VARIATION OF TANNAKA GROUPS OF PERVERSE SHEAVES IN FAMILY

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Abstract. Let k be a field of characteristic 0, let S be a smooth, geometrically connected variety over k, with generic point η, and $f : \mathcal{X} \to S$ a morphism separated and of finite type. Fix a prime ℓ. Let $\mathcal{P}$ be an f-universally locally acyclic relative perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}/S$. We prove that if for some (equivalently, every) geometric point $\bar{\eta}$ over η the restriction $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple as a perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}_{\bar{\eta}}$, then there is a non-empty open subscheme $U \subset S$ such that, for every geometric point $\bar{s}$ on U, the restriction $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple as a perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}_{\bar{s}}$. When $f : \mathcal{X} \to S$ is an abelian scheme, we give applications of this result to the variation with $s \in S$ of the Tannaka group of $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$. As a special case, given a family $\mathcal{Y} \hookrightarrow \mathcal{X}$ of closed subvarieties satisfying mild assumptions, we prove that the locus of all $s \in S$ where $\mathcal{Y}_{\bar{s}}$ adquires additional symmetries (with respect to the structure of the ambient abelian variety $\mathcal{X}_{\bar{s}}$) is not Zariski-dense.

Contents

1. Introduction

Let k be a field of characteristic 0, let S be a smooth, geometrically connected variety over k, with generic point η, and let $g : \mathcal{Y} \to S$ be a smooth projective S-scheme of relative dimension d. A general and central question in algebraic geometry is to understand how the fibers $\mathcal{Y}_s$ vary with $s \in S$. For instance, one may ask when some power $\mathcal{Y}_{\bar{s}}^n$ of $\mathcal{Y}_{\bar{s}}$ carries exceptional algebraic cycles. If $k \subset \mathbb{C}$, the Hodge conjecture predicts that this is the same as asking when the Mumford-Tate group $G(\mathcal{V})_s$ of the

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polarizable $\mathbb{Q}$-Hodge structure $\mathrm{H}^\bullet(\mathcal{Y}_s^{\mathrm{an}}, \mathbb{Q}) \simeq s^* R^\bullet g_*^{\mathrm{an}} \mathbb{Q}$ becomes smaller than the generic Mumford-Tate group $G(\mathcal{V})$ of the polarizable $\mathbb{Q}$-variation of Hodge structures $\mathcal{V} := R^\bullet g_*^{\mathrm{an}} \mathbb{Q}$ on the analytification S^{an} of $S \times_k \mathbb{C}$. Here "becomes smaller" makes sense because the Tannaka categories of polarizable $\mathbb{Q}$-variations of Hodge structures are functorial with respect to pullbacks along morphism of complex analytic spaces so that one can view naturally $G(\mathcal{V})_s$ as a subgroup of $G(\mathcal{V})$. This leads to introduce and study the Hodge locus

$$S_{\mathcal{V}} := \{s \in S \mid G(\mathcal{V})_s \subsetneq G(\mathcal{V})\}$$

of a polarizable $\mathbb{Q}$-variation of Hodge structures $\mathcal{V}$ on S^{an}. If k is finitely generated over $\mathbb{Q}$, similar considerations apply with ℓ-adic étale local systems on S yielding the introduction of the Tate locus $S_{\mathcal{V}_\ell} \subset S$ of such a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$. Under mild assumptions, general heuristics predict that these exceptional loci $S_{\mathcal{V}}$, $S_{\mathcal{V}_\ell}$ are sparse in some precise sense - *e.g.* that the atypical part of the Hodge locus $S_{\mathcal{V}}$ is not Zariski-dense in S [Kl23] or that the set of k-rational points in the Tate locus $S_{\mathcal{V}_\ell}$ is not Zariski-dense in S [C23]. Proving such sparsity results is notoriously challenging as it requires constructing bridges between the Zariski topology of S and the analytic natures of the coefficients $\mathcal{V}$, $\mathcal{V}_\ell$. The results of this article are also partly motivated by the problem of understanding how the fibers $\mathcal{Y}_s$ vary with $s \in S$ and inspired by the above Tannaka approaches, but in a more restricted setting and with a rather different category of coefficients, which makes the sparsity of the exceptional loci more accessible.

Namely, assume $g : \mathcal{Y} \to S$ factors as

$$g : \mathcal{Y} \overset{\iota}{\hookrightarrow} \mathcal{X} \overset{f}{\to} S$$

with $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed immersion and $f : \mathcal{X} \to S$ a smooth proper morphism whose fibers carry an interesting structure (*e.g.* a distinguished automorphism, a group law *etc.*). Then one may ask about the locus of all $s \in S$ where $\mathcal{Y}_s$ acquires additional symmetries with respect to the structure of the ambient variety $\mathcal{X}_s$, and in particular whether this locus is not Zariski-dense. A way to encode this problem is to consider the relative intersection cohomology sheaf $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_\ell[d]$; this is a relative perverse sheaf on $f : \mathcal{X} \to S$ (f-perverse sheaf for short) in the sense of Hansen-Scholze [HS23] that is, for every $s \in S$, $\mathcal{P}|_{\mathcal{X}_s}$ is a perverse sheaf on $\mathcal{X}_s$. As $\mathcal{Y} \to S$ is smooth and $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ is proper, $\mathcal{P}$ is f-universally locally acyclic (f-ULA or simply ULA for short), which roughly means that the perverse sheaves $\mathcal{P}|_{\mathcal{X}_s}$ vary nicely with $s \in S$. In particular, if $s_1 \in S$ specializes to s_0, there is a well-defined specialization $sp_{s_1,s_0}(\mathcal{P}|_{\mathcal{X}_{s_1}})$ of $\mathcal{P}|_{\mathcal{X}_{s_1}}$ which is again a perverse sheaf on $\mathcal{X}_{s_0}$, and can be compared with $\mathcal{P}|_{\mathcal{X}_{s_0}}$. When $f : \mathcal{X} \to S$ is an abelian scheme, the category of perverse sheaves on $\mathcal{X}_s$ (modded out by the full subcategory of negligible ones) can be upgraded to a Tannaka category whose tensor structure is built out from the group law on $\mathcal{X}_s$, and the symmetries of $\mathcal{P}|_{\mathcal{X}_s}$ are encapsulated in its Tannaka group $G(\mathcal{P})_s := G(\mathcal{P}|_{\mathcal{X}_s})$. The construction of the Tannaka category of perverse sheaves on $\mathcal{X}_s$ extends to the category of f-ULA relative perverse sheaves on $f : \mathcal{X} \to S$ compatibly with specialization so that, if $s_1 \in S$ specializes to s_0, one gets a well-defined cospecialization map $csp_{s_1,s_0} : G(\mathcal{P})_{s_0} \to G(\mathcal{P})_{s_1}$. As a result, one gets a well-defined degeneracy locus $S_{\mathcal{P}} \subset S$ and the initial problem is converted into describing the structure of $S_{\mathcal{P}} \subset S$.

More formally, let $f : \mathcal{X} \to S$ be a morphism separated and of finite type. Fix a prime ℓ. Let $D_c^b(\mathcal{X})$ denote the triangulated category of étale $\overline{\mathbb{Q}}_\ell$-sheaves with bounded constructible cohomology on $\mathcal{X}$ and $\mathrm{Perv}(\mathcal{X}/S) \subset D_c^b(\mathcal{X})$ the full subcategory of f-perverse sheaves. Let $D^{\mathrm{ULA}}(\mathcal{X}/S) \subset D_c^b(\mathcal{X})$ denote the full subcategory of those complexes which are f-ULA, which is a triangulated subcategory, and let $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ denote the full category of f-ULA f-perverse sheaves. When $f : \mathcal{X} \to S$ is an abelian scheme, the convolution product built out from the multiplication on $\mathcal{X}$ endows $D^{\mathrm{ULA}}(\mathcal{X}/S)$ with a structure of $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category. This monoidal structure, in turn, induces a structure of $\overline{\mathbb{Q}}_\ell$-Tannaka category on the quotient $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \twoheadrightarrow P^{\mathrm{ULA}}(\mathcal{X}/S)$ of $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ by the Serre subcategory of negligible objects. When S is a point, one recovers the usual construction $\mathrm{Perv}(X) \twoheadrightarrow P(X)$ of the $\overline{\mathbb{Q}}_\ell$-Tannaka category of perverse sheaves on an abelian variety X. In the relative setting, for every $s \in S$ and geometric point $\overline{s}$ over s, the canonical restriction functors

$$\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \overset{|_{\mathcal{X}_s}}{\to} \mathrm{Perv}(\mathcal{X}_s) \overset{|_{\mathcal{X}_{\overline{s}}}}{\to} \mathrm{Perv}(\mathcal{X}_{\overline{s}})$$

induce exact tensor functors

$$P^{\mathrm{ULA}}(\mathcal{X}/S) \overset{|_{\mathcal{X}_s}}{\to} P(\mathcal{X}_s) \overset{|_{\mathcal{X}_{\overline{s}}}}{\to} P(\mathcal{X}_{\overline{s}}).$$

In particular, fixing a fiber functor $\omega_{\bar{s}} : P(\mathcal{X}_{\bar{s}}) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$, one may ask, for $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, how the corresponding Tannaka groups

$$G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \subset G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \subset G(\mathcal{P}, \omega_{\bar{s}})$$

vary[1] with $s \in S$. As S is smooth over k, the canonical functor

$$-|_{\mathcal{X}_\eta} : \langle \mathcal{P} \rangle \to \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle$$

is an equivalence of Tannaka categories and, for every $s \in S$ one gets a natural (up to inner automorphisms) cospecialization diagram:

(1)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{\bar{\eta}}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{\bar{\eta}}) \arrow[r] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] & 1,
\end{tikzcd}$$

where, for $t \in S$, the category $\langle \mathcal{P}|_{\mathcal{X}_t} \rangle_0 \subset \langle \mathcal{P}|_{\mathcal{X}_t} \rangle$ denotes the full subcategory whose objects are of the form $0_{t*}\mathcal{L}$ for $\mathcal{L} \in \mathrm{Perv}(\mathrm{spec}(k(t)))$ and $0_t : \mathrm{spec}(k(t)) \to \mathcal{X}_t$ the zero-section. See Section 3 for more details. As the existence of this cospecialization diagram does not depend on the choice of the fiber functors, we omit them from the notation from now on. In other words, one would like to understand the arithmetico-geometric structure of the following degeneracy loci:

$$S^?_{\mathcal{P}} := \{s \in S \mid G(\mathcal{P}|_{\mathcal{X}_s})^? \subsetneq G(\mathcal{P}|_{\mathcal{X}_\eta})^?\}$$

$$S^{\mathrm{geo},?}_{\mathcal{P}} := \{s \in S \mid G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^? \subsetneq G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^?\},$$

where *e.g.*

?=			
	no decoration	G	
	$\circ$	G°	:= neutral component of G
	der	G^{der}	:= derived subgroup of G
	$\circ$, der	$G^{\circ,\mathrm{der}}$	

For $S^{\mathrm{geo}}_{\mathcal{P}}$, this question has been tackled in Krämer's dissertation thesis [Kr13, 3.7]; in particular Krämer observes that one cannot expect, in general, that $S^{\mathrm{geo}}_{\mathcal{P}}$ be a strict, Zariski-closed subset of S unless the determinant $\det(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ of $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is torsion and uniformly bounded with s [Kr13, Ex. 3.17 a)]. In the converse direction, Krämer proves the following.

Fact 1.1. ([Kr13, Prop. 3.20)], [KW15, Prop. 7.4]) *Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume that for every geometric point $\bar{s}$ over $s \in S$, the restriction $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ with torsion determinant[2] and that the order of $\det(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ is uniformly bounded with $s \in S$. Then $S^{\mathrm{geo}}_{\mathcal{P}}$ is not Zariski-dense in S.*

Our main result is about relaxing the simplicity assumption; it holds for any morphism $f : \mathcal{X} \to S$ separated and of finite type.

Theorem 1.2. *Let S be an integral variety over k with generic point η and let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. For every $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, after possibly replacing S by a non-empty open subscheme (depending on $\mathcal{P}$) the following holds. For every $s \in S$,*

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}).$$

Moreover, if $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$ then $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$.

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[1]Recall that if k is algebraically closed and K/k is an extension of algebraically closed fields then the canonical restriction functor $P(X) \to P(X_K)$, is a fully faithful tensor functor with image stable under subquotients, so that the category $\langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}} \rangle$ and the group $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ do not depend on the choice of the geometric point $\bar{s}$ over $s \in S$ but only on s itself.

[2]As a connected reductive group G over an algebraically closed field Q of characteristic 0 admits an irreducible faithful representation if and only if its center is $\mathbb{G}_{m,Q}$ or finite cyclic, the condition that $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple with torsion determinant imposes that $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^\circ$ is semisimple with finite cyclic center.

Remark 1.3. From [SGA4 1/2, Thm. 2.13, p. 242] and [B24, Lemma 3.10], for every $\mathcal{K} \in D_c^b(\mathcal{X})$ there exists a non-empty open subscheme $U \subset S$ (depending on $\mathcal{K}$) such that $\mathcal{K}|_{\mathcal{X}\times_S U} \in D^{\mathrm{ULA}}(\mathcal{X} \times_S U/U)$ so that Theorem 1.2, and more generally every statement involving finitely many f-ULA (f-perverse) sheaves which holds after possibly replacing S by a non-empty open subscheme, automatically extend to arbitrary (f-perverse) sheaves. For simplicity, we only state the core case of f-ULA (f-perverse) sheaves. Also, for simplicity and because this is the only case we need for our applications, we state our results for varieties S over k but most of them extend to more general base schemes S.

Corollary 1.4. *Assume furthermore $f : \mathcal{X} \to S$ is an abelian scheme. Then for every $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, after possibly replacing S by a non-empty open subscheme (depending on $\mathcal{P}$) the following holds. For every $s \in S$,*

$$\mathrm{length}_{P(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) = \mathrm{length}_{P(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}).$$

Moreover, if $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple (resp. semisimple) in $P(\mathcal{X}_{\bar{\eta}})$ then $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple (resp. semisimple) in $P(\mathcal{X}_{\bar{s}})$.

Theorem 1.2 yields the following generalization of Fact 1.1 to arbitrary semisimple perverse sheaves.

Corollary 1.5. *Let $f : \mathcal{X} \to S$ be an abelian scheme and let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$.*

(1) Assume $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ for every $s \in S$. Then $S_{\mathcal{P}}^{\mathrm{geo}}$ is a countable union of strict, Zariski-closed subvarieties of S.

(2) Assume $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple in $P(\mathcal{X}_{\bar{\eta}})$. Then $S_{\mathcal{P}}^{\mathrm{geo}}$ is contained in a countable union of strict, Zariski-closed subvarieties of S.

If k is countable, we do not know if $S_{\mathcal{P}}^{\mathrm{geo}} \subsetneq S$ in general though we suspect it is true. Still, combined with Fact 1.1 and some Tannaka formalism Theorem 1.2 yields the following. For an algebraic group G over a field Q, let $R(G) \subset G$ denote its solvable radical (*viz* its largest connected normal solvable subgroup) and $G \twoheadrightarrow G^{\mathrm{ss}} := G/R(G)$ its maximal semisimple quotient.

Corollary 1.6. *Let $f : \mathcal{X} \to S$ be an abelian scheme and let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$.*

(1) Assume $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple in $P(\mathcal{X}_{\bar{\eta}})$ with torsion determinant. Then $S_{\mathcal{P}}^{\mathrm{geo}}$ is not Zariski-dense in S.

(2) Up to replacing S by a non-empty open subscheme, one may assume that for all $s \in S$ the canonical morphism induced by cospecialization

$$G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ} \hookrightarrow G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ} \twoheadrightarrow G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{ss}}$$

factors through an isogeny

$$\begin{tikzcd} G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ} \arrow[r, hook] \arrow[d] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ} \arrow[d] \\ G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ,\mathrm{ss}} \arrow[r, dotted] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{ss}}. \end{tikzcd}$$

In particular,

(a) if $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is semisimple (e.g. $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple with torsion determinant in $P(\mathcal{X}_{\bar{\eta}})$) then $S_{\mathcal{P}}^{\mathrm{geo},\circ}$ is not Zariski-dense in S.

(b) if $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is reductive (viz $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple in $P(\mathcal{X}_{\bar{\eta}})$) then $S_{\mathcal{P}}^{\mathrm{geo},\circ,\mathrm{der}}$ is not Zariski-dense in S.

Here is a sample of geometric application of Corollary 1.6.

Corollary 1.7. (Corollary 5.2) *Let $\mathcal{X} \to S$ be an abelian scheme of relative dimension $g \geq 3$ and $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth and geometrically connected over S. Assume $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ has ample normal bundle and trivial stabilizer. Then the following properties are equivalent:*

(i) The set of all $s \in S$ such that $\mathcal{Y}_{\bar{s}} \subset \mathcal{X}_{\bar{s}}$ is a product is Zariski-dense in S;
(ii) After possibly replacing S by a non-empty open subscheme, for every $s \in S$, $\mathcal{Y}_{\bar{s}} \subset \mathcal{X}_{\bar{s}}$ is a product;
(iii) $\mathcal{Y}_{\bar{\eta}} \subset \mathcal{X}_{\bar{\eta}}$ is a product.

Surprisingly, we are not aware of an elementary proof of Corollary 1.7 (see also Remark 5.3). We refer to Section 5 for more details and a variant of Corollary 1.7 for symmetric powers of curves (Corollary 5.4).

As for $S_{\mathcal{P}}$, at least if k is arithmetically rich enough, the non-Zariski density of $S_{\mathcal{P}}^{\mathrm{geo}}$ in S automatically implies that $S_{\mathcal{P}}$ is sparse in the following sense. For an integer $d \geq 1$, write

$$|S|^{\leq d} := \{s \in |S| \mid [k(s) : k] \leq d\}.$$

Proposition 1.8. *Let $f : \mathcal{X} \to S$ be an abelian scheme and let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume $S_{\mathcal{P}}^{\mathrm{geo}}$ is not Zariski-dense in S. Assume furthermore that S has dimension > 0 and that k is Hilbertian (e.g. finitely generated over $\mathbb{Q}$). Then there exists an integer $d \geq 1$ such that $|S|^{\leq d} \setminus (S_{\mathcal{P}} \cap |S|^{\leq d})$ is infinite.*

When S is a curve, k is a number field and $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$ is semisimple, the conclusion of Proposition 1.8 can be strengthened to: for every integer $d \geq 1$ the set $S_{\mathcal{P}}^{\circ} \cap |S|^{\leq d}$ is finite. This applies, for instance, to the intersection complex $\iota_* \overline{\mathbb{Q}}_\ell[d]$ for $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed immersion such that $\mathcal{Y} \to S$ is smooth, geometrically connected of relative dimension d and symmetric in the sense that $[-1]^*\mathcal{Y} = \mathcal{Y}$.

Organization of the paper. In Section 2.1, we briefly review the Tannaka formalism of perverse sheaves on abelian schemes, both in the absolute and relative setting. In Section 3, we elucidate the existence of the specialization diagram (1), giving two constructions. The proofs of Theorem 1.2, its corollaries and Proposition 1.8 are performed in Section 4. The final Section 5 is devoted to a sample of geometric applications of Corollary 1.6. In Appendix A, we gather formal observation about (semi)simplicity in Noetherian and Artinian abelian categories. In Appendix B, we provide an alternative (more elementary) proof of Theorem 1.2, which was communicated to us by Beat Zurbuchen, and which works for base fields k with arbitrary residue characteristic $\neq \ell$. We give an application to hypergeometric sheaves.

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Notation and conventions

For an additive functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ between abelian categories, we write $F : \mathcal{A}_1 \overset{\approx}{\to} \mathcal{A}_2$ if it is fully faithful with image stable under subquotients and $F : \mathcal{A}_1 \overset{\simeq}{\to} \mathcal{A}_2$ if it is an equivalence.

For a rigid symmetric monoidal category $(\mathcal{T}, \otimes)$ with unit $\mathbb{I}$ and an object X in $\mathcal{T}$, let $X^\vee$ denote its dual and, for every integers $m, n \geq 0$, set $T^{m,n}(X) := X^{\otimes m} \otimes X^{\vee \otimes n}$; write $T(X) := \oplus_{m,n \geq 0} T^{m,n}(X)$. If $(\mathcal{T}, \otimes)$ is Tannaka, for every integers $m, n \geq 0$, let also $I^{m,n}(X) \subset T^{m,n}(X)$ denote the sum of all subobjects of $T^{m,n}(X)$ which are isomorphic to $\mathbb{I}$ in $\mathcal{T}$ (so that $\mathrm{Hom}_{\mathcal{T}}(\mathbb{I}, I^{m,n}(X)) \tilde{\to} \mathrm{Hom}_{\mathcal{T}}(\mathbb{I}, T^{m,n}(X)))$. If $\mathcal{T}$ is Tannaka with fiber functor $\omega : \mathcal{T} \to \mathrm{Vect}_Q$, let $G(\mathcal{T}, \omega)$ denote its Tannaka group; recall that $G(\mathcal{T}, \omega)$ may depend on ω but that if Q is algebraically closed then $G(\mathcal{T}, \omega)$ is uniquely determined up to non-canonical isomorphism. For an object X in $\mathcal{T}$ let $\langle X \rangle \subset \mathcal{T}$ denote the smallest Tannaka category containing X and, given a fiber functor $\omega : \langle X \rangle \to \mathrm{Vect}_Q$ set $G(X, \omega) := G(\langle X \rangle, \omega)$.

In the whole paper, we fix a prime ℓ. For a scheme S, let $\mathrm{Loc}(S)$ denote the category of étale $\overline{\mathbb{Q}}_\ell$-local systems on S, $D^b_c(S)$ the triangulated category of étale $\overline{\mathbb{Q}}_\ell$-sheaves with bounded constructible cohomology on S and $D^b_{\mathrm{liss}}(S) \subset D^b_c(S)$ the full subcategory of those objects with cohomology in $\mathrm{Loc}(S)$.

A variety over a field K is a scheme separated and of finite type over K.

When S is a variety, let $\mathrm{Perv}(S) \subset D_c^b(S)$ denote the full subcategory of perverse sheave and, for a morphism $f : X \to S$ of varieties, write

$$D_{X/S}(-) := \mathrm{RHom}(-, Rf^!\overline{\mathbb{Q}}_\ell) : D_c^b(X)^{\mathrm{op}} \to D_c^b(X)$$

for the relative Verdier duality functor. When S is a point, we simply set $D_X(-) := D_{X/S}(-)$.

For morphisms of varieties $S_1 \to S \leftarrow S_2$, one writes

$$\boxtimes_S^L : D_c^b(S_1) \times D_c^b(S_2) \to D_c^b(S_1 \times_S S_2), \ \ (\mathcal{K}_1, \mathcal{K}_2) \mapsto p_1^*\mathcal{K}_1 \otimes^L p_2^*\mathcal{K}_2,$$

for the outer tensor product, where $p_i : S_1 \times_S S_2 \to S_i$ denotes the ith projection, $i = 1, 2$. When $S = \mathrm{spec}(k)$, one simply writes $\boxtimes^L := \boxtimes_S^L$.

2. Preliminaries on (relative) perverse sheaves

2.1. **Tannaka category of (relative) perverse sheaves.**

2.1.1. *Absolute setting.* See [Kr13], [KrW15] for details, and [JKrLM25, §3.1] for a shorter overview. Let K be a field of characteristic 0 and let X be an abelian variety over K with group law $m : X \times_K X \to X$. The convolution product

$$* : D_c^b(X) \times D_c^b(X) \to D_c^b(X), \ \ (\mathcal{K}_1, \mathcal{K}_2) \mapsto \mathcal{K}_1 * \mathcal{K}_2 := Rm_*(\mathcal{K}_1 \boxtimes^L \mathcal{K}_2)$$

endows $D_c^b(X)$ with the structure of a $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category with duality functor

$$(-)^\vee : D_c^b(X) \to D_c^b(X), \ \ \mathcal{K} \mapsto \mathcal{K}^\vee := [-1]^* D_X(\mathcal{K})$$

and with unit the rank one skyscraper sheaf $\delta_0 := \iota_{0*}\overline{\mathbb{Q}}_\ell \in D_c^b(X)$ supported on 0.

The full subcategory $\mathrm{Perv}(X) \subset D_c^b(X)$ is abelian and stable under Verdier duality, but not under convolution. To remedy this, one can mod out by negligible objects. Recall that every $\mathcal{P} \in \mathrm{Perv}(X)$ has non negative Euler-Poincaré characteristic:

$$\chi(X, \mathcal{P}) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\overline{\mathbb{Q}}_\ell}(\mathrm{H}^i(X_{\bar{K}}, \mathcal{P})) \geq 0.$$

Let ${}^p\mathrm{H}^n(-) : D_c^b(X) \to \mathrm{Perv}(X)$, $n \in \mathbb{Z}$ denote the perverse cohomology functors and let $N(X) \subset D_c^b(X)$ denote the full subcategory of all $\mathcal{K} \in D_c^b(X)$ such that $\chi(X, {}^p\mathrm{H}^n(\mathcal{K})) = 0$ for all $n \in \mathbb{Z}$; this is a null system such that the convolution bifunctor $* : D_c^b(X) \times D_c^b(X) \to D_c^b(X)$, the dualization functor $(-)^\vee : D_c^b(X)^{\mathrm{op}} \to D_c^b(X)$ and the perverse cohomology ${}^pH^0(-) : D_c^b(X) \to \mathrm{Perv}(X)$ restrict to

$$* : N(X) \times D_c^b(X) \to N(X), \ \ * : D_c^b(X) \times N(X) \to N(X)$$

$$(-)^\vee : N(X)^{\mathrm{op}} \to N(X)$$

and

$${}^pH^0(-) : N(X) \to N(X) \cap \mathrm{Perv}(X).$$

Consider the quotient functor

$$\mathrm{Perv}(X) \to P(X) := \mathrm{Perv}(X)/(N(X) \cap \mathrm{Perv}(X))$$

so that one gets

$$\begin{tikzcd}
\mathrm{Perv}(X) \times \mathrm{Perv}(X) \arrow[r, "*"] \arrow[d] & D_c^b(X) \arrow[r, "{}^pH^0(-)"] & \mathrm{Perv}(X) \arrow[d] \\
P(X) \times P(X) \arrow[rr, dotted, "*"'] & & P(X).
\end{tikzcd}$$

The abelian category $P(X)$ endowed with

$$* : P(X) \times P(X) \to P(X)$$

is Tannaka with duality functor induced by

$$\begin{tikzcd} \mathrm{Perv}(X)^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d] & \mathrm{Perv}(X) \arrow[d] \\ P(X)^{\mathrm{op}} \arrow[r, dotted, "(-)^\vee"'] & P(X) \end{tikzcd}$$

and unit the image of δ_0 in $P(X)$.

2.1.2. *Relative setting.* See [HS23] for details. Let k be a field of characteristic 0, S a smooth, geometrically connected variety over k with generic point η. Let $f : \mathcal{X} \to S$ be an abelian scheme. Let $D^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^b_c(\mathcal{X})$ denote the full subcategory of f-ULA complexes; this is a triangulated subcategory. As in the absolute setting, the convolution product

$$* : D^{\mathrm{ULA}}(\mathcal{X}/S) \times D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(\mathcal{X}/S), \ \ (\mathcal{K}_1, \mathcal{K}_2) \mapsto \mathcal{K}_1 * \mathcal{K}_2 := Rm_*(\mathcal{K}_1 \boxtimes^L_S \mathcal{K}_2)$$

endows $D^{\mathrm{ULA}}(\mathcal{X}/S)$ with the structure of a $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category with duality functor

$$(-)^\vee : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(\mathcal{X}/S), \ \ \mathcal{K} \mapsto \mathcal{K}^\vee := [-1]^* D_{\mathcal{X}/S}(\mathcal{K})$$

and unit $\delta_{S,0} := 0_*\overline{\mathbb{Q}}_\ell \in D^b_c(\mathcal{X})$, where $0 : S \hookrightarrow \mathcal{X}$ is the 0-section. By construction and proper base change, for every $s \in S$, the pull-back functor $-|_{\mathcal{X}_s} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_s)$ is a tensor functor.

Let $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ denote the full subcategory of f-ULA f-perverse sheaves on $\mathcal{X}$. This is an abelian category, stable under relative Verdier duality $D_{\mathcal{X}/S}(-) : D^b_c(\mathcal{X})^{\mathrm{op}} \to D^b_c(\mathcal{X})$ and such that for every $s \in S$, the pull-back functor $-|_{\mathcal{X}_s} : D^b_c(\mathcal{X}) \to D^b_c(\mathcal{X}_s)$ restricts to an exact functor $-|_{\mathcal{X}_s} : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_s)$ which, when $s = \eta$, is fully faithful with essential image stable under subquotients. Actually, $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ is the heart of a t-structure $D^{\mathrm{ULA},\leq 0}(\mathcal{X}/S), D^{\mathrm{ULA},\geq 0}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ - the relative perverse t-structure with associated truncation functors ${}^{p/S}\tau^{\leq 0} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA},\leq 0}(\mathcal{X}/S)$, ${}^{p/S}\tau^{\geq 0} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA},\geq 0}(\mathcal{X}/S)$ and perverse cohomology functors

$${}^{p/S}\mathrm{H}^n : D^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S), \ \ n \in \mathbb{Z}.$$

As $f : \mathcal{X} \to S$ is proper, $Rf_* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(S)$ factors as $Rf_* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(S/S) = D^b_{\mathrm{liss}}(S)$ [B24, Lem. (ii), p.20, Lem. (i), p.21]. Combined with the fact that, for every $t \in S$, the following diagrams

$$\begin{tikzcd} D^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "-|_{\mathcal{X}_{\bar t}}"] \arrow[d, "{}^{p/S}\mathrm{H}^n(-)"'] & D^b_c(\mathcal{X}_{\bar t}) \arrow[d, "{}^{p}\mathrm{H}^n(-)"] \\ \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "-|_{\mathcal{X}_{\bar t}}"'] & \mathrm{Perv}(\mathcal{X}_{\bar t}) \end{tikzcd} \quad , \ \ n \in \mathbb{Z}$$

commute, one gets that, for $\mathcal{K} \in D^{\mathrm{ULA}}(\mathcal{X}/S)$, the following properties are equivalent:

(i) $\mathcal{K}|_{\mathcal{X}_\eta} \in N(\mathcal{X}_\eta)$;
(ii) For every $s \in S$, $\mathcal{K}|_{\mathcal{X}_s} \in N(\mathcal{X}_s)$;
(iii) There exists $s \in S$ such that $\mathcal{K}|_{\mathcal{X}_s} \in N(\mathcal{X}_s)$.

In other words, for every $s \in S$, one has

$$\begin{aligned} N^{\mathrm{ULA}}(\mathcal{X}/S) \ &:= \ker(D^{\mathrm{ULA}}(\mathcal{X}/S) \stackrel{-|_{\mathcal{X}_\eta}}{\to} D^b_c(\mathcal{X}_\eta) \to D^b_c(\mathcal{X}_\eta)/N(\mathcal{X}_\eta)) \\ &= \ker(D^{\mathrm{ULA}}(\mathcal{X}/S) \stackrel{-|_{\mathcal{X}_s}}{\to} D^b_c(\mathcal{X}_s) \to D^b_c(\mathcal{X}_s)/N(\mathcal{X}_s)). \end{aligned} \tag{2}$$

By construction the functors $* : D^{\mathrm{ULA}}(\mathcal{X}/S) \times D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(\mathcal{X}/S)$, $(-)^\vee : D^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \to D^{\mathrm{ULA}}(\mathcal{X}/S)$, ${}^{p/S}H^0(-) : D^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ and $-|_{\mathcal{X}_s} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_s)$, $s \in S$ restrict to

$$* : N^{\mathrm{ULA}}(\mathcal{X}/S) \times D^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S), \ \ * : D^{\mathrm{ULA}}(\mathcal{X}/S) \times N^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S)$$

$$(-)^\vee : N^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \to N^{\mathrm{ULA}}(\mathcal{X}/S)$$

$${}^{p/S}H^0(-) : N^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S) \cap \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$$

and

$$-|_{\mathcal{X}_s} : N^{\mathrm{ULA}}(\mathcal{X}/S) \to N(\mathcal{X}_s), \quad s \in S.$$

Consider the quotient functor

$$\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to P^{\mathrm{ULA}}(\mathcal{X}/S) := \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)/(\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \cap N^{\mathrm{ULA}}(\mathcal{X}/S))$$

so that one gets

$$\begin{tikzcd} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \times \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "*"] \arrow[d] & D^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "{}^{p/S}H^0(-)"] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[d] \\ P^{\mathrm{ULA}}(\mathcal{X}/S) \times P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[rr, dotted, "*"'] & & P^{\mathrm{ULA}}(\mathcal{X}/S) \end{tikzcd}$$

The abelian category $P^{\mathrm{ULA}}(\mathcal{X}/S)$ endowed with

$$* : P^{\mathrm{ULA}}(\mathcal{X}/S) \times P^{\mathrm{ULA}}(\mathcal{X}/S) \to P^{\mathrm{ULA}}(\mathcal{X}/S)$$

is a $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category with duality functor induced by

$$\begin{tikzcd} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[d] \\ P^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \arrow[r, dotted, "(-)^\vee"'] & P^{\mathrm{ULA}}(\mathcal{X}/S) \end{tikzcd}$$

and unit the image of $\delta_{S,0}$ in $P^{\mathrm{ULA}}(\mathcal{X}/S)$. For every $s \in S$, the exact pull-back functor $-|_{\mathcal{X}_s}$: $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_s)$ induces an exact faithful functor of $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal categories

(3)
$$\begin{tikzcd} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "-|_{\mathcal{X}_s}"] \arrow[d] & \mathrm{Perv}(\mathcal{X}_s) \arrow[d] \\ P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, dotted, "-|_{\mathcal{X}_s}"'] & P(\mathcal{X}_s), \end{tikzcd}$$

which, when $s = \eta$, is fully faithful with essential image stable under subquotients. In particular, $-|_{\mathcal{X}_\eta} : P^{\mathrm{ULA}}(\mathcal{X}/S) \to P(\mathcal{X}_\eta)$ identifies $P^{\mathrm{ULA}}(\mathcal{X}/S)$ with a full Tannaka subcategory of $P(\mathcal{X}_\eta)$ and for every $\mathcal{P} \in P^{\mathrm{ULA}}(\mathcal{X}/S)$, induces an equivalence of Tannaka categories $-|_{\mathcal{X}_\eta} : \langle \mathcal{P} \rangle \xrightarrow{\simeq} \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle$.

2.2. **Extensions of the base field.** Let K/k be a field extension and X a variety over k. In this subsection, we gather remarks about the canonical exact restriction functors $(-)|_{X_K}$: $\mathrm{Perv}(X) \to \mathrm{Perv}(X_K)$.

2.2.1. *At the level of* $\mathrm{Perv}(X)$*.* For later references, in particular in Appendix B, we state the results of this subsection for k of arbitrary characteristic $\neq \ell$.

2.2.1.1. *Normal extensions.* Let K/k be a normal (not necessarily finite) extension - *e.g.* $K = \bar{k}$. The restriction functor $(-)|_{X_K} : \mathrm{Perv}(X) \to \mathrm{Perv}(X_K)$ maps semisimple objects in $\mathrm{Perv}(X)$ to semisimple objects in $\mathrm{Perv}(X_K)$.

Proof. Let K^{perf}/K denote the perfect closures of K, $k^{\mathrm{perf}} \subset K^{\mathrm{perf}}$ the perfect closure of k in K^{perf} and $k^{K\mathrm{perf}} := k^{\mathrm{perf}} \cap K$ the perfect closure of k in K. Consider the canonical Cartesian diagram

$$\begin{tikzcd} & X \times_k K \arrow[dl] \arrow[d] & X \times_k (K \cdot k^{\mathrm{perf}}) \arrow[l, "(*)"'] \arrow[d] \arrow[dl, phantom, "\square"] \\ X & X \times_k k^{K\mathrm{perf}} \arrow[l, "(*)"'] & X \times_k k^{\mathrm{perf}} \arrow[l, "(*)"'] \end{tikzcd}$$

As the arrows $(*)$ are universal homeomorphisms, the corresponding restriction functors

$$\begin{tikzcd} & D^b_c(X\times_k K) \arrow[r, "(*)", "\simeq"'] \arrow[d] \arrow[dr, phantom, "\square"] & D^b_c(X\times_k (K\cdot k^{\mathrm{perf}})) \arrow[d] \\ D^b_c(X) \arrow[ur] \arrow[r, "(*)", "\simeq"'] & D^b_c(X\times_k k^{K\mathrm{perf}}) \arrow[r, "(*)", "\simeq"'] & D^b_c(X\times_k k^{\mathrm{perf}}) \end{tikzcd}$$

are equivalence of categories. So, up to replacing K/k with $K\cdot k^{\mathrm{perf}}/k^{\mathrm{perf}}$, one may assume k is perfect - hence K/k is Galois. It is enough to prove that for a simple object $\mathcal{P}\in \mathrm{Perv}(X)$, $\mathcal{P}|_{X_K}$ is a semisimple object in $\mathrm{Perv}(X_K)$. By [BeBerDG82, Thm. 4.3.1 (ii)], there is an immersion $j: V\hookrightarrow X$ with V irreducible, of dimension d, and smooth over k, and a simple object $\mathcal{F}\in \mathrm{Loc}(V)$ such that $\mathcal{P}=j_{!*}\mathcal{F}[d]$. Let $V_1,\dots,V_n$ the irreducible *viz* connected components of V_K and, for each $i=1,\dots,n$, let $j_i: V_i\hookrightarrow V_K \overset{j_K}{\hookrightarrow} X_K$ denote the canonical immersion. Then $\mathcal{P}|_{X_K}=\oplus_{i=1}^n j_{i!*}(\mathcal{F}|_{V_i}[d])$ so that it is enough to prove that, for each $i=1,\dots,n$, $\mathcal{F}|_{V_i}$ is semisimple in $\mathrm{Loc}(V_i)$. This follows from the fact that the image of $\pi_1(V_i)\to\pi_1(V)$ is a closed *normal* subgroup in an open subgroup of $\pi_1(V)$ as K/k is Galois. More precisely, there is a subextension $k'\subset K$ finite over k, such that the $V_1,\dots,V_n$ are defined over k'. For every $i=1,\dots,n$, the composition $V_{i,k'}\hookrightarrow V_{k'}\to V$ is a finite étale cover of integral schemes, so $\pi_1(V_{i,k'})$ is an open subgroup of $\pi_1(V)$. Since K/k' is Galois, $\pi_1(V_i)\to\pi_1(V_{i,k'})$ is a closed *normal* subgroup. Therefore, $\mathcal{F}|_{V_i}$ is semisimple in $\mathrm{Loc}(V_i)$. □

2.2.1.2. *Extensions of algebraically closed fields.* Let K/k be an extension of algebraically closed fields. Then the canonical restriction functor $(-)|_{X_K}:\mathrm{Perv}(X)\to\mathrm{Perv}(X_K)$ is fully faithful and for every $\mathcal{P}\in\mathrm{Perv}(X)$, $\mathcal{P}$ is simple (resp. semisimple) in $\mathrm{Perv}(X)$ if and only if $\mathcal{P}|_{X_K}$ is simple (resp. semisimple) in $\mathrm{Perv}(X_K)$, and one has

$$\mathrm{length}_{\mathrm{Perv}(X)}(\mathcal{P})=\mathrm{length}_{\mathrm{Perv}(X_K)}(\mathcal{P}|_{X_K}).$$

Proof. To prove that $(-)|_{X_K}:\mathrm{Perv}(X)\to\mathrm{Perv}(X_K)$ is fully faithful, it is enough to prove that $(-)|_{X_K}:D^b_c(X)\to D^b_c(X_K)$ is fully faithful, which readily follows from the functorial identification

$$\mathrm{RHom}(\mathcal{K}_1,\mathcal{K}_2)\tilde{\to}R\Gamma(X,\underline{\mathrm{RHom}}(\mathcal{K}_1,\mathcal{K}_2))$$

and the fact that ℓ-adic cohomology is invariant under extension of algebraically closed fields of characteristic $\neq\ell$ (*e.g.* [St25, Tag 0F0B]).

The observations in Paragraph A.2 (2) then reduce the proof of the second part of the assertion to proving that for every $\mathcal{P}\in\mathrm{Perv}(X)$, $\mathcal{P}$ is simple in $\mathrm{Perv}(X)$ if and only if $\mathcal{P}|_{X_K}$ is simple in $\mathrm{Perv}(X_K)$. The if part follows from the fact that $-|_{X_K}:\mathrm{Perv}(X)\to\mathrm{Perv}(X_K)$ is exact and fully faithful. For the only if part, from [BeBerDG82, Thm. 4.3.1 (ii)] every simple object $\mathcal{S}$ in $\mathrm{Perv}(X)$ is of the form $\mathcal{S}=\iota_*j_{!*}\mathcal{F}[d]$ for some irreducible closed subscheme $\iota: Z\hookrightarrow X$, non-empty open subscheme $j:U\hookrightarrow Z$, smooth over k and pure of dimension d, and simple $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}$ on U. Then

$$\mathcal{S}|_{X_K}=(\iota_*j_{!*}\mathcal{F}[d])|_{X_K}\simeq\iota_{K*}j_{K!*}\mathcal{F}|_{U_K}[d]$$

and the assertion follows from the fact that the restriction functor

$$-|_{U_K}:\mathrm{Loc}(U)\to\mathrm{Loc}(U_K)$$

maps simple objects to simple objects since the canonical morphism $\pi_1(U_K)\to\pi_1(U)$ is surjective (*e.g.* [St25, Tag 0387]). □

2.2.2. *At the level of $P(X)$.* See [JKrLM25, Sec. 4] for details. Assume k has characteristic 0 and X is an abelian variety over k. Let K/k be a field extension and let $k^K\subset K$ denote the algebraic closure of k in K; assume k^K/k is Galois. Let $\mathcal{T}\subset P(X)$ be a full abelian $\otimes$-subcategory and let $\mathcal{T}_K\subset P(X_K)$ denote the full abelian $\otimes$-subcategory generated by the essential image of $\mathcal{T}\hookrightarrow P(X)\overset{-|_{X_K}}{\to}P(X_K)$, namely the full subcategory of all $\mathcal{Q}\in P(X_K)$ such that there exists $\mathcal{P}\in\mathcal{T}$ with $\mathcal{Q}$ a subquotient of $\mathcal{P}|_{X_K}$. For instance, for every $\mathcal{P}\in P(X)$, $\langle\mathcal{P}\rangle_K=\langle\mathcal{P}|_{X_K}\rangle$. The structure of $\mathcal{T}$ is closely related to the structure of $\mathcal{T}_K$ and the structure of the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(k^K/k))$ of finite dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations of the Galois group $\mathrm{Gal}(k^K/k)$ of k^K/k. More precisely,

- The canonical functor
$$-|_{X_K} : \mathrm{Perv}(X) \overset{|_{X_K}}{\to} \mathrm{Perv}(X_K) \to P(X_K)$$
is an exact functor of $\overline{\mathbb{Q}}_\ell$-linear categories which induces a faithful functor of Tannaka categories
$$\begin{tikzcd} \mathrm{Perv}(X) \arrow[r, "|_{X_K}"] \arrow[d] & \mathrm{Perv}(X_K) \arrow[d] \\ P(X) \arrow[r, dotted, "|_{X_K}"] & P(X_K) \end{tikzcd}.$$
- For simplicity, write $\mathrm{Perv}(k) := \mathrm{Perv}(\mathrm{spec}(k))$. The canonical functor
$$0_* : \mathrm{Perv}(k) \overset{0_*}{\to} \mathrm{Perv}(X) \to P(X)$$
is an exact fully faithful functor of Tannaka categories with essential image $P_0(X) \subset P(X)$ stable under subquotients. Precomposing $0_* : \mathrm{Perv}(k) \to P_0(X) \hookrightarrow P(X)$ with the fully faithful exact tensor functor $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(k^K/k)) \hookrightarrow \mathrm{Perv}(k)$, one gets an exact fully faithful functor of Tannaka categories
$$0_*^K : \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(k^K/k)) \to P(X);$$
let $P_0^K(X) \subset P_0(X)$ denote its essential image.

Consider the full Tannaka subcategory $\mathcal{T}_0^K := \mathcal{T} \cap P_0^K(X) \subset \mathcal{T}$. Then, for every fiber functor $\omega : \mathcal{T}_K \to \mathrm{Vect}_{\mathbb{Q}_\ell}$, the sequence of Tannaka categories
$$\mathcal{T}_0^K \to \mathcal{T} \overset{|_{X_K}}{\to} \mathcal{T}_K$$
induces a short exact sequence of proalgebraic groups
$$1 \longrightarrow G(\mathcal{T}_K, \omega) \longrightarrow G(\mathcal{T}, \omega) \longrightarrow G(\mathcal{T}_0^K, \omega) \longrightarrow 1,$$
from which one immediately deduces that

(1) For every $\mathcal{P} \in P(X)$, the sequence of Tannaka categories
$$\langle \mathcal{P} \rangle_0^K \to \langle \mathcal{P} \rangle \overset{|_{X_K}}{\to} \langle \mathcal{P}|_{X_K} \rangle$$
induces a short exact sequence of algebraic groups
$$1 \longrightarrow G(\mathcal{P}|_{X_K}, \omega) \longrightarrow G(\mathcal{P}, \omega) \longrightarrow G(\langle \mathcal{P} \rangle_0^K, \omega) \longrightarrow 1.$$

(2) If k is algebraically closed, the restriction functor $-|_K : \mathcal{T} \overset{\simeq}{\to} \mathcal{T}_K$ is an equivalence of Tannaka categories. In particular, for every $\mathcal{P} \in P(X)$, $G(\mathcal{P}|_{X_K}, \omega) \tilde{\to} G(\mathcal{P}, \omega)$ and one gets the following generalization of Paragraph 2.2.1.2 for $P(X)$, namely the restriction functor $(-)|_{X_K} : P(X) \to P(X_K)$ is fully faithful, and for every $\mathcal{P} \in \mathrm{Perv}(X)$, $\mathcal{P}$ is simple (resp. semisimple) in $P(X)$ if and only if $\mathcal{P}|_{X_K}$ is simple (resp. semisimple) in $P(X_K)$, and one has
$$\mathrm{length}_{P(X)}(\mathcal{P}) = \mathrm{length}_{P(X_K)}(\mathcal{P}|_{X_K}).$$

We drop the superscript $(-)^K$ when $K = \overline{k}$.

3. Specialization

In the following, to simplify notation, given an exact tensor functor of Tannaka categories $F : \mathcal{T}' \to \mathcal{T}$ and a fiber functor ω on $\mathcal{T}$, we will again write $\omega := \omega \circ F$ for the resulting fiber functor on $\mathcal{T}'$; as the functor $F : \mathcal{T}' \to \mathcal{T}$ should always be clear from the context, this should not give rise to confusion.

Let k be a field of characteristic 0, let S a smooth, geometrically connected variety over k with generic point η. Let $f : \mathcal{X} \to S$ be a morphism separated and of finite type and consider the resulting canonical diagram of $\overline{\mathbb{Q}}_\ell$-linear abelian categories.

(4)
$$\begin{tikzcd}
\mathrm{Perv}(\mathcal{X}_s)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{s}}) . \\
& \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & \\
\mathrm{Perv}(\mathcal{X}_\eta)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

If $f : \mathcal{X} \to S$ is an abelian scheme, from (2), the diagram (4) induces canonical diagrams of Tannaka categories (4-1) and for every $\mathcal{P} \in P^{\mathrm{ULA}}(\mathcal{X}/S)$, (4-2)

(4-1)
$$\begin{tikzcd}
P(\mathcal{X}_s)_0 \arrow[r] & P(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & P(\mathcal{X}_{\bar{s}}) , \\
& P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & \\
P(\mathcal{X}_\eta)_0 \arrow[r] & P(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & P(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

(4-2)
$$\begin{tikzcd}
\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_s} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}} \rangle \\
& \langle \mathcal{P} \rangle \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & \\
\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}} \rangle .
\end{tikzcd}$$

For every $t \in S$, fix a fiber functor $\omega_{\bar{t}} : \langle \mathcal{P}|_{\mathcal{X}_{\bar{t}}} \rangle \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$. We claim that for every $s \in S$ and choice of an isomorphism of fiber functors

$$\omega_{\bar{s}} \circ -|_{\mathcal{X}_{\bar{s}}} \tilde{\to} \omega_{\bar{\eta}} \circ -|_{\mathcal{X}_{\bar{\eta}}} : P^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell},$$

diagram (4-2) induces a diagram of algebraic groups with exact lines, which can be completed as indicated by the dotted arrows in diagram (5) below.

(5)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \arrow[r] \arrow[ddd, dotted, "csp_{\bar{\eta},\bar{s}}"] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[ddd, dotted, "(csp_{\bar{\eta},\bar{s}})_0"] & 1 \\
& & G(\mathcal{P}, \omega_{\bar{s}}) \arrow[d, "\simeq"] & & \\
& & G(\mathcal{P}, \omega_{\bar{\eta}}) & & \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{\bar{\eta}}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{\bar{\eta}}) \arrow[r] \arrow[u, "\simeq"] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] & 1
\end{tikzcd}$$

The exactness of the lines is 2.2.2 (1). The fact that $G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \hookrightarrow G(\mathcal{P}, \omega_{\bar{s}})$ is a closed immersion is formal[3].

Note that, in diagram (5), the existence of the dotted arrows is independent of the choice of the isomorphism $\omega_{\bar{s}} \circ -|_{\mathcal{X}_{\bar{s}}} \tilde{\to} \omega_{\bar{\eta}} \circ -|_{\mathcal{X}_{\bar{\eta}}}$ hence of the fiber functors $\omega_{\bar{\eta}}$, $\omega_{\bar{s}}$. So we will be free to choose $\omega_{\bar{\eta}}$, $\omega_{\bar{s}}$. Note also that the existence of the arrow $csp_{\bar{\eta},\bar{s}}$ is equivalent to the one of the arrow $(csp_{\bar{\eta},\bar{s}})_0$. If $f : \mathcal{X} \to S$ admits a section, we provide two constructions of (5), one *via* a construction of $csp_{\bar{\eta},\bar{s}}$ and one *via* a construction of $(csp_{\bar{\eta},\bar{s}})_0$. In both cases, we actually complete diagram (4), and if $f : \mathcal{X} \to S$ is an abelian scheme, diagrams (4-1) and (4-2) by introducing some intermediate categories (to be defined) - $(*)_{\bar{\eta},\bar{s}}$ to construct $csp_{\bar{\eta},\bar{s}}$, and $((*)_{\bar{\eta},\bar{s}})_0$ to construct $(csp_{\bar{\eta},\bar{s}})_0$ as indicated in diagrams (6), (6-1) and (6-2) below.

---

[3]Indeed, by definition of $\langle \mathcal{P}|_{\mathcal{X}_s} \rangle$, every object in $\langle \mathcal{P}|_{\mathcal{X}_s} \rangle$ is a subquotient of some $T^{m,n}(\mathcal{P}|_{\mathcal{X}_s}) \simeq T^{m,n}(\mathcal{P})|_{\mathcal{X}_s}$ for some integers $m, n \geq 0$.

(6)
$$\begin{tikzcd}
\mathrm{Perv}(\mathcal{X}_s)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{s}}) \\
((*)_{\bar{\eta},\bar{s}})_0 \arrow[u, dotted] \arrow[r, dotted] \arrow[d, dotted, "\simeq"] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[r, dotted] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & (*)_{\bar{\eta},\bar{s}} \arrow[u, dotted] \arrow[d, dotted, "\simeq"] \\
\mathrm{Perv}(\mathcal{X}_\eta)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

(6-1)
$$\begin{tikzcd}
P(\mathcal{X}_s)_0 \arrow[r] & P(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & P(\mathcal{X}_{\bar{s}}) \\
((*)_{\bar{\eta},\bar{s}})_0 \arrow[u, dotted] \arrow[r, dotted] \arrow[d, dotted, "\simeq"] & P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[r, dotted] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & (*)_{\bar{\eta},\bar{s}} \arrow[u, dotted] \arrow[d, dotted, "\simeq"] \\
P(\mathcal{X}_\eta)_0 \arrow[r] & P(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & P(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$
,

(6-2)
$$\begin{tikzcd}
\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_s}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}}\rangle \\
((*)_{\bar{\eta},\bar{s}})_0 \arrow[u, dotted] \arrow[r, dotted] \arrow[d, dotted, "\simeq"] & \langle \mathcal{P}\rangle \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[r, dotted] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & (*)_{\bar{\eta},\bar{s}} \arrow[u, dotted] \arrow[d, dotted, "\simeq"] \\
\langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\rangle
\end{tikzcd}$$

3.1. **A construction of** $(csp_{\bar{\eta},\bar{s}})_0$**.** We begin with the following observation.

Lemma 3.1. *Let* $f : \mathcal{X} \to S$ *be a morphism separated, of finite type, with a section* $\iota : S \to \mathcal{X}$*. The following commutative diagram*

$$\begin{tikzcd}
\mathrm{Loc}(S) = \mathrm{Perv}^{\mathrm{ULA}}(S/S) \arrow[r, "\iota_*"] \arrow[d, "\eta^*"'] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[d, "|_{\mathcal{X}_\eta}"] \\
\mathrm{Perv}(\eta) \arrow[r, "\iota_{\eta *}"] & \mathrm{Perv}(\mathcal{X}_\eta)
\end{tikzcd}$$

is cartesian. Namely for every $\mathcal{L}_{[\eta]} \in \mathrm{Perv}(\eta)$*, if there exists* $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ *such that* $\iota_{\eta *}\mathcal{L}_{[\eta]} \simeq \mathcal{P}|_{\mathcal{X}_\eta}$ *then there exists* $\mathcal{L} \in \mathrm{Loc}(S)$ *such that* $\eta^*\mathcal{L} \simeq \mathcal{L}_{[\eta]}$ *and* $\iota_*\mathcal{L} \simeq \mathcal{P}$*.*

Proof. As $f : \mathcal{X} \to S$ is separated, $\iota : S \hookrightarrow \mathcal{X}$ is a closed immersion; let $j : U := \mathcal{X} \setminus \iota(S) \hookrightarrow \mathcal{X}$ denote the complementary open immersion. Then $(j^*\mathcal{P})|_{U_\eta} \simeq j_\eta^*(\mathcal{P}|_{\mathcal{X}_\eta}) \simeq j_\eta^*\iota_{\eta *}\mathcal{L}_{[\eta]} \simeq 0$. As $j^*\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(U/S)$ and $-|_{U_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(U/S) \to \mathrm{Perv}(U_\eta)$ is fully faithful, this forces $j^*\mathcal{P} = 0$. From the distinguished triangle $j_!j^*\mathcal{P} \to \mathcal{P} \to \iota_*\iota^*\mathcal{P} \overset{+1}{\to}$ in $D^b_c(\mathcal{X})$, $\mathcal{P} \tilde{\to} \iota_*\iota^*\mathcal{P}$ hence, by [B24, Lem. 3.6 (iv)], $\iota^*\mathcal{P} \in D^{\mathrm{ULA}}(S/S)$. From [B24, Lem. 3.7 (ii)], $\iota^*\mathcal{P} \in D^b_{\mathrm{liss}}(S)$. But $\eta^*\iota^*\mathcal{P} \simeq \iota_\eta^*(\mathcal{P}|_{\mathcal{X}_\eta}) \simeq \iota_\eta^*\iota_{\eta *}\mathcal{L}_{[\eta]} \simeq \mathcal{L}_{[\eta]}$, so that $\mathcal{L} := \iota^*\mathcal{P}$ lies in $\mathrm{Loc}(S)$ and has the requested property. $\square$

We return to the case where $f : \mathcal{X} \to S$ is a an abelian scheme and $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. From Lemma 3.1, one can complete (4-2) as

(7)
$$\begin{tikzcd}
\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_s}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}}\rangle \\
\langle \mathcal{P}\rangle_0 \arrow[u, "|_{\mathcal{X}_s}"] \arrow[r] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & \langle \mathcal{P}\rangle \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & \\
\langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\rangle,
\end{tikzcd}$$

which, as claimed, formally yields a commutative diagram of algebraic groups:

(8)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \arrow[r] \arrow[ddd, hook, dotted, "csp_{\bar{\eta},\bar{s}}"] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[d] \arrow[ddd, bend left, "(csp_{\bar{\eta},\bar{s}})_0"] & 1 \\
& & G(\mathcal{P}, \omega_{\bar{s}}) \arrow[r] \arrow[d, "\simeq"] & G(\langle \mathcal{P} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[d, "\simeq"] & 1 \\
& & G(\mathcal{P}, \omega_{\bar{\eta}}) \arrow[r] & G(\langle \mathcal{P} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] & 1 \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{\bar{\eta}}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{\bar{\eta}}) \arrow[r] \arrow[u, "\simeq"] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] \arrow[u, "\simeq"] & 1.
\end{tikzcd}$$

3.2. **A construction of** $csp_{\bar{\eta},\bar{s}}$.

3.2.1. *Absolutely integrally closed valuation rings and nearby cycles.* Recall that a valuation ring V is said to be absolutely integrally closed (AIC for short) if it satisfies the following equivalent conditions

(AIC-1) The fraction field of V is algebraically closed;

(AIC-2) Every monic polynomial of degree ≥ 1 in $V[T]$ has a root in V.

In particular, such a valuation ring V is strictly henselian.

Fact 3.2. ([HS23, Thm. 1.7, Thm. 6.1 (ii), Cor. 4.2]) *Let $S = \mathrm{spec}(V)$ be the spectrum of an AIC valuation ring with generic point η and closed point s. Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite presentation and write $\mathcal{X}_\eta \overset{\beta}{\hookrightarrow} \mathcal{X} \overset{\alpha}{\hookleftarrow} \mathcal{X}_s$ for the inclusions of the generic and closed fibers respectively. Then,*

(1) The functor $\beta^* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_\eta)$ is an equivalence of categories with quasi-inverse $R\beta_* : D^b_c(\mathcal{X}_\eta) \to D^{\mathrm{ULA}}(\mathcal{X}/S)$. In particular, $\beta^* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_\eta)$ restricts to an equivalence of categories $\beta^* : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_\eta)$ with quasi-inverse $R\beta_* : \mathrm{Perv}(\mathcal{X}_\eta) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$.

(2) The nearby cycle functor $R\psi_f = \alpha^* R\beta_* : D(\mathcal{X}_\eta) \to D(\mathcal{X}_s)$ restricts to a functor $R\psi_f : D^b_c(\mathcal{X}_\eta) \to D^b_c(\mathcal{X}_s)$ which is t-exact with respect to the perverse t-structures hence induces an exact functor $R\psi_f : \mathrm{Perv}(\mathcal{X}_\eta) \to \mathrm{Perv}(\mathcal{X}_s)$.

(3) Assume furthermore that $f : \mathcal{X} \to S$ is an abelian scheme. Then $R\psi_f : D^b_c(\mathcal{X}_\eta) \to D^b_c(\mathcal{X}_s)$ is a tensor functor and

$$N(\mathcal{X}_\eta) \ = \ker(D^b_c(\mathcal{X}_\eta) \overset{R\psi_f}{\to} D^b_c(\mathcal{X}_s) \to D^b_c(\mathcal{X}_s)/N(\mathcal{X}_s)).$$

In particular, $R\psi_f : \mathrm{Perv}(\mathcal{X}_\eta) \to \mathrm{Perv}(\mathcal{X}_s)$ induces a faithful exact tensor functor $R\psi_f : P(\mathcal{X}_\eta) \to P(\mathcal{X}_s)$.

From [BhM21, Lem. 3.28], for a quasi-compact, quasi-separated scheme T, a specialization $t_1 \rightsquigarrow t$ of points on T, one can always find a morphism $S \to T$ with source the spectrum $S = \mathrm{Spec}(V)$ of an AIC valuation ring V, mapping the generic point η (resp. the closed point s) of S to t_1 (resp. t). We will call such a morphism - usually written as $(S, \eta, s) \to (T, t_1, t)$, a *witness* for $t_1 \rightsquigarrow t$ in T. The proof of [BhM21, Lem. 3.28] shows that, if furthermore one fixes a geometric point $\bar{t}_1$ over t_1, one can choose S in such a way that $\eta \to t_1$ factors as $\eta \to \bar{t}_1 \to t_1$; if we want to specify a geometric point over which $\eta \to t_1$ factors, we will rather write $(S, \eta, s) \to (T, \bar{t}_1, t)$.

3.2.2. We return to the case where S is a smooth, geometrically connected variety over k and $f : \mathcal{X} \to S$ is a morphism separated and of finite type. For every specialization $\eta \rightsquigarrow s$ of points on S, fix a witness $(S', \eta', s') \to (S, \eta, s)$ and geometric points $\eta' \to \bar{\eta} \to \eta$, $s' \to \bar{s} \to s$. Set $f' : \mathcal{X}' := \mathcal{X} \times_S S' \to S'$. From

Fact 3.2, one gets a canonical diagram of $\overline{\mathbb{Q}}_\ell$-linear abelian categories

$$\begin{tikzcd}
 & \mathrm{Perv}(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{s}}) \arrow[r, "|_{\mathcal{X}'_{s'}}"] & \mathrm{Perv}(\mathcal{X}'_{s'}) \\
\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[ur, "|_{\mathcal{X}_s}"] \arrow[r, "|_{\mathcal{X}'}"] \arrow[dr, "\approx", "|_{\mathcal{X}_\eta}"'] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S') \arrow[urr, bend left=10, "|_{\mathcal{X}'_{s'}}"] \arrow[drr, bend right=10, "\simeq", "|_{\mathcal{X}'_{\eta'}}"'] & & \\
 & \mathrm{Perv}(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"'] & \mathrm{Perv}(\mathcal{X}_{\bar{\eta}}) \arrow[r, "|_{\mathcal{X}'_{\eta'}}"'] & \mathrm{Perv}(\mathcal{X}'_{\eta'}) \arrow[uu, dotted, bend right, "R\psi_{f'}"']
\end{tikzcd} \tag{9}$$

and, if $f:\mathcal{X}\to S$ is an abelian scheme, for every $\mathcal{P}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ diagram (9) induces a canonical diagram of Tannaka categories

$$\begin{tikzcd}
 & \langle \mathcal{P}|_{\mathcal{X}_s}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}}\rangle \arrow[r, "|_{\mathcal{X}'_{s'}}", "\simeq"'] & \langle \mathcal{P}|_{\mathcal{X}'_{s'}}\rangle \\
\langle \mathcal{P}\rangle \arrow[ur, "|_{\mathcal{X}_s}"] \arrow[r, "|_{\mathcal{X}'}"] \arrow[dr, "\simeq", "|_{\mathcal{X}_\eta}"'] & \langle \mathcal{P}|_{\mathcal{X}'}\rangle \arrow[urr, bend left=10, "|_{\mathcal{X}'_{s'}}"] \arrow[drr, bend right=10, "\simeq", "|_{\mathcal{X}'_{\eta'}}"'] & & \\
 & \langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"'] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\rangle \arrow[r, "\simeq", "|_{\mathcal{X}'_{\eta'}}"'] & \langle \mathcal{P}|_{\mathcal{X}'_{\eta'}}\rangle \arrow[uu, dotted, bend right, "R\psi_{f'}"']
\end{tikzcd} \quad , \tag{10}$$

which in turn, as claimed, formally yields a commutative diagram of algebraic groups with exact rows

$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}},\omega_{s'}) \arrow[r] \arrow[dddd, dotted, bend right=60, "csp_{\bar{\eta},\bar{s}}"'] & G(\mathcal{P}|_{\mathcal{X}_s},\omega_{s'}) \arrow[r] \arrow[dd, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0,\omega_{s'}) \arrow[r] \arrow[dd] \arrow[dddd, dotted, bend left=60, "(csp_{\bar{\eta},\bar{s}})_0"] & 1 \\
 & G(\mathcal{P}|_{\mathcal{X}'_{s'}},\omega_{\bar{s}}) \arrow[u, "\simeq"'] \arrow[d] \arrow[dd, bend right=60, "R\psi_{f'}"'] & & & \\
 & G(\mathcal{P}|_{\mathcal{X}'},\omega_{s'}) \arrow[r, hook] & G(\mathcal{P},\omega_{s'}) \arrow[r] & G(\langle \mathcal{P}\rangle_0,\omega_{s'}) \arrow[r] & 1 \\
 & G(\mathcal{P}|_{\mathcal{X}'_{\eta'}},\omega_{s'}) \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] & & & \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}},\omega_{s'}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta},\omega_{s'}) \arrow[r] \arrow[uu, "\simeq"'] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle_0,\omega_{s'}) \arrow[r] \arrow[uu, "\simeq"'] & 1
\end{tikzcd} \tag{11}$$

For simplicity, we now omit fiber functors from the notation.

4. Proofs

Unless otherwise stated, in this Section k denotes a field of characteristic 0, S a smooth geometrically connected variety over k with generic point η and $f:\mathcal{X}\to S$ a morphism, separated and of finite type.

4.1. **Proof of Theorem 1.2 and Corollary 1.4.**

4.1.1. *Preliminary reductions.*

4.1.1.1. *Independence of the geometric point.* The following observation will enable us to choose geometric points freely. Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Let $t \in S$ and let $\bar{t}_1, \bar{t}_2$ be two geometric points over t. Then

(1) $\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_1})$ if and only if $\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_2})$ and one has

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{t}_1})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{t}_2})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}).$$

(2) Assume furthermore that $f : \mathcal{X} \to S$ is an abelian scheme. Then $\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}$ is simple (resp. semisimple) in $P(\mathcal{X}_{\bar{t}_1})$ if and only if $\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}$ is simple (resp. semisimple) in $P(\mathcal{X}_{\bar{t}_2})$ and one has

$$\mathrm{length}_{P(\mathcal{X}_{\bar{t}_1})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}) = \mathrm{length}_{P(\mathcal{X}_{\bar{t}_2})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}).$$

Indeed, by considering a geometric point $\bar{t}$ over both $\bar{t}_1$ and $\bar{t}_2$ one immediately reduces to the case where, say, $\bar{t}_2$ is over $\bar{t}_1$. The assertions then follow from Subsections 2.2.1.2 and 2.2.2 (2).

4.1.1.2. Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. Every witness $(S', \eta', s') \to (S, \eta, s)$ induces a canonical exact functor

$$R\psi_{f'} : \mathrm{Perv}(\mathcal{X}_{\eta'}) \xrightarrow{\simeq} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S') \to \mathrm{Perv}(\mathcal{X}_{s'}),$$

where the notation are as follows

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ S' \arrow[r] & S. \end{tikzcd}$$

Assume furthermore $f : \mathcal{X} \to S$ is an abelian scheme. Then

$$N(\mathcal{X}_{\eta'}) \cap \mathrm{Perv}(\mathcal{X}_{\eta'}) = \ker(\mathrm{Perv}(\mathcal{X}_{\eta'}) \xrightarrow{R\psi_{f'}} \mathrm{Perv}(\mathcal{X}_{s'}) \to P(\mathcal{X}_{s'})).$$

So, Paragraph 4.1.1.1 and the observations in Paragraph A.2 (2) (b) applied to

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "p"] \arrow[d, "F"'] & \overline{\mathcal{A}}_1 \arrow[d, "\overline{F}"] \\ \mathcal{A}_2 \arrow[r, "p'"'] & \overline{\mathcal{A}}_2 \end{tikzcd} \quad = \quad \begin{tikzcd} \mathrm{Perv}(\mathcal{X}_{\eta'}) \arrow[r] \arrow[d, "R\psi_{f'}"'] & P(\mathcal{X}_{\eta'}) \arrow[d] \\ \mathrm{Perv}(\mathcal{X}_{s'}) \arrow[r] & P(\mathcal{X}_{s'}). \end{tikzcd}$$

reduce the proof of Theorem 1.2 and Corollary 1.4 to the following statement.

Theorem 4.1. *Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. Let $\mathcal{P}_i \in \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$, $i = 1, \dots, r$ be finitely many simple objects in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$. After possibly replacing S by a non-empty open subscheme S the following holds. For every $s \in S$, there exists a witness $(S', \eta', s') \to (S, \overline{\eta}, s)$ such that $R\psi_{f'}(\mathcal{P}_i|_{\mathcal{X}_{\eta'}})$ is simple in $\mathrm{Perv}(\mathcal{X}_{s'})$, $i = 1, \dots, r$.*

4.1.2. *Proof of Theorem 4.1.*

4.1.2.1. *Intermediate extensions and the ULA property.* Let ℓ be a prime. Let $\mathcal{Z} \to S$ be a separated morphism of finite presentation of quasi-compact quasi-separated schemes over $\mathbb{Z}[1/\ell]$. Assume that S has only finitely many irreducible components, so that by [HS23, Theorem 6.7] the relative perverse t-structure exists on $D^{\mathrm{ULA}}(\mathcal{Z}/S)$. Let $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ be an open immersion of finite presentation. Given $\mathcal{K} \in D^{\mathrm{ULA}}(\mathcal{U}/S)$ with $j_!\mathcal{K}$, $j_!D_{\mathcal{U}/S}\mathcal{K} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$, write $j_{!*/S}\mathcal{K}$ for the image of the natural morphism

$${}^{p/S}H^0(j_!\mathcal{K}) \to {}^{p/S}H^0(D_{\mathcal{Z}/S}j_!D_{\mathcal{U}/S}\mathcal{K})$$

in the abelian category $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{Z}/S)$. Observe the following

(1) When S is the spectrum of a field and $\mathcal{K} \in \mathrm{Perv}(\mathcal{U})$, $j_{!*/S}\mathcal{K}$ is the usual middle extension of $\mathcal{K}$ to $\mathcal{Z}$.

(2) As, for ULA objects, both the formation of relative Verdier duality and $j_!$ commute with base change $S' \to S$ ([HS23, Proposition 3.4 (ii)]), the formation of $j_{!*/S}\mathcal{K}$ also commutes with base-changes $S' \to S$. In particular, for every geometric point $\bar{s}$ on S, if $j_{\bar{s}} : \mathcal{U}_{\bar{s}} \hookrightarrow \mathcal{Z}_{\bar{s}}$ denotes the base change of $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ along $\bar{s} \to S$, one has $(j_{!*/S}\mathcal{K})|_{\mathcal{Z}_{\bar{s}}} = j_{\bar{s},!*}(\mathcal{K}|_{\mathcal{U}_{\bar{s}}})$.

Lemma 4.2. *Consider a diagram*

$$
\begin{tikzcd}
\widetilde{\mathcal{U}} \arrow[r, hook, "\widetilde{j}"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Z}} \arrow[d, "g"] & \\
\mathcal{U} \arrow[r, hook, "j"] & \mathcal{Z} \arrow[r, "f"] & S
\end{tikzcd}
\tag{12}
$$

of quasi-compact quasi-separated schemes. Assume that

- $f : \mathcal{Z} \to S$ *is separated of finite presentation and* $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ *is an open immersion with* $\mathcal{U} \to S$ *smooth;*
- $g : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ *is proper, and* $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ *is finite étale;*
- $\widetilde{\mathcal{Z}} \to S$ *is smooth and* $\widetilde{\mathcal{D}} := \widetilde{\mathcal{Z}} \setminus \widetilde{\mathcal{U}}$ *is a divisor on* $\widetilde{\mathcal{Z}}$ *with strict normal crossings relative to* $\widetilde{\mathcal{Z}} \to S$.

Let ℓ *be a prime invertible on* S. *Let* $\mathcal{F}$ *be a* $\overline{\mathbb{Q}}_\ell$*-local system on* $\mathcal{U}$. *Assume that* $h^*\mathcal{F}$ *is tamely ramified along* $\widetilde{\mathcal{D}}$. *Then* $j_!\mathcal{F}$, $j_!(\mathcal{F}^\vee)$ *are constructible sheaves that are ULA relative to* $f : \mathcal{Z} \to S$. *In particular, if* S *has only finitely many irreducible components and* d *denotes the relative dimension of* $\mathcal{U} \to S$, *then* $j_{!*/S}\mathcal{F}[d]$ *is a well defined object in* $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{Z}/S)$, *whose formation commutes with base-changes.*

Proof. By [St25, Tag 0818], $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ is of finite presentation. From [L81a, Proposition 1.4.4], and as $\mathcal{U} \to S$ is smooth, $\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{U}/S)$. So, since $D_{\mathcal{U}/S}(\mathcal{F}[d]) = \mathcal{F}^\vee[d]$, the second part of Lemma 4.2 follows from the first part.

The sheaves $j_!\mathcal{F}$ and $\widetilde{j}_!h^*\mathcal{F}$ are constructible. From [S17, Lemma 3.14], and as $h^*\mathcal{F}$ is tamely ramified along $\widetilde{\mathcal{D}}$, $\widetilde{j}_!h^*\mathcal{F}$ is ULA relative to $\widetilde{\mathcal{Z}} \to S$. From [HS23, p.643], and as $g : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is proper, $Rg_*\widetilde{j}_!h^*\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$. Hence $j_!h_*h^*\mathcal{F} \simeq Rg_*\widetilde{j}_!h^*\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$.

Since $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is finite étale, the natural morphism $\overline{\mathbb{Q}}_{\ell,\mathcal{U}} \to h_*\overline{\mathbb{Q}}_{\ell,\widetilde{\mathcal{U}}}$ of lisse sheaves is the inclusion of a direct summand. By the projection formula [St25, Tag 0F0G], as $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is proper, one has $h_*(h^*\mathcal{F}) = \mathcal{F} \otimes_{\overline{\mathbb{Q}}_\ell} h_*\overline{\mathbb{Q}}_{\ell,\widetilde{\mathcal{U}}}$. Thus, $\mathcal{F} \hookrightarrow h_*h^*\mathcal{F}$ and hence $j_!\mathcal{F} \hookrightarrow j_!h_*h^*\mathcal{F}$ are direct summands. Since universal local acyclicity is preserved under passing to direct summands, one has $j_!\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$. As $h^*\mathcal{F}^\vee$ is also tamely ramified along $\widetilde{\mathcal{D}}$, one has $j_!\mathcal{F}^\vee \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$. $\square$

4.1.2.2. *Proof of Theorem 4.1.* From [BeBerDG82, Thm. 4.3.1 (ii)], for every $i = 1, \ldots, r$, there exists an integral closed subscheme $\iota_i : Z_i \hookrightarrow \mathcal{X}_{\bar\eta}$ and a non-empty open subscheme $j_i : U_i \hookrightarrow Z_i$, smooth over $k(\bar\eta)$ and pure of dimension d_i, and a simple object $\mathcal{F}_i$ in $\mathrm{Loc}(U_i)$ such that $\mathcal{P}_i = \iota_{i,*}j_{i,!*}\mathcal{F}_i[d_i]$. Fix also a $k(\bar\eta)$-point $u_i \in U_i$ and a smooth normal crossing compactification $U_i \hookrightarrow U_i^{\mathrm{cpt}}$. There exists a finite field extension K_0 of $k(\eta)$ such that, for every $i = 1, \ldots, r$,

$$
\begin{tikzcd}
U_i^{\mathrm{cpt}} & U_i \arrow[l, hook'] \arrow[r, hook, "j_i"] & Z_i \arrow[r, hook, "\iota_i"] & \mathcal{X}_{\bar\eta} \arrow[r] & \mathrm{spec}(k(\bar\eta)) \arrow[llll, bend right, "u_i"', start anchor=north west, end anchor=north east, to path={-- (\tikztotarget)}]
\end{tikzcd}
$$

is defined over K_0 and spread out as

$$
\begin{tikzcd}
U_i^{\mathrm{cpt}} \arrow[d] \arrow[dr, phantom, "\square"] & U_i \arrow[l, hook'] \arrow[r, hook, "j_i"] \arrow[d] \arrow[dr, phantom, "\square"] & Z_i \arrow[r, hook, "\iota_i"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{\eta_0} \arrow[r] \arrow[d] & \mathrm{spec}(K_0) \arrow[d, "\eta_0"] \arrow[lll, bend right, "u_i"'] \\
\mathcal{U}_i^{\mathrm{cpt}} & \mathcal{U}_i \arrow[l, hook'] \arrow[r, hook, "j_i"] & \mathcal{Z}_i \arrow[r, hook, "\iota_i"] & \mathcal{X} \times_S S_0 \arrow[r] & S_0 \arrow[lll, bend left, "u_i"]
\end{tikzcd}
$$

with

- $\iota_i : \mathcal{Z}_i \hookrightarrow \mathcal{X} \times_S S_0$ a closed immersion and $\mathcal{Z}_i \to S_0$ geometrically irreducible;
- $j_i : \mathcal{U}_i \hookrightarrow \mathcal{Z}_i$ an open immersion and $\mathcal{U}_i \to S_0$ is smooth, pure of relative dimension d_i;
- $\mathcal{U}_i \hookrightarrow \mathcal{U}_i^{\mathrm{cpt}}$ is a relative smooth normal crossing compactification over S_0,

where $S_0 \subset \widetilde{S}_0$ is a non-empty open in the normalization $\widetilde{S}_0 \to S$ of S in $\mathrm{spec}(K_0) \to \mathrm{spec}(k(\eta)) \xrightarrow{\eta} S$.

By Lemma 4.3, as $\mathrm{char}(k) = 0$, shrinking S_0 if necessary, one may furthermore assume that there is a diagram

$$\begin{tikzcd} \widetilde{\mathcal{U}}_i \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Z}}_i \arrow[d] & \\ \mathcal{U}_i \arrow[r, hook, "j_i"] & \mathcal{Z}_i \arrow[r] & S_0 \end{tikzcd}$$

satisfying the conditions of Lemma 4.2. Then the diagram base changed along a witness $S' \to S_0$ also satisfies the conditions of Lemma 4.2. From $\mathrm{char}(k) = 0$, the tame ramification condition holds.

As the image of every non-empty open subscheme of S_0 contains a non-empty open subscheme of S and as, for every $s_0 \in S_0$ with image $s \in S$ any witness $(S'_0, \eta'_0, s'_0) \to (S_0, \eta_0, s_0)$ induces a witness $(S'_0, \eta'_0, s'_0) \to (S_0, \eta_0, s_0) \to (S, \eta, s)$, one may freely replace S with S_0 so that we remove the subscripts $(-)_0$ from the notation. Let now $s \in S$ and fix a witness $(S', \eta', s') \to (S, \overline{\eta}, s)$. By invariance of étale fundamental group under extensions of algebraically closed fields in characteristic 0, one has $\pi_1(\mathcal{U}_{i,\eta'}) \tilde{\to} \pi_1(U_i)$ hence $\mathcal{F}_i|_{\mathcal{U}_{i,\eta'}}$ is again irreducible. As S' is strictly henselian, $\pi_1(S') = 1$ and as $\mathcal{U}'_i := \mathcal{U}_i \times_S S' \to S'$ has a section, the canonical morphisms

$$\pi_1(\mathcal{U}'_{i,\eta'}) \tilde{\to} \pi_1(\mathcal{U}'_i) \tilde{\leftarrow} \pi_1(\mathcal{U}'_{i,s'})$$

are both isomorphisms [SGA1, XIII, 4.3, 4.4]. In particular, $\mathcal{F}_i|_{\mathcal{U}_{i,\eta'}}$ extends uniquely to an object $\mathcal{F}'_i$ in $\mathrm{Loc}(\mathcal{U}'_i)$, and $\mathcal{F}'_i|_{\mathcal{U}'_{i,s'}}$ is simple in $\mathrm{Loc}(\mathcal{U}'_{i,s'})$. From [BeBerDG82, Thm. 4.3.1 (ii)], it is thus enough to show that

$$R\psi_{f'}(\mathcal{P}_i|_{\mathcal{X}_{\eta'}}) \simeq \iota_{i,s'*} j_{i,s'!*}(\mathcal{F}'_i|_{\mathcal{U}'_{i,s'}}[d_i]). \tag{13}$$

Consider the commutative diagram

$$\begin{tikzcd} \mathcal{U}'_i \arrow[r, hook, "j'_i"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z}'_i \arrow[r, hook, "\iota'_i"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\ \mathcal{U}_i \arrow[r, hook, "j_i"] & \mathcal{Z}_i \arrow[r, hook, "\iota_i"] & \mathcal{X} \arrow[r, "f"] & S \end{tikzcd}$$

of schemes with cartesian squares. By Lemma 4.2, $j'_{i,!*/S'}\mathcal{F}'_i[d_i] \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{Z}'_i/S')$. Therefore, $\mathcal{K} := \iota'_{i*} j'_{i,!*/S'} \mathcal{F}'_i[d_i]$ is in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S')$. By Observation (2) in Paragraph 4.1.2.1 and the proper base change theorem, as $\iota'_i : \mathcal{Z}'_i \hookrightarrow \mathcal{X}'$ is proper, $\mathcal{K}|_{\mathcal{X}_{\eta'}}$ is the pullback of $\mathcal{P}_i$ along $\mathcal{X}_{\eta'} \to \mathcal{X}_{\overline{\eta}}$, and $\mathcal{K}|_{\mathcal{X}_{s'}}$ is $\iota_{i,s'*} j_{i,s'!*}(\mathcal{F}'|_{\mathcal{U}'_{i,s'}}[d_i])$, which proves (13).

Lemma 4.3. *Let S be an irreducible scheme with generic point η. Assume that $k(\eta)$ is of characteristic 0. Let $f : \mathcal{Z} \to S$ be a morphism separated of finite presentation, with $\mathcal{Z}_\eta$ integral. Let $U \hookrightarrow \mathcal{Z}_\eta$ be an open subset smooth over $k(\eta)$. Then up to shrinking S to an affine open subset, there is a diagram (12) satisfying the conditions of Lemma 4.2, such that $\mathcal{U}_\eta = U$ and $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is an isomorphism.*

Proof. By Hironaka's resolution of singularities (see, e.g., [SGA5, I, 3.1.5 b) a)]), as $k(\eta)$ is of characteristic 0, $\mathcal{Z}_\eta$ is strongly desingularizable. As $\mathcal{Z}_\eta$ is integral, there is a proper morphism $\widetilde{Z} \to \mathcal{Z}_\eta$ with $\widetilde{Z}$ smooth over $k(\eta)$, such that the pullback $\widetilde{U} := U \times_{\mathcal{Z}_\eta} \widetilde{Z} \to U$ is an isomorphism, and that $\widetilde{Z} \setminus \widetilde{U}$ is a strict normal crossing divisor. The result follows by spreading out. $\square$

Remark 4.4. The nearby cycles functor may not preserve simplicity of perverse sheaves nor commute with middle extension. Let S be the spectrum of a strictly Henselian discrete valuation ring with generic point η and closed point s. Let $f : \mathcal{X} \to S$ be a proper semi-stable morphism with geometrically integral fibers of dimension d. Assume that the special fiber $\mathcal{X}_s \hookrightarrow \mathcal{X}$ is a strict normal crossing divisor on $\mathcal{X}$. Then there is an open subset $j : \mathcal{U} \hookrightarrow \mathcal{X}$ smooth over S, such that $\mathcal{U}_\eta = \mathcal{X}_\eta$ and that $\mathcal{U}_s$ is Zariski-dense in $\mathcal{X}_s$. Let $R\psi_f : D^b_c(\mathcal{X}_{\overline{\eta}}) \to D^b_c(\mathcal{X}_s)$ be the nearby cycles functor. By [I94, Thm. 3.2 (c) (i)], $H^0 R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\overline{\eta}}}) \simeq \overline{\mathbb{Q}}_{\ell,\mathcal{X}_s}$. Let $j_s : \mathcal{U}_s \hookrightarrow \mathcal{X}_s$ be the pullback of $j : \mathcal{U} \hookrightarrow \mathcal{X}$ along $s \to S$. Let $\mathrm{IC}_{\mathcal{X}_s} := j_{s,!*}\overline{\mathbb{Q}}_{\ell,\mathcal{U}_s}[d]$ be the intersection cohomology complex on $\mathcal{X}_s$. In general, $H^{-d}\mathrm{IC}_{\mathcal{X}_s}$

is not constant, in which case the perverse sheaf $R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar\eta}}[d])$ is not isomorphic to $\mathrm{IC}_{\mathcal{X}_s}$. Also, from [SGA4-III, XV, Thm 2.1], one has

$$\left(R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar\eta}})\right)|_{\mathcal{U}_s} = R\psi_{f\circ j}(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar\eta}}) = \overline{\mathbb{Q}}_{\ell,\mathcal{U}_s}.$$

Then by [BeBerDG82, Thm. 4.3.1 (ii)], $R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar\eta}}[d])$ is not simple in $\mathrm{Perv}(\mathcal{X}_s)$ (otherwise, it would be isomorphic to the simple object $\mathrm{IC}_{\mathcal{X}_s}$) while $\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar\eta}}[d]$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar\eta})$.

4.2. **Proof of Corollary 1.5.**

4.2.1. Let X is an abelian variety over a field K of characteristic 0. We apply the formalism of Subsection A.3 to the Serre subcategory $N(X)\cap\mathrm{Perv}(X)\hookrightarrow\mathrm{Perv}(X)$; in particular, let $(-)_\neg : \mathrm{Perv}(X)\to N(X)\cap\mathrm{Perv}(X)$ and $(-)^\neg : \mathrm{Perv}(X)\to N(X)\cap\mathrm{Perv}(X)$ denote the "maximal negligible subobject" "maximal negligible quotient object" functors respectively and $(-)^* : \mathrm{Perv}(X)\to\mathrm{Perv}(X)$ the resulting functor $\mathcal{P}\mapsto\ker(\mathcal{P}/\mathcal{P}_\neg\to(\mathcal{P}/\mathcal{P}_\neg)^\neg)$. By Galois descent [Ri14, Lem. A.6], for every $\mathcal{P}\in\mathrm{Perv}(X)$, $(\mathcal{P}_\neg)|_{X_{\bar K}} = (\mathcal{P}|_{X_{\bar K}})_\neg\hookrightarrow\mathcal{P}|_{X_{\bar K}}$ and $\mathcal{P}|_{X_{\bar K}}\twoheadrightarrow(\mathcal{P}^\neg)|_{X_{\bar K}} = (\mathcal{P}|_{X_{\bar K}})^\neg$; in particular

$$(-)^*\circ -|_{X_{\bar K}} \simeq -|_{X_{\bar K}}\circ(-)^* : \mathrm{Perv}(X)\to\mathrm{Perv}(X_{\bar K}). \tag{14}$$

Let $f:\mathcal{X}\to S$ be an abelian scheme, and apply again the formalism of Subsection A.3 to the Serre subcategory $N^{\mathrm{ULA}}(\mathcal{X}/S)\cap\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)\hookrightarrow\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. As for every $s\in S$,

$$N^{\mathrm{ULA}}(\mathcal{X}/S)\cap\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) = \ker(\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)\to P(\mathcal{X}_{\bar s})),$$

and, for $s=\eta$, $-|_{\mathcal{X}_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)\to\mathrm{Perv}(\mathcal{X}_\eta)$ is fully faithful with essential image stable under subquotients, one has

$$(-)^*\circ -|_{\mathcal{X}_\eta}\simeq -|_{\mathcal{X}_\eta}\circ(-)^* : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)\to\mathrm{Perv}(\mathcal{X}_\eta). \tag{15}$$

Combining (14), (15) one gets

$$(-)^*\circ -|_{\mathcal{X}_{\bar\eta}}\simeq -|_{\mathcal{X}_{\bar\eta}}\circ(-)^* : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)\to\mathrm{Perv}(\mathcal{X}_{\bar\eta}).$$

This observation together with Lemma A.1 yields the following result.

Corollary 4.5. *Let $f:\mathcal{X}\to S$ an abelian scheme. Let $\mathcal{P}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume that $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $P(\mathcal{X}_{\bar\eta})$. Then $\mathcal{P}^*|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar\eta})$ with*

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar\eta})}(\mathcal{P}^*|_{\mathcal{X}_{\bar\eta}}) = \mathrm{length}_{P(\mathcal{X}_{\bar\eta})}(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}).$$

Furthermore, for every $s\in S$, $\mathcal{P}^*|_{\mathcal{X}_{\bar s}}\simeq\mathcal{P}|_{\mathcal{X}_{\bar s}}$ in $P(\mathcal{X}_{\bar s})$.

4.2.2. Let $f:\mathcal{X}\to S$ an abelian scheme. Let $\mathcal{P}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ such that $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $P(\mathcal{X}_{\bar\eta})$. From Corollary 4.5, up to replacing $\mathcal{P}$ with $\mathcal{P}^*$, one may assume $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar\eta})$ and, from Theorem 1.2, up to replacing S by a non-empty open subscheme, one may assume that for every $s\in S$, $\mathcal{P}|_{\mathcal{X}_{\bar s}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar s})$. This reduces the proof of Corollary 1.5 (2) to the one of Corollary 1.5 (1). Under the assumptions of Corollary 1.5 (1), $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is a reductive group and, for every $s\in S$, $G(\mathcal{P}|_{\mathcal{X}_{\bar s}})\subset G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is a closed reductive subgroup. Recall that [D82, Prop. 3.1 (c)] for a reductive group G over a field Q of characteristic 0, a finite-dimensional Q-rational faithful representation V of G and a closed reductive subgroup $H\subset G$, one has

$$H = \mathrm{Fix}_G(u_H)\subset G,$$

for some integers $m,n\geq 0$ and $0\neq u_H\in T^{m,n}(V)$. In particular, $H\subsetneq G$ if and only if

$$\dim_Q(I^{m,n}(V)) < \dim_Q(I^{m,n}(V|_H)),$$

for some integers $m,n\geq 0$.

This reduces the proof of Corollary 1.5 (1) to the following.

Corollary 4.6. *Let $f : \mathcal{X} \to S$ be an abelian scheme. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ such that $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ for every $s \in S$. Then for every integers $m, n \geq 0$, there exists a strict closed subscheme $S_{m,n} \hookrightarrow S$ such that for every $s \in S$,*

$$\dim_{\overline{\mathbb{Q}}_\ell}(I^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})) < \dim_{\overline{\mathbb{Q}}_\ell}(I^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}))$$

if and only if $s \in S_{m,n}$.

Proof. Write

$$\mathcal{P}_{m,n} := {}^{p/S}\mathrm{H}^0(T^{m,n}(\mathcal{P})),$$

which is again in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with the properties that, for every $s \in S$,

$${}^{p}\mathrm{H}^0(T^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})) \simeq \mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$$

and, by Lemma 4.7, $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$. For $s \in S$, decompose $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ as

$$\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}} \simeq (\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}})_\neg \oplus \mathcal{S},$$

where $\mathcal{S}$ is the sum of all simple non-negligible subobjects of $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$. As for every $\mathcal{N} \in N(\mathcal{X}_{\bar{s}}) \cap \mathrm{Perv}(\mathcal{X}_{\bar{s}})$ one has

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{N}, \delta_0) = 0,$$

the canonical morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{S}, \delta_0) \to \mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}, \delta_0)$$

is an isomorphism. On the other hand, as for every non-negligible simple objects $\mathcal{S}_1, \mathcal{S}_2$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ the canonical morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_{P(\mathcal{X}_{\bar{s}})}(\mathcal{S}_1, \mathcal{S}_2)$$

is an isomorphism, the canonical morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{S}, \delta_0) \to \mathrm{Hom}_{P(\mathcal{X}_{\bar{s}})}(\mathcal{S}, \delta_0)$$

is also an isomorphism. This proves that

$$\dim_{\overline{\mathbb{Q}}_\ell}(I^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})) = \dim_{\overline{\mathbb{Q}}_\ell}(I^{1,0}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}})) = \dim_{\overline{\mathbb{Q}}_\ell}(\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}, \delta_0)).$$

By Lemma 4.8, there is a quotient $\mathcal{P}_{m,n} \twoheadrightarrow \mathcal{P}_{m,n,\{0\}}$ in $\mathrm{Perv}(\mathcal{X}/S)$ such that for every geometric point $\bar{s}$ on S, $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}} \twoheadrightarrow \mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}$ is the maximal quotient of $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ with support in $\{0\}$. In particular, the canonical injective morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}, \delta_0) \to \mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}, \delta_0)$$

is an isomorphism. But as the full subcategory $\mathrm{Perv}_0(\mathcal{X}_{\bar{s}}) \subset \mathrm{Perv}(\mathcal{X}_{\bar{s}})$ of all objects with support in $\{0\}$ identifies with $\mathrm{Perv}(\bar{0}) \simeq \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ *via*

$$0_* : \mathrm{Perv}(\bar{0}) \tilde{\to} \mathrm{Perv}_0(\mathcal{X}_{\bar{s}}),$$

one has $\mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}} \simeq \delta_0^{\oplus \mu_s}$ with

$$\mu_s := \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}, \delta_0) = \chi(\mathcal{X}_{\bar{s}}, \mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}).$$

This eventually reduces Corollary 4.6 to proving that for every $b \geq 0$ the subset

$$U_{\leq b} := \{s \in S \mid \mu_s \leq b\} \subset S$$

is open. As the map $\mu : S \to \mathbb{Z}_{\geq 0}$ is constructible, it is enough to prove that $U_{\leq b}$ is stable under generization. This essentially follows from the existence of the cospecialization morphism since, for every specialization $t_1 \rightsquigarrow t_0$ of points in S, $csp_{\bar{t}_1, \bar{t}_0}$ identifies $G(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_0}})$ with a subgroup

$$G(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_0}}) \subset G(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_1}}) \subset \mathrm{GL}(\omega_{\bar{t}_1}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_1}})),$$

so that $\mu_{t_0} \geq \mu_{t_1}$. □

Lemma 4.7. *Let k be an algebraically closed field of characteristic 0, let X be an abelian variety over k and let $\mathcal{P} \in \mathrm{Perv}(X)$. Assume $\mathcal{P}$ is semisimple in $\mathrm{Perv}(X)$. Then for every integers $m, n \geq 0$, ${}^{p}\mathrm{H}^0(T^{m,n}(\mathcal{P}))$ is again semisimple in $\mathrm{Perv}(X)$.*

Proof. (Sketch of) This is mentioned as [KrW15, Ex. 5.1]. The fact that $Rm_* : D^b_c(X \times X) \to D^b_c(X)$ preserves direct sums of shifts of simple perverse sheaves follows from Kashiwara's conjecture (Kashiwara's conjecture is reduced to a conjecture of de Jong in [Dr01], and de Jong's conjecture is proved in [BoK06], [Ga07]) while the fact that the exterior tensor product $\mathcal{P}_1 \boxtimes^L \mathcal{P}_2$ of two simple objects $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Perv}(X)$ is a direct sums of shifts of simple perverse sheaves follows from the structure of simple perverse sheaves and the fact that for every immersion $\iota_i : U_i \hookrightarrow X$ and $\mathcal{L}_i \in \mathrm{Loc}(U_i)$, $i = 1, 2$ one has

$$(\iota_1 \times \iota_2)_{!*}(\mathcal{L}_1 \boxtimes^L \mathcal{L}_2) \simeq \iota_{1,!*}(\mathcal{L}_1) \boxtimes^L \iota_{2,!*}(\mathcal{L}_2).$$

See *e.g.* [MS22, Ex. 10.2.31]. $\square$

Lemma 4.8. *Let $f : \mathcal{X} \to S$ be a separated morphism of finite type and let $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed immersion. For every $\mathcal{P} \in \mathrm{Perv}(\mathcal{X}/S)$, there is a quotient $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}$ in $\mathrm{Perv}(\mathcal{X}/S)$ such that for every geometric point $\bar{s}$ on S, $\mathcal{P}|_{\mathcal{X}_{\bar{s}}} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}|_{\mathcal{X}_{\bar{s}}}$ is the maximal quotient of $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ with support in $\mathcal{Z}_{\bar{s}}$.*

Proof. Define $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}$ as the image of the composite

$$\mathcal{P} \xrightarrow{\mathrm{ad}_\iota} \iota_*\iota^*\mathcal{P} \to {}^{p/S}\tau^{\geq 0}(\iota_*\iota^*\mathcal{P})$$

of the adjunction morphism for $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ and the relative perverse truncation with respect to $f : \mathcal{X} \to S$. As $\iota_* : D^b_c(\mathcal{Z}) \to D^b_c(\mathcal{X})$ is t-exact and $\iota^* : D^b_c(\mathcal{X}) \to D^b_c(\mathcal{Z})$ is right t-exact with respect to the relative perverse t-structure on $f : \mathcal{X} \to S$, one has

$${}^{p/S}\tau^{\geq 0}(\iota_*\iota^*-) \simeq \iota_*{}^{p/S}\tau^{\geq 0}(\iota^*-) \simeq \iota_*{}^{p/S}\mathrm{H}^0(\iota^*-) : \mathrm{Perv}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}/S).$$

Furthermore, for every geometric point $\bar{s}$ on S, by proper base-change,

$$-|_{\mathcal{X}_{\bar{s}}} \circ \iota_* \simeq \iota_{\bar{s}*} \circ -|_{\mathcal{Z}_{\bar{s}}} : \mathrm{Perv}(\mathcal{Z}/S) \to \mathrm{Perv}(\mathcal{X}_{\bar{s}}),$$

while, by definition of the relative perverse t-structure,

$$-|_{\mathcal{Z}_{\bar{s}}} \circ {}^{p/S}\mathrm{H}^0(-) \simeq {}^{p}\mathrm{H}^0(-|_{\mathcal{Z}_{\bar{s}}}) : \mathrm{Perv}(\mathcal{Z}/S) \to \mathrm{Perv}(\mathcal{Z}_{\bar{s}}).$$

This proves that for every geometric point $\bar{s}$, the formation of $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}$ commutes with $-|_{\mathcal{X}_{\bar{s}}} : \mathrm{Perv}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_{\bar{s}})$ and reduces the proof of Lemma 4.8 to the case where $S = \mathrm{spec}(\bar{k})$ is the spectrum of an algebraically closed field. By construction $\mathcal{P}_{\mathcal{Z}}$ has support in $\mathcal{Z}$. Let $j : \mathcal{X} \setminus \mathcal{Z} \hookrightarrow \mathcal{X}$ denote the complementary open immersion. Conversely, for every quotient $\mathcal{P} \twoheadrightarrow \mathcal{Q}$ in $\mathrm{Perv}(\mathcal{X})$ with support in $\mathcal{Z}$, the distinguished triangle

$$j_!j^*\mathcal{Q} \to \mathcal{Q} \to \iota_*\iota^*\mathcal{Q} \xrightarrow{+1}$$

ensures that $\mathcal{Q} \tilde{\to} \iota_*\iota^*\mathcal{Q}$ so that, by adjunction, $\mathcal{P} \twoheadrightarrow \mathcal{Q} \tilde{\to} \iota_*\iota^*\mathcal{Q}$ factors as

$$\begin{tikzcd} \mathcal{P} \arrow[rr, two heads] \arrow[dr, "\mathrm{ad}_\iota"'] & & \iota_*\iota^*\mathcal{Q}, \\ & \iota_*\iota^*\mathcal{P} \arrow[ur, dotted] & \end{tikzcd} \tag{16}$$

and, as $\mathcal{Q} \in \mathrm{Perv}(\mathcal{X})$, (16) factors further as

$$\begin{tikzcd} \mathcal{P} \arrow[rr, two heads] \arrow[dr, "\mathrm{ad}_\iota"'] & & \iota_*\iota^*\mathcal{Q} \\ & \iota_*\iota^*\mathcal{P} \arrow[ur] \arrow[r, two heads] & {}^{p/S}\tau^{\geq 0}(\iota_*\iota^*\mathcal{P}), \arrow[u, dotted] \end{tikzcd}$$

which concludes the proof of Lemma 4.8. $\square$

4.3. **Proof of Corollary 1.6.** Let $f : \mathcal{X} \to S$ be an abelian scheme. We begin with the following observation.

Lemma 4.9. *Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ has torsion determinant of order N. Then for every $s \in S$, $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ also has torsion determinant of order dividing N.*

Proof. Let n denote the dimension of $\mathcal{P}$ in $P^{\mathrm{ULA}}(\mathcal{X}/S)$ and $\det(\mathcal{P}) := \wedge^n\mathcal{P}$ its determinant. It follows from the general formalism of Tannaka categories that for every $t \in S$, $\mathcal{P}|_{\mathcal{X}_t}$, $\mathcal{P}|_{\mathcal{X}_{\bar{t}}}$ again have dimension n and that $(\wedge^n\mathcal{P})|_{\mathcal{X}_t} \simeq \wedge^n(\mathcal{P}|_{\mathcal{X}_t})$, $(\wedge^n\mathcal{P})|_{\mathcal{X}_{\bar{t}}} \simeq \wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar{t}}})$. In particular, as for every $s \in S$, one has

$$G(\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})) \subset G(\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})),$$

hence $\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ is also torsion, with order dividing N. □

Note that if $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is semisimple then any object in $\langle\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\rangle$ has torsion determinant of order dividing $|\pi_0(G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}))|$.

We now turn to the proof of Corollary 1.6 itself. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. After possibly replacing $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $\mathcal{P}|_{\mathcal{X}_{\bar{k}}} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}_{\bar{k}}/S_{\bar{k}})$, one may assume $k = \bar{k}$ is algebraically closed.

- Proof of Corollary 1.6 (1). Up to replacing $\mathcal{P}$ with $\mathcal{P}^*$ - see Corollary 4.5, one may assume $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$, and not only in $P(\mathcal{X}_{\bar{\eta}})$. Then Corollary 1.6 (1) immediately follows from Fact 1.1, Theorem 1.2 and Lemma 4.9.
- Proof of Corollary 1.6 (2). Let us first observe that for a connected reductive group G and a closed subgroup $H \subset G$, the following are equivalent
  (1) $H \subset G \twoheadrightarrow G^{\mathrm{ss}}$ factors through an isogeny $H^{\mathrm{ss}} \twoheadrightarrow G^{\mathrm{ss}}$;
  (2) $\dim(R(G) \cap H) = \dim(R(H))$ and $\dim(H) - \dim(R(H)) = \dim(G) - \dim(R(G))$;
  (3) $\dim(R(G) \cap H^\circ) = \dim(R(H^\circ))$ and $\dim(H^\circ) - \dim(R(H^\circ)) = \dim(G) - \dim(R(G))$;
  (4) $H^\circ \subset G \twoheadrightarrow G^{\mathrm{ss}}$ factors through an isogeny $H^{\circ,\mathrm{ss}} \twoheadrightarrow G^{\mathrm{ss}}$.

  In particular, to prove the first part of Corollary 1.6 (2), one can replace $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^\circ$ with $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^\circ$ (which will simplify a bit the notation).

  Replacing $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $[N]_*\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ for some integer $N \geq 1$, one may assume $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is connected (see [W15]). From the short exact sequence

$$1 \to G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) \to G(\mathcal{P}) \to G(\langle\mathcal{P}\rangle_0) \to 1$$

  and the description of $G(\langle\mathcal{P}\rangle_0)$ in terms of representation of $\pi_1(S)$ (see Subsection 4.4 below), replacing $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $\mathcal{P}|_{\mathcal{X}_{S'}} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}_{S'}/S')$ for some connected étale cover $S' \to S$, one may also assume $G(\mathcal{P})$ is connected.

  Let $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}}$ denote the maximal *adjoint* quotient[4] of $G(\mathcal{P})$. The morphism $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}}$ factors as $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ss}} \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}}$. On the other hand, as $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is normal in $G(\mathcal{P})$, $R(G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})) = (R(G(\mathcal{P})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}))^\circ$ so that the morphism $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) \hookrightarrow G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ss}}$ factors through a morphism $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\mathrm{ss}} \to G(\mathcal{P})^{\mathrm{ss}}$ inducing an isogeny onto its image, which is a closed normal subgroup of $G(\mathcal{P})^{\mathrm{ss}}$. By the structure theory of connected semisimple groups, there is a (unique) connected (automatically adjoint) quotient $G(\mathcal{P})^{\mathrm{ad}} \twoheadrightarrow \widetilde{G}$ such that the resulting canonical morphism

$$G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\mathrm{ss}} \to G(\mathcal{P})^{\mathrm{ss}} \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}} \twoheadrightarrow \widetilde{G}$$

  is an isogeny. As $\widetilde{G}$ is adjoint, it admits an irreducible faithful representation corresponding to a simple object $\mathcal{Q} \in \langle\mathcal{P}\rangle$; in particular, $\widetilde{G} = G(\mathcal{Q})$. The commutative diagram of exact tensor functors

(17)
$$\begin{tikzcd}
\langle\mathcal{Q}|_{\mathcal{X}_{\bar{\eta}}}\rangle \arrow[d, hook] \arrow[rrr, bend left, "csp_{\bar{\eta},\bar{s}}"] & \langle\mathcal{Q}\rangle \arrow[l, "|_{\mathcal{X}_{\bar{\eta}}}"'] \arrow[r, "|_{\mathcal{X}_s}"] \arrow[d, hook] & \langle\mathcal{Q}|_{\mathcal{X}_s}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] \arrow[d, hook] & \langle\mathcal{Q}|_{\mathcal{X}_{\bar{s}}}\rangle \arrow[d, hook] \\
\langle\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\rangle \arrow[rrr, bend right, "csp_{\bar{\eta},\bar{s}}"'] & \langle\mathcal{P}\rangle \arrow[l, "|_{\mathcal{X}_{\bar{\eta}}}"'] \arrow[r, "|_{\mathcal{X}_s}"] & \langle\mathcal{P}|_{\mathcal{X}_s}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle\mathcal{P}|_{\mathcal{X}_{\bar{s}}}\rangle
\end{tikzcd}$$

[4] Namely, $G(\mathcal{P})^{\mathrm{ad}} = G(\mathcal{P})^{\mathrm{red}}/Z(G(\mathcal{P})^{\mathrm{red}})$, where $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{red}} := G(\mathcal{P})/R_u(G(\mathcal{P}))$ is the maximal reductive quotient of $G(\mathcal{P})$.

induces a commutative diagram of algebraic groups

$$
\begin{tikzcd}
G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}) \arrow[r, hook] & G(\mathcal{Q}) & G(\mathcal{Q}|_{\mathcal{X}_s}) \arrow[l, hook'] & G(\mathcal{Q}|_{\mathcal{X}_{\bar s}}) \arrow[l, hook'] \arrow[lll, bend right, "csp_{\bar\eta,\bar s}"'] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}} \arrow[r] \arrow[u, two heads] & G(\mathcal{P})^{\mathrm{ss}} \arrow[u, two heads] & & \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}) \arrow[r, hook] \arrow[u, two heads] & G(\mathcal{P}) \arrow[u, two heads] & G(\mathcal{P}|_{\mathcal{X}_s}) \arrow[l, hook'] \arrow[uu, two heads] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}}) \arrow[l, hook'] \arrow[uu, two heads] \arrow[lll, bend left, "csp_{\bar\eta,\bar s}"]
\end{tikzcd}
\tag{18}
$$

As $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}} \twoheadrightarrow G(\mathcal{Q})$ is an isogeny, $G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}) \to G(\mathcal{Q})$ is an isomorphism. In particular, $\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}$ is a simple object in $P(\mathcal{X}_{\bar\eta})$ and every object in $\langle \mathcal{Q}|_{\mathcal{X}_{\bar\eta}} \rangle$ has trivial determinant. By Corollary 1.6 (1) applied to $\mathcal{Q}$ up to replacing S by a non-empty open subscheme, one may assume that for every $s \in S$ the cospecialization morphism $csp_{\bar\eta,\bar s} : G(\mathcal{Q}|_{\mathcal{X}_{\bar s}}) \hookrightarrow G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}})$ is an isomorphism so that one has a canonical commutative diagram

$$
\begin{tikzcd}
G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}) & & G(\mathcal{Q}|_{\mathcal{X}_{\bar s}}) \arrow[ll, "csp_{\bar\eta,\bar s}"', "\simeq"] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}} \arrow[u, two heads] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}})) \arrow[l, hook'] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}} \arrow[u, two heads] \\
 & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))^{\circ} \arrow[u, two heads] \arrow[ur, two heads] & \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}) \arrow[uu, two heads] & & G(\mathcal{P}|_{\mathcal{X}_{\bar s}}). \arrow[ll, hook', "csp_{\bar\eta,\bar s}"] \arrow[uu, two heads] \arrow[ul, two heads]
\end{tikzcd}
\tag{19}
$$

In particular,

$$
\begin{aligned}
\dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}}) \geq \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}}) \quad & \geq \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))) \\
& = \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))^{\circ}) \\
& \geq \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}})
\end{aligned}
$$

which, as $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ - hence $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}}$ - are connected, imposes that the morphisms

$$G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}})) \xrightarrow{\simeq} G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}}$$

and

$$G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))^{\circ} \xrightarrow{\simeq} G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}},$$

are isomorphisms. This concludes the proof of the first part of Corollary 1.6 (2). The second part when $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is semisimple tautologically follows from the first part as, then, $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ} = G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ,\mathrm{ss}}$ while the second part when $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is reductive follows from the first part and the fact that, for every $s \in S$ such that $G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\circ} \subset G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ}$ factors through an isogeny $G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\circ,\mathrm{ss}} \twoheadrightarrow G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ,\mathrm{ss}}$, the arrows $(*\text{-}\bar s)$, $(*\text{-}\bar\eta)$ and the right vertical arrow in the canonical commutative diagram

$$
\begin{tikzcd}
G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\circ,\mathrm{der}} \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "(*\text{-}\bar s)"] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\circ} \arrow[r, two heads] \arrow[d, hook] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\circ,\mathrm{ss}} \arrow[d] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ,\mathrm{der}} \arrow[r, hook] \arrow[rr, bend right, "(*\text{-}\bar\eta)"'] & G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ} \arrow[r, two heads] & G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ,\mathrm{ss}}.
\end{tikzcd}
$$

are isogenies.

4.4. **Proof of Proposition 1.8.** Let k be a field of characteristic 0, S a smooth geometrically connected variety over k with generic point η and $f : \mathcal{X} \to S$ an abelian scheme. Let $P^{\mathrm{ULA}}(\mathcal{X}/S)_0 \subset P^{\mathrm{ULA}}(\mathcal{X}/S)$ denote the essential image of

$$0_* : \mathrm{Loc}(S) = \mathrm{Perv}^{\mathrm{ULA}}(S/S) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to P^{\mathrm{ULA}}(\mathcal{X}/S)$$

and for every $\mathcal{P} \in P^{\mathrm{ULA}}(\mathcal{X}/S)$, consider the full subcategory $\langle \mathcal{P} \rangle_0 := \langle \mathcal{P} \rangle \cap P^{\mathrm{ULA}}(\mathcal{X}/S)_0 \subset \langle \mathcal{P} \rangle$. From Lemma 3.1, the equivalence of Tannaka categories $-|_{\mathcal{X}_\eta} : \langle \mathcal{P} \rangle \tilde{\to} \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle$ restricts to an equivalence

$$-|_{\mathcal{X}_\eta} : \langle \mathcal{P} \rangle_0 \tilde{\to} \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0.$$

This yields an explicit categorical description of the morphism $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0) \to G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0)$ as the composite $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0) \to G(\langle \mathcal{P} \rangle_0) \tilde{\leftarrow} G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0)$ arising from the diagram of Tannaka categories

$$\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0 \overset{|_{\mathcal{X}_s}}{\leftarrow} \langle \mathcal{P} \rangle_0 \overset{|_{\mathcal{X}_\eta}}{\to} \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0.$$

Assume furthermore $S_{\mathcal{P}}^{\mathrm{geo}} = \emptyset$ that is, for every $s \in S$, the cospecialization morphism $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) \to G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is an isomorphism so that the morphism $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0) \hookrightarrow G(\langle \mathcal{P} \rangle_0)$ is a closed immersion. Then every $\otimes$-generator $0_*\mathcal{L}$ of $\langle \mathcal{P} \rangle_0$ yields a $\otimes$-generator $(0_*\mathcal{L})|_{\mathcal{X}_s} \simeq 0_{s*}s^*\mathcal{L}$ of $\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0$ and from the canonical diagram of Tannaka categories

$$\begin{tikzcd}
\langle \mathcal{L} \rangle \arrow[r, hook] \arrow[d, "\simeq"', "0_*"] \arrow[rrr, bend left] & \mathrm{Perv}^{\mathrm{ULA}}(S/S) \arrow[r, "s^*"] \arrow[d, "\simeq"', "0_*"] & \mathrm{Perv}(s) \arrow[d, "\simeq"', "0_{s*}"] & \langle s^*\mathcal{L} \rangle \arrow[l, hook'] \arrow[d, "\simeq"', "0_{s*}"] \\
\langle 0_*\mathcal{L} \rangle \arrow[r, hook] \arrow[rrr, bend right] & P^{\mathrm{ULA}}(\mathcal{X}/S)_0 \arrow[r, "|_{\mathcal{X}_s}"] & P(\mathcal{X}_s)_0 & \langle 0_{s*}s^*\mathcal{L} \rangle \simeq \langle (0_*\mathcal{L})|_{\mathcal{X}_s} \rangle \arrow[l, hook']
\end{tikzcd}$$

the morphism $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0) \hookrightarrow G(\langle \mathcal{P} \rangle_0)$ also describes the functor of Tannaka categories $s^* : \langle \mathcal{L} \rangle \to \langle s^*\mathcal{L} \rangle$ hence corresponds to the embedding

$$G(\mathcal{L})_s \hookrightarrow G(\mathcal{L}) \hookrightarrow \mathrm{GL}(\mathcal{L}_{\bar{s}})$$

of the Zariski-closures of the images

$$\Pi(\mathcal{L})_s \subset \Pi(\mathcal{L}) \subset \mathrm{GL}(\mathcal{L}_{\bar{s}})$$

of $\pi_1(s, \bar{s}) \to \pi_1(S, \bar{s})$ acting on $\mathcal{L}_{\bar{s}}$ respectively.

This observation yields the following.

Lemma 4.10. *Assume S has dimension > 0. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $S_{\mathcal{P}}^{\mathrm{geo}} = \emptyset$. Then,*

(1) if k is Hilbertian, there exists an integer $d \geq 1$ such that $|S|^{\leq d} \setminus S_{\mathcal{P}} \cap |S|^{\leq d}$ is infinite.

(2) if S is a curve, k is finitely generated over $\mathbb{Q}$ and $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$ is semisimple, for every integer $d \geq 1$, $S_{\mathcal{P}}^{\circ} \cap |S|^{\leq d}$ is finite.

Proof. From the exact specialization diagram (5) and the fact that, by our assumptions, for every $s \in |S|$ the morphisms $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) \to G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is an isomorphism and the morphism $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0) \to G(\langle \mathcal{P} \rangle_0)$ is a closed immersion, it is enough to prove that, under the assumptions

- in (1): there exists an integer $d \geq 1$ such that for infinitely many $s \in |S|^{\leq d}$ the closed immersion $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0) \hookrightarrow G(\langle \mathcal{P} \rangle_0)$ is an isomorphism.

  This follows from the defining property of Hilbertian fields and a Frattini argument [Se89, §10.6], which ensures that there exists an integer $d \geq 1$ such that for infinitely many $s \in |S|^{\leq d}$ one has $\Pi(\mathcal{L})_s = \Pi(\mathcal{L})$.

- in (2): for every integer $d \geq 1$ and for all but finitely many $s \in |S|^{\leq d}$ the closed immersion $G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0)^\circ \hookrightarrow G(\langle \mathcal{P} \rangle_0)^\circ$ is an isomorphism.

  This follows from [CT13, Thm. 1], which asserts that if $\rho : \pi_1(S) \to \mathrm{GL}_N(\mathbb{Z}_\ell)$ is a continuous GLP representation then, for every integer $d \geq 1$ and all but finitely many $s \in |S|^{\leq d}$, $\rho(\pi_1(s)) \subset \rho(\pi_1(S))$ is open. The GLP condition means that every open subgroup of $\Pi := \rho(\pi_1(S_{\bar{k}}))$ has finite abelianization or, equivalently, that the Lie algebra $\mathrm{Lie}(\Pi)$ of Π (as an ℓ-adic Lie group) is perfect. This is for instance the case if (*) one can realize Π as a closed subgroup $\Pi \subset H_0(\mathbb{Q}_\ell)$ of the group of $\mathbb{Q}_\ell$-points of an algebraic group H_0 over $\mathbb{Q}_\ell$ such that the Zariski-closure of Π in H_0 is semisimple. The assumption that $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$ is semisimple ensures that one can reduce to this situation. Indeed, as $G(\mathcal{L}|_{S_{\bar{k}}})$ is a quotient of $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$, $G(\mathcal{L}|_{S_{\bar{k}}})$ is semisimple as well and, as there exists a finite Galois extension Q_ℓ of $\mathbb{Q}_\ell$ such that $\mathcal{L}$ arises from a Q_ℓ-local system on S, one may assume $\Pi(\mathcal{L}) \subset \mathrm{GL}(\mathcal{L}_{\bar{s}}) \simeq \mathrm{GL}_r(Q_\ell)$. But as $\mathrm{GL}_r(Q_\ell)$ has a natural structure of Lie group over $\mathbb{Q}_\ell$, so has $\Pi(\mathcal{L})$ [Se65, L.G., Chap. V, §9] hence, as $\Pi(\mathcal{L})$ is also compact being the continuous image of a profinite group, it admits a faithful embedding into $\mathrm{GL}_N(\mathbb{Z}_\ell)$ for some $N \geq 1$ [Lu88, Prop. 4]. To apply [CT13, Thm. 1] to the resulting ℓ-adic representation $\pi_1(S) \twoheadrightarrow \Pi(\mathcal{L}) \subset \mathrm{GL}_N(\mathbb{Z}_\ell)$, it is thus enough to show that $\Pi := \Pi(\mathcal{L}|_{S_{\bar{k}}})$ satisfies the criterion (*). This follows from the claim below, applied with $K/k = Q_\ell/\mathbb{Q}_\ell$, $\Pi := \Pi(\mathcal{L}|_{S_{\bar{k}}})$ and $G_0 := \mathrm{GL}_{r,\mathbb{Q}_\ell}$.

Claim. *Let K/k be a finite Galois extension and write $R := \mathrm{Res}_{K|k} : Sch/_K \to Sch/_k$ for the Weil restriction functor. Let G_0 be an algebraic group over k and set $G := G_{0,K}$. Let $\Pi \subset G(K) = (RG)(k)$ be a subgroup. Let $\iota : H \hookrightarrow G$ denote the Zariski closure of Π in G and $\iota_0 : H_0 \hookrightarrow RG$ the Zariski-closure of Π in RG. Write $ad : G_0 \hookrightarrow RG$ for the adjunction morphism. Then the morphism $c : H \overset{\iota}{\hookrightarrow} G \overset{ad_K}{\hookrightarrow} (RG)_K$ factors through an isomorphism*

$$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[drr, dotted, "c"', "\simeq"] & G \arrow[r, hook, "ad_K"] & (RG)_K \\ & & H_{0,K}. \arrow[u, hook, "\iota_{0,K}"'] \end{tikzcd}$$

Proof of the claim. At the level of K-points, the diagram

$$\begin{tikzcd} H \arrow[r, hook, "\iota"] & G \arrow[r, hook, "ad_K"] & (RG)_K & H_{0,K} \arrow[l, hook', "\iota_{0,K}"'] \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} H(K) \arrow[r, hook] & G(K) \arrow[r, hook] & (RG)_K(K) & H_{0,K}(K) \arrow[l, hook'] \\ & \Pi \arrow[ul, hook] \arrow[rr, bend right, hook] & RG(k) \arrow[ul, equal] \arrow[u, hook] & H_0(k). \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

As Π is Zariski-dense in H, this already shows the existence of the factorization $c : H \hookrightarrow H_{0,K}$. On the other hand, at the level of k-points the diagram

$$\begin{tikzcd} RH \arrow[r, hook, "R\iota"] & RG & H_0 \arrow[l, hook', "\iota_0"'] \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} RH(k) = H(K) \arrow[r, hook] & (RG)(k) = G(K) \\ \Pi \arrow[u, hook] \arrow[r, hook] & H_0(k) \arrow[u, hook] \end{tikzcd}$$

As Π is Zariski-dense in H_0, this shows that $H_0 \overset{\iota_0}{\hookrightarrow} RG$ factors as $\iota_0 : H_0 \overset{d_0}{\hookrightarrow} RH \overset{R\iota}{\hookrightarrow} RG$. One thus gets

$$\begin{tikzcd} RH \arrow[r, hook, "R\iota"] \arrow[drr, bend right, "Rc"'] & RG \arrow[r, hook, "R(ad_K)"] & R((RG)_K) \\ & H_0 \arrow[ul, hook, "d_0"] \arrow[u, hook, "\iota_0"] \arrow[r, hook, "ad"] & R(H_{0,K}) \arrow[u, hook, "R(\iota_{0,K})"'] \end{tikzcd}$$

Let $d : H_{0,K} \to H$ denote the morphism corresponding by functoriality, to $d_0 : H_0 \hookrightarrow RH$. Then, by construction, $c \circ d = Id : H_{0,K} \tilde{\to} H_{0,K}$ and $d \circ c = Id : H \tilde{\to} H$. □

Proposition 1.8 (and its strengthening when S is a curve and k is finitely generated over $\mathbb{Q}$) follows from Lemma 4.10 applied to the restriction of $\mathcal{P}$ to $\mathcal{X} \times_S U$, where $U \subset S$ denotes the complement of the Zariski-closure of $S^{\mathrm{geo}}_{\mathcal{P}}$ in S.

5. Geometric applications

Let $\mathcal{X} \to S$ be an abelian scheme and let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed subscheme, smooth and geometrically connected over S.

5.1. **Preliminaries.** As S is smooth, $\mathcal{X} \to S$ is projective [R70, Thm. XI.1.4] hence $\mathcal{X}$ carries a line bundle $\mathcal{O}_{\mathcal{X}}(1)$ which is very ample with respect to $\mathcal{X} \to S$. Let $P \in \mathbb{Q}[T]$ denote the Hilbert polynomial of $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ with respect to $\mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{X}_{\bar{\eta}}}$ and let $\mathfrak{H}^P_{\mathcal{X}/S} \to S$ be the Hilbert scheme classifying closed subschemes of $\mathcal{X} \times_S T$ which are flat over T and with constant Hilbert polynomial P [Gro61, Thm. 3.2]. By construction, $\mathcal{X}$ acts by translation on $\mathfrak{H}^P_{\mathcal{X}/S}$ over S. Let $[\mathcal{Y}] \in \mathfrak{H}^P_{\mathcal{X}/S}(S)$ be the S-point corresponding to $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ and consider the corresponding morphism of S-schemes

$$\phi_{[\mathcal{Y}]} : \mathcal{X} \to \mathfrak{H}^P_{\mathcal{X}/S} \times_S \mathfrak{H}^P_{\mathcal{X}/S}, \ \ x \mapsto ([\mathcal{Y}], [\mathcal{Y} + x]).$$

Let also

$$\Delta : \mathfrak{H}^P_{\mathcal{X}/S} \hookrightarrow \mathfrak{H}^P_{\mathcal{X}/S} \times_S \mathfrak{H}^P_{\mathcal{X}/S}$$

denote the diagonal embedding, which is a closed immersion as $\mathfrak{H}^P_{\mathcal{X}/S} \to S$ is projective. Define the stabilizer $\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y})$ of $\mathcal{Y}$ in $\mathcal{X}$ as the fiber product

$$\begin{tikzcd}
\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\phi_{[\mathcal{Y}]}"] \\
\mathfrak{H}^P_{\mathcal{X}/S} \arrow[r, hook, "\Delta"'] & \mathfrak{H}^P_{\mathcal{X}/S} \times_S \mathfrak{H}^P_{\mathcal{X}/S}.
\end{tikzcd}$$

By construction $\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y}) \hookrightarrow \mathcal{X}$ is a closed subgroup scheme of $\mathcal{X} \to S$, whose formation commutes with arbitrary Noetherian base change $T \to S$. In particular, for every $t \in S$ one has

$$\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y})_{\bar{t}} = \mathrm{Stab}_{\mathcal{X}_{\bar{t}}}(\mathcal{Y}_{\bar{t}}). \tag{20}$$

Lemma 5.1. *Let $\mathcal{X} \to S$ an abelian scheme and $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth, geometrically connected, and of relative dimension d over S. Then,*

(1) the relative perverse sheaf $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_{\ell,Y}[d]$ lies in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$;

(2) Assume furthermore $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ has

i) ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar{\eta}}/\mathcal{X}_{\bar{\eta}}}$ then, after possibly replacing S by a non-empty open subset, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar{s}} \hookrightarrow \mathcal{X}_{\bar{s}}$ also has ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar{s}}/\mathcal{X}_{\bar{s}}}$

ii) trivial stabilizer $\mathrm{Stab}_{\mathcal{X}_{\bar{\eta}}}(\mathcal{Y}_{\bar{\eta}})$ then, after possibly replacing S by a non-empty open subset, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar{s}} \hookrightarrow \mathcal{X}_{\bar{s}}$ also has trivial stabilizer.

Proof. For (1), as $\mathcal{Y} \to S$ is smooth, $\overline{\mathbb{Q}}_{\ell,\mathcal{Y}} \in D^{\mathrm{ULA}}(\mathcal{Y}/S)$ [B24, 3.6, Lemma (i)] and as $\mathcal{Y} \hookrightarrow \mathcal{X}$ is proper, $\iota_* \overline{\mathbb{Q}}_{\ell,\mathcal{Y}} \in D^{\mathrm{ULA}}(\mathcal{X}/S)$ [B24, 3.6, Lemma (ii)] hence $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_{\ell,\mathcal{Y}}[d] \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assertion (2) ii) follows from (20) and [EGAIV$_3$, Thm. 8.10.5. (i)]. For assertion (2) i), under our assumptions for every $t \in S$ and with the notation in the base-change diagram

$$\begin{tikzcd}
\mathcal{Y}_{\bar{t}} \arrow[r, hook, "\iota_{\bar{t}}"] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[dd, bend right=60, "\iota_{\mathcal{Y}_{\bar{t}}}"'] & \mathcal{X}_{\bar{t}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{spec}(k(\bar{t})) \arrow[d] \\
\mathcal{Y}_t \arrow[r, hook, "\iota_t"] \arrow[d, hook, "\iota_{\mathcal{Y}_t}"'] \arrow[dr, phantom, "\square"] & \mathcal{X}_t \arrow[r] \arrow[d, hook, "\iota_{\mathcal{X}_t}"] \arrow[dr, phantom, "\square"] & \mathrm{spec}(k(t)) \arrow[d] \\
\mathcal{Y} \arrow[r, hook, "\iota"] & \mathcal{X} \arrow[r] & S
\end{tikzcd},$$

the canonical morphisms

$$\iota^*_{\mathcal{Y}_t} \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \to \mathcal{N}_{\mathcal{Y}_t/\mathcal{X}_t}, \ \ \iota^*_{\mathcal{Y}_{\bar{t}}} \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \to \mathcal{N}_{\mathcal{Y}_{\bar{t}}/\mathcal{X}_{\bar{t}}} \tag{21}$$

are isomorphisms. Indeed, applying $\iota_{\mathcal{Y}_t}^*$ to the short exact sequence of locally free $\mathcal{O}_{\mathcal{Y}}$-modules [Li02, §6.3, Prop.. 3.13]

$$0 \to \mathcal{C}_{\mathcal{Y}/\mathcal{X}} \to \iota^*\Omega^1_{\mathcal{X}|S} \to \Omega^1_{\mathcal{Y}|S} \to 0$$

and using the canonical identifications

$$\iota_{\mathcal{Y}_t}^*\Omega^1_{\mathcal{Y}|S} \simeq \Omega^1_{\mathcal{Y}_t|k(t)}, \quad \iota_{\mathcal{Y}_t}^*\iota^*\Omega^1_{\mathcal{X}|S} \simeq \iota_t^*\iota_{\mathcal{X}_t}^*\Omega^1_{\mathcal{X}|S} \simeq \iota_t^*\Omega^1_{\mathcal{X}_t|k(t)},$$

one gets the short exact sequence of locally free $\mathcal{O}_{\mathcal{Y}_t}$-modules

$$0 \to \iota_{\mathcal{Y}_t}^*\mathcal{C}_{\mathcal{Y}/\mathcal{X}} \to \iota_t^*\Omega^1_{\mathcal{X}_t|k(t)} \to \Omega^1_{\mathcal{Y}_t|k(t)} \to 0,$$

which yields $\iota_{\mathcal{Y}_t}^*\mathcal{C}_{\mathcal{Y}/\mathcal{X}} \simeq \mathcal{C}_{\mathcal{Y}_t/\mathcal{X}_t}$, whence the assertion, by dualizing. On the other hand, as $\mathcal{N}_{\mathcal{Y}_{\bar\eta}/\mathcal{X}_{\bar\eta}}(\simeq (\mathcal{N}_{\mathcal{Y}/\mathcal{X}})|_{\mathcal{Y}_{\bar\eta}})$ is ample, by fpqc descent of ampleness, $(\mathcal{N}_{\mathcal{Y}/\mathcal{X}})|_{\mathcal{Y}_\eta}$ is ample [St25, Tag 0D2P]; the assertion thus follows from [Ha66, Prop. 4.4]. □

5.2. **Sample of rigidity phenomena.** We give here two examples of rigidity phenomena, building on the classification results of [JKrLM25] and Corollary 1.6 (2).

For an abelian variety X over a field K of characteristic 0 and a closed subvariety $Y \subset X$, smooth, geometrically connected and of dimension $d \geq 2$ over K, one says that Y is:

- a product if there exist closed subvarieties $Y_1, Y_2 \subset X$, smooth over K and of dimension > 0, such that the sum map $+ : Y_1 \times Y_2 \to X$ induces an isomorphism $+ : Y_1 \times Y_2 \tilde{\to} Y$;
- a symmetric power of a curve if there is a closed smooth irreducible curve $C \hookrightarrow X$ such that the sum morphism $\mathrm{Sym}^d C \to X$ is a closed embedding with image Y.

Note that if K is algebraically closed and L/K is a field extension, then Y is a product (resp. a symmetric power of a curve) if and only if $Y \times_K L$ is. The only if assertion is straightforward and the if one follows from spreading out and specialization, using Hilbert Nullstellensatz.

Corollary 5.2. *Let $\mathcal{X} \to S$ an abelian scheme of relative dimension $g \geq 3$ and $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth and geometrically connected over S. Assume $\mathcal{Y}_{\bar\eta} \hookrightarrow \mathcal{X}_{\bar\eta}$ has ample normal bundle and trivial stabilizer. Then the following properties are equivalent:*

(i) The set of all $s \in S$ such that $\mathcal{Y}_{\bar s} \subset \mathcal{X}_{\bar s}$ is a product is Zariski-dense in S;
(ii) After possibly replacing S by a non-empty open subscheme, for every $s \in S$, $\mathcal{Y}_{\bar s} \subset \mathcal{X}_{\bar s}$ is a product;
(iii) $\mathcal{Y}_{\bar\eta} \subset \mathcal{X}_{\bar\eta}$ is a product.

Proof. From Lemma 5.1 (1), $\mathcal{P} := \iota_*\overline{\mathbb{Q}}_{\ell,\mathcal{Y}}[d] \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ and from Lemma 5.1 (2), up to replacing S by a non-empty open subscheme, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar s} \hookrightarrow \mathcal{X}_{\bar s}$ has ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar s}/\mathcal{X}_{\bar s}}$ and trivial stabilizer. The implication (ii) ⇒ (i) is straightforward and the implication (iii) ⇒ (ii) is by spreading out. To prove the implication (i) ⇒ (iii), observe first that, as $\mathcal{Y}_{\bar\eta}$ is smooth and irreducible $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is simple - hence semisimple [BeBerDG82, Thm. 4.3.1 (ii)] so that Corollary 1.6 (2) (b) applies. The assertion thus follows from [JKrLM25, Thm. 6.1], which asserts that, for any $t \in S$, $\mathcal{Y}_{\bar t}$ is a product if and only if $G(\mathcal{P}|_{\mathcal{Y}_{\bar t}})^{\circ\,\mathrm{der}}$ is not simple. □

Remark 5.3. For $\mathcal{Y} = \mathcal{X} \to S$, in general, it is not true that the set of all $s \in S$ such that $\mathcal{X}_{\bar s}$ is a product is Zariski-dense in S (if and) only if $\mathcal{X}_{\bar\eta}$ is itself a product. For instance, let $k = \mathbb{C}$, $S = M_2$ the moduli space of genus 2 smooth projective curves (with suitable level structures) and $\mathcal{X} := \mathrm{Jac}(\mathcal{C}|S) \to S$ the Jacobian of the universal genus 2 curve $\mathcal{C} \to S$. Then the set of all points $s \in S$ such that $\mathcal{X}_{\bar s}$ is a product of two elliptic curves[5] is supported on infinitely many irreducible curves $C_d \hookrightarrow S$. Let $S^{\mathrm{prod}} \subset S$ denote the Zariski-closure of the union of all C_d. Then the geometric generic fiber $\mathcal{X}_{\bar\xi}$ over the generic point ξ of an irreducible component of S^{prod} of dimension ≥ 2 is not a product of two elliptic curves. See [K16] and the references therein for details.

Corollary 5.4. *Let $\mathcal{X} \to S$ an abelian scheme of relative dimension g and $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth, geometrically connected and of relative dimension $d < \frac{g-1}{2}$ over S. Assume $\mathcal{Y}_{\bar\eta} \hookrightarrow \mathcal{X}_{\bar\eta}$ has ample normal bundle and trivial stabilizer. Then the following properties are equivalent:*

(i) The set of all $s \in S$ such that $\mathcal{Y}_{\bar s} \subset \mathcal{X}_{\bar s}$ is a symmetric power of a curve is Zariski-dense in S;
(ii) After possibly replacing S by a non-empty open subscheme, for every $s \in S$, $\mathcal{Y}_{\bar s} \subset \mathcal{X}_{\bar s}$ is a symmetric power of a curve;

[5] Observe that if $+ : X_1 \times X_2 \tilde{\to} \mathcal{X}_{\bar s}$ is a product then, necessarily, X_1, X_2 are translates of abelian subvarieties of $\mathcal{X}_{\bar s}$.

(iii) $\mathcal{Y}_{\bar{\eta}} \subset \mathcal{X}_{\bar{\eta}}$ *is a symmetric power of a curve.*

Proof. The argument is similar to the one for Corollary 5.2. Again, the difficult implication is (i) $\Rightarrow$ (iii). If $d = 1$ there is nothing to prove so that we may assume $d \geq 2$. Again, $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_{\ell,\mathcal{Y}}[d]$ lies in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ simple - hence semisimple, and, up to replacing S by a non-empty open subscheme, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar{s}} \hookrightarrow \mathcal{X}_{\bar{s}}$ has ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar{s}}/\mathcal{X}_{\bar{s}}}$ and trivial stabilizer. Let $r := \chi(\mathcal{X}_{\bar{t}}, \mathcal{P}) = (-1)^d \chi(\mathcal{Y}_{\bar{t}})$ denote the Euler-Poincaré characteristic of $\mathcal{P}|_{\mathcal{X}_{\bar{t}}}$ for one (equivalently every) $t \in S$. Assume that for some $s \in S$, $\mathcal{Y}_{\bar{s}} \simeq \mathrm{Sym}^d(C)$ is a symmetric power of a curve. Then from [JKrLM25, Lem. 7.2], $G(\mathcal{P}|_{\mathcal{Y}_{\bar{s}}})^{\circ\,\mathrm{der}}$ acting on $\omega(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ identifies with the image of $\mathrm{SL}_{n,\overline{\mathbb{Q}}_\ell}$ acting on a wedge power $\wedge^d \mathrm{Std}_n$ of the standard representation of $\mathrm{SL}_{n,\overline{\mathbb{Q}}_\ell}$ with $n = -\chi(C, \overline{\mathbb{Q}}_\ell) = 2g_C - 2$ (where g_C denotes the genus of C). Note that, as $r = \binom{n}{d}$, $2g_C - 2 = n =: n(r,d) =$ is uniquely determined by r and d hence is independent of $s \in S$. Furthermore, as $W_d(C) \subset \mathrm{Alb}(C)$ is then automatically smooth, it follows from Riemann's singularity theorem (*e.g.* [GrH78, p. 344]) that C has gonality $\geq d+1$ hence genus $g_C \geq 2d-1$. As $d \geq 2$, this imposes $g_C \geq 3$ hence $n(r,d) \geq 4$, whence, $2 \leq d \leq \frac{n(r,d)}{2}$. Under this numerical condition, it follows from [JKrLM25, Thm. 7.3] that, for any $t \in S$, $\mathcal{Y}_{\bar{t}} \simeq \mathrm{Sym}^d(C)$ is a symmetric power of a curve if and only if $G(\mathcal{P}|_{\mathcal{Y}_{\bar{t}}})^{\circ\,\mathrm{der}}$ acting on $\omega(\mathcal{P}|_{\mathcal{X}_{\bar{t}}})$ identifies with the image of $\mathrm{SL}_{n,\overline{\mathbb{Q}}_\ell}$ acting on a wedge power $\wedge^d \mathrm{Std}_{n(r,d)}$ of the standard representation of $\mathrm{SL}_{n(r,d),\overline{\mathbb{Q}}_\ell}$. The assertion thus follows, again, from Corollary 1.6 (2) (b). □

Remark 5.5. Using [KrM25, Thm. 6.1] instead of [JKrLM25, Thm. 7.3], one could probably relax the assumption that $\mathcal{Y} \to S$ is smooth.

Appendix A. (Semi)simplicity in Noetherian and Artinian abelian categories

A.1. **Recollection on artinian and noetherian abelian categories.** Let $\mathcal{A}$ be a an artinian and noetherian abelian category. Then,

(1) For every $A \in \mathcal{A}$ and $\phi \in \mathrm{End}_{\mathcal{A}}(A)$, one has a ϕ-stable direct sum decomposition (Fitting lemma): $A \simeq A_{\phi,0} \oplus A_{\phi,\infty}$, with the property that the induced morphism $\phi : A_{\phi,0} \to A_{\phi,0}$ is nilpotent and $\phi : A_{\phi,\infty} \to A_{\phi,\infty}$ is an automorphism; explicitly $A_{\phi,0} = \ker(\phi^n)$, for $n \gg 0$, $A_{\phi,\infty} = \mathrm{im}(\phi^n)$, for $n \gg 0$. In particular, for every $A \in \mathcal{A}$, A is indecomposable in $\mathcal{A}$ if and only if $\mathrm{End}_{\mathcal{A}}(A)$ is a local ring and every $A \in \mathcal{A}$ admits a Krull-Schmidt decomposition: A decomposes into a direct sum $A = \oplus_{1 \leq i \leq r} A_i$ with $A_1, \ldots, A_r \in \mathcal{A}$ indecomposable and the indecomposable objects $A_1, \ldots, A_r$ (counted with multiplicity) are unique up to isomorphism and called the Krull-Schmidt or indecomposable factors of A. In particular, A is semisimple if and only if its indecomposable factors are simple.

(2) Every $A \in \mathcal{A}$ admits a composition series that is a filtration
$$0 = A_{r+1} \subsetneq A_r \subsetneq \cdots \subsetneq A_2 \subsetneq A_1 := A$$
in $\mathcal{A}$ with $S_i := A_i/A_{i+1}$ a simple object in $\mathcal{A}$, $i = 1, \ldots, r$. Furthermore, the simple objects $S_1, \ldots, S_r$ (counted with multiplicity) are unique up to isomorphism and called the Jordan-Hölder or simple factors of A. In particular the length $\mathrm{length}_{\mathcal{A}}(A) := r$ of A is a well-defined integer.

(3) Let $\mathcal{N} \subset \mathcal{A}$ be a Serre subcategory and let $p : \mathcal{A} \to \overline{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ denote the resulting quotient functor, which is exact and essentially surjective. Then, for every simple object S in $\mathcal{A}$ not lying in $\mathcal{N}$, $p(S)$ is again a simple object in $\overline{\mathcal{A}}$. This follows from the definition of morphisms in $\overline{\mathcal{A}}$. Indeed, consider a diagram
$$X \xrightarrow{f} Y \xleftarrow{s} S$$
in $\mathcal{A}$ with $N := \mathrm{coker}(s) \in \mathcal{N}$ such that the resulting morphism
$$p(X) \xrightarrow{p(f)} p(Y) \xrightarrow[\simeq]{p(s)^{-1}} p(S)$$
is injective. In particular, the morphism $p(S) \xrightarrow[\simeq]{p(s)} p(Y) \to p(Y/X)$ is surjective. So either $p(Y/X) = 0$ and $p(X) \xrightarrow{p(f)} p(Y)$ is an isomorphism in $\overline{\mathcal{A}}$ or the morphism $S \xrightarrow{s} Y \to Y/X$ is non-zero hence injective. But then the morphism $p(S) \xrightarrow[\simeq]{p(s)} p(Y) \to p(Y/X)$ is an isomorphism in $\overline{\mathcal{A}}$ hence so

is $p(Y) \to p(Y/X)$, which imposes $p(X) = 0$. In particular, for every $A \in \mathcal{A}$, if one defines $\mathrm{length}_{\mathcal{A},\mathcal{N}}(A) \leq \mathrm{length}_{\mathcal{A}}(A)$ to be the number of Jordan-Hölder factors of A which lies in $\mathcal{N}$, one has

$$\mathrm{length}_{\overline{\mathcal{A}}}(p(A)) + \mathrm{length}_{\mathcal{A},\mathcal{N}}(A) = \mathrm{length}_{\mathcal{A}}(A).$$

A.2. **Functoriality.** Let now $\mathcal{A}_1$, $\mathcal{A}_2$ be artinian and noetherian abelian categories and let $F : \mathcal{A}_1 \to \mathcal{A}_2$ be an additive functor.

(1) Assume $F : \mathcal{A}_1 \to \mathcal{A}_2$ is fully faithful. Then for every $A_1 \in \mathcal{A}_1$, A_1 is indecomposable in $\mathcal{A}_1$ if and only if $F(A_1)$ is indecomposable in $\mathcal{A}_2$.

(2) Consider the following conditions

$(\mathrm{S}_{F,[??]})$	For every $A_1 \in \mathcal{A}_1$, A_1 is simple in $\mathcal{A}_1$	[??]	$F(A_1)$ is simple in $\mathcal{A}_2$;
$(\mathrm{SS}_{F,[??]})$	For every $A_1 \in \mathcal{A}_1$, A_1 is semisimple in $\mathcal{A}_1$	[??]	$F(A_1)$ is semisimple in $\mathcal{A}_2$.

with [??] one of $\Leftarrow$, $\Rightarrow$, $\Leftrightarrow$.

(a) Then $(\mathrm{S}_{F,\Rightarrow})$ always implies $(\mathrm{SS}_{F,\Rightarrow})$ and,

(i) If $F : \mathcal{A}_1 \to \mathcal{A}_2$ is exact, then for every $A_1 \in \mathcal{A}_1$,

$$\mathrm{length}_{\mathcal{A}_1}(A_1) = \mathrm{length}_{\mathcal{A}_2}(F(A_1)),$$

(ii) If $F : \mathcal{A}_1 \to \mathcal{A}_2$ preserves monomorphisms and epimorphisms, then for every $A_1 \in \mathcal{A}_1$,

$$\mathrm{length}_{\mathcal{A}_1}(A_1) \leq \mathrm{length}_{\mathcal{A}_2}(F(A_1)).$$

Argue by induction on $\mathrm{length}_{\mathcal{A}_1}(A_1)$. If $\mathrm{length}_{\mathcal{A}_1}(A_1) = 1$ then, by assumption, $\mathrm{length}_{\mathcal{A}_2}(F(A_1)) = 1$. If $\mathrm{length}_{\mathcal{A}_1}(A_1) \geq 2$, then there is a short exact sequence

$$0 \to A_1' \xrightarrow{u} A_1 \xrightarrow{v} A_1'' \to 0$$

with A_1'' simple in $\mathcal{A}_1$. Then one gets, in $\mathcal{A}_2$, a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F(A_1') \arrow[r, "F(u)"] \arrow[d, hook] & F(A_1) \arrow[r] \arrow[d, equal] & \mathrm{coker}(F(u)) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \ker(F(v)) \arrow[r] & F(A_1) \arrow[r, "F(v)"] & F(A_1'') \arrow[r] & 0.
\end{tikzcd}$$

Hence,

$$\mathrm{length}_{\mathcal{A}_2}(F(A_1)) = 1+\mathrm{length}_{\mathcal{A}_2}(\ker(F(v))) \geq 1+\mathrm{length}_{\mathcal{A}_2}(F(A_1')) \geq 1+\mathrm{length}_{\mathcal{A}_1}(A_1') = \mathrm{length}_{\mathcal{A}_1}(A_1).$$

(b) If $F : \mathcal{A}_1 \to \mathcal{A}_2$ is fully faithful then $(\mathrm{S}_{F,\Leftarrow})$ implies $(\mathrm{SS}_{F,\Leftarrow})$. Indeed, let $A_1 \in \mathcal{A}_1$ and consider the Krull-Schmidt decomposition $A_1 = \oplus_{1\leq i\leq r} A_{1,i}$ of A_1 in $\mathcal{A}_1$. Then $F(A_1) = \oplus_{1\leq i\leq r} F(A_{1,i})$ is the Krull-Schmidt decomposition of $F(A_1)$ in $\mathcal{A}_2$. So, one has

$$\begin{aligned}
F(A_1) \text{ is semisimple in } \mathcal{A}_2 &\iff F(A_{1,i}) \text{ is simple in } \mathcal{A}_2,\ i = 1, \dots r \\
&\overset{(\mathrm{S}_{F,\Leftarrow})}{\Longrightarrow} A_{1,i} \text{ is simple in } \mathcal{A}_1,\ i = 1, \dots r \\
&\iff A_1 \text{ is semisimple in } \mathcal{A}_1.
\end{aligned}$$

A.3. **Lifting semisimplicity.** Let $\mathcal{A}$ be an artinian and noetherian abelian category, let $\iota : \mathcal{N} \hookrightarrow \mathcal{A}$ be a Serre subcategory and let $p : \mathcal{A} \to \overline{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ denote the resulting quotient functor. The inclusion functor $\iota : \mathcal{N} \hookrightarrow \mathcal{A}$ admits both a right adjoint $(-)_\neg : \mathcal{A} \to \mathcal{N}$ ("maximal subobject in $\mathcal{N}$") and a left adjoint $(-)^\neg : \mathcal{A} \to \mathcal{N}$ ("maximal quotient object in $\mathcal{N}$"). Explicitly, for $A \in \mathcal{A}$, $A_\neg = \sum_{N\in S_\neg(A)} N \hookrightarrow A$, where $S_\neg(A)$ denotes the subset of all subobjects of A in $\mathcal{N}$ and $A \twoheadrightarrow A^\neg = A/\cap_{S\in S^\neg(A)} S$, where $S^\neg(A)$ denotes the subobjects S of A such that $A/S \in \mathcal{N}$. Then for every $A \in \mathcal{A}$,

$$A^* := \ker(A/A_\neg \to (A/A_\neg)^\neg)$$

is a subquotient of A in $\mathcal{A}$ satisfying $(A^*)^\neg = (A^*)_\neg = 0$ and $p(A^*) \simeq p(A)$ in $\overline{\mathcal{A}}$. Observing that for every $A_1, A_2 \in \mathcal{A}$ with $A_1^\neg = 0$ and $A_{2,\neg} = 0$ the canonical morphism

$$\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\overline{\mathcal{A}}}(p(A_1), p(A_2))$$

is an isomorphism, one gets that for every $A_1, A_2 \in \mathcal{A}$,

$$p(A_1) \simeq p(A_2) \text{ in } \overline{\mathcal{A}} \text{ if and only if } A_1^* \simeq A_2^* \text{ in } \mathcal{A}.$$

Lemma A.1. *Let $A \in \mathcal{A}$ such that $p(A)$ is semisimple in $\overline{\mathcal{A}}$. Then A^* is semisimple in $\mathcal{A}$ and*

$$\mathrm{length}_{\mathcal{A}}(A^*) = \mathrm{length}_{\overline{\mathcal{A}}}(p(A)).$$

Proof. Assume first $p(A)$ is simple in $\overline{\mathcal{A}}$. As $p(A^*) \simeq p(A)$ in $\overline{\mathcal{A}}$, A^* has a single Jordan-Hölder factor in $\mathcal{A}$ which is not in $\mathcal{N}$. But as $(A^*)_\neg = (A^*)^\neg = 0$, this forces A^* to be simple in $\mathcal{A}$. In general, let $\overline{S}_1, \dots, \overline{S}_r$ denote the simple factors (counted with multiplicities) of $p(A)$ in $\overline{\mathcal{A}}$ and let $S_1, \dots, S_r \in \mathcal{A}$ with $p(S_i) \simeq \overline{S}_i$ in $\overline{\mathcal{A}}$, $i = 1, \dots, r$. Set

$$S := S_1^* \oplus \cdots \oplus S_r^*.$$

Then, S is semisimple in $\mathcal{A}$, $S = S^*$, and $p(S) \simeq p(A)$ in $\overline{\mathcal{A}}$. This shows $A^* \simeq (S^* \simeq)S$ is semisimple in $\mathcal{A}$ and

$$\mathrm{length}_{\mathcal{A}}(A^*) = \mathrm{length}_{\mathcal{A}}(S) = r = \mathrm{length}_{\overline{\mathcal{A}}}(p(A)).$$

□

Remark.

(1) Actually, the map $A \mapsto A^*$ defines a functor $(-)^* : \mathcal{A} \to \mathcal{A}$ with essential image the full subcategory $\mathcal{A}^* \subset \mathcal{A}$ of all $A \in \mathcal{A}$ with $A^\neg = A_\neg = 0$ and the composite

$$\mathcal{A}^* \hookrightarrow \mathcal{A} \xrightarrow{p} \overline{\mathcal{A}}$$

is an equivalence of categories. By the universal property of $p : \mathcal{A} \to \overline{\mathcal{A}}$, the retraction functor $(-)^* : \mathcal{A} \to \mathcal{A}^*$ factors through

$$(-)^* : \mathcal{A} \xrightarrow{p} \overline{\mathcal{A}} \to \mathcal{A}^*,$$

providing an explicit quasi-inverse $\overline{\mathcal{A}} \to \mathcal{A}^*$ to $\mathcal{A}^* \xrightarrow{\simeq} \overline{\mathcal{A}}$. In particular, one can endow $\mathcal{A}^*$ with the structure of abelian category of $\overline{\mathcal{A}}$ (but be aware that, in general, if $\phi : A_1 \to A_2$ is a morphism in $\mathcal{A}$ with $A_1, A_2 \in \mathcal{A}^*$ then the kernel and cokernel of ϕ computed in $\mathcal{A}$ do not coincide with those computed in $\mathcal{A}^*$).

(2) One can check that the functor $(-)^* : \mathcal{A} \to \mathcal{A}^*$ is canonically isomorphic to the functor defined by

$$A \mapsto \mathrm{coker}(A_\neg \to (\ker(A \to A^\neg) + A_\neg)).$$

Appendix B. Generic simplicity of perverse sheaves in positive characteristic, after Beat Zurbuchen

Let ℓ be a prime. The aim of this appendix is to extend Theorem 1.2 to base field k of characteristic $\neq \ell$.

Theorem B.1. (Zurbuchen) *Let k be a field of characteristic $p \neq \ell$, let S be an integral variety over k with generic point η and let $f : \mathcal{X} \to S$ be a separated morphism of finite type. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Then, after possibly replacing S by a non-empty open subscheme, for every geometric point $\bar{s}$ on S, one has*

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}).$$

Furthermore, if for one (equivalently every) geometric point $\bar{\eta}$ over η the restriction $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$ then $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$.

Proof. The conclusion of Theorem B.1 remain unchanged if one replaces S by a non-empty open subscheme or base change by a finite cover $S' \to S$; for the latter; see Paragraph 2.2.1.2. In particular, up to replacing S by its normalization, one may assume S is normal. The assumption that S is normal, integral variety is then also preserved by replacing S by a non-empty open subscheme and by replacing S by its normalization $S' \to S$ in $\mathrm{spec}(k(\eta')) \to S$ for a finite field extension $k(\eta')/k(\eta)$. We now proceed in several steps.

(1) Reduction to the case where $\mathcal{P}|_{\mathcal{X}_\eta}$ is simple. As $\mathrm{Perv}(\mathcal{X}_\eta)$ is both artinian and noetherian [BeBerDG82, Thm. 4.3.1 (i)], $\mathcal{P}|_{\mathcal{X}_\eta}$ admits a finite decreasing filtration

$$\mathcal{P}|_{\mathcal{X}_\eta} = \mathcal{P}_{[\eta],1} \supset \mathcal{P}_{[\eta],2} \supset \cdots \supset \mathcal{P}_{[\eta],n} \supset \mathcal{P}_{[\eta],n+1} = 0 \tag{22}$$

in $\mathrm{Perv}(\mathcal{X}_\eta)$ with $\mathcal{P}_{[\eta],i}/\mathcal{P}_{[\eta],i+1}$ simple, $i=1,\dots,n$. From [HS23, Thm 1.10 (ii)] (here we use the normality of S), (22) lifts to a finite decreasing filtration

$$\mathcal{P}=\mathcal{P}_1\supset\mathcal{P}_2\supset\cdots\supset\mathcal{P}_n\supset\mathcal{P}_{n+1}=0 \tag{23}$$

in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ *via* the canonical restriction functor $(-)|_{\mathcal{X}_\eta}:\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)\to\mathrm{Perv}(\mathcal{X}_\eta)$. On the other hand, for every $s\in S$, as $(-)|_{\mathcal{X}_{\bar{s}}}:\mathrm{Perv}(\mathcal{X}/S)\to\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ is exact, (23) restricts to a finite decreasing filtration

$$\mathcal{P}|_{\mathcal{X}_{\bar{s}}}=\mathcal{P}_1|_{\mathcal{X}_{\bar{s}}}\supset\mathcal{P}_2|_{\mathcal{X}_{\bar{s}}}\supset\cdots\supset\mathcal{P}_n|_{\mathcal{X}_{\bar{s}}}\supset\mathcal{P}_{n+1}|_{\mathcal{X}_{\bar{s}}}=0$$

in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$. Therefore, one has

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})=\sum_{i=1}^{n}\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_i|_{\mathcal{X}_{\bar{s}}}/\mathcal{P}_{i+1}|_{\mathcal{X}_{\bar{s}}})=\sum_{i=1}^{n}\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}((\mathcal{P}_i/\mathcal{P}_{i+1})|_{\mathcal{X}_{\bar{s}}}).$$

If one furthermore assumes $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple, one has

$$\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\simeq\oplus_{i=1}^{n}(\mathcal{P}_i|_{\mathcal{X}_{\bar{\eta}}}/\mathcal{P}_{i+1}|_{\mathcal{X}_{\bar{\eta}}})\simeq\oplus_{i=1}^{n}(\mathcal{P}_i/\mathcal{P}_{i+1})|_{\mathcal{X}_{\bar{\eta}}}$$

(with $(\mathcal{P}_i/\mathcal{P}_{i+1})|_{\mathcal{X}_{\bar{\eta}}}$ semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$, $i=1,\dots,n$ - but see below). Fix a witness $(S',\eta',s')\to(S,\bar{\eta},s)$, and let $f':\mathcal{X}'\to S'$ denote the base-change of $f:\mathcal{X}\to S$ along $S'\to S$. As $(\mathcal{P}_i/\mathcal{P}_{i+1})|_{\mathcal{X}'}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S')$ [HS23, Prop. 3.4 (i)], and $(-)|_{\mathcal{X}_{s'}}\simeq R\psi_{f'}\circ(-)|_{\mathcal{X}_{\eta'}}$ [HS23, Thm. 1.7], one has

$$\mathcal{P}|_{\mathcal{X}_{s'}}=R\psi_{f'}(\mathcal{P}|_{\mathcal{X}_{\eta'}})\cong\oplus_{i=1}^{n}R\psi_{f'}((\mathcal{P}_i/\mathcal{P}_{i+1})|_{\mathcal{X}_{\eta'}})=\oplus_{i=1}^{n}(\mathcal{P}_i/\mathcal{P}_{i+1})|_{\mathcal{X}_{s'}},$$

Thus, Theorem B.1 holds for $\mathcal{P}$ if and only if it holds for $\mathcal{P}_i/\mathcal{P}_{i+1}$, $i=1,\dots,n$. Therefore, one may assume that $\mathcal{P}|_{\mathcal{X}_\eta}$ is simple in $\mathrm{Perv}(\mathcal{X}_\eta)$. From Subsection 2.2.1.1, this automatically implies $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$. So we are to show that if $\mathcal{P}$ is simple in $\mathrm{Perv}(\mathcal{X}_\eta)$ then $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ with

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})=\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}).$$

(2) Writing $\mathcal{P}|_{\mathcal{X}_\eta}$ as a middle extension. Let $Z_{[\eta]}$ be the Zariski closure in $\mathcal{X}_\eta$ of the support[6] of $\mathcal{P}|_{\mathcal{X}_\eta}$, viewed as a reduced closed subscheme of $\mathcal{X}_\eta$. As $\mathcal{P}|_{\mathcal{X}_\eta}\neq0$, one has $Z_{[\eta]}\neq\emptyset$. Let $U_{[\eta]}$ be the largest open subset of $Z_{[\eta]}$ where all the $H^n(\mathcal{P}|_{\mathcal{X}_\eta})$, $n\in\mathbb{Z}$ are lisse. Let $\mathcal{Z}$ be the Zariski closure in $\mathcal{X}$ of the support of $\mathcal{P}$, viewed as a reduced closed subscheme of $\mathcal{X}$. Then $Z_{[\eta]}\subset\mathcal{Z}_\eta$. Let $\mathcal{U}\subset\mathcal{Z}$ be the largest open subset where all the $H^n(\mathcal{P})$, $n\in\mathbb{Z}$ are lisse. Then $\mathcal{U}$ is dense in $\mathcal{Z}$. Hence, $\mathcal{U}_\eta$ is dense in $\mathcal{Z}_\eta$ (so, in particular, $\mathcal{U}_\eta\neq\emptyset$). As $\mathcal{P}|_{\mathcal{U}_\eta}$ is nonzero with all the $H^n(\mathcal{P}|_{\mathcal{U}_\eta})$, $n\in\mathbb{Z}$ lisse, one has $\mathcal{U}_\eta\subset Z_{[\eta]}$ and hence $\mathcal{U}_\eta\subset U_{[\eta]}$. By [BeBerDG82, Thm. 4.3.1 (ii)], since $\mathcal{P}|_{\mathcal{X}_\eta}$ is simple, $Z_{[\eta]}$ is irreducible, and there is a nonempty open subscheme $V_{[\eta]}\subset U_{[\eta]}$ with $(V_{[\eta],\bar{\eta}})_{\mathrm{red}}$ smooth over $k(\bar{\eta})$, such that $\mathcal{F}_{[\eta]}:=\mathcal{P}|_{V_{[\eta]}}[-d]$ is a simple lisse sheaf and

$$\mathcal{P}|_{\mathcal{X}_\eta}=j_{[\eta]!*}\mathcal{F}_{[\eta]}[d], \tag{24}$$

where $d:=\dim V_{[\eta]}$ and $j_{[\eta]}:V_{[\eta]}\hookrightarrow\mathcal{X}_\eta$ is the immersion.

(3) Spreading out $V_{[\eta]}$. Up to replacing S by a non-empty open subscheme, one may assume that there is an immersion $j:\mathcal{V}\hookrightarrow\mathcal{X}$ over S, whose base change $j_\eta:\mathcal{V}_\eta\hookrightarrow\mathcal{X}_\eta$ along $\eta\to S$ identifies with $j_{[\eta]}:V_{[\eta]}\to\mathcal{X}_\eta$. By [St25, Tag 04KT], there is a finite extension $k(\eta')/k(\eta)$ such that $(\mathcal{V}_{\eta'})_{\mathrm{red}}$ is geometrically reduced over $k(\eta')$. By [St25, Tag 054M], the closed immersion $((\mathcal{V}_{\eta'})_{\mathrm{red}})_{\bar{\eta}}\to\mathcal{V}_{\bar{\eta}}$ factors through an isomorphism

$$((\mathcal{V}_{\eta'})_{\mathrm{red}})_{\bar{\eta}}\xrightarrow{\sim}(\mathcal{V}_{\bar{\eta}})_{\mathrm{red}}.$$

In particular, $(\mathcal{V}_{\eta'})_{\mathrm{red}}$ is smooth over $k(\eta')$. Let $S'\to S$ be the normalization of S in $\mathrm{spec}(k(\eta'))\to S$. Observing that $(\mathcal{V}_{\eta'})_{\mathrm{red}}\simeq((\mathcal{V}\times_S S')_{\mathrm{red}})_{\eta'}$ [St25, Tag 054Z] (where $(\mathcal{V}\times_S S')_{\mathrm{red}}\hookrightarrow\mathcal{V}\times_S S'$ denotes the reduced closed subscheme underlying $\mathcal{V}\times_S S'$), replacing S by S', $U_{[\eta]}$, $\mathcal{U}$ and $\mathcal{V}$ by $(U_{[\eta],\eta'})_{\mathrm{red}}$, $(\mathcal{U}\times_S S')_{\mathrm{red}}$ and $(\mathcal{V}\times_S S')_{\mathrm{red}}$ respectively, one may assume further that $\mathcal{V}_\eta$ is smooth over $k(\eta)$. Replacing further S by a non-empty open subscheme, one may assume that $\mathcal{V}\to S$ is smooth of relative dimension d with irreducible fibers. Then by [St25, Tag 037A], $\mathcal{V}$ is irreducible.

[6] Recall that, for a noetherian scheme X over $\mathbb{Z}[\frac{1}{\ell}]$, the support of $\mathcal{K}\in D_c^b(X)$ is the union of the supports of the constructible sheaves $H^n(\mathcal{K})$, $n\in\mathbb{Z}$.

(4) Spreading out $\mathcal{F}_{[\eta]}$ to a lisse sheaf $\mathcal{F}$ on $\mathcal{V}$. As $U_{[\eta]}$ is irreducible, $\mathcal{U}_\eta \cap \mathcal{V}_\eta$ is a dense open subscheme of $U_{[\eta]}$. By [BeBerDG82, Lemme 4.3.2], as $\mathcal{V}_\eta$ is smooth over $k(\eta)$, replacing replacing $\mathcal{V}_\eta$ with $\mathcal{U}_\eta \cap \mathcal{V}_\eta$ one may assume that $\mathcal{V}_\eta$ is a dense open subscheme of $\mathcal{U}_\eta$. Replacing S and $\mathcal{V}$ by non-empty open subschemes, one may assume that the immersion $j : \mathcal{V} \hookrightarrow \mathcal{X}$ factors as $j : \mathcal{V} \hookrightarrow \mathcal{U} \hookrightarrow \mathcal{X}$ with $\mathcal{V} \hookrightarrow \mathcal{U}$ an open immersion.

In particular, for every $n \in \mathbb{Z}$, $H^n(\mathcal{P}|_\mathcal{V})$ is a lisse sheaf on $\mathcal{V}$. On the other hand, for every $n \neq -d$, one has

$$(H^n(\mathcal{P}|_\mathcal{V}))\,|_{\mathcal{V}_\eta} = H^n(\mathcal{P}|_{\mathcal{V}_\eta}) = H^n(\mathcal{F}_{[\eta]}[d]) = 0.$$

So, as $\mathcal{V}$ is irreducible, $H^n(\mathcal{P}|_\mathcal{V}) = 0$ and hence $\mathcal{F} := \mathcal{P}|_\mathcal{V}[-d]$ is a lisse sheaf on $\mathcal{V}$ with, by construction, $\mathcal{F}|_{\mathcal{V}_\eta} = \mathcal{F}_{[\eta]}$.

(5) Describing $\mathcal{P}$ as the "relative" middle extension of $\mathcal{F}$. From [SGA4 1/2, Finitude, Thm. 1.9 (i)], and as S is Noetherian, up to replacing S by a non-empty open subscheme, one may assume that $Rj_*\mathcal{F} \in D^b_c(\mathcal{X})$. Let $\mathcal{Q}$ be the image of the natural morphism

$${}^{p/S}H^0(j_!\mathcal{F}[d]) \to {}^{p/S}H^0(Rj_*\mathcal{F}[d])$$

in $\mathrm{Perv}(\mathcal{X}/S)$. From the generic base change theorem [SGA4 1/2, Finitude, Cor. 2.9], after replacing further S by a non-empty open subscheme, one may assume that the formation of $j_!\mathcal{F}$ and $Rj_*\mathcal{F}$ commute with any base change of $\mathcal{X}$ hence, in particular, that for every $s \in S$, $\mathcal{Q}|_{\mathcal{X}_{\bar{s}}}$ is the image of

$${}^pH^0(j_{\bar{s}!}\mathcal{F}|_{\mathcal{V}_{\bar{s}}}[d]) \to {}^pH^0(Rj_{\bar{s}*}\mathcal{F}|_{\mathcal{V}_{\bar{s}}}[d])$$

in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$, namely $j_{\bar{s}!*}(\mathcal{F}|_{\mathcal{V}_{\bar{s}}})[d]$. Similarly, (24) implies $\mathcal{Q}|_{\mathcal{X}_\eta} = \mathcal{P}|_{\mathcal{X}_\eta}$. Replacing again S by a non-empty open subscheme, one may assume $\mathcal{Q} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ [SGA4 1/2, Finitude, Thm. 2.13]. Hence, since the functor

$$\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_\eta)$$

is fully faithful [HS23, Theorem 1.10 (ii)], $\mathcal{Q}$ is isomorphic to $\mathcal{P}$ in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. But then, for every geometric point $\bar{s}$ on S, one gets

$$\mathcal{P}|_{\mathcal{X}_{\bar{s}}} \simeq \mathcal{Q}|_{\mathcal{X}_{\bar{s}}} = j_{\bar{s}!*}\mathcal{F}|_{\mathcal{V}_{\bar{s}}}[d]. \tag{25}$$

(6) Reduction to the case where $\mathcal{V} \to S$ is geometrically connected. As $\mathcal{V}_{\bar{\eta}}$ is smooth over $k(\bar{\eta})$, its connected components $\mathcal{V}_{\bar{\eta},1}, \ldots, \mathcal{V}_{\bar{\eta},m}$ are irreducible, and all of dimension d. Let $k(\eta')/k(\eta)$ be a finite field extension which is common field of definition for $\mathcal{V}_{\bar{\eta},1}, \ldots, \mathcal{V}_{\bar{\eta},m}$ so that, up to replacing S by a non-empty open subscheme of the normalization of S in $\mathrm{spec}(k(\eta')) \to S$, one may assume $\mathcal{V} = \sqcup_{1 \leq i \leq m}\mathcal{V}_i$ with $\mathcal{V}_i \to S$ smooth, geometrically connected, of relative dimension d, and such that $\mathcal{V}_{i,\bar{\eta}} \simeq \mathcal{V}_{\bar{\eta},i}$, $i = 1, \ldots, m$. Write $c_i : \mathcal{V}_i \hookrightarrow \mathcal{V}$, $j_i := j \circ c_i : \mathcal{V}_i \hookrightarrow \mathcal{X}$ for the corresponding immersions and set $\mathcal{F}_i := c_i^*\mathcal{F}$ for the resulting lisse sheaf on $\mathcal{V}_i$, $i = 1, \ldots, m$. With these notation, one has

$$\mathcal{F} = \oplus_{i=1}^m c_{i!}\mathcal{F}_i = \oplus_{i=1}^m Rc_{i*}\mathcal{F}_i. \tag{26}$$

(7) Applying specialization argument to each of the $\mathcal{F}_i$, $i = 1, \ldots, m$. As $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$, from (24) and Lemma B.2, $\mathcal{F}|_{\mathcal{V}_{\bar{\eta}}}$ is a semisimple lisse sheaf on $\mathcal{V}_{\bar{\eta}}$ of length

$$\mathrm{length}_{\mathrm{Loc}(\mathcal{V}_{\bar{\eta}})}(\mathcal{F}|_{\mathcal{V}_{\bar{\eta}}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}).$$

From (26), one has

$$\mathcal{F}|_{\mathcal{V}_{\bar{\eta}}} = \oplus_{i=1}^m c_{i,\bar{\eta}*}\mathcal{F}_i|_{\mathcal{V}_{i,\bar{\eta}}}.$$

In particular, for every $i = 1\ldots, m$, the restriction $\mathcal{F}_i|_{\mathcal{V}_{i,\bar{\eta}}}$ is a semisimple lisse sheaf on $\mathcal{V}_{i,\bar{\eta}}$.

Let $\rho_i : \pi_1(\mathcal{V}_i) \to \mathrm{GL}_{n_i}(\overline{\mathbb{Q}}_\ell)$ be the continuous representation corresponding to $\mathcal{F}_i \in \mathrm{Loc}(\mathcal{V}_i)$. From Pink's specialization argument [Kat90, Thm. 8.18.2], (here we use that S is normal, connected and Noetherian and that $\mathcal{V}_i \to S$ is smooth and geometrically connected), up to replacing S by a non-empty open subscheme, one may assume that for every geometric point $\bar{s}$ on S, $\rho_i(\pi_1(\mathcal{V}_{i,\bar{s}}))$ is conjugate in $\mathrm{GL}_{n_i}(\overline{\mathbb{Q}}_\ell)$ to $\rho_i(\pi_1(\mathcal{V}_{i,\bar{\eta}}))$. In particular, $\mathcal{F}_i|_{\mathcal{V}_{i,\bar{s}}}$ is a semisimple lisse sheaf on $\mathcal{V}_{i,\bar{s}}$ of length

$$\mathrm{length}_{\mathrm{Loc}(\mathcal{V}_{i,\bar{s}})}(\mathcal{F}_i|_{\mathcal{V}_{i,\bar{s}}}) = \mathrm{length}_{\mathrm{Loc}(\mathcal{V}_{i,\bar{\eta}})}(\mathcal{F}_i|_{\mathcal{V}_{i,\bar{\eta}}}),$$

and, from (25), one has

$$\mathcal{P}|_{\mathcal{X}_{\bar{s}}} \cong \oplus_{i=1}^m j_{i,\bar{s}!*}\mathcal{F}_i|_{\mathcal{V}_{i,\bar{s}}}[d].$$

From [BeBerDG82, Thm. 4.3.1 (ii)], $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is thus semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$, of length

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) = \sum_{1\leq i\leq m} \mathrm{length}_{\mathrm{Loc}(\mathcal{V}_{i,\bar{s}})}(\mathcal{F}_i|_{\mathcal{V}_{i,\bar{s}}}) = \sum_{1\leq i\leq m} \mathrm{length}_{\mathrm{Loc}(\mathcal{V}_{i,\bar{\eta}})}(\mathcal{F}_i|_{\mathcal{V}_{i,\bar{\eta}}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}).$$

□

Lemma B.2. *Let k be an algebraically closed field of characteristic $\neq \ell$. Let X be an algebraic variety over k, let $j : U \hookrightarrow X$ be an immersion with U smooth over k, of pure dimension d. Let $\mathcal{F} \in \mathrm{Loc}(U)$ and write $\mathcal{P} := j_{!*}\mathcal{F}[d] \in \mathrm{Perv}(X)$. Then*

$$\mathrm{length}_{\mathrm{Loc}(U)}(\mathcal{F}) \leq \mathrm{length}_{\mathrm{Perv}(X)}(\mathcal{P}).$$

Furthermore, $\mathcal{P}$ is semisimple in $\mathrm{Perv}(X)$ if and only if $\mathcal{F}$ is semisimple in $\mathrm{Loc}(U)$, in which case

$$\mathrm{length}_{\mathrm{Loc}(U)}(\mathcal{F}) = \mathrm{length}_{\mathrm{Perv}(X)}(\mathcal{P}).$$

Proof. From [BeBerDG82, Prop. 1.4.26, Lem. 4.3.3], the additive functor $j_{!*} \circ (-)[d] : \mathrm{Loc}(U) \to \mathrm{Perv}(X)$ preserves simple objects, so it preserves semisimple objects. Decompose $j : U \hookrightarrow X$ as $j : U \hookrightarrow Z \overset{\iota}{\hookrightarrow} X$ with $U \hookrightarrow Z$ an open immersion and $\iota : Z \hookrightarrow X$ a closed immersion. As the functor $\iota_* : \mathrm{Perv}(Z) \to \mathrm{Perv}(X)$ is exact and preserves simple objects, it preserves length and semisimple objects. Furthermore, as $\iota^*\iota_* \simeq Id : \mathrm{Perv}(Z) \to \mathrm{Perv}(Z)$, every $\mathcal{P}_Z \in \mathrm{Perv}(Z)$ with $\iota_*\mathcal{P}_Z$ simple (resp. semisimple) in $\mathrm{Perv}(X)$ is simple (resp. semisimple) in $\mathrm{Perv}(Z)$. Indeed, let $\mathcal{P}_Z = \mathcal{P}_{Z,1} \supset \mathcal{P}_{Z,2} \supset \cdots \supset \mathcal{P}_{Z,n} \supset \mathcal{P}_{Z,n+1} = 0$ be a decreasing filtration with $\mathcal{P}_{Z,i}/\mathcal{P}_{Z,i+1}$ simple in $\mathrm{Perv}(Z)$, $i = 1, \ldots, n$ and let $\mathcal{P}_Z^{\mathrm{ss}} := \oplus_{i=1}^n (\mathcal{P}_{Z,i}/\mathcal{P}_{Z,i+1})$ be the semisimplification of $\mathcal{P}_Z$ in $\mathrm{Perv}(Z)$. Then, as $\iota_*\mathcal{P}_Z$ and $\iota_*(\mathcal{P}_Z^{\mathrm{ss}})$ are both semisimple in $\mathrm{Perv}(X)$ with the same factors counted with multiplicites, one as $\iota_*\mathcal{P}_Z \simeq \iota_*(\mathcal{P}_Z^{\mathrm{ss}})$ in $\mathrm{Perv}(X)$ hence $\mathcal{P}_Z \simeq \mathcal{P}_Z^{\mathrm{ss}}$ in $\mathrm{Perv}(Z)$. So to prove Lemma B.2, one may assume $j : U \hookrightarrow X$ is an open immersion. The first part of the assertion then follows from the fact that $j_{!*} \circ (-)[d] : \mathrm{Loc}(U) \to \mathrm{Perv}(X)$ preserves monomorphisms and epimorphisms and Subsection A.2 (2) (a) (ii). For the second part of the assertion, assume $\mathcal{P}$ is semisimple in $\mathrm{Perv}(X)$ and decompose it as a direct sum $\mathcal{P} = \oplus_{i=1}^n \mathcal{P}_i$, with $\mathcal{P}_i$ simple in $\mathrm{Perv}(X)$, $i = 1, \ldots, n$. Then

$$\mathcal{F} = \mathcal{P}|_U[-d] = \oplus_{i=1}^n (\mathcal{P}_i|_U[-d]).$$

As U is smooth over k, of pure dimension d, the shift functor $(-)[d] : \mathrm{Loc}(U) \to \mathrm{Perv}(U)$ is a Serre subcategory [A21, Prop. 3.4.1], hence, for each $i = 1, \ldots, n$, $\mathcal{P}_i|_U[-d] \in Loc(U)$. By [BeBerDG82, Cor 1.4.25], as $\mathcal{P}_i$ is simple in $\mathrm{Perv}(X)$, one has $\mathcal{P}_i = j_{!*}(\mathcal{P}_i|_U)$. Therefore, $\mathcal{P}_i|_U[-d]$ is simple in $\mathrm{Loc}(U)$. □

Remark B.3. Without the semisimplicity assumption on $\mathcal{P}$ in Lemma B.2, one may have

$$\mathrm{length}_{\mathrm{Loc}(U)}(\mathcal{F}) < \mathrm{length}_{\mathrm{Perv}(X)}(\mathcal{P}).$$

For instance, in [dCM09, Example 2.2.5], one has $\mathrm{length}_{\mathrm{Loc}(U)}(\mathcal{F}) = 2$ while $\mathrm{length}_{\mathrm{Perv}(X)}(\mathcal{P}) = 3$.

Remark B.4. The proof of Theorem B.1 is significantly more elementary than the proof of Theorem 4.1 as it avoids Hironaka's desingularization theorem (and as Pink's specialization argument is more elementary that the tame specialization theory of [SGA1, XIII]). On the other hand, Theorem 4.1 is actually stronger as we do not assume that the perverse sheaves $\mathcal{P}_i \in \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$, $i = 1, \ldots, r$ arise from relative perverse sheaves on $f : \mathcal{X} \to S$.

We conclude with an application in characteristic $p > 0$. Let k be an algebraically closed field of characteristic $p > 0$, $\ell \neq p$ be a prime, and let T be a torus over k. In [GLo96, §8], Gabber and Loeser introduce a full subcategory $\mathrm{Hyp}_{\mathrm{int}}(T) \subset \mathrm{Perv}(T)$ of int-hypergeometric geometric sheaves which are built out from elementary sheaves (see [GLo96, Def. 8.1.2] for the precise definition), and give a simple characterization of objects in $\mathrm{Hyp}_{\mathrm{int}}(T)$.

Fact B.5. ([GLo96, Thm. 8.2]) *For every $\mathcal{P} \in \mathrm{Perv}(T)$, $\mathcal{P} \in \mathrm{Hyp}_{\mathrm{int}}(T)$ if and only if $\mathcal{P}$ is irreducible in $\mathrm{Perv}(T)$ with Euler-Poincaré characteristic $\chi(T, \mathcal{P}) = 1$.*

Corollary B.6. *Let S be an integral variety over k with generic point η, and let $f : \mathcal{X} \to S$ a relative torus viz a smooth affine commutative group scheme over S such that for every geometric point $\bar{s}$ on S, $\mathcal{X}_{\bar{s}}$ is a torus. Let $\mathcal{P} \in \mathrm{Perv}(\mathcal{X}/S)$. Then the following are equivalent:*

(i) The set of all $s \in S$ such that $\mathcal{P}|_{\mathcal{X}_{\bar{s}}} \in \mathrm{Hyp}_{\mathrm{int}}(\mathcal{X}_{\bar{s}})$ is Zariski-dense in S;
(ii) After possibly replacing S by a non-empty open subscheme, for every $s \in S$, $\mathcal{P}|_{\mathcal{X}_{\bar{s}}} \in \mathrm{Hyp}_{\mathrm{int}}(\mathcal{X}_{\bar{s}})$;
(iii) $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}} \in \mathrm{Hyp}_{\mathrm{int}}(\mathcal{X}_{\bar{\eta}})$.

Proof. The implication (ii) ⇒ (i) is straightforward and the implication (iii) ⇒ (ii) is by spreading out. We prove (i) ⇒ (iii). Recall that, for every geometric point $\bar{s}$ on S the Euler-Poincaré characteristics with usual cohomology and compact support cohomology coincide [L81b]:

$$\chi(\mathcal{X}_{\bar{s}}, \mathcal{P}) = \chi_c(\mathcal{X}_{\bar{s}}, \mathcal{P}).$$

As $Rf_!\mathcal{P} \in D^b_c(S)$, up to replacing S by a non-empty open subscheme, one may assume that $R^q f_!\mathcal{P} \in \mathrm{Loc}(S)$ for every $q \in \mathbb{Z}$. By the base change theorem [SGA4-III, XVII, Proposition 5.2.8], for every geometric point $\bar{s}$ on S, one has $(R^q f_!\mathcal{P})_{\bar{s}} = \mathrm{H}^q_c(\mathcal{X}_{\bar{s}}, \mathcal{P}|_{\mathcal{X}_{\bar{s}}})$. Thus, the functions

$$S \to \mathbb{Z}_{\geq 0},\ s \mapsto \dim(\mathrm{H}^q_c(\mathcal{X}_{\bar{s}}, \mathcal{P}|_{\mathcal{X}_{\bar{s}}})),\ q \in \mathbb{Z}, \text{ and } S \to \mathbb{Z},\ s \mapsto \chi_c(\mathcal{X}_s, \mathcal{P}|_{\mathcal{X}_s}).$$

are constant. From Theorem B.1, up to replacing further S by a non-empty open subscheme one may assume that the function

$$S \to \mathbb{Z}, \quad s \mapsto \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})} \mathcal{P}|_{\mathcal{X}_{\bar{s}}}$$

is constant as well. The assertion then follows from Fact B.5. □

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