A UNIFORM OPEN IMAGE THEOREM FOR ℓ-ADIC REPRESENTATIONS I.

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ABSTRACT. Let k be a field finitely generated over $\mathbb{Q}$ and let X be a smooth, separated and geometrically connected curve over k. Fix a prime ℓ. A representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is said to be geometrically Lie perfect if the Lie algebra of $\rho(\pi_1(X_{\overline{k}}))$ is perfect. Typical examples of such representations are those arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X or, more generally, from the action of $\pi_1(X)$ on the ℓ-adic etale cohomology groups $\mathrm{H}^i_{et}(Y_{\overline{\eta}}, \mathbb{Q}_\ell)$, $i \geq 0$ of the geometric generic fiber of a smooth proper scheme Y over X. Let G denote the image of ρ. Any k-rational point x on X induces a splitting $x : \Gamma_k := \pi_1(\mathrm{Spec}(k)) \to \pi_1(X)$ of the canonical restriction epimorphism $\pi_1(X) \to \Gamma_k$ so one can define the closed subgroup $G_x := \rho \circ x(\Gamma_k) \subset G$. The main result of this paper is the following uniform open image theorem. Under the above assumptions, for every geometrically Lie perfect representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$, the set X_ρ of all $x \in X(k)$ such that G_x is not open in G is finite and there exists an integer $B_\rho \geq 1$ such that $[G : G_x] \leq B_\rho$ for every $x \in X(k) \smallsetminus X_\rho$.

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1. Introduction

Let ℓ be a prime. A compact ℓ-adic Lie group G is said to be *Lie perfect* if one of the following two equivalent conditions holds:

(i) The Lie algebra $\mathrm{Lie}(G)$ of G is perfect that is its abelianization $\mathrm{Lie}(G)^{ab}$ is trivial;

(ii) For every open subgroup $U \subset G$, the abelianization U^{ab} of U is finite.

Observe that, given an open subgroup $U \subset G$, G is Lie perfect if and only if U is Lie perfect.

Let k be a field and let X be a scheme geometrically connected and of finite type over k. Write $\pi_1(X)$ for the etale fundamental group of X[1]. Then, the structure morphism $X \to \mathrm{Spec}(k)$ induces at the level of etale fundamental groups a short exact sequence of profinite groups (sometimes referred to as the fundamental short exact sequence for $\pi_1(X)$):

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \Gamma_k \to 1.$$

(Here, we identify $\pi_1(\mathrm{Spec}(k))$ with the absolute Galois group Γ_k of k.) An ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is said to be *Lie perfect* (LP for short) if $G := \rho(\pi_1(X)) \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ is Lie perfect, and *geometrically Lie perfect* (GLP for short) if $G^{geo} := \rho(\pi_1(X_{\overline{k}})) \subset G$ is Lie perfect. Note that any subquotient of an LP representation (respectively, of a GLP representation) is again an LP representation (respectively a GLP representation).

Any k-rational point $x : \mathrm{Spec}(k) \to X$ induces a splitting $x : \Gamma_k \to \pi_1(X)$ of the fundamental short exact sequence for $\pi_1(X)$, identifying Γ_k with a closed subgroup of $\pi_1(X)$. Set $G_x := \rho \circ x(\Gamma_k)$ for the corresponding closed subgroup of G.

This paper deals with estimating the dimension of G_x as an ℓ-adic Lie group. From $G = G^{geo} \cdot G_x$, one gets the rough estimates $\dim(G) \geq \dim(G_x) \geq \dim(G) - \dim(G^{geo})$. The main result of this paper is a one-dimensional uniform open image theorem for GLP representations, which ensures that, if X is a curve, then the first inequality is an equality for all but finitely many $x \in X(k)$. More precisely,

Theorem 1.1. *Assume that k is a field finitely generated over $\mathbb{Q}$, that X is smooth of dimension 1 and that $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is a GLP representation. Then the set X_ρ of all $x \in X(k)$ such that*

[1] As the choice of a fiber functor plays no part in the following, we will omit to mention it in our notation for etale fundamental groups.

G_x is not open in G is finite and there exists an integer $B_\rho \geq 1$ such that $[G : G_x] \leq B_\rho$ for every $x \in X(k) \smallsetminus X_\rho$.

As classical examples of GLP representations, let us mention the ones arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X or, more generally, from the action of $\pi_1(X)$ on the ℓ-adic etale cohomology groups $\mathrm{H}^i_{et}(Y_{\overline{\eta}}, \mathbb{Q}_\ell)$, $i \geq 1$ of the geometric generic fiber of a smooth proper scheme Y over X. In the former case, note that when A_η is an elliptic curve, theorem 1.1 follows from ([Se68, Chap. IV, 2.2] and) [A08].

As in the case of [A08], Mordell's conjecture (Faltings' theorem) is one key ingredient of the proof of theorem 1.1. To apply it, the main difficulty to overcome is to show that the genus of certain 'modular' curves $\mathcal{X}_n$ becomes not less than 2 for n large enough. This explains the restriction to 1-dimensional base-schemes X in theorem 1.1. But, anyway, theorem 1.1 clearly does not extend as it is to higher dimensional base schemes[2]. If one believes in the Bombieri-Lang conjecture for the non-density of rational points in varieties of general type over finitely generated fields of characteristic 0 and could prove that the modular schemes $\mathcal{X}_n$ alluded to above become of general type for n large enough then a rough higher-dimensional conjectural generalization of theorem 1.1 could be that the set X_ρ of all $x \in X(k)$ such that G_x is not open in G is not Zariski-dense in X.

The paper is organized as follows. Section 2 generalizes [CT08, Sect. 3] to the case of general ℓ-adic homogeneous spaces for closed subgroups of $\mathrm{GL}_m(\mathbb{Z}_\ell)$; this will be the main ingredient in subsection 3.3. The proof of theorem 1.1 is carried out in section 3. The main group-theoretical tool is described in subsection 3.1 whereas the main geometrical result (theorem 3.4) is stated and proved in subsection 3.3. We conclude the proof of theorem 1.1 in subsection 3.4. In section 4, we prove certain uniform boundedness results (corollary 4.3 (1)(2)) for arbitrary GLP ℓ-adic representations. In section 5, we give classical examples of GLP ℓ-adic representations. In particular, we focus on GLP ℓ-adic representations arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X and deduce uniform boundedness results for the ℓ-primary torsion of abelian varieties varying in one-dimensional families, which strengthen the results of [CT08].

In part II of the present paper, we generalize theorem 1.1 by allowing x to be arbitrary closed point of X with bounded degree.

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2. Reduction modulo ℓ^n of homogeneous spaces for closed subgroups of $\mathrm{GL}_m(\mathbb{Z}_\ell)$

This section generalizes [CT08, Sect. 3] to the case of general ℓ-adic homogeneous spaces for closed subgroups of $\mathrm{GL}_m(\mathbb{Z}_\ell)$.

For every $n \geq 0$, let $(\ \ )_n : \mathrm{GL}_m(\mathbb{Z}_\ell) \to \mathrm{GL}_m(\mathbb{Z}/\ell^n)$ denote the reduction modulo ℓ^n morphism. Given a closed subgroup $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$, write G_n for the image of G by $(\ \ )_n$ and $G(n) := G \cap (Id + \ell^n \mathrm{M}_d(\mathbb{Z}_\ell))$ for the kernel of $(\ \ )_n|_G : G \to G_n$.

Now, let $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ be a closed subgroup (hence, automatically, a compact ℓ-adic Lie group). Let $H \subset G$ be a closed but not open subgroup of G, [3] and, for every open subgroup $U \subset G$, set:

$$K_H(U) := \bigcap_{u \in U} uHu^{-1}$$

[2] Consider for instance $X := A_{2,1,3}$ over $k := \mathbb{Q}(\zeta_3)$, where $A_{d,1,3}$ denotes the fine moduli scheme of principally polarized d-dimensional abelian varieties with full-level-3 structures and ζ_3 is a primitive third root of unity. Let ρ be the natural GLP ℓ-adic representation of $\pi_1(X)$ arising from the (generic) Tate module of the universal abelian scheme over X (see section 5). Then X_ρ contains the image of $(A_{1,1,3} \times A_{1,1,3})(k)$, which is infinite as $Y(3) := A_{1,1,3}$ is a genus 0 curve over k admitting a k-rational point.

[3] The existence of H automatically implies that $\dim(G) > 0$.

for the largest closed subgroup of G contained in H which is normalized by U. Let $I \subset G$ be any closed subgroup of G. Then $I_H := I/I \cap K_H(G)$ acts faithfully on G/H.

We make the following additional assumption:

$$(\#)\ K_H(HG(n)) = K_H(G),\ n \geq 0.$$

This is equivalent to requiring that $K_H(U) = K_H(G)$ for every open subgroup $U \subset G$ containing H. Indeed, for every such U, from $H = \bigcap_{n\geq 0} HG(n)$ and $H \subset U$ one gets:

$$G = U \cup \bigcup_{n\geq 0}(G \setminus HG(n)).$$

Now, it follows from the compactness of G that there exists $n_U \geq 0$ such that $HG(n_U) \subset U$.

Theorem 2.1. *Under the above assumptions, one has:*

$$\lim_{n\mapsto\infty} \frac{|I_n \setminus G_n/H_n|}{|G_n/H_n|} = \frac{1}{|I_H|},$$

where $\frac{1}{\infty} := 0$.

Remark 2.2. Recall that a compact ℓ-adic Lie group is a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_\ell)$ for some integer $m \geq 0$ [L88, Prop. 4] and conversely [Se65, L.G., Chap. V, §9]. Though not required for our purpose, one can give a version of theorem 2.1 which is independent of the embedding $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$. With the notation of theorem 2.1,

$$\lim_{N \underset{\text{open}}{\lhd} G,\ [G:N]\mapsto\infty} \frac{|I_N \setminus G_N/H_N|}{|G_N/H_N|} = \frac{1}{|I_H|}.$$

(Here, note that $K_N(G) = N$, as N is normal in G.) This apparently stronger statement follows from theorem 2.1 since, for every embedding $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ the $G(n)$, $n \geq 0$ form a fundamental system of neighborhoods of 1 hence are cofinal in the open normal subgroups of G.

Proof. We follow the method of [CT08, Sect. 3].

2.1. **Step 1.** Let X be a topological space. We say that a subset $Y \subset X$ is a *thin closed subset (in X)* if it is closed in X and does not contain any nonempty open subset of X. Assumption (#) is there to ensure:

Lemma 2.3. *For every closed subgroup $I \subset G$ such that I_H is non-trivial, $(G/H)^I \subset G/H$ is a thin closed subset in G/H.*

Proof. For every $\gamma \in I$, consider the continuous map: $\Phi_\gamma : G/H \to G/H \times G/H$, $x \mapsto (\gamma x, x)$. Then, by definition,

$$(G/H)^{\langle\gamma\rangle} = \Phi_\gamma^{-1}(\Delta_{G/H}).$$

Thus, $(G/H)^I = \bigcap_{\gamma\in I}(G/H)^{\langle\gamma\rangle}$ is closed in G/H. Now, suppose that $(G/H)^I \subset G/H$ contains an open subset of G/H, that is, there exists an open subgroup $U \subset G$ and $g \in G$ such that $gUH/H \subset (G/H)^I$. Up to replacing U by $K_U(G)H$, one may assume that $H \subset U$. Observe that $gU/H(= gUH/H) \subset (G/H)^I$ is equivalent to $I \subset gK_H(U)g^{-1}$. By assumption (#), one gets $gK_H(U)g^{-1} = gK_H(G)g^{-1} = K_H(G)$. Whence I_H is trivial: a contradiction. □

2.2. **Step 2.**

Lemma 2.4. *For every closed subgroup $I \subset G$ one has*

$$\lim_{n\mapsto\infty} \frac{|(G_n/H_n)^{I_n}|}{|G_n/H_n|} = 0$$

unless I_H is trivial.

Proof. Consider the following canonical commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\pi_H} & G/H \\ \downarrow & & \downarrow \\ G_n & \xrightarrow[\pi_{H_n}]{} & G_n/H_n \end{array}$$

and set:

$$\mathcal{X}_I := \pi_H^{-1}((G/H)^I) = \{g \in G \mid g^{-1}Ig \subset H\}$$

and:

$$\mathcal{X}_{n,I_n} = \pi_{H_n}^{-1}((G_n/H_n)^{I_n}) = \{g_n \in G_n \mid g_n^{-1}I_n g_n \subset H_n\}.$$

Then, since π_{H_n} has fibers with constant cardinality,

$$\frac{|\mathcal{X}_{n,I_n}|}{|G_n|} = \frac{|(G_n/H_n)^{I_n}|}{|G_n/H_n|},$$

so it is enough to prove that:

$$\lim_{n \mapsto \infty} \frac{|\mathcal{X}_{n,I_n}|}{|G_n|} = 0.$$

For this, recall [CT08, Lem. 3.3]:

Lemma 2.5. *Let $A := (A_{n+1} \stackrel{\phi_{n+1,n}}{\to} A_n)_{n \geq 0}$ be a projective system of nonempty finite sets such that for all $n \geq 0$, there exists $d(n) \geq 0$, such that for all $a_n \in A_n$, $|\phi_{n+1,n}^{-1}(a_n)| = d(n)$ and let $B := (B_{n+1} \stackrel{\phi_{n+1,n}}{\to} B_n)_{n \geq 0}$ be a projective subsystem of A. Set $B_\infty := \varprojlim B_n$ and $B_{\infty,n} := p_n(B_\infty) \subset B_n$, where $p_n : B_\infty \to B_n$ is the canonical projection. In particular, $(B_{\infty,n+1} \stackrel{\phi_{n+1,n}}{\to} B_{\infty,n})_{n \geq 0}$ is a projective subsystem of B such that $B_\infty = \varprojlim B_{\infty,n}$. Then*

(1) *$m(B) := \lim_{n \to \infty} \frac{|B_n|}{|A_n|}$ and $m_\infty(B) := \lim_{n \to \infty} \frac{|B_{\infty,n}|}{|A_n|}$ exist and belong to $[0,1]$.*

(2) *$m(B) = m_\infty(B)$.*

We will apply lemma 2.5 to $A = (G_{n+1} \stackrel{\pi_{n+1,n}}{\to} G_n)_{n \geq 0}$ ($d(n) = |\ker(\pi_{n+1,n})|$) and $B = (\mathcal{X}_{n+1,I_{n+1}} \stackrel{\pi_{n+1,n}}{\to} \mathcal{X}_{n,I_n})_{n \geq 0}$. As:

$$\varprojlim \mathcal{X}_{n,I_n} = \mathcal{X}_I$$

one thus gets:

$$\lim_{n \mapsto \infty} \frac{|\mathcal{X}_{n,I_n}|}{|G_n|} = \lim_{n \mapsto \infty} \frac{|\mathcal{X}_{I,n}|}{|G_n|}.$$

Lemma 2.6. *$\mathcal{X}_I \subset G$ is a closed ℓ-adic analytic subset.*

Proof Since $H \subset G$ is a closed subgroup of G (hence an ℓ-adic Lie subgroup) the homogeneous space G/H is an ℓ-adic analytic manifold and the quotient morphism $\pi_H : G \twoheadrightarrow G/H$ is ℓ-adic analytic. So, it is enough to prove that $(G/H)^{I_H} \subset G/H$ is a closed ℓ-adic analytic subset of G/H. But, as I is also a compact ℓ-adic Lie group, it is (topologically) finitely generated. Let $\gamma_1, \dots, \gamma_r$ be a finite set of generators. Then, by definition,

$$(G/H)^{I_H} = (\Phi_{\gamma_1}, \dots, \Phi_{\gamma_r})^{-1}(\Delta_{G/H} \times \cdots \times \Delta_{G/H}).$$

So, since $(\Phi_{\gamma_1}, \dots, \Phi_{\gamma_r}) : G/H \to (G/H \times G/H)^r$ is ℓ-adic analytic and $\Delta_{G/H} \subset G/H \times G/H$ is a closed ℓ-adic analytic subset of G/H, we are done. □

Now, from lemma 2.3, $(G/H)^{I_H} \subset G/H$ is a thin closed subset in G/H so, by definition of the quotient topology, $\mathcal{X}_I \subset G$ is also a thin closed subset in G; in particular $\dim(\mathcal{X}_I) < \dim(G)$, where dim stands for the dimension as ℓ-adic analytic space. Since $\mathrm{GL}_m(\mathbb{Z}_\ell)$ can be regarded as a closed ℓ-adic submanifold of $\mathbb{Z}_\ell^{d^2+1}$ defined by the equation $Y\det(X_{i,j}) - 1 = 0$ and since reduction modulo

ℓ^n of $\mathrm{GL}_m(\mathbb{Z}_\ell)$ as a submanifold of $\mathbb{Z}_\ell^{d^2+1}$ coincides with reduction modulo ℓ^n of $\mathrm{GL}_m(\mathbb{Z}_\ell) \subset \mathrm{M}_d(\mathbb{Z}_\ell)$, [O82] yields:

$$\frac{|\mathcal{X}_{n,I_n}|}{|G_n|} = O(\ell^{n(\dim\ (\mathcal{X}_I) - \dim\ (G))}),$$

whence the announced result.

2.3. **Step 3.** Given any finite group F, write $\mathcal{M}(F)$ for the (finite) set of nontrivial minimal subgroups of F. Equivalently, $\mathcal{M}(F)$ is the set of cyclic subgroups of F with prime order. (Note that $\mathcal{M}(F) = \emptyset$ if and only if $F = \{1\}$). Set $(G_n/H_n)' := \bigcup_{J \in \mathcal{M}((I_n)_{H_n})} (G_n/H_n)^J$, then one has

$$\frac{1}{|(I_n)_{H_n}|}\left(1 - \frac{|(G_n/H_n)'|}{|G_n/H_n|}\right) \leq \frac{|(I_n)_{H_n}\backslash G_n/H_n|}{|G_n/H_n|} \leq \frac{1}{|(I_n)_{H_n}|}\left(1 - \frac{|(G_n/H_n)'|}{|G_n/H_n|}\right) + \frac{|(G_n/H_n)'|}{|G_n/H_n|}.$$

If I_H is finite, then I_H is isomorphic to $(I_n)_{H_n}$ for $n \gg 0$, so

$$0 \leq \frac{|(G_n/H_n)'|}{|G_n/H_n|} \leq \sum_{J \in \mathcal{M}(I_H)} \frac{|(G_n/H_n)^J|}{|G_n/H_n|} \to 0\ (n \to \infty)$$

by lemma 2.4. As $|(I_n)_{H_n}| \to |I_H|$, this yields the desired conclusion.

If I_H is infinite, then I_H admits a torsion-free closed pro-ℓ cyclic subgroup $\langle \gamma \rangle \simeq \mathbb{Z}_\ell$. Since $|I_n \setminus G_n/H_n| \leq |\langle \gamma_n \rangle \setminus G_n/H_n|$, up to replacing I_H by such a subgroup, one may assume $I_H \simeq \mathbb{Z}_\ell$. Fix any $N \geq 0$ and set $J = I_H^{\ell^N} \subset I_H$. Then, for every $x_n \in G_n/H_n$ with stabilizer $(I_H)_{x_n}$ under I_H, note that $x_n \notin (G_n/H_n)^J$ if and only if $J \not\subset (I_H)_{x_n}$. But, since closed subgroups of $I_H \simeq \mathbb{Z}_\ell$ are totally ordered for $\subset$, the latter is also equivalent to $(I_H)_{x_n} \subsetneq J = I_H^{\ell^N}$, or $(I_H)_{x_n} \subset I_H^{\ell^{N+1}}$. So, $x_n \notin (G_n/H_n)^J$ implies $|I_H \cdot x_n| = [I_H : (I_H)_{x_n}] \geq [I_H : I_H^{\ell^{N+1}}] = \ell^{N+1}$. Now,

$$0 \leq \frac{|(I_n)_{H_n}\backslash G_n/H_n|}{|G_n/H_n|} \leq \frac{|(I_n)_{H_n}\backslash (G_n/H_n \smallsetminus (G_n/H_n)^J)|}{|G_n/H_n|} + \frac{|(G_n/H_n)^J|}{|G_n/H_n|} \leq \frac{1}{\ell^{N+1}} + \frac{|(G_n/H_n)^J|}{|G_n/H_n|}.$$

By lemma 2.4, one has $\frac{|(G_n/H_n)^J|}{|G_n/H_n|} \to 0$, so

$$0 \leq \lim_{n\to\infty} \frac{|(I_n)_{H_n}\backslash G_n/H_n|}{|G_n/H_n|} \leq \frac{1}{\ell^{N+1}},$$

which yields the desired conclusion as $N \geq 0$ is arbitrary. $\square$

3. Proof of theorem 1.1

3.1. **The projective system** $(\mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G))_{n\geq 0}$. Let $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ be a closed subgroup. We are going to associate with G a projective system $(\mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G))_{n\geq 0}$ of finite sets of open subgroups of G. This construction will play a crucial part in our proof of theorem 1.1.

3.1.1. *Group-theoretical preliminaries.* Given a profinite group G, write $\Phi(G)$ for its Frattini subgroup (that is the intersection of all maximal open subgroups of G).

Lemma 3.1. *Let $K \lhd G$ be a normal closed subgroup. Then, for every closed subgroup $H \subset G$ such that $H\Phi(K) \supset K$ one has $H \supset K$.*

Proof. Just observe that $H\Phi(K) \supset K$ is equivalent to $(H\Phi(K)) \cap K = K$ but, since $\Phi(K) \subset K$, $(H\Phi(K)) \cap K = (H \cap K)\Phi(K)$. Hence it follows from the Frattini property [RZ00, Lem. 2.8.1] that $H \cap K = K$ or, equivalently, that $H \supset K$. $\square$

Lemma 3.2. $\Phi(G(n)) = G(n+1)$, $n \gg 0$.

Proof. Since $G(n)/G(n+1) \subset (Id + \ell^n \mathrm{M}_d(\mathbb{Z}_p))/(Id + \ell^{n+1}\mathrm{M}_d(\mathbb{Z}_\ell))$, $G(n)/G(n+1)$ is an elementary abelian ℓ-group. Since $G(n)$ is a pro-ℓ group, $\Phi(G(n)) = G(n)^\ell[G(n), G(n)]$ [RZ00, Lem. 2.8.7 (c)] hence $\Phi(G(n)) \subset G(n+1)$.

For the converse inclusion, recall that the $\mathbb{F}_\ell$-dimension of $G(n)/\Phi(G(n))$ is the minimal number $r(G(n))$ of generators of $G(n)$ [DSMS91, 0.9, p.13]. Also, since $G \subset \mathbb{Z}_\ell^{d^2+1}$ is a smooth closed ℓ-adic analytic subspace - say of dimension $\delta > 0$, it follows from [Se81b, Rem. 1, p. 148] that there exists a constant $0 < \mu(G) \leq 1$ such that $[G : G(n)] = |G_n| = \mu(G)\ell^{n\delta}$ for $n \gg 0$. As a result, the $\mathbb{F}_\ell$-dimension of $G(n)/G(n+1)$ is δ for $n \gg 0$ hence $\delta \leq r(G(n))$ for $n \gg 0$. On the other hand, $r(G(n)) \leq \delta$ for $n \gg 0$. Indeed, since G is a compact ℓ-adic Lie group, it contains a uniformly powerful open pro-ℓ subgroup $G_0 \subset G$ [DSMS91, Thm. 9.34, p. 194]. Then, for every open subgroup $H \subset G_0$, one has the following inequalities:

$$r(H) \overset{(1)}{\leq} r(G_0) \overset{(2)}{=} \dim(G_0) = \delta,$$

where (1) follows from [DSMS91, Thm. 3.8, p. 54] and (2) follows from [DSMS91, Thm. 9.38, p. 195]. Thus $r(G(n)) = \delta$ and $\Phi(G(n)) = G(n+1)$ for $n \gg 0$. □

3.1.2. *The projective system* $(\mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G))_{n\geq 0}$. For each $n \geq 1$, let $\mathcal{H}_n(G)$ denote the set of all open subgroups $U \subset G$ such that $\Phi(G(n-1)) \subset U$ but $G(n-1) \not\subset U$ and set $\mathcal{H}_0(G) := \{G\}$. Then the $\mathcal{H}_n(G)$, $n \geq 0$ satisfy the following elementary properties:

Lemma 3.3. (1) *$\mathcal{H}_n(G)$ is finite, $n \geq 0$.*

(2) *The maps $\phi_n : \mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G)$, $U \mapsto U\Phi(G(n-1))$ (with the convention that $\Phi(G(-1)) = G$) endow the $\mathcal{H}_n(G)$, $n \geq 0$ with a canonical structure of projective system $(\mathcal{H}_{n+1}(G) \overset{\phi_n}{\to} \mathcal{H}_n(G))_{n\geq 0}$.*

(3) *For every $\underline{H} := (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_n(G)$,*

$$H[\infty] := \varprojlim H[n] = \bigcap_{n\geq 0} H[n] \subset G$$

is a closed but not open subgroup of G.

(4) *For every closed subgroup $H \subset G$ such that $G(n-1) \not\subset H$ there exists $U \in \mathcal{H}_n(G)$ such that $H \subset U$.*

(5) *For $n \gg 0$, $\mathcal{H}_n(G)$ is the set of all open subgroups $U \subset G$ such that $G(n) \subset U$ but $G(n-1) \not\subset U$.*

Proof. For (1), by definition, $\mathcal{H}_n(G)$ is contained in the set of all open subgroups $U \subset G$ such that $\Phi(G(n-1)) \subset U$. But since $G(n-1)$ is open in G, $\Phi(G(n-1))$ is open in G as well (cf. lemma 3.2). Thus, $\mathcal{H}_n(G)$ can be identified with a set of subgroups of the finite group $G/\Phi(G(n-1))$, whence the assertion follows. For (2), assume that for some $U \in \mathcal{H}_{n+1}(G)$, $U\Phi(G(n-1)) \notin \mathcal{H}_n(G)$. Since $U\Phi(G(n-1)) \supset \Phi(G(n-1))$, this means that $U\Phi(G(n-1)) \supset G(n-1)$. But then, from lemma 3.1, $U \supset G(n-1) \supset G(n)$, which contradicts the definition of $\mathcal{H}_{n+1}(G)$. For (3), let $\underline{H} = (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_n(G)$, then $H[\infty] \subset H[n]$ and, in particular, $G(n-1) \not\subset H[\infty]$, $n \geq 0$. Thus, H cannot be open in G, since the $G(n)$, $n \geq 0$ form a fundamental system of neighborhoods of 1 in G. For (4), it follows again from lemma 3.1 that for every closed subgroup $H \subset G$ such that $G(n-1) \not\subset H$, $U := H\Phi(G(n-1)) \in \mathcal{H}_n(G)$. Finally, (5) follows straightforwardly from lemma 3.2. □

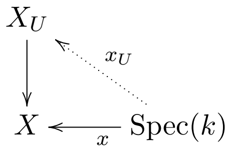
3.2. **Preliminary remarks.** From now on and till the end of section 3, we let k be a field of characteristic 0, X a smooth, separated, geometrically connected curve over k, and $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ a GLP representation. We retain the notation of the introduction. In particular, we set $G := \rho(\pi_1(X))$ and $G^{geo} := \rho(\pi_1(X_{\overline{k}}))$.

(1) Recall that the $G(n)$, $n \geq 0$ form a fundamental system of open neighborhoods of 1 in G. So, for every open subgroup $U \subset G$ such that $G(n) \subset U$ one has $[G : U] \leq [G : G(n)] =: B_n$. Conversely, since G is a finitely generated profinite group, for every integer $B \geq 1$ the set $S(G, B)$ of all open subgroups $U \subset G$ with $[G : U] \leq B$ is finite. So:

$$\bigcap_{U \in S(G,B)} U \subset G$$

is again an open subgroup of G, hence contains $G(n_B)$ for some integer $n_B \geq 0$. As a result, the second assertion of theorem 1.1 is also equivalent to the following: *There exists an integer* $n_\rho \geq 0$ *such that* $G(n_\rho) \subset G_x$, $x \in X(k) \smallsetminus X_\rho$.

(2) Notation: For every open subgroup $U \subset G$ let $X_U \to X$ denote the corresponding etale cover; it is defined over a finite extension k_U of k and it satisfies the following two properties:
   (a) $X_U \times_{k_U} \overline{k} \to X_{\overline{k}}$ is the etale cover corresponding to the inclusion of open subgroups $G^{geo} \cap U \subset G^{geo}$.
   (b) For every k-rational point $x : \mathrm{Spec}(k) \to X$, $G_x \subset U$ (up to conjugacy) if and only if $x : \mathrm{Spec}(k) \to X$ lifts to a k-rational point:



Write g_U for the genus of (the smooth compactification of) X_U. It follows from (a) that $g_U = g_{G^{geo} \cap U}$.

3.3. **A key geometric statement and its corollaries.** The key point to conclude the proof of theorem 1.1 is the following geometric statement:

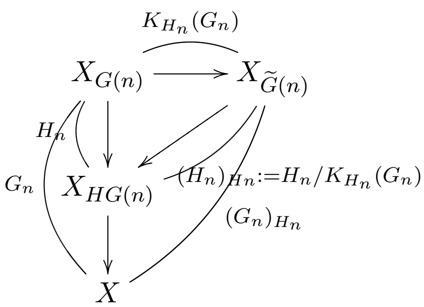
Theorem 3.4. *For every* $H \subset G^{geo}$ *closed but not open subgroup of* G^{geo} *one has*

$$\lim_{n \mapsto \infty} g_{HG^{geo}(n)} = \infty.$$

The strategy of the proof of theorem 3.4 is as follows. First, up to replacing k by $\overline{k}$, we shall assume, without loss of generality, that k is algebraically closed, till the end of subsection 3.3.2. Thus, in particular, $G = G^{geo}$. Let $H \subset G$ be a closed but not open subgroup of G. Reduction modulo ℓ^n induces a canonical morphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_H(G) & \longrightarrow & G & \longrightarrow & G/K_H(G) =: G_H & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_{H_n}(G_n) & \longrightarrow & G_n & \longrightarrow & G_n/K_{H_n}(G_n) =: (G_n)_{H_n} & \longrightarrow & 1. \end{array}$$

Recall that $G(n) \subset G$ denotes the kernel of the reduction modulo ℓ^n-morphism $G \to G_n$. Let $\widetilde{G}(n) \subset G$ and $G_H(n) \subset G_H$ denote the kernels of the morphism $G \to (G_n)_{H_n}$ and the morphism $G_H \to (G_n)_{H_n}$, respectively; since $K_{H_n}(G_n) \subset H_n$, one has $\widetilde{G}(n) \subset HG(n)$. Then, the inclusion of open subgroups $G(n) \subset \widetilde{G}(n) \subset HG(n) \subset G$ corresponds to a commutative diagram of finite etale covers:



First, using that $X_{\widetilde{G}(n)} \to X$ is Galois and that G is Lie perfect we will show that $g_{\widetilde{G}(n)} \to \infty$. Then, resorting to the Riemann-Hurwitz formula, we will compare $g_{\widetilde{G}(n)}$ and $g_{HG(n)}$. It will follow from the fact that $H \subset G$ is closed but not open in G and from theorem 2.1 that, actually, $g_{HG(n)} \to \infty$ if and only if $g_{\widetilde{G}(n)} \to \infty$.

3.3.1. $g_{\widetilde{G}(n)} \to \infty$. Since (i) H is closed but not open in G (so, in particular, G is not finite) and (ii) by definition $K_H(G) \subset H$, one has $|(G_n)_{H_n}| \to \infty$. Hence, by the Riemann-Hurwitz formula, $\sup\{g_{\widetilde{G}(n)}\} < \infty$ if and only if $\sup\{g_{\widetilde{G}(n)}\} \leq 1$.

If $\sup\{g_{\widetilde{G}(n)}\} = 1$, then there exists $n_1 \geq 0$, such that $g_{\widetilde{G}(n)} = 1$, $n \geq n_1$. Then the smooth compactification $X^{cpt}_{\widetilde{G}(n)}$ of $X_{\widetilde{G}(n_1)}$ is an elliptic curve and $G_H(n_1)$ is a quotient of $\pi_1(X^{cpt}_{\widetilde{G}(n_1)})$ ($\twoheadleftarrow \hat{\mathbb{Z}}^2$). But then $G_H(n_1)$ would be an infinite quotient of $\widetilde{G}(n_1)^{ab}$, which contradicts the fact that G is Lie perfect.

If $\sup\{g_{\widetilde{G}(n)})\} = 0$, then $X_{\widetilde{G}(n)} \to X$ is a Galois cover of genus 0 curves with degree $|(G_n)_{H_n}| \to \infty$. So, according to the classification of finite subgroups of PGL_2, there exists $n_0 \geq 0$, such that $G_H(n_0) \simeq \mathbb{Z}_\ell$ (we refer to [CT08, §4.1.3] for more details), which, again, contradicts the fact that G is Lie perfect.

3.3.2. *End of the proof of theorem 3.4.* For every connected finite etale cover $Y \to X$ of degree d, set $\lambda(Y \to X) := \dfrac{2g(Y)-2}{d}$, where $g(Y)$ stands for the genus of the smooth compactification of Y. Then, if $Z \to Y \to X$ are connected finite etale covers, it follows from the Riemann-Hurwitz formula for $Z \to Y$ that $\lambda(Z \to X) \geq \lambda(Y \to X)$.

Now, for each open subgroup $U \subset G$, set $\lambda_U := \lambda(X_U \to X)$. Thus, one has $\lambda_{\widetilde{G}(n+1)} \geq \lambda_{\widetilde{G}(n)}$, $\lambda_{HG(n+1)} \geq \lambda_{HG(n)}$ and $\lambda_{\widetilde{G}(n)} \geq \lambda_{HG(n)}$, $n \geq 0$. In particular, $\lambda := \lim_{n\to\infty} \lambda_{\widetilde{G}(n)}$ and $\lambda_H := \lim_{n\to\infty} \lambda_{HG(n)}$ exist and $\lambda \geq \lambda_H$.

As already noticed, $|(G_n)_{H_n}| \to \infty$ and, as well, $|(G_n)_{H_n}/(H_n)_{H_n}| = |G_n/H_n| \to \infty$. So, $g_{\widetilde{G}(n)} \to \infty$ if and only if $\lambda > 0$, and $g_{HG(n)} \to \infty$ if and only if $\lambda_H > 0$. But from paragraph 3.3.1, $\lambda > 0$ holds. Thus, it is enough to prove that $\lambda = \lambda_H$.

Let g denote the genus of the smooth compactification X^{cpt} of X, $X^{cpt} \setminus X := \{P_1, \dots, P_r\}$ denote the point at infinity on X and $I_1, \dots, I_r$ (resp. $I_{1,n}, \dots, I_{r,n}$) the images of the corresponding inertia groups in G (resp. G_n). By the Riemann-Hurwitz formula, one has:

$$\lambda_{\widetilde{G}(n)} = 2g - 2 + \sum_{1\leq i\leq r} (1 - \frac{1}{|(I_{i,n})_{H_n}|})$$

and

$$\lambda_{HG(n)} = 2g - 2 + \sum_{1\leq i\leq r} (1 - \epsilon_i(n)),$$

where:

$$\epsilon_i(n) = \frac{|I_{i,n} \backslash G_n/H_n|}{|G_n/H_n|} (= \frac{|(I_{i,n})_{H_n} \backslash (G_n)_{H_n}/(H_n)_{H_n}|}{|(G_n)_{H_n}/(H_n)_{H_n}|}),\ i = 1, \dots, r.$$

So,

$$\lambda_{\widetilde{G}(n)} - \lambda_{HG(n)} = \sum_{1\leq i\leq r} (\epsilon_i(n) - \frac{1}{|(I_{i,n})_{H_n}|}).$$

Thus the conclusion will follow from theorem 2.1 provided assumption (#) is satisfied.

Lemma 3.5. *Let $H_0 \subset H_1 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots \subset G$ be an increasing sequence of closed subgroups of G. Assume that for every $0 \leq m \leq n$ H_m is normal in H_n. Then, the sequence $H_0 \subset H_1 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots \subset G$ stabilizes.*

Proof The dimension of the H_n as ℓ-adic analytic spaces stabilizes for $n \gg 0$ or, equivalently, H_n is open in H_{n+1} for $n \gg 0$. So, one may assume without loss of generality that H_0 is open in H_n, $n \geq 0$. Let $K \subset G$ denote the normalizer of H_0 in G. Since H_0 is closed in G, K is closed in G as well and, by assumption, $H_n \subset K$, $n \geq 0$. As K/H_0 is an ℓ-adic analytic group, it contains a torsion-free open normal subgroup. Thus, the order of finite subgroups of K/H_0 is bounded, hence the sequence $H_1/H_0 \subset \cdots \subset H_n/H_0 \subset H_{n+1}/H_0 \subset \cdots \subset K/H_0$ stabilizes. $\square$

See the appendix for a generalization of lemma 3.5 to any increasing sequence of closed subgroups of G.

From lemma 3.5 applied to the increasing sequence

$$K_H(G) \subset \cdots \subset K_H(HG(n)) \subset K_H(HG(n+1)) \subset \cdots \subset G,$$

one gets that $K_H(HG(n)) = K_H(HG(n+1))$, $n \gg 0$. But as we are only interested in the asymptotic behavior of $g_{HG(n)}$, we can replace X by $X_{HG(n)}$ and G by $HG(n)$ (which is still Lie perfect) for some $n \geq 0$. So up to renumbering, one may assume without loss of generality that $K_H(G) = K_H(HG(n))$, $n \geq 0$. $\square$

3.3.3. *Corollaries to theorem 3.4.*

Corollary 3.6. *For every $\underline{H} = (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_n(G^{geo})$, one has $g_{H[n]} \to \infty$ $(n \to \infty)$.*

Proof. Set

$$H[\infty] := \varprojlim H[n] = \bigcap_{n\geq 0} H[n] \subset G^{geo},$$

which is a closed but not open subgroup of G^{geo} by lemma 3.3. By theorem 3.4, one has $g_{H[\infty]G^{geo}(n)} \to \infty$ $(n \to \infty)$. Thus, it suffices to prove that $H[n] = H[\infty]G^{geo}(n)$ for $n \gg 0$.

By the definition of the projective system $(\mathcal{H}_{n+1}(G^{geo}) \to \mathcal{H}_n(G^{geo}))_{n\geq 0}$ and by lemma 3.2, one has

$$H[n] = H[n+1]\Phi(G^{geo}(n-1)) = H[n+1]G^{geo}(n),$$

$n \gg 0$. Since $G^{geo}(n) \supset G^{geo}(n+1)$, iterating this, one gets

$$H[n] = H[N]G^{geo}(n),\ N \geq n \gg 0.$$

Accordingly, one concludes

$$H[n] = \bigcap_{N\geq n} (H[N]G^{geo}(n)) = (\bigcap_{N\geq n} H[N])G^{geo}(n) = H[\infty]G^{geo}(n).$$

Here, the second equality follows from the exactness property of inverse limits of profinite groups (see, e.g., [RZ00, Prop. 2.2.4]). More precisely, since $H[N] \to H[N]G^{geo}(n)/G^{geo}(n) \subset G_n^{geo}$ is surjective for each $N \geq n$, the resulting morphism

$$\bigcap_{N\geq n} H[N] \to \bigcap_{N\geq n} (H[N]G^{geo}(n)/G^{geo}(n)) = (\bigcap_{N\geq n} H[N]G^{geo}(n))/G^{geo}(n)$$

is also surjective. $\square$

Corollary 3.7. *For every integer $c \geq 0$, there exists an integer $N_\rho(c) \geq 0$ such that for every $n \geq N_\rho(c)$ and any $U \in \mathcal{H}_n(G^{geo})$, one has $g_U \geq c$.*

Proof. Else, the subset $\mathcal{H}_{n,<c}(G^{geo}) \subset \mathcal{H}_n(G^{geo})$ of all $U \in \mathcal{H}_n(G^{geo})$ such that $g_U < c$ is non-empty, $n \geq 0$, hence $\varprojlim \mathcal{H}_{n,<c}(G^{geo})$ is non-empty as well. But for every $\underline{H} = (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_{n,<c}(G^{geo})$, $g_{H[n]} \to \infty$ by corollary 3.6: a contradiction. $\square$

Corollary 3.8. *For every integers $c_1 \geq 0$, $c_2 \geq 1$, there exists an integer $N_\rho(c_1, c_2) \geq 0$ such that for every $n \geq N_\rho(c_1, c_2)$ and any $U \in \mathcal{H}_n(G)$, either $g_U \geq c_1$ or $[k_U : k] \geq c_2$.*

Proof. First, by lemma 3.3(5) for G and G^{geo}, there exists an integer $N_\rho > 0$ such that for every $n \geq N_\rho$:

$$\mathcal{H}_n(G) = \{U \subset G \mid G(n) \subset U,\ G(n-1) \not\subset U\},$$
$$\mathcal{H}_n(G^{geo}) = \{U \subset G^{geo} \mid G^{geo}(n) \subset U,\ G^{geo}(n-1) \not\subset U\}.$$

Second, by corollary 3.7, there exists an integer $N_\rho(c_1) \geq N_\rho$ such that for every $n \geq N_\rho(c_1)$ and any $U \in \mathcal{H}_n(G^{geo})$, one has $g_U \geq c_1$. Third, as noticed in remark (1) of subsection 3.2, there exists an integer $N_\rho(c_1, c_2) \geq N_\rho(c_1)$ such that for every open subgroup $U \subset G$ with $[G : U] < c_2[G^{geo} : G^{geo}(N_\rho(c_1) - 1)]$, one has $G(N_\rho(c_1, c_2) - 1) \subset U$.

Now, for every $n \geq N_\rho(c_1, c_2)$ and any $U \in \mathcal{H}_n(G)$, set $U^{geo} = U \cap G^{geo}$. Recall that $U \in \mathcal{H}_n(G)$ is equivalent to saying that $G(n) \subset U$ and $G(n-1) \not\subset U$. Since $G(n) \subset U$, one has $G^{geo}(n) \subset U^{geo}$. Let n_0 be the minimal integer $\geq N_\rho(c_1) - 1$ such that $G^{geo}(n_0) \subset U^{geo}$. If $n_0 \geq N_\rho(c_1)$, then one has $G^{geo}(n_0 - 1) \not\subset U^{geo}$, hence $U^{geo} \in \mathcal{H}_{n_0}(G^{geo})$. Then one has $g_U = g_{U^{geo}} \geq c_1$ by the definition of $N_\rho(c_1)$. Else, $n_0 = N_\rho(c_1) - 1$, that is, $G^{geo}(N_\rho(c_1) - 1) \subset U^{geo}$. Since $G(n-1) \not\subset U$, one has $G(N_\rho(c_1, c_2) - 1) \not\subset U$, a fortiori. Thus, by the definition of $N_\rho(c_1, c_2)$, one has $[G : U] \geq c_2[G^{geo} : G^{geo}(N_\rho(c_1) - 1)]$, hence

$$[k_U : k] = \frac{[G : U]}{[G^{geo} : U^{geo}]} \geq c_2 \frac{[G^{geo} : G^{geo}(N_\rho(c_1) - 1)]}{[G^{geo} : U^{geo}]} \geq c_2,$$

as desired. □

Corollary 3.9 is a reformulation of theorem 3.4 stressing its interest independently of the technical purpose for which we proved it.

Corollary 3.9. *Let k be an algebraically closed field of characteristic 0 and let K/k be a function field of transcendence degree 1. Let L/K be a Galois extension with group G such that:*
(i) G is an ℓ-adic Lie group and $\mathrm{Lie}(G)^{ab} = 0$*;*
(ii) L/K is ramified only over a finite number of places of K.
Then, for every $g \geq 0$, there are only finitely many finite subextensions K'/K of L/K with genus $\leq g$.

Proof. By the first assertion of (i), we may assume that G is a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_\ell)$ for some $m \geq 0$ (cf. remark 2.2). By (ii), G can be regarded as a quotient of $\pi_1(X)$, where X is a suitable (smooth, separated, connected) curve over k whose function field is K. Thus, G can be regarded as the image of an ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$, which is LP by the second assertion of (i). Now, the statement of corollary 3.9 follows from corollary 3.7 and the already mentioned facts:
- There exists an integer $N \gg 0$ such that $G(n+1) = \Phi(G(n))$, $n \geq N$;
- For every open subgroup $U \subset G$, there exists a unique integer $n_U \geq 0$ such that $G(n_U + 1) \subset U$ but $G(n_U) \not\subset U$;
- For each integer $N \geq 0$, there are only finitely many open subgroups $U \subset G$ such that $G(N) \subset U$,
which show that $\bigcup_{n \geq 0} \mathcal{H}_n(G)$ contains all except possibly finitely many open subgroups $U \subset G$. □

3.4. **End of the proof of theorem 1.1.** Now, assume that k is finitely generated over $\mathbb{Q}$. Set:

$$\mathcal{X}_n := \coprod_{U \in \mathcal{H}_n(G)} X_U.$$

Then $(\mathcal{X}_{n+1} \to \mathcal{X}_n)_{n \geq 0}$ is a projective system of (possibly disconnected) etale covers with transition morphisms induced by the maps $\phi_n : \mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G)$, $n \geq 0$. But, from corollary 3.8, there exists an integer $N \geq 0$ such that for every $U \in \mathcal{H}_n(G)$ either $g_U \geq 2$ or $[k_U : k] \geq 2$, $n \geq N$. If $[k_U : k] \geq 2$ then, clearly, $X_U(k) = \emptyset$ and, if $g_U \geq 2$ then, by Mordell's conjecture for fields finitely generated over $\mathbb{Q}$ [FW92, Chap. VI, Thm. 3], $X_U(k)$ is finite. As a result, $\mathcal{X}_n(k)$ is finite for all $n \geq N$. Let $X_{\rho,N}$ denote the image of $\mathcal{X}_N(k)$ in $X(k)$ then:
- $X_{\rho,N}$ is finite since $\mathcal{X}_N(k)$ is;
- No $x \in X(k) \setminus X_{\rho,N}$ lifts to a k-rational point on $\mathcal{X}_N$. So, by the definition of N, $G_x \not\subset U$, for every $U \in \mathcal{H}_N(G)$. But then, by lemma 3.3 (4), $G(N-1) \subset G_x$.

So, $X_\rho \subset X_{\rho,N}$ and, in particular, X_ρ is finite. Finally, by the definition of X_ρ, for each $x \in X(k) \smallsetminus X_\rho$, G_x is open in G, or, equivalently, there exists an integer N_x such that $G(N_x) \subset G_x$. Set $n_\rho := \max\{N, N_x\ (x \in X_{\rho,N} \smallsetminus X_\rho)\}$. Then, for each $x \in X(k) \smallsetminus X_\rho$, one has $G(n_\rho) \subset G_x$, as desired. □

Actually, by lemma 3.3(4), X_ρ coincides with the image of $\varprojlim \mathcal{X}_n(k)$ in $X(k)$.

Remark 3.10. (Characteristic 0 hypothesis) Though we have not checked the details, it is highly probable that theorem 3.4 remains true as it is under the assumption that k has positive characteristic $p \neq \ell$. The only difficulty may come from higher ramification terms in the Riemann-Hurwitz formula which, in the ℓ-adic situation ($p \neq \ell$), should be easy to control just as in [CT08, §4.1.4].

On the contrary, it is not clear that theorem 1.1 extends to positive characteristic $p \neq \ell$. Indeed, one main additional ingredient in the proof of theorem 1.1 is Mordell's conjecture for finitely generated fields of characteristic 0. The analog in positive characteristic $p > 0$ is Samuel's theorem [Sa66], which only ensures finiteness of the set of k-rational points under a certain additional non-isotriviality assumption on the smooth compactification of the curve. In [CT09] (which deals with the special case of bounding uniformly the ℓ-primary torsion in the special fibers of abelian schemes over curves - see also section 5.1 below) we are able to treat the isotrivial case by a specialization argument. More precisely, we prove that if the $\mathcal{X}_n(k)$ are non-empty and have an isotrivial connected component for all $n \geq 0$ then we can reduce (in the restricted setting of [CT09]) to the case that k is finite and that the $\mathcal{X}_n(k)$ are non-empty (hence finite non-empty) for all $n \geq 0$. This imposes that $\varprojlim \mathcal{X}_n(k) \neq \emptyset$, which yields a contradiction in the restricted setting of [CT09] but not in the setting of theorem 1.1.

3.5. **Counterexamples.** Considering the projective system $(\mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G))_{n\geq 1}$, we have proved simultaneously:

1. X_ρ is finite;
2. there exists $N_\rho \geq 0$ such that $G(N_\rho) \subset G_x$ for all $x \in X(k) \smallsetminus X_\rho$.

Remark 3.11. Define $\mathcal{H}_n^+(G)$ to be the set of all open subgroup $U \subset G$ which can be written as $U = H\Phi(G(n-1))$ for some closed but not open subgroup $H \subset G$. Then, just by definition, $(\mathcal{H}_{n+1}^+(G) \stackrel{\phi_n}{\to} \mathcal{H}_n^+(G))_{n\geq 0}$ forms a projective system, where $\phi_n : U \mapsto U\Phi(G(n-1))$. Now, just to prove (1), it is enough to consider this projective system, instead of the projective system $(\mathcal{H}_{n+1}(G) \stackrel{\phi_n}{\to} \mathcal{H}_n(G))_{n\geq 0}$. Then the proof gets slightly easier, since then we do not need to resort to lemmas 3.1 and 3.2. (Actually, by lemma 3.1, we see that $\mathcal{H}_n^+(G) \subset \mathcal{H}_n(G)$ and that $(\mathcal{H}_{n+1}^+(G) \to \mathcal{H}_n^+(G))_{n\geq 0}$ is a projective subsystem of $(\mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G))_{n\geq 0}$.)

For ℓ-adic representations which are not GLP representations, one may thus wonder whether (1) can hold even though (2) does not hold. We give below an example of such a representation as well as an example where neither (1) nor (2) holds.

Let k be a field finitely generated over $\mathbb{Q}$ and set $X = \mathrm{Spec}(k[T, \frac{1}{T}]) = \mathbb{G}_{m,k}$. Note that $\pi_1(X_{\overline{k}}) \simeq \widehat{\mathbb{Z}}$. Any k-rational point $x : \mathrm{Spec}(k) \to X$ induces a splitting $x : \Gamma_k \to \pi_1(X)$ of the fundamental short exact sequence for $\pi_1(X)$, which identifies Γ_k with the decomposition group of $\pi_1(X)$ at x. Let $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$ denote the cyclotomic character. Then $\chi \circ x = \chi : \Gamma_k \to \mathbb{Z}_\ell^\times$ is the cyclotomic character of k. The Kummer short exact sequence:

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \stackrel{\ell^n}{\to} \mathbb{G}_m \to 1$$

on the etale site over X yields an exact sequence:

$$1 \to \mu_{\ell^n}(k) \to k[T, \frac{1}{T}]^\times \stackrel{\ell^n}{\to} k[T, \frac{1}{T}]^\times \to \mathrm{H}^1(\pi_1(X), \mu_{\ell^n}(\overline{k})) \to 1,$$

since $\mathrm{Pic}(X) = 0$ (as $k[T, \frac{1}{T}]$ is a PID). Taking projective limits, we thus get a group morphism:

$$\psi : k[T, \frac{1}{T}]^\times \to \varprojlim\ k[T, \frac{1}{T}]^\times / k[T, \frac{1}{T}]^{\times \ell^n} = \varprojlim\ \mathrm{H}^1(\pi_1(X), \mu_{\ell^n}(\overline{k})) = \mathrm{H}^1(\pi_1(X), \mathbb{Z}_\ell(1)),$$

and we say that $\psi(a)$ is the *Kummer class* of $a \in k[T, T^{-1}]^\times$. Fix a compatible system of ℓ^nth roots of T then the map $\psi_T : \pi_1(X) \to \mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ defined by $\psi_T(\gamma) = \left(\frac{\gamma(T^{\frac{1}{\ell^n}})}{T^{\frac{1}{\ell^n}}}\right)_{n\geq 0}$, $\gamma \in \pi_1(X)$ is a cocycle for the Kummer class $\psi(T)$. In particular, $\psi_T(\gamma\gamma') = \psi_T(\gamma) + \chi(\gamma)\psi_T(\gamma')$, γ, $\gamma' \in \pi_1(X)$. Again, a direct computation shows that, for each $x \in X(k) = \mathbb{G}_m(k) = k^\times$, $\psi_T \circ x =: \psi_x$ is a cocycle for the Kummer class $\psi(x) \in \mathrm{H}^1(\Gamma_k, \mathbb{Z}_\ell(1))$.

3.5.1. *Counterexample to (2).* Let $\rho : \pi_1(X) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ defined by:

$$\rho(\gamma) = \begin{pmatrix} \chi(\gamma) & \psi_T(\gamma) \\ 0 & 1 \end{pmatrix}, \ \gamma \in \pi_1(X).$$

Then its specialization $\rho_x := \rho \circ x : \Gamma_k \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ at $x \in X(k)$ is given by

$$\rho_x(\sigma) = \begin{pmatrix} \chi(\sigma) & \psi_x(\sigma) \\ 0 & 1 \end{pmatrix}, \ \sigma \in \Gamma_k.$$

So, $G = \chi(\Gamma_k) \ltimes \mathbb{Z}_\ell$ and $G_x \subset \chi(\Gamma_k) \ltimes \psi_x(\Gamma_k)$. In particular, for every $x \in (k^\times)^{\ell^n}$, $G_x \subset \chi(\Gamma_k) \ltimes \ell^n\mathbb{Z}_\ell$, hence $[G : G_x] \geq \ell^n$, which shows that (2) is not satisfied since $(k^\times)^{\ell^n}$ is infinite, $n \geq 0$. On the other hand, $G_x \subset G$ is not open if and only if $\psi(x)$ lies in the kernel $\mathrm{H}^1(\mathrm{Gal}(k(\mu_{\ell^\infty})/k), \mathbb{Z}_\ell(1))$ of the restriction morphism $\mathrm{H}^1(\Gamma_k, \mathbb{Z}_\ell(1)) \to \mathrm{H}^1(\Gamma_{k(\mu_{\ell^\infty})}, \mathbb{Z}_\ell(1))$. Observe that $\mathrm{H}^1(\mathrm{Gal}(k(\mu_{\ell^\infty})/k), \mathbb{Z}_\ell(1))$ is torsion and that $\mathrm{H}^1(\Gamma_{k(\mu_{\ell^\infty})}, \mathbb{Z}_\ell(1)) = \mathrm{Hom}(\Gamma_{k(\mu_{\ell^\infty})}, \mathbb{Z}_\ell(1))$ is torsion-free. Thus, $G_x \subset G$ is not open if and only if $\psi(x)$ is torsion, or, equivalently, if and only if $x \in k^\times$ is torsion. But, since k is finitely generated over $\mathbb{Q}$, the torsion subgroup $\mu_\infty(k)$ of $k^\times$ is finite, hence (1) is satisfied.

3.5.2. *Counterexample to (1).* Fix $a \in k^\times \setminus \mu_\infty(k)$ and let now $\rho_a : \pi_1(X) \to \mathrm{GL}_3(\mathbb{Z}_\ell)$ defined by:

$$\rho_a(\gamma) = \begin{pmatrix} \chi(\gamma) & \psi_a(\gamma) & \psi_T(\gamma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \gamma \in \pi_1(X).$$

Then its specialization $\rho_{a,x} := \rho_a \circ x : \Gamma_k \to \mathrm{GL}_3(\mathbb{Z}_\ell)$ at $x \in X(k)$ is given by

$$\rho_{a,x}(\sigma) = \begin{pmatrix} \chi(\sigma) & \psi_a(\sigma) & \psi_x(\sigma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \sigma \in \Gamma_k.$$

So $G \subset \mathbb{Z}_\ell^\times \ltimes \mathbb{Z}_\ell^2$ is open whereas, if $x = a^r$, then $\psi(x) = \psi(a^r) = r\psi(a)$, hence, in particular, $\psi(x)|_{\Gamma_{k(\mu_{\ell^\infty})}} = r\psi(a)|_{\Gamma_{k(\mu_{\ell^\infty})}}$, which is equivalent to: $\psi_x|_{\Gamma_{k(\mu_{\ell^\infty})}} = r\psi_a|_{\Gamma_{k(\mu_{\ell^\infty})}}$, as $\Gamma_{k(\mu_{\ell_\infty})}$ acts trivially on $\mathbb{Z}_\ell(1)$. It follows from this that $G_x \subset \mathbb{Z}_\ell^\times \ltimes \mathbb{Z}_\ell^2$ is not open in G. Since $a \in k^\times \setminus \mu_\infty(k)$, the set $\{a^r \mid r \geq 0\}$ is infinite, hence (1) is not satisfied.

4. Uniform boundedness of ℓ-primary torsion

The techniques involved in the proof of theorem 1.1 generalize the ones developed in [CT08] to prove a certain uniform boundedness result for the ℓ-primary χ-torsion of abelian varieties. And, actually, this result can be generalized to arbitrary GLP ℓ-adic representations, by means of theorem 1.1 (for fields finitely generated over $\mathbb{Q}$). So, let k be a field finitely generated over $\mathbb{Q}$ and X a smooth, separated, geometrically connected curve over k. Let M be a finitely generated free $\mathbb{Z}_\ell$-module of rank $m < \infty$ (i.e., $M \simeq \mathbb{Z}_\ell^m$), and $\rho : \pi_1(X) \to \mathrm{GL}(M)$ a GLP representation. Set $V := M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ ($\simeq \mathbb{Q}_\ell^m$) and $D := V/M = M \otimes_{\mathbb{Z}_\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ ($\simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m$). Thus, we have a natural identification $M/\ell^n = D[\ell^n]$ for each $n \geq 0$.

Then ρ induces $\pi_1(X)$-actions on V and D naturally.

Definition 4.1. *Define $M_{(0)}$ to be the maximal isotrivial submodule of M, or, more precisely, $M_{(0)}$ is the maximal submodule of M on which the geometric part $\pi_1(X_{\overline{k}})$ of $\pi_1(X)$ acts via a finite quotient.*

Recall that each morphism $\xi : \mathrm{Spec}(L) \to X$ (where L is any field) induces a homomorphism $\xi : \Gamma_L \to \pi_1(X)$, hence a representation $\rho_\xi := \rho \circ \xi$ and, for each ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, an ℓ-adic character $\chi_\xi := \chi \circ \xi$. Set

$$\overline{D}_\xi := \{v \in D \mid \rho_\xi(\sigma)v \in \langle v\rangle \text{ for every } \sigma \in \Gamma_L\},$$

$$\overline{M}_\xi := \{v \in M \mid \rho_\xi(\sigma)v \in \langle v\rangle \text{ for every } \sigma \in \Gamma_L\},$$

which are Γ_L-sets, and, for each ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, set

$$D_\xi(\chi) := \{v \in D \mid \rho_\xi(\sigma)v = \chi_\xi(\sigma)v \text{ for every } \sigma \in \Gamma_L\},$$

$$M_\xi(\chi) := \{v \in M \mid \rho_\xi(\sigma)v = \chi_\xi(\sigma)v \text{ for every } \sigma \in \Gamma_L\},$$

which are Γ_L-modules. Next, for each subset $E \subset D$ and $n \geq 0$, set $E[\ell^n] := E \cap D[\ell^n]$ and $E[\ell^n]^* := E \cap (D[\ell^n] \smallsetminus D[\ell^{n-1}])$, where $D[\ell^{-1}] := \emptyset$. For each subset E of M, set $E^* := E \cap (M \smallsetminus \ell M)$. Then one has

$$\varprojlim \overline{D}_\xi[\ell^n] = \overline{M}_\xi, \ \ \varprojlim \overline{D}_\xi[\ell^n]^* = \overline{M}_\xi^*,$$

and

$$\varprojlim D_\xi(\chi)[\ell^n] = M_\xi(\chi), \ \ \varprojlim D_\xi(\chi)[\ell^n]^* = M_\xi(\chi)^*.$$

Definition 4.2. *Let $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$ be an ℓ-adic character. Then χ is said to be non-sub-ρ if χ_x is not isomorphic to a subrepresentation of ρ_x for every $x \in X(k)$.*

Now, the main result of this section, which is a corollary of theorem 1.1, is as follows.

Corollary 4.3. (1) *For every non-sub-ρ ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, there exists an integer $N := N(\rho, \chi)$, such that, for every $x \in X(k)$, the Γ_k-module $D_x(\chi)$ is contained in $D[\ell^N]$.*
(2) *Assume furthermore that $M_{(0)} = 0$. Then there exists an integer $N := N(\rho)$, such that, for every $x \in X(k) \smallsetminus X_\rho$, the Γ_k-set $\overline{D}_x$ is contained in $D[\ell^N]$.*

Proof. From theorem 1.1 applied to the GLP ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}(M)$, the set X_ρ of all $x \in X(k)$ with $G_x \subset G$ not open is finite and there exists an integer $N_0 := N_\rho \geq 0$ such that for all $x \in X(k) \smallsetminus X_\rho$, $G(N_0) \subset G_x$. Let η_{N_0} denote the generic point of the geometrically connected etale cover $X_{G(N_0)} \to X$ corresponding to the open subgroup $G(N_0) \subset G$.

(2) For each $v \in \overline{M}_{\eta_{N_0}} \smallsetminus \{0\}$, one has: $\gamma \cdot v = \lambda_{\gamma,v} v$ for some (unique) $\lambda_{\gamma,v} \in \mathbb{Z}_\ell^\times$. One can easily check that the map $\chi_v : \pi_1(X_{G(N_0)}) \to \mathbb{Z}_\ell^\times$, $\gamma \mapsto \lambda_{\gamma,v}$ is a character. Since $G^{geo} \cap G(N_0)$ has finite abelianization (as ρ is GLP), $\pi_1(X_{G(N_v),\overline{k}})$ acts trivially on v for some $N_v \geq N_0$. Thus, one gets: $\overline{M}_{\eta_{N_0}} \subset M_{(0)}$.

From the inclusion $G_{\eta_{N_0}} = G(N_0) \subset G_x$, one gets the inclusion: $\overline{D}_x \subset \overline{D}_{\eta_{N_0}}$. Now, suppose that $\overline{D}_x$ is infinite. Then $\overline{D}_{\eta_{N_0}}$ is also infinite, hence $\overline{D}_{\eta_{N_0}}[\ell^n]^*$ is nonempty for every $n \geq 0$, and $\overline{M}_{\eta_{N_0}}^* = \varprojlim \overline{D}_{\eta_{N_0}}[\ell^n]^*$ is nonempty. As $\overline{M}_{\eta_{N_0}} \subset M_{(0)}$, this implies that $M_{(0)} \neq 0$, as desired.

(1) First, consider the special case that χ is the trivial character $\mathbf{1}$. In this case, the inclusion $G_{\eta_{N_0}} = G(N_0) \subset G_x$ implies $D_x(\mathbf{1}) \subset D_{\eta_{N_0}}(\mathbf{1})$. Observe that the action of $\pi_1(X)$ on $D_{\eta_{N_0}}(\mathbf{1})$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(X)/\pi_1(X_{G(N_0)}) = G/G(N_0) = G_{N_0}$. Thus, $D_x(\mathbf{1})$ coincides with the module of elements of $D_{\eta_{N_0}}(\mathbf{1})$ fixed by the subgroup $(G_x)_{N_0} \subset G_{N_0}$. As G_{N_0} is a finite group, there are only finitely many subgroups of G_{N_0} that coincide with $(G_x)_{N_0}$ for some $x \in X(k)$. Accordingly, there are only finitely many submodules of $D_{\eta_{N_0}}(\mathbf{1})$ that coincide with $D_x(\mathbf{1})$ for some $x \in X(k)$. Since $D_x(\mathbf{1})$ is finite for each $x \in X(k)$ (as $\mathbf{1}$ is non-sub-ρ), this completes the proof in the special case.

For general χ, define the $\pi_1(X)$-module $M[\chi^{-1}]$ as follows: $M[\chi^{-1}] = M$ as $\mathbb{Z}_\ell$-modules, and the $\pi_1(X)$-action on $M[\chi^{-1}]$ is given by $\rho \cdot \chi^{-1}$. (Thus, $M[\chi^{-1}] = M \otimes \mathbb{Z}_\ell[\chi^{-1}]$.) Set $D[\chi^{-1}] := M[\chi^{-1}] \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell)$, and let $\rho[\chi^{-1}]$ denote the ℓ-adic representation of $\pi_1(X)$ associated to the $\pi_1(X)$-module $M[\chi^{-1}]$. Observe that the trivial character $\mathbf{1}$ is non-sub-$\rho[\chi^{-1}]$, as χ is non-sub-ρ.

We claim that $\rho[\chi^{-1}] = \rho \cdot \chi^{-1}$ is a GLP representation. Indeed, by [CT09, §3, especially, Lem. 3.2], $\chi(\pi_1(X_{\overline{k}}))$ is finite, since k is finitely generated over $\mathbb{Q}$. Thus, for every sufficiently small open subgroup $H \subset \pi_1(X_{\overline{k}})$, one has that $\chi|_H$ is trivial, hence $\rho|_H = \rho[\chi^{-1}]|_H$. From this and the assumption that ρ is GLP, $\rho[\chi^{-1}]$ is also GLP.

Now, applying the preceding argument to $\rho[\chi^{-1}]$, one concludes that there exists an integer N, such that $D[\chi^{-1}]_x(\mathbf{1}) \subset D[\chi^{-1}][\ell^N]$ for every $x \in X(k)$. Here, observe that the identification $M[\chi^{-1}] = M$ (as $\mathbb{Z}_\ell$-modules) induces the identifications $D[\chi^{-1}][\ell^N] = D[\ell^N]$ and $D[\chi^{-1}]_x(\mathbf{1}) = D_x(\chi)$. From this, the assertion follows. □

5. Classical examples of geometrically Lie perfect representations

5.1. Generic Tate modules of abelian schemes.

5.1.1. *Generic Tate modules of abelian schemes.* Let k be a field of characteristic $q \geq 0$, X a smooth, geometrically connected, separated curve over k with generic point η. Let $A \to X$ be an abelian scheme. Fixing a prime $\ell \neq q$, consider the corresponding ℓ-adic representation $\Gamma_{\kappa(\eta)} \to \mathrm{GL}(T_\ell(A_\eta))$. Since A is an abelian scheme over X and $\ell \neq q$, this representation is unramified over X and factors through $\rho_{A,\ell} : \pi_1(X) \to \mathrm{GL}(T_\ell(A_\eta))$. Then:

Theorem 5.1. *$\rho_{A,\ell} : \pi_1(X) \to \mathrm{GL}(T_\ell(A_\eta))$ is a GLP representation.*

Proof. We have to prove that U^{ab} is finite for every open subgroup $U \subset G^{geo}$. Since U contains $G^{geo}(n_U)$ for some $n \gg 0$, it suffices to prove that $G^{geo}(n)^{ab}$ is finite for every $n \geq 1$. So, up to replacing X by the connected etale cover $X_{G(n)} \to X$ corresponding to the open subgroup $G(n) \subset G$, it suffices to prove G^{ab} is finite, under the extra assumption that $G = G(1)$. Note that this last assumption is equivalent to saying that the finite etale group scheme $A[\ell]$ over X is isomorphic to the constant group scheme $(\mathbb{Z}/\ell)^{2\dim(A_\eta)}$. In particular, $G^{geo} = G^{geo}(1)$, and both G and G^{geo} are pro-ℓ. Next, up to replacing $A \to X \to k$ by $A_{k'} \to X_{k'} \to k'$ for a suitable finite extension, one may assume that X admits a smooth compactification X^{cpt} and that $X^{cpt} \smallsetminus X$ consists of r k-rational points $P_1, \dots, P_r$.

Step 1. Reduction to an arithmetic setting. Let $\mathbb{F}$ denote $\mathbb{Z}$ (resp. $\mathbb{F}_q$) when $q = 0$ (resp. $q > 0$). Then, since all the geometric objects are of finite type over the base, the data

$$(X = X^{cpt} \smallsetminus \{P_1, \dots, P_r\},\ A \to X,\ A[\ell] \simeq (\mathbb{Z}/\ell)^{2\dim(A_\eta)})$$

over k admits a model

$$(\mathcal{X} = \mathcal{X}^{cpt} \smallsetminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r),\ \mathcal{A} \to \mathcal{X},\ \mathcal{A}[\ell] \simeq (\mathbb{Z}/\ell)^{2\dim(A_\eta)})$$

over some finitely generated $\mathbb{F}$-subalgebra R of k such that ℓ is invertible in R. More precisely, $\mathcal{X}^{cpt}$ is a proper, smooth, geometrically connected curve over R; $\mathcal{P}_1, \dots, \mathcal{P}_r$ are mutually disjoint sections of $\mathcal{X}^{cpt} \to \mathrm{Spec}(R)$; $\mathcal{X} = \mathcal{X}^{cpt} \smallsetminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r)$ is a smooth, geometrically connected curve over R; $\mathcal{A}$ is an abelian scheme over $\mathcal{X}$; and $\mathcal{A}[\ell] \simeq (\mathbb{Z}/\ell)^{2\dim(A_\eta)}$ is an isomorphism of finite etale group schemes over $\mathcal{X}$. Since ℓ is invertible in R and R is finitely generated over $\mathbb{F}$, any closed point $r \in \mathrm{Spec}(R)$ has finite residue field $\kappa(r)$ of characteristic different from ℓ.

Let k_R denote the field of fractions of R and $\eta_R = \mathrm{Spec}(k_R)$ the generic point of $\mathrm{Spec}(R)$. The following cartesian diagram describes the situation.

(1)
$$\begin{array}{ccccccc}
A & \longrightarrow & \mathcal{A}_{\eta_R} & \longrightarrow & \mathcal{A} & \longleftarrow & \mathcal{A}_r \\
\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
X & \longrightarrow & \mathcal{X}_{\eta_R} & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_r \\
\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
k & \longrightarrow & k_R & \longrightarrow & R & \longleftarrow & \kappa(r)
\end{array}$$

Since r is a specialization of η_R, one gets, at the level of pro-ℓ completions, canonical isomorphisms [SGA1]:

$$\pi_1(X_{\overline{k}})^{(\ell)} \tilde{\to} \pi_1(\mathcal{X}_{\overline{\eta_R}})^{(\ell)} \tilde{\to} \pi_1(\mathcal{X}_{\overline{r}})^{(\ell)}.$$

Similarly, if $\eta_{\mathcal{X}}$ and $\eta_{\mathcal{X}_r}$ denote the generic points of $\mathcal{X}$ and $\mathcal{X}_r$ respectively, then $\eta_{\mathcal{X}_r}$ is a specialization of $\eta_{\mathcal{X}}$, hence, at the level of ℓ-adic Tate modules, one gets a canonical isomorphisms $T_\ell(A_\eta) \tilde{\to} T_\ell(\mathcal{A}_{\eta_{\mathcal{X}}}) \tilde{\to} T_\ell(\mathcal{A}_{\overline{r},\eta_{\mathcal{X}_r}})$ which are compatible with the actions of $\pi_1(X_{\overline{k}})^{(\ell)}$, $\pi_1(\mathcal{X}_{\overline{\eta_R}})^{(\ell)}$ and $\pi_1(\mathcal{X}_{\overline{r}})^{(\ell)}$ via the above isomorphisms $\pi_1(X_{\overline{k}})^{(\ell)} \tilde{\to} \pi_1(\mathcal{X}_{\overline{\eta_R}})^{(\ell)} \tilde{\to} \pi_1(\mathcal{X}_{\overline{r}})^{(\ell)}$. Thus, G^{geo} for $A \to X \to k$ is identified with G^{geo} for $\mathcal{A}_r \to \mathcal{X}_r \to \kappa(r)$.

Step 2. Frobenius weight argument: So, from step 1, one may assume that k is a finite field of characteristic $\neq \ell$, that $G = G(1)$ and that $X^{cpt} \setminus X$ consists of r k-rational points $P_1, \dots, P_r$.

Set $B := (G^{geo})^{ab}/Tors((G^{geo})^{ab})$. Since G^{geo} is a finitely generated pro-ℓ group, one has $B \simeq \mathbb{Z}_\ell^b$ for some integer $b \geq 0$. Let N denote the kernel of the canonical epimorphism $G^{geo} \twoheadrightarrow B$. Then we have the short exact sequence of profinite groups:

$$1 \to N \to G^{geo} \to B \to 1.$$

Note that $\pi_1(X)$ acts on G^{geo} by conjugation. Since $G^{geo} \twoheadrightarrow B$ is a characteristic quotient, this action induces an action on B, hence one on N. Taking Lie algebras, one gets a short exact sequence:

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{b} \to 0.$$

Since G^{geo} is a closed subgroup of $\mathrm{GL}(T_\ell(A_\eta)) \subset \mathrm{GL}(V_\ell(A_\eta))$, $\mathfrak{g}$ canonically embeds in $\mathfrak{gl}(V_\ell(A_\eta)) = \mathrm{End}(V_\ell(A_\eta))$ and $\pi_1(X)$ acts on $\mathfrak{g}$ *via* $Ad \circ \rho_{A,\ell}$. Furthermore, as B is an abelian ℓ-adic Lie group, the logarithm induces a canonical $\pi_1(X)$-equivariant morphism $\log : B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \tilde{\to} \mathfrak{b}$.

Let $\phi \in \Gamma_k$ denotes the Frobenius element. Then ϕ acts on $\pi_1(X_{\overline{k}})^{(\ell),ab} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ with eigenvalues of absolute value $|k|$ or $|k|^{\frac{1}{2}}$, hence also on $B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $\mathfrak{b}$ with eigenvalues of absolute value $|k|$ or $|k|^{\frac{1}{2}}$. But, on the other hand, let $\alpha_1, \dots, \alpha_{2\dim(A_\eta)}$ denotes the eigenvalues of ϕ acting on $V_\ell(A_\eta)$ then $Ad \circ \rho_{A,\ell}(\phi)$ has eigenvalues $\frac{\alpha_i}{\alpha_j}$, $1 \leq i, j \leq 2\dim(A_\eta)$. But, again, $|\alpha_i| = |k|$, $i = 1, \dots, 2\dim(A_\eta)$ so the eigenvalues of ϕ acting on $\mathfrak{b}$ have absolute values 1, which is only possible if $\mathfrak{b} = 0$ or, equivalently $B = 0$, hence $(G^{geo})^{ab} = Tors((G^{geo})^{ab})$ is finite (recall that G^{geo} is finitely generated). □

5.1.2. *Recovering the uniform boundedness of ℓ-primary χ-torsion.* We retain the notation of subsection 5.1.1, and assume furthermore that k is finitely generated over $\mathbb{Q}$.

Corollary 5.2. (1) *For every non-sub-$\rho_{A,\ell}$ ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, there exists an integer $N := N(\rho_{A,\ell}, \chi)$, such that, for every $x \in X(k)$, the Γ_k-module $A_x[\ell^\infty](\chi_x) = (A_\eta[\ell^\infty])_x(\chi)$ is contained in $A_x[\ell^N]$.*

(2) *Assume furthermore that A_η contains no non-trivial isotrivial subvariety. Then there exists an integer $N := N(\rho_{A,\ell})$, such that, for every $x \in X(k) \setminus X_{\rho_{A_\eta,\ell}}$, the Γ_k-set $\overline{A_x[\ell^\infty]}(k) = \overline{(A_\eta[\ell^\infty])}_x$ is contained in $A_x[\ell^N]$.*

Proof. By theorem 5.1, $\rho_{A,\ell}$ is a GLP representation. So, one may apply corollary 4.3 to $\rho_{A,\ell}$. This complete the proof, since $T_\ell(A_\eta)_{(0)} = 0$ if and only if A_η contains no non-trivial isotrivial subvariety ([CT08, Cor. 2.4]). □

Remark 5.3. Recall ([CT08, Sect. 2.2], [CT09, §3]) that an ℓ-adic character $\Gamma_k \to \mathbb{Z}_\ell^\times$ is said to be non-Tate if it does not appear as a subrepresentation of Γ_k acting on the ℓ-adic Tate module of an abelian variety over k. Classical examples of non-Tate characters are the trivial and the cyclotomic characters. By definition, if $\chi : \Gamma_k \to \mathbb{Z}_\ell^\times$ is a non-Tate ℓ-adic character, then $\chi \circ r : \pi_1(X) \to \mathbb{Z}_\ell^\times$ is non-sub-$\rho_{A,\ell}$, where $r : \pi_1(X) \to \Gamma_k$ denotes the canonical restriction homomorphism. Thus, corollary 5.2 recovers and strengthens [CT08, Cor. 1.2, Cor. 4.7].

5.1.3. *A conjecture on ℓ-independence.* When $\dim(A_\eta) = 1$, one recovers Serre's open image theorem [Se68, Chap. IV, 2.2] (and its uniform version [A08]). More precisely, consider the family of elliptic curves $E \to X := \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1728, \infty\}$ defined by:

$$E_j : y^2 + xy - x^3 + \frac{36}{j - 1728}x + \frac{1}{j - 1728} = 0.$$

Lemma 5.4. *$E \to X$ defines an abelian scheme and the image G of the ℓ-adic representation $\rho_{E,\ell} : \pi_1(X) \to \mathrm{GL}(T_\ell(E_\eta))$ is open.*

Proof. To check the first part of the assertion, embed $E \subset \mathbb{P}^2_X$ as usual. Then $E \to X$ is projective, smooth (use the Jacobian criterion or observe that $E \to X$, being a local complete intersection, is flat, hence smooth, since, for all $j \in X = \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1728, \infty\}$, E_j is an elliptic curve), and admits the constant section $[0 : 1 : 0]$. So, the conclusion follows from Grothendieck's rigidity theorem [MF82, Thm. 6.14].

As for the second part of the assertion, take an integer $N_0 \geq 3$ and consider the etale cover $X_{N_0} \to X$ corresponding to the kernel of the mod N_0 representation $\pi_1(X) \to \mathrm{GL}(\mathbb{Z}/N_0)$. Then, by choosing a basis for $E[N_0]$, we get a classifying morphism $X_{N_0} \to Y(N_0)$. This is non-constant, hence the associated homomorphism $\pi_1(X_{N_0}) \to \pi_1(Y(N_0))$ has open image. Thus, we are reduced to the case of the universal elliptic curve over the modular curve $Y(N_0)$. In this case, the desired openness property is well-known. (It follows from the connectedness of the modular curves $Y(N)$ [Se89, §10.4].) □

As a result the subset $X_{\rho_{E,\ell}}$ of all $j \in X(k)$ such that G_j is not open in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ is finite and there exists an integer $B_{\rho_{E,\ell}} \geq 1$ such that $[\mathrm{GL}_2(\mathbb{Z}_\ell) : G_j] \leq B_{\rho_{E,\ell}}$, $j \in X(k) \smallsetminus X_{\rho_{E,\ell}}$. Also, observe that, for $j \in X = \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1728, \infty\}$, the j-invariant $j(E_j)$ of E_j is just j. Now, consider any other elliptic curve $\mathfrak{E}$ over k with j-invariant j. then E_j and $\mathfrak{E}$ become isomorphic over a degree ≤ 2 extension of k (recall that $j \neq 0, 1728$). Hence, for every elliptic curve $\mathfrak{E}$ over k with j-invariant $j \in X(k) \smallsetminus X_{\rho_{E,\ell}}$, the image G of the ℓ-adic representation $\rho_{\mathfrak{E},\ell} : \Gamma_k \to \mathrm{GL}(T_\ell(\mathfrak{E}))$ is open and $[\mathrm{GL}_2(\mathbb{Z}_\ell) : G] \leq 2B_{\rho_{E,\ell}}$.

However, Serre's open image theorem is more precise in the sense that it describes explicitly the exceptional locus $X_{\rho_{E,\ell}}$ as the set of CM elliptic curves defined over k or, equivalently, as the set of elliptic curves defined over k with Mumford-Tate group of dimension 2; in particular $X_{\rho_{E,\ell}} \subset X(k)$ is independent of ℓ. We conjecture that the same holds in higher-dimensional situations, that is:

Conjecture 5.5. *Let k be a field finitely generated over $\mathbb{Q}$, X a smooth, separated, geometrically connected curve over k and A an abelian scheme over X. Then there exists a finite subset $X_A \subset X(k)$ such that for every prime ℓ, $X_{\rho_{A,\ell}} = X_A$.*

This conjecture would follow from the Mumford-Tate conjecture:

Conjecture 5.6. (Mumford-Tate conjecture) *Let k be a field finitely generated over $\mathbb{Q}$ and $\mathfrak{A}$ an abelian variety over k. Then there exists a connected algebraic group $MT(\mathfrak{A})$ over $\mathbb{Q}$ such that for every prime ℓ, the connected component at 1 of the Zariski closure of $\rho_{\mathfrak{A},\ell}(\Gamma_k)$ in the algebraic group $\mathrm{GL}(V_\ell(\mathfrak{A}))$ over $\mathbb{Q}_\ell$ is identified with $MT(\mathfrak{A}) \times_{\mathbb{Q}} \mathbb{Q}_\ell$ (as algebraic group over $\mathbb{Q}_\ell$) and $\rho_{\mathfrak{A},\ell}(\Gamma_k) \cap MT(\mathfrak{A})(\mathbb{Q}_\ell)$ is (ℓ-adically) open in $MT(\mathfrak{A})(\mathbb{Q}_\ell)$.*

Indeed, since the dimension of $MT(\mathfrak{A})(\mathbb{Q}_\ell)$ as an ℓ-adic Lie group is the same as the dimension of $MT(\mathfrak{A})$ as an algebraic group over $\mathbb{Q}$, the Mumford-Tate conjecture implies that:
(i) $d_\eta := \dim(\rho_{A,\ell}(\pi_1(X)))$ and $d_x := \dim(\rho_{A,\ell} \circ x(\Gamma_k))$, $x \in X(k)$ are independent of ℓ ;
(ii) $X_{\rho_{A,\ell}}$ is the set of all $x \in X(k)$ with $d_x < d_\eta$.

Remark 5.7. At the time of revising this paper, C.-Y. Hui [H11] announced that he could prove conjecture 5.5. His proof does not resort to the Mumford-Tate conjecture and does not require the base scheme X to be a curve. More precisely, set $G_{\eta,\ell} := \rho_{A,\ell}(\pi_1(X))$ and $G_{x,\ell} := \rho_{A,\ell} \circ x(\Gamma_k)$ for each closed point $x \in X$. Then combining three main ingredients: the fact that $\mathrm{Lie}(G_{\eta,\ell})$ (resp. $\mathrm{Lie}(G_{x,\ell})$, for every closed point $x \in X$) is algebraic and has rank independent of ℓ ([Se81a]), the Tate and the semisimplicity conjectures for abelian varieties proved by G. Faltings [FW92] and a sufficient condition given by Y. Zarhin for two reductive algebraic Lie subalgebras of $\mathfrak{gl}_n(\mathbb{Q}_\ell)$ to be equal [Z92], he shows that for every closed point $x \in X$ if $G_{x,\ell}$ is open in $G_{\eta,\ell}$ for some prime ℓ then $G_{x,\ell}$ is open in $G_{\eta,\ell}$ for all prime ℓ.

5.2. **ℓ-adic cohomology groups.** Essentially, the proof of theorem 5.1 only uses:

(1) The specialization isomorphism for fundamental groups;
(2) The fact that Frobenius acts with constant weight on the Tate module of an abelian variety and acts with non-zero weights on $B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Hence theorem 5.1 extends to any such representations. The most classical example is the following. Let again k be a field of characteristic $q \geq 0$, X a smooth, separated and geometrically connected curve over k with generic point η. Let $Y \to X$ be a smooth proper scheme over X. for every prime $\ell \neq q$ and smooth $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on X_{et} and any geometric point $\overline{x}$ on X, $\pi_1(X, \overline{x})$ acts on the stalk $\mathcal{F}_{\overline{x}}$. In particular, for every smooth $\mathbb{Q}_\ell$-sheaf $\mathcal{G}$ on Y_{et}, $\pi_1(X, \overline{x})$ acts on $(R^i_{et} f_* \mathcal{G})_{\overline{x}} = \mathrm{H}^i_{et}(Y_{\overline{x}}, \mathcal{G}|_{Y_{\overline{x}}})$. This yields representations: $\rho_i : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^i_{et}(Y_{\overline{\eta}}, \mathbb{Q}_\ell))$, $i \geq 0$ and, again:

Theorem 5.8. *$\rho_i : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^i_{et}(Y_{\overline{\eta}}, \mathbb{Q}_\ell))$ is a GLP representation, $i \geq 0$.*

Proof. The pattern of the proof is exactly the same as the one of the proof of theorem 5.1. The specialization argument of step 1 follows again from the specialization theorem for etale fundamental groups [SGA1, Exp. XIII] and for ℓ-adic cohomology groups [SGA4, Exp. XVI]. The Frobenius weight argument of step 2 follows from the celebrated Weil conjecture [D74, Thm. 1.6]. □

Remark 5.9.

(1) The statement of theorem 5.8 (hence also 5.1) remains true when X is a smooth, geometrically connected k-scheme of arbitrary dimension. To show this, we may assume, up to shrinking X, that X is affine, hence quasi-projective. Then, by using a version of Bertini's theorem [J83, Thm. 6.10], one can construct an extension K of k (which is the function field of a suitable Grassmannian variety) and a K-curve $C \subset X_K$, such that the induced morphism $\pi_1(C_{\overline{K}}) \twoheadrightarrow \pi_1(X_{\overline{k}})$ at the level of geometric fundamental groups be surjective. This, together with the specialization theorem for ℓ-adic cohomology groups [SGA4, Exp. XVI], reduces the GLP property for the variety X over k to the GLP property for the curve C over K, which is available from theorem 5.8.
(2) The argument we give works both for characteristic 0 and positive characteristic $q \neq \ell$. In characteristic 0, Y. André suggested another proof: by comparison between ℓ-adic and Betti cohomology, Deligne's semisimplicity theorem [D71, Cor. (4.2.9) a)] ensures that (the connected component of) the Zariski closure of G^{geo} in the algebraic group $\mathrm{GL}(\mathrm{H}^i_{et}(Y_{\overline{\eta}}, \mathbb{Q}_\ell))$ over $\mathbb{Q}_\ell$ is semisimple. This, in turn, together with Chevalley's theory of algebraic Lie algebras [B91, Chap. II, §7], implies that the Lie algebra of G^{geo} (as ℓ-adic Lie group) coincides with the Lie algebra of its Zariski closure (as algebraic group over $\mathbb{Q}_\ell$) hence, in particular, is semisimple. André's proof does not extend to positive characteristic. However, in the special case of abelian schemes and positive characteristic $q > 2$, an alternative argument can be found in [Z77].

6. Appendix: Noetherian property of compact ℓ-adic Lie groups

In this section, we generalize lemma 3.5 to any sequence of closed subgroups.

Theorem 6.1. *Let G be a compact ℓ-adic Lie group. Then any increasing sequence $H_0 \subset H_1 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots \subset G$ of closed subgroups of G stabilizes.*

Proof. We first prove theorem 6.1 when G is standard (according to the terminology of [DSMS91, Def. 9.25, p. 188]) and then deduce the general case from this special case.

As the $\dim(H_n)$, $n \geq 0$ stabilize, up to renumbering, one may assume that $\dim(H_0) = \dim(H_n)$, $n \geq 0$ or, equivalently, that H_0 is open in H_n, $n \geq 0$.

Step 1. Assume that G is standard, and for each closed subgroup $H \subset G$ set:

$$H^{\ell^{-n}} := \{g \in G \mid g^{\ell^n} \in H\},\ n \geq 0 \text{ and } H^{\ell^{-\infty}} := \bigcup_{n \geq 0} H^{\ell^{-n}}.$$

Then one has:

Lemma 6.2. (1) *There exists $N \geq 0$ such that $H^{\ell^{-\infty}} = H^{\ell^{-N}}$;*
(2) *$H^{\ell^{-\infty}}$ is closed in G;*
(3) *H is open in $H^{\ell^{-\infty}}$.*

Proof. As H is closed in G and G is an ℓ-adic Lie group, H is also an ℓ-adic Lie group. Hence, from [DSMS91, Thm. 9.31, p. 191], H contains an open subgroup H_0, which is also standard. Since a standard ℓ-adic Lie group is automatically a pro-ℓ group, one can write $[H : H_0] = \ell^{m_0}$. So, one has the inclusions $H_0^{\ell^{-n}} \subset H^{\ell^{-n}} \subset H_0^{\ell^{-n-m_0}}$, $n \geq 0$ from which one deduces $H_0^{\ell^{-\infty}} = H^{\ell^{-\infty}}$ and $H^{\ell^{-\infty}} = H^{\ell^{-n}}$ for $n \gg 0$ if and only if $H_0^{\ell^{-\infty}} = H_0^{\ell^{-n}}$ for $n \gg 0$. So, one can assume without loss of generality that H is also standard. Then, in particular, it follows from [DSMS91, Thm. 9.33, p. 193] that G and H are uniformly powerful pro-ℓ groups.

(1) This implies that G and H can be endowed with an additive structure $+$ (defined by $x + y = \lim_{n\to\infty} (x^{\ell^n} y^{\ell^n})^{\ell^{-n}}$) for which they are free $\mathbb{Z}_\ell$-modules of finite rank [DSMS91, §4.3, p. 68]. Write $\mathfrak{g}$ and $\mathfrak{h}$ for $(G, +)$ and $(H, +)$ respectively and $\log : G \tilde{\to} \mathfrak{g}$, $\log : H \tilde{\to} \mathfrak{h}$ for the corresponding logarithms (which, here, coincide with the identity). Observe that $\log(g^m) = m\log(g)$, $g \in G$, $m \in \mathbb{Z}$. So

$$\log(H^{\ell^{-m}}) = \{x \in \mathfrak{g} \mid \ell^n x \in \mathfrak{h}\} = \pi_{\mathfrak{h}}^{-1}((\mathfrak{g}/\mathfrak{h})[\ell^n]) \text{ and } \log(H^{\ell^{-\infty}}) = \pi_{\mathfrak{h}}^{-1}((\mathfrak{g}/\mathfrak{h})[\ell^\infty])$$

where $\pi_{\mathfrak{h}} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ denotes the canonical quotient morphism. But $\mathfrak{g}/\mathfrak{h}$ is a finitely generated $\mathbb{Z}_\ell$-module so its torsion subgroup $\mathfrak{g}/\mathfrak{h}[\ell^\infty]$ is finite that is $(\mathfrak{g}/\mathfrak{h})[\ell^\infty] = (\mathfrak{g}/\mathfrak{h})[\ell^n]$, $n \gg 0$ or, equivalently, $H^{\ell^{-\infty}} = H^{\ell^{-n}}$, $n \gg 0$.

(2) follows from (1) since by definition the $H^{\ell^{-n}}$ are closed subsets of G, $n \geq 0$.

As for (3), since G is uniformly powerful, the map $e_n : g \mapsto g^{\ell^n}$ induces an ℓ-adic analytic homeomorphism $G \to G^{\ell^n}$ (where G^{ℓ^n} denotes the closed subgroup of G generated by the g^{ℓ^n}, $g \in G$) and G^{ℓ^n} is open in G. (e_n is clearly ℓ-adic analytic and since G is compact, it is enough to prove that e_n is bijective. The surjectivity follows from [DSMS91, Cor. 3.5, p. 53] and the injectivity from [DSMS91, §4.3, p. 68] since $g^{\ell^n} = g'^{\ell^n}$ if and only if $\ell^n\log(g) = \ell^n\log(g')$ which, in turn, is equivalent to $\log(g) = \log(g')$ hence $g = g'$ since $\mathfrak{g}$ is torsion-free). Since H is powerful as well, $e_n : G \to G^{\ell^n}$ restricts to an ℓ-adic analytic homeomorphism $e_n : H \to H^{\ell^n}$ and H^{ℓ^n} is open in H hence in $H \cap G^{\ell^n}$. From which one gets that $H = e_n^{-1}(H^{\ell^n})$ is open in $H^{\ell^{-n}} = e_n^{-1}(G^{\ell^n} \cap H)$. □

We now conclude the proof. Set $H_\infty := \bigcup_{n \geq 0} H_n \subset G$. Since, for all $n \geq 0$ $[H_n : H_0] = \ell^{m_n}$ for some $m_n \geq 0$, one gets $H_0 \subset H_\infty \subset H_0^{\ell^{-\infty}}$. But, from lemma 6.2 (2), $H_0^{\ell^{-\infty}}$ is closed in G so $\overline{H_\infty} \subset H_0^{\ell^{-\infty}}$. Also, from lemma 6.2 (3) H_0 is open in $H_0^{\ell^{-\infty}}$ so H_0 is open in $\overline{H_\infty}$ or, equivalently, $[\overline{H_\infty} : H_0]$ is finite. But this implies that $[H_\infty : H_0]$ is finite as well that is that the sequence stabilizes.

Step 2. Assume now that G is any compact ℓ-adic Lie group. Then it contains an open standard subgroup $G_0 \subset G$ [DSMS91, Thm. 9.31, p. 191]. From step 1, the sequence $G_0 \cap H_n$, $n \geq 0$ stabilizes so, up to renumbering, one may assume that $G_0 \cap H_0 = G_0 \cap H_n$, $n \geq 0$. But $[H_n : H_0] \leq [H_n : G_0 \cap H_0] = [H_n : G_0 \cap H_n] \leq [G : G_0]$ so, since $[G : G_0]$ is finite, the sequence stabilizes. □

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