A UNIFORM OPEN IMAGE THEOREM FOR ℓ-ADIC REPRESENTATIONS II.

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Abstract. Let k be a field finitely generated over $\mathbb{Q}$ and let X be a curve over k. Fix a prime ℓ. A representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is said to be geometrically Lie perfect if any open subgroup of $\rho(\pi_1(X_{\overline{k}}))$ has finite abelianization. Let G denote the image of ρ. Any closed point x on X induces a splitting $x : \Gamma_{\kappa(x)} := \pi_1(\mathrm{Spec}(\kappa(x))) \to \pi_1(X_{\kappa(x)})$ of the restriction epimorphism $\pi_1(X_{\kappa(x)}) \to \Gamma_{\kappa(x)}$ (here, $\kappa(x)$ denotes the residue field of X at x) so one can define the closed subgroup $G_x := \rho \circ x(\Gamma_{\kappa(x)}) \subset G$. The main result of this paper is the following uniform open image theorem. Under the above assumptions, for any geometrically Lie perfect representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ and any integer $d \geq 1$, the set $X_{\rho,d}$ of all closed points $x \in X$ such that G_x is not open in G and $[\kappa(x) : k] \leq d$ is finite and there exists an integer $B_{\rho,d} \geq 1$ such that $[G : G_x] \leq B_{\rho,d}$ for any closed point $x \in X \smallsetminus X_{\rho,d}$ with $[\kappa(x) : k] \leq d$.

A key ingredient of our proof is that, for any integer $\gamma \geq 1$ there exist an integer $\nu = \nu(\gamma) \geq 1$ such that, given any projective system $\cdots \to Y_{n+1} \to Y_n \to \cdots \to Y_0$ of curves (over an algebraically closed field of characteristic 0) with the same gonality γ and with $Y_{n+1} \to Y_n$ a Galois cover of degree > 1, one can construct a projective system of genus 0 curves $\cdots \to B_{n+1} \to B_n \to \cdots \to B_\nu$ and degree γ morphisms $f_n : Y_n \to B_n$, $n \geq \nu$ such that Y_{n+1} is birational to $B_{n+1} \times_{B_n, f_n} Y_n$, $n \geq \nu$. This, together with the case for $d = 1$ (which is the main result of part I of this paper), gives the proof for general d.

Our method also yields the following unconditional variant of our main result. With the above assumptions on k and X, for any ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ and integer $d \geq 1$, the set of all closed points $x \in X$ such that G_x is of codimension at least 3 in G and $[\kappa(x) : k] \leq d$ is finite.

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1. Introduction

Let ℓ be a prime. A compact ℓ-adic Lie group G is said to be *Lie perfect* if one of the following two equivalent conditions holds:

(i) $Lie(G)^{ab} = 0$;

(ii) For any open subgroup $U \subset G$, U^{ab} is finite.

Observe that, given an open subgroup $U \subset G$, G is Lie perfect if and only if U is Lie perfect.

Let k be a field and let X be a scheme geometrically connected and of finite type over k. Then, the structure morphism $X \to \mathrm{Spec}(k)$ induces at the level of etale fundamental groups a short exact sequence of profinite groups (sometimes referred to as the fundamental short exact sequence for $\pi_1(X)$):

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \Gamma_k \to 1.$$

An ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is said to be *Lie perfect* (LP for short) if $G := \rho(\pi_1(X)) \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ is Lie perfect and *geometrically Lie perfect* (GLP for short) if $G^{geo} := \rho(\pi_1(X_{\overline{k}})) \subset G$ is Lie perfect.

Any closed point $x \in X$ induces a splitting $x : \Gamma_{\kappa(x)} \to \pi_1(X_{\kappa(x)})$ of the fundamental short exact sequence for $\pi_1(X_{\kappa(x)})$, identifying $\Gamma_{\kappa(x)}$ with a closed subgroup of $\pi_1(X_{\kappa(x)}) \subset \pi_1(X)$. Set $G_x := \rho \circ x(\Gamma_{\kappa(x)})$ for the corresponding closed subgroup of G.

We will use the following notation: X^{cl} for the set of closed points of X and, for any integer $d \geq 1$,

$$X^{cl, \leq d} := \{x \in X^{cl} \mid [\kappa(x) : k] \leq d\}.$$

Also, when, in addition, X is smooth, separated of dimension 1 over k, we will say that X is a k-curve and denote by X^{cpt} the smooth compactification of X (if any); we will then denote the genus of $X^{cpt}_{\overline{k}}$ by g_X and the gonality of $X^{cpt}_{\overline{k}}$ by γ_X.

The main result of this paper is the following one-dimensional uniform open image theorem for GLP representations.

Theorem 1.1. *Assume that k is a field finitely generated over $\mathbb{Q}$, that X is a k-curve and that $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is a GLP representation. Then, for any integer $d \geq 1$ the set $X_{\rho,d}$ of all $x \in X^{cl,\, \leq d}$ such that G_x is not open in G is finite and there exists an integer $B_{\rho,d} \geq 1$ such that $[G : G_x] \leq B_{\rho,d}$ for any $x \in X^{cl,\, \leq d} \smallsetminus X_{\rho,d}$.*

As classical examples of GLP representations, let us mention the ones arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X or, more generally, from the action of $\pi_1(X)$ on the ℓ-adic etale cohomology groups $\mathrm{H}^i_{et}(Y_{\overline{\eta}}, \mathbb{Q}_\ell)$, $i \geq 0$ of the geometric generic fiber of a smooth proper scheme Y over X [CT12c, Sect. 4].

Theorem 1.1 is a strong version of [CT12c, Thm. 1.1], which only deals with the case $d = 1$. A crucial step in the proof of [CT12c, Thm. 1.1] consists in showing that the genus g_{X_n} of a projective system $(X_{n+1} \to X_n)_{n \geq 0}$ of certain etale covers of X is becoming larger than 2 for n large enough, which allows us to resort to Mordell's conjecture [FW84]. The strategy to obtain bounds depending no longer on $\kappa(x)$ but only on $[\kappa(x) : k]$ is to prove that the gonality γ_{X_n} of the covers $(X_{n+1} \to X_n)_{n \geq 0}$ goes to infinity with n and to replace Mordell's conjecture by the following corollary (see [Fr94]) of Lang's conjecture [Fa91] (see also [Mc95]):

Theorem 1.2. *Let k be a field finitely generated over $\mathbb{Q}$ and X a smooth, proper, geometrically connected curve over k of k-gonality γ $(\geq \gamma_X)$. Then, for any integer $1 \leq d \leq \left[\frac{\gamma-1}{2}\right]$ the set $X^{cl,\, \leq d}$ is finite.*

The main technical tool we resort to is that, for any integer $\gamma \geq 1$ there exists an integer $\nu = \nu(\gamma) \geq 1$ such that, given any projective system $\cdots \to Y_{n+1} \to Y_n \to \cdots \to Y_0$ of curves with the same gonality γ and with $Y_{n+1} \to Y_n$ a Galois cover of degree > 1, one can construct a projective system of genus 0 curves $\cdots \to B_{n+1} \to B_n \to \cdots \to B_\nu$ and degree γ morphisms $f_n : Y_n \to B_n$, $n \geq \nu$ such that Y_{n+1} is birational to $B_{n+1} \times_{B_n, f_n} Y_n$, $n \geq \nu$. We apply this general construction to the projective system $(X_{n+1} \to X_n)_{n \geq 0}$ in order to show that $\gamma_n \to +\infty$.

Modifying slightly the definition of the projective system $(X_{n+1} \to X_n)_{n \geq 0}$, our method yields the following unconditional variant of theorem 1.1.

Theorem 1.3. *Assume that k is a field finitely generated over $\mathbb{Q}$, that X is a k-curve and that $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is any ℓ-adic representation. Then, for any integer $d \geq 1$ the set of all $x \in X^{cl,\, \leq d}$ such that G_x is of codimension ≥ 3 in G is finite.*

The paper is organized as follows. Section 2 is devoted to the general construction of the B_n and $f_n : Y_n \to B_n$, $n \geq \nu$. In section 3, we carry out the proof of theorem 1.1 following the strategy of [CT12c] with gonality replacing genus. The group-theoretical construction of [CT12c] is recalled in subsection 3.1 whereas the main geometrical result (theorem 3.3) is stated and proved in subsection 3.2. Eventually, we conclude the proof of theorem 1.1 in subsection 3.3. In section 4, we give some applications of theorem 1.1. In subsection 4.1, we prove certain strong uniform boundedness results (corollary 4.2 (1)(2)) for arbitrary GLP ℓ-adic representations. In subsection 4.1.2, we state them more specificaly for GLP ℓ-adic representations arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X; this yields strong uniform boundedness results for the ℓ-primary torsion of abelian varieties varying in one-dimensional families. In subsection 4.2, we also show how to recover the Hecke-Deuring-Heilbronn theorem (cf. [Si35]) and the finiteness of CM elliptic curves defined over a number field of degree $\leq d$ from our results. Eventually, section 5 is devoted to theorem 1.3, which we prove in subsection 5.1. In subsection 5.2, we exhibit an ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ such that the set of all $x \in X(k)$ such that G_x is of codimension ≥ 2 in G is infinite.

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2. Gonality

Let k be an algebraically closed field of characteristic 0. The aim of this section is to prove the following result about the growth of gonality along projective systems of proper k-curves.

Theorem 2.1. *Let:*

$$\cdots \xrightarrow{\pi_{n+1}} Y_n \xrightarrow{\pi_n} Y_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 \tag{1}$$

be a projective system of proper k-curves. Assume that:

(i) $\pi_n : Y_n \to Y_{n-1}$ is a Galois cover with Galois group G_n, $n \geq 1$.
(ii) $\gamma_{Y_n} = \gamma$, $n \gg 1$;
(iii) G_n is cyclic of order ≥ 3 for infinitely many $n \geq 1$.

Then there exists $N \geq 0$ such that diagram (1) can be completed as follows:

$$\begin{tikzcd}
\cdots \arrow[r, "\pi_{n+1}"] & Y_n \arrow[r, "\pi_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\tilde{\square}"] & Y_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d, "f_{n-1}"] \arrow[dr, phantom, "\tilde{\square}"] & \cdots \arrow[r, "\pi_{N+1}"] \arrow[dr, phantom, "\tilde{\square}"] & Y_N \arrow[r, "\pi_N"] \arrow[d, "f_N"] & Y_{N-1} \arrow[r, "\pi_{N-1}"] & \cdots \\
\cdots \arrow[r, "\pi'_{n+1}"'] & B_n \arrow[r, "\pi'_n"'] & B_{n-1} \arrow[r, "\pi'_{n-1}"'] & \cdots \arrow[r, "\pi'_{N+1}"'] & B_N,
\end{tikzcd} \tag{2}$$

where $\pi'_n : B_n \to B_{n-1}$ is a Galois cover with Galois group G_n, each square:

$$\begin{tikzcd}
Y_n \arrow[r, "\pi_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\tilde{\square}"] & Y_{n-1} \arrow[d, "f_{n-1}"] \\
B_n \arrow[r, "\pi'_n"'] & B_{n-1}
\end{tikzcd}$$

is cartesian (up to normalization[1]) and G_n-equivariant and $g_{B_n} = 0$, $\deg(f_n) = \gamma$ or $g_{B_n} = 1$, $\deg(f_n) = \frac{\gamma}{2}$, $n \geq N$.

Theorem 2.1 is enough to carry out the proof of theorem 1.1. In subsection 2.4, we will give a more general statement about gonality (corollary 2.8). In subsection 2.1, we collect some elementary facts about gonality. In subsection 2.2, we perform the construction of theorem 2.1 when the projective system is finite (and without condition (iii) on the G_n). Eventually, in subsection 2.3, we pass to the projective limit using a classical finiteness argument; this is where condition (iii) is required.

2.1. One step.

Given a diagram of proper k-curves:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\pi"'] & B \\
Y',
\end{tikzcd} \tag{3}$$

where $f : Y \to B$ is a non-constant morphism of proper k-curves and $\pi : Y \to Y'$ is a Galois cover with Galois group G, we will say that (3) is *equivariant* if for any $\sigma \in G$ there exists $\sigma_B \in \mathrm{Aut}_k(B)$ such that $f \circ \sigma = \sigma_B \circ f$ and that (3) is *primitive* if for any commutative diagram of morphisms of proper k-curves:

$$\begin{tikzcd}
Y \arrow[r, "f'"] \arrow[rr, bend left, "f"] \arrow[d, "\pi"'] & B' \arrow[r, "j"] & B \\
Y'
\end{tikzcd} \tag{4}$$

[1]That is, Y_n is the normalization of the fiber product $B_n \times_{\pi'_n, B_{n-1}, f_{n-1}} Y_{n-1}$.

with f' of degree ≥ 2 the diagram:

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[d, "\pi"'] & B' \\ Y', & \end{tikzcd}$$

is not equivariant.

We will resort to the following result from [T04, §2].

Lemma 2.2. ([T04, Thm. 2.4]). *If (3) is primitive then*

$$\deg(f) \geq \sqrt{\frac{g_Y+1}{g_B+1}}$$

and if, furthermore, $g_{Y'} \geq 2$ then

$$\deg(f) \geq \sqrt{\frac{|G|(g_{Y'}-1)+2}{g_B+1}} \geq \sqrt{\frac{|G|+2}{g_B+1}}.$$

For any diagram (3), consider a decomposition:

(5)
$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] \arrow[rr, bend left, "f"] & C \arrow[r] & B \\ Y', & & \end{tikzcd}$$

with

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] & C \\ Y' & \end{tikzcd}$$

equivariant and $Y \to C$ of degree maximal for such a property. Then, by definition, the action of G on Y induces that on C, hence we obtain a homomorphism $G \to \mathrm{Aut}_k(C)$. Set $K := \mathrm{Ker}(G \to \mathrm{Aut}_k(C))$ and $\overline{G} := G/K$. Then (5) can be decomposed as follows:

(6)
$$\begin{tikzcd} Y \arrow[r] \arrow[r, bend right, "K"'] \arrow[rrr, bend left, "f"] & Z := Y/K \arrow[r] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\tilde{\square}"] & C \arrow[r] \arrow[d, "\overline{G}"] & B \\ & Y' \arrow[r, "f'"'] & B' := C/\overline{G} & \end{tikzcd}$$

with, by construction,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & C \\ Y' & \end{tikzcd} \quad \text{equivariant and} \quad \begin{tikzcd} C \arrow[r] \arrow[d] & B \\ B' & \end{tikzcd} \quad \text{primitive.}$$

We will say that (6) is an *equivariant-primitive decomposition (an E-P decomposition for short) of (3)*. Note that such a decomposition is not unique. We will use the following notation:

$$\begin{aligned}
&a = |K|;\\
&b := |\overline{G}|;\\
&c := |G|;\\
&e := \deg(C \to B);\\
&d := \deg(f);\\
&d' := \deg(f') = \deg(Z \to C).
\end{aligned}$$

Then, one has:

Lemma 2.3. *If $c > 1$ and $g_{B'} \geq 2$, then*

$$d \geq \sqrt{2\left(1 - \frac{1}{g_{B'}+1}\right)}\sqrt{\frac{g_{B'}+1}{g_B+1}}d' \geq \sqrt{\frac{4}{3}}\sqrt{\frac{g_{B'}+1}{g_B+1}}d'.$$

Proof. This is a direct computation. From lemma 2.2, $e \geq \sqrt{\frac{b(g_{B'}-1)+2}{g_B+1}}$ hence:

$$d = d'ae \geq d'a\sqrt{\frac{b(g_{B'}-1)+2}{g_B+1}} = d'\sqrt{\frac{a^2b(g_{B'}-1)+2a^2}{g_{B'}+1}}\sqrt{\frac{g_{B'}+1}{g_B+1}}.$$

But, also:

$$\frac{a^2b(g_{B'}-1)+2a^2}{g_{B'}+1} = \frac{ac(g_{B'}-1)+2a^2}{g_{B'}+1} \geq \frac{c(g_{B'}-1)+2}{g_{B'}+1} \geq \frac{2(g_{B'}-1)+2}{g_{B'}+1} = 2 - \frac{2}{g_{B'}+1} \geq \frac{4}{3}. \square$$

2.2. **Finitely many steps.** Let:

(7)
$$\begin{tikzcd}
Y_N \arrow[r, "\pi_N"] \arrow[d, "f_N"'] & Y_{N-1} \arrow[r, "\pi_{N-1}"] & \cdots & Y_1 \arrow[r, "\pi_1"] & Y_0 \\
B_N & & & &
\end{tikzcd}$$

be a diagram of proper k-curves, where $f_N : Y_N \to B_N$ is a non-constant morphism of proper k-curves and $\pi_k : Y_k \to Y_{k-1}$ is a Galois cover with Galois group G_k, $k = 1, \ldots, N$.

Using successive E-P decompositions, we construct from (7) a huge diagram of proper k-curves:

(8)

$$\begin{tikzcd}
Y_N \arrow[r] \arrow[dr, bend right, "/K_N"'] \arrow[rrrrrr, bend left, "f_N"] & Z_N \arrow[rrrr] \arrow[d, "/G_N"] & & & & C_N \arrow[r] \arrow[d, "/G_N"] & B_N \\
& Y_{N-1} \arrow[r] \arrow[dr, bend right, "/K_{N-1}"'] \arrow[rrrr, bend left, "f_{N-1}"] & Z_{N-1} \arrow[rr] \arrow[d, "/G_{N-1}"] & & C_{N-1} \arrow[r] \arrow[d, "/G_{N-1}"] & B_{N-1} & \\
& & Y_{N-2} \arrow[rr, bend left, "f_{N-2}"] & \cdots \quad \cdots & B_{N-2} & &
\end{tikzcd}$$

$$\cdots \qquad \cdots$$

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[dr, bend right, "/K_1"'] \arrow[rrr, bend left, "f_1"] & Z_1 \arrow[r] \arrow[d, "/G_1"] \arrow[dr, phantom, "\widetilde{\square}"] & C_1 \arrow[r] \arrow[d, "/G_1"] & B_1 \\
& Y_0 \arrow[r, "f_0"] & B_0, &
\end{tikzcd}$$

where:

$$\begin{tikzcd}
Y_k \arrow[r, "K_k"'] \arrow[rrr, bend left, "f_k"] & Z_k \arrow[r] \arrow[d, "\overline{G_k}"'] \arrow[dr, phantom, "\widetilde{\Box}"] & C_k \arrow[r] \arrow[d, "\overline{G_k}"] & B_k \\
& Y_{k-1} \arrow[r, "f_{k-1}"'] & B_{k-1} &
\end{tikzcd}$$

is an E-P decomposition of $f_k : Y_k \to B_k$, $k = 1, \dots, N$. We will use the following notation:

$$\begin{aligned}
&a_k = |K_k|;\\
&b_k := |\overline{G_k}|;\\
&c_k := |G_k|;\\
&e_k := \deg(C_k \to B_k);\\
&d_k := \deg(f_k),
\end{aligned}$$

and assume that $c_k > 1$ for all $k = 1, \dots, N$. Also, set $A_{k,N} := \prod_{k \le l \le N} a_l$, $E_{k,N} := \prod_{k \le l \le N} e_l$, $k = 1, \dots, N$ and $A_{N+1,N} = E_{N+1,N} = 1$.

Lemma 2.4. *Assume that $\gamma_{Y_k} = \gamma$, $k = 0, \dots, N$, $d_N = \gamma$ and $g_{B_N} = 0$. Then there exist two integers $0 \le N_1 \le N_0 \le N$ such that:*
- *$g_{B_k} = 0$, $A_{k+1,N} = E_{k+1,N} = 1$, $N_0 \le k \le N$;*
- *$g_{B_k} = 1$, $A_{k+1,N} = 1$, $E_{k+1,N} = 2$, $N_1 \le k < N_0$;*
- *$\gamma_{B_k} > 2$ (hence, in particular, $g_{B_k} \ge 3$), $E_{k+1,N} > 2$, $0 \le k < N_1$.*

Furthermore,
- *$g_{C_{N_0}} = 1$, unless $N_0 = N_1$;*
- *$N_1 \le \dfrac{\log(\gamma\sqrt{2})}{\log(\sqrt{\frac{3}{2}})}$ is bounded only in terms of γ;*
- *one can complete diagram (8) as follows:*

(9)
$$\begin{tikzcd}
Y_N \arrow[r, "\pi_N"] \arrow[dd] & \cdots \arrow[r] & Y_{N_0} \arrow[r, "\pi_{N_0}"] \arrow[d] \arrow[dr, phantom, "\widetilde{\Box}"] & Y_{N_0-1} \arrow[r, "\pi_{N_0-1}"] \arrow[d] \arrow[dr, phantom, "\widetilde{\Box}"] & \cdots \arrow[r, "\pi_{N_1+1}"] \arrow[dr, phantom, "\widetilde{\Box}"] & Y_{N_1} \arrow[d] \\
& & C_{N_0} \arrow[r, "\pi'_{N_0}"'] \arrow[d] \arrow[dr, phantom, "\widetilde{\Box}"] & B_{N_0-1} \arrow[r, "\pi'_{N_0-1}"'] \arrow[d] \arrow[dr, phantom, "\widetilde{\Box}"] & \cdots \arrow[r, "\pi'_{N_1+1}"'] \arrow[dr, phantom, "\widetilde{\Box}"] & B_{N_1} \arrow[d] \\
B_N \arrow[r, "\pi'_N"'] & \cdots \arrow[r, "\pi'_{N_0+1}"'] & B_{N_0} \arrow[r] & B'_{N_0-1} \arrow[r] & \cdots \arrow[r] & B'_{N_1}
\end{tikzcd}$$

where $g_{B'_k} = 0$ for $N_1 \le k < N_0$.

Proof. We begin with the following elementary lemma.

Lemma 2.5. *Let*

(10)
$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\pi"'] & B \\
B' &
\end{tikzcd}$$

be a diagram of proper k-curves with $\deg(C \to B) = 2$ and $\pi : C \to B'$ a Galois cover with Galois group G.

(1) *If $g_B = 0$ and $g_C \ge 2$ then (10) is equivariant.*

(2) *If $g_B = 0$ and $g_C = g_{B'} = 1$ then there exists a smooth proper k-curve B'' with $g_{B''} = 0$ and a cartesian square (up to normalization)*

(11)
$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\tilde{\square}"] & B \arrow[d] \\ B' \arrow[r] & B''. \end{tikzcd}$$

Proof. (1) Since $g_B = 0$ and $\deg(C \to B) = 2$, C is hyperelliptic. Let $i : C \tilde{\to} C$ denote the hyperelliptic involution. Then $C \to B$ is just $C \to C/\langle i \rangle$. Since $g_C \geq 2$, i is a unique hyperelliptic involution, hence lies in the center of $\mathrm{Aut}_k(C)$. In particular, $G \subset \mathrm{Aut}_k(C)$ normalizes $\langle i \rangle$, as required.

(2) Since $\deg(C \to B) = 2$, $C \to B$ is Galois with group say $\langle i \rangle$. As $g_B = 0$ and $g_C = 1$, $C \to B$ has exactly four ramified points $Q_1, \ldots, Q_4 \in C$. Regarding C as an elliptic curve with origin $Q_1 \in C$, one may identify $i : C \tilde{\to} C$ with the multiplication by -1 automorphism $[-1]_C : C \tilde{\to} C$. Let P'_1 denote the image of Q_1 in B' and regard B' as an elliptic curve with origin $P'_1 \in B'$. Then $\pi : C \to B'$ becomes an isogeny and, in particular, $\pi \circ [-1]_C = [-1]_{B'} \circ \pi$. Hence $B'' := B'/\langle [-1]_{B'} \rangle$ works. $\square$

Now, we carry out the proof of lemma 2.4. Set:
$$N_0 := \min\{0 \leq k \leq N \mid E_{k+1,N} = 1\}$$
and
$$N_1 := \min\{0 \leq k \leq N \mid E_{k+1,N} \leq 2\}.$$
Since $E_{N+1,N} = 1$, these are well-defined and satisfy $0 \leq N_1 \leq N_0 \leq N$. Since $E_{k+1,N}$ is monotonically non-increasing in k, one has $E_{k+1,N} = 1$ (resp. $= 2$, resp. > 2) for $N_0 \leq k \leq N$ (resp. $N_1 \leq k < N_0$, resp. $0 \leq k < N_1$).

For any $0 \leq k \leq N$ one has:
$$A_{k+1,N} E_{k+1,N} d_k = d_N = \gamma = \gamma_{Y_k} \leq d_k \gamma_{B_k},$$
hence:
$$A_{k+1,N} E_{k+1,N} \leq \gamma_{B_k}.$$

First, we shall prove that the following (I-i)-(I-iv) are all equivalent: (I-i) $N_0 \leq k \leq N$ (i.e., $E_{k+1,N} = 1$); (I-ii) $g_{B_k} = 0$; (I-iii) $\gamma_{B_k} = 1$; and (I-iv) $A_{k+1,N} = E_{k+1,N} = 1$. Indeed, if $N_0 \leq k \leq N$, or, equivalently, $E_{k+1,N} = 1$, then $C_l \tilde{\to} B_l$ for $k+1 \leq l \leq N$. Thus, B_N dominates B_k. Since $g_{B_N} = 0$, one has $g_{B_k} = 0$. Thus, (I-i) $\Longrightarrow$ (I-ii). It is clear that (I-ii) $\Longleftrightarrow$ (I-iii). If $\gamma_{B_k} = 1$, then $A_{k+1,N} E_{k+1,N} \leq 1$. Thus, (I-iii) $\Longrightarrow$ (I-iv). It is clear that (I-iv) $\Longrightarrow$ (I-i).

Next, we shall prove that the following (II-i)-(II-iv) are all equivalent: (II-i) $N_1 \leq k < N_0$ (i.e., $E_{k+1,N} = 2$); (II-ii) $g_{B_k} = 1$; (II-iii) $\gamma_{B_k} = 2$; and (II-iv) $A_{k+1,N} = 1$, $E_{k+1,N} = 2$. Indeed, if $N_1 \leq k < N_0$, one has $E_{l+1,N} = 2$ for $k \leq l < N_0$, hence $C_l \tilde{\to} B_l$ for $k \leq l < N_0$ and $C_{N_0} \to B_{N_0}$ is a double cover. Thus, C_{N_0} dominates B_k and $\gamma_{C_{N_0}} \leq 2$, since $g_{B_{N_0}} = 0$. Therefore, $\gamma_{B_k} \leq 2$. Since $\gamma_{B_k} > 1$ (as $k < N_0$), one has $\gamma_{B_k} = 2$. Moreover, by construction,
$$\begin{tikzcd} C_{N_0} \arrow[r] \arrow[d] & B_{N_0} \\ B_{N_0-1} & \end{tikzcd}$$
is not equivariant. So, from lemma 2.5 (1), $g_{C_{N_0}} = 1$, hence $g_{B_k} \leq 1$. But, since $\gamma_{Y_k} = \gamma$, one necessarily has $g_{B_k} \geq 1$, hence $g_{B_k} = 1$. Thus, (II-i) $\Longrightarrow$ (II-ii). If $\gamma_{B_k} = 2$, then $A_{k+1,N} E_{k+1,N} \leq 2$. However, if $E_{k+1,N} = 1$, that is, $N_0 \leq k \leq N$, then $\gamma_{B_k} = 1$: a contradiction. So, one must have $A_{k+1,N} = 1$, $E_{k+1,N} = 2$. Thus, (II-iii) $\Longrightarrow$ (II-iv). It is clear that (II-ii) $\Longrightarrow$ (II-iii) and (II-iv) $\Longrightarrow$ (II-i).

From these, the following (III-i)-(III-iii) are also all equivalent: (III-i) $0 \leq k < N_1$ (i.e., $E_{k+1,N} > 2$); (III-ii) $g_{B_k} \geq 2$; and (III-iii) $\gamma_{B_k} \geq 3$. (In particular, $g_{B_k} \geq 2$ automatically implies $g_{B_k} \geq 3$.)

The above proof of (II-i) $\Longrightarrow$ (II-ii) already shows that $g_{C_{N_0}} = 1$ if $N_0 > N_1$. (Take any $N_1 \leq k <$

N_0, say, $k = N_1$.) The bound for N_1 follows from the first inequality of lemma 2.3. Indeed, since $g_{B_k} \geq 3$, $0 \leq k < N_1$, one has:

$$\gamma \geq d_{N_1} \geq \left(\frac{3}{2}\right)^{\frac{N_1}{2}} \sqrt{\frac{g_{B_0}+1}{g_{B_{N_1}}+1}}\, d_0 \geq \frac{1}{\sqrt{2}} \left(\frac{3}{2}\right)^{\frac{N_1}{2}}.$$

Eventually, the last part of lemma 2.4 follows from lemma 2.5 (2). □

2.3. **Infinitely many steps.** We now carry out the proof of theorem 2.1. Recall that we start with a projective system of proper k-curves

$$\cdots \rightarrow Y_n \stackrel{\pi_n}{\rightarrow} Y_{n-1} \stackrel{\pi_{n-1}}{\rightarrow} \cdots \stackrel{\pi_1}{\rightarrow} Y_0$$

satisfying conditions (i), (ii), (iii) of theorem 2.1. According to condition (ii) and up to renumbering, one may assume that $\gamma_{Y_n} = \gamma$ for all $n \geq 0$ and G_n is cyclic of order ≥ 3 for all $n \geq 1$. Set:

$$\nu(\gamma) := \left[\frac{\log(\gamma\sqrt{2})}{\log(\sqrt{\frac{3}{2}})}\right].$$

For each $n > \nu(\gamma)$ define $\mathcal{F}_n$ to be the set of all diagrams (modulo isomorphism) of proper k-curves:

$$\begin{tikzcd}
Y_n \arrow[r, "\pi_n"] \arrow[dd, "f_n"'] \arrow[ddr, phantom, "\widetilde{\square}"] & Y_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[dd, "f_{n-1}"] \arrow[ddr, phantom, "\widetilde{\square}"] & \cdots \arrow[r, "\pi_{N_0+1}"'] \arrow[ddr, phantom, "\widetilde{\square}"] & Y_{N_0} \arrow[r, "\pi_{N_0}"] \arrow[d] \arrow[dd, bend right, "f_{N_0}"'] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{N_0-1} \arrow[r, "\pi_{N_0-1}"] \arrow[d, "f_{N_0-1}"] \arrow[dr, phantom, "\widetilde{\square}"] & \cdots \arrow[r, "\pi_{N_1+1}"] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{N_1} \arrow[r, "\pi_{N_1}"] \arrow[d, "f_{N_1}"] & Y_{N_1-1} \arrow[r, "\pi_{N_1-1}"] & \cdots \\
 & & & C_{N_0} \arrow[r, "\pi'_{N_0}"'] \arrow[d] & B_{N_0-1} \arrow[r, "\pi'_{N_0-1}"'] & \cdots \arrow[r, "\pi'_{N_1+1}"'] & B_{N_1} & & \\
B_n \arrow[r, "\pi'_n"'] & B_{n-1} \arrow[r, "\pi'_{n-1}"'] & \cdots \arrow[r, "\pi'_{N_0+1}"'] & B_{N_0} & & & & &
\end{tikzcd} \tag{12}$$

of the type constructed in subsection 2.2. (Strictly speaking, in the extreme case $N_0 = N_1$ (resp. $N_0 = n$), we do not consider C_{N_0}, $Y_{N_0} \rightarrow C_{N_0}$, $C_{N_0} \rightarrow B_{N_0}$ (resp. B_{N_0}, $f_{N_0} : Y_{N_0} \rightarrow B_{N_0}$, $C_{N_0} \rightarrow B_{N_0}$) as part of the data.) More precisely, $0 \leq N_1 \leq N_0 \leq n$; $N_1 \leq \nu(\gamma)$; $g_{B_k} = 0$, $\deg(f_k) = \gamma$, for $N_0 \leq k \leq n$; $g_{B_k} = 1$, $\deg(f_k) = \frac{\gamma}{2}$, for $N_1 \leq k < N_0$; $g_{C_{N_0}} = 1$, if $N_1 < N_0$; the square

$$\begin{tikzcd}
Y_k \arrow[r, "\pi_k"] \arrow[d, "f_k"'] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{k-1} \arrow[d, "f_{k-1}"] \\
B_k \arrow[r, "\pi'_k"'] & B_{k-1}
\end{tikzcd}$$

is cartesian up to normalization and G_k-equivariant, for $N_1 < k \leq n$, $k \neq N_0$; and the square

$$\begin{tikzcd}
Y_{N_0} \arrow[r, "\pi_{N_0}"] \arrow[d] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{N_0-1} \arrow[d, "f_{N_0-1}"] \\
C_{N_0} \arrow[r, "\pi'_{N_0}"'] & B_{N_0-1}
\end{tikzcd}$$

is cartesian up to normalization and G_{N_0}-equivariant, if $N_0 > N_1$. Then the maps $\phi_n : \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n$ defined by deleting the last vertical arrow f_n endows the $\mathcal{F}_n$, $n > \nu(\gamma)$ with a canonical structure of projective system $(\mathcal{F}_{n+1} \stackrel{\phi_n}{\rightarrow} \mathcal{F}_n)_{n > \nu(\gamma)}$. From lemma 2.4, $\mathcal{F}_n \neq \emptyset$, $n > \nu(\gamma)$. Since a projective system of non-empty finite sets is non-empty, to obtain the desired result, it would be enough to prove that $\mathcal{F}_n$ is finite, $n \gg \nu(\gamma)$. But this follows from condition (iii) and the two finiteness lemmas below.

2.3.1. *Finiteness lemmas.*

Lemma 2.6. (Genus 0 case) *Let c be an integer ≥ 3. Given a Galois cover $Y \to Z$ of proper k-curves with Galois group $\mathbb{Z}/c$ and such that $\gamma_Y = \gamma_Z = \gamma$, there are only finitely many isomorphism classes of equivariant diagrams:*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\mathbb{Z}/c"'] \arrow[dr, phantom, "\tilde{\square}"] & \mathbb{P}^1_k \arrow[d, "\mathbb{Z}/c"] \\ Z \arrow[r, "g"'] & \mathbb{P}^1_k \end{tikzcd}$$

with $\deg(f) = \deg(g) = \gamma$.

Proof. Since $\mathbb{P}^1_k \to \mathbb{P}^1_k$ is Galois with Galois group $\mathbb{Z}/c$, one can assume that it is given by $x \mapsto x^c$ (for some choice of the coordinate x of $\mathbb{P}^1_k$) and that the data of a diagram of covers:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\mathbb{Z}/c"'] \arrow[dr, phantom, "\tilde{\square}"] & \mathbb{P}^1_k \arrow[d, "\mathbb{Z}/c"] \\ Z \arrow[r, "g"'] & \mathbb{P}^1_k \end{tikzcd}$$

is equivalent to the data of field extensions:

$$\begin{tikzcd} k(Y) = k(Z)(g^{\frac{1}{c}}) & k(g^{\frac{1}{c}}) \arrow[l, hook'] \\ k(Z) \arrow[u, hook, "\mathbb{Z}/c"] \arrow[ur, phantom, "\tilde{\square}"] & k(g). \arrow[l, hook'] \arrow[u, hook, "\mathbb{Z}/c"'] \end{tikzcd}$$

From Kummer theory, the cyclic subgroup $\langle g \rangle \subset k(Z)^\times / k(Z)^{\times c}$ is uniquely determined by the extension $k(Z) \hookrightarrow k(Y)$. So, it is enough to prove that there are only finitely many $g' \in k(Z)^\times / k^\times$ such that (i) $\deg(g') = \gamma$ and (ii) $g' = g^i h^c$, for some $0 \leq i < c$, $(i, c) = 1$ and $h \in k(Z)^\times$. So, fix such i, and, for $j = 1,\ 2$, assume that $g'_j = g^i h_j^c$, $h_j \in k(Z)^\times$ with $\deg(g'_j) = \gamma$. Then $g'_1 {g'_2}^{-1} = (h_1 h_2^{-1})^c$. Assume that $g'_1 {g'_2}^{-1} \notin k^\times$, then $h_1 h_2^{-1} \notin k^\times$ and $\deg(g'_1 {g'_2}^{-1}) = c\deg(h_1 h_2^{-1})$. But, since $h_1 h_2^{-1} \notin k^\times$, $\deg(h_1 h_2^{-1}) \geq \gamma (= \gamma_Z)$, whereas $\deg(g'_1 {g'_2}^{-1}) \leq \deg(g'_1) + \deg(g'_2) = 2\gamma$. Whence a contradiction, since $c \geq 3$ by assumption. As a result, the number of isomorphism classes of diagrams as in lemma 2.6 is at most $\varphi(c) \leq c - 1$. □

Lemma 2.7. (Genus 1 case) *Let Y be a proper k-curve and let d be an integer ≥ 1. Denote by $\mathcal{E}_{Y,d}$ the set of all pairs (E, f), where E is an elliptic curve over k and $f : Y \to E$ is a finite morphism of degree d, regarded up to isomorphism of E as k-scheme. Then $\mathcal{E}_{Y,d}$ is finite.*

Proof. Fix $P \in Y$ and let $\mathcal{E}_{Y,d,P}$ be the set of all pairs (E, f), where E is an elliptic curve over k and $f : Y \to E$ is a finite morphism of degree d, regarded up to isomorphism of E as pointed k-scheme (or, equivalently, as k-abelian variety), such that $f(P) = 0$. Then $\mathcal{E}_{Y,d}$ is canonically in bijection with $\mathcal{E}_{Y,d,P}$ so it is enough to prove that $\mathcal{E}_{Y,d,P}$ is finite. Denote by $j_P : Y \to J_{Y|k}$ the canonical morphism induced by P from Y into its jacobian variety $J_{Y|k}$. Then, from the albanese property of $j_P : Y \to J_{Y|k}$, $\mathcal{E}_{Y,d,P}$ is canonically in bijection with the set of all pairs (E, ϕ), where E is an elliptic curve over k and $\phi : J_{Y|k} \twoheadrightarrow E$ is an epimorphism of abelian varieties, regarded up to isomorphism of E as k-abelian variety, such that $\phi \circ j_P : Y \to E$ has degree d. (Here, note that $\mathcal{E}_{Y,d,P} \neq \emptyset$ implies that $g_Y \geq 1$, hence $j_P : Y \to J_{Y|k}$ is a closed immersion.) For any such pair (E, ϕ), consider the factorization:

$$\begin{tikzcd} Y \arrow[r, hook, "j_P"] & J_{Y|k} \arrow[r, two heads] \arrow[d, "\phi"'] & J_{Y|k}/\ker(\phi)^0 \arrow[dl, "\phi^0"] \\ & E \simeq J_{Y|k}/\ker(\phi), & \end{tikzcd}$$

where $\ker(\phi)^0$ denotes the connected component of 0 in $\ker(\phi)$. Then, since $\phi \circ j_P : Y \to E$ has degree d, $\phi^0 : J_{Y|k}/\ker(\phi)^0 \twoheadrightarrow J_{Y|k}/\ker(\phi)$ has degree $[\ker(\phi) : \ker(\phi)^0] \leq d$.

So, let $\mathcal{E}^0_{Y,d,P}$ denote the set of all pairs $(E, \phi) \in \mathcal{E}_{Y,d,P}$ such that $\ker(\phi) = \ker(\phi)^0$ is connected. Assume that for any $d'|d$, $\mathcal{E}^0_{Y,d',P}$ is finite. Then, for any $(E, \phi) \in \mathcal{E}_{Y,d,P}$, there are only finitely many possibilities for $\ker(\phi)^0$. But since $[\ker(\phi) : \ker(\phi)^0] \leq d$, for each $\ker(\phi)^0$ there are only finitely many possibilities for $\ker(\phi)$. Thus one only has to prove that $\mathcal{E}^0_{Y,d,P}$ is finite.

But $\mathcal{E}^0_{Y,d,P}$ is in canonical bijective correspondence with the set of all abelian subvarieties $K \subset J_{Y|k}$ of codimension 1 such that $Y \overset{j_P}{\hookrightarrow} J_{Y|k} \twoheadrightarrow J_{Y|k}/K$ has degree d. As $\deg(Y \to J_{Y|k}/K) = K \cdot Y$ and, since Y is numerically equivalent to $\frac{1}{(g-1)!}\Theta^{g-1}$ (Poincaré's formula, cf. [GH78, Chap. 2, Sect. 7]), where Θ stands for the usual theta divisor on $J_{Y|k}$, one has, actually, $d = \deg(Y \to J_{Y|k}/K) = K \cdot Y = \frac{1}{(g-1)!} K \cdot \Theta^{g-1} = \frac{1}{(g-1)!}\deg_\Theta(K)$. Whence $\deg_\Theta(K) = d(g-1)!$. But it is classically known[2] that, given an abelian variety, there are only finitely many abelian subvarieties with bounded degree (with respect to a fixed ample divisor). $\square$

2.3.2. *End of the proof.* To conclude the proof, observe first that for a fixed arrow $Y_n \to B_n$ (resp. $Y_n \to C_n$) when $N_0 < n$ (resp. $N_0 = n$), there are only finitely many diagrams of the type (12) in $\mathcal{F}_n$ which contains this arrow. Indeed, this follows from the fact that a diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\tilde{\square}"] & Y \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

that is cartesian up to normalization is uniquely determined (modulo isomorphism) by:

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \\ B' & \end{tikzcd}$$

(birationally, $k(B) = k(B') \cap k(Y) \subset k(Y')$), and (if $N_1 < N_0 < n$) the fact that $Y_{N_0} \to C_{N_0} \to B_{N_0}$ is determined up to finite possibilities (modulo isomorphism) by $Y_{N_0} \to B_{N_0}$, since the latter is a finite cover of curves over an algebraically closed field.

So, let $n > \nu(\gamma)$. Note that $n > N_1$, since $N_1 \leq \nu(\gamma)$. If $N_0 < n$, then, from lemma 2.6, there are only finitely many choices for the arrow $f_n : Y_n \to B_n$. If $N_0 = n$, then from lemma 2.7, there are only finitely many choices for the arrow $Y_n \to C_n$. In any case, from the above considerations, there are only finitely many such diagram in $\mathcal{F}_n$.

[2]For lack of suitable reference we recall the main argument. Let A be an abelian variety over an algebraically closed field k and Θ an ample divisor of A (hence, 3Θ is very ample). Let K be an abelian subvariety of A and set $e = \dim(K)$ and $\delta = \deg_\Theta(K)$. Then the Hilbert polynomial of K embedded in $\mathbb{P}^N$ *via* $(3\Theta)|_K$ is $3^e\delta T^e$ (cf. [CT12b, Lem. 1.4]) and, in particular, depends only on δ and e. Classical theory of Hilbert schemes shows that there exists a scheme $S_{\delta,e}$ of finite type over k and an abelian subscheme $\mathcal{K}$ of $A \times_k S_{\delta,e}$ over $S_{\delta,e}$ such that any abelian subvariety K of A whose embedding in $\mathbb{P}^N$ *via* $(3\Theta)|_K$ has Hilbert polynomial $3^e\delta T^e$ is the pull-back of $\mathcal{K} \to S_{\delta,e}$ by a unique morphism $f_K : \mathrm{Spec}(k) \to S_{\delta,e}$. But then, it follows from [Mi86, Prop. 20.3] (taking care that the statement there requires the base scheme S to be connected) that for each of the *finitely many* connected components C of $S_{\delta,e}$ there exists an abelian subvariety $K_C \hookrightarrow A$ such that $\mathcal{K} \times_{S_{\delta,e}} C = K_C \times_k C$.

2.4. **A general statement.** It follows from the classification of finite subgroups of $\mathrm{PGL}_2(k)$ that the only possible automorphism groups for a Galois cover of a genus 0 curve are the cyclic groups $\mathbb{Z}/m$, $m \geq 1$, the dihedral groups D_{2m}, $m \geq 2$, the alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ and the symmetric group $\mathcal{S}_4$. Also, since a Galois cover of a genus 1 curve is automatically etale, the only possible automorphism groups for a Galois cover of a genus 1 curve E are the finite quotients of $\pi_1(E) \simeq \widehat{\mathbb{Z}}^2$. We will say that a finite group that appears as the automorphism group of a Galois cover of genus 0 curves or of genus 1 curves is *exceptional*. Observe that an exceptional finite group G admits a cyclic subgroup of order ≥ 3, unless $G \simeq (\mathbb{Z}/2)^r$ with $r \leq 2$. We will say that a finite group is *very exceptional*, if $G \simeq (\mathbb{Z}/2)^r$ with $r \leq 2$.

Corollary 2.8. *Let* $\cdots \stackrel{\pi_{n+2}}{\to} Y_{n+1} \stackrel{\pi_{n+1}}{\to} Y_n \stackrel{\pi_n}{\to} \cdots \stackrel{\pi_1}{\to} Y_0$ *be a projective system of proper k-curves with* $\pi_n : Y_n \to Y_{n-1}$ *a Galois cover of group* G_n, $n \geq 1$. *Suppose that one does not have* $\gamma_{Y_n} \to +\infty$ $(n \to \infty)$. *(Or, equivalently, suppose that γ_{Y_n} is constant for $n \gg 0$.) Then:*

(1) *For all but finitely many $n \geq 0$, G_n is exceptional.*
(2) $\underleftarrow{\lim}\mathcal{F}_n \neq \emptyset$, *where $\mathcal{F}_n$ $(n \gg 0)$ is as in subsection 2.3.*

Proof. Assertion (1) follows from the definition of exceptional finite groups, lemma 2.4 and the fact that $N_1 \leq \nu(\gamma)$ is bounded only in terms of γ in lemma 2.4.

As for assertion (2), one may assume, up to renumbering, that, for all $n \geq 0$, $|G_n| > 1$, G_n is exceptional, and $\gamma_n = \gamma$ (constant). Then, from lemma 2.4, one always has $\mathcal{F}_n \neq \emptyset$, $n > \nu(\gamma)$. Define $\mathcal{F}'_n$ to be the image of $\mathcal{F}_{n+2}$ in $\mathcal{F}_n$, which is also nonempty. Then $\mathcal{F}'_n$, $n > \nu(\gamma)$ form a projective subsystem of $(\mathcal{F}_{n+1} \to \mathcal{F}_n)_{n>\nu(\gamma)}$, and $\underleftarrow{\lim}\mathcal{F}_n = \underleftarrow{\lim}\mathcal{F}'_n$. Thus, it suffices to prove that $\mathcal{F}'_n$ is finite for $n > \nu(\gamma)$.

First, if G_n is (exceptional but) not very exceptional, G_n admits a cyclic subgroup of order ≥ 3. So, in this case, $\mathcal{F}_n$ is finite by lemmas 2.6 and 2.7, hence $\mathcal{F}'_n$ is finite, a fortiori. Similarly, if G_{n+1} (resp. G_{n+2}) is not very exceptional, then $\mathcal{F}_{n+1}$ (resp. $\mathcal{F}_{n+2}$) is finite, hence $\mathcal{F}'_n$ is finite, a fortiori. Next, define $\mathcal{F}^{g=1}_n$ to be the subset of elements (12) of $\mathcal{F}_n$ with $N_0 = n$. Then $\mathcal{F}^{g=1}_n$ is finite by lemma 2.7.

So, it suffices to prove the finiteness of $\mathcal{F}'_n \setminus \mathcal{F}^{g=1}_n$ under the extra assumption that G_n, G_{n+1} and G_{n+2} are all (non-trivial and) very exceptional. This follows from (2) of the following finiteness lemma.

Lemma 2.9. (1) *Let* $Y'' \stackrel{(\mathbb{Z}/2)^{r''}}{\to} Y' \stackrel{(\mathbb{Z}/2)^{r'}}{\to} Y$ *be a sequence of Galois covers of proper k-curves with* $r', r'' \in \{1, 2\}$, $\gamma_{Y''} = \gamma_{Y'} = \gamma_Y = \gamma$. *Then either there are only finitely many isomorphism classes of equivariant diagrams:*

$$\begin{tikzcd}
Y'' \arrow[r, "(\mathbb{Z}/2)^{r''}"] \arrow[d, "f''"'] \arrow[dr, phantom, "\tilde{\square}"] & Y' \arrow[r, "(\mathbb{Z}/2)^{r'}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\tilde{\square}"] & Y \arrow[d, "f"] \\
\mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r''}"'] & \mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r'}"'] & \mathbb{P}^1_k,
\end{tikzcd}$$

or $Y'' \to Y$ is Galois with Galois group $(\mathbb{Z}/2)^2$ (hence, in particular, $r' = r'' = 1$ in the latter case).

(2) *Let* $Y''' \stackrel{(\mathbb{Z}/2)^{r'''}}{\to} Y'' \stackrel{(\mathbb{Z}/2)^{r''}}{\to} Y' \stackrel{(\mathbb{Z}/2)^{r'}}{\to} Y$ *be a sequence of Galois covers of proper k-curves with* $r', r'', r''' \in \{1, 2\}$, $\gamma_{Y'''} = \gamma_{Y''} = \gamma_{Y'} = \gamma_Y = \gamma$. *Then there are only finitely many isomorphism classes of equivariant diagrams:*

$$\begin{tikzcd}
Y''' \arrow[r, "(\mathbb{Z}/2)^{r'''}"] \arrow[d, "f'''"'] \arrow[dr, phantom, "\tilde{\square}"] & Y'' \arrow[r, "(\mathbb{Z}/2)^{r''}"] \arrow[d, "f''"'] \arrow[dr, phantom, "\tilde{\square}"] & Y' \arrow[r, "(\mathbb{Z}/2)^{r'}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\tilde{\square}"] & Y \arrow[d, "f"] \\
\mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r'''}"'] & \mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r''}"'] & \mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r'}"'] & \mathbb{P}^1_k.
\end{tikzcd}$$

Proof. (1) First, consider the case where $Y'' \to Y$ is Galois with Galois group G. Then it follows again from the classification of finite subgroups of $\mathrm{PGL}_2(k)$ that, necessarily, only one of the three following cases can occur:
- $r' = r'' = 1$ and $G \simeq \mathbb{Z}/4$;
- $(r', r'') = (1, 2)$ or $(2, 1)$ and $G \simeq D_8$;
- $r' = r'' = 1$ and $G \simeq (\mathbb{Z}/2)^2$.

In the first two cases, G admits a cyclic subgroup of order 4. Thus, the finiteness follows again from lemma 2.6.

Next, consider the case where $Y'' \to Y$ is not Galois. Assume first that $r'' = r' = 1$ and fix a diagram

$$\begin{tikzcd}
Y'' \arrow[r, "(\mathbb{Z}/2)^{r''}"] \arrow[d, "f''"'] \arrow[dr, phantom, "\tilde{\Box}"] & Y' \arrow[r, "(\mathbb{Z}/2)^{r'}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\tilde{\Box}"] & Y \arrow[d, "f"] \\
B'' = \mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r''}"'] & B' = \mathbb{P}^1_k \arrow[r, "(\mathbb{Z}/2)^{r'}"'] & B = \mathbb{P}^1_k.
\end{tikzcd}$$

Let $\hat{Y} \to Y$ denote the Galois closure of $Y'' \to Y$ and $\hat{B} \to B$ the Galois closure of $B'' \to B$. Note that $\hat{Y} \to Y$ depends only on the data $Y'' \overset{(\mathbb{Z}/2)^{r''}}{\to} Y' \overset{(\mathbb{Z}/2)^{r'}}{\to} Y$.

Then Y'' has at most one distinct conjugate - say Y''_1 under the automorphism group of $\hat{Y} \to Y$. Similarly, B'' has at most one distinct conjugate - say B''_1 under the automorphism group of $\hat{B} \to B$. As, by assumption, $Y'' \neq \hat{Y}$, this implies that $\hat{Y} \to Y''$ has degree 2 and, as the natural restriction morphism $\mathrm{Aut}(\hat{Y}/Y'') \hookrightarrow \mathrm{Aut}(\hat{B}/B'')$ is a monomorphism, one obtains that $\hat{B} \to B''$ has degree 2 as well and that the square:

$$\begin{tikzcd}
\hat{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\tilde{\Box}"] & Y'' \arrow[d] \\
\hat{B} \arrow[r] & B''
\end{tikzcd}$$

is cartesian up to normalization. Now, consider the commutative square:

$$\begin{tikzcd}
 & \hat{B} \arrow[dl, "\mathbb{Z}/2"'] \arrow[dr, "\mathbb{Z}/2"] & \\
B'' \arrow[dr, "\mathbb{Z}/2"'] & & B''_1 \arrow[dl, "\mathbb{Z}/2"] \\
 & B' &
\end{tikzcd}$$

Since $g_{B'} = g_{B''} = g_{B''_1} = 0$, each cover $B'' \to B'$ and $B''_1 \to B'$ is ramified at exactly two points. Let R and R_1 denote the branch locus of $B'' \to B'$ and $B''_1 \to B'$ respectively. There are three possible cases:
- $|R \cap R_1| = 2$. Then $B'' = B''_1$ hence $\hat{B} = B''$: a contradiction;
- $|R \cap R_1| = 0$. Then $g_{\hat{B}} = 1$ so, from lemma 2.7, there are only finitely many possibilities for the arrow $\hat{Y} \to \hat{B}$ hence for the arrow $Y'' \to B''$.
- $|R \cap R_1| = 1$. Let P' denote the common branch point in $R \cap R_1$ and P''_1 its lifting to B''_1. From Abhyankar's lemma, $\hat{B} \to B''_1$ is unramified at P''_1 so it is ramified at exactly two points. Hence $g_{\hat{B}} = 0$. but, then $\hat{B} \to B$ is a degree 8 Galois cover of genus 0 curves. So, it follows once again from the classification of finite subgroups of $\mathrm{PGL}_2(k)$ that the only possibilities for its automorphism group are $\mathbb{Z}/8$ or D_8, which both contain a cyclic subgroup of order ≥ 3. So, from lemma 2.6, there are only finitely many possibilities for the arrow $\hat{Y} \to \hat{B}$. (In fact, the abelian group $\mathbb{Z}/8$ does not occur, since $B'' \to B$ is non-Galois.)

The three other cases reduce to the case $r'' = r' = 1$. Indeed,
- If $r' = 2$, $r'' = 1$. Since $Y'' \to Y$ is not Galois, there exists an automorphism σ' of $Y' \to Y$ such

that ${}^{\sigma'}Y'' \neq Y''$ hence $Y'' \to Y' \to Y'/\langle\sigma'\rangle$ satisfies the hypotheses of the $r'' = r' = 1$ case. So, if σ'_B denotes the automorphism of $B' \to B$ corresponding to σ, there are only finitely many possibilities for the diagram:

$$\begin{tikzcd}
Y'' \arrow[r, "\mathbb{Z}/2"] \arrow[d, "f''"'] \arrow[dr, phantom, "\tilde{\square}"] & Y' \arrow[r, "\mathbb{Z}/2"] \arrow[d, "f'"'] \arrow[dr, phantom, "\tilde{\square}"] & Y'/\langle\sigma'\rangle \arrow[d, "\overline{f}'"] \\
B'' \arrow[r, "\mathbb{Z}/2"'] & B' \arrow[r, "\mathbb{Z}/2"'] & B'/\langle\sigma'_B\rangle,
\end{tikzcd}$$

hence, in particular, for the arrow $Y'' \to B''$.
- If $r' = 1$ or 2, $r'' = 2$. If $r' = 2$, by the same argument as in the case $r' = 2$, $r'' = 1$, one may reduce to the case $r' = 1$. If $r' = 1$, since $Y'' \to Y$ is not Galois, ${}^{\sigma'}Y'' \neq Y''$, where σ' denotes the non-trivial automorphism of $Y' \to Y$. So, if Z_1, Z_2, Z_3 denote the three non-trivial intermediate covers of $Y'' \to Y'$, there exists $1 \leq i \leq 3$ such that ${}^{\sigma'}Z_i \neq Z_i$ hence $Z_i \to Y' \to Y$ satisfies the hypotheses of the $r'' = r' = 1$ case and one can conclude as in the $r' = 2$, $r'' = 1$ case.

(2) Apply (1) to $Y''' \to Y'' \to Y'$, then the only remaining case is that $Y''' \to Y'$ is Galois with Galois group $(\mathbb{Z}/2)^2$. In this case, however, the proof is completed by applying (1) to $Y''' \to Y' \to Y$. □

3. Proof of theorem 1.1

3.1. **Preliminaries.** In this subsection, we recall some of the basic notation and results introduced in [CT12c] and that we will re-use in the proof of theorem 1.1. We refer to [CT12c, Sect. 3.1 and Sect. 3.2] for more details.

3.1.1. *Group-theoretical preliminaries.* Let $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ be a closed subgroup. For any $n \geq 0$, let $(\ )_n : \mathrm{GL}_m(\mathbb{Z}_\ell) \to \mathrm{GL}_m(\mathbb{Z}/\ell^n)$ denote the reduction modulo ℓ^n morphism and write $G(n) := G \cap (Id + \ell^n \mathrm{M}_m(\mathbb{Z}_\ell))$ for the kernel of $G \twoheadrightarrow G_n$. Recall that the $G(n)$, $n \geq 0$ form a fundamental system of open neighborhoods of 1 in G. Also, write $\Phi(G)$ for its Frattini subgroup (that is the intersection of all maximal open subgroups of G) and d_G for its dimension as ℓ-adic analytic space.

Lemma 3.1. (1) *$[G(n) : G(n+1)] = \ell^{d_G}$, $n \gg 0$;*
(2) *If $r(G(n))$ denotes the minimal number of topological generators of $G(n)$ then $r(G(n)) = d_G$, $n \gg 0$;*
(3) *$G(n+1)$ is the Frattini subgroup $\Phi(G(n))$ of $G(n)$, $n \gg 0$.*

We are going to associate with G a projective system $(\mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G))_{n\geq 0}$ of finite sets of open subgroups of G. For each $n \geq 1$, let $\mathcal{H}_n(G)$ denotes the set of all open subgroups $U \subset G$ such that $\Phi(G(n-1)) \subset U$ but $G(n-1) \not\subset U$ and set $\mathcal{H}_0(G) := \{G\}$. Then the $\mathcal{H}_n(G)$, $n \geq 0$ satisfy the following elementary properties:

Lemma 3.2. (1) *$\mathcal{H}_n(G)$ is finite, $n \geq 0$.*
(2) *The maps $\phi_n : \mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G)$, $U \mapsto U\Phi(G(n-1))$ (with the convention that $\Phi(G(-1)) = G$) endow the $\mathcal{H}_n(G)$, $n \geq 0$ with a canonical structure of projective system $(\mathcal{H}_{n+1}(G) \stackrel{\phi_n}{\to} \mathcal{H}_n(G))_{n\geq 0}$.*
(3) *For any $\underline{H} := (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_n(G)$,*

$$H[\infty] := \varprojlim H[n] = \bigcap_{n\geq 0} H[n] \subset G$$

is a closed but not open subgroup of G.
(4) *For any closed subgroup $H \subset G$ such that $G(n-1) \not\subset H$ there exists $U \in \mathcal{H}_n(G)$ such that $H \subset U$.*
(5) *For $n \gg 0$, $\mathcal{H}_n(G)$ is the set of all open subgroups $U \subset G$ such that $G(n) \subset U$ but $G(n-1) \not\subset U$.*

3.1.2. *General remarks.* From now on and till the end of section 3, we let k be a field of characteristic 0, X a smooth, separated, geometrically connected curve over k, and $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ a GLP representation. We retain the notation of the introduction. In particular, we set $G := \rho(\pi_1(X))$, $G^{geo} := \rho(\pi_1(X_{\overline{k}}))$, and $G_x := \rho \circ x(\Gamma_{\kappa(x)})$, $x \in X^{cl}$.

(1) For any open subgroup $U \subset G$ such that $G(n) \subset U$ one has $[G : U] \leq [G : G(n)] =: B_n$. Conversely, since G is a finitely generated profinite group, for any integer $B \geq 1$ the set $S(G, B)$ of all open subgroups $U \subset G$ with $[G : U] \leq B$ is finite. So:

$$\bigcap_{U \in S(G,B)} U \subset G$$

is again an open subgroup of G, hence contains $G(n_B)$ for some integer $n_B \geq 0$. As a result, the second assertion of theorem 1.1 is also equivalent to the following: *There exists an integer $n_{\rho,d} \geq 0$ such that $G(n_{\rho,d}) \subset G_x$, $x \in X^{cl,\, \leq d} \smallsetminus X_{\rho,d}$.*

(2) For any open subgroup $U \subset G$ let $X_U \to X$ denote the corresponding etale cover; it is defined over a finite extension k_U of k and it satisfies the following two properties:
  (a) $X_U \times_{k_U} \overline{k} \to X_{\overline{k}}$ is the etale cover corresponding to the inclusion of open subgroups $G^{geo} \cap U \subset G^{geo}$.
  (b) For any closed point $x \in X$, $G_x \subset U$ (up to conjugacy) if and only if $x : \mathrm{Spec}(\kappa(x)) \to X$ lifts to a $\kappa(x)$-rational point:

$$\begin{tikzcd} X_U \arrow[d] & \\ X & \mathrm{Spec}(\kappa(x)) \arrow[l, "s"] \arrow[ul, dotted, "x_U"'] \end{tikzcd}$$

Write g_U for the genus of (the smooth compactification of) U and γ_U for its ($\overline{k}$-)gonality. It follows from (a) that $g_U = g_{G^{geo}\cap U}$ and that $\gamma_U = \gamma_{G^{geo}\cap U}$.

3.2. **A key geometrical result and its corollaries.** One of the main results of [CT12c] (theorem 3.4) is that, *for any closed but not open subgroup $H \subset G^{geo}$,*

$$\lim_{n \mapsto \infty} g_{HG^{geo}(n)} = +\infty.$$

Our aim here is to improve this statement, as follows.

Theorem 3.3. *For any closed but not open subgroup $H \subset G^{geo}$ one has*

$$\lim_{n \mapsto \infty} \gamma_{HG^{geo}(n)} = +\infty.$$

Remark 3.4. Given a cover of proper k-curves $f : Y \to Z$ one always has $\gamma_Z \leq \gamma_Y \leq \deg(f)\gamma_Z$. So, given a sequence of cartesian squares of covers of proper k-curves

$$\begin{tikzcd} \cdots \arrow[r] & Y'_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\tilde{\square}"] & Y'_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\tilde{\square}"] & \cdots \arrow[r] \arrow[dr, phantom, "\tilde{\square}"] & Y'_0 \arrow[d] \\ \cdots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & \cdots \arrow[r] & Y_0, \end{tikzcd}$$

one has $\gamma_{Y_n} \to \infty$ if and only if $\gamma_{Y'_n} \to \infty$. This will allow us to perform finitely many arbitrary base changes in our argument below.

To prove theorem 3.3, we make a few reductions. First, up to replacing k by $\overline{k}$, we shall assume, without loss of generality, that k is algebraically closed, till the end of subsection 3.2.2. Thus, in particular, $G = G^{geo}$.

For technical reasons (see Lemma 3.6), we also need to ensure that the (solvable) radical of $\mathrm{Lie}(G)$ is abelian (in the following, we will simply say that an ℓ-adic representation with this property has abelian radical). So, set $\mathfrak{g} := \mathrm{Lie}(G)$ and write $r(\mathfrak{g})$ for the radical of $\mathfrak{g}$. One can find[3] a (non-unique) sequence of closed normal subgroups of G

$$D \subset R \subset G$$

corresponding under the Lie algebra functor to the inclusion of ideals of $\mathfrak{g}$

$$[r(\mathfrak{g}), r(\mathfrak{g})] \subset r(\mathfrak{g}) \subset \mathfrak{g}.$$

Then, by construction, the ℓ-adic representation

$$\overline{\rho} : \pi_1(X) \to G \to G/D \ (\hookrightarrow \mathrm{GL}_{\overline{m}}(\mathbb{Z}_\ell) \text{ for some } \overline{m})$$

has abelian radical and is still (G)LP. We claim that it is enough to prove Theorem 3.3 for $\overline{\rho}$. This is a consequence of the following three elementary observations. Given a subgroup $U \subset G$, write $\overline{U} := UD/D \subset G/D$.

(1) As $HG(n) \subset HDG(n)$ and $\rho^{-1}(HDG(n)) = \overline{\rho}^{-1}(\overline{H}\overline{G(n)})$ one has

$$\gamma_{HG(n)} \geq \gamma_{HDG(n)} = \gamma_{\overline{H}\overline{G(n)}}.$$

Thus, it is enough to prove that $\lim\limits_{n \mapsto \infty} \gamma_{\overline{H}\overline{G(n)}} = \infty$. Note that $\overline{G(n)}$, $n \geq 0$ is a fundamental system of open normal neighborhoods of 1 in $\overline{G}$.

(2) In Theorem 3.3, the property $\lim\limits_{n \mapsto \infty} \gamma_{HG(n)} = +\infty$ is equivalent to $\lim\limits_{n \mapsto \infty} \gamma_{HU_n} = +\infty$ for any fundamental system $\{U_n\}_{n \geq 0}$ of open normal neighborhoods of 1 in G. In terms of Lie agebras, this is equivalent to saying that In particular, to prove that $\lim\limits_{n \mapsto \infty} \gamma_{\overline{H}\overline{G(n)}} = \infty$ it is enough to prove that $\lim\limits_{n \mapsto \infty} \gamma_{\overline{H}\,\overline{G}(n)} = \infty$, where $\{\overline{G}(n)\}_{n \geq 0}$ is the fundamental system of open normal neighborhoods of 1 in $\overline{G}$ defined by any choice of an embedding of ℓ-adic Lie groups $\overline{G} \subset \mathrm{GL}_{\overline{m}}(\mathbb{Z}_\ell)$.

(3) If $H \subset G$ is closed but *not open* in G then $\overline{H} \subset \overline{G}$ is not open in $\overline{G}$ as well. In terms of Lie algebras, this is equivalent to saying that for a Lie subalgebra $\mathfrak{h} \subsetneq \mathfrak{g}$, one has $\mathfrak{h} + [r(\mathfrak{g}), r(\mathfrak{g})] \subsetneq \mathfrak{g}$. But, as $\mathfrak{g}^{\mathrm{ab}} = 0$ (i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$), $r(\mathfrak{g}) = r(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent [B72a, §5.3, Theorem 1 and Remark 2 to Definition 3], hence its Frattini subalgebra is $[r(\mathfrak{g}), r(\mathfrak{g})]$ [M67, §2, Corollary 2]. Now, suppose $\mathfrak{h} + [r(\mathfrak{g}), r(\mathfrak{g})] = \mathfrak{g}$. Then one has $(\mathfrak{h} \cap r(\mathfrak{g})) + [r(\mathfrak{g}), r(\mathfrak{g})] = r(\mathfrak{g})$, which in turn implies $r(\mathfrak{g}) = \mathfrak{h} \cap r(\mathfrak{g}) \subset \mathfrak{h}$, hence $\mathfrak{g} = \mathfrak{h}$. This contradicts the assumption $\mathfrak{h} \subsetneq \mathfrak{g}$.

So, from now on, we also assume that $\mathfrak{g}$ has abelian radical.'

Next, from remark 3.4, one can replace X by $X_{G(n)}$ for any $n \geq 0$. In particular, we shall assume that $G = G(n_0)$ with $n_0 \geq 1$ (resp. $n_0 \geq 2$) for $\ell \neq 2$ (resp. $\ell = 2$). Thus, in particular, G is a pro-ℓ group.

---

[3]More precisely, the following always holds. Let G be a compact ℓ-adic Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g} \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_n$ be a finite decreasing sequence of ideals which are stable under every Lie algebra automorphism of $\mathfrak{g}$. Then there exists a decreasing sequence $G \supset H_1 \supset \cdots \supset H_n$ of *normal* closed subgroups of G such that $\mathfrak{h}_i$ is the Lie algebra of H_i for $i = 1, \ldots, n$. Indeed, since the Lie algebra functor commutes with finite intersections, it is enough to prove the statement for $n = 1$. So let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal which is stable under every Lie algebra automorphism of $\mathfrak{g}$. From [**?**, Chapter 3, §7, Proposition 2], there exists an open subgroup $G' \subset G$ and a normal closed subgroup $H' \subset G'$ such that $\mathfrak{h}$ is the Lie algebra of H'. Set $H := \cap_{g \in G/G'} gH'g^{-1}$. This is a closed normal subgroup of G contained in H' so it is enough to show that H is open in H'. But for every $g \in G$ one has $\mathrm{Lie}(gH'g^{-1}) = \mathrm{Ad}(g)(Lie(H')) = \mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h} = \mathrm{Lie}(H')$. So the conclusion follows from [**?**, Chapter 3, §7, Theorem 2].

3.2.1. *A group-theoretical lemma.* We begin with a group-theoretical lemma.

Lemma 3.5. *Consider the sequence of open subgroups*

$$\cdots \subset HG(i+1) \subset HG(i) \subset \cdots \subset HG(1) \subset G \tag{13}$$

Then,

(1) $HG(n+n_0)$ *is normal in* $HG(n)$, $n \geq 0$;
(2) $HG(n)/HG(n+n_0) \simeq (\mathbb{Z}/\ell^{n_0})^{\Delta}$, $n \gg 0$ *where* $\Delta = \dim(G) - \dim(H) > 0$.

Proof. (1) For short, write $\mathrm{GL} := \mathrm{GL}_m(\mathbb{Z}_\ell)$. A direct computation shows that $[\mathrm{GL}(n), \mathrm{GL}(n')] \subset \mathrm{GL}(n+n')$, $n, n' \geq 0$. Since G is a closed subgroup of GL one also has $[G(n), G(n')] \subset G(n+n')$, $n, n' \geq 0$. For any $h, h' \in H$, $g_n \in G(n)$, $g_{n+n_0} \in G(n+n_0)$, $(h'g_n)(hg_{n+n_0})(h'g_n)^{-1} = h'(g_n h g_n^{-1})(g_n g_{n+n_0} g_n^{-1})h'^{-1} \in HG(n+n_0)$ if and only if $g_n h g_n^{-1} \in HG(n+n_0)$. But $g_n h g_n^{-1} = h[h^{-1}, g_n] \in H[G(n_0), G(n)] \subset HG(n+n_0)$ (recall that $G = G(n_0)$ by assumption).

(2) Since $G(n+n_0) \subset G(n)$, one has $(HG(n+n_0)) \cap G(n) = H(n)G(n+n_0)$, $n \geq 0$. Set:

$$Q_n := HG(n)/HG(n+n_0) \simeq G(n)/((HG(n+n_0) \cap G(n)) = G(n)/(H(n)G(n+n_0)),\ n \geq 0.$$

One has canonical isomorphisms:

$$(G(n)/G(n+n_0))/(H(n)/H(n+n_0)) \simeq G(n)/(H(n)G(n+n_0)) \simeq Q_n,\ n \geq 0.$$

Let d_G and d_H denote the dimensions of G and H as ℓ-adic analytic spaces.

Assume now that n is large enough so that conditions (1), (2), (3) of lemma 3.1 are fulfilled for both G and H. Since $[G(n), G(n)] \subset [G(n), G(n_0)] \subset G(n+n_0)$ (use, again, that $G = G(n_0)$), the group $G(n)/G(n+n_0)$ is abelian. Also, from (3) of lemma 3.1, $G(n+n_0) = \Phi^{n_0}(G(n))$. But $G = G(n_0)$ is a pro-ℓ group and the Frattini subgroup of a pro-ℓ group L being generated by $L^{\ell}[L, L]$, $G(n)/G(n+n_0)$ has exponent $\leq \ell^{n_0}$. So, from (2) of lemma 3.1, one gets an epimorphism $(\mathbb{Z}/\ell^{n_0})^{d_G} \twoheadrightarrow G(n)/G(n+n_0)$ which, from (1) of lemma 3.1, is actually an isomorphism $(\mathbb{Z}/\ell^{n_0})^{d_G} \tilde{\rightarrow} G(n)/G(n+n_0)$. Similarly, $(\mathbb{Z}/\ell^{n_0})^{d_H} \tilde{\rightarrow} H(n)/H(n+n_0)$. These imply that $Q_n \simeq (\mathbb{Z}/\ell^{n_0})^{\Delta}$, as desired, since the exact sequence

$$0 \to H(n)/H(n+n_0) \to G(n)/G(n+n_0) \to Q_n \to 0$$

splits (as $H(n)/H(n+n_0) \simeq (\mathbb{Z}/\ell^{n_0})^{d_H}$ is an injective $\mathbb{Z}/\ell^{n_0}$-module). $\square$

So, after replacing X by $X_{G(n)}$ for some $n \gg 0$, one may assume that $HG(n+n_0)$ is normal in $HG(n)$ and that $HG(n)/HG(n+n_0) \simeq (\mathbb{Z}/\ell^{n_0})^{\Delta}$, $n \geq 0$. Extract from (13) the sequence:

$$H \subset \cdots HG((i+1)n_0) \subset HG(in_0) \subset \cdots \subset HG(n_0) \subset G. \tag{14}$$

It is enough to prove that

$$\lim_{i \mapsto \infty} \gamma_{HG(in_0))} = +\infty$$

For each $i \geq 0$ decompose $HG((i+1)n_0) \subset HG(in_0)$ into a subsequence

$$HG((i+1)n_0) = U_{i,\Delta} \subset U_{i,\Delta-1} \subset \cdots \subset U_{i,j+1} \subset U_{i,j} \subset \cdots \subset HG(in_0) = U_{i,0} \tag{15}$$

such that $U_{i,j+1}$ is normal in $U_{i,j}$ and $U_{i,j}/U_{i,j+1} \simeq \mathbb{Z}/\ell^{n_0}$, $j = 0, \cdots \Delta - 1$. Then (15) gives rise to a sequence of connected etale covers

$$\cdots \to X_{U_{i,\Delta}} = X_{U_{i+1,0}} \to X_{U_{i,\Delta-1}} \to \cdots \to X_{U_{i,1}} \to X_{U_{i,0}} = X_{U_{i-1,\Delta}} \to \cdots \to X, \tag{16}$$

which, for simplicity, we rewrite as:

$$\cdots \to X_{i+1} \to X_i \to \cdots \to X_{i-1} \to \cdots \to X_0 = X, \tag{17}$$

with $X_i \to X_{i-1}$ a Galois cover with Galois group $G_i = \mathbb{Z}/\ell^{n_0}$, $i \geq 1$. It is enough to prove that

$$\lim_{i \mapsto \infty} \gamma_{X_i} = +\infty$$

For this, we are going to apply the techniques of section 2 to (17).

3.2.2. *End of the proof of theorem 3.3.* Assume that the gonalities γ_{X_i}, $i \geq 0$ are bounded. Then, up to replacing X by X_i and G by $G(i)$ for some i large enough, one may assume that $\gamma_{X_i} = \gamma_X = \gamma$, $i \geq 0$. So, according to theorem 2.1, one may assume (again up to replacing X by X_i and G by $G(i)$ for some i large enough) that (17) can be completed as follows:

(18)
$$\begin{tikzcd}
\cdots \arrow[r] & X_{i+1}^{cpt} \arrow[r, "G_{i+1}"] \arrow[d, "f_{i+1}"'] \arrow[dr, phantom, "\tilde{\square}"] & X_i^{cpt} \arrow[r, "G_i"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\tilde{\square}"] & \cdots \arrow[r, "G_2"] \arrow[dr, phantom, "\tilde{\square}"] & X_1^{cpt} \arrow[r, "G_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\tilde{\square}"] & X_0^{cpt} = X^{cpt} \arrow[d] \\
\cdots \arrow[r] & B_{i+1} \arrow[r, "G_{i+1}"'] & B_i \arrow[r, "G_i"'] & \cdots \arrow[r, "G_2"'] & B_1 \arrow[r, "G_1"'] & B_0 =: B
\end{tikzcd}$$

where $B_{i+1} \to B_i$ is a Galois cover of proper k-curves with Galois group G_{i+1}, each square:

$$\begin{tikzcd}
X_{i+1}^{cpt} \arrow[r, "G_{i+1}"] \arrow[d, "f_{i+1}"'] \arrow[dr, phantom, "\tilde{\square}"] & X_i^{cpt} \arrow[d, "f_i"] \\
B_{i+1} \arrow[r, "G_{i+1}"'] & B_i
\end{tikzcd}$$

is cartesian up to normalization and equivariant, $\deg(f_i) = \gamma'$ (constant), $i \geq 0$, and either $\gamma' = \gamma$, $g_{B_i} = 0$, $i \geq 0$ or $\gamma' = \frac{\gamma}{2}$, $g_{B_i} = 1$, $i \geq 0$.

Let $S \subset B$ denote the ramification locus of $f := f_0 : X^{cpt} \to B$, and set $C := B \smallsetminus (S \cup f(X^{cpt} \smallsetminus X))$. Then, up to replacing X by $f^{-1}(C) \subset X^{cpt}$, one may assume that $X \to C$ is finite etale, hence that $\pi_1(X) \to \pi_1(C)$ is injective. Define C_i to be the inverse image of C in B_i. Then it is easy to see that $C_{i+1} \to C_i$ and $f_i : X_i \to C_i$ are finite etale, $i \geq 0$.

Let $\hat{X}_i \to C_i$ denote the Galois closure of $X_i \to C_i$. Then $k(\hat{X}_i) = k(\hat{X}_0) \cdot k(C_i)$ and, in particular, $[k(\hat{X}_{i+1}) : k(C_{i+1})] \leq [k(\hat{X}_i) : k(C_i)]$. So, up to replacing X by X_i for $i \gg 0$, one may assume that $[k(\hat{X}_{i+1}) : k(C_{i+1})] = [k(\hat{X}_i) : k(C_i)]$ or, equivalently, that $\hat{X}_i$ is the normalization of the fiber product $\hat{X}_0 \times_{C_0} C_i$, $i \geq 0$. Then, up to replacing X_0 by $\hat{X}_0$ and γ' by some integer $\gamma_0 \leq (\gamma')!$, one may assume that $f_i : X_i \to C_i$ is Galois, $i \geq 0$.

Write $M := \mathbb{Z}_\ell^m$ for the $\pi_1(X)$-module associated with our (G)LP representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$, $N := \mathrm{Ind}_{\pi_1(C)}^{\pi_1(X)}(M)$ for the $\pi_1(C)$-module induced from M and $\rho_0 : \pi_1(C) \to \mathrm{GL}_{m\gamma_0}(\mathbb{Z}_\ell)$ for the corresponding representation.

Lemma 3.6. *$\rho_0 : \pi_1(C) \to \mathrm{GL}_{m\gamma_0}(\mathbb{Z}_\ell)$ is, again, a (G)LP representation.*

Proof. Let $\Delta : \pi_1(X) \hookrightarrow \pi_1(X)^{\gamma_0}$ denote the diagonal embedding and fix a system $T \subset \pi_1(C)$ of representatives of $\pi_1(C)/\pi_1(X)$. Then (*) $\rho_0|_{\pi_1(X)} = \prod_{t \in T} \rho|_{\pi_1(X)}(t \ \ t^{-1}) \circ \Delta$. As a result, to prove Lemma 3.6, it is enough to prove that, given two ℓ-adic (G)LP representations $\rho_i : \pi_1(X) \to \mathrm{GL}_{m_i}(\mathbb{Z}_\ell)$ with abelian radical, $i = 1, 2$, the direct sum representation $\rho := \rho_1 \oplus \rho_2 : \pi_1(X) \to \mathrm{GL}_{m_1+m_2}(\mathbb{Z}_\ell)$ is again (G)LP with abelian radical.

So, let $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ denote the Lie algebras of the images of ρ_1, ρ_2 and ρ respectively. By definition $\mathfrak{g} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For $i = 1, 2$, write $p_i : \mathfrak{g} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2 \twoheadrightarrow \mathfrak{g}_i$ for the composite of this inclusion with the ith projection. Then $\mathfrak{g}$ surjects onto $\mathfrak{g}_i$ *via* p_i, which yields the following commutative diagram of Lie algebras with split short exact rows

$$\begin{tikzcd}
0 \arrow[r] & r(\mathfrak{g}) \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, "p_i"] & s(\mathfrak{g}) \arrow[r] \arrow[d, "\overline{p}_i"] \arrow[l, bend right, dotted] & 0 \\
0 \arrow[r] & r(\mathfrak{g}_i) \arrow[r] & \mathfrak{g}_i \arrow[r] & s(\mathfrak{g}_i) \arrow[r] \arrow[l, bend right, dotted] & 0,
\end{tikzcd}$$

in which $s(\mathfrak{g}) := \mathfrak{g}/r(\mathfrak{g})$ and $s(\mathfrak{g}_i) := \mathfrak{g}_i/r(\mathfrak{g}_i)$ are semisimple. In particular, $r(\mathfrak{g}) \subset r(\mathfrak{g}_1) \oplus r(\mathfrak{g}_2)$ (which already shows that ρ has abelian radical); $\mathfrak{g}^{\mathrm{ab}} = r(\mathfrak{g})_{s(\mathfrak{g})}$, $\mathfrak{g}_i^{\mathrm{ab}} = r(\mathfrak{g}_i)_{s(\mathfrak{g}_i)}$ (here $(-)_{s(\mathfrak{g})}$, $(-)_{s(\mathfrak{g}_i)}$ stand for the coinvariant functors); and $r(\mathfrak{g}_1)$, $r(\mathfrak{g}_2)$ are equipped with a structure of $s(\mathfrak{g})$-modules *via* $\overline{p}_i : s(\mathfrak{g}) \twoheadrightarrow s(\mathfrak{g}_i)$. With this structure, the embedding $r(\mathfrak{g}) \subset r(\mathfrak{g}_1) \oplus r(\mathfrak{g}_2)$ is an embedding of $s(\mathfrak{g})$-modules. As $s(\mathfrak{g})$ is semisimple, the coinvariant functor is left exact hence one obtains an embedding

$$r(\mathfrak{g})_{s(\mathfrak{g})} \subset r(\mathfrak{g}_1)_{s(\mathfrak{g})} \oplus r(\mathfrak{g}_2)_{s(\mathfrak{g})} = r(\mathfrak{g}_1)_{s(\mathfrak{g}_1)} \oplus r(\mathfrak{g}_2)_{s(\mathfrak{g}_2)},$$

where the last equality comes from the surjectivity of $s(\mathfrak{g}) \twoheadrightarrow s(\mathfrak{g}_i)$. Now, one gets

$$\mathfrak{g}^{\mathrm{ab}} = r(\mathfrak{g})_{s(\mathfrak{g})} \subset r(\mathfrak{g}_1)_{s(\mathfrak{g}_1)} \oplus r(\mathfrak{g}_2)_{s(\mathfrak{g}_2)} = \mathfrak{g}_1^{\mathrm{ab}} \oplus \mathfrak{g}_2^{\mathrm{ab}} = 0,$$

as desired. □

Now, set $U_i := \rho_0(\pi_1(C_i))$ and let $C_{U_i} \to C$ be, as usual, the etale connected cover corresponding to the inclusion of open subgroups $U_i \subset U_0$, $i \geq 0$. From the inclusions

$$\pi_1(C_i) \subset \pi_1(C_{U_i}) \subset \pi_1(C),$$

one actually has a sequence of etale covers $C_i \to C_{U_i} \to C$. In particular, $g_{C_{U_i}} \leq g_{C_i} \leq 1$, $i \geq 0$. Assume for a while that $U_\infty := \bigcap_{i\geq 0} U_i \subset U_0$ is (closed but) not open in U_0. Then, it follows from [CT12c, Thm. 3.4] that $\lim_{i\mapsto\infty} g_{C_{U_i}} = +\infty$: a contradiction.

So, it only remains to prove that $U_\infty = \bigcap_{i\geq 0} U_i \subset U_0$ is (closed but) not open in U_0. This will follow from:

Lemma 3.7. *$[U_i : U_{i+1}] \geq \ell$, $i \gg 0$.*

Proof. Set $V_i := \rho_0(\pi_1(X_i)) \subset U_i$. From (*), one has $\rho(\pi_1(X_i)) \simeq V_i$ hence $V_i/V_{i+1} \simeq G_i = \mathbb{Z}/\ell^{n_0}$. Let J denote the set of all $j \geq 0$ such that $U_{j+1} = U_j$ and set $J_i := J \cap \{0, \cdots, i\}$, $i \geq 0$. Then, on the one hand:

$$[U_0 : V_i] = [U_0 : U_i][U_i : V_i] \leq [U_0 : U_i]\gamma_0 \leq \ell^{n_0(i-|J_i|)}\gamma_0.$$

And, on the other hand:

$$[U_0 : V_i] = [U_0 : V_0] \prod_{0\leq j\leq i-1} [V_j : V_{j+1}] \geq \ell^{n_0 i}.$$

Hence $|J_i| \leq \frac{\log(\gamma_0)}{n_0 \log(\ell)}$, $i \geq 0$ so, as well, $|J| \leq \frac{\log(\gamma_0)}{n_0 \log(\ell)}$. □

Remark 3.8. For simplicity, we have given a uniform proof using theorem 2.1. (Or, alternatively, one can apply directly corollary 2.8.) However, it is worth noticing that this is only necessary if $\Delta = 1$. Else, the result of subsection 2.2 is enough to conclude. Indeed, if $\Delta = 2$ then, from the classification of finite subgroups of $\mathrm{PGL}_2(k)$, $(\mathbb{Z}/\ell^{n_0})^2$ cannot appear as the Galois group of a Galois cover of genus 0 curves. But the result of subsection 2.2 then implies that $g_{X_i} = 1$, $i \gg 0$, which is ruled out by the GLP assumption. If $\Delta \geq 3$ then one can conclude directly from the result of subsection 2.2 since then $(\mathbb{Z}/\ell^{n_0})^\Delta$ cannot appear as the Galois group of a Galois cover of genus 0 curves or of a Galois cover of genus 1 curves.

3.2.3. *Corollaries to theorem 3.3.*

Corollary 3.9. *For any $\underline{H} = (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_n(G^{geo})$, $\gamma_{H[n]} \to +\infty$.*

Proof. (cf. [CT12c, Cor. 3.6].) Set

$$H[\infty] := \varprojlim H[n] = \bigcap_{n\geq 0} H[n] \subset G^{geo},$$

which is a closed but not open subgroup of G^{geo} by lemma 3.2. By theorem 3.3, one has $\gamma_{H[\infty]G^{geo}(n)} \to \infty$ $(n \to \infty)$. Thus, it suffices to prove that $H[n] = H[\infty]G^{geo}(n)$ for $n \gg 0$. However, this is already proved in the proof of [CT12c, Cor. 3.6]. □

Corollary 3.10. *For any integer $c \geq 1$, there exists an integer $N_\rho(c) \geq 0$ such that for any $n \geq N_\rho(c)$ and any $U \in \mathcal{H}_n(G^{geo})$, one has $\gamma_U \geq c$.*

Proof. (cf. [CT12c, Cor. 3.7].) Else, the subset $\mathcal{H}_{n,<c}(G^{geo}) \subset \mathcal{H}_n(G^{geo})$ of all $U \in \mathcal{H}_n(G^{geo})$ such that $\gamma_U < c$ is non-empty, $n \geq 0$ hence $\varprojlim \mathcal{H}_{n,<c}(G^{geo})$ is non-empty as well. But for any $\underline{H} = (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_{n,<c}(G^{geo})$, $\gamma_{H[n]} \to +\infty$ by corollary 3.9: a contradiction. □

Corollary 3.11. *For any integers $c_1 \geq 1$, $c_2 \geq 1$, there exists an integer $N_\rho(c_1, c_2) \geq 0$, such that, for any $n \geq N_\rho(c_1, c_2)$ and any $U \in \mathcal{H}_n(G)$, either $\gamma_U \geq c_1$ or $[k_U : k] \geq c_2$.*

Proof. (cf. [CT12c, Cor. 3.8].) First, by lemma 3.2(5) for G and G^{geo}, there exists an integer $N_\rho > 0$ such that for any $n \geq N_\rho$:

$$\mathcal{H}_n(G) = \{U \subset G \mid G(n) \subset U,\ G(n-1) \not\subset U\},$$

$$\mathcal{H}_n(G^{geo}) = \{U \subset G^{geo} \mid G^{geo}(n) \subset U,\ G^{geo}(n-1) \not\subset U\}.$$

Second, by theorem 3.3, there exists an integer $N_\rho(c_1) \geq N_\rho$ such that for any $n \geq N_\rho(c_1)$ and any $U \in \mathcal{H}_n(G^{geo})$, one has $\gamma_U \geq c_1$. Third, as noticed in remark (1) of subsection 3.1.2, there exists an integer $N_\rho(c_1, c_2) \geq N_\rho(c_1)$ such that for any open subgroup $U \subset G$ with $[G : U] < c_2[G^{geo} : G^{geo}(N_\rho(c_1) - 1)]$, one has $G(N_\rho(c_1, c_2) - 1) \subset U$.

Now, for any $n \geq N_\rho(c_1, c_2)$ and any $U \in \mathcal{H}_n(G)$, set $U^{geo} = U \cap G^{geo}$. Recall that $U \in \mathcal{H}_n(G)$ is equivalent to saying that $G(n) \subset U$ and $G(n-1) \not\subset U$. Since $G(n) \subset U$, one has $G^{geo}(n) \subset U^{geo}$. Let n_0 be the minimal integer $\geq N_\rho(c_1) - 1$ such that $G^{geo}(n_0) \subset U^{geo}$. If $n_0 \geq N_\rho(c_1)$, then one has $G^{geo}(n_0 - 1) \not\subset U^{geo}$, hence $U^{geo} \in \mathcal{H}_{n_0}(G^{geo})$. Then one has $\gamma_U = \gamma_{U^{geo}} \geq c_1$ by the definition of $N_\rho(c_1)$. Else, $n_0 = N_\rho(c_1) - 1$, that is, $G^{geo}(N_\rho(c_1) - 1) \subset U^{geo}$. Since $G(n-1) \not\subset U$, one has $G(N_\rho(c_1, c_2) - 1) \not\subset U$, a fortiori. Thus, by the definition of $N_\rho(c_1, c_2)$, one has $[G : U] \geq c_2[G^{geo} : G^{geo}(N_\rho(c_1) - 1)]$, hence

$$[k_U : k] = \frac{[G : U]}{[G^{geo} : U^{geo}]} \geq c_2 \frac{[G^{geo} : G^{geo}(N_\rho(c_1) - 1)]}{[G^{geo} : U^{geo}]} \geq c_2,$$

as desired. □

Corollary 3.12. *Let k be an algebraically closed field of characteristic 0 and let K/k be a function field of transcendence degree 1. Let L/K be a Galois extension with Galois group G such that:*
(i) G is an ℓ-adic Lie group and $\mathrm{Lie}(G)^{ab} = 0$;
(ii) L/K is ramified only over a finite number of places of K.
Then, for any $\gamma \geq 1$, there are only finitely many finite subextensions K'/K of L/K with gonality $\leq \gamma$.

Proof. See [CT12c, Cor. 3.9]. □

3.3. **End of the proof of theorem 1.1.** Now, assume that k is finitely generated over $\mathbb{Q}$. Set:

$$\mathcal{X}_n := \coprod_{U \in \mathcal{H}_n(G)} X_U.$$

Then $(\mathcal{X}_{n+1} \to \mathcal{X}_n)_{n\geq 0}$ is a projective system of (possibly disconnected) etale covers with transition morphisms induced by the maps $\phi_n : \mathcal{H}_{n+1}(G) \to \mathcal{H}_n(G)$, $n \geq 0$. But, from corollary 3.11, for any integer $d \geq 1$, there exists an integer $N \geq 0$ such that for any $U \in \mathcal{H}_n(G)$ either $\gamma_U \geq 2d+1$ or $[k_U : k] \geq d+1$, $n \geq N$. If $[k_U : k] \geq d+1$ then, clearly, $X_U^{cl,\, \leq d} = \emptyset$ and, if $\gamma_U \geq 2d+1$ then, from theorem 1.2, $X_U^{cl,\, \leq d}$ is finite. As a result, $\mathcal{X}_n^{cl,\, \leq d}$ is finite for all $n \geq N$. Let $X_{\rho,d,N}$ denote the image of $\mathcal{X}_N^{cl,\, \leq d}$ in $X^{cl,\, \leq d}$ then:
- $X_{\rho,d,N}$ is finite since $\mathcal{X}_N^{cl,\, \leq d}$ is;
- No $x \in X^{cl,\, \leq d} \setminus X_{\rho,d,N}$ lifts to a $\kappa(x)$-rational point on $\mathcal{X}_N$. So, by the definition of N, $G_x \not\subset U$, for

any $U \in \mathcal{H}_N(G)$. But then, by lemma 3.2 (4), $G(N-1) \subset G_x$.

So, $X_{\rho,d} \subset X_{\rho,d,N}$ and, in particular, $X_{\rho,d}$ is finite. Finally, by the definition of $X_{\rho,d}$, for each $x \in X^{cl,\,\leq d} \smallsetminus X_{\rho,d}$, G_x is open in G, or, equivalently, there exists an integer N_x such that $G(N_x) \subset G_x$. Set $n_\rho := \max\{N, N_x\ (x \in X_{\rho,N,d} \smallsetminus X_{\rho,d})\}$. Then, for each $x \in X^{cl,\,\leq d} \smallsetminus X_{\rho,d}$, one has $G(n_\rho) \subset G_x$, as desired. □

Actually, by lemma 3.2 (4), $X_{\rho,d}$ coincides with the image of $\varprojlim \mathcal{X}_n^{cl\,\leq d}$ in $X^{cl,\,\leq d}$.

4. Applications

4.1. Strong uniform boundedness of ℓ-primary torsion.

4.1.1. *General formulation.* As in the case of [CT12c], theorem 1.1 yields a certain uniform boundedness of ℓ-primary χ-torsion for arbitrary GLP ℓ-adic representations (defined over fields finitely generated over $\mathbb{Q}$). So, let k be a field finitely generated over $\mathbb{Q}$ and X a smooth, separated, geometrically connected curve over k. Let M be a finitely generated free $\mathbb{Z}_\ell$-module of rank $m < \infty$ (i.e., $M \simeq \mathbb{Z}_\ell^m$), and $\rho : \pi_1(X) \to \mathrm{GL}(M)$ a GLP representation. Set $V := M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ ($\simeq \mathbb{Q}_\ell^m$) and $D := V/M = M \otimes_{\mathbb{Z}_\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ ($\simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m$). Thus, we have a natural identification $M/\ell^n = D[\ell^n]$ for each $n \geq 0$.

Then ρ induces $\pi_1(X)$-actions on V and D naturally.

Denote by $M_{(0)}$ the maximal isotrivial submodule of M, or, more precisely, $M_{(0)}$ is the maximal submodule of M on which the geometric part $\pi_1(X_{\overline{k}})$ of $\pi_1(X)$ acts via a finite quotient.

Recall that each morphism $\xi : \mathrm{Spec}(L) \to X$ (where L is any field) induces a homomorphism $\xi : \Gamma_L \to \pi_1(X)$, hence a representation $\rho_\xi := \rho \circ \xi$ and, for each ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, an ℓ-adic character $\chi_\xi := \chi \circ \xi$. Set

$$\overline{D}_\xi := \{v \in D \mid \rho_\xi(\sigma)v \in \langle v\rangle \text{ for any } \sigma \in \Gamma_L\},$$
$$\overline{M}_\xi := \{v \in M \mid \rho_\xi(\sigma)v \in \langle v\rangle \text{ for any } \sigma \in \Gamma_L\},$$

which are Γ_L-sets, and, for each ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, set

$$D_\xi(\chi) := \{v \in D \mid \rho_\xi(\sigma)v = \chi_\xi(\sigma)v \text{ for any } \sigma \in \Gamma_L\},$$
$$M_\xi(\chi) := \{v \in M \mid \rho_\xi(\sigma)v = \chi_\xi(\sigma)v \text{ for any } \sigma \in \Gamma_L\},$$

which are Γ_L-modules. Next, for each subset $E \subset D$ and $n \geq 0$, set $E[\ell^n] := E \cap D[\ell^n]$ and $E[\ell^n]^* := E \cap (D[\ell^n] \smallsetminus D[\ell^{n-1}])$, where $D[\ell^{-1}] := \emptyset$. For each subset E of M, set $E^* := E \cap (M \smallsetminus \ell M)$. Then one has

$$\varprojlim \overline{D}_\xi[\ell^n] = \overline{M}_\xi,\ \ \varprojlim \overline{D}_\xi[\ell^n]^* = \overline{M}_\xi^*,$$

and

$$\varprojlim D_\xi(\chi)[\ell^n] = M_\xi(\chi),\ \ \varprojlim D_\xi(\chi)[\ell^n]^* = M_\xi(\chi)^*.$$

Let $d \geq 1$.

Definition 4.1. *Let $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$ be an ℓ-adic character. Then χ is said to be d-non-sub-ρ if χ_x is not isomorphic to a subrepresentation of ρ_x for any $x \in X^{cl,\,\leq d}$.*

Now, the main result of this section, which is a corollary of theorem 1.1, is as follows.

Corollary 4.2.
(1) *For any d-non-sub-ρ ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$, there exists an integer $N := N(\rho, \chi, d)$, such that, for any $x \in X^{cl,\,\leq d}$, the $\Gamma_{\kappa(x)}$-module $D_x(\chi)$ is contained in $D[\ell^N]$.*
(2) *Assume furthermore that $M_{(0)} = 0$. Then there exists an integer $N := N(\rho, d)$, such that, for any $x \in X^{cl\,\leq d} \smallsetminus X_{\rho,d}$, the $\Gamma_{\kappa(x)}$-set $\overline{D}_x$ is contained in $D[\ell^N]$.*

Proof. (cf. [CT12c, Cor. 4.3].) From theorem 1.1 applied to the GLP ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}(M)$, the set $X_{\rho,d}$ of all $x \in X^{cl,\,\leq d}$ with $G_x \subset G$ not open is finite and there exists an integer $N_0 := N_{\rho,d} \geq 0$ such that for all $x \in X^{cl,\,\leq d} \smallsetminus X_{\rho,d}$, $G(N_0) \subset G_x$. Let η_{N_0} denote the generic point of the geometrically connected etale cover $X_{G(N_0)} \to X$ corresponding to the open subgroup $G(N_0) \subset G$.

(2) For each $v \in \overline{M}_{\eta_{N_0}} \smallsetminus \{0\}$, one has: $\gamma \cdot v = \lambda_{\gamma,v} v$ for some (unique) $\lambda_{\gamma,v} \in \mathbb{Z}_\ell^\times$. One can easily check that the map $\chi_v : \pi_1(X_{G(N_0)}) \to \mathbb{Z}_\ell^\times$, $\gamma \mapsto \lambda_{\gamma,v}$ is a character. Since $G^{geo} \cap G(N_0)$ has finite abelianization (as ρ is GLP), $\pi_1(X_{G(N_v),\bar{k}})$ acts trivially on v for some $N_v \geq N_0$. Thus, one gets: $\overline{M}_{\eta_{N_0}} \subset M_{(0)}$.

From the inclusion $G_{\eta_{N_0}} = G(N_0) \subset G_x$, one gets the inclusion: $\overline{D}_x \subset \overline{D}_{\eta_{N_0}}$. Now, suppose that $\overline{D}_x$ is infinite. Then $\overline{D}_{\eta_{N_0}}$ is also infinite, hence $\overline{D}_{\eta_{N_0}}[\ell^n]^*$ is nonempty for any $n \geq 0$, and $\overline{M}^*_{\eta_{N_0}} = \varprojlim \overline{D}_{\eta_{N_0}}[\ell^n]^*$ is nonempty. As $\overline{M}_{\eta_{N_0}} \subset M_{(0)}$, this implies that $M_{(0)} \neq 0$, as desired.

(1) First, consider the special case that χ is the trivial character $\mathbf{1}$. In this case, the inclusion $G_{\eta_{N_0}} = G(N_0) \subset G_x$ implies $D_x(\mathbf{1}) \subset D_{\eta_{N_0}}(\mathbf{1})$. Observe that the action of $\pi_1(X)$ on $D_{\eta_{N_0}}(\mathbf{1})$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(X)/\pi_1(X_{G(N_0)}) = G/G(N_0) = G_{N_0}$. Thus, $D_x(\mathbf{1})$ coincides with the module of elements of $D_{\eta_{N_0}}(\mathbf{1})$ fixed by the subgroup $(G_x)_{N_0} \subset G_{N_0}$. As G_{N_0} is a finite group, there are only finitely many subgroups of G_{N_0} that coincide with $(G_x)_{N_0}$ for some $x \in X^{cl,\, \leq d}$. Accordingly, there are only finitely many submodules of $D_{\eta_{N_0}}(\mathbf{1})$ that coincide with $D_x(\mathbf{1})$ for some $x \in X(k)$. Since $D_x(\mathbf{1})$ is finite for each $x \in X^{cl,\, \leq d}$ (as $\mathbf{1}$ is d-non-sub-ρ), this completes the proof in the special case.

For general χ, define the $\pi_1(X)$-module $M[\chi^{-1}]$ as follows: $M[\chi^{-1}] = M$ as $\mathbb{Z}_\ell$-modules, and the $\pi_1(X)$-action on $M[\chi^{-1}]$ is given by $\rho \cdot \chi^{-1}$. (Thus, $M[\chi^{-1}] = M \otimes \mathbb{Z}_\ell[\chi^{-1}]$.) Set $D[\chi^{-1}] := M[\chi^{-1}] \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell)$, and let $\rho[\chi^{-1}]$ denote the ℓ-adic representation of $\pi_1(X)$ associated to the $\pi_1(X)$-module $M[\chi^{-1}]$. Observe that the trivial character $\mathbf{1}$ is d-non-sub-$\rho[\chi^{-1}]$, as χ is d-non-sub-ρ. Also, as shown in the proof of [CT12c, Cor. 4.3], $\rho[\chi^{-1}] = \rho \cdot \chi^{-1}$ is a GLP representation.

Now, applying the preceding argument to $\rho[\chi^{-1}]$, one concludes that there exists an integer N, such that $D[\chi^{-1}]_x(\mathbf{1}) \subset D[\chi^{-1}][\ell^N]$ for any $x \in X^{cl,\, \leq d}$. Here, observe that the identification $M[\chi^{-1}] = M$ (as $\mathbb{Z}_\ell$-modules) induces the identifications $D[\chi^{-1}][\ell^N] = D[\ell^N]$ and $D[\chi^{-1}]_x(\mathbf{1}) = D_x(\chi)$. From this, the assertion follows. □

4.1.2. *Generic Tate modules of abelian schemes.* Let $A \to X$ be an abelian scheme over X. Then, from [CT12c, Thm. 4.1], for any prime ℓ the corresponding ℓ-adic representation $\rho_{A,\ell} : \pi_1(X) \to \mathrm{GL}(T_\ell(A_\eta))$ is a GLP representation. In this particular case, one can derive from theorem 1.1 strong uniform boundedness results for the ℓ-primary torsion in the special fibers of $A \to X$.

Corollary 4.3. *Fix an integer $d \geq 1$.*

1. *For any d-non-sub-$\rho_{A,\ell}$ ℓ-adic character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^\times$ there exists an integer $N := N(\rho_{A,\ell}, \chi, d)$, such that, for any $x \in X^{cl,\, \leq d}$, the $\Gamma_{\kappa(x)}$-module $A_x[\ell^\infty](\chi_x) := \{v \in A_x[\ell^\infty] \mid {}^\sigma v = \chi_x(\sigma)v,\ \sigma \in \Gamma_{\kappa(x)}\}$ is contained in $A_x[\ell^N]$.*
2. *Assume furthermore that A_η contains no non-trivial isotrivial subvariety. Then there exists an integer $N := N(\rho_{A,\ell}, d)$, such that, for any $x \in X^{cl,\, \leq d} \smallsetminus X_{\rho_{A,\ell},d}$, the $\Gamma_{\kappa(x)}$-set $\overline{A_x[\ell^\infty]}(\kappa(x)) := \{v \in A_x[\ell^\infty] \mid \sigma \cdot v \in \langle v \rangle,\ \sigma \in \Gamma_{\kappa(x)}\}$ is contained in $A_x[\ell^N]$.*

Proof. (cf. [CT12c, Cor. 5.2]) One may apply corollary 4.2 to $\rho_{A,\ell}$. This completes the proof, since $A_x[\ell^\infty](\chi_x)$ (resp. $\overline{A_x[\ell^\infty]}(\kappa(x))$) is identified with $(A_\eta[\ell^\infty])_x(\chi)$ (resp. $\overline{(A_\eta[\ell^\infty])}_x$) via the specialization isomorphism $A_x[\ell^\infty] \simeq A_\eta[\ell^\infty]$ and since $T_\ell(A_\eta)_{(0)} = 0$ if and only if A_η contains no non-trivial isotrivial subvariety ([CT12a, Cor. 2.4]). □

Remark 4.4. Recall ([CT12a, Sect. 2.2], [CT09, §3]) that an ℓ-adic character $\Gamma_k \to \mathbb{Z}_\ell^\times$ is said to be non-Tate if it does not appear as a subrepresentation of Γ_k acting on the ℓ-adic Tate module of an abelian variety over k. By [CT12a, Lem. 2.1], if $\chi : \Gamma_k \to \mathbb{Z}_\ell^\times$ is a non-Tate ℓ-adic character, then $\chi \circ r : \pi_1(X) \to \mathbb{Z}_\ell^\times$ is d-non-sub-$\rho_{A,\ell}$ for any $d \geq 1$, where $r : \pi_1(X) \to \Gamma_k$ denotes the canonical restriction homomorphism. Thus, corollary 4.3 applies to such χ (like the trivial character or the cyclotomic character).

Remark 4.5. As in [CT12a, Sect. 5], corollary 4.3 for the cyclotomic character yields the following strong version of the 1-dimensional modular tower conjecture. For any 1-dimensional modular tower $\underline{H} = (H_{n+1} \to H_n)$ defined over a number field k, any curve X over k, any $\zeta : X \to H_0$ and any integer $d \geq 1$, there exists an integer $N = N(\underline{H}, X, \zeta, d)$ such that $X_n^{cl, \leq d} = \emptyset$, $n \geq N$. Here, we set $X_n = X \times_{\zeta, H_0} H_n$. For more details about the 1-dimensional modular tower conjecture, see [CT12a, Sect. 5].

4.2. **Finiteness of the number of CM elliptic curves over number fields.** Let $d \geq 1$ be an integer and let $\mathcal{J}(d) \subset (\mathbb{P}^1_{\mathbb{Q}})^{cl, \leq d}$ denote the set of j-invariants with $[\mathbb{Q}(j) : \mathbb{Q}] \leq d$ such that the corresponding curve $E_j/\overline{\mathbb{Q}}$ (which is defined over $\mathbb{Q}(j)$) has complex multiplication. Then $|\mathcal{J}(d)|$ is known to be finite.[4] We can recover this result from theorem 1.1. Indeed, consider for instance the family of elliptic curves $E \to X := \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1728, \infty\}$ defined by:

$$E_j : y^2 + xy - x^3 + \frac{36}{j - 1728}x + \frac{1}{j - 1728} = 0.$$

It defines an abelian scheme and the image of the corresponding ℓ-adic representation $\rho_{E,\ell} : \pi_1(X) \to \mathrm{GL}_2(T_\ell(E_\eta))$ is open [CT12c, Lemma 5.4]. Observe that, for $j \in X = \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1728, \infty\}$, the j-invariant $j(E_j)$ of E_j is just j.

For each $j \in \mathcal{J}(d)$, the Galois image G_j at j is almost abelian (as E_j has complex multiplication), hence not open in $\mathrm{GL}(T_\ell(E_\eta)) \simeq \mathrm{GL}_2(\mathbb{Z}_\ell)$. Thus, the finiteness of $\mathcal{J}(d)$ follows from theorem 1.1.

One can also recover the classical Hecke-Deuring-Heilbronn theorem (cf. [Si35]), to the effect that there are only finitely many imaginary quadratic fields with bounded class number, from theorem 1.1. Indeed, suppose that for some integer $h \geq 1$ there are infinitely many imaginary quadratic fields k with class number $h(\mathcal{O}_k) \leq h$. Then, for each such k, one can construct an elliptic curve E_k (say, over $\mathbb{C}$) with CM by $\mathcal{O}_k$. Let $j(E_k)$ denote the j-invariant of E_k. As $k(j(E_k))$ is the Hilbert class field of k, one has $[\mathbb{Q}(j(E_k)) : \mathbb{Q}] \leq [k(j(E_k)) : \mathbb{Q}] = 2[k(j(E_k)) : k] = 2h(\mathcal{O}_k) \leq 2h$. But this contradicts theorem 1.1 for $d = 2h$ applied to the above abelian scheme $E \to X = \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1728, \infty\}$, as $E_{j(E_k)} \simeq E_k$ (say, over $\mathbb{C}$).

5. Proof of theorem 1.3

5.1. **Proof.**

5.1.1. *Group-theoretical preliminaries.* For any closed subgroup $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$, it is known that $|G_n| = O(\ell^{n\dim(G)})$ [Se81]. Here, we describe more precisely the hidden constant in $O(\ell^{n\dim(G)})$.

Lemma 5.1. *Let n_0 be any integer ≥ 1 (resp. ≥ 2) if $\ell \neq 2$ (resp. $\ell = 2$). Then, for any closed subgroup $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)(n_0)$, one has $|G_n| \leq \ell^{(n-n_0)\dim(G)}$ for any $n \geq n_0$.*

Proof. Set $\Gamma := \mathrm{GL}_m(\mathbb{Z}_\ell)$. Since $\Gamma(n_0)$ is a powerful pro-ℓ group and $\Phi(\Gamma(n_0 + n)) = \Gamma(n_0 + n + 1)$ [DSMS91, Thm. 5.2] the map $\Gamma(n_0 + n) \tilde{\to} \Gamma(n_0 + n + 1)$, $x \mapsto x^\ell$ is a homeomorphism and induces a bijection:

$$\pi_n : \Gamma(n_0 + n)/\Gamma(n_0 + n + 1) \tilde{\to} \Gamma(n_0 + n + 1)/\Gamma(n_0 + n + 2),$$

which is actually a group isomorphism [DSMS91, Lemmas 2.4 (ii) and 4.10]. The restriction of π_n to $G(n_0 + n)/G(n_0 + n + 1) \subset \Gamma(n_0 + n)/\Gamma(n_0 + n + 1)$ induces a group monomorphism:

$$\pi_n : G(n_0 + n)/G(n_0 + n + 1) \hookrightarrow G(n_0 + n + 1)/G(n_0 + n + 2).$$

As a result, one has:

$$|G(n_0 + n)/G(n_0 + n + 1)| \leq |G(n_0 + n + 1)/G(n_0 + n + 2)|, \ n \geq 0.$$

[4] For instance, $|\mathcal{J}(1)| = 13$. More generally, Let E be an elliptic curve defined over a number field of degree $\leq d$ and with CM by an order $C_E = \mathcal{O}_\kappa^f := \mathbb{Z} + f\mathcal{O}_\kappa$ ($f \in \mathbb{Z}_{\geq 1}$) in a imaginary quadratic field κ; here $\mathcal{O}_\kappa$ stands for the ring of integers of κ. Then, one has [L87, Chap. 10, §3, Thm. 5] $h(\mathcal{O}_\kappa) \leq h(C_E) = [\kappa(j(E)) : \kappa] \leq d$. So, from the Hecke-Deuring-Heilbronn theorem (cf. [Si35]), there are only finitely many possibilities for κ. But, for each such κ, one has [L87, Chap. 8, §1, Thm. 7]: $d \geq h(\mathcal{O}_\kappa^f) \geq \frac{h(\kappa)}{3}\varphi(f)$, where $\varphi(f) = |(\mathbb{Z}/f)^\times| \to +\infty$. So, there are only finitely many possible orders $\mathcal{O}_\kappa^f$ in a fixed κ. Finally, for any such κ and $\mathcal{O}_\kappa^f$, the number of elliptic curves with CM by $\mathcal{O}_\kappa^f$ is exactly $h(\mathcal{O}_\kappa^f)$.

But, from lemma 3.5 (2), $|G(n_0+n)/G(n_0+n+1)| = \ell^{\dim(G)}$ for $n \gg 0$. So:

$$|G_n| = |G/G(n)| = |G(n_0)/G(n_0+(n-n_0))| = \prod_{i=0}^{n-n_0-1} |G(n_0+i)/G(n_0+i+1)| \leq \ell^{(n-n_0)\dim(G)},\ n \geq n_0. \square$$

Corollary 5.2. *There exists an absolute constant $C_{m,\ell} > 0$ depending only on ℓ and m such that for any closed subgroup $G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ one has $|G_n| \leq C_{m,\ell}\ell^{n\dim(G)}$ for any $n \geq 0$.*

Proof. Just observe that, with $n_0 = 2$, one has:

$$|G_n| \leq |G(n_0)_n||G_{n_0}| \leq \ell^{n\dim(G)}|\mathrm{GL}_m(\mathbb{Z}/\ell^{n_0})|. \square$$

Corollary 5.3. *Let $C \subset G \subset \mathrm{GL}_m(\mathbb{Z}_\ell)$ be closed subgroups and fix an integer $0 \leq k \leq \dim(G)$. Assume that there exists a closed subgroup $C[n] \subset G$ of codimension $\geq k$ such that $CG(n) = C[n]G(n)$, $n \gg 0$. Then C has codimension $\geq k$ as well.*

Proof. From [Se81], there exists a constant $\mu(C) > 0$ such that $|C_n| = \mu(C)\ell^{n\dim(C)}$, $n \gg 0$. Hence, one has:

$$\mu(C)\ell^{n\dim(C)} = |C_n| = |C[n]_n| \leq C_{m,\ell}\ell^{n\dim(C[n])} \leq C_{m,\ell}\ell^{n(\dim(G)-k)},\ n \gg 0.$$

Thus, $\dim(C) \leq \dim(G) - k$, as desired. $\square$

5.1.2. *Proof of theorem 1.3.* We can now adapt the proof of theorem 1.1 to prove its unconditional variant. The key point consists in replacing the projective system:

$$(\mathcal{H}_{n+1}(G) \overset{\phi_n}{\to} \mathcal{H}_n(G))_{n\geq 0}$$

by the projective system:

$$(\mathcal{H}_{k,n+1}(G) \overset{\phi_{k,n}}{\to} \mathcal{H}_{k,n}(G))_{n\geq 0},$$

where $\mathcal{H}_{k,n}(G)$ denotes the set of all open subgroups $U \subset G$ which can be written as $U = CG(n)$ for some closed subgroup $C \subset G$ of codimension $\geq k$ and the transition map $\phi_{k,n} : \mathcal{H}_{k,n+1}(G) \to \mathcal{H}_{k,n}(G)$ is $U \mapsto UG(n)$.

From corollary 5.3 and the proof of [CT12c, Cor. 3.6], for any $\underline{H} := (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_{k,n}(G)$, the group:

$$H[\infty] := \varprojlim H[n] = \bigcap_{n\geq 0} H[n] \subset G$$

is closed of codimension $\geq k$.

Now, if $k \geq 3$, from remark 3.8, theorem 3.3 still holds under the assumption that H is of codimension $\geq k$ in G^{geo}. Accordingly, corollaries 3.9 and 3.10 still hold with $(\mathcal{H}_{n+1}(G^{geo}) \to \mathcal{H}_n(G^{geo}))_{n\geq 0}$ replaced by $(\mathcal{H}_{k,n+1}(G^{geo}) \to \mathcal{H}_{k,n}(G^{geo}))_{n\geq 0}$. Corollary 3.11 holds as well but its proof has to be modified as follows:

(1) Assume that for some integers $c_1 \geq 1$, $c_2 \geq 1$ the statement of corollary 3.11 no longer holds. Then, for any $n \geq 0$ the subset $\mathcal{H}_{k,n,<c_1,<c_2}(G)$ of all $U \in \mathcal{H}_{k,n}(G)$ such that $\gamma_U < c_1$ and $[k_U : k] < c_2$ is non-empty. Furthermore, the projective system structure on the $\mathcal{H}_{k,n}(G)$, $n \geq 0$ induces a projective system structure on the $\mathcal{H}_{k,n,<c_1,<c_2}(G)$, $n \geq 0$. In particular, $\varprojlim \mathcal{H}_{k,n,<c_1,<c_2}(G) \neq \emptyset$. Take any $\underline{H} := (H[n])_{n\geq 0} \in \varprojlim \mathcal{H}_{k,n,<c_1,<c_2}(G)$. Recall from the proof of [CT12c, Cor. 3.6] that $H[\infty]G(n) = H[n]$ for $n \gg 0$.

(2) Then, in particular, $H[\infty]G^{geo}G(n) = H[n]G^{geo}$ for $n \gg 0$, hence $[G : H[\infty]G^{geo}G(n)] = [G : H[n]G^{geo}] < c_2$ for $n \gg 0$. Thus, the sequence

$$H[\infty]G^{geo}G(0) \supset H[\infty]G^{geo}G(1) \supset H[\infty]G^{geo}G(2) \supset \cdots$$

of open subgroups stabilizes. This implies that $H[\infty]G^{geo} = \bigcap_{n\geq 0} H[\infty]G^{geo}G(n)$ is open in G. (Here, the equality follows from the fact that $\{G(n)\}_{n\geq 0}$ forms a fundamental open neighborhood of $1 \in G$ and that $H[\infty]G^{geo}$ is closed in G.) Now, since $H[\infty]G^{geo} \subset G$ is open, the codimension of $H[\infty] \cap G^{geo}$ in G^{geo} is the same as the codimension of $H[\infty]$ in G and, in particular, is $\geq k \geq 3$. So,

$$\gamma_{(H[\infty]\cap G^{geo})G^{geo}(n)} \to +\infty.$$

Thus, to prove that $\gamma_{(H[\infty]G(n))\cap G^{geo}} \to +\infty$ (which contradicts $\gamma_{H[n]} < c_1$, $n \geq 0$), it is enough to prove that the degree of the cover:

$$X_{(H[\infty]\cap G^{geo})G^{geo}(n)} \to X_{(H[\infty]G(n))\cap G^{geo}}$$

is bounded independently of n.

(3) For this, write $A_n := (H[\infty]G(n)) \cap G^{geo}$ and $B_n := (H[\infty] \cap G^{geo})G^{geo}(n)$. Then:

$$[B_n : G^{geo}(n)] = [(H[\infty] \cap G^{geo})G^{geo}(n) : G^{geo}(n)] = |(H[\infty] \cap G^{geo})_n|$$

so:

$$[G^{geo} : B_n] = \frac{[G^{geo} : G^{geo}(n)]}{[B_n : G^{geo}(n)]} = \frac{|G_n^{geo}|}{|(H[\infty] \cap G^{geo})_n|}$$

and:

$$\begin{aligned}
[G^{geo} : A_n] &= [G^{geo} : (H[\infty]G(n)) \cap G^{geo}] \\
&= [(H[\infty]G(n))G^{geo} : H[\infty]G(n)] \\
&= \frac{[G : G(n)]}{[G : (H[\infty]G(n))G^{geo}][H[\infty]G(n) : G(n)]} \\
&\geq \frac{|G_n|}{[G : H[\infty]G^{geo}]|H[\infty]_n|}.
\end{aligned}$$

Hence:

$$[A_n : B_n] = \frac{[G^{geo} : B_n]}{[G^{geo} : A_n]} \leq \frac{|G_n^{geo}|}{|(H[\infty] \cap G^{geo})_n|} \frac{[G : H[\infty]G^{geo}]|H[\infty]_n|}{|G_n|}.$$

But, from [Se81], for $n \gg 0$, the right-hand term is:

$$K \frac{\ell^{n\dim(G^{geo})}}{\ell^{n\dim(H[\infty]\cap G^{geo})}} \frac{\ell^{n\dim(H[\infty])}}{\ell^{n\dim(G)}} = K,$$

where K is a constant > 0 depending only on G, $H[\infty]$, G^{geo} and $H[\infty] \cap G^{geo}$.

Now, one can conclude the proof of theorem 1.3 exactly as in subsection 3.3.

5.2. **A counterexample.** In this last subsection, we construct an ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_4(\mathbb{Z}_\ell)$ such that the set of all $x \in X(k)$ such that G_x is of codimension ≥ 2 in G is infinite. The idea is to generalize the counterexample to (1) in [CT12c, §3.5.2], with $X = \mathbb{G}_m$ replaced by a non-CM elliptic curve $X = E$ of positive (Mordell-Weil) rank.

So, let k be a field finitely generated over $\mathbb{Q}$ and let E be a non-CM elliptic curve over k with positive rank; let η denote its generic point. Since E is non-CM, the image of $\rho_{E,\ell} : \Gamma_k \to \mathrm{GL}(T_\ell(E)) \simeq \mathrm{GL}_2(\mathbb{Z}_\ell)$ is open. For any field extension F/k, the Kummer short exact sequence:

$$0 \to E[\ell^n] \to E \xrightarrow{[\ell^n]} E \to 0$$

on $(\mathrm{Spec}(F))_{et}$ yields an exact sequence:

$$0 \to E[\ell^n](F) \to E(F) \stackrel{[\ell^n]}{\to} E(F) \to \mathrm{H}^1(\Gamma_F, E[\ell^n]).$$

Taking projective limits one thus gets a monomorphism:

$$E(F)^\wedge_\ell := \varprojlim E(F)/[\ell^n]E(F) \hookrightarrow \mathrm{H}^1(\Gamma_F, T_\ell(E)).$$

Given any $\zeta \in E(F)$, we will write $\psi(\zeta)$, $\tilde{\psi}(\zeta)$ and $\overline{\psi}(\zeta)$ for the image of ζ in $\mathrm{H}^1(\Gamma_F, T_\ell(E))$, $\mathrm{H}^1(\Gamma_{Fk(E[\ell^\infty])}, T_\ell(E))$ and $\mathrm{H}^1(\Gamma_{F\overline{k}}, T_\ell(E))$ respectively (where, as usual, $k(E[\ell^\infty])$ denotes the subfield fixed by $\ker(\rho_{E,\ell})$ in $\overline{k}$). Also, by fixing a compatible system $(\frac{1}{[\ell^n]}\zeta)_{n\geq 0}$, we define a map $\psi_\zeta : \Gamma_F \to T_\ell(E)$ by $\psi_\zeta(\gamma) = (\gamma(\frac{1}{[\ell^n]}\zeta) - \frac{1}{[\ell^n]}\zeta)_{n\geq 0}$, which is a cocycle for the Kummer class $\psi(\zeta) \in \mathrm{H}^1(\Gamma_F, T_\ell(E))$. In particular $\psi_\zeta(\gamma\gamma') = \psi_\zeta(\gamma) + \gamma \cdot \psi_\zeta(\gamma')$, γ, $\gamma' \in \Gamma_F$. By a suitable choice of the above compatible systems, one may assume that the cocycle $\psi_\eta : \Gamma_{k(\eta)} \to T_\ell(E)$ induces a cocycle $\pi_1(E) \to T_\ell(E)$ (denoted again by ψ_η) and that for any closed point $x \in E^{cl}$, $\psi_\eta \circ x = \psi_x$.

Fix a point $a \in E(k)$ and let $\rho : \Gamma_{k(\eta)} \to \mathrm{GL}_4(\mathbb{Z}_\ell)$ defined by:

$$\rho(\gamma) = \left(\begin{array}{ll} \rho_{E,\ell}(\gamma|_{\overline{k}}) & \psi_{a,\eta}(\gamma) \\ 0_2 & I_2 \end{array} \right), \ \gamma \in \Gamma_{k(\eta)},$$

where:

$$\psi_{a,\eta}(\gamma) = (\psi_a(\gamma|_{\overline{k}}), \psi_\eta(\gamma)) \in \mathrm{M}_2(\mathbb{Z}_\ell).$$

Note that, by construction, $\rho : \Gamma_{k(\eta)} \to \mathrm{GL}_4(\mathbb{Z}_\ell)$ factors through:

$$\rho : \pi_1(E) \to \mathrm{GL}_4(\mathbb{Z}_\ell).$$

Proposition 5.4. *Assume that $a \in E(k)$ is not torsion. Then G has dimension 8 whereas $G_{[n]a}$ has dimension ≤ 6 for any $n \in \mathbb{Z}$.*

Proof The second part of the assertion follows from the identity $\psi([n]a) = [n]\psi(a)$. Indeed, this particularly implies $\psi([n]a)|_{\Gamma_{k(E[\ell^\infty])}} = [n]\psi(a)|_{\Gamma_{k(E[\ell^\infty])}}$, which is equivalent to: $\psi_{[n]a}|_{\Gamma_{k(E[\ell^\infty])}} = [n]\psi_a|_{\Gamma_{k(E[\ell^\infty])}}$, as $\Gamma_{k(E[\ell^\infty])}$ acts trivially on $T_\ell(E)$.

As for the first part, $\dim(G) \leq 8$ is clear. Consider the filtration:

$$G \supset \rho(\pi_1(E_{k(E[\ell^\infty])})) \supset \rho(\pi_1(E_{\overline{k}})) \supset \{1\}.$$

We are going to prove that:

(1) $\dim(G/\rho(\pi_1(E_{k(E[\ell^\infty])}))) = 4$;
(2) $\dim(\rho(\pi_1(E_{k(E[\ell^\infty])}))/\rho(\pi_1(E_{\overline{k}}))) \geq 2$;
(3) $\dim(\rho(\pi_1(E_{\overline{k}}))) = 2$.

Proof of (1):

One has a profinite group isomorphism:

$$G/\rho(\pi_1(E_{k(E[\ell^\infty])})) \tilde{\to} \rho_{E,\ell}(\Gamma_k).$$

But, by assumption, $\rho_{E,\ell}(\Gamma_k) \subset \mathrm{GL}_2(\mathbb{Z}_\ell)$ is open. Hence $\dim(G/\rho(\pi_1(E_{k(E[\ell^\infty])}))) = 4$.

Proof of (2):

Again, one has a profinite group epimorphism:

$$\rho(\pi_1(E_{k(E[\ell^\infty])}))/\rho(\pi_1(E_{\overline{k}})) \twoheadrightarrow \tilde{\psi}(a)(\Gamma_{k(E[\ell^\infty])}).$$

So it is enough to prove that $\dim(\tilde{\psi}(a)(\Gamma_{k(E[\ell^\infty])})) = 2$. Note that the image of $\mathrm{H}^1(\Gamma_k, T_\ell(E))$ in

$$\mathrm{H}^1(\Gamma_{k(E[\ell^\infty])}, T_\ell(E)) = \mathrm{Hom}(\Gamma_{k(E[\ell^\infty])}, T_\ell(E))$$

lies in $\mathrm{Hom}_{\mathrm{Gal}(k(E[\ell^\infty])/k)}(\Gamma_{k(E[\ell^\infty])}, T_\ell(E))$. In particular, $\tilde{\psi}(a)(\Gamma_{k(E[\ell^\infty])}) \subset T_\ell(E)$ is a $\mathrm{Gal}(k(E[\ell^\infty])/k)$-submodule. But $\mathrm{Gal}(k(E[\ell^\infty])/k) \simeq \rho_{E,\ell}(\Gamma_k) \subset \mathrm{GL}(T_\ell)(E)$ is open, hence $\tilde{\psi}(a)(\Gamma_{k(E[\ell^\infty])}) \subset T_\ell(E)$ is open if and only if $\tilde{\psi}(a) \neq 0$.

Lemma 5.5. $\mathrm{H}^m(\mathrm{Gal}(k(E[\ell^\infty])/k), T_\ell(E))$ *is torsion for all* $m \geq 0$.

Proof. Set $H := \mathrm{Gal}(k(E[\ell^\infty])/k)$. By assumption, $H \subset \mathrm{GL}(T_\ell(E))$ is open so $H \cap \mathbb{Z}_\ell^\times I_2 \subset \mathbb{Z}_\ell^\times I_2$ is open as well and, in particular, contains an open subgroup $Z \simeq \mathbb{Z}_\ell$. Note that Z is central, hence normal, in H. Consider the Hochschild-Serre spectral sequence:

$$\mathrm{H}^i(H/Z, \mathrm{H}^j(Z, T_\ell(E))) \Rightarrow \mathrm{H}^{i+j}(H, T_\ell(E)).$$

Let:

$$\{F^i(\mathrm{H}^m(H, T_\ell(E)))\}_{0\leq i\leq m}$$

denote the filtration on $\mathrm{H}^m(H, T_\ell(E))$ induced by the spectral sequence. Then:

$$F^i(\mathrm{H}^m(H, T_\ell(E)))/F^{i+1}(\mathrm{H}^m(H, T_\ell(E)))$$

is a subquotient of $\mathrm{H}^i(H/Z, \mathrm{H}^{m-i}(Z, T_\ell(E)))$, $i = 0, \dots, m$ and, thus, it is enough to prove that $\mathrm{H}^i(H/Z, \mathrm{H}^j(Z, T_\ell(E)))$ is torsion. Since Z acts on $T_\ell(E)$ by scalar multiplication, one has:

$$\begin{aligned} \mathrm{H}^j(Z, T_\ell(E)) \quad &= T_\ell(E)^Z = 0 \quad &&\text{if } j = 0;\\ &= T_\ell(E)_Z &&\text{if } j = 1;\\ &= 0 &&\text{if } j > 1.\end{aligned}$$

The second equality comes from the classical computation of cohomology of cyclic groups and the third one from the fact that $Z \simeq \mathbb{Z}_\ell$ has ℓ-cohomological dimension 1. Eventually, since $Z \subset \mathbb{Z}_\ell^\times I_2$ is open, $Z \supset (1 + \ell^N \mathbb{Z}_\ell)I_2$ for some $N \geq 1$. Hence $T_\ell(E)_Z$ is of ℓ^N-torsion. So $\mathrm{H}^i(H/Z, \mathrm{H}^{m-i}(Z, T_\ell(E)))$ is 0 for $i \neq m-1$ and $\mathrm{H}^{m-1}(H/Z, \mathrm{H}^1(Z, T_\ell(E)))$ is of ℓ^N-torsion. Hence $\mathrm{H}^m(H, T_\ell(E))$ is of ℓ^N-torsion as well. □

Remark 5.6.

(1) The statement of Lemma 5.5 remains true for any abelian variety A over a finitely generated field extension k of $\mathbb{Q}$ since $\mathrm{Gal}(k(A[\ell^\infty])/k)$ always contains an open subgroup of $\mathbb{Z}_\ell^\times Id$ [Bo80, Cor. 1].

(2) One can actually prove that the group $\mathrm{H}^m(\mathrm{Gal}(k(E[\ell^\infty])/k), T_\ell(E))$ is finite for all $m \geq 0$. Indeed, as $T_\ell(E)_Z$ is finite, the kernel of $H/Z \to \mathrm{GL}(T_\ell(E)_Z)$ is a normal open subgroup $K \triangleleft_{op} H/Z$. As K is a compact (recall that H is) ℓ-adic Lie group, it contains a characteristic open subgroup which is torsion-free (for instance uniform powerful). So let U be a normal open torsion-free subgroup of H/Z acting trivially on $T_\ell(E)_Z$. The Hoschild-Serre spectral sequence

$$\mathrm{H}^i((H/Z)/U, \mathrm{H}^i(U, T_\ell(E)_Z)) \Rightarrow \mathrm{H}^{i+j}(H/Z, T_\ell(E)_Z)$$

shows that to prove the finiteness of $\mathrm{H}^{m-1}(H/Z, T_\ell(E)_Z)$, it is enough to prove the finiteness of the $\mathrm{H}^i(U, T_\ell(E)_Z)$. By an elementary devissage, one is reduced to the case where $T_\ell(E)_Z$ is the trivial U-module $\mathbb{F}_\ell$. Now the conclusion follows from [Laz65], where it is shown that a compact torsion-free ℓ-adic Lie group is a Poincaré group in the sense of Serre [Se00, I, §4.5].

Let $c : E(k) \to E(k)^\wedge_\ell$ denote the canonical morphism. Then, from the injectivity of:

$$E(k)^\wedge_\ell \hookrightarrow \mathrm{H}^1(\Gamma_k, T_\ell(E)),$$

to prove that $\tilde{\psi}(a) \neq 0$ it is enough to prove that:

$$c(a) \notin \ker(\mathrm{H}^1(\Gamma_k, T_\ell(E)) \to \mathrm{H}^1(\Gamma_{k(E[\ell^\infty])}, T_\ell(E))).$$

This kernel is given by the inflation-restriction exact sequence:

$$0 \to \mathrm{H}^1(\mathrm{Gal}(k(E[\ell^\infty])/k), T_\ell(E)) \to \mathrm{H}^1(\Gamma_k, T_\ell(E)) \to \mathrm{H}^1(\Gamma_{k(E[\ell^\infty])}, T_\ell(E)).$$

Since:

$$\mathrm{H}^1(\Gamma_{k(E[\ell^\infty])}, T_\ell(E)) = \mathrm{Hom}(\Gamma_{k(E[\ell^\infty])}, T_\ell(E))$$

is torsion free whereas, from lemma 5.5, $\mathrm{H}^1(\mathrm{Gal}(k(E[\ell^\infty])/k), T_\ell(E))$ is torsion, one has:

$$\mathrm{H}^1(\mathrm{Gal}(k(E[\ell^\infty])/k), T_\ell(E)) = \mathrm{H}^1(\Gamma_k, T_\ell(E))_{tors}.$$

Since $E(k)$ is finitely generated, one has $c^{-1}((E(k)^\wedge_\ell)_{tors}) = E(k)_{tors}$, which shows that $c(a) \notin \mathrm{H}^1(\Gamma_k, T_\ell(E))_{tors} \cap E(k)^\wedge_\ell$, as required.

Proof of (3):

As in (2), the image of $\mathrm{H}^1(\pi_1(E), T_\ell(E))$ in

$$\mathrm{H}^1(\pi_1(E_{\overline{k}}), T_\ell(E)) = \mathrm{Hom}(\pi_1(E_{\overline{k}}), T_\ell(E))$$

lies in $\mathrm{Hom}_{\Gamma_k}(\pi_1(E_{\overline{k}}), T_\ell(E))$. Thus, again, $\overline{\psi}(\eta)(\pi_1(E_{\overline{k}})) \subset T_\ell(E)$ is a Γ_k-submodule. But $\rho_{E,\ell}(\Gamma_k) \subset \mathrm{GL}(T_\ell)(E)$ is open hence $\overline{\psi}(\eta)(\pi_1(E_{\overline{k}})) \subset T_\ell(E)$ is open if and only if $\overline{\psi}(\eta) \neq 0$. To prove that $\overline{\psi}(\eta) \neq 0$, consider the commutative diagram:

$$\begin{tikzcd}
E(k(\eta)) \arrow[r, "c"] \arrow[d, hook] & E(k(\eta))^\wedge_\ell \arrow[r, hook] \arrow[d] & \mathrm{H}^1(\pi_1(E), T_\ell(E)) \arrow[d] \\
E(k(\eta)\overline{k}) \arrow[r, "c"] & E(k(\eta)\overline{k})^\wedge_\ell \arrow[r, hook] & \mathrm{H}^1(\pi_1(E_{\overline{k}}), T_\ell(E)).
\end{tikzcd}$$

Here, note that the image of $\eta \in E(k(\eta))$ in $E(k(\eta)\overline{k}) = E_{\overline{k}}(k(\eta)\overline{k})$ coincides with $\eta_{\overline{k}}$. From the injectivity of $E(k(\eta)\overline{k})^\wedge_\ell \hookrightarrow \mathrm{H}^1(\pi_1(E_{\overline{k}}), T_\ell(E))$, it is enough to prove that $c(\eta_{\overline{k}}) \neq 0$ that is,

$$\eta_{\overline{k}} \notin \ker(c) = \bigcap_{n \geq 0} [\ell^n] E(k(\eta)\overline{k}).$$

But, one has:

$$\begin{aligned}
E(k(\eta)\overline{k}) &= \mathrm{Hom}_{Sch/\overline{k}}(k(\eta)\overline{k}, E_{\overline{k}}) \\
&= \mathrm{Hom}_{Sch/\overline{k}}(E_{\overline{k}}, E_{\overline{k}}) \\
&= \mathrm{Hom}_{AV/\overline{k}}(E_{\overline{k}}, E_{\overline{k}}) \oplus E(\overline{k}),
\end{aligned}$$

with $\mathrm{Hom}_{AV/\overline{k}}(E_{\overline{k}}, E_{\overline{k}})$ a finitely generated free $\mathbb{Z}$-module of positive rank. So, on the one hand, $\ker(c) = E(\overline{k})$ and, one the other hand, $\eta_{\overline{k}} = Id_{E_{\overline{k}}} \neq 0$ in $E(k(\eta)\overline{k})/E(\overline{k})$, as required. □

Remark 5.7.

(1) The above construction can be extended to any d-dimensional abelian variety A over k of positive (Mordell-Weil) rank and such that $\rho_{A,\ell}(\Gamma_k) \subset \mathrm{GSp}_{2d}(\mathbb{Z}_\ell)$ is open. In particular, it shows that over a d-dimensional basis X, $k = 3$ in theorem 1.3 should be replaced, at least, by $k = 2d + 1$.

(2) At the time of writing this paper, the authors do not know whether or not we could replace $X^{cl, \leq d}$ by X^{cl} in the assertion of theorem 1.3.

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