THE FUNDAMENTAL THEOREM OF WEIL II FOR CURVES WITH ULTRAPRODUCT COEFFICIENTS

ANNA CADORET
IMJ-PRG, SORBONNE UNIVERSITÉ

This is a report on [C18].

Let k_0 be a finite field of characteristic $p>0$ with geometric Frobenius F_0. Fix an algebraic closure k of k_0. In this note a variety over k_0 always means a reduced scheme separated and of finite type over k_0. For a variety X_0 over k_0, write $X := X_0 \times_{k_0} k$.

Given a field K of characteristic 0, an embedding $\iota : K \hookrightarrow \mathbb{C}$ and $q \in \mathbb{R}_{>1}$, one defines the ι-weights (with respect to q) of an automorphism F of a finite-dimensional K-vector space V to be the $w \in \mathbb{R}$ such that $|\iota\alpha| = q^{\frac{w}{2}}$ for α describing the set of eigenvalues of F acting on $V \otimes \overline{K}$.

Given a prime $\ell(\neq p)$, we always denote by Q_ℓ a finite extension of $\mathbb{Q}_\ell$ and by Z_ℓ, λ_ℓ and F_ℓ the corresponding ring of integers, uniformizer and residue field .

0.1. Fix an infinite set of primes $\mathcal{L}$ not containing p. For a map $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$, $\ell \to n_\ell$, set $\underline{F}_{\underline{n}} := \prod_{\ell \in \mathcal{L}} \mathbb{F}_{\ell^{n_\ell}}$, $\underline{F} := \prod_{\ell\in\mathcal{L}} \overline{\mathbb{F}}_\ell = \varinjlim \underline{F}_{\underline{n}}$. Given a (non principal)[1] ultrafilter $\mathcal{U}$ on $\mathcal{L}$, let $\underline{F}_{\underline{n}} \twoheadrightarrow F_{\underline{n},\mathcal{U}}$ and $\underline{F} \twoheadrightarrow F_{\mathcal{U}} = \varinjlim F_{\underline{n},\mathcal{U}}$ denote the corresponding ultraproducts. One has the following parallelism

	$\overline{\mathbb{Q}}_\ell$	$F_{\mathcal{U}}$
torsion coefficients	Z_ℓ/λ_ℓ^n, $n \geq 1$	$\mathbb{F}_{\ell^{n_\ell}}$, $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$
$\varprojlim$ (to char 0 ring)	Z_ℓ	$\underline{F}_{\underline{n}}$
localization (exact) (to char 0 field)	$Z_\ell \hookrightarrow Q_\ell$	$\underline{F}_{\underline{n}} \twoheadrightarrow F_{\underline{n},\mathcal{U}}$
$\varinjlim$ (to alg. closed char 0 field $\simeq \mathbb{C}$)	$Q_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell$	$F_{\underline{n},\mathcal{U}} \hookrightarrow F_{\mathcal{U}}$

Recall also that the kernel of $\underline{F}_{\underline{n}} \to \prod_{\mathcal{U}} F_{\underline{n},\mathcal{U}}$ is the ideal of elements with finite support. This translates to the general principal that a property which, for every ultrafilter $\mathcal{U}$, holds over a set $S \in \mathcal{U}$ actually holds for all but finitely many $\ell \in \mathcal{L}$.

0.2. Let X_0 be a smooth and geometrically connected variety over k_0. For $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$, let $S_{lcc,\underline{n}}(X_0)$ denote the abelian (not full!) subcategory of étale torsion sheaves whose objects $\underline{\mathcal{F}}$ are direct products of locally constant constructible (lcc for short) sheaves $\mathcal{F}_\ell$ of $\mathbb{F}_{\ell^{n_\ell}}$-modules and let $S_{lcc}(X_0) :=$2-$\varinjlim S_{lcc,\underline{n}}(X_0)$. For every ultrafilter $\mathcal{U}$ on $\mathcal{L}$ let $S^t_{\mathcal{U}}(X_0) \subset S_{lcc}(X_0)$ denote the full subcategory of *almost $\mathcal{U}$-tame* sheaves that is of those $\underline{\mathcal{F}}$ such that (1) $\underline{\mathcal{F}}_{\overline{x},\mathcal{U}} := \underline{\mathcal{F}}_{\overline{x}} \otimes F_{\mathcal{U}}$ has finite $F_{\mathcal{U}}$-rank and (2) there exists a connected étale cover $X'_0 \to X_0$ for which the set of primes $\ell \in \mathcal{L}$ such that $\mathcal{F}_\ell|_{X'_0}$ is curve-tame is in $\mathcal{U}$. Here, 'curve-tame' means that for every smooth curve C over k and morphism $C \to X$, $\mathcal{F}_\ell|_C$ is tamely ramified in the usual sense. $S^t_{\mathcal{U}}(X_0)$ is an abelian category admitting internal Hom, $\otimes$, stable under arbitrary pull-back and finite direct image and $S^t_{\mathcal{U}}(X_0) \otimes F_{\mathcal{U}}$ is Tannakian with fiber functor $\underline{\mathcal{F}} \to \underline{\mathcal{F}}_{\overline{x},\mathcal{U}}$. In contrast $S^t_{\mathcal{U}}(X_0)$ is not stable under higher direct image by smooth-proper morphisms.

0.3. The finiteness condition (1) allows to define Frobenius weights. Given an isomorphism $\iota : F_{\mathcal{U}} \tilde{\to} \mathbb{C}$, the $(\mathcal{U},\iota)$-weights of $\underline{\mathcal{F}}$ at $x_0 \in |X_0|$ are the ι-weights with respect to $|k(x_0)|$ of F_{x_0} acting on $\underline{\mathcal{F}}_{x,\mathcal{U}}$. If for every $x_0 \in |X_0|$ the $(\mathcal{U},\iota)$-weights of $\underline{\mathcal{F}}$ at x_0 are all equal to $w \in \mathbb{R}$ one says that $\underline{\mathcal{F}}$ is $(\mathcal{U},\iota)$-pure of weight w.

0.4. The tameness condition (2) ensure that the $F_{\mathcal{U}}$-vector spaces $\mathrm{H}^i_{?,\mathcal{U}}(X,\underline{\mathcal{F}}) := (\prod_{\ell\in\mathcal{L}} \mathrm{H}^i_c(X,\mathcal{F}_\ell)) \otimes F_{\mathcal{U}}$, $? = c, \emptyset$, $i \geq 0$ are finite-dimensional, which is enough to get the cohomological interpretation of attached L-functions:

$$\prod_{x_0\in|X_0|} \det(1 - TF_{x_0}|\underline{\mathcal{F}}_{x,\mathcal{U}})^{-1} = \prod_{i\geq 0} \det(1 - TF_0|\mathrm{H}^i_{c,\mathcal{U}}(X,\underline{\mathcal{F}}))^{(-1)^{i+1}}.$$

Date: Arithmetic Geometry and Symmetries around Galois and Fundamental Groups - MFO, April 15th-21st.

[1] In this note, an ultrafilter always means a non principal ultrafilter.

Condition (2) also ensures that the global and local unipotent monodromy theorems hold and that the canonical functor $S^t_{\mathcal{U}}(X_0)\to S^t_{\mathcal{U}}(X_0)\otimes F_{\mathcal{U}}$ is essentially surjective[2].

0.5. With these tools in hands, one can adjust Deligne's proof of [D80, Thm. (3.2.1)] to $F_{\mathcal{U}}$-coefficients. Fix an ultrafilter $\mathcal{U}$ on $\mathcal{L}$ and an isomorphism $\iota : F_{\mathcal{U}}\tilde{\to}\mathbb{C}$. Let X_0 be a smooth curve over k_0 and $\underline{\mathcal{F}}\in S^t_{\mathcal{U}}(X_0)$.

Theorem. *If $\underline{\mathcal{F}}$ is $(\mathcal{U},\iota)$-pure of weight w then, for every $i\geq 0$, $\mathrm{H}^i_{c,\mathcal{U}}(X,\underline{\mathcal{F}})$ has ι-weights $\leq w+i$. Equivalently, $\mathrm{H}^i_{\mathcal{U}}(X,\underline{\mathcal{F}})$ has ι-weights $\geq w+i$.*

0.6. Combined with geometric method (Bertini, Lefschetz pencils), Theorem 0.5 is enough for most applications. Let X_0 be a smooth, geometrically connected variety and let $\underline{\mathcal{F}}\in S^t_{\mathcal{U}}(X_0)$ be $(\mathcal{U},\iota)$-pure of weight w.
- **(Purity)** Assume furthermore X_0 is proper over k_0. Then for every $i\geq 0$, $H^i(X,\underline{\mathcal{F}})$ is $(\mathcal{U},\iota)$-pure of weights $w+i$.
- **(Geometric semisimplicity)** $\pi_1(X)$ acts semisimply on $\underline{\mathcal{F}}_{x,\mathcal{U}}$ (equivalently, the set of primes $\ell\in\mathcal{L}$ such that $\mathcal{F}_\ell|_X$ is semisimple is in $\mathcal{U}$).
- **(Cebotarev)** Let $\underline{\mathcal{F}},\underline{\mathcal{F}}'\in S^t_{\mathcal{U}}(X_0)$ be $(\mathcal{U},\iota)$-pure. Assume that for every closed point $x_0\in|X_0|$, $tr(F_{x_0},\underline{\mathcal{F}}_{x,\mathcal{U}})= tr(F_{x_0},\underline{\mathcal{F}}_{x,\mathcal{U}})$. Then the set of primes $\ell\in\mathcal{L}$ such that $\mathcal{F}^{ss}_\ell\simeq\mathcal{F}'^{ss}_\ell$ is in $\mathcal{U}$.

0.7. **Integral models in E-RCS.** Let X_0 be a smooth variety. Sheaves $\underline{\mathcal{F}}$ in $S^t_{\mathcal{U}}(X_0)$ naturally arise when taking Z_ℓ-models and reducing modulo λ_ℓ in pure E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves.

0.7.1. E-**RCS.** Given a number field E, an E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves on X_0 is a system $\mathcal{F}_{\ell^\infty}$, $\ell\in\mathcal{L}$ of lcc $\overline{\mathbb{Q}}_\ell$-sheaves on X_0 such that for every $x_0\in|X_0|$ the characteristic polynomial of F_{x_0} acting on $\mathcal{F}_{\ell^\infty,x}$ is in $E[T]$ and independent of ℓ. If for every $x_0\in|X_0|$ and isomorphism $\iota:\overline{\mathbb{Q}}_\ell\tilde{\to}\mathbb{C}$, the ι-weights with respect to $|k(x_0)|$ of F_{x_0} acting on $\mathcal{F}_{\ell^\infty,x}$ are all equal to w one says that $\mathcal{F}_{\ell^\infty}$ is pure of weight w.

By definition, a lcc $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_{\ell^\infty}$ on X_0 is obtained as $\mathcal{F}_{\ell^\infty}=\mathcal{H}_{\ell^\infty}\otimes\overline{\mathbb{Q}}_\ell$ for some lcc sheaf of Z_ℓ-modules $\mathcal{H}_{\ell^\infty}$. Call such an $\mathcal{H}_{\ell^\infty}$ a Z_ℓ-model for $\mathcal{F}_{\ell^\infty}$ and write $\mathcal{H}_\ell:=\mathcal{H}_{\ell^\infty}\otimes F_\ell$ for its reduction modulo λ_ℓ. Given an E-RCS $\mathcal{F}_{\ell^\infty}$, $\ell\in\mathcal{L}$ and a choice of Z_ℓ-models $\mathcal{H}_{\ell^\infty}$, $\ell\in\mathcal{L}$, write $\underline{\mathcal{H}}=(\mathcal{H}_\ell)\in S_{lcc}(X_0)$. Then for every ultrafilter $\mathcal{U}$ on $\mathcal{L}$, $\underline{\mathcal{H}}$ is in $S^t_{\mathcal{U}}(X_0)$ and for every closed point $x_0\in|X_0|$, the characteristic polynomials of F_{x_0} acting on $\underline{\mathcal{F}}_{x,\mathcal{U}}$ and $\mathcal{F}_{\ell^\infty,x}$ coincide.

0.7.2. **Arbitrary coefficients.** E-RCS provide the right setting to define 'arbitrary coefficients' in the category of lcc $\overline{\mathbb{Q}}_\ell$-sheaves. More precisely, consider the following isomorphism classes

(G) irreducible lcc $\overline{\mathbb{Q}}_\ell$-sheaf of rank r with finite determinant on X_0;
(RCS) irreducible E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves pure of weight 0.

If X_0 is a curve

(A) cuspidal automorphic representations of $\mathrm{GL}_r(\mathbb{A})$, unramified on X_0 and whose central character is of finite order (where $\mathbb{A}$ denotes the adèle ring of $k(X_0)$)

Then
- (1) Up to semisimplification, twist and isomorphism every lcc $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 is a direct sum of objects in (G).
- (2) There is a (necessarily unique) 1-1 correspondance $(G)\longleftrightarrow(RCS)$ and, if X_0 is a smooth curve, there is also a (necessarily unique) 1-1 correspondance $(A)\longleftrightarrow(G)$. Both correspondances are characterized by the fact that the local factors of the involved objects coincide.

While (1) is formal, (2) when X_0 is a curve follows from the Langlands correspondance for GL_r [L02]; the higher dimensional case of (2) reduces to the case of curves by geometric arguments [Dr12]. To sum it up, one has the following parallelism

	(G)	E-RCS	Z_ℓ-models	reduction
constant coefficients	$\mathbb{Q}_\ell$	$\mathbb{Q}_\ell$, $\ell\in\mathcal{L}$	$\mathbb{Z}_\ell$, $\ell\in\mathcal{L}$	$\mathbb{F}_\ell$, $\ell\in\mathcal{L}$
arbitrary coefficients	$\mathcal{F}_{\ell^\infty}$	$\mathcal{F}_{\ell^\infty}$, $\ell\in\mathcal{L}$	$\mathcal{H}_{\ell^\infty},\ell\in\mathcal{L}$	$\mathcal{H}_\ell$, $\ell\in\mathcal{L}$

Note that in the case of arbitrary coefficients, there is *a priori* no canonical choice for the Z_ℓ-models hence for their reduction. But see 0.7.3 (5) below.

[2]The delicate point is to show that subobjects lift; this requires the fact that the tame étale fundamental group in the sense of Kerz-Schmidt is topologically finitely generated.

0.7.3. The following summarizes the main applications of Theorem 0.5 to E-RCS. Let $\mathcal{F}_{\ell^\infty}$, $\ell \in \mathcal{L}$ be a pure E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves on X_0. There exists a prime ℓ_0 such that for $\ell \geq \ell_0$ and every system of integral models $\mathcal{H}_{\ell^\infty}$, $\ell \in \mathcal{L}$ the following holds.

- (1) $\mathcal{H}_\ell|_X$ is semisimple on X;
- (2) $\mathcal{H}_{\ell^\infty,\eta}^{\pi_1(X)} \otimes_{Z_\ell} F_\ell = \mathcal{H}_{\ell,\eta}^{\pi_1(X)}$ (equivalently, $\mathrm{H}^1(X, \mathcal{H}_{\ell^\infty})$ is torsion-free);
- (3) The Zariski-closure of $\pi_1(X)$ acting on $\mathcal{H}_{\ell^\infty,\eta}$ is a semisimple group-scheme over Z_ℓ;
- (4) $\mathcal{F}_{\ell^\infty}|_X$ is irreducible (resp. $\mathcal{F}_{\ell^\infty}$ is semisimple, resp. irreducible) (if and) only if $\mathcal{H}_\ell \otimes \overline{\mathbb{F}}_\ell|_X$ is irreducible (resp. $\mathcal{H}_\ell$ is semisimple, resp. $\mathcal{H}_\ell \otimes \overline{\mathbb{F}}_\ell$ is irreducible).
- (5) (Resp. if $\mathcal{F}_{\ell^\infty}$ is semisimple for $\ell \gg 0$) for any two Z_ℓ-models $\mathcal{H}_{\ell^\infty}$, $\mathcal{H}'_{\ell^\infty}$ of $\mathcal{F}_{\ell^\infty}$, $\mathcal{H}_{\ell^\infty}|_X \simeq \mathcal{H}'_{\ell^\infty}|_X$ (resp. $\mathcal{H}_{\ell^\infty} \simeq \mathcal{H}'_{\ell^\infty}$).
- (6) For every Z_ℓ-model $\mathcal{H}_{\ell^\infty}$ of $\mathcal{F}_{\ell^\infty}$, $\mathrm{H}^i(X, \mathcal{H}_{\ell^\infty})$ is torsion-free, $i \geq 0$.

(1), (2), (3) (resp. (6)) (reprove and) extend the main results of [CHT17a] (resp. of Gabber's torsion-freeness theorem [G83]) to arbitrary coefficients. The fact that ℓ_0 can be taken independently of the choice of system of Z_ℓ-models and the asymptotic unicity of such in (5) are formal output of the definition of ultraproducts. (5) show in particular that the correspondance (G) $\longleftrightarrow$ (RCS) automatically extends at the level of systems of integral models modulo 'asymptotic' isomorphisms and that for every ultrafilter $\mathcal{U}$ on $\mathcal{L}$ the sheaf $\underline{\mathcal{F}} := \underline{\mathcal{H}} \in S^t_{\mathcal{U}}(X_0)$ is well-defined independently of the choice of the system of Z_ℓ-models. When X_0 is a smooth curve, this combined with (5) and **??** this provides a unique and well-defined injective map (A) $\longrightarrow$ (G) satisfying the expected local compatibility conditions of a Langlands correspondance for GL_r with $F_{\mathcal{U}}$-coefficients.

0.8. **Questions.**

- (1) Simplify the proof of Theorem 0.5 following Laumon's strategy [Lau87].
- (2) Prove the Langlands correspondance for GL_r with $F_{\mathcal{U}}$-coefficients (namely that the injective map (A) $\longrightarrow$ (G) is surjective);
- (3) Define a good notion of constructibility (see [O17]) so that one can develop a systematic formalism of ultraproduct coefficients paralleling the one of $\overline{\mathbb{Q}}_\ell$-coefficients (and in particular, get a relative theory of Frobenius weights).

References

[C18] A. Cadoret, *The fundamental theorem of Weil II for curves with ultraproduct coefficients*. Preprint (available under preliminary version on https://webusers.imj-prg.fr/ anna.cadoret/Travaux.html)

[CHT17a] A. Cadoret, C.Y. Hui and A. Tamagawa, *Geometric monodromy - semisimplicity and maximality*, Annals of Math.**186**, p. 205–236, 2017.

[D80] P. Deligne, *La conjecture de Weil: II*, Inst. Hautes Études Sci. Publ. Math. **52**, p. 137–252, 1980.

[Dr12] V. Drinfeld, *On a conjecture of Deligne*, Mosc. Math. J. **12**, 2012, p. 515–542.

[G83] O. Gabber, *Sur la torsion dans la cohomologie ℓ-adique d'une variété*, C.R. Acad. Sci. Paris Ser. I Math. **297**, 1983, p. 179–182.

[L02] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147**, 2002, p.1–241.

[Lau87] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Publ. Math. I.H.E.S **65**, 1995, p. 561–579.

[O17] F. Orgogozo, *Constructibilité et modération uniformes en cohomologie étale*, Preprint, 2017. ArXive:arXiv:1703.10850v1

anna.cadoret@imj-prg.fr
IMJ-PRG – Sorbonne Université, Paris, FRANCE