DEGENERATION OF ℓ-ADIC LOCAL SYSTEMS ON PRODUCTS OF TWO CURVES

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Abstract. Let X be a smooth, geometrically connected variety over a field k of characteristic 0 and let $\mathcal{V}$ be a $\mathbb{Q}_\ell$-local system on X. A central problem is to understand the arithmetico-geometric structure of the degeneration locus $X_\mathcal{V}$ of $\mathcal{V}$. To $\mathcal{V}$ one can attach a projective system of level varieties

$$\underline{X}_\mathcal{V} = \cdots \to X_{\mathcal{V},n+1} \to X_{\mathcal{V},n} \to \cdots \to X_{\mathcal{V},1} \to X_{\mathcal{V},0} = X$$

whose transition morphisms are (non-connected) étale covers with the property that for every $x \in X$, $\mathcal{V}$ degenerates at x if and only if x lifts to a projective system of $k(x)$-points on $\underline{X}_\mathcal{V}$. When k is algebraically closed, $X = X_1 \times X_2$ is the product of two hyperbolic curves and the algebraic monodromy of $\mathcal{V}$ is , we show that for every integers $g \geq 1$, $\gamma \geq 1$ there exists an integer $N = N(g,\gamma) \geq 1$ such that for $n \geq N$ every connected component $X^\circ_{\mathcal{V},n}$ of $X_{\mathcal{V},n}$ is either birational to a smooth projective surface of general type and contains only finitely many closed integral curves with geometric genus $\leq g$ or dominates a smooth connected curve of gonality $\geq \gamma$. This implies that for every integer $g \geq 0$ the degeneration locus $X_\mathcal{V} \subset X$ of $\mathcal{V}$ contains only finitely many transverse curves (that is closed integral curves $C \hookrightarrow X$ dominating both X_1 and X_2) with geometric genus $\leq g$. When k is a number field and assuming the Bombieri-Lang conjecture for (product quotient) surfaces of general type, this also implies (uniform) boundedness results of arithmetic nature - for instance the following generalization of the ℓ-primary part of a question of Mazur for elliptic curves: let C be a smooth curve over k and $A \to C$ an abelian scheme then there exists an integer $N = N(A,\ell)$ such that for every $c_1, c_2 \in C(k)$, $A_{c_1}(\overline{k})[\ell^N] \simeq A_{c_1}(\overline{k})[\ell^N]$ as $\pi_1(k)$-modules implies that A_{c_1} and A_{c_1} are k-isogenous. The proof of our main result, which is inspired from a strategy initiated by A. Tamagawa and the author when X is a curve, combines three main ingredients: the combinatorial description of the Chern numbers of product quotient surfaces in terms of ramification data, asymptotic estimates for the reduction modulo-ℓ^n of homogeneous spaces for ℓ-adic Lie groups and, ultimately, a celebrated theorem of Bogolomov (the geometric Lang conjecture for surfaces of general type with $c_1^2 - c_2 > 0$).

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1. Introduction

Let k be a field of characteristic 0. A variety over k (or a k-variety) is a separated scheme of finite type over k.

1.1. Let X be a smooth, geometrically connected variety of dimension d over k and let $f : Y \to X$ be a smooth proper morphism. An important problem in arithmetic geometry is to describe the locus X_f of all $x \in X$ where the powers $(Y_x)^r$, $r \geq 1$ of the fiber Y_x at x carry more algebraic cycles (modulo homological equivalence) than the powers $(Y_\eta)^r$, $r \geq 1$ of the fiber Y_η at the generic point η of X. Fix a prime ℓ and consider the $\mathbb{Q}_\ell$-local system $\mathcal{V} := \oplus_{i\geq 0} R^{2i} f_* \mathbb{Q}_\ell(i)$. For every $x \in X$, fix a geometric point $\overline{x}$ over x and an étale path from $\overline{x}$ to $\overline{\eta}$ inducing a canonical isomorphism $\mathcal{V}_{\overline{x}} \tilde{\to} \mathcal{V}_{\overline{\eta}} =: V$ equivariant with respect to the canonical isomorphism of étale fundamental groups $\pi_1(X,\overline{x}) \tilde{\to} \pi_1(X,\overline{\eta})$. Let $\Pi_\mathcal{V} \subset \mathrm{GL}(V)$ denote the image of $\pi_1(X,\overline{\eta})$ acting on V and $\Pi_{\mathcal{V},x} \subset \Pi_\mathcal{V}$ the image of $\pi_1(x,\overline{x})$ acting on V *via* $\pi_1(x,\overline{x}) \to \pi_1(X,\overline{x}) \tilde{\to} \pi_1(X,\overline{\eta})$. Assume k is finitely generated over $\mathbb{Q}$. Then, conjecturally, the degeneration locus $X_\mathcal{V}$ of $\mathcal{V}$ - that is the locus of all $x \in X$ where $\Pi_{\mathcal{V},x}$ has codimension ≥ 1 in Π - coincides with X_f and, if instead of algebraic cycles one considers motivated cycles in the sense of André [An96] then (unconditionnally) $X_f \subset X_\mathcal{V}$.

1.2. This prompts more generally the question of describing the degeneration locus $X_\mathcal{V} \subset X$ of an arbitrary $\mathbb{Q}_\ell$-local system $\mathcal{V}$ over X. Under mild assumptions on $\mathcal{V}$, one expects that $X(k) \cap X_\mathcal{V}$ is not Zariski-dense in X. Indeed, every fundamental system[1] $\Pi_\mathcal{V}(n)$, $n \geq 0$ of neighbourhoods of 1 in $\Pi_\mathcal{V}$ gives rise to a projective system of "level varieties" (See Section 2.1.2 for details)

$$\underline{X}_\mathcal{V} = \cdots \to X_{\mathcal{V},n+1} \to X_{\mathcal{V},n} \to \cdots \to X_{\mathcal{V},1} \to X_{\mathcal{V},0} = X$$

[1]In the following, given a profinite group Γ, a fundamental system of neighbourhoods of 1 in Γ means a decreasing sequence of normal open subgroups $\Gamma = \Gamma(0) \supset \Gamma(1) \supset \cdots \supset \Gamma(n) \supset \Gamma(n+1) \supset \cdots$ such that $\cap_{n\geq 0} \Gamma(n) = 1$ (or, equivalently, for every open subgroup $U \subset \Gamma$ there exists $n \geq 0$ such that $\Gamma(n) \subset U$).

whose transition morphisms are (non-connected) étale covers of X with the property that for every $x \in X$, $x \in X_{\mathcal{V}}$ if and only if x lifts to a projective system of $k(x)$-points on $\underline{X}_{\mathcal{V}}$. In particular, $X(k) \cap X_{\mathcal{V}}$ is exactly the image of

$$\lim_n X_{\mathcal{V},n}(k) \to X(k).$$

The rough expectation is that for $n \gg 0$, every connected component $X^\circ_{\mathcal{V},n}$ of $X_{\mathcal{V},n}$ is either defined[2] over an extension k_U of k of degree ≥ 2 (in which case $X^\circ_{\mathcal{V},n}(k) = \emptyset$) or is "complicated enough" so that $X^\circ_{\mathcal{V},n}(k)$ is not Zariski-dense in $X^\circ_{\mathcal{V},n}$.

For $d = 1$, the Mordell conjecture [FW84] gives a sufficient geometric condition for the non-Zariski-density (*viz* the finiteness) of $X^\circ_{\mathcal{V},n}(k)$ in $X^\circ_{\mathcal{V},n}$, namely that $X^\circ_{\mathcal{V},n}$ has geometric genus ≥ 2. In turn, a sufficient condition on $\mathcal{V}$ so that every connected component of $X_{\mathcal{V},n}$ defined over k has geometric genus ≥ 2 for $n \gg 0$ is that $\mathcal{V}$ be geometrically Lie-perfect (GLP for short) that is the Lie algebra $\mathrm{Lie}(\overline{\Pi}_{\mathcal{V}})$ of the image $\overline{\Pi}_{\mathcal{V}}$ of $\pi_1(X_{\overline{k}}, \overline{\eta})$ acting on V is perfect - see[3][CT12b, Thm. 1.1].

For $d \geq 2$ the picture is drastically more complicated. A main obstruction is that there is no unconditional geometric criterion like the Mordell conjecture ensuring the non-Zariski density of k-points on higher dimensional varieties. Classically, a higher-dimensional conjectural substitute for the Mordell conjecture is the following. For a geometrically integral variety Y over k define the special locus $sp_Y \subset Y$ as the Zariski-closure of the union of the images of all rational maps from a variety birational to a smooth projective variety of Kodaira dimension ≤ 0 over k to X. Then,

Conjecture 1. (Bombieri-Lang - *e.g.* [A09, 1.5], [HS00, F.5.2]) *Let k be a number field and let Y be a geometrically integral variety over k, birational to a smooth projective variety of general type over k. Then $sp_Y \subset Y$ is of codimension ≥ 1 in Y and $(Y \setminus sp_Y)(k)$ is finite.*

Say that $\mathcal{V}$ is geometrically curve-Lie-perfect (GCLP for short) if for every connected smooth curve C over $\overline{k}$ and non-constant morphism $\phi : C \to X$, $\phi^*\mathcal{V}$ is GLP. Define the notion of being geometrically Lie-semisimple (GLS for short) and geometrically curve-Lie-semisimple (GCLS for short) similarly, replacing the condition of being perfect by the condition of being semisimple; of course, GLS (resp. GCLS) implies GLP (resp. GCLP). Note that as a quotient of a perfect (resp. semisimple) Lie algebra is again perfect (resp. semisimple) the properties of being GLP, GCLP (resp. GLS, GCLS) are preserved by quotient. Pure $\mathbb{Q}_\ell$-local systems - in particular those lying in the Tannakian category generated by the $R^i f_* \mathbb{Q}_\ell(j)$, $i \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}$ for $f : Y \to X$ a smooth proper morphism - are GLS [D80, (3.4.12), (3.4.13)]. For examples of GLP $\mathbb{Q}_\ell$-local systems which are not GLS in general, see [C21].

Though Conjecture 1 seems currently out of reach even for surfaces, it motivates the following possibly more tractable geometric conjecture.

Conjecture 2. ([C22, Conj. 16])*Assume $k = \overline{k}$ and $\mathcal{V}$ is GCLP. Then every connected component of $X_{\mathcal{V},n}$ dominates a variety of general type for $n \gg 0$.*

1.3. To investigate Conjecture 2 as well as the non-Zariski-density of $X(k) \cap X_{\mathcal{V}}$, one can freely replace X by a non-empty open subscheme. In particular, one may assume that X is a (resp. strongly) hyperbolic Artin neighbourhood *i.e.* that it decomposes into a sequence

$$X = X_d \to X_{d-1} \to \cdots \to X_1 \to X_0 = \mathrm{spec}(k)$$

of elementary fibrations $X_n \to X_{n-1}$ into hyperbolic curves[4] (resp. with the additionnal property that X_n embeds into a product of hyperbolic curves), $n = 1, \ldots, d$ [SSt16, Cor. 5.9]. Strongly hyperbolic Artin neighbourhood are anabelian in the sense that they can be reconstructed from their étale homotopy type [SSt16, Thm. 1.2]. As the projective systems $\underline{X}_{\mathcal{V}}$ are also constructed from the representation $\mathcal{V}_{\overline{\eta}}$ of $\pi_1(X, \overline{\eta})$, they should be easier to control when X has some strong anabelian features.

In this note we consider the simplest case of 2-dimensional strongly hyperbolic Artin neighbourhoods - namely the direct product $X = X_1 \times X_2$ of two smooth, geometrically connected hyperbolic curves X_1, X_2 over k.

[2]The field of definition of a connected variety Y over k is the finite field extension K/k corresponding to the image of $\pi_1(Y) \to \pi_1(k)$ by the morphism of profinite groups induced by the structural morphism $Y \to spec(k)$.

[3]Actually, one can show that the minimum of the geometric gonality of the irreducible components of $X_{\mathcal{V},n}$ defined over k goes to $+\infty$ with n which, by the Mordell-Lang conjecture [Fa91], ensures that for every integer $\delta \geq 1$ and $n \gg 0$ depending on δ, the set of closed points $x \in X_{\mathcal{V},n}$ with degree $[k(x) : k] \leq \delta$ is finite; hence, in particular, that the set of closed points $x \in X_{\mathcal{V}}$ with degree $[k(x) : k] \leq \delta$ is finite. See [CT13] for details.

[4]Let $g, r \geq 0$ be integers such that $2g - 2 + r > 0$. An hyperbolic curve of type (g, r) over S is a S-scheme $C \to S$ such that there exists a smooth proper curve $C^+ \to S$ of genus g and a closed subscheme $D \hookrightarrow C^+$ with $D \hookrightarrow C^+ \to S$ finite étale of degree r and $C = C^+ \setminus D$.

1.4. Assume $k=\overline{k}$ and $X=X_1\times_k X_2$ with X_i a smooth, connected curve over k of genus g_i, $i=1,2$. Let $\mathcal{V}$ be a (G)LP $\mathbb{Q}_\ell$-local system on X. From now on we omit base points from the notation as well as the subscripts $(-)_{\mathcal{V}}$. Set $V:=\mathcal{V}_{\overline{\eta}}$. The projections $pr_i: X\to X_i$, $i=1,2$ induce an isomorphism $\pi_1(X)\tilde{\to}\pi_1(X_1)\times\pi_1(X_2)$. Write $\Pi_1:=\rho(\pi_1(X_1)\times 1), \Pi_2:=\rho(1\times\pi_1(X_2))\subset\Pi$ and let $G_i\subset G$ denote the Zariski-closure of Π_i in $\mathrm{GL}(V)$, $i=1,2$. Let $p:\Pi_1\times\Pi_2\twoheadrightarrow\Pi$ and $p_i:\Pi_1\times\Pi_2\to\Pi_i$, $i=1,2$ denote the canonical projections; for a closed subgroup $H\subset\Pi$, write $\widetilde{H}:=p^{-1}(H)\subset\Pi_1\times\Pi_2$ and for a closed subgroup $H\subset\Pi_1\times\Pi_2$ write $H_i:=p_i(H)\subset\Pi_i$, $i=1,2$. One says that $H\subset\Pi$ is transverse if both $\widetilde{H}_1\subset\Pi_1$ and $\widetilde{H}_2\subset\Pi_2$ are open subgroups.

For an open subgroup $U\subset\Pi$, let $X_U\to X$ denote the connected étale cover corresponding to the inverse image of U in $\pi_1(X)$. Similarly, for $i=1,2$ and $U_i\subset\Pi_i$ an open subgroup, let $X_{i,U_i}\to X_i$ denote the connected étale cover corresponding to the inverse image of U_i in $\pi_1(X_i)$. Fix a fundamental system $\Pi(n)$, $n\geq 0$ of open neighbourhoods of 1 in Π. Every closed subgroup $H\subset\Pi$ of codimension ≥ 1 gives rise to a projective system of connected étale covers

$$\underline{X}_H=\cdots\to X_{H\Pi(n+1)}\to X_{H\Pi(n)}\to\cdots\to X_{H\Pi(1)}\to X_{H\Pi(0)}=X.$$

We can now state our main result, which answers (and refines) Conjecture 2 - see Subsection 2.1.2.

Theorem 3. *Let $\mathcal{V}$ be a (G)LP $\mathbb{Q}_\ell$-local system on X. Then one has the following dichotomy*

(1) *$H\subset\Pi$ is not transverse that is, for one of $i=1,2$, $\widetilde{H}_i\subset\Pi_i$ is a closed subgroup of codimension ≥ 1. Then the projective system $\underline{X}_H$ fits into a projective commutative diagram*

$$\begin{tikzcd}\cdots \arrow[r] & X_{H\Pi(n+1)} \arrow[r]\arrow[d] & X_{H\Pi(n)} \arrow[r]\arrow[d] & \cdots \arrow[r] & X \arrow[d]\\ \cdots \arrow[r] & B_{i,H,n+1} \arrow[r] & B_{i,H,n} \arrow[r] & \cdots \arrow[r] & X_i\end{tikzcd}$$

with $B_{i,H,n}\to X_i$ a connected étale cover with gonality (hence, a fortiori, geometric genus) going to $+\infty$ with n;

(2) *$H\subset\Pi$ is transverse and $X_{H\Pi(n)}$ is birational to a smooth projective surface of general type and, for every integer $g\geq 0$, contains only finitely many closed integral curves with geometric genus $\leq g$ for $n\gg 0$.*

1.5. We refer to Section 2 for a sample of (unconditional) geometric applications and, assuming Conjecture 1 for (product-quotient surfaces), arithmetic applications of Theorem 3. Let us just illustrate these by considering a concrete example. For a closed integral subvariety $Z\hookrightarrow X$, write $\overline{\Pi}_Z:=\overline{\Pi}_{\mathcal{V}|_Z}$, $\Pi_Z:=\Pi_{\mathcal{V}|_Z}$ and $\overline{G}_Z, G_Z\subset\mathrm{GL}_V$ for their Zariski-closure. Say that $Z\hookrightarrow X$ is $\mathcal{V}$-weakly special (resp. $\mathcal{V}$-special) if Z is maximal among closed integral subvarieties with connected geometric (resp. arithmetic) monodromy $\overline{G}_Z^\circ$ (resp. G_Z°); in particular, every closed point is weakly special and, special implies weakly special. Let k be a number field. Fix an integer $d\geq 1$, for $i=1,2$, let $f_i: A_i\to X_i$ be an abelian scheme of relative dimension d and let $f:=f_1\times f_2: A_1\times A_2\to X_1\times X_2=X$ denote the resulting product abelian scheme. Set $\mathcal{V}:=R^1f_*\mathbb{Q}_\ell=R^1f_{1*}\mathbb{Q}_\ell\oplus R^2f_{1*}\mathbb{Q}_\ell$. Say that a closed integral curve $Z\hookrightarrow X$ is transverse if the resulting morphisms $Z\hookrightarrow X\stackrel{pr_i}{\to}X_i$, $i=1,2$ are dominant. Then the following holds.

(1) (Corollary 5) There are only finitely many transverse $\mathcal{V}$-weakly special curves $Z\hookrightarrow X$ with bounded geometric genus; furthermore for a given integer $g\geq 0$, there exists an integer $B\geq 1$ such that for every transverse non $\mathcal{V}$-weakly special curve $Z\hookrightarrow X$, $[\overline{\Pi}:\overline{\Pi}_Z]\leq B$.

(2) (Corollary 6) Assume Conjecture 1 holds for (product-quotient) surfaces of general type over k, then $X_{\mathcal{V}}(k)$ is not Zariski-dense in X and there exists an integer $B\geq 1$ such that for every $x\in X(k)\setminus X_{\mathcal{V}}\cap X(k)$, $[\Pi:\Pi_x]\leq B$;

(3) (Corollary 8) Assume Conjecture 1 holds for (product-quotient) surfaces of general type over k, then there exists an integer $N=N(A_1,A_2,\ell)\geq 0$ such that for every $(x_1,x_2)\in X(k)$ the following properties are equivalent:

  (i) $A_{1,x_1}(\overline{k})[\ell^N]\simeq A_{2,x_2}(\overline{k})[\ell^N]$ as $\pi_1(k)$-modules;
  (ii) $A_{1,x_1}(\overline{k})[\ell^n]\simeq A_{2,x_2}(\overline{k})[\ell^n]$ as $\pi_1(k)$-modules for some $n\geq N$;
  (iii) $A_{1,x_1}(\overline{k})[\ell^n]\simeq A_{2,x_2}(\overline{k})[\ell^n]$ as $\pi_1(k)$-modules for every $n\geq N$;
  (iv) A_{1,x_1} and A_{2,x_2} are k-isogenous.

For instance, when $f_1 : A_1 \to X_1 = f_2 : A_2 \to X_2$ is "the" universal elliptic scheme $f : \mathcal{E} \to Y(1)$, the transverse $\mathcal{V}$-weakly special curves $Z \hookrightarrow Y(1) \times Y(1)$ are exactly the images $C(N)$ of the Hecke correspondances $Y_0(N) \to Y(1) \times Y(1)$ so that (1) recovers the classical fact (which can of course be deduced directly from the explicit Hodge-theoretic description of $C(N)$) that the geometric genus of $C(N)$ goes to $+\infty$ with N. Also, as there are only finitely many $\mathcal{V}$-special points of bounded degree on $Y(1)$, (2) amounts to saying that $\cup_N C(N)(k)$ is not Zariski-dense in $Y(1) \times Y(1)$, which is still an open question (See *e.g.* [U04, Sec. 4.1]). (3) follows from (2) and is the ℓ-primary part of a - still widely open - question of Mazur [M78, p. 133]. For some closely related explicit results using the specific feature of the moduli of elliptic curves see [KS98].

1.6. **Organization of the paper and brief sketch of the strategy for Theorem 3.** We begin by giving some applications of Theorem 3 in Section 2. The geometric applications (Subsection 2.1.2) are unconditional but the arithmetic applications (Subsection 2.2) require the assumption that Conjecture 1 holds for surfaces of general type over k. The remaining part of the paper is devoted to the proof of Theorem 3. In Section 3, we give the general architecture of the proof and reduces it to a core geometric statement - Theorem 13 - for certain projective systems $\underline{Y} = \cdots \to Y_{n+1} \to Y_n \to \cdots \to Y_1 \to Y_0 = X$ of product-quotient surfaces. In a preliminary step (Subsection 3.2.1), we reduce to the case where $\mathrm{Lie}(\Pi)$ has abelian radical, which ensures that $\mathcal{V}$ is not only GLP but GCLP (Corollary 10). The GCLP assumption is used here through Fact 12. The proof of Theorem 13 in turn is carried out in two steps. For Theorem 13, one easily reduces to the case where $\mathcal{V}$ is GCLS. The first step (Section 4) consists in checking that the Chern classes of our product-quotient surfaces satisfy some numerical relations ensuring that they are of general type and, by a celebrated theorem of Bogomolov, that their set of closed integral curves of geometric genus $\leq g$ are bounded provided $n \gg 0$ (depending on g). This step uses heavily the specific geometric features of product-quotient surfaces and technical results about reduction modulo-ℓ^n of homogeneous spaces for ℓ-adic Lie groups as developped in [CT12b], [CT13] (Fact 20). The GCLS assumption is used there to perform a technical reduction (Lemma 15) ensuring one can apply Fact 20. The second step (Section 5) consists in showing that the boundedness of the set of closed integral curves of geometric genus $\leq g$ implies its finiteness provided $n \gg 0$ (depending on g). Ultimately (See proof of Corollary 23) this uses again the GCLS assumption.

The study of the degeneration locus of a GLP $\mathbb{Q}_\ell$-local system is the étale counterpart of the study of the Hodge locus of a polarizable $\mathbb{Z}$-variation of Hodge structures, which, in the recent years, has drawn lots of attention with spectacular results based on tame geometry and the use of complex analytic period maps (See [Kl22] for a survey). One important feature of the proof of Theorem 3 is that it is purely algebraic and works under a very mild assumption - namely that $\mathcal{V}$ be GLS; in particular it does not require $\mathcal{V}$ to satisfy any of the additional properties of $\mathbb{Q}_\ell$-local systems arising from geometry (such as purity, rationality or, given an embedding $k \hookrightarrow \mathbb{C}$, that $\mathcal{V}$ admit a singular incarnation underlying a $\mathbb{Z}$-VHS on the analytification X^{an} of $X_{\mathbb{C}}$).

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2. Applications

2.1. Geometric Applications.

2.1.1. For a profinite group Π, let $\Phi(\Pi) \subset \Pi$ denote its Frattini subgroup (that is the intersection of all the maximal open subgroups of Π).

Let Π be a compact ℓ-adic Lie group and let $\Pi(n)$, $n \geq 0$ be a fundamental system of neighbourhoods of 1 in Π. For every $n \geq 1$, let $\mathcal{H}_n(\Pi)$ denote the (finite) set of open subgroups $U \subset \Pi$ such that $\Phi(\Pi(n-1)) \subset U$ but $\Pi(n-1) \not\subset U$ and let $\mathcal{H}_0(\Pi) := \{\Pi\}$. Then ([CT12b, Lem. 3.3, (Proof of Cor. 3.6)],

(1) $\mathcal{H}_n(\Pi)$ is finite, $n \geq 0$.
(2) The maps $\mathcal{H}_{n+1}(\Pi) \to \mathcal{H}_n(\Pi)$, $U \mapsto U\Phi(\Pi(n-1))$ (with the convention that $\Phi(\Pi(-1)) = \Pi$) endow the $\mathcal{H}_n(\Pi)$, $n \geq 0$ with a canonical structure of projective system $(\mathcal{H}_{n+1}(\Pi) \stackrel{\phi_n}{\to} \mathcal{H}_n(\Pi))_{n\geq 0}$.
(3) For every $\underline{H} := (H[n])_{n\geq 0} \in \lim\limits_{n} \mathcal{H}_n(\Pi)$,

$$H[\infty] := \lim_{n} H[n] = \cap_{n\geq 0} H[n] \subset \Pi$$

is a closed subgroup of codimension ≥ 1 in Π and $H[n] = H[\infty]\Pi(n)$, $n \gg 0$.

(4) For every closed subgroup $H \subset \Pi$ such that $\Pi(n-1) \not\subset H$ there exists $U \in \mathcal{H}_n(\Pi)$ such that $H \subset U$.

2.1.2. We retain the notation and assumption of Subsection 1.4. From 2.1.1 (1),

$$X_n := \bigsqcup_{U \in \mathcal{H}_n(\Pi)} X_U \to X$$

is a (non-connected) étale cover of X and, by functoriality of étale fundamental group, the maps $\mathcal{H}_{n+1}(\Pi) \to \mathcal{H}_n(\Pi)$, $n \geq 0$ of (2.1.1.2) endow the X_n, $n \geq 0$ with a structure of projective system

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

whose transition morphisms are étale covers.

Corollary 4. *For every integers $g \geq 1$, $\gamma \geq 1$ there exists an integer $N = N(g, \gamma) \geq 1$ such that for every $n \geq N$ every connected component X_n° of X_n is either birational to a smooth projective surface of general type and contains only finitely many closed integral curves with geometric genus $\leq g$ or, for one of $i = 1, 2$, fits into a commutative square of non-constant morphisms*

$$\begin{tikzcd} X_n^\circ \arrow[r] \arrow[d] & X \arrow[d, "pr_i"] \\ B_{i,n} \arrow[r] & X_i \end{tikzcd}$$

with $B_{i,n}$ a smooth connected curve of gonality $\geq \gamma$.

Proof. (See also [C22, Sec. 4.2]) Let $\mathcal{H}_{n,<g,\gamma}(\Pi) \subset \mathcal{H}_n(\Pi)$ denote the subset of all $U \in \mathcal{H}_n(\Pi)$ such that X_U is not of the type described in Corollary 4. Then the projective system $\mathcal{H}_{n+1}(\Pi) \to \mathcal{H}_n(\Pi)$, $n \geq 0$ restricts to a projective system $\mathcal{H}_{n+1,<g,\gamma}(\Pi) \to \mathcal{H}_{n,<g,\gamma}(\Pi)$, $n \geq 0$. Assume $\mathcal{H}_{n,<g,\gamma}(\Pi) \neq \emptyset$, $n \geq 1$. By 2.1.1 (1), $\mathcal{H}_{n,<g,\gamma}(\Pi)$ is finite, $n \geq 0$ hence $\lim\limits_{n} \mathcal{H}_{n,<g,\gamma}(\Pi) \neq \emptyset$. Let $\underline{H} := (H[n])_{n\geq 1} \in \lim\limits_{n} \mathcal{H}_{n,<g,\gamma}(\Pi)$. By 2.1.1 (3), $H[\infty] := \cap_{n\geq 1} H[n]$ is a closed subgroup of codimension ≥ 1 in Π and $H[n] = H[\infty]\Pi(n)$, for $n \gg 0$, which contradicts Theorem 3 for $H = H[\infty]$. □

2.1.3. Recall that a closed integral curve $Z \hookrightarrow X$ is said to be transverse if the resulting morphisms $Z \hookrightarrow X \stackrel{pr_i}{\to} X_i$, $i = 1, 2$ are dominant. If $Z \hookrightarrow X$ is transverse then the image $\Pi_Z \subset \Pi$ of $\pi_1(Z) \to \pi_1(X) \twoheadrightarrow \Pi$ is transverse in the sense of Subsection 1.4. The following can be regarded as a uniform variant of Bertini theorem.

Corollary 5. *With the same notation and assumptions as in Theorem 3, for every integer $g \geq 0$ and for all but finitely many transverse closed integral curves $Z \hookrightarrow X$ of geometric genus $\leq g$, $\Pi_Z \subset \Pi$ is open and there exists an integer $B \geq 1$ such that for all transverse closed integral curves $Z \hookrightarrow X$ with geometric genus $\leq g$ and $\Pi_Z \subset \Pi$ open, $[\Pi : \Pi_Z] \leq B$.*

Proof. Let $\mathcal{H}_n^+(\Pi) \subset \mathcal{H}_n(\Pi)$ denote the subset of all $U \in \mathcal{H}_n(\Pi)$ such that X_U is birational to a smooth projective surface of general type and contains only finitely many closed integral curves with geometric genus $\leq g$; set

$$X_n^+ := \bigsqcup_{U \in \mathcal{H}_n^+(\Pi)} X_U \to X$$

for the corresponding (non-connected) étale cover. Let $S[n] \subset X$ denote the union of the images of the finitely many closed integral curves $Z \hookrightarrow X_n^+ \to X$ with geometric genus $\leq g$. By construction $S[n+1] \subset S[n]$ hence, as X is Noetherian, $S[\infty] := S[N_0] = S[n]$, $n \geq N_0$ for some integer $N_0 \geq 1$. Let $Z \hookrightarrow X$ be a closed integral transverse curve with geometric genus $\leq g$. By 2.1.1 (4), $Z \not\subset S[\infty]$ implies that $\Pi_Z \subset \Pi$ is open, which already shows the first part of Corollary 5. For the second part, fix an integer $\gamma > \frac{g+3}{2}$ and let $N_1 = N(g, \gamma) \geq 1$ be as in Corollary 4; write $N := \max\{N_0, N_1\}$. Let $Z \hookrightarrow X$ be a closed integral transverse curve with geometric genus $\leq g$ and such that $Z \not\subset S[\infty]$. Assume $Z \hookrightarrow X$ lifts to $X_N \to X$. By the Brill-Noether inequality, Z has gonality $\leq \lfloor \frac{g+3}{2} \rfloor < \gamma$ hence $Z \hookrightarrow X$ necessarily lifts to $X_N^+ \to X$. But this contradicts the assumption that $Z \not\subset S[\infty]$. So $Z \hookrightarrow X$ does not lift to $X_N \to X$ or, equivalently 2.1.1 (4), $\Pi(N-1) \subset \Pi_Z$. Hence, in particular, $[\Pi : \Pi_Z] \leq [\Pi : \Pi(N-1)]$. □

Example. (Geometric torsion conjecture for abelian scheme over X) Let $A \to X$ be an abelian scheme. Then, for every prime ℓ and integer $g \geq 0$ there exists an integer $B = B(A, \ell, g) \geq 1$ such that for every closed integral curve $Z \hookrightarrow X$ with geometric genus $\leq g$ either $A \times_X Z \to Z$ is k-isotrivial or $|A[\ell^\infty](Z)| \leq B$.

2.2. **(Conjectural) arithmetic implications.** We now assume k is a number field. For a GLS $\mathbb{Q}_\ell$-local system $\mathcal{V}$ on $X = X_1 \times X_2$ we let again $\Pi_x \subset \Pi \subset GL(V)$ denote respectively the image of $\pi_1(x)$ and $\pi_1(X)$ acting on $\mathcal{V}_{\overline{x}} \tilde{\to} \mathcal{V}_{\overline{\eta}} =: V$. We refer to [C22, Sec. 4.2] for the proof of how Corollary 6 below, which is the main arithmetic motivation for Theorem 3, follows from Corollary 4.

Corollary 6. *Assume Conjecture 1 holds for (product-quotient) surfaces of general type*[5]. *Let $\mathcal{V}$ be a GLS $\mathbb{Q}_\ell$-local system on X. Then $X_{\mathcal{V}} \cap X(k)$ is not Zariski-dense in X and*

$$\sup\{[\Pi : \Pi_x] \mid x \in X(k) \setminus X_{\mathcal{V}} \cap X(k)\} < +\infty.$$

and to [C22, Sec. 3] for the following applications of Corollary 6.

Corollary 7. *Assume Conjecture 1 holds for (product-quotient) surfaces of general type over k. Let $f : Y \to X$ be a smooth projective morphism.*

(1) *The set of all $x \in X(k)$ such that the motivated motivic Galois group of $Y_{\overline{x}}$ is of codimension ≥ 1 in the motivated motivic Galois group of $Y_{\overline{\eta}}$ is not Zariski-dense in X;*
(2) *The set of all $x \in X(k)$ such that the (injective) specialization map of Neron-Severi groups $sp_{\overline{x}} : NS(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to NS(Y_{\overline{x}}) \otimes \mathbb{Q}$ is not an isomorphism is not Zariski-dense in X.*
(3) *There exists an integer $n = n(f, \ell, k) \geq 0$ such that for every $x \in X(k)$ with Y_x satisfying the Tate conjecture for divisors $|Br(Y_{\overline{x}})^{\pi_1(k)}[\ell^\infty]| \leq \ell^n$.*

Assume furthermore $f : Y \to X$ is an abelian scheme, then

(3) *There exists an integer $n = n(f, \ell, k) \geq 0$ such that $|Y_x[\ell^\infty](k)| \leq \ell^n$, $x \in X(k)$.*
(4) *There exists an integer $n = n(f, \ell, k) \geq 0$ such that $|Y_x[\ell^\infty](k)| \leq \ell^n$, $x \in X(k)$.*

Remark. The proof of Corollary 7 (1), (2) only requires Corollary 6 (1) while the proof of Corollary 7 (3), (4) requires the full strength of Corollary 6. Again, see [C22, Sec. 3] for details.

The following application, which - modulo Conjecture1 - answers positively a higher-dimensional generalization of the ℓ-primary part of a question of Mazur [M78, p. 133] is specific to the direct product situation and is not treated in [C22]. Let $\mathcal{V}_i$ be a rank-r GLS $\mathbb{Q}_\ell$-local system on X_i with torsion-free $\mathbb{Z}_\ell$-model $\mathcal{T}_i$, $i = 1, 2$. Consider the GLS $\mathbb{Q}_\ell$-local system $\mathcal{V} = p_1^*\mathcal{V}_1 \oplus p_2^*\mathcal{V}_2$ on $X = X_1 \times X_2$.

Corollary 8. *Assume that Conjecture 1 holds for (product-quotient) surfaces of general type over k. Then, there exists an integer $N = N(\mathcal{V}, k) \geq 1$ such that for every $x = (x_1, x_2) \in X(k)$ the following properties are equivalent:*

$$\begin{array}{ll} (i) & x_1^*(\mathcal{T}_1/\ell^N) \simeq x_2^*(\mathcal{T}_2/\ell^N); \\ (ii) & x_1^*(\mathcal{T}_1/\ell^n) \simeq x_2^*(\mathcal{T}_2/\ell^n) \text{ for some } n \geq N; \\ (iii) & x_1^*(\mathcal{T}_1/\ell^n) \not\simeq x_2^*(\mathcal{T}_2/\ell^n) \text{ for every } n \geq N; \\ (iv) & x_1^*\mathcal{V}_1 \simeq x_2^*\mathcal{V}_2. \end{array}$$

.

Proof. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. We prove (i) $\Rightarrow$ (ii) that is, equivalently, $x_1^*\mathcal{V}_1 \not\simeq x_2^*\mathcal{V}_2 \Rightarrow x_1^*(\mathcal{T}_1/\ell^N) \not\simeq x_2^*(\mathcal{T}_2/\ell^N)$. Set $T_i := \mathcal{T}_{i,\overline{\eta}_i}$, $T := T_1 \oplus T_2$ and let $\Pi(n)$ denote the kernel of the morphism $\Pi \subset \mathrm{GL}(T) \twoheadrightarrow \mathrm{GL}(T/\ell^n)$ induced by reduction modulo-ℓ^n. Observe first that for every closed subgroup $H \subset \Pi$ reduction modulo-ℓ^n induces a structure of projective system on the set $\mathrm{Isom}_H(T_1/\ell^n, T_2/\ell^n)$ of H-equivariants isomorphisms, whence a canonical morphism

$$\lim_{\substack{\leftarrow \\ n}} \mathrm{Isom}_H(T_1/\ell^n, T_2/\ell^n) \to \mathrm{Isom}_H(T_1, T_2) \to \mathrm{Isom}_H(V_1, V_2)$$

[5]The proof of Theorem 3 indeed shows that it is enough to assume Conjecture 1 for product quotient surfaces of general type. A product quotient surface is a minimal resolution of singularity $q : \widetilde{Y} \to Y$ of a (singular) surface of the form $Y = (X_1 \times X_2)/\Delta_\alpha$, where X_1, X_2 are smooth, projective, geometrically connected curves endowed with a faithfull action of a finite group Γ and $\Gamma \tilde{\to} \Delta_\alpha \subset \Gamma \times \Gamma$ is the graph of an automorphism $\alpha : \Gamma \tilde{\to} \Gamma$. Let $g_{i,0}$ denote the genus of $X_{i,0} := X_i/\Gamma$, $i = 1, 2$. Then the irregularity of $\widetilde{Y}$ is $g_{1,0} + g_{2,0}$. As Theorem 3 is very sensitive to base change, we cannot ensure that the product quotient surfaces $\widetilde{Y}_n^+$ that appear as smooth compactifications of level varieties in Subsection 3.3.2 have irregularity > 0 (and actually, the most difficult case is precisely the one where $g_{1,0} + g_{2,0} = 0$). In particular we cannot ensure they fall in the class of surfaces of general type for which Conjecture 1 is known - namely essentially those which can be embedded into abelian varieties [Fa91].

As $\mathrm{Isom}_H(T_1/\ell^n, T_2/\ell^n)$ is finite for every $n \geq 0$, $\mathrm{Isom}_H(V_1, V_2) = \emptyset$ implies that there exists an integer $n_H \geq 0$ such that $\mathrm{Isom}_H(T_1/\ell^n, T_2/\ell^n) = \emptyset$, $n \geq n_H$.

From Corollary 6, $X_{\mathcal{V}} \cap X(k)$ is not Zariski-dense in X and there exists an integer $N = N(\mathcal{V}, k) \geq 1$ such that for every $x \in X(k) \setminus X_{\mathcal{V}}$, $\Pi(N) \subset \Pi_x$. In particular, letting $\Pi(N) \subset U_1, \ldots, U_r \subset \Pi$ denote the finitely many open subgroups of Π containing $\Pi(N)$, for every $x = (x_1, x_2) \in X(k) \setminus X_{\mathcal{V}}$, $\Pi_x = U_{i_x}$ for some $i_x \in \{1, \ldots, r\}$. Let $I \subset \{1, \ldots, r\}$ denote the subset of $1 \leq i \leq r$ such that $\mathrm{Isom}_{U_i}(V_1, V_2) = \emptyset$ and $n_i := n_{U_i} \geq 0$ the corresponding integer. Then

- Either $i_x \notin I$ hence $\mathrm{Isom}_{\Pi_x}(V_1, V_2) \neq \emptyset$ and there is nothing to say;
- or $i_x \in I$ and $\mathrm{Isom}_{\Pi_x}(T_1/\ell^n, T_2/\ell^n) = \emptyset$, $n \geq n_{i_x}$.

As a result, with $N_0 = \max\{n_i \mid i \in I\}$, for every $x \in X(k) \setminus X_{\mathcal{V}}$ and $n \geq N_0$, one has: $x_1^*\mathcal{V}_1 \not\simeq x_2^*\mathcal{V}_2 \Rightarrow x_1^*(\mathcal{T}_1/\ell^n) \not\simeq x_2^*(\mathcal{T}_2/\ell^n)$. Next, let $Z_1, \ldots, Z_r$ denote the 1-dimensional irreducible components of the Zariski-closure $Z_{\mathcal{V}}$ of $X_{\mathcal{V}}(k)$ in X, $Z_1^{\mathrm{sm}}, \ldots, Z_s^{\mathrm{sm}}$ their smooth loci and $Z_{\mathcal{V},0} := \{z_1, \ldots, z_t\}$ the finitely many k-rational points which are either a 0-dimensional component of $Z_{\mathcal{V}}$ or in one of the $Z_i \setminus Z_i^{\mathrm{sm}}$, $i = 1, \ldots, s$. As $\mathcal{V}$ is GLS hence GCLS (See Remark 11 below), $\mathcal{W}_i := \mathcal{V}|_{Z_i}$ is GLS as well so that by [CT12b, Thm. 1], $Z_{i\mathcal{W}_i}(k)$ is finite and there exists an integer $N = N(\mathcal{W}_i, k) \geq 1$ such that for every $x \in Z_i(k) \setminus Z_{i\mathcal{W}_i}$, $\Pi_{Z_i}(N) \subset \Pi_x$, where $\Pi_{Z_i} \subset \Pi$ is the image of $\pi_1(Z_i)$ acting on V and $\Pi_{Z_i}(N) := \Pi_{Z_i} \cap \Pi(N)$. One can thus argue as above considering the finitely many open subgroups $\Pi_{Z_i}(N) \subset U_1, \ldots, U_r \subset \Pi_{Z_i}$ to obtain an integer $n_i \geq 0$ such that for every $x = (x_1, x_2) \in Z_i(k) \setminus Z_{i\mathcal{W}_i}$ and $n \geq n_i$, one has: $x_1^*\mathcal{V}_1 \not\simeq x_2^*\mathcal{V}_2 \Rightarrow x_1^*(\mathcal{T}_1/\ell^n) \not\simeq x_2^*(\mathcal{T}_2/\ell^n)$. Hence, setting $N_1 := \max\{N_0, n_1, \ldots, n_s\}$, for every $x = (x_1, x_2) \in X(k) \setminus (Z_{\mathcal{V},0} \cup Z_{1\mathcal{W}_1} \cup \cdots Z_{s\mathcal{W}_s})$ and $n \geq N_1$, one has $x_1^*\mathcal{V}_1 \not\simeq x_2^*\mathcal{V}_2 \Rightarrow x_1^*(\mathcal{T}_1/\ell^n) \not\simeq x_2^*(\mathcal{T}_2/\ell^n)$. It remains to treat the finitely many k-rational points in $Z_{\mathcal{V},0} \cup Z_{1\mathcal{W}_1} \cup \cdots Z_{s\mathcal{W}_s}$ but, for each $z \in Z_{\mathcal{V},0} \cup Z_{1\mathcal{W}_1} \cup \cdots Z_{s\mathcal{W}_s}$,

- Either $\mathrm{Isom}_{\Pi_z}(V_1, V_2) \neq \emptyset$ and there is nothing to say;
- or $\mathrm{Isom}_{\Pi_z}(V_1, V_2) \neq \emptyset$ and $\mathrm{Isom}_{\Pi_z}(T_1/\ell^n, T_2/\ell^n) = \emptyset$, $n \geq n_z := n_{\Pi_z}$.

So, if $I \subset Z_{\mathcal{V},0} \cup Z_{1\mathcal{W}_1} \cup \cdots Z_{s\mathcal{W}_s}$ denotes the subset of $z \in Z_{\mathcal{V},0} \cup Z_{1\mathcal{W}_1} \cup \cdots Z_{s\mathcal{W}_s}$ such that $\mathrm{Isom}_{\Pi_z}(V_1, V_2) \neq \emptyset$, $N_2 := \max\{N_1, n_z,\ z \in I\}$ has the requested property, namely: for every $x = (x_1, x_2) \in X(k)$ and $n \geq N_2$, one has $x_1^*\mathcal{V}_1 \not\simeq x_2^*\mathcal{V}_2 \Rightarrow x_1^*(\mathcal{T}_1/\ell^n) \not\simeq x_2^*(\mathcal{T}_2/\ell^n)$. □

Example: Let $A_i \to X_i$ be an abelian scheme, $i = 1, 2$. By the Tate conjecture for abelian varieties [FW84], Corollary 8 applied with $V_i = V_\ell(A_{i,\overline{\eta}_i})$ and $T_i = T_\ell(A_{i,\overline{\eta}_i})$ yields the following ℓ-primary generalization of a question of Mazur for elliptic curves [M78, p. 133]. Assume that Conjecture 1 holds for (product-quotient) surfaces of general type over k. Then there exists an integer $N = n(A_1, A_2, \ell, k) \geq 0$ such that for every $x_1 \in X_1(k)$, $x_2 \in X_2(k)$, if the $\pi_1(k)$-modules $A_{\overline{x}_1}[\ell^n]$, $A_{\overline{x}_2}[\ell^n]$ are isomorphic then the abelian varieties A_{x_1}, A_{x_2} are k-isogenous.

* * *

From now on and till the end of the paper, let k be an *algebraically closed field of characteristic* 0. Before diving in the proof of Theorem 3, we begin by the following observation. Let $\mathfrak{g}$ be a Lie algebra with solvable radical $\mathfrak{r} \subset \mathfrak{g}$. Recall that $\mathfrak{g}/\mathfrak{r}$ is a semisimple Lie algebra and that the short exact sequence of Lie algebras

$$0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \to 0$$

splits; choose a Levi complement $\mathfrak{g} \supset \mathfrak{s} \tilde{\to} \mathfrak{g}/\mathfrak{r}$. Set

$$\mathfrak{r}_{\mathfrak{s}} := \mathfrak{r}/[\mathfrak{s}, \mathfrak{r}], \quad \mathfrak{r}^{\mathfrak{s}} := \{r \in \mathfrak{r} \mid [s, r] = 0,\ s \in \mathfrak{s}\}.$$

Lemma 9. *The following hold.*

(1) *Assume $\mathfrak{r}$ is abelian. Then $\mathfrak{g}^{\mathrm{ab}} = \mathfrak{r}_{\mathfrak{s}}$ and $\mathfrak{g}$ is perfect if and only if $\mathfrak{r}^{\mathfrak{s}} = 0$.*

(2) *Assume there exists commuting Lie subalgebras $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$. Let $\mathfrak{r}_i \subset \mathfrak{g}_i$ denote the solvable radical of $\mathfrak{g}_i$ and $\mathfrak{s}_i \subset \mathfrak{g}_i$ a Levi complement, $i = 1, 2$. Then $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2$ and, if $\mathfrak{g}$ is perfect with $\mathfrak{r}$ abelian, the following hold.*
   (a) *$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$;*
   (b) *$\mathfrak{g}_1, \mathfrak{g}_2$ are perfect;*
   (c) *Every Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that the induced morphisms $\mathfrak{g}_0 \hookrightarrow \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \stackrel{pr_i}{\to} \mathfrak{g}_i$, $i = 1, 2$ are surjective is perfect.*

Proof. The equality $\mathfrak{g}^{\mathrm{ab}}=\mathfrak{r}_{\mathfrak{s}}$ in (1) follows from $[\mathfrak{r},\mathfrak{r}]=0$ (as $\mathfrak{r}$ is abelian), $[\mathfrak{s},\mathfrak{s}]=0$ (as $\mathfrak{s}$ is semisimple) and $\mathfrak{g}=\mathfrak{r}+\mathfrak{s}$. For the second assertion in (1), as $\mathfrak{s}$ is semisimple, the short exact sequences of $\mathfrak{s}$-modules

$$0\rightarrow \mathfrak{r}'\rightarrow \mathfrak{r}\rightarrow \mathfrak{r}_{\mathfrak{s}}\rightarrow 0$$

and

$$0\rightarrow \mathfrak{r}^{\mathfrak{s}}\rightarrow \mathfrak{r}\rightarrow \mathfrak{r}''\rightarrow 0$$

split that is $\mathfrak{r}\simeq\mathfrak{r}'\oplus\mathfrak{r}_{\mathfrak{s}}\simeq\mathfrak{r}^{\mathfrak{s}}\oplus\mathfrak{r}''$ as $\mathfrak{s}$-modules hence $\mathfrak{r}_{\mathfrak{s}}\subset\mathfrak{r}^{\mathfrak{s}}$ and $\mathfrak{r}_{\mathfrak{s}}\twoheadrightarrow\mathfrak{r}^{\mathfrak{s}}$. For $i=1,2$, as $\mathfrak{g}_i\subset\mathfrak{g}$ is an ideal and $\mathfrak{r}_i\subset\mathfrak{g}_i$ is characteristic in $\mathfrak{g}_i$, $\mathfrak{r}_i\subset\mathfrak{g}$ is a solvable ideal hence $\mathfrak{r}_i\subset\mathfrak{r}$. This already shows $\mathfrak{r}_1+\mathfrak{r}_2\subset\mathfrak{r}$. On the other hand, the canonical morphism $\mathfrak{g}_1/\mathfrak{r}_1\oplus\mathfrak{g}_2/\mathfrak{r}_2\twoheadrightarrow\mathfrak{g}/(\mathfrak{r}_1+\mathfrak{r}_2)$ is surjective hence, as $\mathfrak{g}_1/\mathfrak{r}_1\oplus\mathfrak{g}_2/\mathfrak{r}_2$ is semisimple, $\mathfrak{g}/(\mathfrak{r}_1+\mathfrak{r}_2)$ is semisimple as well, which shows $\mathfrak{r}\subset\mathfrak{r}_1+\mathfrak{r}_2$ and proves the first part of (2). Assume furthermore $\mathfrak{g}$ is perfect. Write $\mathfrak{u}:=\mathfrak{r}_1\cap\mathfrak{r}_2$. Then $[\mathfrak{u},\mathfrak{g}_i]\subset[\mathfrak{g}_1,\mathfrak{g}_2]=0$, $i=1,2$ so that $[\mathfrak{u},\mathfrak{g}]=0$ hence *a fortiori* $[\mathfrak{u},\mathfrak{s}]=0$. In other words, $\mathfrak{u}\subset\mathfrak{r}^{\mathfrak{s}}$ but, from (1), $\mathfrak{r}^{\mathfrak{s}}=0$. This already shows $\mathfrak{r}=\mathfrak{r}_1\oplus\mathfrak{r}_2$. But then, $\mathfrak{g}_1\cap\mathfrak{g}_2\subset Z(\mathfrak{g})\subset\mathfrak{g}_1\cap\mathfrak{g}_2\cap\mathfrak{r}=\mathfrak{r}_1\cap\mathfrak{r}_2=0$, which concludes the proof of (2) (a). For the proof of (2) (b), observe that as $\mathfrak{r}_i\subset\mathfrak{r}$, $\mathfrak{r}_i$ is abelian hence, from (1), $\mathfrak{g}_i$ is perfect if and only if $\mathfrak{r}_i^{\mathfrak{s}_i}=0$. But $\mathfrak{r}=\mathfrak{r}_1\oplus\mathfrak{r}_2$ implies $\mathfrak{r}^{\mathfrak{s}}=\mathfrak{r}_1^{\mathfrak{s}}\oplus\mathfrak{r}_2^{\mathfrak{s}}=\mathfrak{r}_1^{\mathfrak{s}_1}\oplus\mathfrak{r}_2^{\mathfrak{s}_2}$ and from (1), $\mathfrak{r}^{\mathfrak{s}}=0$, which implies $\mathfrak{r}_i^{\mathfrak{s}_i}=0$, $i=1,2$ and proves (2) (b). Let $\mathfrak{r}_0\subset\mathfrak{g}_0$ denote the solvable radical of $\mathfrak{g}_0$ and $\mathfrak{s}_0\subset\mathfrak{g}_0$ a Levi complement. Consider the canonical projection $pr:\mathfrak{g}\twoheadrightarrow\mathfrak{g}/\mathfrak{r}\simeq\mathfrak{s}_1\oplus\mathfrak{s}_2$. By assumption the canonical projection $\mathfrak{s}_1\oplus\mathfrak{s}_2\twoheadrightarrow\mathfrak{s}_i$ restricts to a surjective morphism $pr(\mathfrak{g}_0)\twoheadrightarrow\mathfrak{s}_i$, $i=1,2$ hence, as $\mathfrak{s}_1$, $\mathfrak{s}_2$ are semisimple, $pr(\mathfrak{r}_0)\subset\mathfrak{s}_1\cap\mathfrak{s}_2=0$ that is $\mathfrak{r}_0\subset\mathfrak{r}_1\oplus\mathfrak{r}_2=\mathfrak{r}$. In particular, $\mathfrak{r}_0$ is abelian and $\mathfrak{r}_0=\mathfrak{g}_0\cap\mathfrak{r}$ so that $\mathfrak{s}_0\simeq\mathfrak{g}_0/\mathfrak{r}_0\hookrightarrow\mathfrak{p}_1/\mathfrak{r}_1\oplus\mathfrak{p}_2/\mathfrak{r}_2\simeq\mathfrak{s}_1\oplus\mathfrak{s}_2$. This shows $\mathfrak{r}_0^{\mathfrak{s}_0}\subset\mathfrak{r}_1^{\mathfrak{s}_0}\oplus\mathfrak{r}_2^{\mathfrak{s}_0}=\mathfrak{r}_1^{\mathfrak{s}_1}\oplus\mathfrak{r}_2^{\mathfrak{s}_2}=\mathfrak{r}^{\mathfrak{s}}=0$ hence, by (1), $\mathfrak{g}_0$ is perfect. □

Corollary 10. *Let $\mathcal{V}$ be a $\mathbb{Q}_\ell$-local system on X. Assume that the solvable radical of* Lie(Π) *is abelian. Then $\mathcal{V}$ is (G)CLP if and only if $\mathcal{V}$ is (G)LP.*

Proof. The direct implication follows from the Bertini theorem and does not use the assumption that X is a product of two hyperbolic curves. Conversely, we retain the notation of Subsection 1.4. Let C be a connected, smooth curve over k and $\phi:C\rightarrow X$ a non-constant morphism. Write $\Pi_C\subset\Pi$ for the image of $\pi_1(C)$ acting on V *via* $\pi_1(C)\stackrel{\phi}{\rightarrow}\pi_1(X)$. Set $\mathfrak{p}:=\mathrm{Lie}(\Pi)$, $\mathfrak{p}_i:=\mathrm{Lie}(\Pi_i)$, $i=1,2$ and $\mathfrak{p}_C:=\mathrm{Lie}(\Pi_C)$. By assumption $\mathfrak{p}$ has abelian solvable radical and $\mathfrak{p}_1,\mathfrak{p}_2\subset\mathfrak{p}$ are commuting Lie subalgebras such that $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_2$ so that we are in the setting of Lemma 9. Assume the image of $\phi:C\rightarrow X$ is non-transverse that is there exists $i=1,2$ - say $i=1$ - such that $pr_1\circ\phi:C\rightarrow X_1$ is constant with image x_1 so that $\phi:C\rightarrow X$ factors through a dominant morphism $\phi:C\rightarrow X_{x_1}=\{x_1\}\times X_2$, which identifies Π_C with Π_2. Then $\mathfrak{p}_C=\mathfrak{p}_2$ and the assertion follows from Lemma 9 (2) (b). Assume the image of $\phi:C\rightarrow X$ is transverse. Then the induced morphisms $\mathfrak{p}_C\hookrightarrow\mathfrak{p}\simeq\mathfrak{p}_1\oplus\mathfrak{p}_2\stackrel{pr_i}{\twoheadrightarrow}\mathfrak{p}_i$, $i=1,2$ are both surjective and the assertion follows from Lemma 9 (2) (c). □

Remark 11. For the GLS property, one easily shows that $\mathcal{V}$ is (G)CLS if and only if $\mathcal{V}$ is (G)LS. Indeed, the proof is exactly the same as for Corollary 10 observing that, in Lemma 9 (2), if $\mathfrak{g}$ is furthermore assumed to be semisimple (equivalently, $\mathfrak{r}=0$) then

- $\mathfrak{r}=\mathfrak{r}_1=\mathfrak{r}_2=0$ hence $\mathfrak{g}_1$, $\mathfrak{g}_2$ are both semisimple;
- Every Lie subalgebra $\mathfrak{g}_0\subset\mathfrak{g}$ as in Lemma (2) (c) is semismple (as its solvable radical $\mathfrak{r}_0$ is contained in $\mathfrak{r}=0$).

3. Proof of Theorem 3 - general strategy

For every variety X over k, $\mathbb{Q}_\ell$-local system $\mathcal{V}$ on X and open subgroup $U\subset\Pi_{\mathcal{V}}$, write $X_U\rightarrow X$ for the connected étale cover corresponding to the inverse image of U in $\pi_1(X)$.

3.1. Theorem 3 in the non-transverse case easily reduces to the main result of [CT13]. Namely, let X be a smooth connected curve over k. Let $\mathcal{V}$ be a $\mathbb{Q}_\ell$-local system on X. Fix a fundamental system $\Pi(n)$, $n\geq 0$ of neighbourhoods of 1 in $\Pi:=\Pi_{\mathcal{V}}$.

Fact 12. ([CT13, Thm. 3.3])*Assume $\mathcal{V}$ is (G)LP. Then, for every closed subgroup $H\subset\Pi$ of codimension ≥ 1, the gonality of $X_{H\Pi(n)}$ goes to $+\infty$ with n.*

Theorem 3 in the transverse case reduces to the case of a projective system of "product-quotient surfaces", which is encapsulated in the following statement. Let Γ be a positive dimensional compact ℓ-adic Lie group and let $\Gamma(n)\subset\Gamma$, $n\geq 0$ be a fundamental system of neighborhoods of 1 in Γ. Write $(-)_n:\Gamma\twoheadrightarrow\Gamma_n:=\Gamma/\Gamma(n)$ for the reduction-modulo-$\Gamma(n)$ morphism, $n\geq 0$. For $i=1,2$, let X_i be a smooth, connected curve over

k with smooth compactification $X_i \hookrightarrow X_i^+$ and assume we are given a continuous surjective representation $\rho_i : \pi_1(X_i) \twoheadrightarrow \Gamma$. Let $Y_{i,n} \to X_i$ denote the étale cover corresponding to $\rho_i^{-1}(\Gamma(n)) \subset \pi_1(X_i)$ and $Y_{i,n} \hookrightarrow Y_{i,n}^+$ its smooth compactification, which is also the normalization of X_i^+ in $Y_{i,n} \to X_i \hookrightarrow X_i^+$. By construction, Γ_n acts faithfully on $Y_{i,n}^+$. Let $\Delta_n \tilde{\to} \Gamma_n \subset \Gamma_n \times \Gamma_n$ denote the diagonal subgroup and $Y_n^+ := (Y_{1,n}^+ \times Y_{2,n}^+)/\Delta_n$ the resulting "product-quotient surface", $q_n : \widetilde{Y}_n^+ \to Y_n^+$ a minimal resolution of singularities.

Theorem 13. *Assume Lie(Γ) is perfect. Then, for every integer $g \geq 0$, $\widetilde{Y}_n^+$ is of general type and contains only finitely many closed integral curves $C \hookrightarrow \widetilde{Y}_n^+$ with geometric genus $\leq g$ for $n \gg 0$ (depending on g).*

3.2. **Proof of Theorem 3 - preliminary reductions.**

3.2.1. *Reduction to abelian radical.* (See [CT13, §3.2] for details) Write $\mathfrak{p} := \mathrm{Lie}(\Pi)$ and let $\mathfrak{r} \subset \mathfrak{p}$ denote the solvable radical of $\mathfrak{p}$. One can always find closed normal subgroups $R' \subset R \subset \Pi$ of Π corresponding *via* the Lie algebra functor to the inclusions of Lie ideals $\mathfrak{r}' := [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r} \subset \mathfrak{p}$; let $p : \Pi \twoheadrightarrow \overline{\Pi} := \Pi/R'$ denote the canonical projection and for a closed subgroup $U \subset \Pi$, write $\overline{U} := p(U) \subset \overline{\Pi}$. Then one easily checks the following:

(1) For every closed subgroup $H \subset \Pi$ of codimension ≥ 1, $\overline{H} \subset \overline{\Pi}$ is again a closed subgroup of codimension ≥ 1 (see item (3) after Rem. 3.4 in [CT13, §3.2]);

(2) For every closed subgroup $H \subset \Pi$ of codimension ≥ 1, the inclusion $H\Pi(n) \subset HR'\Pi(n) = p^{-1}(\overline{H\Pi(n)}) \subset \Pi$ of open subgroups corresponds to a connected étale cover
$$X_{H\Pi(n)} \to X_{HR'\Pi(n)} = X_{\overline{H\Pi(n)}},$$
so that it is enough to prove Theorem 3 for the projective system
$$\cdots \to X_{\overline{H\Pi(n)}} \to X_{\overline{H\Pi(n)}} \to \cdots X_{\overline{H\Pi(1)}} \to X.$$

(3) $\overline{\Pi(n)}$, $n \geq 0$ is again a fundamental system of neighbourhoods of 1 in $\overline{\Pi}$ so that, fixing a closed embedding $\overline{\Pi} \hookrightarrow \mathrm{GL}_r(\mathbb{Z}_\ell)$ of ℓ-adic Lie groups [L88, Prop. 4], it is enough to prove Theorem 3 for the $\mathbb{Q}_\ell$-local system $\overline{\mathcal{V}}$ on X corresponding to the representation $\pi_1(X) \twoheadrightarrow \overline{\Pi} \hookrightarrow \mathrm{GL}_r(\mathbb{Z}_\ell)$.

So, from now on, assume the solvable radical of Lie(Π) is abelian. From Lemma 10, this ensures $\mathcal{V}$ is (G)CLP.

3.2.2. *Reduction to direct product.* From now on, we retain the notation and assumptions of Subsection 1.4. We identify $\Pi_1 \tilde{\to} \Pi_1 \times 1 \subset \Pi_1 \times \Pi_2$ and $\Pi_2 \tilde{\to} 1 \times \Pi_2 \subset \Pi_1 \times \Pi_2$. By definition the restriction of the product map $\mu : \Pi_1 \times \Pi_2 \twoheadrightarrow \Pi$, $(\pi_1, \pi_2) \mapsto \pi_1\pi_2$ to Π_i is injective so that one can also identify $\Pi_i \tilde{\to} \mu(\Pi_i) \subset \Pi$. With these conventions, and as $\mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$ (see (2) (a) of the claim in the proof of Lemma 10), $\Pi_1 \cap \Pi_2 \tilde{\to} \ker(p)$, $z \mapsto (z, z^{-1})$ is finite.

To prove Theorem 3 one can choose the fundamental system $\Pi(n)$, $n \geq 0$ of open neighbourhoods of 1 in Π freely. So that, for $i = 1, 2$, we fix a fundamental system $\Pi_i(n)$, $n \geq 0$ of open neighbourhoods of 1 in Π_i and take $\Pi(n) = \Pi_1(n)\Pi_2(n)(= \mu(\Pi_1(n) \times \Pi_2(n))) \subset \Pi$, $n \geq 0$ (as $\Pi_1(n) \times \Pi_2(n)$, $n \geq 0$ is a fundamental system of open neighbourhoods of 1 in $\Pi_1 \times \Pi_2$ and $\mu : \Pi_1 \times \Pi_2 \twoheadrightarrow \Pi$ is surjective, $\Pi(n)$, $n \geq 0$ is indeed a fundamental system of open neighbourhoods of 1 in Π).

Let $H \subset \Pi$ be a closed subgroup of codimension ≥ 1. By definition, one has a commutative diagram of profinite groups

$$\begin{tikzcd}
\Pi_1 \times \Pi_2 \arrow[d] & (\widetilde{H}_1\Pi_1(n)) \times (\widetilde{H}_2\Pi_2(n)) \arrow[l, hook'] \arrow[dr, phantom, "\square"] & \widetilde{H}(\Pi_1(n) \times \Pi_2(n)) = \widetilde{H\Pi(n)} \arrow[l, hook'] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\Pi(n)} \arrow[l, hook'] \arrow[d] & \Pi_1(n) \times \Pi_2(n) \arrow[l, hook'] \arrow[d] \\
\Pi & & H\Pi(n) \arrow[ll, hook'] & \Pi(n) \arrow[l, hook'] \arrow[r, equal] & \Pi(n)
\end{tikzcd}$$

corresponding to a commutative diagram of connected étale covers

$$\begin{tikzcd}
X \arrow[d, equal] & X_{1,\widetilde{H}_1\Pi_1(n)} \times X_{2,\widetilde{H}_2\Pi_2(n)} \arrow[l] & X_{\widetilde{H\Pi(n)}} \arrow[l] \arrow[d, equal] & X_{\widetilde{\Pi(n)}} \arrow[l] \arrow[d, equal] & X_{1,\Pi_1(n)} \times X_{2,\Pi_2(n)} \arrow[l] \arrow[d] \\
X & & X_{H\Pi(n)} \arrow[ll] & X_{\Pi(n)} \arrow[l] \arrow[r, equal] & X_{\Pi(n)}
\end{tikzcd}$$

This shows that Theorem 3 for $\rho : \pi_1(X) \twoheadrightarrow \Pi$ and $H \subset \Pi$ follows from Theorem 3 for $\rho_1 \times \rho_2 : \pi_1(X) \twoheadrightarrow \Pi_1 \times \Pi_2$ and $\widetilde{H} \subset \Pi_1 \times \Pi_2$. So, in the following, we will assume $\rho = \rho_1 \times \rho_2 : \pi_1(X) \twoheadrightarrow \Pi_1 \times \Pi_2$ and consider only closed subgroups $\widetilde{H} \subset \Pi_1 \times \Pi_2$ containing the finite abelian antidiagonal subgroup $Z := \Pi_1 \cap \Pi_2$.

3.3. **Proof of Theorem 3 assuming Theorem 13.**

3.3.1. *Non transverse case.* Assume that for one of $i = 1, 2$, $\widetilde{H}_i \subset \Pi_i$ is a closed subgroup of codimension ≥ 1. It is enough to show that the gonality of $X_{i,\widetilde{H}_i \Pi_i(n)}$ is unbounded. But since $\widetilde{H}_i \subset \Pi_i$ is a closed subgroup of codimension ≥ 1, and the $\mathbb{Q}_\ell$-local system $\mathcal{V}|_{X_i}$ is GLP (Lemma 10) this follows from Fact 12.

3.3.2. *Transverse case.* Assume now $\widetilde{H}_i \subset \Pi_i$ is an open subgroup, $i = 1, 2$. As we are interested only in the asymptotic behavior of $X_{H\Pi(n)}$, without loss of generality one may replace X_i with $X_{\widetilde{H}_i}$ hence assume that $\widetilde{H}_i = \Pi_i$, $i = 1, 2$.

By the structure of closed subgroups of $\Pi_1 \times \Pi_2$, there exists closed normal subgroups $N_i \subset \Pi_i$, $i = 1, 2$ and a profinite group isomorphism $\varphi : \Pi_1/N_1 \tilde{\to} \Pi_2/N_2$ such that $\widetilde{H} \subset \Pi_1 \times \Pi_2$ is the inverse image of the graph $\mathrm{Graph}(\varphi) \subset \Pi_1/N_1 \times \Pi_2/N_2$ of φ *via* the canonical projection $q : \Pi_1 \times \Pi_2 \twoheadrightarrow \Pi_1/N_1 \times \Pi_2/N_2$. In particular, $N_1 \times N_2 \subset \widetilde{H}$. Hence up to replacing $\rho : \pi_1(X) \twoheadrightarrow \Pi_1 \times \Pi_2$ with $q \circ \rho : \pi_1(X) \twoheadrightarrow \Pi_1 \times \Pi_2 \twoheadrightarrow \Pi_1/N_1 \times \Pi_2/N_2$ (which corresponds again to a (G)LP $\mathbb{Q}_\ell$-local system on X), $\widetilde{H}$ with $q(\widetilde{H})$ and $\Pi_1(n) \times \Pi_2(n)$ with $q(\Pi_1(n) \times \Pi_2(n))$, one may assume $N_i = 1$, $i = 1, 2$. But then, up to replacing again $\rho = \rho_1 \times \rho_2 : \pi_1(X) \twoheadrightarrow \Pi_1 \times \Pi_2$ with $\rho = \varphi\rho_1 \times \rho_2 : \pi_1(X) \twoheadrightarrow \Pi_2 \times \Pi_2$ one may assume $\Pi_1 = \Pi_2 =: \Gamma$ and $\widetilde{H} := \Delta \tilde{\to} \Gamma \subset \Gamma \times \Gamma = \Pi_1 \times \Pi_2$ is the diagonal subgroup. We may also assume $\Pi_1(n) = \Pi_2(n) =: \Gamma(n)$. Let $(-)_n : \Gamma \twoheadrightarrow \Gamma_n := \Gamma/\Gamma(n)$ denote the canonical projection and set $Y_{i,n} := X_{i,\Pi_i(n)} := X_{i,\Gamma(n)} \to X_i$, $i = 1, 2$. Let $X_i \hookrightarrow X_i^+$, $Y_{i,n} \hookrightarrow Y_{i,n}^+$ denote the smooth compactification of X_i, $Y_{i,n}$ respectively and let $p'_n : Y_{1,n}^+ \times Y_{2,n}^+ \to Y_n^+ := (Y_{1,n}^+ \times Y_{2,n}^+)/\Delta_n$ denote the quotient of $Y_{1,n}^+ \times Y_{2,n}^+$ by the action of the diagonal subgroup $\Delta_n \tilde{\to} \Gamma_n \subset \Gamma_n \times \Gamma_n$. By construction $Y_n := X_{\widetilde{H}(\Pi_1(n) \times \Pi_2(n))} := X_{\Delta(\Gamma(n) \times \Gamma(n))}$ identifies with the dense open subset $(Y_{1,n} \times Y_{2,n})/\Delta_n \hookrightarrow Y_n^+$. It is thus enough to show that if $q_n : \widetilde{Y}_n^+ \to Y_n^+$ is a minimal resolution of singularities then $\widetilde{Y}_n^+$ is of general type and contains only finitely many irreducible curves with geometric genus $\leq g$. But this is precisely the statement of Theorem 13.

3.4. **Proof of Theorem 13 - general strategy.** The remaining part of the paper is devoted to the proof of Theorem 13.

3.4.1. Recall that the surfaces $\widetilde{Y}_n^+$ are constructed from a ℓ-adic Lie group Γ with $\mathrm{Lie}(\Gamma)$ perfect and continuous surjective morphisms $\pi_1(X_i) \twoheadrightarrow \Gamma$, $i = 1, 2$. Recall that for every continuous surjective morphisms $(-)^\flat$: $\Gamma \twoheadrightarrow \Gamma^\flat$, $\mathrm{Lie}(\Gamma^\flat)$ is again perfect.

Lemma 14. *Let $(-)^\flat : \Gamma \twoheadrightarrow \Gamma^\flat$ be a continuous surjective morphism. Assume that Theorem 13 holds for the surfaces $\widetilde{Y}_n^+$ constructed from $\pi_1(X_i) \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma^\flat$, $i = 1, 2$. Then Theorem 13 for the surfaces $\widetilde{Y}_n^+$ constructed from $\pi_1(X_i) \twoheadrightarrow \Gamma$, $i = 1, 2$.*

In other words, to prove Theorem 13, one may freely replace Γ by a continuous quotient $\Gamma \twoheadrightarrow \Gamma^\flat$.

Proof. Set $\Gamma^\flat(n) := \Gamma(n)^\flat \subset \Gamma^\flat$, $n \geq 0$, which form again a fundamental system of neighbourhoods of 1 in $\Gamma^\flat$. For $i = 1, 2$, consider the resulting representation $\rho_i^\flat : \pi_1(X_i) \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma^\flat$. As before, let $Y_{i,n}^\flat \to X_i$ denote the étale cover corresponding to $\rho_i^{\flat\,-1}(\Gamma^\flat(n)) \subset \pi_1(X_i)$ and $Y_{i,n}^\flat \hookrightarrow Y_{i,n}^{\flat,+}$ its smooth compactification, which is also the normalization of X_i^+ in $Y_{i,n}^\flat \to X_i \hookrightarrow X_i^+$. By construction, $\Gamma_n^\flat := \Gamma^\flat/\Gamma^\flat(n)$ acts faithfully on $Y_{i,n}^{\flat,+}$. Let $\Delta_n^\flat \simeq \Gamma_n^\flat \subset \Gamma_n^\flat \times \Gamma_n^\flat$ denote the diagonal subgroup and $Y_n^{\flat,+} := (Y_{1,n}^{\flat,+} \times Y_{2,n}^{\flat,+})/\Delta_n^\flat$ the resulting product-quotient surface, $q_n^\flat : \widetilde{Y}_n^{\flat,+} \to Y_n^{\flat,+}$ a minimal resolution of singularities. Write $N := \ker(\Gamma \twoheadrightarrow \Gamma^\flat)$. Corresponding to the morphism of short exact sequences with exact columns

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma(n) \cap N & \longrightarrow & N & \longrightarrow & N_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma(n) & \longrightarrow & \Gamma & \longrightarrow & \Gamma_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma^\flat(n) & \longrightarrow & \Gamma^\flat & \longrightarrow & \Gamma_n^\flat & \longrightarrow & 1
\end{array}$$

one has a commutative diagram

$$\begin{tikzcd}
Y^+_{1,n}\times Y^+_{2,n} \arrow[r, "\Delta_n"] \arrow[d, "N_n\times N_n"'] \arrow[dd, bend right=60, "\Gamma_n\times\Gamma_n"'] & Y^+_n \arrow[d, "N_n\times N_n/N_n\simeq N_n"] & \widetilde{Y}^+_n \arrow[l, "q_n^\flat"'] \arrow[d, dotted] \\
Y^{\flat,+}_{1,n}\times Y^{\flat,+}_{2,n} \arrow[r, "\Delta_n^\flat"'] \arrow[d, "\Gamma_n^\flat\times\Gamma_n^\flat"'] & Y^{\flat,+}_n & \widetilde{Y}^{\flat,+}_n \arrow[l, "q_n"] \\
X^+_1\times X^+_2, & &
\end{tikzcd}$$

(where the right vertical arrow is only a rational map). If the $\widetilde{Y}^{\flat,+}_n$, $n\geq 0$ satisfy the cionclusion of Theorem 13 then so do the $\widetilde{Y}^+_n$, $n\geq 0$. $\square$

In particular, if $R\subset\Gamma$ is a closed normal subgroup of Π corresponding *via* the Lie algebra functor to the inclusion of the solvable radical of $\mathrm{Lie}(\Gamma)$ in $\mathrm{Lie}(\Gamma)$, replacing Γ with $\Gamma\twoheadrightarrow\Gamma^\flat:=\Gamma/R$, one may assume $\mathrm{Lie}(\Gamma)$ is semisimple.

Lemma 15. *Let Γ be a compact ℓ-adic Lie group such that $Lie(\Gamma)$ is semisimple. Then Γ fits into a short exact sequence of profinite groups*

$$1\to N\to\Gamma\to\Gamma^\flat\to 1,$$

with N finite and $\Gamma^\flat$ such that for every non-trivial abelian subgroup $\Lambda\subset\Gamma^\flat$, the normalizer $N_{\Gamma^\flat}(\Lambda)\subset\Gamma^\flat$ is a closed subgroup of codimension ≥ 1.

Proof. Fix a finite dimensional faithful continuous $\mathbb{Q}_\ell$-representation V of Γ [L88, Prop. 4] and let $G\subset GL(V)$ denote the Zariski-closure of Γ. As $\mathrm{Lie}(\Gamma)$ is semisimple, G is semisimple as well and $\mathrm{Lie}(\Gamma)=\mathrm{Lie}(G)$. It is enough to show that there is an isogeny $G\twoheadrightarrow G^\flat$ with $G^\flat$ containing no non-trivial abelian subgroup normalized by the neutral component $G^{\flat\circ}$ of $G^\flat$. Then one can take $\Gamma^\flat$ to be the image of $\Gamma\to G(\mathbb{Q}_\ell)\to G^\flat(\mathbb{Q}_\ell)$. Indeed, let $\Lambda\subset G^\flat(\mathbb{Q}_\ell)$ be a non-trivial abelian subgroup with Zariski-closure $L\subset G^\flat$. Then L is again abelian and $N_{\Gamma^\flat}(\Lambda)\subset N_{G^\flat}(L)$. In particular, if $N_{\Gamma^\flat}(\Lambda)\subset\Gamma^\flat$ is open then $G^{\flat\circ}\subset N_{G^\flat}(L)$: a contradiction. To construct the isogeny $G\twoheadrightarrow G^\flat$, observe first that replacing G by $G/Z(G^\circ)$ one may assume G° is adjoint. In particular, if $L\subset G$ is an abelian subgroup normalized by G° then, $L\cap G^\circ=1$. Indeed, $(L\cap G^\circ)^\circ\subset G^\circ$ is then a connected normal solvable subgroup hence contained in the solvable radical of G°, which is trivial as G° is semisimple so that $L\cap G^\circ\subset G^\circ$ is finite, hence central since G° is connected, hence trivial since G° is adjoint. This shows that L is finite so, as G° is connected, L is contained in the centralizer $Z_G(G^\circ)$ of G° in G. But as G° is normal in G, $Z_G(G^\circ)$ is also normal in G and as G° is adjoint, $G^\circ\cap Z_G(G^\circ)=1$. To sum it up, identifying L and $Z_G(G^\circ)$ with their image *via* the canononical projection $G\twoheadrightarrow\pi_0(G):=G/G^\circ$, one has $L\subset Z_G(G^\circ)\lhd\pi_0(G)$. One can thus argue by induction on $|\pi_0(G)|$: if $G=G^\circ$ then the above shows that G contains no non-trivial abelian subgroup. If G contains no non-trivial abelian subgroup normalized by G°, take $G=G^\flat$. Otherwise, replace G by $G_1:=G/Z_G(G^\circ)$ so that $|\pi_0(G_1)|<|\pi_0(G)|$ and apply the induction hypothesis to G_1. $\square$

So, replacing further Γ with $\Gamma\twoheadrightarrow\Gamma^\flat:=\Gamma/N$, where $N\subset\Gamma$ is as in Lemma 15, one may assume that $\mathrm{Lie}(\Gamma)$ is semisimple and that the normalizer $N_\Gamma(\Lambda)\subset\Gamma$ of every non-trivial abelian subgroup $\Lambda\subset\Gamma$ is a closed subgroup of codimension ≥ 1.

3.4.2. Our starting point is the following celebrated theorem of Bogomolov. Let Y be a smooth projective surface over k. For a divisor D on Y, write $h^0(D):=dim(H^0(Y,\mathcal{O}_X(D)))$ for the k-dimension of its global sections, $P_m(D):=h^0(mD)$ for its mth plurigenus, $\kappa(D)$ for its Kodaira dimension and $c_i(D)$, $i=1,2$ for its first and second Chern class. For $D=K_Y$ a canonical divisor, we simply write $P_m:=P_m(K_Y)$, $\kappa:=\kappa(K_Y)$, $c_i:=c_i(K_Y)$, $i=1,2$.

Fact 16. ([B77], see also [D04, Thm. 10, Thm. 11]) *Assume Y is of general type with $c_1^2 - c_2 > 0$. Then Y contains only finitely many closed integral curves $C \hookrightarrow Y$ with geometric genus ≤ 1 and for every integer $g \geq 0$, closed integral curves $C \hookrightarrow Y$ with geometric genus $\leq g$ form a bounded family*[6].

Lemma 17. *Assume $c_1^2 + c_2 > 24$. Then Y is of general type.*

Proof. Since $c_1^2 + c_2 > 24$, the Euler-Poincaré characteristic $\chi = \frac{1}{12}(c_1^2 + c_2)$ of Y is > 2 hence $P_1 \geq \chi - 1 > 1$. From the Enriques-Kodaira classification this imposes $\kappa \geq 1$ since a smooth projective variety with Kodaira dimension $-\infty$ (resp. 0) has geometric genus 0 (resp. ≤ 1). But $\kappa = 1$ is impossible as well. Indeed, as the set of all $m \geq 0$ such that $P_m \neq 0$ is a semigroup, $P_1 > 0$ implies $P_m > 0$ for every $m \geq 1$ and that there exists $\alpha_-, \alpha_+ > 0$ such that $\alpha_- m^\kappa \leq P_m \leq \alpha_+ m^\kappa$, $m \geq 0$ [I81, Thm. 10.2]. By Riemann-Roch and Serre duality for mK_Y,

$$P_m + h^0(-(m-1)K_Y) \geq \chi + \frac{m(m-1)}{2} c_1^2 = c_1^2 m^2 + O(m).$$

As a result, $\kappa = 1$ would force $\kappa(-K_Y) = 2$ that is, $-K_Y$ is big. Write $-K_Y = A + E$ as the sum of an ample divisor A and an effective divisor E and choose $m \geq 1$ such that mA is very ample. Then, as a high enough multiple of a very ample divisor is effective, $-NmK_Y$ is both big and effective for $N \gg 0$. This in turn would imply $P_{Nm} = h^0(-(-NmK_Y)) = 0$ for $N \gg 0$: a contradiction. Hence $\kappa = 2$ as claimed. □

3.4.3. We now retain the notation of Subsection 3.1. Recall that we may assume (Subsection 3.4.1) that $\mathrm{Lie}(\Gamma)$ is semisimple and that the normalizer $N_\Gamma(\Lambda) \subset \Gamma$ of every non-trivial abelian subgroup $\Lambda \subset \Gamma$ is a closed subgroup of codimension ≥ 1. Write $c_{i,n} := c_i(K_{\widetilde{Y}_n^+})$, $i = 1, 2$, $n \geq 0$. From Lemma 17, to show that $\widetilde{Y}_n^+$ satisfies the assumptions of Fact 16 it is enough to prove the following.

Theorem 18. *Let $\epsilon = \pm 1$. Then $\lim_{n \to +\infty} c_{1,n}^2 + \epsilon c_{2,n} = +\infty$.*

The proof of Theorem 18 is carried out in Section 4. It uses the fact that the size of the Chern classes of $\widetilde{Y}_n^+$ can be controlled asymptotically by the representation theoretic data attached to $\rho_1 \times \rho_2 : \pi_1(X_1) \times \pi_1(X_2) \twoheadrightarrow \Gamma \times \Gamma$. This, in turn, relies on the specific features of product-quotient surfaces and asymptotic estimates from [CT12b] about the number of points on the reduction modulo-ℓ^n of certain homogeneous spaces for compact ℓ-adic Lie groups. It is to apply these asymptotic estimates that we need the technical assumption that the normalizer $N_\Gamma(\Lambda) \subset \Gamma$ of every non-trivial abelian subgroup $\Lambda \subset \Gamma$ is a closed subgroup of codimension ≥ 1.

From Fact 16 and Lemma 17, Theorem 18 already implies

Corollary 19. *Assume $Lie(\Gamma)$ is semisimple. Then, for every integer $g \geq 0$ and $n \gg 0$ (depending on g), $\widetilde{Y}_n^+$ is of general type, contains only finitely many closed integral curves $C \hookrightarrow \widetilde{Y}_n^+$ with geometric genus ≤ 1, and closed integral curves $C \hookrightarrow \widetilde{Y}_n^+$ with geometric genus $\leq g$ form a bounded family.*

Roughly Corollary 19 ensures the existence of an integral variety $\mathcal{H}$ of dimension ≥ 1 over k and a morphism $C \to X \times \mathcal{H}$ such that $C \to \mathcal{H}$ is a relative smooth hyperbolic curve with geometric genus ≥ 2, $C \to X$ is dominant and that for every $n \geq 0$ there exists $h_n \in \mathcal{H}(k)$ such that $C_{h_n} \to X$ lifts along $Y_n \to X$. But the existence of such a $C \to X \times \mathcal{H}$ should impose that the diagonal Lie subalgebra $\mathrm{Lie}(\Gamma) \tilde{\to} \mathrm{Lie}(\Delta) \subset \mathrm{Lie}(\Gamma \times \Gamma) = \mathrm{Lie}(\Gamma) \oplus \mathrm{Lie}(\Gamma)$ is an ideal, which is only possible if $Lie(\Gamma)$ is abelian. The details are carried out in Section 5.

4. Proof of Theorem 18

4.1. **Product-quotient surfaces.** We briefly recall basics about product-quotient surfaces, following [BCGP12, §2] and [BP12, §1]. Let Γ be a finite group and Y_i^+ a smooth projective curve endowed with a faithful action of Γ; write $p_i : Y_i^+ \to X_i^+ := Y_i^+/\Gamma$ $i = 1, 2$ and let g_i and $g_{i,0}$ denote the genus of Y_i^+ and X_i^+ respectively.

[6] That is for one (equivalently every) projective embedding $Y \hookrightarrow \mathbb{P}_k^N$ there exists a constant $d(g)$ (depending on the projective embedding) such that for every closed irreducible curve $C \hookrightarrow Y$ with geometric genus g, $deg(\mathcal{O}_{\mathbb{P}_k^N(1)}|_C) \leq d(g)$.

Consider the following canonical commutative diagram

$$\begin{tikzcd}
 & Y_1^+ \times Y_2^+ \arrow[d, "p'"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] & \\
 & Y^+ := (Y_1^+ \times Y_2^+)/\Delta \arrow[dl, "p_1''"'] \arrow[d, "p''"] \arrow[dr, "p_2''"] & \\
X_1^+ & X_1^+ \times X_2^+ \arrow[l] \arrow[r] & X_2^+,
\end{tikzcd}$$

where $\Gamma \tilde{\to} \Delta \subset \Gamma \times \Gamma$ denotes the diagonal subgroup. Set $p = (p_1, p_2) = p'' \circ p' : Y_1^+ \times Y_2^+ \to X_1^+ \times X_2^+$. Let $q : \widetilde{Y}^+ \to Y^+$ denote a minimal resolution of singularities. Eventually, for $i = 1, 2$, let $\partial X_i = \{x_{i,1}, \ldots, x_{i,r_i}\} \subset X_i^+$ denote the branch locus of $p_i : Y_i^+ \to X_i^+$ and for $x_{i,j} \in \partial X_i$, let $e_{i,j} \geq 2$ denote the corresponding ramification index. Write $\Sigma := \partial X_1 \times \partial X_2$.

4.1.1. Since Δ acts freely on $Y_1^+ \times Y_2^+ \setminus p^{-1}(\Sigma)$, the singular locus $Y^{+,sing}$ of Y^+ is contained in $p''^{-1}(\Sigma)$ hence is finite. Furthermore every $y = p'(y_1, y_2) \in Y^{+,sing}$ is a cyclic quotient singularity of type $\frac{1}{m_y}(1, a_y)$, for some $1 \leq a_y \leq m_y - 1$ with $m_y \wedge a_y = 1$, where $m_y := |Stab_\Delta(y_1, y_2)|$ that is, locally analytically around y, Y^+ is isomorphic to $\mathbb{C}^2/\mu_{m_y}$, where μ_{m_y} acts on $\mathbb{C}^2$ *via* $\mu_{m_y} \times \mathbb{C}^2 \to \mathbb{C}^2$, $(\zeta, (z_1, z_2)) \mapsto (\zeta z_1, \zeta^{a_y} z_2)$. Note that the type of the singularity is only defined up to the equivalence relation $\frac{1}{m_y}(1, a_y) \sim \frac{1}{m_y}(1, a'_y)$, where a'_y is the unique integer $1 \leq a'_y \leq m_y - 1$ such that $a_y a'_y \equiv 1[m_y]$. The type $\frac{1}{m_y}(1, a_y)$ encodes the geometry of the exceptional divisor $E_y := q^{-1}(y) \subset \widetilde{Y}^+$ as follows. Consider the continuous fraction

$$\frac{m_y}{a_y} = b_{y,1} - \frac{1}{b_{y,2} - \frac{1}{\cdots - b_{y,l_y}}} =: [b_{y,1}, \ldots, b_{y,l_y}],$$

where $b_{y,i} \in \mathbb{Z}_{\geq 2}$, $i = 1, \ldots, l_y$. Then E_y decomposes as a connected union $E_y = \sum_{1 \leq i \leq l_y} Z_{y,i}$ of smooth rational curves $Z_{y,1}, \ldots, Z_{y,l_y}$ with $Z_{y,i} \cdot Z_{y,i} = -b_{y,i}$, $Z_{y,i} \cdot Z_{y,i+1} = 1$ and $Z_{y,i} \cdot Z_{y,j} = 0$ for $|i - j| \geq 2$.

4.1.2. For every $y \in Y^{+,sing}$, consider the 'correction terms'

$$\begin{array}{ll}
\kappa_y := -2 + \frac{2 + a_y + a'_y}{m_y} + \sum_{1 \leq i \leq l_y} (b_{y,i} - 2) & , \kappa := \sum_{y \in Y^{+,sing}} \kappa_y; \\
e_y := 1 + l_y - \frac{1}{m_y} & , e := \sum_{y \in Y^{+,sing}} e_y; \\
B_y := 2e_y + \kappa_y & , B := \sum_{y \in Y^{+,sing}} B_y.
\end{array}$$

Note that, by construction, $\kappa_y, e_y, B_y \geq 0$. These data compute the Chern classes $c_i := c_i(\widetilde{Y}^+)$, $i = 1, 2$ of $\widetilde{Y}^+$

$$\begin{array}{l}
c_1^2 = K_{\widetilde{Y}^+}^2 = \frac{8}{|\Gamma|}(g_1 - 1)(g_2 - 1) - \kappa; \\
c_2 = \frac{4}{|\Gamma|}(g_1 - 1)(g_2 - 1) + e = \frac{1}{2}(K_{\widetilde{Y}^+}^2 + B);
\end{array}$$

hence its Euler-Poincaré characteristic

$$\chi(\widetilde{Y}^+) = h^0(\widetilde{Y}^+, \mathcal{O}_{\widetilde{Y}^+}) - h^1(\widetilde{Y}^+, \mathcal{O}_{\widetilde{Y}^+}) + h^2(\widetilde{Y}^+, \mathcal{O}_{\widetilde{Y}^+}) = \frac{1}{12}(c_1^2 + c_2) = \frac{1}{24}(3K_{\widetilde{Y}^+}^2 + B).$$

4.1.3. Given integers $g \geq 0$, $e_1, \ldots, e_r \geq 1$, let $\pi(g; e_1, \ldots, e_r)$ denote the profinite completion of the group

$$\langle c_1, c'_1, \ldots, c_g, c'_g, \gamma_1, \ldots, \gamma_r \mid [c_1, c'_1] \cdots [c_g, c'_g] \cdot \gamma_1 \cdots \gamma_r = 1,\ \gamma_i^{e_i} = 1,\ i = 1, \ldots, r \rangle.$$

For $i = 1, 2$, fixing canonical generators $\gamma_1^{(i)}, \ldots, \gamma_{r_i}^{(i)}$ of the inertia for $\pi_1(X_i)$, the finite cover $Y_i \to X_i$ corresponds to a continuous epimorphism $\psi_i : \pi(g_{i,0}, e_{i,1}, \ldots, e_{i,r_i}) \twoheadrightarrow \Gamma$. Write $\gamma_{i,j} := \psi_i(\gamma_j^{(i)})$ and $I_{i,j} := \langle \gamma_{i,j} \rangle \subset \Gamma$ for the corresponding inertia group. Group-theoretically, for $\underline{x}_{i,j} = (x_{1,i}, x_{2,j}) \in \Sigma$, the fiber $p^{-1}(\underline{x}_{i,j})$ is in bijection with $\Gamma/I_{1,i} \times \Gamma/I_{2,j}$ while the fiber $p''^{-1}(\underline{x}_{i,j})$ is in bijection with

$$I_{1,i} \setminus \Gamma/I_{2,j} (\tilde{\to} (\Gamma/I_{1,i} \times \Gamma/I_{2,j})/\Delta,\ I_{1,i} g I_{2,j} \mapsto \Delta \cdot ((1, g) \cdot (I_{1,i} \times I_{2,j})).$$

For $y \in p''^{-1}(\underline{x}_{i,j})$ corresponding to a class $I_{1,i} g_y I_{2,j} \in I_{1,i} \setminus \Gamma/I_{2,j}$, one can compute the corresponding type as follows. One has $m_y = |I_{1,i} \cap g_y I_{2,j,n} g_y^{-1}|$. Hence

$$\langle \gamma_{1,i}^{e_{1,i}/m_y} \rangle = I_{1,i} \cap g_y I_{2,j} g_y^{-1} = \langle g_y \gamma_{2,j}^{e_{2,j}/m_y} g_y^{-1} \rangle$$

and a_y is the unique $1 \leq a_y \leq m_y - 1$ such that

$$\gamma_{1,i}^{e_{1,i}/m_y} = (g_y \gamma_{2,j}^{e_{2,j}/m_y} g_y^{-1})^{a_y}.$$

4.2. **Group-theoretical preliminaries.** For a profinite group Γ, an open subgroup $U \subset \Gamma$ and closed subgroups $I, J \subset \Gamma$, write $K_I(U) := \cap_{g\in U} gIg^{-1} \subset I$ for the largest subgroup of I normalized by U. Then $J \cap K_I(\Gamma)$ is the kernel of the canonical morphism $J \to Aut(\Gamma/I)$ induced by left translation; write also $J_I := J/J \cap K_I(\Gamma)$ for the image of $J \to Aut(\Gamma/I)$.

4.2.1. Assume now Γ is a compact ℓ-adic Lie group. Let $\Gamma(n) \subset \Gamma$, $n \geq 0$ be a fundamental system of neighborhoods of 1. Write $(-)_n : \Gamma \twoheadrightarrow \Gamma_n := \Gamma/\Gamma(n)$ for the reduction-modulo-$\Gamma(n)$ morphism, $n \geq 0$. Let $I \subset \Gamma$ be a closed subgroup. The technical condition

$$(I, \Gamma) \;\; K_I(U) = K_I(\Gamma) \;\text{ for every open subgroup } U \subset \Gamma$$

is motivated by the following statement.

Fact 20. ([CT12b, Thm. 2.1, Lemma 2.4]) *Let $I_1, I_2 \subset \Gamma$ be closed subgroups. Assume $I_2 \subset \Gamma$ is of codimension ≥ 1 and (I_2, Γ) holds. Then the following holds.*

(1) *For every closed subgroup $J \subset I_1$, $\lim\limits_{n\to+\infty} \dfrac{|(\Gamma_n/I_{2,n})^{J_n}|}{|\Gamma_n/I_{2,n}|} = 0$ unless $J_{I_2} = 1$;*

(2) $\lim\limits_{n\to+\infty} \dfrac{|I_{1,n} \setminus \Gamma_n/I_{2,n}|}{|\Gamma_n/I_{2,n}|} = \dfrac{1}{|(I_1)_{I_2}|}$.

4.2.2. Let $I_1, I_2 \subset \Gamma$ be closed, procyclic subgroups contained in Γ. For every $g \in \Gamma_n$, write

$$I_{n,g} := I_{1,n} \cap gI_{2,n}g^{-1} (= Stab_{I_{1,n}}(gI_{2,n})).$$

Let $(\Gamma_n/I_{2,n})' \subset \Gamma_n/I_{2,n}$ denote the $I_{1,n}$-subset of $\Gamma_n/I_{2,n}$ where $I_{1,n}$ does not act freely and $(I_{1,n}\setminus\Gamma_n/I_{2,n})' \subset I_{1,n} \setminus \Gamma_n/I_{2,n}$ denote the subset of all $I_{1,n}gI_{2,n} \in I_{1,n} \setminus \Gamma_n/I_{2,n}$ such that $|I_{n,g}| \geq 2$ so that $(I_{1,n} \setminus \Gamma_n/I_{2,n})' = I_{1,n} \setminus (\Gamma_n/I_{2,n})'$. For every integer $n \geq 1$ write

$$\kappa(n) := \frac{1}{|\Gamma_n|} \sum_{g\in(I_{1,n}\setminus\Gamma_n/I_{2,n})'} |I_{n,g}|.$$

Lemma 21. *Assume (I_2, Γ) holds. Then $\lim\limits_{n\to+\infty} \kappa(n) = 0$ unless $I_1 = I_2 =: I$ is normal in Γ.*

Proof. One has $(I_i)_{I_j} = 1$ if and only if $I_i \subset K_{I_j}(\Gamma)(\subset I_j)$ so that the following are equivalent

(i) $(I_1)_{I_2} = (I_2)_{I_1} = 1$;
(ii) $I_1 \subset K_{I_2}(\Gamma) \subset I_2 \subset K_{I_1}(\Gamma) \subset I_1$;
(iii) $I_1 = K_{I_2}(\Gamma) = I_2 \subset K_{I_1}(\Gamma) = I_1$;
(iv) $I_1 = I_2 =: I$ is normal in Γ.

As a result, unless $I_1 = I_2 =: I$ is normal in Γ, one has $(I_1)_{I_2} \neq 1$ or $(I_1)_{I_2} \neq 1$.

Assume $(I_1)_{I_2}$ is both finite and non-trivial. Let $n \gg 0$ so that the canonical morphisms $(I_1)_{I_2} \twoheadrightarrow (I_{1,n})_{I_{2,n}}$ are isomorphisms. Let $\mathcal{M}((I_1)_{I_2})$ denote the subset of minimal subgroups $1 \subsetneq J \subset (I_1)_{I_2}$ (equivalently, the subset of subgroups of $(I_1)_{I_2}$ of prime order). Then

$$\kappa(n) \leq \frac{1}{|\Gamma_n|} \sum_{J\in\mathcal{M}((I_1)_{I_2})} \sum_{g\in I_{1,n}\setminus(\Gamma_n/I_{2,n})^J} |I_{n,g}| \leq \sum_{J\in\mathcal{M}((I_1)_{I_2})} \frac{|(\Gamma_n/I_{2,n})^J|}{|\Gamma_n/I_{2,n}|}$$

and the conclusion follows from Lemma 20 (2).

Assume $(I_1)_{I_2}$ is infinite. Then the conclusion follows from the rough upper bound

$$\kappa(n) \leq \frac{1}{|\Gamma_n|} |I_{1,n} \setminus \Gamma_n/I_{2,n}||I_{2,n}| = \frac{|I_{1,n} \setminus \Gamma_n/I_{2,n}|}{|\Gamma_n/I_{2,n}|}$$

and from Lemma 20 (1). $\square$

4.3. **Proof of Theorem 18.** Recall that, by assumption, $\mathrm{Lie}(\Gamma)$ is semisimple and the normalizer $N_\Gamma(\Lambda)\subset\Gamma$ of every non-trivial closed abelian subgroup $\Lambda\subset\Gamma$, is a closed subgroup of codimension ≥ 1. This ensures that for every non-trivial closed abelian subgroup $J\subset\Gamma$ and open subgroup $U\subset\Gamma$, $K_J(U)=1$ hence, in particular, that the technical assumption (J,Γ) always holds.

For $i=1,2$, set $\partial X_i:=X_i^+\setminus X_i:=\{x_{i,1},\dots,x_{i,r_i}\}$, fix corresponding generators of inertia $\gamma_{i,1},\dots,\gamma_{i,r_i}$ with $\gamma_{i,1}\cdots\gamma_{i,r_i}=1$. Write again $\gamma_{i,j}:=\rho_i(\gamma_{i,j})\subset\Gamma$ and $I_{i,j}:=\langle\gamma_{i,j}\rangle\subset\Gamma$ for the corresponding inertia group. We estimate the $c_{i,n}:=c_i(\widetilde{Y}_n^+)$, $i=1,2$ using the observations of Subsection 4. Let $g_{i,n}$ denote the geometric genus of $Y_{i,n}^+$. Since $\mathrm{Lie}(\Gamma)$ is semisimple, it follows from the case $H=1$ of (the proof of) [CT12b, Thm. 3.4] that

$$\lim_{n\to+\infty}\frac{2(g_{i,n}-1)}{|\Gamma_n|}>0.$$

so that, from the expression for $c_{i,n}$ given in Subsection 4.1.2, it is enough to show that, setting

$$\kappa_{i,j}(n):=\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}(-2+\frac{2+a_g+a_g'}{|I_{n,g}|}+\sum_{1\leq i\leq l_g}(b_g-2))$$

and

$$e_{i,j}(n):=\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}(1+l_g-\frac{1}{|I_{n,g}|})$$

one has

$$\lim_{n\to+\infty}\kappa_{i,j}(n)=0,\quad \lim_{n\to+\infty}e_{i,j}(n)=0.$$

Here we write $I_{n,g}:=I_{1,i,n}\cap gI_{2,j,n}g^{-1}$ and a_g, a_g', b_g, l_g, $m_g:=|I_{n,g}|$ are the invariants introduced in Subsection 4.1.1 and attached to the singular point of Y_n corresponding to $g\in(I_{1,i,n}\setminus\Gamma_n/I_{2,j,n})'$ while the notation $(I_{1,i,n}\setminus\Gamma_n/I_{2,j,n})'$ is the same as in Subsection 21.

For $\kappa_{i,j}(n)$, since

$$-2\leq -2+\frac{2+a_g+a_g'}{|I_{n,g}|}\leq -2+\frac{2+2(|I_{n,g}|-1)}{|I_{n,g}|}=0,$$

one has

$$-2\frac{|(I_{1,i,n}\setminus\Gamma_n/I_{2,j,n})'|}{|\Gamma_n|}+\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}\sum_{1\leq i\leq l_g}(b_g-2)\leq\kappa_{i,j}(n)\leq\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}\sum_{1\leq i\leq l_g}(b_g-2)$$

hence it is enough to show that

$$(*)\quad\lim_{n\to+\infty}\frac{|(I_{1,i,n}\setminus\Gamma_n/I_{2,j,n})'|}{|\Gamma_n|}=0$$

and

$$(**)\quad\lim_{n\to+\infty}\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}\sum_{1\leq i\leq l_g}(b_g-2)=0.$$

From the claim below, $(0\leq)\sum_{1\leq i\leq l_g}(b_g-2)\leq|I_{n,g}|$. Hence it is enough to show that

$$\lim_{n\to+\infty}\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}|I_{n,g}|=0,$$

which follows from Lemma 21 since (I_2,Γ) holds for every closed abelian (in particular pro-cyclic) subgroup $I_2\subset\Gamma$.

For $e_{i,j}(n)$, using again the claim below, we have

$$0\leq e_{i,j}(n)\leq\frac{|(I_{1,i,n}\setminus\Gamma_n/I_{2,j,n})'|}{|\Gamma_n|}+\frac{1}{\Gamma_n}\sum_{g\in(I_{1,i,n}\backslash\Gamma_n/I_{2,j,n})'}|I_{n,g}|$$

and the conclusion follows again from $(*)$, $(**)$.

Claim. *Let $n\geq 2$ an integer $1\leq a\leq n-1$; write $\frac{n}{a}=[b_1,\dots,b_l]$. Then $l\leq n$ and*

$$\sum_{1\leq i\leq l}(b_i-2)<n-a.$$

Proof. Write $n_1 := n$, $n_2 = a$. By definition of $b_1, \dots, b_l$, these are determined inductively by the conditions
$$n_i = b_i n_{i+1} - n_{i+2} \ 0 \leq n_{i+2} \leq n_{i+1} - 1$$
and l is the smallest integer such that $n_{l+1} \neq 0$ so that one has a decreasing sequence
$$1 \leq n_{l+1} < n_l < \cdots < n_2 = a < n_1 = n$$
and $l \leq n$. Adding the relations $n_i = b_i n_{i+1} - n_{i+2}$, $i = 1, \dots, l$, one gets
$$n - a = \sum_{1 \leq i \leq l} (b_i - 2) n_{i+1} + n_{l+1} \geq n_{l+1}(1 + \sum_{1 \leq i \leq l} (b_i - 2)) \geq 1 + \sum_{1 \leq i \leq l} (b_i - 2),$$
where the first inequality follows from $b_i \geq 2$ (indeed, b_i is an integer and $b_i = \frac{n_i}{n_{i+1}} + \frac{n_{i+2}}{n_{i+1}} > \frac{n_i}{n_{i+1}} \geq 1$) and $n_{i+1} \geq n_{l+1}$, and the second inequality follows from $n_{l+1} \geq 1$. □

5. Curves of bounded geometric genus in projective systems

Recall $k = \bar{k}$ is algebraically closed of characteristic 0.

5.1. **A general construction.** We consider the following situation.

Let
$$\begin{tikzcd}
\cdots \arrow[r, "\phi_{n+1}"] \arrow[d, hook] & Y_{n+1} \arrow[r, "\phi_n"] \arrow[d, hook] \arrow[l, phantom, "\square"] & Y_n \arrow[r, "\phi_{n-1}"] \arrow[d, hook] & \cdots \arrow[r, "\phi_1"] & Y_1 \arrow[r, "\phi_0"] \arrow[d, hook] & Y_0 =: Y \arrow[d, hook] \\
\cdots \arrow[r, "\phi_{n+1}"] & Y^+_{n+1} \arrow[r, "\phi_n"] & Y^+_n \arrow[r, "\phi_{n-1}"] & \cdots \arrow[r, "\phi_1"] & Y^+_1 \arrow[r, "\phi_0"] & Y^+_0 =: Y^+
\end{tikzcd}$$
be a projective system of generically finite morphisms between irreducible, smooth varieties Y_n over k with smooth compactification Y_n^+, $n \geq 0$. Assume Y^+ contains only finitely many closed integral curves $C^+ \hookrightarrow Y^+$ with geometric genus ≤ 1 and that for every integer $g \geq 2$ and $n \geq 0$, the set of closed integral curves $C^+ \hookrightarrow Y_n^+$ of geometric genus g form a bounded family. Assume that for some integer $g \geq 2$ and every $n \geq 0$, Y_n^+ contains infinitely many closed integral curves $C^+ \hookrightarrow Y_n^+$ of geometric genus g and let $g \geq 2$ be the smallest such integer.

Lemma 22. *There exists an integral variety $\mathcal{H}$ of dimension ≥ 1 over k and a morphism $C \to Y \times \mathcal{H}$ such that*
1. *$C \to \mathcal{H}$ is a smooth hyperbolic curve of geometric genus $\geq g$;*
2. *For every $n \geq 0$ there exists one (actually a Zariski-dense subset of) $h \in \mathcal{H}(k)$ such that $C_h \to Y$ lifts along $Y_n^+ \times_{Y^+} Y \to Y$;*
3. *$C \to Y$ is dominant.*

Proof. Set $\phi_{n,m} := \phi_m \circ \cdots \circ \phi_{n-1} : Y_n^+ \to Y_m^+$, $n \geq m$ (with the convention that $\phi_{n,n} = Id$). We proceed in two steps.

Step 1: Up to ignoring finitely many $n \geq 0$, we show that there exists an integer $g^{ar} \geq g$, a sequence $\mathcal{H}_n$, $n \geq 0$ of reduced varieties over k and a sequence of divisors $C_n^+ \hookrightarrow Y_n^+ \times \mathcal{H}_n$, $n \geq 0$ such that the resulting morphism $C_n^+ \to \mathcal{H}_n$ is flat, pure of relative dimension 1, with integral fibers of arithmetic genus g^{ar} and that there exists a projective system $\varphi_n : \mathcal{H}_{n+1}(g) \to \mathcal{H}_n(g)$ of infinite subsets $\mathcal{H}_n(g) \subset \mathcal{H}_n(k)$ with the following properties:
1. For every $h \in \mathcal{H}_n(g)$, $C_{n,h}^+ \hookrightarrow Y_n^+$ is a closed integral curve with geometric genus g, and for every $h \neq h' \in \mathcal{H}_n(g)$, $C_{n,h}^+ \neq C_{n,h'}^+ \hookrightarrow Y_n$[?;
2. $\phi_{n,m}(C_{n,h}^+) = C_{m,\varphi_{n,m}(h)}^+$, $h \in \mathcal{H}_n(g)$, $n \geq m$;
3. $\mathcal{H}_n(g) \subset \mathcal{H}_n$ is Zariski-dense, $n \geq 0$;
4. $\varphi_{n,0}(\mathcal{H}_n(g)) \subset \mathcal{H}_0$ is Zariski-dense, $n \geq 0$;
5. $\mathcal{H}_0$ is integral.

Note that since $\mathcal{H}_n(g)$ is infinite, $\mathcal{H}_n$ has dimension ≥ 1. Note also that replacing $\mathcal{H}_0$ by a dense open subscheme $U_0 \subset \mathcal{H}_0$, $\mathcal{H}_n(g)$ by $\mathcal{H}_n(g) \cap \varphi_{n,0}^{-1}(U_0)$ and $\mathcal{H}_n$ by the Zariski-closure of $\mathcal{H}_n(g) \cap \varphi_{n,0}^{-1}(U_0) \subset \mathcal{H}_n$ does not affect the properties (1) - (5) nor the fact that $\mathcal{H}_n(g)$ is infinite.

Fix $n \geq 0$. By the theory of Chow and Hilbert moduli schemes, there exists a reduced variety H_n over k

and a divisor $C_n^+ \hookrightarrow Y_n^+ \times H_n$ such that the resulting morphism $C_n^+ \to H_n$ is flat, pure of relative dimension 1, with (geometrically) reduced fibers and such that every closed irreducible curve $C^+ \hookrightarrow Y_n^+$ of geometric genus g is of the form $C^+ = C_{n,h}^+$ for one (and only one) $h \in H_n(k)$. Let $H_n(g) \subset H_n(k)$ denote the - infinite by assumption - subset of all $h \in H_n(k)$ such that $C_{n,h}^+$ is an (a geometrically) integral curve with geometric genus g. Up to replacing H_n by the Zariski-closure of $H_n(g)$, one may assume that $H_n(g)$ is Zariski-dense in H_n. By [EGAIV.3, (9.7.8)], up to replacing H_n by a dense open subscheme $U \hookrightarrow H_n$ and $H_n(g)$ with the - still infinite - subset $H_n(g) \cap U$, one may assume that $C_n^+ \to H_n$ has geometrically integral fibers (hence, in particular, is separable). Also, since the function $H_n \to \mathbb{Z}_{\geq 1}$, $h \to dim H^1(C_{n,h}^+, \mathcal{O}_{C_{n,h}^+})$ is upper semicontinous, up to replacing further H_n by a dense open subscheme $U \hookrightarrow H_n$ and $H_n(g)$ with the - still infinite - subset $H_n(g) \cap U$, one may assume that the fibers of $C_n^+ \to H_n$ have constant arithmetic genus g_n^{ar}. By minimality of g and since the geometric genus can only decrease under non-constant morphisms of irreducible curves, the subset $H_n^{ex}(g) \subset H_n(g)$ of all $h \in H_n(g)$ such that $\phi_{n,m}(C_{n,h}^+)$ is a point or an integral curve with geometric genus $\leq g-1$ for some $0 \leq m \leq n-1$ is finite. Hence replacing again H_n with the open subscheme $U := H_n \setminus H_n^{ex}(g)$ and $H_n(g)$ with the - still infinite - subset $H_n(g) \cap U$, one may assume $H_n^{ex}(g)$ is empty, $n \geq 0$. Then for every $n \geq m \geq 0$ one gets a well-defined set-theoretic map $\varphi_{n,m} : H_n(g) \to H_m(g)$ such that $\phi_{n,m}(C_{n,h}^+) = C_{m,\varphi_{n,m}(h)}^+$, $h \in H_n(g)$. Let $\widetilde{C}_{n,h}^+ \to C_{n,h}^+$ and $\widetilde{C}_{m,\varphi_{n,m}(h)}^+ \to C_{m,\varphi_{n,m}(h)}^+$ denote the normalization of $C_{n,h}^+$ and $C_{m,\varphi_{n,m}(h)}^+$ respectively. The universal property of normalization yields a canonical commutative square of dominant morphisms of projective curves

$$\begin{tikzcd}
\widetilde{C}_{n,h}^+ \arrow[r] \arrow[d] & C_{n,h}^+ \arrow[d, "\phi_{n,m}"] \\
\widetilde{C}_{m,\varphi_{n,m}(h)}^+ \arrow[r] & C_{m,\varphi_{n,m}(h)}^+
\end{tikzcd}$$

By Riemann-Hurwitz, the morphism $\widetilde{C}_{n,h}^+ \to \widetilde{C}_{m,\varphi_{n,m}(h)}^+$ is an isomorphism. Since the normalization morphisms $\widetilde{C}_{n,h}^+ \to C_{n,h}^+$, $\widetilde{C}_{m,\varphi_{n,m}(h)}^+ \to C_{m,\varphi_{n,m}(h)}^+$ are birational, the morphism $C_{n,h}^+ \to C_{m,\varphi_{n,m}(h)}^+$ is proper birational. In particular [Stacks, 50.18.4],

$$g_m^{ar} = dim H^1(C_{m,\varphi_{n,m}(h)}^+, \mathcal{O}_{C_{m,\varphi_{m,n}(h)}^+}) \geq g_n^{ar} = dim H^1(C_{n,h}^+, \mathcal{O}_{C_{n,h}^+}) \geq g = dim H^1(\widetilde{C}_{n,h}^+, \mathcal{O}_{\widetilde{C}_{n,h}^+})$$

so that g_n^{ar} becomes constant for $n \gg 0$. Up to ignoring finitely many n, one may assume g_n^{ar} is constant, $n \geq 0$. This, in turn, implies that the morphism $C_{n,h}^+ \to C_{m,\varphi_{n,m}(h)}^+$ is an isomorphism, $n \geq m \geq 0$ [Stacks, 50.18.4]. Set $H_{0,n}(g) := \bigcap_{0 \leq m \leq n} \varphi_{m,0}(H_m(g)) \subset H_{0,0}(g) = H_0(g)$, $n \geq 0$. By construction the sets $H_{0,n}(g)$, $n \geq 0$ are infinite and form a decreasing sequence

$$\cdots \subset H_{0,n+1}(g) \subset H_{0,n}(g) \subset \cdots \subset H_{0,0}(g) = H_0(g).$$

Taking their Zariski-closure in H_0, one gets a decreasing sequence of closed subvarieties

$$\cdots \subset \overline{H_{0,n+1}(g)^{zar}} \subset \overline{H_{0,n}(g)^{zar}} \subset \cdots \subset \overline{H_{0,0}(g)^{zar}} = H_0,$$

each of them having at least one irreducible component of dimension ≥ 1. For each $n \geq 0$, let $\mathcal{I}_n$ denote the set of chains $X_n \subset \cdots \subset X_0$ with $X_m \subset \overline{H_{0,m}(g)^{zar}}$ an irreducible component of dimension ≥ 1, $m = 0, \ldots, n$. The sets $\mathcal{I}_n$, $n \geq 0$ are non-empty, finite and endowed with a projective system structure $\mathcal{I}_{n+1} \to \mathcal{I}_n$, $(X_{n+1} \subset X_n \subset \cdots \subset X_0) \mapsto (X_n \subset \cdots \subset X_0)$. Fix $\underline{X} := \cdots \subset X_n \subset X_{n-1} \subset \cdots \subset X_0 \in \varprojlim \mathcal{I}_n$. Since the dimension of X_n stabilizes for $n \gg 0$, there exists $N \geq 0$ such that $X_n = X_N$, $n \geq N$. For every $n \geq 0$, let $\mathcal{H}_n \subset H_n$ denote the Zariski-closure of $\mathcal{H}_n(g) := \varphi_{n,0}^{-1}(X_N(k)) \cap H_n(g)$ in H_n; by construction these have the announced properties (1) to (5).

Step 2: For simplicity, write $Y^+ := Y_0^+$, $\mathcal{H} := \mathcal{H}_0$ and $C^+ := C_0^+$. Let η denote the generic point of $\mathcal{H}$. Let $Y \hookrightarrow Y^+$ be a dense open subscheme and write $C := C^+ \times_{Y^+} Y \hookrightarrow C^+$. The normalization $\nu : \widetilde{C_\eta^+} \to C_\eta^+$ is a smooth projective curve over η and $\nu^{-1}(C_\eta) \hookrightarrow \widetilde{C_\eta^+}$ is a dense open subscheme which identifies with the normalization $\nu : \widetilde{C_\eta} \to C_\eta$ of C_η. Then $D_\eta := \widetilde{C_\eta^+} \setminus \widetilde{C_\eta}$ is a disjoint union of a finite number of sections $\sigma_{i,\eta} : \eta \to \widetilde{C_\eta^+}$, $i = 1, \ldots, s$. By [EGAIV.3, (8.8.2) (ii), (8.10.5) (xiii), (12.2.4)], [EGAIV.2, (6.9.1)], up to replacing $\mathcal{H}$ by a dense open subscheme, one may assume $\widetilde{C_\eta^+} \to \eta$ is the generic fiber of a smooth projective, geometrically connected morphism $\widetilde{C}^+ \to \mathcal{H}$ of relative dimension 1. By [EGAIV.3, (8.8.2) (i)], up to replacing again $\mathcal{H}$ by a dense open subscheme, the commutative diagram of η-schemes

$$\begin{tikzcd}
 & \eta \arrow[dl, bend right, "\sigma_{i,\eta}"'] & \\
\widetilde{C}^+_\eta \arrow[ur] \arrow[r] & C^+_\eta \arrow[r, hook] & Y^+_\eta \arrow[ul] \\
\widetilde{C}_\eta \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\square"] & C_\eta \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\square"] & Y_\eta \arrow[u, hook]
\end{tikzcd}$$

extends uniquely to a commutative diagram of $\mathcal{H}$-schemes

$$\begin{tikzcd}
 & \mathcal{H} \arrow[dl, bend right, "\sigma_i"'] & \\
\widetilde{C}^+ \arrow[ur] \arrow[r] & C^+ \arrow[r, hook] & Y^+ \times \mathcal{H} \arrow[ul] \\
\widetilde{C} \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\square"] & C \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\square"] & Y \times \mathcal{H} \arrow[u, hook]
\end{tikzcd}$$

with $\widetilde{C}^+ \to C^+$, $\widetilde{C} \to C$ finite birational morphisms and $\sigma_i : \mathcal{H} \to \widetilde{C}^+$, $i = 1, \dots, s$ disjoint sections such that $\widetilde{C} = \widetilde{C^+} \setminus \sqcup_{1 \leq i \leq s} \sigma_i(\mathcal{H})$. The morphism $\widetilde{C} \to Y \times \mathcal{H}$ has the expected property. $\square$

5.2. **End of the proof of Theorem 13.** We now return to the setting of Subsection 3.1, of which we retain the notation and assumptions.

Corollary 23. *For every integer $g \geq 0$ there exists $n_g \geq 0$ such that $\widetilde{Y}^+_n$ contains only finitely many closed integral curves $C \hookrightarrow \widetilde{Y}^+_n$ with geometric genus $\leq g$, $n \geq n_g$.*

Proof. From Corollary 19, up to replacing $\widetilde{Y}^+_0$ with $\widetilde{Y}^+_n$ for some $n \gg 0$, one may assume that the projective system

$$\dots \longrightarrow \widetilde{Y}^+_{n+1} \longrightarrow \widetilde{Y}^+_n \longrightarrow \dots \longrightarrow \widetilde{Y}^+_1 \longrightarrow \widetilde{Y}^+_0 = X^+_1 \times X^+_2 =: X^+$$

satisfies the assumptions of Subsection 5.1 so that there exists an integral scheme $\mathcal{H}$ of dimension ≥ 1 and of finite type over k, and a morphism $C \to Y \times \mathcal{H}$ with properties (1), (2), (3) of Lemma 22. Write $Y_n := Y_{1,n} \times Y_{2,n}/\Delta_n$, which is a dense open subscheme of $Y^+_n = Y^+_{1,n} \times Y^+_{2,n}/\Delta_n$ contained in the smooth locus of Y^+_n so that one can identify Y_n with the dense open subscheme $Y \times_{\widetilde{Y}^+} \widetilde{Y}^+_n \hookrightarrow \widetilde{Y}^+_n$ of the minimal resolution of singularities $q_n : \widetilde{Y}^+_n \to Y^+_n$. For every $n \geq 0$ by (2) in Lemma 22, there exists $h_n \in \mathcal{H}(k)$ such that $C_{h_n} \to Y$ lifts along $Y_n \to X := X_1 \times X_2$. By (1) in Lemma 22 and [SGA1, XIII], one gets a short exact sequence of profinite groups

$$1 \to \pi_1(C_{h_n}) \to \pi_1(C) \to \pi_1(\mathcal{H}) \to 1.$$

On the other hand, by (3) in Lemma 22, $C \to X$ is dominant hence, since X is normal, the resulting morphism of profinite groups $\pi_1(C) \to \pi_1(X)$ has open image. Let $U \subset \Pi = \Gamma \times \Gamma$ denote the image of $\pi_1(C) \to \pi_1(X) \twoheadrightarrow \Pi$ and $N \subset U$ the image of $\ker(\pi_1(C) \twoheadrightarrow \pi_1(\mathcal{H})) \hookrightarrow \pi_1(C) \twoheadrightarrow U$. By construction $U \subset \Pi$ is an open subgroup and $N \subset U$ is a closed normal subgroup which also coincides with the image $\Pi(h_n)$ of $\pi_1(C_{h_n}) \to \pi_1(C) \twoheadrightarrow U$. But as $C_{h_n} \to Y$ lifts along $Y_n \to Y$, $N = \Pi(h_n) \subset \Delta\Pi(n)$, $n \geq 0$. Hence $N \subset \Delta = \cap_{n \geq 0}(\Delta\Pi(n))$. But as at least one of the two maps $C_{h_n} \to X \stackrel{pr_i}{\to} X_i$, $i = 1, 2$ is dominant, at least one of the resulting morphisms of profinite groups $\Pi(h_n) \hookrightarrow \Pi = \Gamma \times \Gamma \stackrel{pr_i}{\to} \Pi_i = \Gamma$ has open image. This forces $N = \Pi(h_n) \subset \Delta$ to be open. But this contradicts the fact that N is normal in U since then the diagonal Lie subalgebra $\mathrm{Lie}(\Delta) \tilde{\to} \mathrm{Lie}(\Gamma) \hookrightarrow \mathrm{Lie}(\Gamma) \oplus \mathrm{Lie}(\Gamma)$ would be an ideal of $\mathrm{Lie}(\Gamma) \oplus \mathrm{Lie}(\Gamma)$, which can happen (if and) only if $\mathrm{Lie}(\Gamma)$ is abelian (since $[(x, 0), (y, y)] = ([x, y], 0) \in \mathrm{Lie}(\Delta)$ if and only if $[x, y] = 0$, $x, y \in Lie(\Gamma)$). This contradicts the fact that we now assume $\mathrm{Lie}(\Gamma)$ is semisimple (Subsection 3.4.1). $\square$

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