k]\dim(A')$. Then the adjoint property of Weil restriction implies that χ appears in $T_p(A)$. Indeed, more specifically, let M_n denote the finite etale commutative group scheme over k that corresponds to the Galois module $\mathbb{Z}/p^n(\chi_n)$ (i.e., the module $\mathbb{Z}/p^n$ on which Γ_k acts via χ_n). Then

$$\mathrm{Hom}_{\Gamma_k}(\mathbb{Z}_p(\chi), T_p(A)) = \varprojlim \mathrm{Hom}_k(M_n, A) = \varprojlim \mathrm{Hom}_{k'}(M_n \times_k k', A') = \mathrm{Hom}_{\Gamma_{k'}}(\mathbb{Z}_p(\chi|_{\Gamma_{k'}}), T_p(A')).$$

In the latter case, take $N \geq 0$ such that $(k')^{q^N} \subset k$ and set $A := A' \times_{k'} k$, where $k' \to k$ is given by $a \mapsto a^{q^N}$. Then, by using the fact that the q^Nth power map $k \to k$ induces the identity on Γ_k, one can show that χ appears in $T_p(A)$.

Finally, assume that k' is an arbitrary finitely generated extension of k and that $\chi|_{\Gamma_{k'}}$ appears in $T_p(A')$ for some abelian variety A' over k'. Then, taking a model of $A' \to k'$ over the spectrum of a finitely generated k-subalgebra of k' whose fraction field coincides with k' and specializing it at a closed point, one can show that there exist a finite extension k'' of k and an abelian variety A'' over k'' such that $\chi|_{\Gamma_{k''}}$ appears in $T_p(A'')$. Thus, by the preceding argument, χ appears in $T_p(A)$ for some abelian variety over k. □

2.3. **Almost-abelian p-adic representations.** Fix a prime p. Let V be a finite-dimensional $\mathbb{Q}_p$-vector space. For a property P of subgroups of $\mathrm{Aut}_{\mathbb{Q}_p}(V)$ (e.g., trivial, abelian, unipotent, pro-p, *etc.*), say that a subgroup G of $\mathrm{Aut}_{\mathbb{Q}_p}(V)$ is *almost-P* if there exists a closed subgroup $H \subset G$ of finite index such that H has P. Further, given a group Π and a representation $\rho : \Pi \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$, say that $\rho : \Pi \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ is almost-P if the image of ρ is almost-P (though the standard terminology for 'almost-unipotent' may be 'quasi-unipotent').

Let k be an algebraically closed field of characteristic $q \neq p$ and let $S \to k$ be a connected normal scheme of finite type over k. Denote by $\eta : k(S) \to S$ the generic point of S. Let $A \to S$ be an abelian scheme with generic fiber $A_\eta \to \eta$, and consider the natural p-adic representation on the Tate module $\rho : \pi_1(S) \to \mathrm{Aut}_{\mathbb{Z}_p}(T_p(A_\eta)) \subset \mathrm{Aut}_{\mathbb{Q}_p}(V_p(A_\eta))$.

Proposition 2.2. *Let M be a $\pi_1(S)$-submodule of $T_p(A_\eta)$ and and consider the corresponding representation $\rho_M : \pi_1(S) \to \mathrm{Aut}_{\mathbb{Z}_p}(M) \subset \mathrm{Aut}_{\mathbb{Q}_p}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Then the following are all equivalent.*

(1) *ρ_M is almost-trivial (i.e., ρ_M has finite image).*
(2) *ρ_M is almost-abelian.*
(3) *ρ_M is almost-unipotent.*
(4) *$M \subset T_p((A_\eta)_0)$.*

Proof. (1)⇒(2). Clear.

(2)⇒(3). First observe $\mathrm{Aut}_{\mathbb{Z}_p}(M) \simeq \mathrm{GL}_m(\mathbb{Z}_p)$, where m is the rank of the free $\mathbb{Z}_p$-module M. Thus, from the short exact sequence

$$1 \to 1 + p\mathrm{M}_m(\mathbb{Z}_p) \to \mathrm{GL}_m(\mathbb{Z}_p) \to \mathrm{GL}_m(\mathbb{Z}/p) \to 1,$$

the group $\mathrm{Aut}_{\mathbb{Z}_p}(M)$ is almost-pro-p. This, together with the assumption that ρ_M is almost-abelian, assures that one may assume that $\rho_M(\pi_1(S))$ is an abelian pro-p group, up to replacing S by a finite etale cover.

Since S is of finite type over k, there exists a subfield k_1 of k, finitely generated over its prime field, such that $A \to S \to k$ admits a model $A_1 \to S_1 \to k_1$, where $S_1 \to k_1$ is a geometrically connected, geometrically normal scheme of finite type over k_1 and $A_1 \to S_1$ is an abelian scheme. Denote by $\eta_1 : k_1(S_1) \to S_1$ the generic point of S_1 and by $A_{1,\eta_1} \to \eta_1$ the generic fiber of $A_1 \to S_1$. One has a canonical surjection $\pi_1(S) \twoheadrightarrow \pi_1((S_1)_{\overline{k_1}})$ ([SGA1, Exp. X, proof of Cor. 1.6]). At the level of Tate modules, one has a canonical isomorphism $T_p(A_\eta) \tilde{\to} T_p(A_{1,\eta_1})$ which can be taken to be compatible with the actions of $\pi_1(S)$ and $\pi_1((S_1)_{\overline{k_1}})$. Thus, one may assume that $k = \overline{k_1}$. Here, observe that $T_p(A_\eta) = T_p(A_{1,\eta_1})$ admits a natural action of $\pi_1(S_1)$, extending the action of $\pi_1(S) = \pi_1((S_1)_k)$.

Denote by Δ the maximal abelian pro-p quotient of $\pi_1(S)$ and by N the kernel of the natural surjection $\pi_1(S) \twoheadrightarrow \Delta$. Set $\Pi := \pi_1(S_1)/N$. Thus, one gets the following exact sequence

$$1 \to \Delta \to \Pi \to \Gamma_{k_1} \to 1.$$

Since Δ is abelian, this exact sequence induces a natural action of Γ_{k_1} on Δ.

Lemma 2.3. *For any finite extension k_1' of k_1, the coinvariant quotient $\Delta_{\Gamma_{k_1'}}$ is finite.*

Proof. This is essentially widely known, and follows, for example, from the fact that the (Frobenius) weights that may appear in $H^1_{\mathrm{et}}(S, \mathbb{Q}_p)$ are 1 and 2. □

As $\rho_M(\pi_1(S))$ is an abelian pro-p group, $\rho_M : \pi_1(S) \to \mathrm{Aut}_{\mathbb{Z}_p}(M)$ factors through Δ, or, equivalently, M is contained in $T_p(A_\eta)^N$. Now, up to replacing M by $T_p(A_\eta)^N$, one may assume that $M \subset T_p(A_\eta)$ is a $\pi_1(S_1)$-submodule and that the resulting representation $\rho_M : \pi_1(S_1) \to \mathrm{Aut}_{\mathbb{Z}_p}(M)$ factors through Π.

Now, define $\overline{\Delta}$ and $\overline{\Pi}$ to be the images of Δ and Π in $\mathrm{Aut}_{\mathbb{Z}_p}(M)$. Set $R_0 := \mathbb{Z}_p[\overline{\Delta}] \subset \mathrm{End}_{\mathbb{Z}_p}(M)$ and $R := R_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. As R_0 is a free $\mathbb{Z}_p$-module, the canonical morphism $R_0 \to R$ is injective, hence so is $R_0^{red} \to R^{red}$, where $R_0^{red} = R_0/\sqrt{0_{R_0}}$ and $R^{red} := R/\sqrt{0_R}$. We shall denote the natural surjection $R \twoheadrightarrow R^{red}$ by $f \mapsto f^{red}$. The action of $\overline{\Pi}$ by conjugation on $\overline{\Delta}$ extends to one on R_0 by $\mathbb{Z}_p$-linearity and to one on R by $\mathbb{Q}_p$-linearity. Note that $\overline{\Pi}$ acts on R not only as $\mathbb{Q}_p$-module automorphisms but also as $\mathbb{Q}_p$-algebra automorphisms. In particular, the action of $\overline{\Pi}$ on R induces one on R^{red}. Denote by $i : \Pi \twoheadrightarrow \overline{\Pi} \to \mathrm{Aut}_{\mathbb{Q}_p\text{-}alg}(R)$ and $i^{red} : \overline{\Pi} \to \mathrm{Aut}_{\mathbb{Q}_p\text{-}alg}(R^{red})$ the resulting representations. Observe that i (hence also i^{red}) factors through $\Pi \twoheadrightarrow \Gamma_{k_1}$, since Δ is abelian.

Since R^{red} is a finite reduced commutative $\mathbb{Q}_p$-algebra, it is isomorphic to a product $\prod_{1 \le i \le r} L_i$ of finite field extensions $L_i/\mathbb{Q}_p$, $i = 1, ..., r$ and its $\mathbb{Q}_p$-algebra automorphism group is finite. In particular, i^{red} has finite image. So, up to replacing k_1 by a finite extension, one may assume that i^{red} is trivial. Then, if we denote the image of Δ in $(R_0^{red})^* \subset (R^{red})^*$ by $\overline{\Delta}^{red}$, it follows that the natural surjection $\Delta \twoheadrightarrow \overline{\Delta}^{red}$ factors through $\Delta_{\Gamma_{k_1}}$. Thus, by lemma 2.3, $\overline{\Delta}^{red}$ is finite.

Now, up to replacing S by a finite etale cover, Δ has trivial image in $(R^{red})^*$, or, equivalently, has unipotent image in $R^* \subset \mathrm{Aut}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, as desired.

(3)⇒(4). Up to replacing S by a finite etale cover, we may assume that ρ_M is unipotent. Denote by M^0 the image of M in $T_p(A_\eta)/T_p((A_\eta)_0) = T_p((A_\eta)^0)$, and suppose $M^0 \neq \{0\}$. Then $\rho_{M^0} : \pi_1(S) \to \mathrm{Aut}(M^0)$ is also unipotent. From this, there exists a $v \in M^0 \smallsetminus \{0\}$ such that $\pi_1(S)$ acts trivially on v. In particular, this forces the p-primary torsion of $(A_\eta)^0(\eta)$ to be infinite. But this contradicts the fact that $(A_\eta)^0(\eta)$ is a finitely generated abelian group by the Lang-Néron theorem ([LN59]). Thus, $M^0 = \{0\}$, as desired.

(4)⇒(1). Immediate from the definition of $(A_\eta)_0$. □

Corollary 2.4. *The following are all equivalent.*

(1) $T_p(A_\eta)$ *admits no nonzero* $\pi_1(S)$*-submodule* M *with* ρ_M *almost-trivial.*
(2) $T_p(A_\eta)$ *admits no nonzero* $\pi_1(S)$*-submodule* M *with* ρ_M *almost-abelian.*
(3) $T_p(A_\eta)$ *admits no nonzero* $\pi_1(S)$*-submodule* M *with* ρ_M *almost-unipotent.*
(4) $(A_\eta)_0 = 0$.

Proof. (4) is equivalent to $T_p((A_\eta)_0) = \{0\}$, hence to saying that $T_p(A_\eta)$ admits no nonzero $\pi_1(S)$-submodule M with $M \subset T_p((A_\eta)_0)$. Thus, corollary 2.4 immediately follows from proposition 2.2. □

Corollary 2.5. *Consider the following conditions:*
(1) ρ *is almost-trivial.*
(2) ρ *is almost-abelian.*
(3) ρ *is almost-unipotent.*
(4) $(A_\eta)_0 = A_\eta$.
(5) A_η *is isotrivial.*

Then we have (1) $\Leftrightarrow$ *(2)* $\Leftrightarrow$ *(3)* $\Leftrightarrow$ *(4)* $\Leftarrow$ *(5). (In particular, if* $\pi_1(S)$ *itself is almost-abelian (i.e., admits an abelian open subgroup), then (4) automatically holds.)*

If, morever, $q = 0$*, (1)–(5) are all equivalent.*

Proof. Applying proposition 2.2 to $M = T_p(A_\eta)$, one gets the equivalence of (1)–(4). For (5)$\Rightarrow$(4) and (4)$\Rightarrow$(5) for $q = 0$, see 2.1. □

Remark 2.6. In corollary 2.5, the equivalences (1)$\Leftrightarrow$(4) and (1)$\Leftrightarrow$(5) for $q = 0$ directly follow from [G66, Prop. 4.3 and Prop. 4.4].

Remark 2.7. Let $X \to S$ be a proper, smooth, geometrically connected curve and apply corollary 2.5 to $A = \mathrm{Pic}^0_{X/S}$, the jacobian of $X \to S$. Further, consider the following condition: (6) X_η is isotrivial. Then we have (5)$\Leftrightarrow$(6). Indeed, (6)$\Rightarrow$(5) is clear and (5)$\Rightarrow$(6) essentially follows from a version of Torelli's theorem. We omit the details.

3. Reduction modulo p^n of p-adic analytic homogeneous spaces

The aim of this section is to prove theorem 3.1 concerning reduction modulo p^n of certain p-adic analytic homogeneous spaces, which is the technical core of our proofs of the main results in §4. The readers who would like to start the proofs of the main results quickly may skip this section, after glancing at the statement of theorem 3.1.

Let p be a prime. Let M be a free $\mathbb{Z}_p$-module of rank $m \geq 1$ and set $M_n := M/p^n$, $n \geq 0$, and $W := M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Set $\mathrm{GL}(M) := \mathrm{Aut}_{\mathbb{Z}_p}(M)$ and $\mathrm{GL}(M_n) := \mathrm{Aut}_{\mathbb{Z}/p^n}(M_n)$, $n \geq 0$. Let $M \twoheadrightarrow M_n$, $v \mapsto v_n$ denote reduction modulo $p^n M$ and $\mathrm{GL}(M) \twoheadrightarrow \mathrm{GL}(M_n)$, $g \mapsto g_n$ the induced morphism. As projective versions of these, set $\mathbb{P}(M) := (M \smallsetminus pM)/\mathbb{Z}_p^*$ and $\mathbb{P}(M_n) := (M_n \smallsetminus pM_n)/(\mathbb{Z}/p^n)^*$, $n \geq 0$. Set $\mathrm{PGL}(M) := \mathrm{GL}(M)/\mathbb{Z}_p^* Id$ and $\mathrm{PGL}(M_n) := \mathrm{GL}(M_n)/(\mathbb{Z}/p^n)^* Id$, $n \geq 0$. Eventually, denote by $M \smallsetminus pM \twoheadrightarrow \mathbb{P}(M)$, $v \mapsto \overline{v}$; $M_n \smallsetminus pM_n \twoheadrightarrow \mathbb{P}(M_n)$, $v_n \mapsto \overline{v}_n$; $\mathrm{GL}(M) \twoheadrightarrow \mathrm{PGL}(M)$, $\gamma \mapsto \overline{\gamma}$; and $\mathrm{GL}(M_n) \twoheadrightarrow \mathrm{PGL}(M_n)$, $\gamma_n \mapsto \overline{\gamma}_n$ the canonical projectivization morphisms.

Let G be a closed subgroup of $\mathrm{GL}(M)$, which is automatically a compact p-adic analytic Lie group. Consider an element $v \in M \smallsetminus pM$ with the following property:

(*) *For any open subgroup* $H \subset G$ *one has* $W = \mathbb{Q}_p[H]v$.

Then:

Theorem 3.1. *For any closed subgroup* $I \subset G$*, one has:*

(1) $\displaystyle\lim_{n\to\infty} \frac{|I_n \backslash G_n v_n|}{|G_n v_n|} = \frac{1}{|I|}$;

(2) $\displaystyle\lim_{n\to\infty} \frac{|\overline{I}_n \backslash G_n \overline{v}_n|}{|G_n \overline{v}_n|} = \frac{1}{|\overline{I}|}$,

where $\frac{1}{\infty} := 0$.

To prove theorem 3.1, we start with:

Theorem 3.2. *For any closed subgroup $J \subset G$, one has:*

(1) $\lim_{n\to\infty} \frac{|(G_nv_n)^J|}{|G_nv_n|} = 0$, *unless J is trivial;*
(2) $\lim_{n\to\infty} \frac{|(G_n\overline{v}_n)^J|}{|G_n\overline{v}_n|} = 0$, *unless $\overline{J}$ is trivial.*

Proof. We begin with a general lemma about projective limits of finite sets.

Lemma 3.3. *Let $A := (A_{n+1} \stackrel{\phi_{n+1,n}}{\to} A_n)_{n\geq 0}$ be a projective system of nonempty finite sets such that*

$$(\#) \text{ for all } n \geq 0, \text{ there exists } d(n) \geq 0, \text{ such that for all } x_n \in A_n,\ |\phi_{n+1,n}^{-1}(x_n)| = d(n)$$

and let $B := (B_{n+1} \stackrel{\phi_{n+1,n}}{\to} B_n)_{n\geq 0}$ be a projective subsystem of A. Set $B_\infty := \varprojlim B_n$ and $B_{\infty,n} := p_n(B_\infty) \subset B_n$, where $p_n : B_\infty \to B_n$ is the canonical projection. In particular, $(B_{\infty,n+1} \stackrel{\phi_{n+1,n}}{\to} B_{\infty,n})_{n\geq 0}$ is a projective subsystem of B such that $B_\infty = \varprojlim B_{\infty,n}$. Then

(1) $m(B) := \lim_{n\to\infty} \frac{|B_n|}{|A_n|}$ *and* $m_\infty(B) := \lim_{n\to\infty} \frac{|B_{\infty,n}|}{|A_n|}$ *exist and belong to* $[0,1]$.

(2) $m(B) = m_\infty(B)$.

Proof of lemma 3.3. For (1), set $\delta(n) := \max\{|B_{n+1} \cap \phi_{n+1,n}^{-1}(x_n)|\}_{x_n\in B_n} \leq d(n)$. Then, by (#), one gets $\frac{|B_{n+1}|}{|A_{n+1}|} \leq \frac{\delta(n)}{d(n)}\frac{|B_n|}{|A_n|} \leq \frac{|B_n|}{|A_n|}$. Thus, $m(B)$ exist and belongs to $[0,1]$ since, by definition, $\frac{|B_n|}{|A_n|} \in [0,1]$, $n \geq 0$. The same argument obviously works for $m_\infty(B)$.

For (2), the inequality $m_\infty(B) \leq m(B)$ follows from the inclusion $B_{\infty,n} \subset B_n$, $n \geq 0$. For the opposite inequality, observe that for any $n \geq 0$ there exists $m(n) \geq n$ such that $\phi_{m(n),n}(B_{m(n)}) = B_{\infty,n}$. Indeed, else, set $\mathcal{B}_{m,n} := \phi_{m,n}^{-1}(B_{\infty,n}) \subset B_m$ and $\mathcal{B}'_{m,n} := B_m \smallsetminus \mathcal{B}_{m,n}$ for $m \geq n$. Then $(\mathcal{B}'_{m+1,n} \to \mathcal{B}'_{m,n})_{m\geq n}$ gives a well-defined projective subsystem of B. So, if $\mathcal{B}'_{m,n} \neq \emptyset$ for all $m \geq n$, then

$$\emptyset \neq \varprojlim_m \mathcal{B}'_{m,n} \subset (\varprojlim B_n) \smallsetminus (\varprojlim B_{n,\infty}) = B_\infty \smallsetminus B_\infty = \emptyset,$$

which is a contradiction. Thus, one has

$$m(B) \leq \frac{|B_{m(n)}|}{|A_{m(n)}|} \leq \frac{\delta(m(n)-1)\cdots\delta(n)}{d(m(n)-1)\cdots d(n)} \frac{|\phi_{m(n),n}(B_{m(n)})|}{|A_n|} \leq \frac{|\phi_{m(n),n}(B_{m(n)})|}{|A_n|} = \frac{|B_{\infty,n}|}{|A_n|}.$$

Hence $m(B) \leq m_\infty(B)$. □

Now, observe that $A := (G_{n+1}v_{n+1} \to G_nv_n)_{n\geq 0}$ satisfies assumption (#). Indeed, for any $x_n = g_nv_n \in G_nv_n$ and any $x_{n+1} = g_{n+1}v_{n+1} \in \phi_{n+1,n}^{-1}(x_n) \subset G_{n+1}v_{n+1}$ one has

$$|\phi_{n+1,n}^{-1}(x_n)| = [\mathrm{Stab}_{G_{n+1}}(x_n) : \mathrm{Stab}_{G_{n+1}}(x_{n+1})] = [\mathrm{Stab}_{G_{n+1}}(v_n) : \mathrm{Stab}_{G_{n+1}}(v_{n+1})],$$

which only depends on n but not on $x_n \in G_nv_n$.[1]

So, to deduce (1) of theorem 3.2 from lemma 3.3, take $B := ((G_{n+1}v_{n+1})^J \to (G_nv_n)^J)_{n\geq 0}$. Then $B_\infty = (Gv)^J$ and $B_{\infty,n} = ((Gv)^J)_n$, $n \geq 0$. From lemma 3.3, it is enough to show that $\lim_{n\to\infty} \frac{|((Gv)^J)_n|}{|G_nv_n|} = 0$. Denote by δ and δ_J the dimension (as p-adic analytic space) of Gv and $(Gv)^J$ respectively. By [O82, Th. 2], there exist real (even rational) numbers $\mu_\delta > 0$ and $\mu_{\delta_J} \geq 0$ such that $\lim_{n\to\infty} p^{-n\delta}|G_nv_n| = \mu_\delta$ and $\lim_{n\to\infty} p^{-n\delta_J}|((Gv)^J)_n| = \mu_{\delta_J}$, respectively. (Observe that $(Gv)_n = G_nv_n$.) So, it is enough to prove that $\delta_J < \delta$. Suppose otherwise, then there would exist an open subgroup

[1]Since $Gv \subset M$ is a smooth closed p-adic analytic subspace, say, of dimension δ, it even follows from [Se81, Rem. 1, p.148] that the canonical projections $G_{n+1}v_{n+1} \to G_nv_n$ are p^δ-to-1 for $n \gg 0$.

$H \subset G$ and an element $g \in G$ such that $gHv \subset (Gv)^J$. But, from assumption (*), this implies that J acts trivially on W hence on M, as desired.

Similarly, to deduce (2) of theorem 3.2, define $B_n \subset G_n v_n$, $n \geq 0$, to be the inverse image of $(G_n\overline{v}_n)^J \subset G_n\overline{v}_n$ under the canonical projection $G_n v_n \twoheadrightarrow G_n\overline{v}_n$. Then $B_\infty = \{x \in Gv \mid \gamma\overline{x} = \overline{x},\ \gamma \in J\}$ and, by lemma 3.3, it is enough to show that $\lim_{n\to\infty} \frac{|B_{\infty,n}|}{|G_n v_n|} = 0$. Since J is a compact p-adic Lie group, it is finitely generated. So, fixing a finite set $\gamma^{(1)}, \dots, \gamma^{(r)}$ of generators, $B_\infty = \bigcap_{1\leq i\leq r} B_\infty^{(i)}$, where $B_\infty^{(i)} := \{x \in Gv \mid \gamma^{(i)}\overline{x} = \overline{x}\}$. Now, $B_\infty^{(i)}$ is the finite disjoint union of p-adic analytic closed subspaces indexed by the set $\mathrm{spec}(\gamma^{(i)})$ of eigenvalues of $\gamma^{(i)}$ in W:

$$B_\infty^{(i)} = \coprod_{\lambda\in\mathrm{spec}(\gamma^{(i)})} K(\gamma^{(i)}, \lambda),$$

where $K(\gamma^{(i)}, \lambda) := \{x \in Gv \mid \gamma^{(i)}x = \lambda x\}$. Since $\mathrm{spec}(\gamma^{(i)})$ is finite, the $K(\gamma^{(i)}, \lambda)$ are also open in $B_\infty^{(i)}$. As in the proof of (1), by [O82, Th. 2], it is enough to prove that $\dim(B_\infty) < \delta$. Suppose otherwise, i.e., $\dim(B_\infty) = \delta$, then $\dim(B_\infty^{(i)}) = \delta$ must hold for all $i = 1, \dots, r$. Thus, for each $i = 1, \dots, r$, there exists $\lambda^{(i)} \in \mathrm{spec}(\gamma^{(i)})$, such that $K(\gamma^{(i)}, \lambda^{(i)})$ has dimension δ, hence contains an open subset gHv (for some open subgroup $H \subset G$ and some element $g \in G$). But then assumption (*) yields that $\gamma^{(i)} = \lambda^{(i)} Id$ on W, hence on M. This means that $\overline{J}$ is trivial, as desired. □

Proof of theorem 3.1. Given any finite group F, write $\mathcal{M}(F)$ for the (finite) set of nontrivial minimal subgroups of F. Equivalently, $\mathcal{M}(F)$ is the set of cyclic subgroups of F with prime order. (Note that $\mathcal{M}(F) = \emptyset$ if and only if $F = \{1\}$.)

We first prove (1) by using (1) of theorem 3.2. Set $(G_n v_n)' := \bigcup_{J\in\mathcal{M}(I_n)} (G_n v_n)^J$, then one has

$$\frac{1}{|I_n|}\left(1 - \frac{|(G_n v_n)'|}{|G_n v_n|}\right) \leq \frac{|I_n\backslash G_n v_n|}{|G_n v_n|} \leq \frac{1}{|I_n|}\left(1 - \frac{|(G_n v_n)'|}{|G_n v_n|}\right) + \frac{|(G_n v_n)'|}{|G_n v_n|}.$$

Since $|I_n| \to |I|$, the only thing to prove is $\frac{|(G_n v_n)'|}{|G_n v_n|} \to 0$.

If I is finite, then I is isomorphic to I_n for $n \gg 0$, so

$$0 \leq \frac{|(G_n v_n)'|}{|G_n v_n|} \leq \sum_{J\in\mathcal{M}(I)} \frac{|(G_n v_n)^J|}{|G_n v_n|} \to 0\ (n \to \infty)$$

by (1) of theorem 3.2.

If I is infinite, then I admits an infinite pro-p cyclic subgroup. Indeed, set $K_p := Id + p^{i(p)}\mathrm{M}_m(\mathbb{Z}_p)$, where $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then the p-adic exponential defines an homeomorphism $exp : p^{i(p)}\mathrm{M}_m(\mathbb{Z}_p) \simeq K_p$ such that $exp(aM) = (exp(M))^a$, $M \in p^{i(p)}\mathrm{M}_m(\mathbb{Z}_p)$, $a \in \mathbb{Z}$. So, K_p is a torsion-free open normal pro-p subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$ and $I \cap K_p$ is an infinite pro-p torsion-free subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$. Now, for any $g \in (I \cap K_p) \smallsetminus \{Id\}$, one has $\langle g\rangle \simeq \mathbb{Z}_p$, as desired.

So, up to replacing I by such a subgroup, one may assume $I \simeq \mathbb{Z}_p$. Fix any $N \geq 0$ and set $J = I^{p^N} \subset I$. Then, for any $x_n \in G_n v_n$ with stabilizer I_{x_n} under I, note that $x_n \notin (G_n v_n)^J$ if and only if $J \not\subset I_{x_n}$. But, since closed subgroups of $I \simeq \mathbb{Z}_p$ are totally ordered for $\subset$, the latter is also equivalent to $I_{x_n} \subsetneq J = I^{p^N}$, or $I_{x_n} \subset I^{p^{N+1}}$. So, $x_n \notin (G_n v_n)^J$ implies $|Ix_n| = [I : I_{x_n}] \geq [I : I^{p^{N+1}}] = p^{N+1}$. Now,

$$0 \leq \frac{|I_n\backslash G_n v_n|}{|G_n v_n|} \leq \frac{|I_n\backslash(G_n v_n \smallsetminus (G_n v_n)^J)|}{|G_n v_n|} + \frac{|(G_n v_n)^J|}{|G_n v_n|} \leq \frac{1}{p^{N+1}} + \frac{|(G_n v_n)^J|}{|G_n v_n|}.$$

By (1) of theorem 3.2, one has $\frac{|(G_n v_n)^J|}{|G_n v_n|} \to 0$, so

$$0 \leq \lim_{n\to\infty} \frac{|I_n\backslash G_n v_n|}{|G_n v_n|} \leq \frac{1}{p^{N+1}},$$

which yields the desired conclusion as $N \geq 0$ is arbitrary.

Finally, the proof for the projective situation (2) is exactly similar: use (2) (instead of (1)) of theorem 3.2. □

4. Proofs

4.1. **Proof of theorem 1.1.** [2] Denote the canonical epimorphism $A_\eta \twoheadrightarrow (A_\eta)^0 = A_\eta/(A_\eta)_0$ by $v \mapsto v^0$. It induces a canonical epimorphism of Π-module $A_\eta[p^\infty] \twoheadrightarrow (A_\eta)^0[p^\infty]$ hence, for any $v \in A_\eta[p^\infty]$ an inclusion of opens subgroups $\Pi_{\langle v\rangle} \subset \Pi_{\langle v^0\rangle} \subset \Pi$ corresponding to a commutative diagram of finite etale covers

$$\begin{array}{ccc} S_{\langle v\rangle} & \longrightarrow & S_{\langle v^0\rangle} \\ \downarrow & \swarrow & \\ S & & \end{array} .$$

But then, by the Riemann-Hurwitz formula, $S_{\langle v\rangle}$ has genus larger than $S_{\langle v^0\rangle}$. The same observation obviously works for v, v^0 as well instead of $\langle v\rangle$, $\langle v^0\rangle$. So we can restrict to the case when $(A_\eta)_0$ is trivial (that is, A_η contains no nontrivial abelian subvariety isogenous to an isotrivial abelian variety), which we assume from now on and till the end of this section.

In particular, to prove theorem 1.1, it is enough to prove the following.

Theorem 4.1. *Assume that A_η contains no nontrivial abelian subvariety isogenous to an isotrivial abelian variety. Let $v \in T_p(A_\eta)^*$ and set $v_n := v \bmod p^n T_p(A_\eta) \in A_\eta[p^n]^*$, $n \geq 0$. Then $g_{v_n} \to \infty$ and $g_{\langle v_n\rangle} \to \infty$ ($n \to \infty$).*

Indeed, assume that, say, the assersion for $g_{\langle v\rangle}$ in theorem 1.1 does not hold. That is, there exists a $c \geq 0$ such that for all $n \geq 0$ there exists $v_n \in A_\eta[p^\infty]$ with $g_{\langle v_n\rangle} \leq c$ but $p^n v_n \neq 0$. From the inclusion of open subgroups $\Pi_{\langle v_n\rangle} \subset \Pi_{\langle p^{-n}|\langle v_n\rangle| v_n\rangle} \subset \Pi$, one gets $g_{\langle p^{-n}|\langle v_n\rangle|v_n\rangle} \leq g_{\langle v_n\rangle} \leq c$ so, one gets:

$$A_\eta[p^n]^*_c := \{v \in A_\eta[p^n]^* \mid g_{\langle v\rangle} \leq c\} \neq \emptyset.$$

Furthermore, for any $v \in A_\eta[p^n]^*_c$ the inclusion of open subgroups $\Pi_{\langle v\rangle} \subset \Pi_{\langle pv\rangle} \subset \Pi$ yields, again, $g_{\langle pv\rangle} \leq g_{\langle v\rangle}$. So $\{A_\eta[p^n]^*_c\}_{n\geq 0}$ forms a projective subsystem of nonempty (finite) subsets of $(A_\eta[p^{n+1}]^* \to A_\eta[p^{n+1}]^*)_{n\geq 0}$ hence:

$$\emptyset \neq \varprojlim A_\eta[p^n]^*_c \subset \varprojlim A_\eta[p^n]^* = T_p(A_\eta)^*,$$

which contradicts theorem 4.1 for $g_{\langle v_n\rangle}$. The case of g_{v_n} is just similar.

So, the rest of this section will be devoted to proving theorem 4.1.

4.1.1. *The Π-module $W(v)$.* For any $v \in T_p(A_\eta)^*$, define:

$$W(v) := \mathbb{Q}_p[\Pi]v \subset V_p(A_\eta).$$

Then $W(v)$ contains the $\mathbb{Q}_p$-submodule:

$$W_\infty(v) := \bigcap_{H\subset\Pi} \mathbb{Q}_p[H]v,$$

where the intersection is over all open subgroups $H \subset \Pi$. By definition $v \in W_\infty(v)$ so $W_\infty(v)$ is nonzero. Also, one can write $W_\infty(v) = \mathbb{Q}_p[H_0]v$ for some open subgroup $H_0 \subset \Pi$. Indeed, for any open subgroups H_1, $H_2 \subset \Pi$ observe that $\mathbb{Q}_p[H_1]v \cap \mathbb{Q}_p[H_2]v \supset \mathbb{Q}_p[H_1 \cap H_2]v$. So any open subgroup $H_0 \subset \Pi$ such that $\mathbb{Q}_p[H_0]v$ has minimal $\mathbb{Q}_p$-dimension works.

In particular, the closed subgroup

$$\Pi_\infty(v) := \{\gamma \in \Pi \mid \gamma \cdot W_\infty(v) = W_\infty(v)\} \subset \Pi$$

[2]As $g_v \geq g_{\langle v\rangle}$, the assertion for g_v is implied by the one for $g_{\langle v\rangle}$. However, here we treat both cases in parallel, partly because the proof for g_v is somewhat simpler than the one for $g_{\langle v\rangle}$, and the former may help the readers to understand the latter.

contains H_0 hence is also open in Π.

Furthermore, from the inclusion:

$$\bigcap_{n\geq 0} \Pi_{\langle v_n\rangle} = \Pi_{\langle v\rangle} \subset \Pi_\infty(v),$$

one obtains:

$$\Pi = \Pi_\infty(v) \bigcup \bigcup_{n\geq 0} (\Pi \smallsetminus \Pi_{\langle v_n\rangle}).$$

So it follows from the compactness of Π that there exists an integer $N \geq 0$ such that:

$$\Pi = \Pi_\infty(v) \bigcup \bigcup_{n\leq N} (\Pi \smallsetminus \Pi_{\langle v_n\rangle}),$$

hence:

$$\Pi_{\langle v_N\rangle} = \bigcap_{n\leq N} \Pi_{\langle v_n\rangle} \subset \Pi_\infty(v).$$

In particular $\Pi_{\langle v_n\rangle} \subset \Pi_\infty(v)$, $n \gg 0$. Similarly, $\Pi_{v_n} \subset \Pi_\infty(v)$, $n \gg 0$.

But, since we are interested in estimating g_{v_n}, $g_{\langle v_n\rangle}$ for $n \gg 0$, one may replace Π by $\Pi_\infty(v)$ and $W(v)$ by $W_\infty(v)$. So, from now on we assume that:

(*) *For any open subgroup $H \subset \Pi$, one has $W(v) = \mathbb{Q}_p[H]v$.*

4.1.2. *The full level-p^n structure $S(n) \to S$.* Set $M := W(v) \cap T_p(A_\eta) (\simeq \mathbb{Z}_p^m)$. We retain the notation of section 3 and denote by $\rho : \Pi \to \mathrm{GL}(M)$ the corresponding representation, and by $G \subset \mathrm{GL}(M)$ its image. Then G is a compact subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$. So it is an almost-pro-p compact p-adic infinite (cf. corollary 2.4) Lie-group.

Denote by $\Pi(n) \lhd \Pi$ and $G_n \simeq \Pi/\Pi(n) \subset \mathrm{GL}(M_n)$ the kernel and the image of $\Pi \xrightarrow{\rho} \mathrm{GL}(M) \twoheadrightarrow \mathrm{GL}(M_n)$ respectively. Finally, set $G(n) := \rho(\Pi(n)) \subset G$ and $G_\# := \rho(\Pi_\#)$ ($\# = v$, $\langle v\rangle$, *etc*). The inclusion of open subgroups $\Pi(n) \subset \Pi_{v_n} \subset \Pi_{\langle v_n\rangle} \subset \Pi$ corresponds to the sequence of finite etale covers

$$S(n) \to S_{v_n} \to S_{\langle v_n\rangle} \to S$$

where $S(n) \to S$ is Galois with group G_n. Write $g(n)$ for the genus of the smooth compactification of $S(n)$.

4.1.3. $g(n) \to \infty$. As already noticed, $|G_n| \to \infty$ by corollary 2.4, so, by the Riemann-Hurwitz formula, $\sup\{g(n)\} < \infty$ if and only if $\sup\{g(n)\} \leq 1$.

If $\sup\{g(n)\} = 1$, then there exists $n_0 \geq 0$ such that $g(n) = 1$ for $n \geq n_0$. Then the smooth compactification $\overline{S(n_0)}$ of $S(n_0)$ is an elliptic curve and that $G(n_0) = \varprojlim (\Pi(n_0)/\Pi(n))$ is a quotient of $\pi_1(\overline{S(n_0)})$ ($\twoheadleftarrow \hat{\mathbb{Z}}^2$), which contradicts corollary 2.4.

If $\sup\{g(n)\} = 0$, then $S(n) \to S$ is a Galois cover of genus 0 curve with degree $|G_n| \to \infty$. Set $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then, replacing S by $S(i(p))$ if necessary, we may assume: (i) G_n is a p-group; and (ii) any element of $G_n = G/G(n)$ of order p is contained in $G(n-1)/G(n)$, hence in the center of G_n. Indeed, (i) is clear and (ii) follows from a standard argument involving the p-adic exponential, as in the proof of theorem 3.1. Now, we resort to the classification of finite subgroups of $\mathrm{PGL}_2(k) = \mathrm{Aut}(\mathbb{P}^1_k)$. More specifically, it follows from [Su82, Th. 6.17, Case I] and condition (i) above that either G_n is a cyclic p-group or $p = 2$ and G_n is a dihedral group of order 2^{m+1}. Moreover, condition (ii) above forces $m \in \{0, 1\}$ in the latter case. So, as $|G_n| \to \infty$, $(G_{n+1} \to G_n)_{n\geq 0}$ must be a projective system of cyclic p-groups. Thus, G is abelian, which contradicts corollary 2.4. □

Remark 4.2. When $k = \mathbb{C}$ and M is the whole p-adic Tate module $T_p(A_\eta)$ (with A_η principally polarized), J.-M. Hwang and W.-K. To proved that a uniform bound (i.e., depending only on $\dim(A_\eta)$) for the growth of $g(n)$ exists [HT06]. By classical arguments (Zarhin's trick and specialization), such a uniform bound for $T_p(A_\eta)$ also exists only under the assumption that k has characteristic 0.

4.1.4. $g_{v_n} \to \infty$. First, for any connected finite etale cover $T \to S$ of degree d, set $\lambda(T \to S) := \frac{2g(T)-2}{d}$, where $g(T)$ stands for the genus of the smooth compactification of T. Then, if $U \to T \to S$ are connected finite etale covers, it follows from the Riemann-Hurwitz formula for $U \to T$ that $\lambda(U) \geq \lambda(T)$.

Now, set $\lambda_n := \lambda(S(n) \to S)$ and $\lambda_{v_n} := \lambda(S_{v_n} \to S)$. Thus, one has $\lambda_{n+1} \geq \lambda_n$, $\lambda_{v_{n+1}} \geq \lambda_{v_n}$ and $\lambda_n \geq \lambda_{v_n}$, $n \geq 0$. In particular, $\lambda := \lim_{n\to\infty} \lambda_n$ and $\lambda_v := \lim_{n\to\infty} \lambda_{v_n}$ exist and $\lambda \geq \lambda_v$.

As already noticed, $|G_n| \to \infty$ (by corollary 2.4), and, as well, $|G_n v_n| \to \infty$. Indeed, else, since $G_n v_n \simeq G/G_{v_n}$, Π_v would be a closed subgroup of finite index in Π, hence open in Π. Thus, by assumption (*), $W(v) = \mathbb{Q}_p[\Pi_v]v = \mathbb{Q}_p v$ is 1-dimensional, which contradicts corollary 2.4. So, $g_n \to \infty$ if and only if $\lambda > 0$, and $g_{v_n} \to \infty$ if and only if $\lambda_v > 0$. But from paragraph 4.1.3, $\lambda > 0$ holds. Thus, it is enough to prove that $\lambda = \lambda_v$.

To do this, we shall rewrite λ_n and λ_{v_n} in group-theoretic terms, by means of the Riemann-Hurwitz formula. More specifically, for $i = 1, ..., r$, write $I_{i,n} \subset G_n$ for the image of the inertia group at P_i in G_n and denote by $d_{i,n}$ and $e_{i,n}$ (resp. $d_n(P)$ and $e_n(P)$) the exponent of the different and the ramification index of any place of $S(n)$ (resp. of the place P of S_{v_n}) above P_i in $S(n) \to S$ (resp. in $S_{v_n} \to S$). Eventually, for any place P of S_{v_n}, write $d^n(P)$ and $e^n(P)$ for the exponent of the different and the ramification index of any place of $S(n)$ above P in $S(n) \to S_{v_n}$. Recall that these data are linked by the relations

$$e_{i,n} = e^n(P)e_n(P), \quad d_{i,n} = e^n(P)d_n(P) + d^n(P).$$

Now, by the Riemann-Hurwitz formula, one has:

$$\lambda_n := 2g(0) - 2 + \sum_{1\leq i\leq r} \frac{d_{i,n}}{e_{i,n}}, \quad \lambda_{v_n} = 2g(0) - 2 + \frac{1}{|G_n v_n|} \sum_{1\leq i\leq r} \sum_{P\in S_{v_n},\, P|P_i} d_n(P).$$

Set $\epsilon_n := \lambda_n - \lambda_{v_n} \geq 0$. Then observe

$$\epsilon_n = \sum_{1\leq i\leq r} \epsilon_{i,n}, \quad \epsilon_{i,n} := \frac{1}{|G_n v_n|} \sum_{P\in S_{v_n},\, P|P_i} \frac{d^n(P)}{e^n(P)}.$$

Moreover, it is classically known ([Se68, Chap. IV, Prop. 4]) that

$$\epsilon_{i,n} = \frac{1}{|G_n v_n|} \sum_{P\in S_{v_n},\, P|P_i} \sum_{j\geq 0} \frac{e^n(P)_j - 1}{e^n(P)},$$

where $e^n(P)_j$ denotes the order of the jth ramification group $I_n(P)_j$ (with "lower numbering") of any place of $S(n)$ above P in $S(n) \to S_{v_n}$, $j \geq 0$. (Thus, $e^n(P)_0 = e^n(P)$, and $e^n(P)_j = 1$ for $j \gg 0$.)

As $G(1)/G(n) = ker(G_n \to G_1)$ is a p-group, so is $I_{i,n}(1) := I_{i,n} \cap (G(1)/G(n)) = Ker(I_{i,n} \to I_{i,1})$. Thus, (since $p \neq char(k)$) the natural surjection $I_{i,n} \to I_{i,1}$ induces an isomorphism $(I_{i,n})_+ \to (I_{i,1})_+$, where $(I_{i,n})_+$ denotes the wild inertia subgroup of $I_{i,n}$, i.e., $(I_{i,n})_+$ is the trivial (resp. unique q-Sylow) subgroup of $I_{i,n}$ if the characteristic of k is 0 (resp. q). Now, for integers $j \geq 0$ and $n \geq 0$, define a subgroup $(\tilde{I}_{i,n})_j$ of $I_{i,n}$ as follows: $(\tilde{I}_{i,n})_0 := I_{i,n}$ and, for $j > 0$, $(\tilde{I}_{i,n})_j$ is the inverse image of the jth ramification group $(I_{i,1})_j$ of $I_{i,1}$ under the above isomorphism $(I_{i,n})_+ \to (I_{i,1})_+$. (Observe, in particular, that there exists $j_0 \geq 0$ such that $(\tilde{I}_{i,n})_j = \{1\}$ for $j > j_0$.) For $j > 0$, the jth ramification group $(I_{i,n})_j$ of $I_{i,n}$ maps surjectively onto $(I_{i,1})_{\lceil j/e_{i,n}(1)\rceil}$ by the above isomorphism $(I_{i,n})_+ \to (I_{i,1})_+$, where $e_{i,n}(1) := |I_{i,n}(1)|$ ([Se68, Chap. IV, Lem. 5]). From this, one obtains $(I_{i,n})_j = (\tilde{I}_{i,n})_{\lceil j/e_{i,n}(1)\rceil}$ for $j \geq 0$. (Note that this is clear for $j = 0$.)

Moreover, define $(\tilde{I}_i)_j$ to be the projective limit of $\{(\tilde{I}_{i,n})_j\}_{n\geq 0}$. Thus, $(\tilde{I}_i)_0$ coincides with the inertia subgroup $I_i \subset G$ at P_i, while, for $j > 0$, $(\tilde{I}_i)_j$ is a finite, normal q-subgroup of I_i which is projected isomorphically onto $(\tilde{I}_{i,n})_j$ for each $n > 0$.

Lemma 4.3. *Let I be a finite group and J a subgroup of I. Let X be a finite set on which I acts. Consider the natural surjection of quotient sets: $J\backslash X \to I\backslash X$. Then, for any $x \in X$, the fiber of* *this map at $Ix \in I\backslash X$ consists of Jsx, where s runs over the representatives of $J\backslash I/I_x$. (Here I_x stands for the stabilizer of x in I.) In particular, if J is normal in I, then the cardinality of this fiber coincides with* $\dfrac{|I||I_x \cap J|}{|J||I_x|}$.

Proof. Easy. □

Now, there exists a natural bijection $I_{i,n}\backslash G_n \simeq \{Q \in S(n) \mid Q|P_i\}$, hence one obtains

$$I_{i,n}\backslash G_n v_n \simeq I_{i,n}\backslash G_n/(G_n)_{v_n} \simeq \{P \in S_{v_n} \mid P|P_i\}.$$

More explicitly, fix $Q_0 \in S(n)$ with $Q_0|P_i$ such that $I_{i,n}$ is the inertia group at Q_0. Then, for $s \in G_n$, The orbit $I_{i,n}sv_n$ corresponds to the double coset $I_{i,n}s(G_n)_{v_n}$, which corresponds to the image $P(s)$ of Q_0^s in S_{v_n}. (Here, we have adopted the action from the right.) For any normal subgroup J of $I_{i,n}$, consider the natural surjection $J\backslash G_n v_n \to I_{i,n}\backslash G_n v_n$. Then, by Lemma 4.3, the cardinality of the fiber of this surjection at the orbit $I_{i,n}sv_n$ is equal to

$$\frac{|I_{i,n}||J \cap (G_n)_{sv_n}|}{|J||I_{i,n} \cap (G_n)_{sv_n}|} = \frac{|I_{i,n}||s^{-1}Js \cap (G_n)_{v_n}|}{|J||s^{-1}I_{i,n}s \cap (G_n)_{v_n}|}.$$

If $J = \{1\}$, this coincides with $\dfrac{e_{i,n}}{e^n(P(s))}$, while, if $J = (I_{i,n})_j$, it coincides with $\dfrac{e_{i,n}}{(e_{i,n})_j} \cdot \dfrac{e^n(P(s))_j}{e^n(P(s))}$, where $(e_{i,n})_j := |(I_{i,n})_j|$. From these, one obtains

$$\sum_{P \in S_{v_n},\, P|P_i} \frac{e_{i,n}}{e^n(P)} = \sum_{s \in I_{i,n}\backslash G_n/(G_n)_{v_n}} \frac{e_{i,n}}{e^n(P(s))} = |G_n v_n|$$

and

$$\sum_{P \in S_{v_n},\, P|P_i} \frac{e_{i,n}}{(e_{i,n})_j} \cdot \frac{e^n(P)_j}{e^n(P)} = \sum_{s \in I_{i,n}\backslash G_n/(G_n)_{v_n}} \frac{e_{i,n}}{(e_{i,n})_j} \cdot \frac{e^n(P(s))_j}{e^n(P(s))} = |(I_{i,n})_j\backslash G_n v_n|.$$

Therefore, one has

$$\sum_{P \in S_{v_n},\, P|P_i} \frac{e^n(P)_j - 1}{e^n(P)} = \frac{(e_{i,n})_j}{e_{i,n}}|(I_{i,n})_j\backslash G_n v_n| - \frac{1}{e_{i,n}}|G_n v_n|$$

for each $j \geq 0$ and

$$\epsilon_{i,n} = \sum_{j \geq 0} \left(\frac{(e_{i,n})_j}{e_{i,n}} \frac{|(I_{i,n})_j\backslash G_n v_n|}{|G_n v_n|} - \frac{1}{e_{i,n}} \right).$$

Now, since

$$(I_{i,n})_j = (\tilde{I}_{i,n})_{\lceil j/e_{i,n}(1) \rceil} = \begin{cases} (\tilde{I}_{i,n})_0, & j = 0, \\ (\tilde{I}_{i,n})_k, & (k-1)e_{i,n}(1) \leq j \leq ke_{i,n}(1) \ (0 < \exists k \leq j_0), \\ \{1\}, & j > j_0 e_{i,n}(1). \end{cases}$$

one has

$$\begin{aligned} \epsilon_{i,n} &= \left(\frac{|(\tilde{I}_{i,n})_0\backslash G_n v_n|}{|G_n v_n|} - \frac{1}{e_{i,n}} \right) + e_{i,n}(1) \sum_{0 < k \leq j_0} \left(\frac{|(\tilde{I}_{i,n})_k|}{e_{i,n}} \frac{|(\tilde{I}_{i,n})_k\backslash G_n v_n|}{|G_n v_n|} - \frac{1}{e_{i,n}} \right) \\ &= \left(\frac{|I_{i,n}\backslash G_n v_n|}{|G_n v_n|} - \frac{1}{|I_{i,n}|} \right) + \frac{1}{e_{i,1}} \sum_{0 < k \leq j_0} |(\tilde{I}_i)_k| \left(\frac{|(\tilde{I}_{i,n})_k\backslash G_n v_n|}{|G_n v_n|} - \frac{1}{|(\tilde{I}_i)_k|} \right). \end{aligned}$$

Now, by applying theorem 3.1 to I_i and $(\tilde{I}_i)_k$, $0 < k \leq j_0$, one has $\epsilon_{i,n} \to 0$ $(n \to \infty)$, as desired. □

4.1.5. $g_{\langle v_n \rangle} \to \infty$. Here, the outline of proof is similar to the case of g_{v_n}, but the details are slightly more complicated. With the notation of section 3, one has the following canonical epimorphism of short exact sequences.

(1)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_p^* & \longrightarrow & \mathrm{GL}(M) & \longrightarrow & \mathrm{PGL}(M) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (\mathbb{Z}/p^n)^* & \longrightarrow & \mathrm{GL}(M_n) & \longrightarrow & \mathrm{PGL}(M_n) & \longrightarrow & 1. \end{array}$$

Consider the projectivizations $\overline{\rho} : \Pi \to \mathrm{PGL}(M)$, $\overline{\rho}_n : \Pi \to \mathrm{PGL}(M_n)$ of ρ and ρ_n respectively. Denote by $\overline{G} \subset \mathrm{PGL}(M)$ and $\overline{\Pi}(n) \lhd \Pi$, $\overline{G}_n \subset \mathrm{PGL}(M_n)$ the image of $\overline{\rho}$ and $\overline{\rho}_n$ respectively. Finally, set $\overline{G}(n) := \overline{\rho}(\overline{\Pi}(n)) \subset \overline{G}$.

With this notation, (2) induces the epimorphism of short exact sequences

(2)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_p^* \cap G & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (\mathbb{Z}/p^n)^* \cap G_n & \longrightarrow & G_n & \longrightarrow & \overline{G}_n & \longrightarrow & 1. \end{array}$$

The inclusion of open subgroups :

(3)
$$\begin{array}{ccccc} \Pi(n) & \subset & \overline{\Pi}(n) & & \\ \cap & & \cap & & \\ \Pi_{v_n} & \subset & \Pi_{\langle v_n \rangle} & \subset & \Pi \end{array}$$

correponds to a commutative diagram of finite etale covers:

(4)
$$\begin{array}{ccccc} S(n) & \longrightarrow & \overline{S}(n) & & \\ \downarrow & & \downarrow & & \\ S_{v_n} & \longrightarrow & S_{\langle v_n \rangle} & \longrightarrow & S, \end{array}$$

where $\overline{S}(n) \to S$ is Galois with group $\overline{G}_n$ and $S(n) \to \overline{S}(n)$ is Galois with group $(\mathbb{Z}/p^n)^* \cap G_n$. Write $\overline{g}(n)$ for the genus of $\overline{S}(n)$.

4.1.5.1. $\overline{g}(n) \to \infty$. First, observe that $|\overline{G}_n| \to \infty$ (indeed, else, $\overline{G} = \varprojlim \overline{G}_n$ would be finite hence G would be almost-abelian, which contradicts corollary 2.4). So, by the Riemann-Hurwitz formula, $\sup\{\overline{g}(n)\} < \infty$ if and only if $\sup\{\overline{g}(n)\} \leq 1$.

If $\sup\{\overline{g}(n)\} = 1$, then, up to replacing Π by $\overline{\Pi}(n)$, S by $\overline{S}(n)$ for some $n \geq 0$ large enough, one may assume that $\overline{g}(n) = 1$, $n \geq 0$, hence that $\overline{S}$ is an elliptic curve and that $\overline{G}$ is a quotient of $\pi_1(\overline{S})$ ($\twoheadleftarrow \hat{\mathbb{Z}}^2$). In particular the normal subgroup $I \subset G$ generated by the inertia subgroups lies in $\ker(G \twoheadrightarrow \overline{G}) = \mathbb{Z}_p^* \cap G$. But, by the semistable reduction theorem [SGA7, Exp. IX, Th.3.6], each inertia group is almost-unipotent, hence, being also scalar, it is finite. Thus, I is also finite, as an abelian group generated by a finite number of finite abelian groups. Now, consider the short exact sequence

$$1 \to I \to G \to G/I \to 1.$$

Since G/I is a quotient of $\hat{\mathbb{Z}}^2$ and I is finite, G is almost-abelian. (Consider any open subgroup U of G such that $U \cap I = \{1\}$.) This contradicts corollary 2.4.

If $\sup\{\overline{g}(n)\} = 0$, then $\overline{S}(n) \to S$ is a Galois cover of genus 0 curve with degree $|\overline{G}_n| \to \infty$. So, according to the consideration of paragraph 4.1.3, up to replacing S by a finite Galois cover, one may assume $\overline{G} \simeq \mathbb{Z}_p$. Now, consider the short exact sequence

$$1 \to \mathbb{Z}_p^* \cap G \to G \to \overline{G} \to 1.$$

Since $\overline{G} \simeq \mathbb{Z}_p$ is a projective profinite group, this exact sequence must split. Then, since $\mathbb{Z}_p^* \cap G$ lies in the center of G and $\overline{G}$ is abelian, G is also abelian. This contradicts corollary 2.4.

4.1.5.2. $g_{\langle v_n \rangle} \to \infty$. The proof is exactly similar to the one of paragraph 4.1.4 resorting to (2) of theorem 3.1 instead of (1).

4.2. **Proof of corollary 1.2.** For each $n \geq 1$, set $\chi_n := \chi \bmod p^n\mathbb{Z}_p : \Gamma_k \to (\mathbb{Z}/p^n)^*$. Set $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then, up to replacing k by the fixed field of $\ker(\chi_{i(p)})$ in $\overline{k}$, one may assume that $\chi_{i(p)} : \Gamma_k \to (\mathbb{Z}/p^{i(p)})^*$ is trivial. (Here, we have used lemma 2.1.) This technical reduction ensures that $\mathrm{Im}(\chi_n) \subset (\mathbb{Z}/p^n)^*$ is contained in the order $p^{n-i(p)}$ cyclic subgroup $1 + p^{i(p)}\mathbb{Z}/p^n\mathbb{Z}$ of $(\mathbb{Z}/p^n)^*$; this will be used in the proof of lemma 4.6.

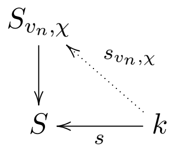
4.2.1. *The curves $S_{v_n,\chi}$.* For each $v_n \in A_\eta[p^n]^*$, consider the restriction morphism $pr_{\langle v_n \rangle} : \pi_1(S_{\langle v_n \rangle}) \to \Gamma_k$ (whose image coincides with $\Gamma_{k_{\langle v_n \rangle}}$) and the natural representation $\rho_{\langle v_n \rangle} : \pi_1(S_{\langle v_n \rangle}) \to \mathrm{Aut}_{\mathbb{Z}/p^n}(\langle v_n \rangle)$. These define together with $\chi_n := \chi \bmod p^n\mathbb{Z}_p$, a representation

$$\rho_{v_n,\chi} : \pi_1(S_{\langle v_n \rangle}) \to \mathrm{Aut}_{\mathbb{Z}/p^n}(\langle v_n \rangle),\ \gamma \mapsto \chi_n(pr_{\langle v_n \rangle}(\gamma))^{-1}\rho_{\langle v_n \rangle}(\gamma).$$

Then let $S_{v_n,\chi} \to S_{\langle v_n \rangle}$ be the etale Galois cover (defined over a finite extension $k_{v_n,\chi}/k$) corresponding to the normal open subgroup $\ker(\rho_{v_n,\chi}) \subset \pi_1(S_{\langle v_n \rangle})$.

Lemma 4.4. *For any character $\chi : \Gamma_k \to \mathbb{Z}_p^*$ and $v_n \in A_\eta[p^n]^*$ the finite etale cover $S_{v_n,\chi} \to S$ has the following properties:*

1. *$S_{v_n,\chi} \times_{k_{v_n,\chi}} \overline{k} \simeq S_{v_n} \times_{k_{v_n}} \overline{k}$, and, in particular, the genus of $S_{v_n,\chi}$ is independent of χ and equal to the genus of S_{v_n}.*
2. *For any k-rational point $s : k \to S$, consider the specialization isomorphism of $\pi_1(S)$-modules*
$$sp_s : A_\eta[p^\infty] \tilde{\to} A_s[p^\infty].$$
Then $sp_s(v_n) \in A_s[p^\infty](\chi)$ if and only if $s : k \to S$ lifts to a k-rational point:



Proof. For 1, just observe that
$$\begin{aligned}\pi_1(S_{v_n,\chi} \times_{k_{v_n,\chi}} \overline{k}) &= \ker(pr_{\langle v_n \rangle}) \cap \ker(\rho_{v_n,\chi})\\ &= \ker(\rho_{v_n,\chi})|_{\ker(pr_{\langle v_n \rangle})}\\ &= \ker(\rho_{v_n}) \cap \ker(pr_{\langle v_n \rangle})\\ &= \pi_1(S_{v_n} \times_{k_{v_n}} \overline{k})\end{aligned}$$

For 2, denote again by s the section $\Gamma_k \hookrightarrow \pi_1(S)$ of $\pi_1(S) \twoheadrightarrow \Gamma_k$ induced (up to conjugacy) by $s : k \to S$, which identifies Γ_k with the decomposition group at s. Then the existence of the lift $s_{v_n,\chi} : k \to S_{v_n,\chi}$ of $s : \Gamma_k \hookrightarrow \pi_1(S)$ is equivalent to the inclusion $s(\Gamma_k) \subset \pi_1(S_{v_n,\chi})(= \ker(\rho_{v_n,\chi}))$, which can be rewritten as $s(\sigma) \cdot v_n = \chi(\sigma)v_n$ $(\sigma \in \Gamma_k)$ or, applying the specialization isomorphism, $\sigma \cdot sp_s(v_n) = \chi(\sigma)sp_s(v_n)$. $\square$

4.2.2. *The projective system $(\mathcal{S}_{n+1,\chi} \to \mathcal{S}_{n,\chi})$.* For each $n \geq 0$, define

$$\mathcal{S}_{n,\chi} := \coprod_{v_n \in A_\eta[p^n]^*} S_{v_n,\chi}.$$

From theorem 1.1, there exists an integer $N \geq 0$ such that for all $v \in A_\eta[p^\infty]$ either $p^N v^0 = 0$ or $S_{v,\chi}$ has genus ≥ 2. So $\mathcal{S}_{n,\chi}$ can be written as a disjoint union $\mathcal{S}_{n,\chi} = \mathcal{S}_{n,\chi}^{(1)} \coprod \mathcal{S}_{n,\chi}^{(2)}$, where

$$\mathcal{S}_{n,\chi}^{(1)} := \coprod_{v_n \in A_\eta[p^n]^*,\ p^N v_n^0 = 0} S_{v_n,\chi} \text{ and } \mathcal{S}_{n,\chi}^{(2)} := \coprod_{v_n \in A_\eta[p^n]^*,\ p^N v_n^0 \neq 0} S_{v_n,\chi}.$$

Then $(\mathcal{S}_{n+1,\chi}(k) \to \mathcal{S}_{n,\chi}(k))_{n\geq 0}$ is a projective system with transition maps induced by the canonical morphisms $S_{v_n,\chi} \to S_{pv_n,\chi}$. They behave with respect to the decomposition $\mathcal{S}_{n,\chi} = \mathcal{S}^{(1)}_{n,\chi} \coprod \mathcal{S}^{(2)}_{n,\chi}$ as follows.

(5)
$$\begin{array}{ccc} \mathcal{S}^{(1)}_{n+1,\chi}(k) & & \mathcal{S}^{(2)}_{n+1,\chi}(k) \\ \downarrow & \overset{(*)}{\swarrow} & \downarrow \\ \mathcal{S}^{(1)}_{n,\chi}(k) & & \mathcal{S}^{(2)}_{n,\chi}(k) \end{array}$$

Namely, $\{\mathcal{S}^{(1)}_{n,\chi}(k)\}_{n\geq 0}$ forms a projective subsystem of $(\mathcal{S}_{n+1,\chi}(k) \to \mathcal{S}_{n,\chi}(k))_{n\geq 0}$, while $\{\mathcal{S}^{(2)}_{n,\chi}(k)\}_{n\geq 0}$ may not (since some element of $\mathcal{S}^{(2)}_{n+1,\chi}(k)$ may map into $\mathcal{S}^{(1)}_{n,\chi}(k)$, as shown by the diagonal arrow with (*) in the above diagram).

Now, suppose that corollary 1.2 does not hold. Then, for all $n \geq 0$, there exists $s_n \in S(k)$ and there exists $\nu_n \in A_{s_n}[p^\infty](\chi)$ such that $p^n\nu_n \neq 0$. Up to replacing ν_n by $|\langle \nu_n \rangle| p^{-n}\nu_n$ one may thus assume that $\mathcal{S}_{n,\chi}(k) \neq \emptyset$, $n \geq 0$. But this yields a contradiction.

Indeed, by the non-Tate assumption on χ and lemma 4.4 (ii), one has

$$\underset{\longleftarrow}{\lim}\ \mathcal{S}_{n,\chi}(k) = \emptyset.$$

On the other hand, assuming lemma 4.5 below, the arrows (*) in (5) disappear for $n \gg 0$. So, the projective system $(\mathcal{S}_{n+1,\chi}(k) \to \mathcal{S}_{n,\chi}(k))_{n\geq 0}$ restricts to a projective system $(\mathcal{S}^{(2)}_{n+1,\chi}(k) \to \mathcal{S}^{(2)}_{n,\chi}(k))_{n\gg 0}$. But, from the Mordell conjecture for finitely generated fields of characteristic 0 [Fa92, Chap. VI, Th.3], $S^{(2)}_{v_n,\chi}(k)$ is finite, $n \geq 0$. Hence $S_{v_n,\chi}(k)$ is finite, $n \gg 0$, which yields the contradiction, using the fact that a projective system of nonempty finite sets is nonempty.

Lemma 4.5. *$\mathcal{S}^{(1)}_{n,\chi}(k) = \emptyset$, $n \gg 0$.*

Proof. Assume that for all $n \geq 0$ there exists $v_n \in A_\eta[p^n]^*$ such that $p^N v_n^0 = 0$ and $S_{v_n,\chi}(k) \neq \emptyset$. Then, up to replacing v_n by $p^N v_{n+N}$, one may assume that $v_n \in (A_\eta)_0[p^n]^*$, $n \geq 0$.

Also, denote by $s_n : k \to S$ the image of a k-rational point $k \to S_{v_n,\chi}$; it yields a splitting $s_n : \Gamma_k \hookrightarrow \pi_1(S)$ of the restriction epimorphism $\pi_1(S) \twoheadrightarrow \Gamma_k$.

As $(A_\eta)_0 \to \eta$ is (isogeneous to) an isotrivial abelian variety, the representation $\rho : \pi_1(S \times_k \overline{k}) \to \mathrm{Aut}_{\mathbb{Z}_p}((A_\eta)_0[p^\infty])$ factors through a finite group Δ and the representation $\rho : \pi_1(S) \to \mathrm{Aut}_{\mathbb{Z}_p}((A_\eta)_0[p^\infty])$ factors through a group Γ *via* $\overline{\rho} : \Gamma \to \mathrm{Aut}_{\mathbb{Z}_p}((A_\eta)_0[p^\infty])$ so that we obtain an epimorphism of short exact sequences:

(6)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S \times_k \overline{k}) & \longrightarrow & \pi_1(S) & \underset{\overset{s_n}{\curvearrowleft}}{\longrightarrow} & \Gamma_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \underset{\overset{s_n}{\curvearrowleft}}{\longrightarrow} & \Gamma_k & \longrightarrow & 1, \end{array}$$

which, apart from s_n, only depends on $(A_\eta)_0 \to \eta$. Set $\Sigma_{s_n} := s_n(\Gamma_k)$ and let $\overline{\Delta}$, $\overline{\Gamma}$ and $\overline{\Sigma}_{s_n}$ denote the images of Δ, Γ and Σ_{s_n} respectively *via* $\overline{\rho} : \Gamma \to \mathrm{Aut}_{\mathbb{Z}_p}((A_\eta)_0[p^\infty])$. Then $\overline{\Gamma} \subset \mathrm{Aut}_{\mathbb{Z}_p}((A_\eta)_0[p^\infty]) \simeq \mathrm{GL}_{2d}(\mathbb{Z}_p)$ ($d := \dim(A)$) is a compact p-adic Lie group so, in particular, it is finitely generated. But a finitely generated profinite group admits only finitely many open subgroups of given bounded index. So, as $[\overline{\Gamma} : \overline{\Sigma}_{s_n}] \leq |\Delta|$ is finite, there are only finitely many possibilities for the $\overline{\Sigma}_{s_n}$, $n \geq 0$. Thus, there exists $s \in S(k)$ such that for infinitely many $n \geq 0$ $\overline{\Sigma}_{s_n} = \overline{\Sigma}_s$.

Write $|\overline{\Delta}| = p^a m$ with $p \not| m$. Then, it follows from lemma 4.6 below that, up to replacing v_n by $p^a v_{n+a}$, one may assume that $s : k \to S$ lifts to a k-rational point $s_{v_n} : k \to S_{v_n,\chi}$, hence $\underset{\longleftarrow}{\lim}\ \mathcal{S}^{(1)}_{n,\chi}(k) \neq \emptyset$: a contradiction.

Lemma 4.6. *Let s, $t \in S(k)$ with $\overline{\Sigma}_s = \overline{\Sigma}_t$, $n \geq 0$, and $v_n \in (A_\eta)_0[p^n]$. If $sp_t(v_n) \in A_t[p^n](\chi)$, then $sp_s(p^a v_n) \in A_s[p^n](\chi)$.*

Proof of lemma 4.6. The statement is trivial for $n \leq a$, so assume that $n > a$ and write $\delta_s(\sigma) := \overline{\rho}(s(\sigma)t(\sigma)^{-1}) \in \overline{\Delta}$. Also, since $\overline{\rho}(s(\sigma)) \in \overline{\Sigma}_s = \overline{\Sigma}_t$, there exists $\tau_{s,\sigma} \in \Gamma_k$ such that $\overline{\rho}(s(\sigma)) = \overline{\rho}(t(\tau_{s,\sigma}))$. As a result, one obtains $\delta_s(\sigma)(v_n) = \chi_n(\tau_{s,\sigma})\chi_n(\sigma^{-1})v_n$. In particular, the order of $\chi_n(\tau_{s,\sigma})\chi_n(\sigma^{-1}) \in (\mathbb{Z}/p^n)^*$ divides the order of $\delta_s(\sigma) \in \overline{\Delta}$, hence divides the order $p^a m$ of $\overline{\Delta}$. On the other hand, by the assumption on χ put at the very beginning of the proof of corollary 1.2, $\chi_n(\tau_{s,\sigma})\chi_n(\sigma^{-1})$ lies in the order $p^{n-i(p)}$ cyclic subgroup $1+p^{i(p)}\mathbb{Z}/p^n\mathbb{Z}$ of $(\mathbb{Z}/p^n)^*$. Thus, it follows that $\chi_n(\tau_{s,\sigma})\chi_n(\sigma^{-1}) \in 1+p^{n-a}\mathbb{Z}/p^n$, which yields the desired result. □

Note that the above proof of corollary 1.2 only resorts to the (weaker) statement for g_v in theorem 1.1 Using the (stronger) statement for $g_{\langle v\rangle}$, one gets the following variant of corollary 1.2.

Corollary 4.7. *Assume that k is finitely generated over $\mathbb{Q}$ and that the set $S^{\#}$ of all $s \in S(k)$ such that $T_p(A_s)$ admits a rank 1 Γ_k-submodule is finite. Then there exists $N = N(A, S, k, p) \geq 0$ such that, for all $s \in S(k) \smallsetminus S^{\#}$, and $T \in A_s[p^\infty]$ if $\langle T\rangle$ is Γ_k-stable then $p^N T = 0$.*

Proof. It follows the lines of the proof of corollary 1.2. Up to replacing $A \to S$ by $A \times_S (S \smallsetminus S^{\#}) \to S \smallsetminus S^{\#}$, one can assume that $S^{\#} = \emptyset$. As above, for each $n \geq 0$, define

$$\mathcal{S}_n := \coprod_{v_n \in A_\eta[p^n]^*} S_{\langle v_n\rangle}.$$

From theorem 1.1, there exists an integer $N \geq 0$ such that, for all $v \in A_\eta[p^\infty]$, either $p^N v^0 = 0$ or $S_{\langle v\rangle}$ has genus ≥ 2. So $\mathcal{S}_n$ can be written as a disjoint union $\mathcal{S}_n = \mathcal{S}_n^{(1)} \coprod \mathcal{S}_n^{(2)}$, where

$$\mathcal{S}_n^{(1)} := \coprod_{v_n \in A_\eta[p^n]^*,\, p^N v^0 = 0} S_{\langle v_n\rangle} \text{ and } \mathcal{S}_n^{(2)} := \coprod_{v_n \in A_\eta[p^n]^*,\, p^N v^0 \neq 0} S_{\langle v_n\rangle}.$$

Then $(\mathcal{S}_{n+1}(k) \to \mathcal{S}_n(k))_{n\geq 0}$ is a projective system with transition maps induced by the canonical morphisms $S_{\langle v_n\rangle} \to S_{\langle pv_n\rangle}$. They behave with respect to the decomposition $\mathcal{S}_n = \mathcal{S}_n^{(1)} \coprod \mathcal{S}_n^{(2)}$ as follows.

(7)
$$\begin{array}{ccc} \mathcal{S}_{n+1}^{(1)}(k) & & \mathcal{S}_{n+1}^{(2)}(k) \\ \downarrow & \swarrow^{(*)} & \downarrow \\ \mathcal{S}_n^{(1)}(k) & & \mathcal{S}_n^{(2)}(k) \end{array}$$

Suppose that corollary 4.7 does not hold. Then for all $n \geq 0$ there exists $s_n \in S(k)$ and there exists $\nu_n \in A_{s_n}[p^\infty]$ such that $p^n \nu_n \neq 0$ but $\langle \nu_n\rangle$ is Γ_k-stable. Up to replacing ν_n by $|\langle \nu_n\rangle| p^{-n}\nu_n$ one may thus assume that $\mathcal{S}_n(k) \neq \emptyset$, $n \geq 0$. Since $S^{\#} = \emptyset$, one has

$$\varprojlim \mathcal{S}_n(k) = \emptyset.$$

Then, assuming that $\mathcal{S}_n^{(1)}(k) = \emptyset$, $n \gg 0$, one obtains, by the same argument as in the proof of corollary 1.2 (i.e., resorting to theorem 1.1 and the Modell conjecture), that

$$\mathcal{S}_n(k) = \emptyset,\ n \gg 0,$$

as desired.

To prove that $\mathcal{S}_n^{(1)}(k) = \emptyset$, $n \gg 0$, suppose, on the contrary, that, for all $n \geq 0$, there exists $v_n \in A_\eta[p^n]^*$ such that $p^N v_n^0 = 0$ and $S_{\langle v_n\rangle}(k) \neq \emptyset$. Then, up to replacing v_n by $p^N v_{n+N}$, one may assume that $v_n \in (A_\eta)_0[p^n]^*$, $n \geq 0$. Also, denote by $s_n : k \to S$ the image of a k-rational point $k \to S_{\langle v_n\rangle}$; it yields a section $s_n : \Gamma_k \hookrightarrow \pi_1(S)$ of the restriction epimorphism $\pi_1(S) \twoheadrightarrow \Gamma_k$. Following the notations of the proof of lemma 4.5 for Δ, Γ, Σ_{s_n}, *etc.*, one can choose, as in the proof of lemma 4.5, a k-rational point $s : k \to S$ such that, for infinitely many $n \geq 0$, $\overline{\Sigma}_{s_n} = \overline{\Sigma}_s$. Then $\langle v_n\rangle$ is stable under $\overline{\Sigma}_s = \overline{\Sigma}_{s_n}$, which implies that $sp_s(\langle v_n\rangle) = \langle sp_s(v_n)\rangle$ is stable under Γ_k. Thus, $s : k \to S$ itself lifts to a k-rational point $k \to S_{\langle v_n\rangle}$, hence $\varprojlim \mathcal{S}_n^{(1)}(k) \neq \emptyset$: a contradiction. □

Remark 4.8. *(Variant in positive characteristic)* To state a variant of corollary 1.2 in characteristic $q > 0$, $q \neq p$, one has to take care of the fact that Samuel's theorem [Sa66], which is to play the part of the Mordell conjecture, involves a certain non-isotriviality assumption. For more details, see a subsequent paper.

5. Application to the 1-dimensional modular tower conjecture

5.1. **G-curves and their stacks.**

5.1.1. *Notation.* The main reference for this preliminary section is [BR07, §§1-6]. Let S be a connected scheme. An S-curve of genus g is a smooth, projective, geometrically connected S-scheme of dimension 1 the geometric fibers of which have genus g.

Given a finite group G of order prime to the characteristics of S, an S-G-curve with group G is a pair (Y, α), where Y is an S-curve and $\alpha : G \hookrightarrow \mathrm{Aut}_S(Y)$ is a group monomorphism. Two S-G-curves (Y_i, α_i), $i = 1, 2$ with the same group G are S-G-isomorphic if there exists an S-scheme isomorphism $u : Y_1 \to Y_2$ such that $u\alpha_1(g)u^{-1} = \alpha_2(g)$, $g \in G$. An S-G-cover with group G is a pair $(f : Y \to X, \alpha)$, where $f : Y \to X$ is a Galois cover of S-curves and $\alpha : G \tilde{\to} \mathrm{Aut}_X(Y)$ is a group isomorphism. Two S-G-covers $(Y_i \to X, \alpha_i)$, $i = 1, 2$ of a given S-curve $X \to S$ with the same group G are S-G-isomorphic if there exists an X-scheme isomorphism $u : Y_1 \to Y_2$ such that $u\alpha_1(g)u^{-1} = \alpha_2(g)$, $g \in G$. Two S-G-covers $(Y_i \to X_i, \alpha_i)$, $i = 1, 2$ with the same group G are weakly S-G-isomorphic if there exists an S-scheme isomorphism $v : X_1 \to X_2$ such that the S-G-covers $(v \circ f_1 : Y_1 \to X_2, \alpha_1)$ and $(f_2 : Y_2 \to X_2, \alpha_2)$ are S-G-isomorphic. The groupoid of S-G-curves with group G and S-G-isomorphisms is then equivalent to the groupoid of S-G-covers with group G and weak S-G-isomorphisms. In the following, we will drop the α in our notation though it remains part of the data.

Fix a finite group G and denote by $\mathcal{C}(G)$ the set of conjugacy classes of G. Given an element $C \in \mathcal{C}(G)$, write $o(C)$ for the common order of the elements in C. Define also two maps

$$\deg : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} \to \mathbb{Z}_{\geq 0},\ \mathbf{C} \mapsto \sum_{C \in \mathcal{C}(G)} \mathbf{C}(C),$$

$$\delta eg : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} \to \mathbb{Z}_{\geq 0},\ \mathbf{C} \mapsto \sum_{C \in \mathcal{C}(G)} \mathbf{C}(C)\frac{|G|}{o(C)},$$

and, for any integer $g \geq 0$, a map

$$\gamma_g : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} \to \frac{1}{2}\mathbb{Z},\ \mathbf{C} \mapsto 1 + |G|(g-1) + \frac{1}{2}(|G|\deg(\mathbf{C}) - \delta eg(\mathbf{C})).$$

Given any integer $g \geq 0$ and $\mathbf{C} \in \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$ such that $2 - 2g - \deg(\mathbf{C}) < 0$, one can consider the category fibered in groupoids $\mathcal{H}_{g,G,\mathbf{C}} \to \mathbb{Z}[\frac{1}{|G|}]$ of genus $\gamma_g(\mathbf{C})$ G-curves Y with group G such that the resulting G-cover $Y \to Y/G$ has inertia canonical invariant $\mathbf{C}$. (In case $\gamma_g(\mathbf{C}) \notin \mathbb{Z}_{\geq 0}$, set $\mathcal{H}_{g,G,\mathbf{C}} = \emptyset$.) Equivalently, $\mathcal{H}_{g,G,\mathbf{C}} \to \mathbb{Z}[\frac{1}{|G|}]$ is the category fibered in groupoids of G-covers of genus g curves with group G and inertia canonical invariant $\mathbf{C}$. (More precisely, the genus g curve is assumed to be equipped with an etale divisor of degree $\deg(\mathbf{C})$ and the inertia canonical invariant $\mathbf{C}$ is assumed to be one for the points on this etale divisor.) Finally, for any integer $r \geq 0$ such that $2 - 2g - r < 0$, write $\mathcal{H}_{g,G,r} \to \mathbb{Z}[\frac{1}{|G|}]$ for the disjoint union of the $\mathcal{H}_{g,G,\mathbf{C}} \to \mathbb{Z}[\frac{1}{|G|}]$, $\deg(\mathbf{C}) = r$.

Proposition 5.1. *$\mathcal{H}_{g,G,\mathbf{C}} \to \mathbb{Z}[\frac{1}{|G|}]_{et}$ is a Deligne-Mumford stack (with finite diagonal), smooth and of finite type over $\mathbb{Z}[\frac{1}{|G|}]_{et}$. Its coarse moduli space $\mathrm{H}_{g,G,\mathbf{C}}$ is a scheme normal and of finite type over $\mathbb{Z}[\frac{1}{|G|}]$.*

5.1.2. *Functoriality properties.* Any group epimorphism $p : \tilde{E} \twoheadrightarrow E$ defines a morphism of monoids $\mu : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(\tilde{E}))} \to \mathbb{Z}_{\geq 0}^{(\mathcal{C}(E))}$, sending $\tilde{C} \in \mathcal{C}(\tilde{E})$ to $p(\tilde{C}) \in \mathcal{C}(E)$. Geometrically, if $\tilde{f} : Y \to X$ is a G-cover with group $\tilde{E}$ and inertia canonical invariant $\tilde{\mathbf{C}}$ then $\mu(\tilde{\mathbf{C}})$ is the inertia canonical invariant of the G-cover $Y/\ker(p) \to X$ with group E.

Similarly, any group monomorphism $i : G \hookrightarrow \tilde{E}$ defines a morphism of monoids $\nu : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(\tilde{E}))} \to \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$ as follows. Consider the canonical map $c_G : G \to \mathcal{C}(G)$, sending $g \in G$ to its conjugacy class $c_G(g)$ in G and let $s : \mathcal{C}(G) \hookrightarrow G$, $C \mapsto s(C)$ a section of it. Then ν sends $\tilde{C} \in \mathcal{C}(\tilde{E})$ to $\sum_{\tilde{e} \in G\backslash\tilde{E}/\langle s(\tilde{C})\rangle} c_G(\tilde{e}s(\tilde{C})^{\frac{o(\tilde{C})}{|G\cap\tilde{e}\langle s(\tilde{C})\rangle\tilde{e}^{-1}|}}\tilde{e}^{-1}) \in \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$. Geometrically, if $\tilde{f} : Y \to X$ is a G-cover with group $\tilde{E}$ and inertia canonical invariant $\tilde{\mathbf{C}}$ then $\nu(\tilde{\mathbf{C}})$ is the inertia canonical invariant of the G-cover $Y \to Y/G$ with group G.

In the following, given an extension of finite groups $1 \to K \to F \overset{p}{\to} Q \to 1$, we will say that Q *acts trivially on* K *if there exists a set-theoretic section* $s : Q \to F$ *of* p *such that* $s(q)ks(q)^{-1} = k$, $q \in Q$, $k \in K$. (This is equivalent to requiring the canonical representation $Q \to \mathrm{Out}(K)$ be trivial.) Then note that for an extension of finite groups $1 \to G \overset{i}{\to} \tilde{E} \overset{p}{\to} E \to 1$ where E acts trivially on G and for any $\tilde{C} \in \mathcal{C}(\tilde{E})$, the set $\{g^{o(p(\tilde{C}))} |\ g \in \tilde{C}\}$ forms a conjugacy class of G, which we denote by $\tilde{C}^{o(p(\tilde{C}))}$. Then $\nu(\tilde{C})$ is simply $\frac{|E|}{o(p(\tilde{C}))}\tilde{C}^{o(p(\tilde{C}))} \in \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$.

Proposition 5.2. *With the above notation, one has:*

1. μ *induces a stack morphism* $\mathcal{H}_{g,\tilde{E},\mathbf{C}} \to \mathcal{H}_{g,E,\mu(\mathbf{C})} \times_{\mathbb{Z}[\frac{1}{|E|}]} \mathbb{Z}[\frac{1}{|\tilde{E}|}]$ *(again denoted by* μ*), which is proper and etale* [3] *(but representable if and only if* $Z(\tilde{E})$ *injects in* $Z(E)$*). In particular,* μ^* *sends each connected component surjectively onto a connected component.*
2. ν *induces a stack morphism* $\mathcal{H}_{g,\tilde{E},\mathbf{C}} \to \mathcal{H}_{\gamma_g(\mu(\mathbf{C})),G,\nu(\mathbf{C})} \times_{\mathbb{Z}[\frac{1}{|G|}]} \mathbb{Z}[\frac{1}{|\tilde{E}|}]$ *(again denoted by* ν*), which is representable, finite and unramified.*

Proof. Standard and similar to [BR07, Prop. 6.14]. □

In particular, corresponding to $p : G \twoheadrightarrow \{1\}$ and $i : \{1\} \hookrightarrow G$, one gets a canonical morphisms $\mu : \mathcal{H}_{g,G,\mathbf{C}} \to \mathcal{M}_{g,[\deg(\mathbf{C})]} \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}]$ and $\nu : \mathcal{H}_{g,G,\mathbf{C}} \to \mathcal{M}_{\gamma_g((C)),[\delta eg(\mathbf{C})]} \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}]$ respectively, where $\mathcal{M}_{g,[r]}$ denotes the stack of genus g proper smooth curves equipped with degree r etale divisors. (In case $g \notin \mathbb{Z}_{\geq 0}$, set $\mathcal{M}_{g,[r]} = \emptyset$.) From the former morphism, one obtains in particular that $\dim(\mathrm{H}_{g,G,r}) = 3g-3+r$, unless $\mathrm{H}_{g,G,r} = \emptyset$. Thus, the only cases when $\mathrm{H}_{g,G,r}$ is a curve correspond to $(g,r) = (0,4),\ (1,1)$.

So far, we have considered all (i.e., both trivial and nontrivial) conjugacy classes. A variant in which nontrivial conjugacy classes are only considered will be also needed later. So, let $\mathcal{C}(G)^* \subset \mathcal{C}(G)$ denote the set of nontrivial conjugacy classes of G. Then one has a natural injection $\mathbb{Z}_{\geq 0}^{(\mathcal{C}(G)^*)} \hookrightarrow \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$ (denoted by $\mathbf{C} \mapsto \mathbf{C}$) and a natural projection $\mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} \twoheadrightarrow \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G)^*)}$ (denoted by $\mathbf{C} \mapsto \mathbf{C}^*$). Any group epimorphism $p : \tilde{E} \twoheadrightarrow E$ defines $\mu^* : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(\tilde{E})^*)} \to \mathbb{Z}_{\geq 0}^{(\mathcal{C}(E)^*)}$ by $\mu^*(\mathbf{C}) = \mu(\mathbf{C})^*$. Similarly, any group monomorphism $i : G \hookrightarrow \tilde{E}$ defines $\nu^* : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(\tilde{E})^*)} \to \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G)^*)}$ by $\nu^*(\mathbf{C}) = \nu(\mathbf{C})^*$. Then one has the following variant of Proposition 5.2:

Proposition 5.3. *With the above notation, one has:*

1. *Assume* $2-2g-\deg(\mu^*(\mathbf{C})) < 0$. *Then* μ^* *induces a stack morphism* $\mathcal{H}_{g,\tilde{E},\mathbf{C}} \to \mathcal{H}_{g,E,\mu^*(\mathbf{C})} \times_{\mathbb{Z}[\frac{1}{|E|}]} \mathbb{Z}[\frac{1}{|\tilde{E}|}]$ *(again denoted by* μ^**), which is smooth (but representable if and only if* $Z(\tilde{E})$ *injects in* $Z(E)$*) and sends each connected component surjectively onto a connected component.*

[3] For the notion of (not necessarily representable) proper and etale morphisms between Deligne-Mumford stacks, see [DM69, §4].

(2) *Assume* $2 - 2\gamma_g(\mu^*(\mathbf{C})) - \deg(\nu^*(\mathbf{C})) < 0$. *Then* ν^* *induces a stack morphism* $\mathcal{H}_{g,\tilde{E},\mathbf{C}} \to \mathcal{H}_{\gamma_g(\mu^*(\mathbf{C})),G,\nu^*(\mathbf{C})} \times_{\mathbb{Z}[\frac{1}{|G|}]} \mathbb{Z}[\frac{1}{|\tilde{E}|}]$ *(again denoted by* ν^**), which is representable and finite.*

Proof. (1) follows from (1) of proposition 5.2, together with standard properties of moduli spaces $\mathcal{M}_{g,[r]}$. The representability assertion of (2) follows from (2) of proposition 5.2, together with standard properties of moduli spaces $\mathcal{M}_{g,[r]}$. The finiteness assertion essentially follows from (the proof of) [BR07, Prop. 6.14]. (See also the proof of [CT06, Lem. 4.3].) □

5.2. **The representability problem.** So far, the base scheme has been (an open subscheme of) the spectrum of $\mathbb{Z}$. From now on, we take the spectrum of a field k of characteristic ≥ 0 as a base scheme. In particular, various moduli stacks are considered to be ones over k.

5.2.1. *Automorphism group of objects in* $\mathcal{H}_{g,G,r}$. Throughout this subsection, let $f : Y \to X$ be a G-cover corresponding to a geometric point $x : \Omega \to \mathcal{H}_{g,G,r}$, where Ω is a separably closed field. For each $v \in \mathrm{Aut}(X)$, $v \circ f$ gives another G-cover $Y \to X$ with group G. So, let E_f denote the stabilizer of the G-isomorphism class of f in $\mathrm{Aut}(X)$ and refer to it as the *base group* of f. Set $B := X/E_f$ and consider the resulting cover $\tilde{f} : Y \overset{f}{\to} X \overset{q}{\to} B$. Then, by the definition of E_f, one sees that $\tilde{f}$ is a G-cover, say, with group $\tilde{E}_f$. Let $\overline{R}_f$ be the image in B of the r marked points in X (given as a part of data of $x : \Omega \to \mathcal{H}_{g,G,r}$), and R_q the ramification locus of the G-cover $q : X \to B$ in B. Moreover, define R to be the union of $\overline{R}_f$ and R_q in B. If we define r_B (resp. $\overline{r}_f$, resp. r_q) to be the cardinality of R (resp. $\overline{R}_f$, resp. R_q), we have a trivial estimate: $r_B \leq \overline{r} + r_q \leq r + r_q$. We shall regard R, $\overline{R}_f$, R_q as reduced closed subschemes of B.

There exists a natural exact sequence $1 \to G \to \tilde{E}_f \overset{p}{\to} E_f \to 1$, where E_f acts trivially on G. Indeed, by the definition of E_f, any $e \in E_f$ admits a lift ϵ of e to Y which is compatible with $\mathrm{Aut}(f) \overset{\alpha^{-1}}{\to} G$ that is, $\alpha^{-1}(\epsilon g \epsilon^{-1}) = \alpha^{-1}(g)$, $g \in\mathrm{Aut}(f)$ and, hence, $\epsilon g \epsilon^{-1} = g$, $g \in G$. In other words, $\tilde{E}_f$ can be identified canonically with the subgroup of $\mathrm{Aut}(Y)$ generated by G and the centralizer of G in $\mathrm{Aut}(Y)$.

Since $\mathcal{H}_{g,G,r}$ are Deligne-Mumford stacks, the representability condition can be explicitly stated in terms of the automorphism groups of the geometric objects.

Lemma 5.4. *A nonempty open substack* $\mathcal{U} \hookrightarrow \mathcal{H}_{g,G,r}$ *is representable if and only if* G *has trivial center and, for any geometric point* $f : Y \to X$ *of* $\mathcal{U}$*, the corresponding* E_f *is trivial.*

In general, $\mathcal{H}_{g,G,r}$ is not representable, and the (rational) points on the coarse moduli space $\mathrm{H}_{g,G,r}$ are not in one-to-one correspondence with the geometric objects classified by $\mathcal{H}_{g,G,r}$. However, we can associate each rational point with a certain geometric object, as follows.

Let K be a field (containing k), and assume that $\Omega = K^{sep}$. Assume, moreover, that the K^{sep}-rational point of $\mathrm{H}_{g,G,r}$ defined by x is K-rational.

Lemma 5.5. (1) $(B; \overline{R}_f, R_q, R)$ *admits a natural model* $(B_K; \overline{R}_{f,K}, R_{q,K}, R_K)$ *defined over* K.

(2) B*,* X *and* Y *are Galois over* B_K*. (Note that* $\mathrm{Aut}(B/B_K) = \Gamma_K$*.) In particular, if we set* $\tilde{E}_{f,K} := \mathrm{Aut}(Y/B_K)$ *and* $E_{f,K} := \mathrm{Aut}(X/B_K)$*, we obtain the following natural commutative diagram:*

$$\begin{array}{rccc} \rho: & \tilde{E}_{f,K} & \to & \mathrm{Aut}(G) \\ & \downarrow & & \downarrow \\ \overline{\rho}: & E_{f,K} & \to & \mathrm{Out}(G) \end{array}$$

via conjugation.

(3) *In fact,* $\overline{\rho}$ *is trivial.*

Proof. (1) The proof of the existence of a natural model B_K of B is just similar to [DE99, proof of Th. 3.1] (*cf.* Rem. 3.2(b), *loc.cit.*) and done via descent theory. The descent of $\overline{R}_f$, R_q and R is automatically done at the same time.

(2) $B = B_K \times_K K^{sep}$ is clearly Galois over B_K with Galois group Γ_K. The assertion that X and Y are Galois over B_K follows essentially from the fact that K is the field of moduli of f as a G-cover.

(3) Again, this follows essentially from the fact that K is the field of moduli of f as a G-cover. □

We shall refer to B_K (resp. $(B_K;\overline{R}_{f,K},R_{q,K},R_K)$) as the canonical model of B (resp. $(B;\overline{R}_f,R_q,R)$) with respect to the G-cover f. These models have the following functorial property:

Lemma 5.6. *For each $i=0,1$, let $f_i:Y_i\to X$ be a G-cover with group G_i (corresponding to a K^{sep}-rational point on $\mathrm{H}_{g,G_i,r}$), and assume that we are given an X-morphism $g:Y_1\to Y_0$. (Observe that then g naturally induces a surjection $G_1\to G_0$ compatible with g.) Moreover, set $B_i:=X/E_{f_i}$ and let $B_{i,K}$ be the canonical model of B_i with respect to the G-cover f_i. Then we have an inclusion $E_{f_1}\subset E_{f_0}$, and the natural K^{sep}-morphism $B_1\to B_0$ induced by this inclusion descends to a K-morphism $B_{1,K}\to B_{0,K}$.*

If, moreover, $E_{f_1}=E_{f_0}$, we have $B_{1,K}=B_{0,K}$.

Proof. Immediate from the various definitions. □

5.2.2. *The abelian case.* We continue the investigation in the previous subsection in the case that G is abelian.

Lemma 5.7. *In the situation of lemma 5.5, assume moreover that G is abelian. In the G-cover $\tilde{f}:Y\to B$ with group $\tilde{E}_f$, consider the subcover $Y^{ab}\to B$ corresponding to the quotient $\tilde{E}_f\to(\tilde{E}_f)^{ab}$. Moreover, for each positive integer b, consider the subcover $Z_b\to B$ of $Y^{ab}\to B$ corresponding to the quotient $(\tilde{E}_f)^{ab}\to(\tilde{E}_f)^{ab}/(\tilde{E}_f)^{ab}[b]=:N_b$. Set $e:=|E_f|$ and define e' to be the exponent of the abelian group $(E_f)^{ab}$. Then:*

1. *N_b admits a subquotient isomorphic to $G/G[eb]$.*
2. *If b is divisible by e', then N_b is isomorphic to a subquotient of G and the natural action of Γ_K on N_b (via conjugation) is trivial.*
3. *If b is divisible by e, the ramification locus in B of the G-cover $Z_b\to B$ is contained in $\overline{R}_f$.*
4. *Assume, moreover, that $|G|$ is a power of a prime p and that the image of the cyclotomic character $\chi:\Gamma_K\to\mathbb{Z}_p^*$ has finite index c in $\mathbb{Z}_p^*$. Set $\nu:=\nu(p,c,r):=[\log_p(crp/(p-1))]$. Then, if b is divisible by both e and $p^\nu e'$, then the G-cover $Z_b\to B$ is unramified everywhere.*

Proof. By applying the Hochschild-Serre spectral sequence to the exact sequence

$$1\to G\to\tilde{E}_f\to E_f\to 1$$

of finite groups, one gets a natural exact sequence

$$0\to H^1(E_f,\mathbb{Q}/\mathbb{Z})\to H^1(\tilde{E}_f,\mathbb{Q}/\mathbb{Z})\to H^1(G,\mathbb{Q}/\mathbb{Z})^{E_f}\to H^2(E_f,\mathbb{Q}/\mathbb{Z}),$$

of finite abelian groups. By the duality between abelian groups M and their character groups $M^\vee:=\mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})$ (or, equivalently, by the Pontryagin duality), one gets a natural exact sequence

$$H^2(E_f,\mathbb{Q}/\mathbb{Z})^\vee\to G\to(\tilde{E}_f)^{ab}\to(E_f)^{ab}\to 0.$$

(Here, we have used the fact that G is abelian and that the action of E_f on G is trivial.) Thus, for the natural morphism $G\to(\tilde{E}_f)^{ab}$, the cokernel is killed by e' and the kernel is kiled by e.

(1) Let $\overline{G}$ denote the image of G in $(\tilde{E}_f)^{ab}$. By the preceding argument, the quotient $\overline{G}$ is bigger than $G/G[e]$, that is, we have $\overline{G}\twoheadrightarrow G/G[e]$ as quotients of G. Now,

$$N_b=(\tilde{E}_f)^{ab}/(\tilde{E}_f)^{ab}[b]\supset\overline{G}/\overline{G}[b]\twoheadrightarrow(G/G[e])/(G/G[e])[b]=G/G[eb],$$

as desired.

(2) By the preceding argument, if b is divisible e', we have $b(\tilde{E}_f)^{ab}\subset\overline{G}\twoheadleftarrow G$. On the other hand, the b-multiplication map induces a natural isomorphism $N_b=(\tilde{E}_f)^{ab}/(\tilde{E}_f)^{ab}[b]\simeq b(\tilde{E}_f)^{ab}$. Thus, the first assertion follows. Since all the construction is natural and Γ_K-equivariant, the second assertion follows from lemma 5.5(2)(3).

(3) The ramification locus in X of the G-cover $f:Y\to X$ with group G is contained in R_f. Thus, for the G-cover $\tilde{f}:Y\to B$ with group $\tilde{E}_f$, the inertia group $I\subset\tilde{E}_f$ at any point of $B\smallsetminus\overline{R}_f$ is

injectively mapped into E_f. In particular, any element of such I is killed by $e = |E_f|$. So, if we denote the image of I in $(\tilde{E}_f)^{ab}$ by $\overline{I}$, we have $\overline{I} \subset (\tilde{E}_f)^{ab}[e]$, as desired.

(4) Take any (closed) point P in $\overline{R}_{f,K} \subset B_K$. Then we have $[K(P) : K] \leq \overline{r} \leq r$, hence

$$[\mathbb{Z}_p^* : \chi(\Gamma_{K(P)})] = [\mathbb{Z}_p^* : \chi(\Gamma_K)][\chi(\Gamma_K) : \chi(\Gamma_{K(P)})] \leq [\mathbb{Z}_p^* : \chi(\Gamma_K)][K : K(P)] \leq cr.$$

Suppose, moreover, that $\chi(\bmod p^n) : \Gamma_{K(P)} \to (\mathbb{Z}/p^n)^*$ is trivial for some $n > 0$. Then we have

$$cr \geq [\mathbb{Z}_p^* : \chi(\Gamma_{K(P)})] \geq |(\mathbb{Z}/p^n)^*| = (p-1)p^{n-1},$$

hence $n \leq \nu$. (Observe that this last estimate is also available for $n = 0$.)

Now, consider the inertia group $I_b \subset N_b$ at P for the abelian G-cover $Z_b \to X$. (In particular, $I_1 \subset N_1 = (\tilde{E}_f)^{ab}$.) I_b admits a natural action of $\Gamma_{K(P)}$, which is via the cyclotomic character. So, by (2), the $\Gamma_{K(P)}$-action on $I_{e'}$ is via the p-adic cyclotomic character on the one hand and trivial on the other hand. By the preceding argument, this implies that $I_{e'}$ is killed by p^ν, hence I_1 is killed by $e'p^\nu$. This, together with (3), proves the assertion. □

Corollary 5.8. *In the situation of lemma 5.7, fix a prime p and assume that the image of the cyclotomic character $\chi : \Gamma_K \to \mathbb{Z}_p^*$ has finite index c in $\mathbb{Z}_p^*$. Then there exist integers $\beta := \beta(p, c, r, e) \geq 1$ and $\delta := \delta(p, c, r, e) \geq 0$, such that, if G is cyclic of order p^m, then N_β is cyclic of order p^μ with $\mu \geq m - \delta$; $Z_\beta \to B$ is a G-cover with group N_β and unramified everywhere; Z_β is Galois over B_K; and the natural action of Γ_K on N_β is trivial.*

Proof. This is just summarizing (a part of) lemma 5.7. More specifically, set $\beta := p^\nu e$ and define μ to be the largest integer that satisfies $p^\delta | e\beta$. Then, N_β is isomorphic to a subquotient of G by (2), hence it is a cyclic p-group. So, set $|N_\beta| = p^\mu$. N_β admits a subquotient isomorphic to $G/G[e\beta] = G/G[p^\delta]$ by (1), hence $\mu \geq m - \delta$. $Z_\beta \to B$ is a G-cover with group N_β by definition, and it is unramified everywhere by (2). Z_β is Galois over B_K by construction, and, finally, the natural action of Γ_K on N_β is trivial by (4). □

5.3. The 1-dimensional modular tower conjecture.

5.3.1. *Statements.* For Fried's modular towers and related conjectures, see [Fr95], [FK97], [D06]. Let p be a prime, and (g, r) a pair of non-negative integers with $2 - 2g - r < 0$. Let $\mathbf{G} = \{G_{n+1} \to G_n\}_{n \geq 0}$ be a projective system of finite groups and assume that $G := \varprojlim G_n$ is p-obstructed in the sense that G contains an open subgroup that admits a quotient isomorphic to $\mathbb{Z}_p$. Denote by Σ_G the set of prime numbers which divide the order of (some finite quotient of) G. Also, fix an r-tuple $\mathbf{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G, and define $\mathbf{C}_n = (C_{n,1}, \ldots, C_{n,r})$ to be the image of $\mathbf{C} = (C_1, \ldots, C_r)$ in G_n for each $n \geq 0$. By [C07, Cor. 3.6] (see also [BF02], [K04]), one has:

Lemma 5.9. *For any field k finitely generated over the prime field of characteristic $q \notin \Sigma_G$ (hence, in particular, $q \neq p$),*

$$\varprojlim \mathrm{H}_{g,G_n,\mathbf{C}_n}(k) = \emptyset.$$

Now, a generalized variant of Fried's modular tower conjecture is the following [4].

Conjecture 5.10. *For any field k finitely generated over the prime field of characteristic $q \notin \Sigma_G$, there exists an integer $N := N(p, k, g, \mathbf{G}, \mathbf{C})$ such that $\mathrm{H}_{g,G_n,\mathbf{C}_n}(k) = \emptyset$, $n \geq N$.*

We formulate the d-dimensional version of this conjecture in characteristic $q \geq 0$, as follows.

($\mathrm{MT}_d(q)$) For any field k finitely generated over the prime field of characteristic $q \notin \Sigma_G$, of characteristic 0, any k-scheme $S \to k$ of finite type with $\dim(S) \leq d$ and any k-morphism $\xi : S \to \mathrm{H}_{g,G_0,\mathbf{C}_0}$, there exists an integer $N := N(p, k, g, \mathbf{G}, \mathbf{C}, S, \xi)$ such that $S_n(k) = \emptyset$, $n \geq N$. Here, we set $S_n := S \times_{\mathrm{H}_{g,G_0,\mathbf{C}_0}} \mathrm{H}_{g,G_n,\mathbf{C}_n}$.

[4]This is a weak version in the sense that p is fixed. For the formulation of a version of the modular tower conjecture which allows p to vary, see [Fr06, §6]

First, we have the following corollary of lemma 5.9:

Corollary 5.11. *$(MT_0(q))$ holds for all $q \geq 0$. In particular, conjecture 5.10 holds for $(g, r) = (0, 3)$.*

Proof. Since S_n is of finite type over k and 0-dimensional, $S_n(k)$ is a finite set for any $n \geq 0$. Thus, the assertion follows from lemma 5.9. □

Now, the main result in this section is:

Theorem 5.12. *$(MT_1(0))$ holds. In particular, conjecture 5.10 holds for $(g, r) = (0, 4), (1, 1)$ and $q = 0$.*

5.3.2. *Proof.* To clarify the structure of the proof of theorem 5.12, we shall also introduce the d-dimensional generalization of corollary 1.2 in characteristic $q \geq 0$:

$(\mathrm{UB}_d(q))$ Let k be a field finitely generated over the prime field of characteristic q, S a scheme of finite type over k with $\dim(S) \leq d$, and A an abelian scheme over S. Let p be a prime and $\chi : \Gamma_k \to \mathbb{Z}_p^*$ a non-Tate character. Then there exists an integer $N := N(A, S, k, p, \chi)$, such that $A_s[p^\infty](\chi) \subset A_s[p^N]$ holds for any $s \in S(k)$.

Theorem 5.13. *$(UB_d(0))$ holds for $d \leq 1$.*

Proof. First, $(\mathrm{UB}_0(q))$ for any $q \geq 0$ follows from the definition of non-Tate characters. (Indeed, if S is of finite type over k and $\dim(S) = 0$, $S(k)$ is a finite set. So, we may treat only finitely many abelian varieties A_s $(s \in S(k))$.) Next, to prove $(\mathrm{UB}_1(0))$, assume that S is of finite type over k and $\dim(S) = 1$. By replacing S by S^{red}, we may assume that S is reduced. By $(\mathrm{UB}_0(0))$, we may replace S by an open dense subscheme freely. So, we may assume that S is regular and separated, hence smooth and separated over k. Finally, treating S componentwise, we may assume that S is connected. Suppose that S is not geometrically conneceted. Then $S(k) = \emptyset$ and there remains nothing to prove. Thus, in summary, we may assume that S is a smooth, separated, geometrically connected curve over k. Now, the assertion follows from corollary 1.2. □

Now, theorem 5.12 formally follows from corollary 5.11, theorem 5.13 and the following (for $d = 1$ and $q = 0$):

Proposition 5.14. *Assume that $(UB_d(q))$ and $(MT_{d-1}(q))$ hold. Then $(MT_d(q))$ holds.*

Proof. Denote by $G(n)$ the kernel of $G \twoheadrightarrow G_n$. Since $G = \varprojlim G_n$ is p-obstructed, there exists an open subgroup U of G which admits a quotient isomorphic to $\mathbb{Z}_p$. For some $n_0 \geq 0$, $G(n_0)$ is contained in U. As $G(n_0)$ is open in U, its image in the quotient is open, hence $G(n_0)$ also admits a quotient Z isomorphic to $\mathbb{Z}_p$. For each $n \geq 0$, denote by $Z(n)$ the image of $G(n+n_0)$ in Z, and set $Z_n := Z/Z(n)$. Thus, $Z_n \simeq \mathbb{Z}/p^{m(n)}\mathbb{Z}$, where $0 \leq m(0) \leq m(1) \leq \cdots \leq m(n) \leq \cdots \to \infty$. Now, by applying the functoriality properties in 5.1.2 to $G_{n+n_0} = G/G(n+n_0) \hookleftarrow G(n_0)/G(n+n_0) \twoheadrightarrow Z/Z(n) = Z_n$, we may reduce the problem to the case where G is isomorphic to $\mathbb{Z}_p$ and G_n is isomorpic to $\mathbb{Z}/p^{m(n)}$.

As in the assertion of $(\mathrm{MT}_d(q))$, let S be a scheme of finite type over k and $\dim(S) \leq d$. Let S'_n be the closure of $S_n(k)$ in S_n. Denote by $\mathcal{I}_n$ the (finite) set of irreducible components of S'_n of dimension d. Since $S_{n+1} \to S_n$ is finite, $\{\mathcal{I}_n\}_{n \geq 0}$ forms a projective system. Thus, one gets the following dichotomy: either $\mathcal{I}_n = \emptyset$ for $n \gg 0$ or $\varprojlim \mathcal{I}_n \neq \emptyset$. The first case can be reduced easily to $(\mathrm{MT}_{d-1}(q))$. In the second case, take an element of $\varprojlim \mathcal{I}_n \neq \emptyset$, and, for each $n \geq 0$, let $T_n \subset S'_n \subset S_n$ be the corresponding irreducible component of dimension d, which is regarded as a reduced closed subscheme. (Observe that $T_n(k)$ is dense in T_n.) Let F_n denote the function field of the integral scheme T_n and K_n the perfect closure of F_n. (That is, $K_n = F_n$ for $q = 0$ and $K_n = \cup_{s \geq 0} F_n^{q^{-s}}$ for $q > 0$.) Set $F := F_0$.

First, we claim that the natural map $\Gamma_{K_n} \to \Gamma_k$ is surjective, which is equivalent to saying that the natural map $\Gamma_{F_n} \to \Gamma_k$ is surjective. Indeed, for the latter surjectivity, take a nonempty open subscheme U_n of T_n which is normal. Then $\Gamma_{F_n} \to \Gamma_k$ factors through $\Gamma_{F_n} \twoheadrightarrow \pi_1(U_n)$. So, it suffices to prove that $\pi_1(U_n) \to \Gamma_k$ is surjective. But this is clear, as $U_n(k) \neq \emptyset$. In particular, since the cyclotomic character $\chi : \Gamma_{F_n} \to \mathbb{Z}_p^*$ is the composite of $\Gamma_{F_n} \twoheadrightarrow \Gamma_k$ and the cyclotomic character $\Gamma_k \to \mathbb{Z}_p^*$, the index c of $\chi(\Gamma_{F_n})$ in $\mathbb{Z}_p^*$ is independent of n.

Next, as $T_{n+1} \to T_n$ is a finite morphism between d-dimensional schemes, we may regard F_{n+1} as a finite extension of F_n. So, we may take an algebraic closure $\Omega = \overline{F}$ of F, so that $F_n \subset \Omega$. Observe that $\Omega = \overline{K_n} = K_n^{sep}$ for each $n \geq 0$. Thus, S_n admits a natural Ω-rational point, which induces $(x_n)_{n\geq 0} \in \varprojlim \mathrm{H}_{g,G_n,\mathbf{C}_n}(\Omega)$. Let $f_n : Y_n \to X_n$ be a G-cover with group G_n over Ω corresponding to x_n. (Note that Ω is not only separably closed but also algebraically closed. Thus, by the very definition of coarse space, such f_n exists.) Since $(x_n)_{n\geq 0}$ is an element of the projective limit, we may assume that $X_n = X_0 =: X$ and that $f_n : Y_n \to X$ is a subcover of $f_{n+1} : Y_{n+1} \to X$ (compatibly with $G_{n+1} \twoheadrightarrow G_n$). Let $E_{f_n} \subset \mathrm{Aut}(X)$ be the base group for f_n. As $E_{f_0} \supset E_{f_1} \supset \cdots$ are inclusions of finite groups, there exists $n_1 \geq 0$ such that $E_{f_n} = E_{f_{n_1}}$ for $n \geq n_1$. So, by renumbering if necessary, we may assume that $E_{f_n} = E_{f_0} =: E$. Let B_{n,K_n} be the canonical model over K_n of $B = X/E$ with respect to f_n, as in lemma 5.5. Then, by lemma 5.6, we may identify B_{n,K_n} with $B_{0,K_0} \times_{K_0} K_n$. So, we shall write $B_{n,K_n} = B_{K_n}$.

Let $1 \to G_n \to \tilde{E}_{f_n} \to E \to 1$ be the exact sequence as in 5.2. In the G-cover $\tilde{f}_n : Y_n \to B$ with group $\tilde{E}_{f_n}$, consider the subcover $Y_n^{ab} \to B$ corresponding to the quotient $\tilde{E}_{f_n} \to (\tilde{E}_{f_n})^{ab}$. Moreover, for each positive integer b, consider the subcover $Z_{n,b} \to B$ of $Y_n^{ab} \to B$ corresponding to the quotient $(\tilde{E}_{f_n})^{ab} \to (\tilde{E}_{f_n})^{ab}/(\tilde{E}_{f_n})^{ab}[b] =: N_{n,b}$.

Now, take $\beta = \beta(p,c,r,e) \geq 1$ and $\delta = \delta(p,c,r,e) \geq 0$ as in corollary 5.8. Then, since G_n is a cyclic group of order $p^{m(n)}$, $N_{n,\beta}$ is cyclic of order $p^{\mu(n)}$ with $\mu(n) \geq m(n) - \delta$; $Z_{n,\beta} \to B$ is a G-cover with group $N_{n,\beta}$ and unramified everywhere; $Z_{n,\beta}$ is Galois over B_{K_n}; and the natural action of Γ_{K_n} on $N_{n,\beta}$ is trivial. The natural surjection $\tilde{E}_{f_{n+1}} \twoheadrightarrow \tilde{E}_{f_n}$ induces a surjection $N_{n+1,\beta} \twoheadrightarrow N_{n,\beta}$. Moreover, we may consider $Z_{n,\beta} \to B$ as a subcover of $Z_{n+1,\beta} \to B$, compatibly with the latter surjection.

There exist a finite extension F' of F included in K_0 and a proper, smooth, geometrically connected curve $B_{F'}$ over F', such that $B_{F'} \times_{F'} K_0$ is K_0-isomorphic to B_{K_0}. When $q = 0$, we have $K_0 = F' = F$. when $q > 0$, we may assume that $F' = F^{q^{-s}}$ for some $s \geq 0$, up to replacing F' by an extension. Now, set $k' := k$ for $q = 0$ and $k' := k^{q^{-s}}$ for $q > 0$. Also, set $F'_n := F_n$ for $q = 0$ and $F'_n := F_n^{q^{-s}}$ for $q > 0$.

When $q = 0$, set $T'_n := T_n$, and, when $q > 0$, set $T'_n = T_n^{q^{-s}}$. Note that T'_n is an integral scheme of finite type over k' with function field F'_n. There exists a nonempty open subscheme U of T'_0 such that $B_{F'} \to F'$ extends to a proper, smooth, geometrically connected scheme $B_U \to U$. Let $J := \mathrm{Pic}^0_{B_U/U}$ be the jacobian. Now, applying $(\mathrm{UB}_d(q))$ to the abelian scheme J over U and the cyclotomic character $\chi : \Gamma_{k'} \to \mathbb{Z}_p^*$, we see that there exists an integer $N := N(J, U, k', p, \chi)$, such that $J_s[p^\infty](\chi) \subset J_s[p^N]$ holds for any $s \in U(k')$.

We have a surjection $T_p(J_{F'}) \twoheadrightarrow N_{n,\beta} \simeq \mathbb{Z}/p^{\mu(n)}$ as modules over $\Gamma_{F'_n} = \Gamma_{K_n}$, hence, by duality, an injection $\mathbb{Z}/p^{\mu(n)}(1) \simeq (N_{n,\beta})^\vee \hookrightarrow J_{F'}[p^\infty]$ as $\Gamma_{F'_n}$-modules. Since $m(n) \to \infty$ as $n \to \infty$, there exists an n_0 such that $\mu(n_0) > N$. Take a nonempty, open, normal subscheme V_{n_0} of $\pi^{-1}(U)$, where $\pi : T'_{n_0} \to T'_0$ denotes the natural morphism induced by $T_{n_0} \to T_0$. Since $J_{V_{n_0}} := J \times_U V_{n_0}$ is an abelian scheme over V_{n_0}, the action of $\Gamma_{F'_{n_0}}$ on $J_{F'}[p^\infty]$ factors through $\Gamma_{F'_n} \twoheadrightarrow \pi_1(V_{n_0})$. Thus, the injection $(N_{n_0,\beta})^\vee \hookrightarrow J_{F'}[p^\infty]$ as $\Gamma_{F'_{n_0}}$-modules extends to an injection $(N_{n_0,\beta})^\vee_{V_{n_0}} \hookrightarrow J_{V_{n_0}}[p^\infty]$ as (ind-)finite-etale group schemes over V_{n_0}.

Recall that $T_{n_0}(k)$ is dense in T_{n_0}, hence $T'_{n_0}(k')$ is dense in T'_{n_0}. In particular, $V_{n_0}(k') \neq \emptyset$. Take $s_{n_0} \in V_{n_0}(k')$ and let s denote the image of s_{n_0} in U. Considering the fiber at s_{n_0} of the injection $(N_{n_0,\beta})^\vee_{V_{n_0}} \hookrightarrow J_{V_{n_0}}[p^\infty]$, one gets the injection $(N_{n_0,\beta})^\vee_{k'} \hookrightarrow J_s[p^\infty]$. Since $(N_{n_0,\beta})^\vee_{k'} \simeq \mathbb{Z}/p^{\mu(n)}(1)$ and $\mu(n) > N$. This contradicts the choice of N.

Thus, the proof of proposition 5.14 is completed. □

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