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Anna Cadoret

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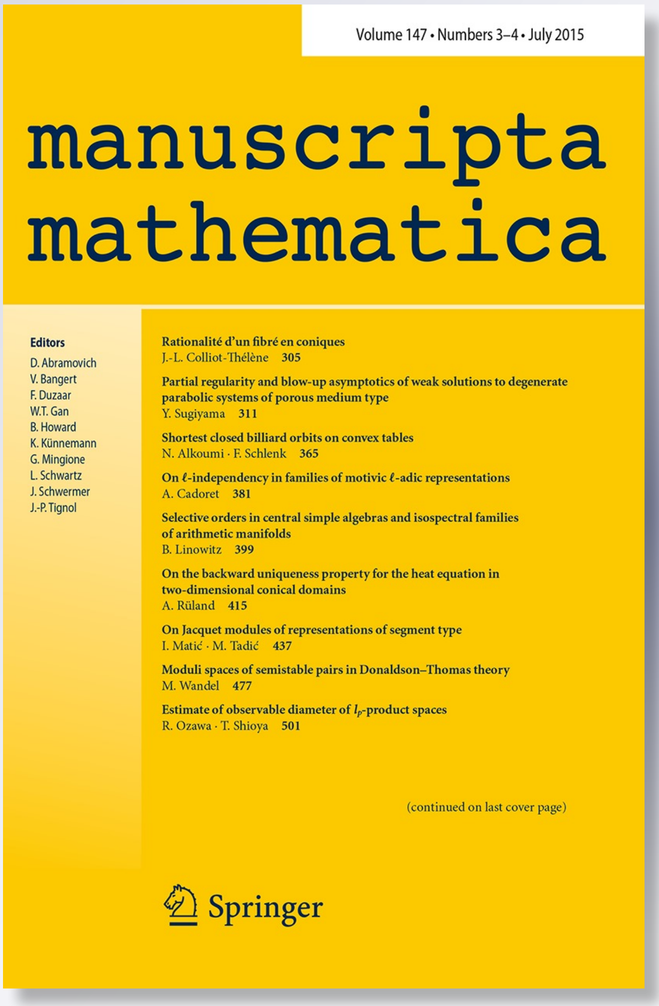
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Abstract. Let k be a finitely generated field of characteristic 0 embedded into $\mathbb{C}$, X a smooth, separated and geometrically connected scheme over k with generic point η and $f : Y \to X$ a smooth proper morphism. Let $f_{\mathbb{C}}^{an} : Y_{\mathbb{C}}^{an} \to X_{\mathbb{C}}^{an}$ denote the associated morphism of complex analytic spaces. For $x \in X(\mathbb{C})$, write H for the Betti cohomology of $Y_{\mathbb{C},x}^{an}$ with coefficients in $\mathbb{Q}$ and for $x \in X$, write H_ℓ for the ℓ-adic cohomology of $Y_{\overline{x}}$ with coefficients in $\mathbb{Q}_\ell$ (Under our assumptions on $f : Y \to X$, H and H_ℓ are independent of x). For every prime ℓ, let X_ℓ^{ex} be the set of all $x \in X$ where the Zariski closure $G_{\ell,x}$ of the image of the Galois representation $\Gamma_{k(x)} \to \mathrm{GL}(\mathrm{H}_\ell)$ has dimension strictly smaller than the dimension of $G_{\ell,\eta}$. By previous works of A. Tamagawa and the author, X_ℓ^{ex} is 'small' in the sense that if X is a curve then for every integer $\delta \geq 1$ the set of all $x \in X_\ell^{ex}$ with $[k(x) : k] \leq \delta$ is finite. Set $X^{ex} := \bigcap_\ell X_\ell^{ex}$. The Tate conjectures predict that for every $x \in X$ the $G_{\ell,x}$ are defined over $\mathbb{Q}$, reductive and independent of ℓ hence, in particular, that the sets X_ℓ^{ex} are independent of ℓ. Let $\overline{G}$ denote the Zariski closure of the image of the monodromy representation $\pi_1(X_{\mathbb{C}}^{an}; x) \to \mathrm{GL}(\mathrm{H})$. Then $\overline{G}$ is a semi-simple algebraic group of rank—say—r. The main result of this note is that for $x \notin X^{ex}$, $G_{\ell,x} \cap \overline{G}_{\mathbb{Q}_\ell}$ is a semi-simple algebraic group of rank r. This implies in particular that: (1) If $\overline{G}_{\overline{\mathbb{Q}}}$ has only simple factors of type A_n then X_ℓ^{ex} is independent of ℓ; (2) For every prime ℓ and $x \notin X^{ex}$ the unipotent radical of $G_{\ell,x}$ coincides with the unipotent radical of $G_{\ell,\eta}$ and, in particular, is independent of $x \notin X^{ex}$; (3) For every prime ℓ, if there exists $x_\ell \in X$ such that G_{ℓ,x_ℓ} is reductive then for every $x \notin X^{ex}$, $G_{\ell,x}$ is reductive. (3) applies in particular when H is a geometrically irreducible $\overline{G}$-module. This implies, for instance, that apart from a few exceptional cases, for every r-tuple $\underline{d} = (d_1, \ldots, d_r)$ of integers ≥ 2 there exists a non-singular complete intersection in $\mathbb{P}_{\mathbb{Q}}^{n+r}$ with multi-degree $\underline{d}$ for which the Tate semi-simplicity conjecture holds (for *every* prime ℓ).

1. Introduction

Given a field k, write Γ_k for the absolute Galois group of k, which we also identify with the étale fundamental group of $\mathrm{spec}(k)$.

Let k be a finitely generated field of characteristic 0 and S a scheme of finite type over k. Given an algebraic closure $k \hookrightarrow \overline{k}$ and a complex embedding $\overline{k} \hookrightarrow \mathbb{C}$, write

A. Cadoret (✉): Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique, 91128 Palaiseau, France. e-mail: anna.cadoret@math.polytechnique.fr

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$\mathrm{H}(S)$ for the Betti cohomology with coefficients in $\mathbb{Q}$ of the associated complex analytic space $S^{an}_{\mathbb{C}}$ and $\mathrm{H}_\ell(S)$ for the ℓ-adic cohomology with coefficients in $\mathbb{Q}_\ell$ of $\overline{S} := S_{\overline{k}}$.

Given an algebraic group G (over a field F of characteristic 0), let $Z(G)$, $D(G)$, G°, $\pi_0(G)$, $R_u(G)$ and $R(G)$ denote its center, derived subgroup, connected component of identity, group of connected components, unipotent radical and radical respectively. By convention, a reductive (resp. semi-simple) group is a (not necessarily connected) algebraic group with trivial unipotent radical (resp. trivial radical).

1.1. The (motivated) Tate conjectures

The main motivation for the study of ℓ-independency of the Galois image in families of ℓ-adic Galois representations arising from ℓ-adic cohomology stems from the (motivated) Tate conjectures. More precisely, let $\mathcal{P}(k)$ denote the category of smooth projective schemes over k and let $M(k)$ and $M_\ell(k)$ denote the categories of pure motivated homological motives built on Betti cohomology and ℓ-adic cohomology respectively (see [1]; here, we take the auxilliary category $\mathcal{V}$ to be $\mathcal{P}(k)$). Recall that $M(k)$ [resp. $M_\ell(k)$] is a neutral semi-simple Tannakian category with coefficients in $\mathbb{Q}$ (resp. $\mathbb{Q}_\ell$) and that the motivic cohomology functor $\mathrm{H} : \mathcal{P}(k) \to \mathrm{vect}_{\mathbb{Q}}$ (resp. $\mathrm{H}_\ell : \mathcal{P}(k) \to \mathrm{vect}_{\mathbb{Q}_\ell}$) factors through the enriched category $\mathrm{HS}_{\mathbb{Q}}$ of $\mathbb{Q}$-Hodge structures (resp. the enriched category $\mathrm{vect}_{\mathbb{Q}_\ell}(\Gamma_k)$ of finite-dimensional, continous $\mathbb{Q}_\ell$-representations of Γ_k).

For every $Y \in \mathcal{P}(k)$, the sub-Tannakian category $\langle Y\rangle^{\otimes}$ of $M(k)$ [resp. of $M_\ell(k)$] generated by Y is equivalent to the category of finite-dimensional $\mathbb{Q}$-representations of a reductive algebraic group[1] $G(Y)$ over $\mathbb{Q}$ (resp. of finite-dimensional $\mathbb{Q}_\ell$-representations of a reductive algebraic group $G_\ell(Y)$ over $\mathbb{Q}_\ell$). Comparison between Betti and ℓ-adic cohomology implies that $G(Y)_{\mathbb{Q}_\ell} \simeq G_\ell(Y)$ and, in particular that $G_\ell(Y)$ is independent of ℓ.

(Motivated) Tate conjectures: *Let $Y \in \mathcal{P}(k)$. Then for every prime ℓ one has*
(Semi-simplicity) : *The representation $\rho_{Y,\ell} : \Gamma_k \to GL(H_\ell(Y))$ is semi-simple;*
(Fullness) : *The functor $H_\ell : \langle Y\rangle^{\otimes} \to vect_{\mathbb{Q}_\ell}(\Gamma_k)$ is full.*

The Tate conjectures can be reformulated as follows: for every prime ℓ the Zariski closure $G_{\ell,Y}$ of the image of $\rho_{\ell,Y} : \Gamma_k \to \mathrm{GL}(\mathrm{H}_\ell(Y))$ in $\mathrm{GL}_{\mathrm{H}_\ell(Y)}$ coincides with $G_\ell(Y)(\simeq G(Y)_{\mathbb{Q}_\ell})$. In particular $G_{\ell,Y}$ should be *reductive* and *independent of* ℓ.

Over finitely generated fields of characteristic 0, apart from partial results for abelian motives, very little is known about the Tate conjectures, *not even whether they are independent of* ℓ. For a survey on the Tate conjectures, see [2, §7.3].

The idea underlying this note is to consider ℓ-independency questions not only for a given $Y \in \mathcal{P}(k)$ but for a family of such Y that is for the fibers of a smooth proper[2] morphism $f : Y \to X$, where X is a scheme separated, smooth and geometrically connected over k with generic point η. Then, outside a (small) exceptional

[1] Explicitly, the group $G(Y)$ [resp. $G_\ell(Y)$] is the closed subgroup of $\mathrm{GL}_{\mathrm{H}(Y)}$ [resp. $\mathrm{GL}_{\mathrm{H}_\ell(Y)}$] fixing all motivated homological cycles on all powers of Y.

[2] The results of this note do not require the projectivity assumption.

locus $X^{ex} \subset X$ the situation is 'rigidified' by the geometric monodromy group and this yields refined results about the comparison between the G_{ℓ,Y_x}, $x \notin X^{ex}$ and G_{ℓ,Y_η}.

1.2. Families of ℓ-adic representations of the étale fundamental group

Let k be a finitely generated field of characteristic 0 and let X be a smooth, separated and geometrically connected scheme over k with generic point η. Since X is geometrically connected over k, the sequence of profinite groups[3]

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \Gamma_k \to 1,$$

induced by functoriality of étale fundamental group from the sequence of morphisms $\overline{X} \to X \to \operatorname{spec}(k)$ is short exact. Still by functoriality, every point $x \in X$ with residue field $k(x)$, regarded as a morphism $x : \operatorname{spec}(k(x)) \to X$ over $\operatorname{spec}(k)$ induces a commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(\overline{X}) \arrow[r] & \pi_1(X) \arrow[r] & \Gamma_k \arrow[r] & 1, \\ & & & \Gamma_{k(x)} \arrow[ul, "\sigma_x"] \arrow[u] & \end{tikzcd}$$

where the image of $\Gamma_{k(x)} \to \Gamma_k$ is *open* in Γ_k.

Fix an integer $d \geq 1$, an infinite set L of primes, and for every $\ell \in L$ a d-dimensional $\mathbb{Q}_\ell$-vector space V_ℓ over which $\pi_1(X)$ acts continuously. Write $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ for the resulting family of ℓ-adic representations and for every $x \in X$, write

$$\rho_{\ell,x} := \rho_\ell \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell),\ \ell \in L$$

for the corresponding 'local' family. We will use the following notation.

$$\Gamma_\ell := \rho_\ell(\pi_1(X)) \subset \mathrm{GL}(V_\ell);\ \ \overline{\Gamma}_\ell := \rho_\ell(\pi_1(\overline{X})) \lhd \Gamma_\ell;\ \ \Gamma_{\ell,x} := \rho_{\ell,x}(\Gamma_{k(x)}) \subset \Gamma_\ell,\ x \in X$$

$G_\ell,\ \overline{G}_\ell,\ G_{\ell,x}$: Zariski closures of $\Gamma_\ell,\ \overline{\Gamma}_\ell,\ \Gamma_{\ell,x}$ in GL_{V_ℓ} respectively.

Note that, $\Gamma_\ell,\ \overline{\Gamma}_\ell,\ \Gamma_{\ell,x},\ x \in X$ are closed (for the ℓ-adic topology) subgroups of $\mathrm{GL}(V_\ell)$ hence are ℓ-adic Lie groups. We will sometimes write $\Gamma_{\ell,\eta},\ G_{\ell,\eta}$ instead of $\Gamma_\ell,\ G_\ell$.

[3] In the following, we always omit the fiber functor in the notation for étale fundamental group unless it helps understand the situation (see Sect. 3).

We will say that a family of ℓ-adic representations $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ is an *abstract motivic family* if

(RC) $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ is a rational compatible family;
(HT) For every $\ell \in L$, the representation $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$ is Hodge–Tate;
(SSG) There exists a $\mathbb{Q}$-vector space V and a connected semi-simple subgroup $\overline{G} \hookrightarrow \mathrm{GL}_V$ such that for every $\ell \in L$ one has $V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq V_\ell$ and modulo this identification $\overline{G}_\ell^\circ = \overline{G}_{\mathbb{Q}_\ell}$.

We refer to Sect. 3.1 for the definitions and properties of rational compatible families of representations and Hodge–Tate representations of étale fundamental group.

The families of ℓ-adic representations arising from ℓ-adic cohomology (see Sect. 3.2), which we call *geometric motivic families* in the following, are abstract motivic families (Theorem 3.5).

To every family of ℓ-adic representations, one associates the family of ℓ-adic exceptional loci

$$X_\ell^{ex} := \left\{x \in X \mid \dim_\ell(\Gamma_{\ell,x}) < \dim_\ell(\Gamma_\ell)\right\},\ \ell \in L,$$

where, here, the notation $\dim_\ell(-)$ stands for the dimension as an ℓ-adic Lie group. In the case of abstract motivic families, the associated exceptional locus X_ℓ^{ex} can also be defined (Proposition 3.1) as

$$X_\ell^{ex} := \left\{x \in X \mid \dim(G_{\ell,x}) < \dim(G_\ell)\right\},$$

where, here, the notation $\dim(-)$ stands for the dimension as an algebraic group over $\mathbb{Q}_\ell$.

By definition $X \smallsetminus X_\ell^{ex}$ contains the generic point η of X and, in the case of abstract motivic families, is 'small' in the sense that if X is a curve then for every integer $\delta \geq 1$ the set of all $x \in X_\ell^{ex}$ with $[k(x) : k] \leq \delta$ is finite [7,8]. Eventually, set $X^{ex} := \bigcap_\ell X_\ell^{ex}$.

As recalled in Sect. 1.1, for geometric motivic families, the Tate conjectures predict that $G_\ell \simeq G(Y_\eta)_{\mathbb{Q}_\ell}$ for every prime ℓ and $G_{\ell,x} \simeq G(Y_x)_{\mathbb{Q}_\ell}$ for every prime ℓ and $x \in X$. In particular, they predict that G_ℓ and $G_{\ell,x},\ x \in X$ are defined over $\mathbb{Q}$, reductive, independent of ℓ and describe X_ℓ^{ex} as the set *independent of* ℓ of all points $x \in X$ where $G(Y_x)$ degenerates. This motivates the following 'variational' ℓ-independency conjecture.

Conjecture 1.1. *Let $\rho_\ell : \pi_1(X) \to GL(V_\ell),\ \ell \in L$ be a geometric motivic family. Then, $X_\ell^{ex} = X^{ex}$ for every $\ell \in L$.*

Conjecture 1.1 can also be formulated by saying that for every $x \notin X^{ex}$ one has $G_{\ell,x}^\circ = G_\ell^\circ$.

Currently, Conjecture 1.1 is only known when the family $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ arises from the degree 1 cohomology (viz. the ℓ-adic Tate module) of the generic fiber of an abelian scheme $Y \to X$ (see [14], which derives it from the Tate conjectures 'for abelian varieties' and the Borel–de Siebenthal Theorem). The main result of this note is the following, which provides evidences for Conjecture 1.1.

Theorem 1.2. *Let $\rho_\ell : \pi_1(X) \to GL(V_\ell)$, $\ell \in L$ be an abstract motivic family.*

(1) *If $\overline{G}^\circ_{\overline{\mathbb{Q}}_\ell}$ has only simple factors of type A_n then X^{ex}_ℓ is independent of ℓ (i.e. Conjecture 1.1 holds);*
(2) *For every $\ell \in L$ and $x \notin X^{ex}$ one has $R_u(G_{\ell,x}) = R_u(G_\ell)$ (i.e. Conjecture 1.1 holds for the unipotent radical) and, in particular, is independent of $x \notin X^{ex}$;*
(3) *For every $\ell \in L$ if there exists $x_\ell \in X$ such that G_{ℓ,x_ℓ} is reductive then for every $x \notin X^{ex}$, the group $G_{\ell,x}$ is reductive.*

To distinguish the inputs from algebraic group theory from those from arithmetic geometry in the proof of Theorem 1.2, we introduce the notion of generating motivic triple, which only involves algebraic groups. The defining conditions of an abstract motivic family $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$, $\ell \in L$ ensure that for every $x \notin X^{ex}$ and prime $\ell \in L$, the associated triple $(G^\circ_\ell, \overline{G}^\circ_\ell, G^\circ_{\ell,x})$ is generating motivic over $\mathbb{Q}_\ell$. Theorem 1.2 then follows formally from general results about generating motivic triples.

Theorem 1.2 (3) applies in particular in the 'large monodromy case' that is when V is a geometrically irreducible $\overline{G}$-module, a condition which can be checked in practice. This, together with the 'smallness' of X^{ex}, gives a variational method to construct varieties over number fields for which the Tate semi-simplicity conjecture holds (for *every* prime ℓ).

The paper is organized as follows. In Sect. 2, we introduce motivic triples and prove the main technical results about them (Theorem 2.1, Corollary 2.2). In Sect. 3, we apply our results about motivic triples to motivic family. In Sect. 3.1, we review Conditions (RC), (HT) and recall the ℓ-independency properties they ensure. We also explain there why these conditions give rise to generating motivic triples (Corollary 3.3) and conclude the proof of Theorem 1.2. In Sect. 3.2, we explain why geometric motivic families are abstract motivic (Theorem 3.5). In Sect. 4, we give equivalent formulations of Conjecture 1.1 deduced from Theorem 1.2 (2) (Corollary 4.2), which might be of some use for further investigations and, in the final Sect. 5, we apply our results to construct 'lots of' non-singular complete intersections over number fields for which the semi-simplicity Tate conjecture holds (for *every* prime ℓ) (Proposition 5.1).

2. Motivic triples

2.1. Motivic triples

Let F be a field of characteristic 0 and let G be an algebraic group over F. The reductive rank $\mathrm{rd}(G)$ of G is the dimension of a maximal torus in G (or, equivalently, in $G/R_u(G)$) and the semi-simple rank of G is

$$\mathrm{ss}(G) := \mathrm{rd}(G/R(G)) = \mathrm{rd}(D(G/R_u(G))).$$

Note that

$$\mathrm{rd}(G) - \mathrm{ss}(G) = \dim(Z(G/R_u(G))).$$

If G is semi-simple then $\mathrm{rd}(G) = \mathrm{ss}(G)$ and we simply call it the rank of G.

We will say that a triple $(G, \overline{G}, H)$ is *motivic* of rank r over F if G is a connected algebraic group over F and $\overline{G}, H \hookrightarrow G$ are closed connected algebraic subgroups such that

(i) $\overline{G}$ is semi-simple of rank r and normal in G;
(ii) $\mathrm{rd}(H) = \mathrm{rd}(G)$;
(iii) $\mathrm{ss}(H) = \mathrm{ss}(G)$,

We will say that a motivic triple $(G, \overline{G}, H)$ is *generating* if, furthermore, $G = \overline{G}H$. Given a motivic triple $(G, \overline{G}, H)$ over F, set $\overline{H} := (H \cap \overline{G})^\circ$. The main results of this section are the following

Theorem 2.1. *Let $(G, \overline{G}, H)$ be a motivic triple of rank r over F. Then $\overline{H} \hookrightarrow \overline{G}$ is a semi-simple subgroup of rank r.*

Corollary 2.2. *Let $(G, \overline{G}, H)$ be a generating motivic triple of rank r over F. Then,*

(1) *If $\overline{G}_{\overline{F}}$ has only simple factors of type A_n then $H = G$;*
(2) *$R_u(H) = R_u(G)$ and $R(H) = R(G)$. In particular, H is reductive (resp. semi-simple) if G is.*

2.2. Proofs of Theorem 2.1 and Corollary 2.2

We will use repeatedly the fact (see [4, 14.11]) that if $\varphi : G \twoheadrightarrow G'$ is a surjective morphism of algebraic groups then

$$R_u(G') = \varphi(R_u(G)) \text{ and } R(G') = \varphi(R(G))$$

and, in particular, that

- if G' is reductive (resp. semi-simple) then $R_u(G) \subset \ker(\varphi)$ [resp. $R(G) \subset \ker(\varphi)$];
- a quotient (and a normal subgroup) of a reductive group is reductive;
- an extension of reductive groups is reductive.

Let $(G, \overline{G}, H)$ be a motivic triple of rank r over F. Fix a Levi factor $L \hookrightarrow H$ for H [4, 11.22]; note that $\mathrm{rd}(H) = \mathrm{rd}(L)$ and $\mathrm{ss}(H) = \mathrm{ss}(L)$. Set $\overline{L} := (L \cap \overline{G})^\circ$. As $\overline{L}$ is a normal connected subgroup of L, it is also reductive and, actually, it is a Levi factor for $\overline{H}$. We begin with the following elementary lemma.

Let $\pi : G \to G/R_u(G)$ denote the canonical projection.

Lemma 2.3. *$\pi(L)$ is isogenous to a Levi factor of $\pi(H)$ and the triple $(\pi(G), \pi(\overline{G}), \pi(H))$ is motivic of rank r over F (and generating if $(G, \overline{G}, H)$ is).*

Proof. By definition $\mathrm{rd}(G) = \mathrm{rd}(\pi(G))$ and $\mathrm{ss}(G) = \mathrm{ss}(\pi(G))$. Also, as $\overline{G}$ and L are reductive, $\pi : G \to G/R_u(G)$ induces isogenies from $\overline{G}$ to $\pi(\overline{G})$ and from L to $\pi(L)$ (so, in particular, $\pi(L)$ is isogenous to a Levi factor of $\pi(H)$). Thus $\pi(\overline{G})$ is again a semi-simple normal subgroup of rank r in $\pi(G)$ and

$$\mathrm{rd}(\pi(H)) = \mathrm{rd}(\pi(L)) = \mathrm{rd}(L) = \mathrm{rd}(H) = \mathrm{rd}(G) = \mathrm{rd}(\pi(G)),$$
$$\mathrm{ss}(\pi(H)) = \mathrm{ss}(\pi(L)) = \mathrm{ss}(L) = \mathrm{ss}(H) = \mathrm{ss}(G) = \mathrm{ss}(\pi(G)).$$
□

2.2.1. Proof of Theorem 2.1 First, let us show that

Lemma 2.4. *$\overline{L} \hookrightarrow \overline{G}$ is a semi-simple subgroup of rank r.*

Proof. We proceed in two steps.

(1) $rd(\overline{L}) = rd(\overline{G})$.
(2) $\overline{L}$ *is semi-simple.*

□

Proof of (1). By assumption $\mathrm{rd}(L) = \mathrm{rd}(G)$. In particular, every maximal torus Θ in L is also a maximal torus in G hence [4, Prop. 11.14 (1)] maps onto a maximal torus $\Theta_{L/\overline{L}}$ in $L/\overline{L}$ and $\Theta_{G/\overline{G}}$ in $G/\overline{G}$. As $\Theta_{L/\overline{L}}$ maps onto $\Theta_{G/\overline{G}}$ *via* $L/\overline{L} \to G/\overline{G}$, one deduces that $\mathrm{rd}\,(L/\overline{L}) \geq \mathrm{rd}\,(G/\overline{G})$ and as $L/\overline{L} \to G/\overline{G}$ has finite kernel, one deduces that $\mathrm{rd}\,(L/\overline{L}) = \mathrm{rd}\,(G/\overline{G})$. So

$$\mathrm{rd}(L) - \mathrm{rd}(\overline{L}) = \mathrm{rd}\,(L/\overline{L}) = \mathrm{rd}\,(G/\overline{G}) = \mathrm{rd}(G) - \mathrm{rd}(\overline{G})$$

hence

$$\mathrm{rd}(\overline{L}) = \mathrm{rd}(\overline{G}).$$

Proof of (2). As the center of the reductive group $\pi(G)$ is the kernel of the restriction of the adjoint representation Ad: $\pi(G) \to \mathrm{GL}_{\mathrm{Lie}(\pi(G))}$ to any maximal torus in $\pi(G)$, choosing a maximal torus of $\pi(G)$ lying in $\pi(L)$ (Lemma 2.3), one sees that $Z(\pi(G)) \subset \pi(L)$ hence that

$$Z(\pi(G)) \subset Z(\pi(L)).$$

Also one has (Lemma 2.3)

$$\dim(Z(\pi(G))) = \mathrm{rd}(\pi(G)) - \mathrm{ss}(\pi(G)) = \mathrm{rd}(\pi(L)) - \mathrm{ss}(\pi(L)) = \dim(Z(\pi(L)))$$

Hence $Z(\pi(L))/Z(\pi(G))$ is finite. But since $\pi(\overline{L})$ is a normal reductive subgroup in $\pi(L)$ one also has $Z(\pi(\overline{L})) \subset Z(\pi(L))$; this follows from the so-called rigidity property of groups of multiplicative type (Apply for instance [4, 8.10, Proposition] with $H = H' = Z(\pi(\overline{L}))$ and $V = \pi(L)$). In particular, if $Z(\pi(\overline{L}))$ were not finite then $\pi(\overline{G})$ would contain a subgroup of $Z(\pi(L))$ of dimension ≥ 1 hence a subgroup of $Z(\pi(G))$ of dimension ≥ 1. This would contradict the semi-simplicity of $\pi(\overline{G})$. Thus $Z(\pi(\overline{L}))$ is finite hence $\pi(\overline{L})$ is semi-simple. But as $\overline{L}$ is reductive, $\pi : G \to G/R_u(G)$ also induces an isogeny from $\overline{L}$ to $\pi(\overline{L})$. So $\overline{L}$ is semi-simple as well. □

It remains to show that $R_u(\overline{H})$ is trivial. Otherwise, it follows from[4] [5, Cor. 3.2] that there exists a strict parabolic subgroup $P \subsetneq \overline{G}$ containing $\overline{H}$ and whose

[4] More precisely, if $R_u(\overline{H})$ is non-trivial, one has the following inclusions

$$R_u(\overline{H}) \subset \overline{H} \subset N_{\overline{G}}(R_u(\overline{H})) \subsetneq \overline{G},$$

where the last one is strict since $\overline{G}$ is reductive. If $\overline{H} = N_{\overline{G}}(R_u(\overline{H}))$ then, from [5, Cor. 3.2], $\overline{H}$ is parabolic. Otherwise, set $P_1 := N_{\overline{G}}(R_u(\overline{H}))$. Then, one has

$$R_u(P_1) \supset R_u(\overline{H}) \subset \overline{H} \subsetneq P_1 \subset N_{\overline{G}}(R_u(P_1)) \subsetneq \overline{G}.$$

Again, either $P_1 = N_{\overline{G}}(R_u(P_1))$ and P_1 is parabolic or set $P_2 := N_{\overline{G}}(R_u(P_1))$ and iterate the construction. By noetherianity, the process stops after finitely many steps.

unipotent radical $R_u(P)$ contains $R_u(\overline{H})$. In particular,

$$R_u(\overline{H}) = (R_u(P) \cap \overline{H})^\circ.$$

So the morphism $\overline{L} \simeq \overline{H}/R_u(\overline{H}) \to P/R_u(P)$ has finite kernel. Since both $\overline{L}$ and $P/R_u(P)$ are reductive, one gets $\mathrm{ss}(\overline{L}) = \mathrm{rank}(D(\overline{L})) \leq \mathrm{rank}(D(P/R_u(P))) = \mathrm{ss}(P/R_u(P))$. But as $P \subsetneq \overline{G}$ is a strict parabolic subgroup, one has $\mathrm{ss}(P/R_u(P)) < r$ (see [16, (End of) 30.2]), which contradicts Lemma 2.4.

This concludes the proof of Theorem 2.1.

2.2.2. Proof of Corollary 2.2

Proof of (1). As a semi-simple group which has only simple factors of type A_n contains no strict semi-simple subgroup of the same rank (See for instance [6, Table p. 14]), the assumption that $\overline{G}_{\overline{F}}$ has only simple factors of type A_n and Theorem 2.1 imply that $\overline{H} = \overline{G}$ hence, since $(G, \overline{G}, H)$ is generating, that $G = \overline{G}H = H$.

Proof of (2). As $(G, \overline{G}, H)$ is generating, the restriction of the canonical projection $\pi(G) \to \pi(G)/\pi(\overline{G})$ to $\pi(H)$ remains surjective so, as $\pi(G)/\pi(\overline{G})$ is reductive, one has $R_u(\pi(H)) \subset \pi(\overline{G})$. But then, $R_u(\pi(H))$ is contained in $R_u(\pi(H)\cap\pi(\overline{G}))$, which is trivial by Theorem 2.1 [applied to the triple $(\pi(G), \pi(\overline{G}), \pi(H))$]. This shows that $R_u(H) \subset R_u(G)$. So to prove that $R_u(H) = R_u(G)$, it is enough to prove that $\dim(R_u(H)) = \dim(R_u(G))$. For this, observe that the canonical projection $G \to G/\overline{G}$ induces isogenies from $R_u(G)$ to $R_u(G/\overline{G})$ and from $R_u(H)$ to $R_u(G/\overline{G})$. Indeed, the fact that $R_u(G) \to R_u(G/\overline{G})$ has finite kernel follows from the semi-simplicity of $\overline{G}$ and the surjectivity of $R_u(G) \to R_u(G/\overline{G})$ follows from [4, 14.11]. The fact that $R_u(H) \to R_u(G/\overline{G})$ has finite kernel follows from the inclusion $(R_u(H) \cap \overline{G})^\circ \subset R_u(\overline{H})$ and Theorem 2.1 (applied to the triple $(G, \overline{G}, H)$) and the surjectivity of $R_u(H) \to R_u(G/\overline{G})$ follows from the fact that $(G, \overline{G}, H)$ is generating and [4, 14.11]. The fact that $R_u(H) = R_u(G)$ implies that $\pi(H)$ is reductive hence that $\pi(H) = \pi(L)$, where, as before, $L \hookrightarrow H$ denotes a Levi factor for H. But then, as already observed in step (2) of the proof of Lemma 2.4, one has $\dim(Z(\pi(G))) = \dim(Z(\pi(H)))$ and $Z(\pi(G)) \subset Z(\pi(H))$ so

$$R(\pi(G)) = Z(\pi(G))^\circ = Z(\pi(H))^\circ = R(\pi(H)).$$

whence $R(G) = \pi^{-1}(R(\pi(G))) = \pi^{-1}(R(\pi(H))) = R(H)$. □

3. Motivic families

3.1. Conditions (RC), (HT) and ℓ-independency properties

The introduction of conditions (RC), (HT) and their applications to the study of ℓ-independency properties in families of ℓ-adic Galois representations was originally carried out over number fields. We recall the main results of this theory in Sect. 3.1.1; note that condition (HT) is a local condition, which is used to relate properties of the ℓ-adic image to properties of its Zariski closure (Proposition 3.1) whereas condition (RC) is a local-global condition, ensuring 'weak' ℓ-independency properties (Theorem 3.2). The extension of these results to families of ℓ-adic Galois representations over finitely generated fields of characteristic 0 and families of ℓ-adic representations of étale fundamental group is then essentially formal provided one adjusts the standard definitions to these settings. This is done at the end of Sect. 3.1.1 and in Sect. 3.1.2.

3.1.1. Over number fields. In this section, k is a *number field*. Let Σ_k denote the set of all finite places of k and, for a prime ℓ, let $\Sigma_{k,\ell} \subset \Sigma_k$ denote the subset of all finite places dividing ℓ. For $v \in \Sigma_k$, write $k(v)$ for the corresponding residue field. Recall that for every $v \in \Sigma_k$ the decomposition groups of v in Γ_k are all conjugated. If D_v is such a decomposition group then one has a canonical epimorphism of profinite groups $D_v \twoheadrightarrow \Gamma_{k(v)} \simeq \widehat{\mathbb{Z}}$ whose kernel I_v is the inertia group at v in D_v. Let k_v^{unr} denote the maximal algebraic extension of k in $\overline{k}$ unramified at v that is the subfield of $\overline{k}$ fixed by all the Γ_k-conjugates of I_v and let $D_v^{unr} \twoheadleftarrow D_v$ denote the decomposition group of v in $\mathrm{Gal}(k_v^{unr}|k)$. Then the induced morphism $D_v^{unr} \tilde{\to} \Gamma_{k(v)}$ is an isomorphism and there exists a unique $F_v \in D_v^{unr}$ lifting the Frobenius element $\varphi_v : x \mapsto x^{|k(v)|}$ in $\Gamma_{k(v)}$.

Given an ℓ-adic Galois representation $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$, set $\Delta_\ell := \rho_\ell(\Gamma_k) \subset \mathrm{GL}(V_\ell)$ and let D_ℓ denote the Zariski-closure of Δ_ℓ in GL_{V_ℓ}.

For every $v \in \Sigma_k$, one says that $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ is *unramified at* v if one (hence every) inertia group I_v lies in $\ker(\rho_\ell)$. In that case, the restriction $\rho_\ell|_{D_v} : D_v \to \mathrm{GL}(V_\ell)$ factors through $D_v \twoheadrightarrow D_v^{unr}$ hence it makes sense to talk about the Δ_ℓ-conjugacy class of $\rho_\ell(F_v)$ and the characteristic polynomial

$$P_{\rho_\ell,v} = \det\left(1 - \rho_\ell(F_v)T\right)$$

is well-defined and independent of the choice of the representative $\rho_\ell(F_v)$ in its Δ_ℓ-conjugacy class. One says that $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ is *rational* if there exists a finite subset $S_{\rho_\ell} \subset \Sigma_k$ such that ρ_ℓ is unramified outside $S_{\rho_\ell} \cup \Sigma_{k,\ell}$ and $P_{\rho_\ell,v} \in \mathbb{Q}[T]$ for all $v \in \Sigma_k \smallsetminus S_{\rho_\ell} \cup \Sigma_{k,\ell}$.

For every $v \in \Sigma_{k,\ell}$ let $\widehat{k}_v$ denote the completion of k at v and let $\mathbb{C}_v$ denote the completion of an algebraic closure of $\widehat{k}_v$. This is an algebraically closed field over which D_v acts continuously. Let also $\chi_\ell : \Gamma_k \to \mathbb{Z}_\ell^\times$ denote the ℓ-adic cyclotomic character and for each $i \in \mathbb{Z}$ introduce

$$V_\ell^i := \left\{ v \in V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}_v \mid (\rho_\ell(\sigma) \otimes \sigma)(v) = \chi_\ell(\sigma)^i v,\ \sigma \in D_v \right\},$$

which is a $\widehat{k}_v$-submodule of $V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}_v$. Set $V_\ell(i) := V_\ell^i \otimes_{\widehat{k^v}} \mathbb{C}_v$. The inclusion $V_\ell^i \hookrightarrow V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}_v$ induces a $\mathbb{C}_v$-linear D_v-equivariant morphism $V_\ell(i) \to V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}_v$ and the resulting morphism

$$\alpha_v : \bigoplus_{i \in \mathbb{Z}} V_\ell(i) \hookrightarrow V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}_v$$

is injective. One says that $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ is *Hodge–Tate at* v if α_v is an isomorphism and that it is *Hodge–Tate* if it is Hodge–Tate at all $v \in \Sigma_{k,\ell}$.

Hodge–Tate representations are algebraic in the following sense.

Proposition 3.1. *Let $\rho_\ell : \Gamma_k \to GL(V_\ell)$ be an ℓ-adic Hodge–Tate representation. Then Δ_ℓ is open in $D_\ell(\mathbb{Q}_\ell)$.*

Proof. See [3]. □

One says that a family $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell) \simeq \mathrm{GL}_{r_\ell}(\mathbb{Q}_\ell),\ \ell \in L$ of rational ℓ-adic representations is *compatible* if for all primes $\ell \neq \ell'$ and $v \in \Sigma_k \setminus S_{\rho_\ell} \cup S_{\rho_{\ell'}} \cup \Sigma_{k,\ell} \cup \Sigma_{k,\ell'}$,

$$P_{\rho_\ell,v}(T) = P_{\rho_{\ell'},v}(T).$$

(In particular the $\mathbb{Q}_\ell$-dimension of V_ℓ is independent of ℓ).

Compatibility and rationality imply that the Zariski closure of the image of the characteristic polynomial morphism

$$\Gamma_k \to \mathrm{GL}(V_\ell) \stackrel{ch(\mathbb{Q}_\ell)}{\to} \left(\mathbf{G}_m \times \mathbb{A}^{d-1}\right)(\mathbb{Q}_\ell)$$

is defined over $\mathbb{Q}$ and independent of ℓ. (Here, $ch : \mathrm{GL}_{V_\ell} \to \mathbf{G}_{m,\mathbb{Q}_\ell} \times \mathbb{A}^{d-1}_{\mathbb{Q}_\ell}$ is the map—actually defined over $\mathbb{Q}$—sending an element $g \in \mathrm{GL}_{V_\ell}$ to the coefficients $(a_d(g), \ldots, a_1(g))$ of its characteristic polynomial $\det(1 - gT) = 1 + \sum_{1\leq i\leq d} a_i(g)T^i$). This has some striking consequences.

Theorem 3.2. *Let $\rho_\ell : \Gamma_k \to GL(V_\ell),\ \ell \in L$ be a compatible family of ℓ-adic rational representations. Then*

(1) *$rd(D_\ell)$;*
(2) *$ss(D_\ell)$;*
(3) *the kernel of the morphism $\kappa_\ell : \Gamma_k \to \pi_0(D_\ell)$*
are independent of ℓ.

Proof. For (1) see [22], for (2) see [15, Thm. 3.19] and for (3) see [21, §3]. □

Now, let X be a smooth, separated and geometrically connected scheme over k and let $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ be a family of ℓ-adic representations. We retain the notation $\Gamma_\ell,\ \overline{\Gamma}_\ell,\ \Gamma_{\ell,x},\ G_\ell,\ \overline{G}_\ell,\ G_{\ell,x}$ of Sect. 1.2. Write $|X|$ for the set of closed points in X. The Frattini/Hilbert-irreducibility argument of [21, §1] shows that for every finite set of primes P there exists (infinitely many) $x \in |X|$ such that $G_\ell = G_{\ell,x}$ for all $\ell \in P$. Consequently, any structural or ℓ-independency result for the local families $(\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell),\ \ell \in L),\ x \in |X|$ transfers to the family $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$. For instance,

- If $\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell)$ is Hodge–Tate for every $x \in |X|$ then Γ_ℓ is open in $G_\ell(\mathbb{Q}_\ell)$.
- If $\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell),\ \ell \in L$ is a compatible family of ℓ-adic rational representations for every $x \in |X|$ then

(1) $\mathrm{rd}(G_\ell)$;
(2) $\mathrm{ss}(G_\ell)$;
(3) the kernel of $\kappa_\ell : \pi_1(X) \to \pi_0(G_\ell)$

are independent of ℓ.

As a result, we will say that an ℓ-adic representation $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$ is *rational* (resp. *Hodge–Tate*) if the local representations $\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell)$, $x \in |X|$ are, and that a family of ℓ-adic representations $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$, $\ell \in L$ is *rational compatible* if the local families $(\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell),\ \ell \in L)$, $x \in |X|$ are.

3.1.2. Over finitely generated fields of characteristic 0. We come back to the general situation, where k is only assumed to be a finitely generated field of characteristic 0.

We will say that an ℓ-adic Galois representation $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ admits a model over a number field $k^\#$ contained in k if there exists a scheme X, smooth, separated and geometrically connected over $k^\#$ with function field k such that the ℓ-adic representation $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ factors through $\Gamma_k \twoheadrightarrow \pi_1(X)$. We call the resulting pair $(X, \overline{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell))$ a model for $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ over $k^\#$.

We will say that an ℓ-adic Galois representation $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ is *Hodge–Tate* (resp. *rational*) if it admits a model over a number field $k^\#$ which is Hodge–Tate (resp. rational).

We will say that a family of ℓ-adic Galois representations $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$, $\ell \in L$ is *rational compatible* if for every $\ell \in L$, $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ admits a model $(X_\ell, \overline{\rho}_\ell : \pi_1(X_\ell) \to \mathrm{GL}(V_\ell))$ over a number field $k^\#_\ell$ such that for every $\ell \neq \ell'$ there exists a connected étale cover $X_{\ell,\ell'}$ of both X_ℓ and X'_ℓ, defined over a number field $k^\#_{\ell,\ell'}$ and such that the restricted representations $\rho_\ell|_{\pi_1(X_{\ell,\ell'})} : \pi_1(X_{\ell,\ell'}) \to \mathrm{GL}(V_\ell)$ and $\rho_{\ell'}|_{\pi_1(X_{\ell,\ell'})} : \pi_1(X_{\ell,\ell'}) \to \mathrm{GL}(V_{\ell'})$ are rational compatible.[5]

If an ℓ-adic Galois representation $\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$ admits a model $(X, \overline{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell))$ over a number field $k^\#$, one can again apply the Frattini/Hilbert-irreducibility argument of [21, §1] to $\overline{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$ and shows that Proposition 3.1 and Theorem 3.2 extend to ℓ-adic representations of Γ_k when k is finitely generated field of characteristic 0.

Eventually, let X be a smooth, proper and geometrically connected scheme over k. We will say that an ℓ-adic representation $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$ is *rational* (resp. *Hodge–Tate*) if the local representations $\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell)$, $x \in |X|$ are, and that a family of ℓ-adic representations $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell)$, $\ell \in L$ is *rational compatible* if the local families $(\rho_{\ell,x} : \Gamma_{k(x)} \to \mathrm{GL}(V_\ell),\ \ell \in L)$, $x \in |X|$ are. Again, with these definitions, Proposition 3.1 and Theorem 3.2 extend to ℓ-adic representations of $\pi_1(X)$. In particular,

Corollary 3.3. *Let $\rho_\ell : \pi_1(X) \to GL(V_\ell)$, $\ell \in L$ be an abstract motivic family. Then for every $x \notin X^{ex}$ and $\ell \in L$, $(G^\circ_\ell, \overline{G}^\circ_\ell, G^\circ_{\ell,x})$ is a generating motivic triple over $\mathbb{Q}_\ell$. Furthermore, $rd(\overline{G}_\ell)$, $rd(G_\ell)$, $ss(G_\ell)$ are independent of ℓ.*

Proof. As for every $x \in X$ the groups $\overline{\Gamma}_\ell$ and $\Gamma_{\ell,x}$ generate an open subgroup of Γ_ℓ, one always has $G^\circ_\ell = \overline{G}^\circ_\ell G^\circ_{\ell,x}$. Indeed, since $G_\ell \supset \overline{G}_\ell G_{\ell,x}$ and $\overline{G}^\circ_\ell G^\circ_{\ell,x}$ is

[5] In practice, for families of ℓ-adic representations arising from geometry, one can take X_ℓ independently of ℓ. See Example 3.4.

connected, the inclusion $G_\ell^\circ \supset \overline{G}_\ell^\circ G_{\ell,x}^\circ$ is straightforward. On the other hand, one has

$$\dim(\overline{G}_\ell G_{\ell,x}) \geq \dim_\ell(\overline{\Gamma}_\ell \Gamma_{\ell,x}) = \dim_\ell(\Gamma_\ell) \stackrel{(*)}{=} \dim(G_\ell),$$

where the equality $(*)$ follows from the étale fundamental group variant of Proposition 3.1.

Next, fix $x \notin X^{ex}$ that is (Proposition 3.1 and its étale fundamental group variant) there exists $\ell \in L$ such that $G_{\ell,x}^\circ = G_\ell^\circ$. In particular, $\mathrm{rd}(G_{\ell,x}) = \mathrm{rd}(G_\ell),\ \mathrm{ss}(G_{\ell,x}) = \mathrm{ss}(G_\ell)$. Hence, for every $\ell' \in L$ one has

$$\mathrm{rd}(G_{\ell',x}) \stackrel{(*)}{=} \mathrm{rd}(G_{\ell,x}) = \mathrm{rd}(G_\ell) \stackrel{(*)}{=} \mathrm{rd}(G_{\ell'}),$$

where the equalities $(*)$ follow from Conditions (RC), (HT) and Theorem 3.2 (1) (and its étale fundamental group variant), and

$$\mathrm{ss}(G_{\ell',x}) \stackrel{(*)}{=} \mathrm{ss}(G_{\ell,x}) = \mathrm{ss}(G_\ell) \stackrel{(*)}{=} \mathrm{ss}(G_{\ell'}),$$

where the equalities $(*)$ follow from Conditions (RC), (HT) and Theorem 3.2 (2) (and its étale fundamental group variant). The condition that $\overline{G}_\ell^\circ$ is a semi-simple subgroup of G_ℓ° is Condition (SSG). □

We can now complete the proof of Theorem 1.2.

3.1.3. Proof of Theorem 1.2 From Corollary 3.3, for every $x \notin X^{ex}$ and $\ell \in L$, $(G_\ell^\circ, \overline{G}_\ell^\circ, G_{\ell,x}^\circ)$ is a generating motivated triple over $\mathbb{Q}_\ell$. So Theorem 1.2 (1) follows from Corollary 2.2 (1) and Theorem 1.2 (2) follows from Corollary 2.2 (2). For Theorem 1.2 (3), observe that G_ℓ° is an extension of $G_\ell^\circ/\overline{G}_\ell^\circ$ by $\overline{G}_\ell^\circ$. But since $G_\ell^\circ = \overline{G}_\ell^\circ G_{\ell,x_\ell}^\circ$, the group $G_\ell^\circ/\overline{G}_\ell^\circ$ is a quotient of G_{ℓ,x_ℓ}° hence it is reductive. So G_ℓ° is reductive as well (as an extension of reductive groups) and Theorem 1.2 (3) follows from Corollary 2.2 (2).

3.2. Geometric motivic families

Let k be a finitely generated field of characteristic 0 and let X be a smooth, separated and geometrically connected scheme over k with generic point η. Fix a separable closure $k(\eta) \hookrightarrow \overline{k(\eta)}$ of the function field of X containing $\overline{k}$ and write $\overline{\eta} : \mathrm{spec}(\overline{k(\eta)}) \to X$ for the corresponding geometric point. The families of ℓ-adic representations we consider in this section are those arising from ℓ-adic cohomology.

3.2.1. ℓ-Adic cohomology More precisely, let $f : Y \to X$ be a smooth, proper morphism. By the smooth-proper base-change theorem, for every prime ℓ and integer $n \geq 1$, the étale sheaf $Rf_*\mathbb{Z}/\ell^n$ is locally constant constructible hence, for every $x \in X$, correspond to a representation of $\pi_1(X;\overline{x})$ on $(Rf_*\mathbb{Z}/\ell^n)_{\overline{x}} \simeq \mathrm{H}(Y_{\overline{x}}, \mathbb{Z}/\ell^n)$. Taking projective limit and tensoring with $\mathbb{Q}_\ell$, one obtains a continuous representation of $\pi_1(X;\overline{x})$ on $\mathrm{H}_\ell(Y_x)$, which we will denote by

$$\rho_\ell^{(x)} : \pi_1\left(X;\overline{x}\right) \to \mathrm{GL}\left(H_\ell(Y_x)\right).$$

For every $x_0, x_1 \in X$, the choice of an étale path from $\overline{x}_0$ to $\overline{x}_1$ yields an isomorphism

$$\mathrm{H}_\ell(Y_{x_0}) \tilde{\to} \mathrm{H}_\ell\left(Y_{x_1}\right)$$

which is compatible with the corresponding isomorphism of étale fundamental groups

$$\pi_1(X;\overline{x}_0) \tilde{\to} \pi_1\left(X;\overline{x}_1\right).$$

When $(x_0, x_1) = (\eta, x)$, the local representation $\rho_{\ell,x}^{(\eta)} : \Gamma_{k(x)} \to \mathrm{GL}(\mathrm{H}_\ell(Y_\eta))$ is identified *via* these isomorphisms with the 'usual' Galois representation $\rho_{\ell,x}^{(x)} : \Gamma_{k(x)} \to \mathrm{GL}(\mathrm{H}_\ell(Y_x))$. Thus, in the following, we will identify $\mathrm{H}_\ell(Y_\eta)$ and $\mathrm{H}_\ell(Y_x)$, $\pi_1(X;\overline{\eta})$ and $\pi_1(X;\overline{x})$, $\rho_\ell^{(\eta)}$ and $\rho_\ell^{(x)}$ and will denote them by V_ℓ, $\pi_1(X)$ and ρ_ℓ respectively.

By a slight abuse of language, we call families of ℓ-adic representations of the above form *geometric motivic families*.

Example 3.4. Let k be a finitely generated field of characteristic 0 and let T be a smooth, proper and geometrically connected scheme over k. One can always find a number field $k^\#$, a scheme S, smooth, separated and geometrically connected over $k^\#$ with generic point η and a smooth proper morphism $\mathcal{T} \to S$ such that $T \simeq \mathcal{T}_\eta$ [over $\mathrm{spec}(k) = \mathrm{spec}(k^\#(\eta))$]. So, by the smooth-proper base change theorem, the family of ℓ-adic Galois representations $\pi_1(S) \to \mathrm{GL}(\mathrm{H}_\ell(\mathcal{T}_\eta))$, ℓ: prime provides a model over $k^\#$ for the family $\Gamma_k \to \mathrm{GL}(\mathrm{H}_\ell(T))$, ℓ: prime. Using the results of [17, Cor. 2.6] (see also [18]), this result extends to schemes T which are only assumed to be a smooth, separated and geometrically connected over k.

3.2.2. Comparison with Betti cohomology Fix a complex embedding $\overline{k} \hookrightarrow \mathbb{C}$. The same formalism as above holds in the analytic setting. Namely, the analytic sheaf $Rf^{an}_{\mathbb{C}*}\mathbb{Q}$ is a local system hence, for every $x \in X(\mathbb{C})$, corresponds to a representation of $\pi_1(X^{an}_{\mathbb{C}}; x)$ on $(Rf^{an}_{\mathbb{C}*}\mathbb{Q})_x \simeq \mathrm{H}(Y_x)$, which we will denote by

$$\rho^{(x)} : \pi_1^{top}(X^{an}_{\mathbb{C}}; x) \to \mathrm{GL}(\mathrm{H}(Y_x)).$$

For every $x_0, x_1 \in X^{an}_{\mathbb{C}}$, the choice of a topological path from x_0 to x_1 yields an isomorphism

$$\mathrm{H}(Y_{x_0}) \tilde{\to} \mathrm{H}(Y_{x_1})$$

which is compatible with the corresponding isomorphism of topological fundamental groups

$$\pi_1^{top}(X_{\mathbb{C}}^{an}; x_0)\tilde{\to}\pi_1^{top}(X_{\mathbb{C}}^{an}; x_1).$$

Thus, in the following, we will identify $\mathrm{H}(Y_{x_0})$ and $\mathrm{H}(Y_{x_1})$, $\pi_1^{top}(X_{\mathbb{C}}^{an}; x_0)$ and $\pi_1^{top}(X_{\mathbb{C}}^{an}; x_1)$, $\rho^{(x_0)}$ and $\rho^{(x_1)}$ and will denote them by V, $\pi_1^{top}(X_{\mathbb{C}}^{an})$ and ρ respectively.

Let $(-)^\wedge$ denote profinite completion. The comparison isomorphism between Betti and ℓ-adic cohomology $V\otimes_{\mathbb{Q}}\mathbb{Q}_\ell\tilde{\to}V_\ell$ is compatible with the comparison isomorphism

$$\pi_1^{top}(X_{\mathbb{C}}^{an})^\wedge\tilde{\to}\pi_1(X_{\mathbb{C}})\tilde{\to}\pi_1(\overline{X}).$$

Let $\overline{G}$ denote the Zariski closure in GL_V of the image $\overline{\Gamma}$ of $\rho:\pi_1^{top}(X_{\mathbb{C}}^{an})\to\mathrm{GL}(V)$. Then the base change of $\overline{G}$ together with its tautological representation $\overline{G}\hookrightarrow\mathrm{GL}_V$ to $\mathbb{Q}_\ell$ identifies with $\overline{G}_\ell$ together with its tautological representation $\overline{G}_\ell\hookrightarrow\mathrm{GL}_{V_\ell}$.

3.2.3. Geometric motivic versus abstract motivic families

Theorem 3.5. *Every geometric motivic family $\rho_\ell:\pi_1(X)\to GL(V_\ell)$, $\ell\in L$ is an abstract motivic family. In particular, for every $x\notin X^{ex}$ and $\ell\in L$, $(G_\ell^\circ,\overline{G}_\ell^\circ,G_{\ell,x}^\circ)$ is a generating motivated triple over $\mathbb{Q}_\ell$ and $rd(\overline{G}_\ell)$, $rd(G_\ell)$, $ss(G_\ell)$ are independent of ℓ.*

Proof. Condition (RC) follows from the Weil conjectures [10] and Condition (HT) follows from works of Fontaine and Messing [12], Faltings [11] and Tsuji [23] (See for instance [20, §1], for an overview). Conditions (SSG) follows from the comparison between Betti and ℓ-adic cohomologies as explained above and the semi-simplicity theorem of Deligne [9, Cor. 4.2.9], which ensures that $\overline{G}$ is a semi-simple algebraic group. □

Remark 3.6. Let us point out that the notion of geometric motivic family is a priori [6] more restrictive than the notion of abstract motivic family. For instance, if $\rho_\ell:\pi_1(X)\to\mathrm{GL}(V_\ell)$, $\ell\in L$ is a geometric motivic family and for every $x\in X$, $(X_x,\overline{\rho}_{\ell,x}:\pi_1(X_x)\to\mathrm{GL}(V_\ell))$, $\ell\in L$ is a model of $\rho_{\ell,x}:\Gamma_{k(x)}\to\mathrm{GL}(V_\ell)$, $\ell\in L$ over a number field $k(x)^\#$ then for every $t\in X_x$, the representations $\overline{\rho}_{\ell,x,t}:\Gamma_{k(x)^\#(t)}\to\mathrm{GL}(V_\ell)$, $\ell\in L$ are rational, *strictly* compatible (that is the sets $S_{\overline{\rho}_{\ell,x,t}}$ can be taken independently of ℓ), satisfy the Weil conjectures (in particular the Riemann hypothesis), are de Rham etc. Taking into account these additional arithmetic-geometric properties may lead to refined versions of Theorem 1.2 for geometric motivic families.

[6] Though, as pointed out by the referee, when the base field is a number field, it does not seem easy to construct abstract motivic families which are not geometric.

4. Reformulation of Conjecture 1.1

Let $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ be an abstract motivic family. From the étale fundamental group variant of Theorem 3.2 (3), after replacing X by a connected étale cover, we may and will assume that G_ℓ is connected for every prime ℓ. Then, for every $x \notin X^{ex}$ there exists $\ell \in L$ such that $x \notin X_\ell^{ex}$, which implies that $G_{\ell,x} = G_\ell$ is connected as well. Then, from Theorem 3.2 (3), this implies that $G_{\ell',x}$ is connected for every prime ℓ'.

From Theorem 1.2 (2), to prove Conjecture 1.1, one can replace the $\rho_\ell : \pi_1(X) \to \mathrm{GL}(V_\ell),\ \ell \in L$ by their $\pi_1(X)$-semi-simplification hence assume that the $G_\ell,\ G_{\ell,x},\ \ell \in L,\ x \notin X^{ex}$ *are reductive*. Under this assumption, Conjecture 1.1 reduces to a simple numerical statement ((iv) of Corollary 4.2 below). This is a direct consequence of the following variant of the Borel–de Siebenthal Theorem which, recall, asserts that if G is a connected reductive group over a field F of characteristic $\neq 2, 3$ and $H \subset G$ is a connected reductive subgroup such that $\mathrm{rd}(H) = \mathrm{rd}(G)$ then H is the connected component of identity of the centralizer of $Z(H)$ in G—See for instance [13].

For a reductive subgroup $G \subset \mathrm{GL}_V$, set

$$m(G, V) := \dim\left((V \otimes V^\vee)^G\right).$$

Lemma 4.1. *Let F be a field of characteristic 0, let V be a finite-dimensional F-vector space and let $H \subset G \subset GL_V$ be connected reductive subgroups. Assume that $rd(H) = rd(G)$ and $ss(H) = ss(G)$. Then the following assertions are equivalent*

(i) $Z(H) \subset Z(G)$;
(ii) $Z(H) = Z(G)$;
(iii) $|\pi_0(Z(H))| = |\pi_0(Z(G))|$;
(iv) *H is normal in G;*
(v) $m(G, V) = m(H, V)$;
(vi) $H = G$.

Proof. We prove

$$\begin{tikzcd} (v) \arrow[rrr, Rightarrow, bend left] & (vi) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (iv) \arrow[r, Rightarrow] & (i) \arrow[r, Leftrightarrow] & (iii) \arrow[r, Leftrightarrow] & (ii) \arrow[r, Rightarrow] & (vi). \end{tikzcd}$$

$(vi) \Rightarrow (iv)$, (v) is straightforward.

Also, note that as the center of the reductive group G is the kernel of the restriction of the adjoint representation Ad: $G \to \mathrm{GL}_{\mathrm{Lie}(G)}$ to any maximal torus in G, choosing a maximal torus of G lying in H (recall that $\mathrm{rd}(H) = \mathrm{rd}(G)$) one sees that $Z(G) \subset H$ hence that $Z(G) \subset Z(H)$. Whence $(i) \Leftrightarrow (ii)$ and one has a canonical morphism

$$\pi_0(Z(G)) \to \pi_0(Z(H)).$$

Furthermore, since both G and H are reductive, one has

$$\dim(Z(G)) = \mathrm{rd}(G) - \mathrm{ss}(G) = \mathrm{rd}(H) - \mathrm{ss}(H) = \dim(Z(H))$$

hence $Z(G)^\circ = Z(H)^\circ$ and the canonical morphism $\pi_0(Z(G)) \to \pi_0(Z(H))$ is injective. As $\pi_0(Z(G))$, $\pi_0(Z(H))$ are both finite, this shows $(ii) \Leftrightarrow (iii)$.

$(iv) \Rightarrow (i)$: Since H is normal in G, G acts by conjugation on $Z(H)$. The induced morphism $G \to \mathrm{Aut}(Z(H)/Z(G))$ is trivial since G is connected and $Z(H)/Z(G)$ is finite. Thus, for every $z \in Z(H)$ one has a morphism $[-, z] : G \to D(G) \cap Z(G)$. Again, as G is connected and $D(G) \cap Z(G)$ is finite (recall that G is reductive), this morphism is trivial that is $Z(H) \subset Z(G)$.

$(v) \Rightarrow (i)$: As $H \subset G$, one has $(V \otimes V^\vee)^G \subset (V \otimes V^\vee)^H$ hence (v) is equivalent to $(V \otimes V^\vee)^G = (V \otimes V^\vee)^H =: E$. But, then

$$Z(H) = H \cap E \subset G \cap E = Z(G).$$

$(ii) \Rightarrow (vi)$: this is the Borel–de Siebenthal Theorem. $\square$

Corollary 4.2. *Assume that the G_ℓ, $\ell \in L$ are connected reductive. Then Conjecture 1.1 is equivalent to one of the following four assertions: For every $x \notin X^{ex}$ and prime ℓ,*

(i) $Z(G_{\ell,x}) \subset Z(G_\ell)$;
(ii) $|\pi_0(Z(G_{\ell,x}))| = |\pi_0(Z(G_\ell))|$;
(iii) $G_{\ell,x}$ *is normal in* G_ℓ;
(iv) $m(G_\ell, V_\ell) = m(G_{\ell,x}, V_\ell)$;

5. Non-singular complete intersections

In practice, one can often check that the G_ℓ, $\ell \in L$ are reductive by showing that V_ℓ decomposes as a direct sum of G_ℓ-modules over which $\overline{G}_\ell$ acts geometrically irreducibly. A standard example is given by non-singular complete intersections. Indeed, let

$$\begin{tikzcd} Y_{\underline{d}} \arrow[r, "f"] \arrow[d, hook] & X_{\underline{d}} \\ \mathbb{P}^{n+r}_{X_{\underline{d}}} \arrow[ur] & \end{tikzcd}$$

be the universal family of non-singular complete intersections with multi-degree $\underline{d} = (d_1, \ldots, d_r)$ ($d_i \geq 2$, $n \geq 1$). More precisely, set

$$\tilde{X}_{\underline{d}} := \mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d_1)\right)\right) \times \cdots \times \mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d_r)\right)\right)$$

and let $\tilde{Y}_{\underline{d}} \hookrightarrow \mathbb{P}^{n+r}_{\tilde{X}_{\underline{d}}}$ be the closed subscheme whose fiber at $([s_1], \ldots, [s_r]) \in \tilde{X}_{\underline{d}}$ is given by $s_1 = \cdots = s_r = 0$. Eventually, let $Y_{\underline{d}} \hookrightarrow \mathbb{P}^{n+r}_{X_{\underline{d}}} \to X_{\underline{d}}$ denote the base-change of $\tilde{Y}_{\underline{d}} \hookrightarrow \mathbb{P}^{n+r}_{\tilde{X}_{\underline{d}}} \to \tilde{X}_{\underline{d}}$ to the open locus $X_{\underline{d}} \subset \tilde{X}_{\underline{d}}$ over which $\tilde{Y}_{\underline{d}} \to \tilde{X}_{\underline{d}}$ is smooth. For $m \neq n$, one has $\dim(\mathrm{H}^m(Y_{\underline{d},\overline{\eta}}, \mathbb{Q}_\ell)) = 0$ (m odd) or 1 (m even). For $m = n$, let V (resp. V_ℓ) denote the middle cohomology $\mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x}, \mathbb{Q})$ (resp. $\mathrm{H}^n(Y_{\underline{d},\overline{\eta}}, \mathbb{Q}_\ell)$) if n is odd and the primitive cohomology

$\mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x},\mathbb{Q})_{prim} \hookrightarrow \mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x},\mathbb{Q})$ [resp. $\mathrm{H}^n(Y_{\underline{d},\overline{\eta}},\mathbb{Q}_\ell)_{prim} \hookrightarrow \mathrm{H}^n(Y_{\underline{d},\overline{\eta}},\mathbb{Q}_\ell)$] if n is even. Then $V \hookrightarrow \mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x},\mathbb{Q})$ and $V_\ell \hookrightarrow \mathrm{H}^n(Y_{\underline{d},\overline{\eta}},\mathbb{Q}_\ell)$ are pure motivated submotives with $V_{\mathbb{Q}_\ell} \simeq V_\ell$ (*via* the comparison between Betti and ℓ-adic cohomologies). When n is even, $\mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x},\mathbb{Q})$ decomposes as

$$\mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x},\mathbb{Q}) = V \oplus D$$

(with D the $\pi_1^{top}(X^{an}_{\underline{d},\mathbb{C}})$-fixed part of $\mathrm{H}^n(Y^{an}_{\underline{d},\mathbb{C},x},\mathbb{Q})$ and $\dim_{\mathbb{Q}}(D) = 1$) as $\pi_1^{top}(X^{an}_{\underline{d},\mathbb{C}})$-modules and $\mathrm{H}^n(Y_{\underline{d},\overline{\eta}},\mathbb{Q}_\ell)$ decomposes as

$$\mathrm{H}^n(Y_{\underline{d},\overline{\eta}},\mathbb{Q}_\ell) = V_\ell \oplus D_{\mathbb{Q}_\ell}$$

as $\pi_1(X_{\underline{d}})$-modules.

When n is odd, $\overline{G}$ is the symplectic group acting on V through its natural representation[7] and, when n is even, $\overline{G}$ is the orthogonal group acting on V through its natural representation[4] except in the following exceptional cases: quadric hypersurfaces, cubic surfaces and even-dimensional intersections of two quadrics (where it is finite). See for instance [19, (6.9)] for details. Assume we are not in one of the exceptional cases listed above. Then,

Proposition 5.1.

(1) *For every $s \notin X^{ex}_{\underline{d}}$ and every prime ℓ the group $G_{\ell,x}$ is reductive (and non-abelian) that is the semi-simplicity conjecture holds for $Y_{\underline{d},x}$ (and every prime ℓ).*

(2) *For every prime ℓ one has*

$$X^{ex}_{\underline{d},\ell} = \left\{x \in X_{\underline{d}} \mid m(G_{\ell,x},V_\ell) > 1\right\}.$$

In particular, it follows from Proposition 5.1 (1) and the 'smallness' of the exceptional locus that there exists a (infinitely many) non-singular complete intersection in $\mathbb{P}^{n+r}_{\mathbb{Q}}$ with multi-degree $\underline{d} = (d_1,\dots,d_r)$ for which the Tate semi-simplicity conjecture holds (for every prime) and it follows from Proposition 5.1 (2) that Conjecture 1.1 holds if and only if for every $x \notin X^{ex}_{\underline{d}}$ and prime ℓ, one has $m(G_{\ell,d},V_\ell) = 1$.

Proof. Assertion (1) [resp. (2)] is a consequence of Theorem 1.2 (2) (resp. Theorem 1.2 (2) and Corollary 4.2) and of the following inclusions

$$\begin{array}{ll} G_\ell = G_\ell(Y_{\underline{d},\eta}) \simeq \mathbb{G}^{n+2}_{m,\mathbb{Q}_\ell} \times \mathrm{Sp}_{V_\ell} & \text{if } n \text{ is odd;}\\ \mathbb{G}^{n+1}_{m,\mathbb{Q}_\ell} \times \mathrm{SO}_{V_\ell} \subset G_\ell \subset G_\ell(Y_{\underline{d},\eta}) \subset \mathbb{G}^{n+2}_{m,\mathbb{Q}_\ell} \times \mathrm{SO}_{V_\ell} & \text{if } n \text{ is even.}\end{array}$$

These follow from the fact that $\overline{G}_{\mathbb{Q}_\ell} \simeq \overline{G}_\ell \subset G_\ell$ and the above description of $\overline{G}$ (for the semi-simple part) and from Proposition 3.1, Riemann hypothesis [10] and the description of the Zariski-closure of an element in a linear algebraic group [4, 7.3] (for the toric part). □

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[7] Once a polarization is fixed.

References

[1] André, Y.: Pour une théorie inconditionnelle des motifs. Publ. Math. I.H.E.S. **83**, 5–49 (1996)
[2] André, Y.: Une introduction aux motifs (motifs purs, motifs mixtes, périodes), Panorama et synthèse, vol. 17. S.M.F. (2004)
[3] Bogomolov, F.: Sur l'algébricité des représentations ℓ-adiques. C.R. Acad. Sci. Paris **290**, 103–701 (1980)
[4] Borel, A.: Linear Algebraic Groups—2nd Enlarged ed., G.T.M., vol. 126, Springer, Berlin (1991)
[5] Borel, A., Tits, J.: Eléments unipotents et sous-groupes paraboliques de groupes réductifs I. Invent. Math. **12**, 95–104 (1971)
[6] Breuillard, E., Green, B., Guralnick, R., Tao, T.: Strongly dense free subgroups of semi-simple algebraic groups. Isr. J. Math. **192**, 347–379 (2012)
[7] Cadoret, A., Tamagawa, A.: A uniform open image theorem for ℓ-adic representations I. Duke Math. J. **161**, 2605–2634 (2012)
[8] Cadoret, A., Tamagawa, A.: A uniform open image theorem for ℓ-adic representations II. Duke Math. J. **162**, 2301–2344 (2013)
[9] Deligne, P.: Théorie de Hodge, II. Inst. Hautes Etudes Sci. Publ. Math. **40**, 5–57 (1971)
[10] Deligne, P.: La conjecture de Weil, II. Inst. Hautes Études Sci. Publ. Math. **52**, 137–252 (1980)
[11] Faltings, G.: Crystalline cohomology and p-adic Galois representations. In: Algebraic Analysis, Geometry, and Number Theory (Baltimore, 1988), pp. 191–224. John Hopkins University Press (1989)
[12] Fontaine, J.-M., Messing, W.: p-Adic periods and p-adic étale cohomology. Contemp. Math. **67**, 179–207 (1987)
[13] Gille, P.: The Borel–de Siebenthal Theorem, Preprint Available on www.math.ens.fr/~gille/prenotes/bds (2010)
[14] Hui, C.-Y.: Specialization of monodromy and ℓ-independence. C.R. Acad Sci. Paris **350**, 5–7 (2012)
[15] Hui, C.-Y.: Monodromy of Galois representations and equal rank subalgebra equivalence. Math. Res. Lett. **20**, 1–24 (2013)
[16] Humphreys, J.E.: Linear Algebraic Groups. G.T.M., vol. 21. Springer (1975)
[17] Illusie L.: Constructibilité générique et uniformité en ℓ, Preprint. Available on http://www.math.upsud.fr/~illusie/constructible3.pdf (2010)
[18] Katz, N., Laumon, G.: Transformation de Fourier et majoration de sommes exponentielles. Publ. Math. I.H.E.S. **62**, 361–418 (1985)
[19] Moonen, B.: An introduction to Mumford–Tate groups, Preprint, 2004 available on http://staff.science.uva.nl/~bmoonen/MTGps
[20] Olsson, M.: On Faltings' method of almost étale extensions. In: Algebraic Geometry—Seattle 2005, Proceedings of the Symposium in Pure Mathematics, vol. 80 Part 2, pp. 811–936, A.M.S. (2009)
[21] Serre, J.-P.: Letter to Ken Ribet (1 Jan 1981). Collected works IV (1985–1988). Springer (2000)
[22] Serre, J.-P.: Letter to Ken Ribet (29 Jan 1981). Collected works IV (1985–1988). Springer (2000)
[23] Tsuji, T.: p-Adic étale cohomology and crystalline cohomology in the semistable reduction case. Invent. Math. **137**, 233–411 (1999)