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CHAPTER 1

ELLIPTIC FUNCTIONS AND THE THEOREMS OF PICARD

§ 1. INTRODUCTION: THE THEOREMS OF PICARD

The theorems of Picard are about the values of holomorphic functions. Let us begin by stating them.

Little Picard Theorem **(1.1)**. — *Let $f\colon \mathbf{C} \to \mathbf{C}$ be an entire function. If f is not constant, then f omits at most one value: the cardinality of $\mathbf{C} \smallsetminus f(\mathbf{C})$ is at most equal to* 1.

Observe that non constant polynomials take every value, but the exponential function takes every value except for 0.

Great Picard Theorem **(1.2)**. — *Let r be a positive real number and let $\dot{\mathbf{D}}(0,r)$ be the complement of the origin in the disk $\mathbf{D}(0,r)$ of radius r centered at the origin. Let $f\colon \dot{\mathbf{D}}(0,r) \to \mathbf{C}$ be any holomorphic function with an essential singularity at the origin. Then f omits at most one value: the cardinality of $\mathbf{C} \smallsetminus f(\dot{\mathbf{D}}(0,r))$ is at most equal to* 1.

The "Little" theorem is indeed a consequence of the "Great" one. Let indeed $f\colon \mathbf{C} \to \mathbf{C}$ be an entire function and let us consider the holomorphic function g on $\dot{\mathbf{D}}(0,1)$ defined by $g(z) = f(1/z)$. Unless f is a polynomial, the function g has an essential singularity at the origin. According to the Great Picard Theorem, g omits at most one value, and so does the restriction of f to any neighborhood of infinity. A fortiori, f omits at most one value.

1.3. For any open subset Ω of $\mathbf{C}$, or any Riemann surface, one denotes with $\mathscr{O}(\Omega)$ the set of all holomorphic functions on Ω and with $\mathscr{M}(\Omega)$ the set of all meromorphic functions on Ω.

1.4. *A trichotomy.* — Let $a \in \mathbf{C}$, let r be a positive real number and let f be a holomorphic function on the disk $\dot{\mathbf{D}}(a,r)$ deprived of a. Then f can be developed as a Laurent series: there are complex numbers a_n, for $n \in \mathbf{Z}$, such that for any $z \in \dot{\mathbf{D}}(a,r)$, $f(z)$ is the sum of the converging series

$$f(z) = \sum_{n\in\mathbf{Z}} a_n (z-a)^n,$$

the convergence being in fact locally uniform.

There are three mutually exclusive possibilities.

a) *The function f is bounded in a neighborhood of a.* Then $a_n = 0$ for all negative n, and f extends to a holomorphic function on the disk $\mathbf{D}(a,r)$.

b) *When $z \to a$, $|f(z)|$ tends to $+\infty$.* Then, there exists a negative integer n_0 such that $a_{n_0} \neq 0$ but $a_n = 0$ for every $n < n_0$, and f extends to a meromorphic function on $\mathbf{D}(a,r)$, with a pole of order $-n_0$ at a.

c) In the remaining case, the set of negative integers n such that $a_n \neq 0$ is infinite and one says that *f has an essential singularity at a*. Moreover, *for any real number $s \in (0,r)$, $f(\dot{\mathbf{D}}(a,s))$ is dense in $\mathbf{C}$* (Theorem of Casorati–Weierstraß).

Let us give some details. The first case relies on Riemann's extension theorem. If f is bounded in a neighborhood of a, then the function g on $\mathbf{D}(a,r)$ defined by $g(z) = (z-a)^2 f(z)$ for $z \neq a$ and $g(a) = 0$ is holomorphic since it is $\mathbf{C}$-differentiable on $\Omega \smallsetminus \{a\}$, as well as at a (with derivative 0). Let $g(z) = \sum_{n\geq 0} b_n (z-a)^n$ be the Taylor expansion of g at a. Since $g(a) = g'(a) = 0$, one has $b_0 = b_1 = 0$ hence $f(z) = \sum_{n\geq 0} b_{n+2} z^n$.

The rest is mainly algebra. Assume that $|f(z)| \to \infty$ for $z \to a$. Then, there exists a real number $s \in (0,r)$ such that $f(z) \neq 0$ for any $z \in \dot{\mathbf{D}}(a,s)$; let g be the holomorphic function on $\dot{\mathbf{D}}(a,s)$ given by $g(z) = 1/f(z)$. One has $g(z) \to 0$ for $z \to a$); by the first case, g extends to a holomorphic function on $\dot{\mathbf{D}}(a,s)$. Let n_0 be the order of vanishing of g at a; the function h on $\dot{\mathbf{D}}(a,s)$ defined by $h(z) = g(z)/(z-a)^{n_0}$ is holomorphic and does not vanish at $z = a$, nor at any $z \in \dot{\mathbf{D}}(a,s)$. Then, the function $z \mapsto (z-a)^{n_0} f(z) = h(z)^{-1}$ is holomorphic around a, as well as on $\dot{\mathbf{D}}(a,r)$. Let $\sum_{n\geq 0} b_n (z-a)^n$ be its Taylor expansion; we obtain that $f(z) = \sum_{n\geq -n_0} b_{n+n_0} (z-a)^n$. Consequently, $a_n = 0$ for $n < -n_0$, $a_{-n_0} = b_0 = h(0)^{-1} \neq 0$. This shows that f is meromorphic at a, with a pole of order $-n_0$ at a.

The remaining case is the definition of an essential singularity. Let $s \in (0,r)$ and let us show that $f(\dot{\mathbf{D}}(a,s))$ is dense in $\mathbf{C}$. Otherwise, there would exist a complex number b and a positive real number δ such that $f(z) - b \geq \delta$ for any $z \in \dot{\mathbf{D}}(a,s)$. Then, the function $z \mapsto 1/(f(z)-b)$ is holomorphic and nonzero on $\dot{\mathbf{D}}(a,s)$, and bounded. By the first case, it extends to a holomorphic function g on $\mathbf{D}(a,s)$. One has $f(z) = b + 1/g(z)$ for any $z \in \dot{\mathbf{D}}(a,s)$. if $g(0) \neq 0$, then f extends to a holomorphic function on $\mathbf{D}(a,s)$; if g has a zero of order n_0 at a, then f is meromorphic at a with a pole of order n_0.

1.5. *"Little" Picard and uniformization.* — The modern point of view proves the Little Picard theorem as a consequence of the uniformization of Riemann surfaces. So let $f\colon \mathbf{C} \to \mathbf{C}$ be any entire function omitting at least two values; we need to show that f is constant. We may assume that $f(\mathbf{C}) \subset \mathbf{C} \smallsetminus \{0,1\}$. The universal cover of $\mathbf{C} \smallsetminus \{0,1\}$, being a simply connected Riemann surface, is isomorphic either to the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, or to the complex plane $\mathbf{C}$, or to the unit disk $\mathbf{D} = \mathbf{D}(0,1)$.

It cannot be $\mathbf{P}^1(\mathbf{C})$ because $\mathbf{C} \smallsetminus \{0,1\}$ is not compact.

It cannot be $\mathbf{C}$ neither. Indeed, let Γ be the fundamental group of $\mathbf{C} \smallsetminus \{0,1\}$ (at some base-point u). Then, Γ can be viewed as a subgroup of the group G of holomorphic diffeomorphisms of $\mathbf{C}$, which is the group of affine transformations $z \mapsto az + b$. No nontrivial element of Γ can have a fixed point, because its orbit in $\mathbf{C}$, which is discrete, would have a limit point. Consequently, Γ acts by *translations* on $\mathbf{C}$, and must be isomorphic to $\{0\}$, $\mathbf{Z}$

or $\mathbf{Z}^2$. However, $\pi_1(\mathbf{C} \smallsetminus \{0,1\})$ is the free group on two generators, and in particular, is not abelian.

So the universal cover of $\mathbf{C} \smallsetminus \{0,1\}$ is the open unit disk $\mathbf{D}$. The map f lifts to a holomorphic function $\tilde{f} \colon \mathbf{C} \to \mathbf{D}$, which is therefore bounded. By Liouville theorem, $\tilde{f}$ is constant, and so is f.

1.6. The classical proof of the theorems of Picard predate the uniformization theorem. In fact, the universal cover of $\mathbf{C} \smallsetminus \{0,1\}$ is constructed explicitly, via elliptic and modular functions. This construction is the topic of the next sections, and we prove the theorems of Picard in Section 11.

§ 2. LATTICES OF THE COMPLEX PLANE

2.1. *Bases of the complex plane.* — Let $\mathscr{B}$ be the set of $\mathbf{R}$-bases of $\mathbf{C}$, and let $\mathscr{B}^+$ be its subset consisting of oriented bases. A pair (ω_1, ω_2) of complex numbers belongs to $\mathscr{B}$ if and only if ω_1, ω_2 are non zero, and ω_2/ω_1 is not a real number. The oriented area of the parallelogram drawn on a pair of vectors (ω_1, ω_2) is equal to

(2.1.1) $$\operatorname{Area}(\omega_1, \omega_2) = \mathfrak{I}(\omega_2/\omega_1)\,|\omega_1|\,.$$

Consequently, a pair belongs to $\mathscr{B}^+$ if, moreover, the imaginary part of ω_2/ω_1 is positive.

Observe also that $\mathscr{B}$ and $\mathscr{B}^+$ are open subsets of $\mathbf{C}^2$.

2.2. The group $\mathrm{GL}_2(\mathbf{R})$ acts on $\mathscr{B}$ by multiplication on the right. Namely, for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$, and $(\omega_1, \omega_2) \in \mathscr{B}$, set

$$(\omega_1, \omega_2) \cdot g = (\omega_1, \omega_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\omega_1 + c\omega_2, b\omega_1 + d\omega_2);$$

this is again a basis of $\mathbf{C}$. Assume that (ω_1, ω_2) is oriented; then $(\omega_1, \omega_2) \cdot g$ is oriented if and only if g belongs to the subgroup $\mathrm{GL}_2(\mathbf{R})^+$ of matrices with positive determinant. We have thus defined a right-action of $\mathrm{GL}_2(\mathbf{R})$ on $\mathscr{B}$, and a right-action of $\mathrm{GL}_2(\mathbf{R})^+$ on $\mathscr{B}^+$. We transfer this right-action to a left-action by transposing matrices: namely, we set, for $g \in \mathrm{GL}_2(\mathbf{R})$ and $(\omega_1, \omega_2) \in \mathscr{B}$,

$$g \cdot (\omega_1, \omega_2) = (\omega_1, \omega_2) \cdot g^{\mathrm{T}}.$$

Explicitly,

(2.2.1) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\omega_1, \omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2).$$

Lemma **(2.3)**. — *The group* $\mathrm{GL}_2(\mathbf{R})$ *acts simply transitively on* $\mathscr{B}$*; the group* $\mathrm{GL}_2(\mathbf{R})^+$ *acts simply transitively on* $\mathscr{B}^+$.

Proof. — The pair $(1, i)$ is obviously an oriented basis of $\mathbf{R}$. Observe that for any matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$,

$$g \cdot (1, i) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (1, i) = (1, i) \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (a + ib, c + id).$$

This shows that for any basis $(\omega_1, \omega_2) \in \mathscr{B}$, there exists a unique matrix $g \in \mathrm{GL}_2(\mathbf{R})$ such that $g \cdot (1, i) = (\omega_1, \omega_2)$. If (ω_1, ω_2) is oriented, then $g \in \mathrm{GL}_2(\mathbf{R})^+$. □

2.4. *Lattices.* — Any subgroup of $\mathbf{C}$ generated by a basis of $\mathbf{C}$ as a real vector space is called a *lattice*. By definition, a subgroup Λ of $\mathbf{C}$ is a lattice if and only if there exists a basis $(\omega_1, \omega_2) \in \mathscr{B}$ such that $\Lambda = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2$. We shall say that (ω_1, ω_2) is a basis of Λ, and an oriented basis if it is oriented. Exchanging ω_1 and ω_2 if necessary, we see that any lattice possesses an oriented basis. The absolute value of the area the parallelogram drawn on (ω_1, ω_2) is then independent of the chosen basis; it is called the covolume of the lattice and denoted $\operatorname{covol}(\Lambda)$.

Let $\mathscr{R}$ be the set of all lattices in $\mathbf{C}$. We have seen that there is a natural surjective map $\mathscr{B}^+ \to \mathscr{R}$, which associates to an oriented basis the lattice it generates. We shall endow $\mathscr{R}$ with the quotient topology.

Lemma (**2.5**). — *Let Λ be a lattice of $\mathbf{C}$, let (ω_1, ω_2) be an oriented basis of Λ.*

Let $g \in \mathrm{GL}_2(\mathbf{R})^+$. Then $g \cdot (\omega_1, \omega_2)$ is a basis of Λ if and only if $g \in \mathrm{SL}_2(\mathbf{Z})$, and $g \cdot (\omega_1, \omega_2)$ generates a sublattice of Λ if and only if $g \in \mathrm{M}_2(\mathbf{Z})$.

In particular, for any oriented basis (ω_1', ω_2') of Λ, there exists a unique matrix $g \in \mathrm{SL}_2(\mathbf{Z})$ such that $(\omega_1', \omega_2') = g \cdot (\omega_1', \omega_2')$.

Proof. — Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$. We have

$$g \cdot (\omega_1, \omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2).$$

By definition of a basis of a lattice, $a\omega_1 + b\omega_2$ belongs to Λ if and only if $(a, b) \in \mathbf{Z}^2$, and $c\omega_1 + \omega_2$ belongs to Λ if and only if $(c, d) \in \mathbf{Z}^2$. It follows that the lattice Λ' generated by $g \cdot (\omega_1, \omega_2)$ is contained in Λ if and only if $g \in \mathrm{M}_2(\mathbf{Z})$. Conversely, using that $(\omega_1, \omega_2) = g^{-1} \cdot (g \cdot (\omega_1, \omega_2))$ we see that $\Lambda' = \Lambda$ if and only if both g and g^{-1} have integral coefficients, that is, if and only if $g \in \mathrm{GL}_2(\mathbf{Z})$. The lemma follows from that, since $g \in \mathrm{GL}_2(\mathbf{R})^+$. □

§ 3. THE UPPER HALF PLANE

Definition (**3.1**). — *The set Π of all complex numbers z such that $\Im(z) > 0$ is called the Poincaré upper half plane*

3.2. The upper half plane appears naturally when one considers bases, or lattices, of $\mathbf{C}$ only up to homothety. Let indeed Λ be a lattice and let (ω_1, ω_2) be a basis of Λ. One can write

$$\Lambda = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2 = \omega_1(\mathbf{Z} + \frac{\omega_2}{\omega_1}\mathbf{Z}) = \omega_1(\mathbf{Z} + \tau\mathbf{Z}),$$

where $\tau = \omega_2/\omega_1$ is a non-real complex number. The basis (ω_1, ω_2) is oriented if and only if $\mathfrak{I}(\tau) > 0$, that is if $\tau \in \Pi$.

This shows that the map $(\omega_1, \omega_2) \mapsto \omega_2/\omega_1$ identifies the space $\mathscr{B}^+/\mathbf{C}^*$ of bases modulo homothety with the upper half plane Π.

3.3. *Action of* $\mathrm{SL}_2(\mathbf{R})$ *on the upper half plane.* — The action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathscr{B}$ and $\mathscr{B}^+$ commutes with homotheties, hence it induces an action of $\mathrm{SL}_2(\mathbf{R})$ on Π.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$. From the computation

$$g \cdot (1, \tau) = (a + b\tau, c + d\tau),$$

we see that g acts on $\tau \in \Pi$ by an homography:

$$g \cdot \tau = \frac{c + d\tau}{a + b\tau}. \tag{3.3.1}$$

The scalar matrices act trivially on Π, so that we get an action of $\mathrm{PSL}_2(\mathbf{R})$.

3.4. Let $\tau_1, \tau_2 \in \Pi$. The two corresponding lattices $\Lambda_1 = \mathbf{Z} + \mathbf{Z}\tau_1$ and $\Lambda_2 = \mathbf{Z} + \mathbf{Z}\tau_2$ are homothetic if and only if there exists $g \in \mathrm{PSL}_2(\mathbf{Z})$ such that $\tau_2 = g \cdot \tau_1$.

We have shown that the set $\mathscr{R}$ of lattices of $\mathbf{C}$ is isomorphic to the quotient space $\mathrm{SL}_2(\mathbf{Z})\backslash\mathscr{B}^+$. The set of lattices up to homothety is thus isomorphic to $\mathrm{SL}_2(\mathbf{Z})\backslash\Pi$.

Remark (3.5). — Usually, the action of $\mathrm{PSL}_2(\mathbf{R})$ on Π is defined by the formula:

$$g * \tau = \frac{a\tau + b}{c\tau + b}, \tag{3.5.1}$$

which does not coincide with the action of $\mathrm{PSL}_2(\mathbf{R})$ on Π we have introduced. However, these actions are conjugate one to the other via an automorphism of $\mathrm{SL}_2(\mathbf{R})$. Let indeed w be the element $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{GL}_2(\mathbf{Z})$. The inner automorphism of $\mathrm{GL}_2(\mathbf{R})$ given by $g \mapsto g^w = wgw^{-1}$ is an involution and induces automorphisms of $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{Z})$. Moreover, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $g^w = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$, so that

$$g^w * \tau = g \cdot \tau. \tag{3.5.2}$$

3.6. *Reduced bases.* — A *reduced basis* of a lattice Λ such that ω_1 is an element of minimal norm in $\Lambda \smallsetminus \{0\}$ and ω_2 is an element of minimal norm among all elements $\omega_2 \in \Lambda \smallsetminus \mathbf{Z}\omega_1$ such that (ω_1, ω_2) is an oriented basis of Λ.

Lemma (3.7). — *Any lattice possesses a reduced basis.*

Proof. — Let z be any nonzero element of Λ. Since Λ is discrete, there are finitely many nonzero elements $\lambda \in \Lambda$ such that $|\lambda| \leq |z|$. Hence there exists one, ω_1, of smallest absolute value.

The vector ω_1 is primitive in the lattice Λ, so can be completed in a basis (ω_1, ω_2) of Λ, a basis which we may even assume to be oriented. The set of $\lambda \in \Lambda$ such that (ω_1, λ) is an oriented basis and $|\lambda| \leq |\omega_2|$ is finite, so contains an element ω_2' of smallest absolute value, and (ω_1, ω_2') is a reduced basis of Λ. □

Proposition **(3.8)**. — *Let* (ω_1, ω_2) *be a basis of a lattice* Λ *and let* $\tau = \omega_2/\omega_1$. *Then* (ω_1, ω_2) *is a reduced basis if and only if* $|\tau| \geq 1$ *and* $|\Re(\tau)| \leq \frac{1}{2}$.

Proof. — Assume that (ω_1, ω_2) is reduced. By definition of ω_1, one has $|\omega_2| \geq |\omega_1|$, whence $|\tau| \geq 1$. Moreover, $(\omega_1, \omega_2 \pm \omega_1)$ is also an oriented basis of Λ. by definition of a reduced basis, $|\omega_2 \pm \omega_1| \geq |\omega_2|$, hence $|\tau \pm 1| \geq |\tau|$. This means than τ is closer to 0 than to 1 or -1, hence $-\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}$.

Conversely, assume that these inequalities are satisfied. Let $(m, n) \in \mathbf{Z}^2 \smallsetminus \{0\}$. Let us prove that $|m + n\tau| \geq 1$, and that $|m + n\tau| \geq \tau$ if $m \neq 0$.

If $n = 0$, then $|m + n\tau| = |m| \geq 1$. Assume that $n = 1$. If $m = 0$, we have $|m + \tau| = |\tau| \geq 1$; otherwise, we see geometrically that $|m + \tau| \geq |\tau - 1| \geq |\tau|$ if $\Re(\tau) \geq 0$, and $|m + \tau| \geq |\tau + 1| \geq |\tau|$ if $\Re(\tau) \leq 0$. The case $n = -1$ is analogous. Let us now assume that $|n| \geq 2$. Then, $|m + n\tau| \geq |n|\,\Im(\tau) \geq 2\Im(\tau)$. Moreover, $\Im(\tau) \geq \frac{1}{2}\sqrt{3}$ and $|\Re(\tau)| \leq \frac{1}{2}$, so that $\Im(\tau) \geq \sqrt{3}\,|\Re(\tau)|$. This implies $4\Im(\tau)^2 \geq 3\,|\tau|^2$, hence $|m + n\tau| \geq \sqrt{3}\,|\tau| > |\tau|$.

This implies that $|\omega_1|$ is a nonzero element of Λ of shortest absolute value, while any element of $\Lambda \smallsetminus \mathbf{Z}$ has absolute value at least $|\tau|\,|\omega_1| = |\omega_2|$, hence the proposition. □

3.9. Let $\mathfrak{F}$ be the subset of Π given by the inequalities of Proposition 3.8. It is called the *fundamental domain* of Π. One has $\mathrm{PSL}_2(\mathbf{Z}) \cdot \mathfrak{F} = \Pi$.

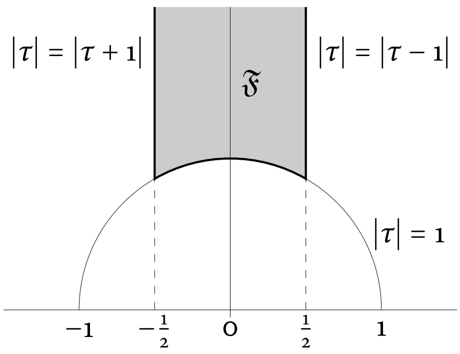

FIGURE 1. The fundamental domain $\mathfrak{F}$ for the action of $\mathrm{SL}_2(\mathbf{Z})$ on Π

§ 4. BI-PERIODIC MEROMORPHIC FUNCTIONS

Definition **(4.1)**. — *Let* Λ *be a lattice in* $\mathbf{C}$. *An* elliptic function *with respect to the lattice* Λ *is a meromorphic function on* $\mathbf{C}$ *which is invariant by translations under every element of* Λ.

We write $\mathscr{M}(\mathbf{C}/\Lambda)$ for the set of elliptic functions with respect to Λ.

Proposition **(4.2)**. — $\mathscr{M}(\mathbf{C}/\Lambda)$ *is a field extension of* $\mathbf{C}$.

Remark **(4.3)**. — Let Λ be a lattice in $\mathbf{C}$ and let f be an elliptic function with respect to Λ.

Let $p \in \mathbf{C}$. By definition of a periodic function, $f(p+\lambda) = f(p)$ for every $\lambda \in \Lambda$. This allows to write $f(u) = f(p)$ if $u \in \mathbf{C}/\Lambda$ is the class of p.

For $\lambda \in \Lambda$, the relation $f(z+\lambda) = f(z)$ implies that the order of vanishing $v_p(f)$ of f at p equals the order of vanishing of f at $p+\lambda$. If $u \in \mathbf{C}/\Lambda$ is the class of p, we may thus write $v_u(f)$ for $v_p(f)$.

Similarly, the residue $\operatorname{Res}_p(f)$ of f at p is equal to the residue of f at $p+\lambda$, so that we may set $\operatorname{Res}_u(f) = \operatorname{Res}_p(f)$.

Proposition **(4.4)**. — *Let Λ be a lattice in $\mathbf{C}$ and let f be an elliptic function with respect to Λ.*

a) *One has* $\sum_{u\in\mathbf{C}/\Lambda} \operatorname{Res}_u(f) = 0$;

b) *If $f \neq 0$, then* $\sum_{u\in\mathbf{C}/\Lambda} v_u(f) = 0$.

Proof. — Let (ω_1, ω_2) be an oriented basis of Λ. To prove the first relation, we integrate f along the boundary ∂F of a fundamental parallelogram. Since f is meromorphic, it has finitely many poles in any compact subset of $\mathbf{C}$ and there exists $a \in \mathbf{C}$ such that f has no pole on the boundary ∂F of the fundamental parallelogram $F = a + [0,1]\omega_1 + [0,1]\omega_2$. By the residue theorem, one has

$$\int_{\partial F} f = 2\pi \mathrm{i} \sum_{p\in \mathring{F}} \operatorname{Res}_p(f).$$

Since f has no pole on ∂F, any pole of f is congruent modulo Λ to a unique pole of f contained in $\mathring{F}$, so that the right-hand-side of the previous formula is equal to $2\pi\mathrm{i}\sum_{u\in\mathbf{C}/\Lambda} \operatorname{Res}_u(f)$.

On the other hand,

$$\begin{aligned}\int_{\partial F} f = \int_0^1 f(a+t\omega_1)\,\mathrm{d}t + \int_0^1 f(a+\omega_1+t\omega_2)\,\mathrm{d}t \\ + \int_0^1 f(a+(1-t)\omega_1+\omega_2)\,\mathrm{d}t + \int_0^1 f(a+(1-t)\omega_2)\,\mathrm{d}t.\end{aligned}$$

Since f is an elliptic function, $f(z+\omega_1) = f(z+\omega_2) = f(z)$, hence

$$\begin{aligned}\int_0^1 f(a+t\omega_1)\,\mathrm{d}t + \int_0^1 f(a+(1-t)\omega_1+\omega_2)\,\mathrm{d}t \\ = \int_0^1 f(a+t\omega_1)\,\mathrm{d}t + \int_0^1 f(a+(1-t)\omega_1)\,\mathrm{d}t = 0\end{aligned}$$

and

$$\begin{aligned}\int_0^1 f(a+\omega_1+t\omega_2)\,\mathrm{d}t + \int_0^1 f(a+(1-t)\omega_2)\,\mathrm{d}t \\ = \int_0^1 f(a+t\omega_2)\,\mathrm{d}t + \int_0^1 f(a+(1-t)\omega_2)\,\mathrm{d}t = 0.\end{aligned}$$

Consequently, $\int_{\partial F} f = 0$, which proves the first equality.

If $f \neq 0$, we can apply this formula to the elliptic function f'/f. For any $p \in F$, one has $\operatorname{Res}_p(f) = v_p(f'/f)$, hence the second equality. □

4.5. Let $a \in \mathbf{C}$. If f is not the constant function a, then the set of elements $u \in \mathbf{C}/\Lambda$ such that $f(u) = a$ is finite. We can thus define $n(f,a) = \sum_{u\in f^{-1}(a)} v_u(f-a)$. Similarly, the set of $u \in \mathbf{C}/\Lambda$ such that $f(u) = \infty$ (the poles of f) is finite and we define $n(f,\infty) = \sum_{u\in f^{-1}(\infty)}(-v_u(f))$.

Proposition (**4.6**). — *Let Λ be a lattice in $\mathbf{C}$ and let f be a non-constant elliptic function with respect to Λ. For any $a \in \mathbf{C}$, $n(f,a) = n(f,0) = n(f,\infty)$.*

Proof. — Proposition 4.4, applied to the function f, implies readily that $n(f,0)-n(f,\infty) = 0$. If $a \in \mathbf{C}$, replacing f by $f-a$, we get that $n(f,a) = n(f-a,0) = n(f-a,\infty) = n(f,\infty)$. ☐

Corollary (**4.7**). — *Any non-constant elliptic function has at least one pole. If a non-constant elliptic function has a single pole modulo Λ, then its order is at least 2.*

Proof. — Let f be a non-constant elliptic function, let a be some value taken by f. One has $n(f,a) > 0$. Consequently, $n(f,\infty) > 0$ which shows that f cannot be holomorphic everywhere. Assume that u is the only pole of f modulo Λ. Then, $\operatorname{Res}_u(f) = 0$; this implies that $v_u(f) \neq 1$, whence the result since $v_u(f) > 0$. ☐

§ 5. THE $\wp$-FUNCTION

Lemma (**5.1**). — *Let Λ be a lattice in $\mathbf{C}$. For any $z \in \mathbf{C} \smallsetminus \Lambda$, the series*

(5.1.1)
$$\frac{1}{z^2} + \sum_{\lambda\in\Lambda\smallsetminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

converges to a complex number $\wp_\Lambda(z)$, the convergence being locally uniform.

The function $\wp_\Lambda$ is an elliptic function with respect to Λ.

Proof. — For $\lambda \to \infty$, we have the following asymptotic expansion

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\left(1-\frac{z}{\lambda}\right)^{-2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\left(1+\mathrm{O}(1/\lambda)\right) - \frac{1}{\lambda^2} = \mathrm{O}(1/\lambda^3).$$

More precisely, for any positive real number R, there exists a real number B such that

$$\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| \leq B/\left|\lambda^3\right|$$

for any pair (z,λ) such that $|z| \leq R$ and $|\lambda| > 2R$. This implies the convergence of the series

$$\sum_{|\lambda|>2R} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

for any $z \in \mathbf{C}$ such that $|z| \leq R$. The convergence being uniform, the limit is a holomorphic function on the open disk $\dot{\mathbf{D}}(0,R)$. Adding the finitely many missing terms, we see that the given series converges to a meromorphic function on that disk, whose only poles are in Λ. Since R is arbitrary, $\wp_\Lambda$ is a meromorphic function on $\mathbf{C}$, holomorphic on $\mathbf{C} \setminus \Lambda$.

According to Cauchy's theory, its derivative is given by the termwise derivative of the series. Therefore, for any $z \in \mathbf{C} \setminus \Lambda$,

$$\wp'_\Lambda(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}.$$

This formula shows that $\wp'_\Lambda$ is an elliptic function.

Let (ω_1, ω_2) be a basis of Λ. Since $\wp'_\Lambda(z+\omega_1) = \wp'_\Lambda(z)$ for any $z \in \mathbf{C} \setminus \Lambda$, $\wp_\Lambda(z+\omega_1) - \wp_\Lambda(z)$ is a constant c_1. oreover, $\pm\omega_1/2$ is not a pole of $\wp_\Lambda$, because $\omega_1/2$ does not belong to Λ. We thus get $c_1 = \wp_\Lambda(\omega_1/2) - \wp_\Lambda(-\omega_1/2) = 0$ since, by the very definition of $\wp_\Lambda$, this is an even function. Consequently, ω_1 is a period of $\wp_\Lambda$. Similarly, ω_2 is a period of $\wp_\Lambda$. This proves that $\wp_\Lambda$ is an elliptic function. □

5.2. We see that $u = 0$ is the only pole of $\wp_\Lambda$ in $\mathbf{C}/\Lambda$, and that $v_0(\wp_\Lambda) = -2$. This gives $n(\wp_\Lambda, \infty) = 2$. Consequently, $n(\wp_\Lambda, a) = 2$ for any $a \in \mathbf{C}$.

Similarly, $n(\wp'_\Lambda, a) = 3$ for any $a \in \mathbf{C} \cup \{\infty\}$. Moreover, the function $\wp'_\Lambda$ is odd. Let (ω_1, ω_2) be an oriented basis of Λ. The three points $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$ are the (only) three non-zero elements of $\mathbf{C}/\Lambda$ which are equal to their opposite. The function $\wp'_\Lambda$ must vanish at any of them. Since $n(\wp'_\Lambda, 0) = 3$, $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$ are the only zeroes of $\wp'_\Lambda$, and these zeroes are simple.

Proposition **(5.3)**. — *For any pair (u, v) of points in $\mathbf{C}/\Lambda$, $\wp_\Lambda(u) = \wp_\Lambda(v)$ if and only if $u = v$ of $u = -v$.*

Proof. — Since $\wp_\Lambda$ is even, $\wp_\Lambda(u) = \wp_\Lambda(v)$ if $u = v$ or $u = -v$. Conversely, fix $u \in \mathbf{C}/\Lambda$; we have seen that $n(\wp_\Lambda, \wp_\Lambda(u)) = n(\wp_\Lambda, \infty) = 2$. If $u \notin \frac{1}{2}\Lambda/\Lambda$, $z = u$ and $z = -u$ are two distinct elements of $\mathbf{C}/\Lambda$ where $\wp_\Lambda$ takes the value $\wp_\Lambda(u)$; consequently, they are the only ones. If $u = 0$, we observe that $\wp_\Lambda$ has a pole of order 2 at 0, and no other pole modulo Λ. Finally, if $u \in \frac{1}{2}\Lambda/\Lambda$ but $u \notin \Lambda$, the relation $\wp'_\Lambda(u) = 0$ shows that $z = u$ is a double root of $\wp_\Lambda(z) - \wp_\Lambda(u)$. Consequently, there is no other element of $\mathbf{C}/\Lambda$ at which $\wp_\Lambda$ takes the value $\wp_\Lambda(u)$. □

§ 6. THE FIELD OF ELLIPTIC FUNCTIONS

6.1. *Laurent expansions of $\wp_\Lambda$ and $\wp'_\Lambda$.* — Let Λ be a lattice in $\mathbf{C}$; let $\Lambda^* = \Lambda \setminus \{0\}$. Let (ω_1, ω_2) be a reduced basis of Λ; in particular, $|\lambda| \geq |\omega_1|$ for any $\lambda \in \Lambda^*$. The meromorphic functions $\wp_\Lambda$ and $\wp'_\Lambda$ are meromorphic on the open disk $\dot{\mathbf{D}}(0, |\omega_1|)$ with 0 as only pole. We compute here their Laurent expansion.

For every $z \in \mathbf{C}$ and $\lambda \in \Lambda$ such that $|z| < |\lambda|$, one can write

$$\frac{1}{\lambda - z} = \frac{1}{\lambda} \frac{1}{1 - z/\lambda} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{z}{\lambda}\right)^n.$$

Differentiating, we obtain

$$\frac{1}{(\lambda - z)^2} = -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\lambda - z}\right) = \frac{1}{\lambda}\sum_{n=1}^{\infty} n\frac{z^{n-1}}{\lambda^n},$$

so that

$$\frac{1}{(\lambda - z)^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda}\sum_{n=2}^{\infty} n\frac{z^{n-1}}{\lambda^n} = \sum_{n=1}^{\infty}(n+1)\frac{z^n}{\lambda^{n+2}}.$$

In particular, for any $z \in \mathbf{C}$ such that $0 < |z| < |\omega_1|$, we get

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)\left(\sum_{\lambda\in\Lambda\smallsetminus\{0\}} \frac{1}{\lambda^{n+2}}\right) z^n.$$

For any positive integer k such $k \geq 3$, let us define the Eisenstein series $G_k(\Lambda)$ of weight k by the converging series:

(6.1.1) $$G_k(\Lambda) = \sum_{\lambda\in\Lambda\smallsetminus\{0\}} \frac{1}{\lambda^k}.$$

We also observe that for any $u \in \mathbf{C}^*$,

(6.1.2) $$G_k(u\Lambda) = u^{-k}G_k(\Lambda).$$

For odd k, one has $G_k(\Lambda) = 0$. We thus have, for any $z \in \dot{\mathbf{D}}(0, |\omega_1|)$,

(6.1.3) $$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}(\Lambda)z^{2k}.$$

By termwise differentiation, we also get

(6.1.4) $$\wp'_\Lambda(z) = \frac{-2}{z^3} + \sum_{k=1}^{\infty} 2k(2k+1)G_{2k+2}(\Lambda)z^{2k-1}.$$

Theorem **(6.2)**. — *For any $z \in \mathbf{C} \smallsetminus \Lambda$, one has*

(6.2.1) $$(\wp'_\Lambda)^2(z) = 4\wp^3_\Lambda(z) - 60G_4(\Lambda)\wp_\Lambda(z) - 140G_6(\Lambda).$$

To shorten the notation, we set

(6.2.2) $$g_2(\Lambda) = 60\, G_4(\Lambda), \quad g_3(\Lambda) = 140\, G_6(\Lambda).$$

The differential equation of $\wp_\Lambda$ can then be written

(6.2.3) $$(\wp'_\Lambda)^2(z) = 4\wp^3_\Lambda(z) - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda).$$

Proof. — Modulo Λ, the origin is the only pole of the six elliptic functions $\wp_\Lambda$, $\wp^2_\Lambda$, $\wp^3_\Lambda$, $\wp'_\Lambda$, $\wp_\Lambda\wp'_\Lambda$ and $(\wp'_\Lambda)^2$, and the order of this pole is at most 6. Consequently, their must be a nontrivial linear combination of them which is an elliptic function without a pole of order at most 1 at the origin, hence is constant. Since $\wp_\Lambda$, $\wp^2_\Lambda$, $\wp^3_\Lambda$, and $(\wp'_\Lambda)^2$ are even, while $\wp'_\Lambda$ and $\wp_\Lambda\wp'_\Lambda$ are odd, there is already such a linear combination among those functions. Let us compute it explicitly.

We first compute the terms of low order in the Laurent expansion of these functions. To shorten the notation, we write G_k for $G_k(\Lambda)$. We have

$$\begin{aligned}
\wp_\Lambda(z) &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + O(z^6) \\
\wp'_\Lambda(z) &= -2\frac{1}{z^3} + 6G_4z + 20G_6z^3 + O(z^5) \\
\wp^2_\Lambda(z) &= \frac{1}{z^4}\left(1 + 3G_4z^4 + O(z^6)\right)^2 \\
&= \frac{1}{z^4} + 6G_4 + O(z^2) \\
(\wp'_\Lambda)^2(z) &= 4\frac{1}{z^6}\left(1 - 3G_4z^4 - 10G_6z^6 + O(z^8)\right)^2 \\
&= 4\frac{1}{z^6} - 24G_4\frac{1}{z^2} - 80G_6 + O(z^2) \\
\wp^3_\Lambda(z) &= \frac{1}{z^6}\left(1 + 3G_4z^4 + 5G_6z^6 + O(z^8)\right)^3 \\
&= \frac{1}{z^6}\left(1 + 9G_4z^4 + 15G_6z^6 + O(z^8)\right) \\
&= \frac{1}{z^6} + 9G_4\frac{1}{z^2} + 15G_6 + O(z^2).
\end{aligned}$$

Consequently,

$$\begin{aligned}
(\wp'_\Lambda)^2(z) - 4\wp^3_\Lambda(z) &= -24G_4\frac{1}{z^2} - 80G_6 + O(z^2) - 36G_4\frac{1}{z^2} - 60G_6 + O(z^2) \\
&= -60G_4\frac{1}{z^2} - 140G_6 + O(z^2),
\end{aligned}$$

so that

$$\begin{aligned}
(\wp'_\Lambda)^2(z) - 4\wp^3_\Lambda(z) + 60G_4\wp_\Lambda(z) &= -60G_4\frac{1}{z^2} - 140G_6 + O(z^2) + 60G_4\frac{1}{z^2} + O(z^2) \\
&= -140G_6 + O(z^2)
\end{aligned}$$

and

$$(\wp'_\Lambda)^2(z) - 4\wp^3_\Lambda(z) + 60G_4\wp_\Lambda(z) + 140G_6 = O(z^2).$$

The left-hand-side is an elliptic function with respect to Λ whose poles are contained in Λ. It extends to a holomorphic function at 0, with value 0. By periodicity, it is holomorphic everywhere, hence must be a constant, necessarily zero. □

Differentiating the preceding relation and simplifying by the non-zero elliptic function $\wp'_\Lambda$, we get the following corollary.

Corollary **(6.3)**. — *For any* $z \in \mathbf{C} \setminus \Lambda$, *one has*

$$\wp''_\Lambda(z) = 6\wp^2_\Lambda(z) - 30G_4(\Lambda).$$

6.4. We have already proven that $\wp'$ vanishes at $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$, and that the values of $\wp$ at these points,

(6.4.1) $$e_1 = \wp(\omega_1/2), \quad e_2 = \wp(\omega_2/2), \quad \text{and} \quad e_3 = \wp((\omega_1+\omega_2)/2)$$

are distinct. By Theorem 6.2, e_1, e_2 and e_3 are the roots of the polynomial of degree 3,

$$4X^3 - g_2(\Lambda)X - g_3(\Lambda).$$

Therefore, its discriminant

(6.4.2) $$\Delta(\Lambda) = 16(e_1-e_2)^2(e-1-e_3)^2(e_2-e_3)^2 = g_2(\Lambda)^3 - 27g_2(\Lambda)^2$$

does not vanish.

Proposition **(6.5)**. — *Let Λ be a lattice in* **C**. *Any even elliptic function with respect to Λ can be expressed as a rational function in $\wp_\Lambda$, any elliptic function can be expressed as a rational function in $\wp_\Lambda$ and $\wp'_\Lambda$. More precisely, the minimal polynomial of $\wp'_\Lambda$ over the subfield $\mathbf{C}(\wp_\Lambda)$ generated by $\wp_\Lambda$ in $\mathscr{M}(\mathbf{C}/\Lambda)$ is equal to $T^2-4\wp_\Lambda^3+g_2(\Lambda)\wp_\Lambda+g_3(\Lambda)$. There is an isomorphism of field extensions of* **C**,

$$\mathbf{C}(X)[Y]/(Y^2-4X^3+g_2(\Lambda)X+g_3(\Lambda)) \xrightarrow{\sim} \mathscr{M}(\mathbf{C}/\Lambda),$$

which maps X to $\wp_\Lambda$ and Y to $\wp'_\Lambda$.

Proof. — By Theorem 6.2, $\wp'_\Lambda$ is a root of the polynomial $P(T) = T^2 - (4\wp_\Lambda^3 - g_2(\Lambda)\wp_\Lambda - g_3(\Lambda))$ with coefficients in the subfield $\mathbf{C}(\wp_\Lambda)$ generated by $\wp_\Lambda$. The function $\wp'_\Lambda$ does not belong to this subfield because it is odd, while any elliptic function in $\mathbf{C}(\wp_\Lambda)$ is even. This shows that this polynomial $P(T)$ is irreducible. Since $\wp_\Lambda$ is non-constant, $\mathbf{C}(\wp_\Lambda)$ is isomorphic to the field of rational functions $\mathbf{C}(X)$, and the field $\mathbf{C}(\wp_\Lambda,\wp'_\Lambda)$ is isomorphic to $\mathbf{C}(X)[Y]/(Y^2-4X^3+g_2(\Lambda)X+g_3)$, as claimed.

It remains to show that $\wp_\Lambda$ and $\wp'_\Lambda$ generate the field of elliptic functions $\mathscr{M}(\mathbf{C}/\Lambda)$, and that $\wp_\Lambda$ generates the subfield of even elliptic functions. Let f be any non-constant elliptic function. The formula

$$f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}$$

expresses f as the sum of an even and of an odd elliptic function. Observing that $f\wp'_\Lambda$ is even if f is an odd elliptic function, it suffices to show that any even elliptic function belongs to $\mathbf{C}(\wp_\Lambda)$.

Let thus f be an even elliptic function. If $p\in\mathbf{C}$ is a zero (resp. a pole) of f, then so is $-p$, with the same order. Moreover, if $p\equiv -p \pmod{\Lambda}$, then this order is even. Indeed, let $\lambda\in\Lambda$ such that $-p = p+\lambda$. For any integer k, one has $f^{(k)}(-z) = (-1)^k f^{(k)}(z)$ and $f^{(k)}(z+\lambda) = f^{(k)}(z)$, so that

$$f^{(k)}(p) = (-1)^k f^{(k)}(-p) = (-1)^k f^{(k)}(p+\lambda) = (-1)^k f^{(k)}(p).$$

In particular, $f^{(k)}(p) = 0$ if k is odd.

Let $a_1,\dots,a_n$ be complex numbers not belonging to Λ, pairwise distinct modulo Λ, and such for any zero (reps. any pole) p of f with $p\notin\Lambda$, there is an integer $i\in\{1,\dots,n\}$ such that $p=\pm a_i$. For any $i\in\{1,\dots,n\}$, set $d_i=2$ if $a_i\equiv -a_i \pmod{\Lambda}$ and $d_i=1$ otherwise.

The function $\wp(z) - \wp(a_i)$ has a pole of order 2 at $z = 0$, and a zero of order d_i at $\pm a_i$. Let us define an elliptic function $g \in \mathbf{C}(\wp_\Lambda)$ by

$$g(z) = \prod_{i=1}^{n} (\wp(z) - \wp(a_i))^{v_{a_i}(f)/d_i}.$$

Since for every i, $a_i \notin \Lambda$, the function g has a zero (or pole) of order $v_{a_i}(f)$ at $\pm a_i$. At the origin, it has a zero of order

$$-2\sum_{i=1}^{n} v_{a_i}(f)/d_i = -\sum_{\substack{u\in\mathbf{C}/\Lambda\\ u\neq 0}} v_u(f) = v_0(f).$$

Consequently, the quotient $f(z)/g(z)$ is an elliptic function without zeroes nor poles, hence is constant. This shows that $f \in \mathbf{C}(\wp_\Lambda)$ and concludes the proof of the proposition. □

§ 7. ELLIPTIC CURVES VIEWED AS RIEMANN SURFACES

7.1. Let Λ be a lattice in $\mathbf{C}$. Let $\pi\colon \mathbf{C} \to \mathbf{C}/\Lambda$ be the natural projection. If Ω is any open subset of $\mathbf{C}/\Lambda$, we say that a function $f\colon \Omega \to \mathbf{C}$ is holomorphic if the function $f \circ \pi$ on $\pi^{-1}(\Omega)$ is holomorphic. This endowed $\mathbf{C}/\Lambda$ with the structure of a Riemann surface.

Proposition **(7.2)**. — *Let Λ_1 and Λ_2 be lattices in* $\mathbf{C}$. *Let $\pi_1\colon \mathbf{C} \to \mathbf{C}/\Lambda_1$ and $\pi_2\colon \mathbf{C} \to \mathbf{C}/\Lambda_2$ be the natural projections.*

Let $a \in \mathbf{C}$ such that $a\Lambda_1 \subset \Lambda_2$ and let $b \in \mathbf{C}$. There exists a unique map $f\colon \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ such that $f(\pi_1(z)) = \pi_2(az + b)$. It is holomorphic.

Conversely, any morphism of Riemann surfaces $f\colon \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ is of this form, for some unique pair $(a, b) \in \mathbf{C} \times \mathbf{C}/\Lambda$.

Proof. — The first part is obvious. Let $b \in \mathbf{C}$ be any point in $\pi_2^{-1}(f(0))$. Since $\mathbf{C}$ is simply connected, the theory of coverings shows the existence of a unique holomorphic map $\tilde f\colon \mathbf{C} \to \mathbf{C}$ such that $\pi_2(\tilde f(z)) = f(\pi_1(z))$ and $\tilde f(0) = b$. For any $\lambda \in \Lambda_1$, let $\varphi(\lambda) = \tilde f(\lambda) - \tilde f(0)$. Then, $z \mapsto \tilde f(z + \lambda) - \varphi(\lambda)$ is a lift of $f \circ \pi_1$ which takes the value $b = \tilde f(0)$ at the origin. Consequently, $\tilde f(z + \lambda) = \tilde f(z) + phi(\lambda)$ for every $z \in \mathbf{C}$. This implies that the function $\tilde f'$ is an elliptic function. It is entire, hence takes a constant value a, and $\tilde f(z) = az + b$. For $\lambda \in \Lambda_1$,

$$f(0) = f(\lambda) = \pi_2(\tilde f(\lambda)) = \pi_2(a\lambda + b) = \pi_2(a\lambda) + \pi_2(\tilde f(0)) = \pi_2(a\lambda),$$

so that $a\lambda \in \Lambda_2$. □

7.3. An elliptic curve is a Riemann surface endowed with a base-point which is isomorphic to the quotient $\mathbf{C}/\Lambda$ of $\mathbf{C}$ by a lattice Λ, with base-point 0.

Morphism of elliptic curves are supposed to respect the base-points. By the proposition, any morphism of elliptic curves $f\colon (\mathbf{C}/\Lambda_1, [0]) \to (\mathbf{C}/\Lambda_2, [0])$ is of the form $\pi_1(z) \mapsto a\pi_2(z)$ for some unique $a \in \mathbf{C}$ such that $a\Lambda_2 \subset \Lambda_1$.

Corollary (**7.4**). — *Any elliptic curve has a unique structure of a complex Lie group for which the origin is the neutral element. A morphism of elliptic curves is a morphism of groups.*

§ 8. ELLIPTIC CURVES AS ALGEBRAIC CURVES

8.1. Let $P \in \mathbf{C}[X]$ be a polynomial of degree 3 with simple roots. Let $F_P = X_0X_2^2 - X_0^3P(X_1/X_0)$; this is a homogeneous polynomial of degree 3 in the indeterminates X_0, X_1, X_2. Let C_P be the projective algebraic curve defined by the polynomial F_P in the complex projective space $\mathbf{P}^2(\mathbf{C})$, namely the set of points $[x_0 : x_1 : x_2] \in \mathbf{P}^2(\mathbf{C})$ such that $F_P(x_0, x_1, x_2) = 0$.

Lemma (**8.2**). — *The curve C_P is irreducible and nonsingular.*

Proof. — We study this question in affine charts of $\mathbf{P}^2(\mathbf{C})$. On the open set U_0 given by $x_0 \neq 0$, we can write $[x_0 : x_1 : x_2] = [1 : x : y]$ and the equation of C_P is $F_0(X, Y) = F(1, X, Y) = Y^2 - P(X)$. Since the polynomial P has odd degreen it is not a square, hence F_0 is irreducible in $\mathbf{C}[X, Y]$. Therefore, the curve $C_P \cap U_0$ is irreducible. Since $\deg(P) = 3$, the coefficient of X_1^3 in F_P is nonzero, so that F_P is prime to X_0. This implies that the polynomial F_P is irreducible, so that C_P is an irreducible curve.

A singular point of $C_P \cap U_0$ is a point (x, y) at which F_0 vanishes, as well as the partial derivatives of F_0. One has $\frac{\partial}{\partial X}F_0 = -P'(X)$ and $\frac{\partial}{\partial Y}F_0 = 2Y$. The equations $F_0(x, y) = 2y = -P'(x) = 0$ imply that x is a common root of P and of P'; since all roots of P are distinct, $C_P \cap U_0$ is nonsingular.

The only point of C_P which does not belong to U_0 is the point $o = [0 : 0 : 1]$ and it remains to show that C_P is nonsingular there. On the open set U_2 given by $x_2 \neq 1$, we can write $[x_0 : x_1 : x_2] = [t : x : 1]$ and $C_P \cap U_1$ is defined by the polynomial $F_2(T, X) = T - P(X/T)T^3$. Since $P(X/T)T^3$ is a homogeneous polynomial of degree 3 in X, T, its partial derivatives at $(0, 0)$ vanish; consequently, $\frac{\partial}{\partial T}F_0(0, 0) = 1 \neq 0$. Consequently, o is a nonsingular point of C_P. □

Proposition (**8.3**). — *Let* $i_\Lambda \colon \mathbf{C}/\Lambda \to \mathbf{P}^2(\mathbf{C})$ *be the map given by*

$$i_\Lambda(z) = [1 : \wp_\Lambda(z) : \wp'_\Lambda(z)]$$

for $z \neq 0$ *and* $i_\Lambda(0) = [0 : 0 : 1]$. *It is holomorphic and induces an isomorphism of Riemann surfaces from* $\mathbf{C}/\Lambda$ *to the curve* C_P *defined by the polynomial* $P = 4X^3 - g_2(\Lambda)X - g_3(\Lambda)$.

Proof. — The map i_Λ is well-defined and holomorphic on $\mathbf{C}/\Lambda \smallsetminus \{0\}$. In a neighborhood of 0, we may write $i_\Lambda(z) = [z^3 : z^3\wp_\Lambda(z) : z^3\wp'_\Lambda(z)]$. The three functions z^3, $z^3\wp_\Lambda$ and $z^3\wp'_\Lambda$ are holomorphic there and equal $0, 0, -2$ at $z = 0$, so they do not vanish simultaneously. This shows that i_Λ is holomorphic.

Let us show that i_Λ is injective. First of all, if $z \notin \Lambda$, then $i_\Lambda(z) \neq i_\Lambda(0)$. So let $z, w \in \mathbf{C} \smallsetminus \Lambda$ be such that $i_\Lambda(z) = i_\Lambda(w)$. This means $\wp_\Lambda(z) = \wp_\Lambda(w)$ and $\wp'_\Lambda(z) = \wp'_\Lambda(w)$. By Proposition 5.3, $z \equiv w \pmod{\Lambda}$ or $z \equiv -w \pmod{\Lambda}$. In the latter case, we then have $\wp'_\Lambda(z) = -\wp'_\Lambda(w)$ so that $\wp'_\Lambda(z) = \wp'_\Lambda(w) = 0$. If (ω_1, ω_2) is a basis of Λ, we have seen that necessarily z and w belong to $\{\omega_1/2, \omega_2/2, , \omega_1 + \omega_2)/2$ modulo Λ. Since $\wp_\Lambda$ takes distinct values at these three points, we get $z = w$.

It follows that i_Λ is étale. Indeed, if the differential of i_Λ would vanish at some point of $\mathbf{C}/\Lambda$, then i_Λ would not be injective in a neighborhood of that point.

By Theorem 6.2, the image of i_Λ is contained in C. Let us show that its image is the whole of C. Since $i_\Lambda(0) = [0 : 0 : 1]$, it suffices to show that for any point $[1 : x : y] \in C$, there exists $z \in \mathbf{C} \smallsetminus \Lambda$ such that $x = \wp_\Lambda(z)$ and $y = \wp'_\Lambda(z)$. The elliptic function $\wp_\Lambda$ is surjective, so there exists $z \in \mathbf{C}$ such that $\wp_\Lambda(z) = x$. It then follows from Theorem 6.2 that $y = \pm\wp'_\Lambda(z)$. Up to replacing z by $-z$ if necessary, we thus have $x = \wp_\Lambda(z)$ and $y = \wp'_\Lambda(z)$. □

8.4. Let $P \in \mathbf{C}[X]$ be a polynomial of degree 3 with simple roots. Let us show that the curve C_P in $\mathbf{P}^2(\mathbf{C})$, endowed with the base-point $o = [0 : 0 : 1]$, is an elliptic curve. Let e_1, e_2, e_3 be the three roots of P, and let c be its leading coefficient, so that

$$P(X) = c(X - e_1)(X - e_2)(X - e_3) = c(X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3),$$

with

$$\sigma_1 = e_1 + e_2 + e_3, \quad \sigma_2 = e_1e_2 + e_2e_3 + e_3e_1, \quad \sigma_3 = e_1e_2e_3.$$

We consider points of $\mathbf{P}^2(\mathbf{C})$ of the form $[1 : x : y]$. If we make the affine change of variables $x = x' + \frac{1}{3}\sigma_1$ and $y = \frac{\sqrt{c}}{2}y'$, the equation $y^2 = P(x)$ of $C_P \cap U_0$ can be rewritten as $(y')^2 = Q(x')$, with

$$Q(x') = 4(x' - e'_1)(x' - e'_2)(x' - e'_3) = 4(x')^3 - ax' - b,$$

where $e'_i = e_i - \frac{1}{3}\sigma_1$ for $i \in \{1, 2, 3\}$ and a, b are complex numbers. (Indeed, $e'_1 + e'_2 + e'_3 = 0$.) The discriminant $a^3 - 27b^2$ of Q is given by

$$\Delta(Q) = 16(e'_1 - e'_2)^2(e'_2 - e'_3)^2(e'_1 - e'_3)^2 = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_1 - e_3)^2,$$

so is nonzero. In fact,

$$a = -4(e'_1e'_2 + e'_1e'_3 + e'_2e'_3) = -4(e_1 - \frac{1}{3}\sigma_1)(e_2 - \frac{1}{3}\sigma_1) - \text{symmetric terms} = -4(\sigma_2 - \frac{1}{3}\sigma_1^2)$$

and

$$b = 4e'_1e'_2e'_3 = 4(e_1 - \frac{1}{3}\sigma_1)(e_2 - \frac{1}{3}\sigma_1)(e_3 - \frac{1}{3}\sigma_1) = -\frac{4}{c}P(\frac{1}{3}\sigma_1).$$

We shall prove in Theorem 9.3 that there exists a (unique) lattice Λ in $\mathbf{C}$ such that $g_2(\Lambda) = a$ and $g_3(\Lambda) = b$, so that (C_Q, o) is an elliptic curve.

The affine transformation $(x', y') \mapsto (x' + \frac{1}{3}\sigma 1, \frac{\sqrt{c}}{2}y')$ of $\mathbf{C}^2$ is induced by the automorphism $[x'_0 : x'_1 : x'_2] \mapsto [x'_0 : x'_1 + \frac{1}{3}\sigma_1 x'_0 : \frac{\sqrt{c}}{2}x'_2]$ of $\mathbf{P}^2(\mathbf{C})$ which maps the curve C_Q to the curve C_P, and the point $o = [0 : 0 : 1]$ to itself. This implies that the pointed curves (C_P, o) and (C_Q, o) are isomorphic. Consequently, (C_P, o) is an elliptic curve.

§ 9. THE MODULI SPACE OF ELLIPTIC CURVES

9.1. One defines the j-invariant $j(\Lambda) \in \mathbf{C}$ of a lattice Λ in $\mathbf{C}$ by the formula

$$j(\Lambda) = 12^3 \frac{g_2(\Lambda)^3}{\Delta(\Lambda)} = 1728 \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27 g_3(\Lambda)^2}. \tag{9.1.1}$$

Recall that $\Delta(\Lambda)$ is the discriminant of the polynomial $4X^3 - g_2(\Lambda)X - g_3(\Lambda)$ whose roots are distinct, hence $j(\Lambda)$ is well-defined as a complex number. For any nonzero complex number $u \in \mathbf{C}^*$, one has

$$j(u\Lambda) = j(\Lambda). \tag{9.1.2}$$

Consequently, the function j descends to a function, still denoted j, on Π:

$$j\colon \Pi \to \mathbf{C}, \quad \tau \mapsto j(\mathbf{Z} + \mathbf{Z}\tau). \tag{9.1.3}$$

It is constant along the orbits of the action of $\mathrm{SL}_2(\mathbf{Z})$ on Π.

Lemma **(9.2)**. — *Let Λ be a lattice in $\mathbf{C}$ and let $(a, b) \in \mathbf{C}^2$ be complex numbers such that $a^3 - 27b^2 \neq 0$. Assume that $j(\Lambda) = 12^3 a^3/(a^3 - 27b^2)$. Then, there exists $u \in \mathbf{C}^*$ such that $g_2(u\Lambda) = a$ and $g_3(u\Lambda) = b$.*

Proof. — Assume first that $j(\Lambda) = 0$, so that $g_2(\Lambda) = a = 0$. Necessarily, $g_3(\Lambda)$ and b are nonzero and there exists $u \in \mathbf{C}^*$ such that $g_3(\Lambda) = u^6 b$. Consequently, $g_3(u\Lambda) = u^{-6} g_3(\Lambda) = b$ and $g_2(u\Lambda) = 0$.

If $j(\Lambda) \neq 0$, then $g_2(\Lambda)$ and a are nonzero too, so that there exists $u \in \mathbf{C}^*$ such that $g_2(\Lambda) = u^4 a$. From the relation $j(u\Lambda) = j(\Lambda)$ and the definition of j, it then follows that $g_3^2(u\Lambda) = b^2$. If $g_3(u\Lambda) = b$, we are done. Otherwise, one has $b = -g_3(u\Lambda) = g_3(iu\Lambda)$ while $a = g_2(u\Lambda) = g_2(iu\Lambda)$. □

Theorem **(9.3)**. — *Let $\mathscr{R}$ be the set of lattices in $\mathbf{C}$. The map (g_2, g_3) from $\mathscr{R}$ to $\mathbf{C}^2$ given by $\Lambda \mapsto (g_2(\Lambda), g_3(\Lambda))$ induces a bijection from $\mathscr{R}$ to the set of points $(a, b) \in \mathbf{C}^2$ such that $a^3 - 27b^2 \neq 0$.*

Remark **(9.4)**. — Let $\mathscr{B}$ be the open subset of $\mathbf{C}^2$ consisting of pairs (ω_1, ω_2) of nonzero complex numbers such that $\omega_2/\omega_1 \notin \mathbf{R}$. Let $\pi\colon \mathscr{B} \to \mathscr{R}$ be the map that associates to a pair $(\omega_1, \omega_2) \in \mathscr{B}$ the lattice $\mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2$ generated by ω_1 and ω_2. One can show that the set $\mathscr{R}$ possesses a unique structure of a complex manifold of dimension 2 for which the map π is a local holomorphic diffeomorphism. When $\mathscr{R}$ is endowed with this structure, the map (g_2, g_3) is actually a biholomorphic diffeomorphism.

Lemma **(9.5)**. — *The functions j are holomorphic on $\mathscr{B}$ and on Π.*

Proof. — We show that for any even integer $k \geq 4$, the function G_k on $\mathscr{B}$ defined by

$$(\omega_1, \omega_2) \mapsto G_k(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2) = \sum_{m \in \mathbf{Z}^2 \setminus \{0\}} \frac{1}{(m_1\omega_1 + m_2\omega_2)^k}$$

is holomorphic. Let S be the set of $(x_1, x_2) \in \mathbf{R}^2$ such that $|x_1| + |x_2| = 1$. The function from $S \times \mathscr{B}$ to $\mathbf{R}$ defined by $(x_1, x_2, \omega_1, \omega_2) \mapsto |x_1\omega_1 + x_2\omega_2|$ is continuous and takes positive values. Therefore, it has a positive lower bound on any compact subset. Since S is compact, for any compact subset K of $\mathscr{B}$, there exists a positive real number C_K such that

$$|m_1\omega_1 + m_2\omega_2| \geq C_K \min(|m_1|, |m_2|)$$

for any $(m_1, m_2) \in \mathbf{Z}^2$. This implies that the series defining G_k converges uniformly on any compact subset of $\mathscr{B}$, so defines a holomorphic function on $\mathscr{B}$.

Moreover, we have seen that for any lattice Λ, $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \neq 0$. Therefore, the function $(\omega_1, \omega_2) \mapsto j(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2)$ on $\mathscr{B}$ is also holomorphic.

In particular, the function $\tau \mapsto j(\mathbf{Z} \oplus \mathbf{Z}\tau)$ on Π is holomorphic. □

Theorem (**9.6**). — *The map $j\colon \mathrm{PSL}_2(\mathbf{Z})\backslash\Pi \to \mathbf{C}$ is bijective.*

Remark (**9.7**). — This map is not a local homeomorphism at the points of Π for which the action of $\mathrm{SL}_2(\mathbf{Z})$ has fixed points. These points of Π correspond to lattices with more symmetries than usual.

9.8. *Injectivity.* — Let us first show that the map (g_2, g_3) from Theorem 9.3 is injective. Let Λ and Λ' be lattices in $\mathbf{C}$ such that $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$. Let $f\colon \mathbf{C}/\Lambda \to \mathbf{C}/\Lambda'$ be the map given by $i_{\Lambda'} \circ f = i_\Lambda$. It is an isomorphism of Riemann surfaces. Since $i_\Lambda(0) = i_{\Lambda'}(0) = [0 : 0 : 1]$, f maps 0 to 0. Consequently, there exists a complex number $a \in \mathbf{C}^*$ such that $a\Lambda \subset \Lambda'$ and $f(z) = az$ for any $z \in \mathbf{C}$. Reversing the roles of Λ and Λ', we obtain that $a\Lambda = \Lambda'$ so that the lattices Λ and Λ' are homothetic.

Since $i_\Lambda(0) = [0 : 0 : 1]$, the image of a neighborhood of 0 in $\mathbf{C}/\Lambda$ is contained in in the open subset of $\mathbf{P}^2(\mathbf{C})$ where $x_2 \neq 0$. There, i_Λ is expressed as the map $z \mapsto \tilde{i}_\Lambda(\wp'_\Lambda(z)^{-1}, \wp_\Lambda(z)\wp'_\Lambda(z)^{-1})$. Since

$$\wp_\Lambda(z) = \frac{1}{z^2} + \mathrm{O}(1), \quad \wp'_\Lambda(z) = \frac{-2}{z^3} + \mathrm{O}(1),$$

one has $\tilde{i}_\Lambda(z) = (\mathrm{O}(z^3), -\frac{1}{2}z + \mathrm{O}(z))$ and the differential of $\tilde{i}_\Lambda$ at the origin is given by $\tilde{i}'_\Lambda(0) = (0, -\frac{1}{2})$. The same result holds for the differential of $\tilde{i}_{\Lambda'}$. Since the differential of f at 0 is the multiplication by a, we obtain $a = 1$, and $\Lambda = \Lambda'$.

This implies that the map j is injective too. Let indeed $\tau, \tau' \in \Pi$ be such that $j(\tau) = j(\tau')$ and let us show that the lattices $\Lambda = \mathbf{Z} + \mathbf{Z}\tau$ and $\Lambda' = \mathbf{Z} + \mathbf{Z}\tau'$ are homothetic. By Lemma 9.2 applied to the lattice Λ and to $a = g_2(\Lambda')$, $b = g_3(\Lambda')$, there exists $u \in \mathbf{C}^*$ such that $g_2(u\Lambda) = g_2(\Lambda')$ and $g_3(u\Lambda) = g_3(\Lambda')$. This implies that the lattices Λ' and $u\Lambda$ are equal, so that the lattices Λ and Λ' are homothetic.

Remark (**9.9**). — Prove the injectivity by showing how to recover the period lattice through elliptic integrals.

Lemma (**9.10**). — *Let $\mathfrak{F}$ be the fundamental domain of* Π. *One has*

$$\lim_{\substack{\Im(\tau)\to\infty \\ \tau\in\Pi}} |j(\tau)| = +\infty.$$

Proof. — Since $j(\tau + n) = j(\tau)$ for any $\tau \in \Pi$ and any $n \in \mathbf{Z}$, it suffices to prove this limit formula under the assumption that $-\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}$. We may thus suppose that τ belongs to the fundamental domain $\mathfrak{F}$.

For given $(x, y) \in \mathbf{R}^2$ and $\tau \in \Pi$ of given real part, $|x + y\tau|$ is an increasing function of $\mathfrak{I}(\tau)$, so that

$$|x + y\tau| \geq |x + \mathfrak{R}(\tau)y| + \frac{\sqrt{3}}{2}|y|.$$

Then, $|x + \mathfrak{R}(\tau)y| \geq |x| - \frac{1}{2}|y|$, hence

$$|x + y\tau| \geq |x| + \frac{\sqrt{3}-1}{2}|y| \geq \frac{\sqrt{3}-1}{2}(|x| + |y|). \tag{9.10.1}$$

This inequality implies that for $k \geq 3$, the convergence of the series

$$G_k(\mathbf{Z} + \tau\mathbf{Z}) = \sum_{(m,n)\in\mathbf{Z}^2\setminus\{0\}} \frac{1}{(m + n\tau)^k}$$

is *uniform* for $\tau \in \mathfrak{F}$. Therefore, we can pass to the limit termwise. All terms with $n \neq 0$ converge to 0 when $\tau \to \infty$. As a consequence, for any integer $k \geq 3$,

$$\lim_{\substack{|\tau|\to\infty \\ \tau\in\mathfrak{F}}} G_k(\mathbf{Z} + \tau\mathbf{Z}) = \sum_{m\in\mathbf{Z}\setminus\{0\}} \frac{1}{m^K} = 2\zeta(k),$$

where ζ is Riemann's zeta function.

In particular, the numerator of $j(\tau)$ has a finite positive limit when $\tau \to \infty$ in $\mathfrak{F}$.

It is known that $\zeta(4) = \pi^4/90$ and $\zeta(6) = \pi^6/945$. Consequently, when $|\tau| \to \infty$, while $\tau \in \mathfrak{F}$, $g_2(\mathbf{Z} + \mathbf{Z}\tau) = 60G_4(\mathbf{Z} + \mathbf{Z}\tau) \to \frac{4}{3}\pi^4$ and $g_3(\mathbf{Z} + \mathbf{Z}\tau) = 140G_6(\mathbf{Z} + \mathbf{Z}\tau) \to \frac{8}{27}\pi^6$. Consequently, $\Delta(\tau)$ converges to

$$\left(\frac{2^6}{3^3} - 27\frac{2^6}{3^6}\right)\pi^{12} = 0.$$

Finally, $|j(\tau)| \to \infty$.

Using that $j(\tau + n) = j(\tau)$ for $\tau \in \Pi$ and $n \in \mathbf{Z}$, we deduce the slightly stronger result that

$$\lim_{\mathfrak{I}(\tau)\to+\infty} |j(\tau)| = +\infty.$$

□

Remark **(9.11)**. — Let us give an alternative argument which does not make use of the computation of the values of Riemann's zeta function at even integers.

Let $\Lambda = \mathbf{Z} + \mathbf{Z}\tau$. One has $\Delta(\tau) = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_1 - e_3)^2$, where $e_1 = \wp_\Lambda(1/2)$, $e_2 = \wp_\Lambda(\tau/2)$ and $e_3 = \wp_\Lambda((1+\tau)/2)$. When $\tau \in \mathfrak{F}$,

$$e_1 = 4 + \sum_{(m,n)\in\mathbf{Z}^2\setminus 0} \left(\frac{1}{(m + n\tau + \frac{1}{2})^2} - \frac{1}{(m + n\tau)^2}\right)$$

remains bounded. Moreover, when $|\tau| \to \infty$,

$$e_2 = \frac{4}{\tau^2} + + \sum_{(m,n)\in\mathbf{Z}^2\setminus 0} \left(\frac{1}{(m + n\tau + \frac{1}{2}\tau)^2} - \frac{1}{(m + n\tau)^2}\right)$$

converges to $-2\zeta(2)$. Indeed, the convergence is uniform, so that we can compute the limit termwise. The term of index (m,n) goes to 0 if $n \neq 0$, and to $-1/m^2$ if $n = 0$ and $m \neq 0$. By the same argument,

$$e_3 = \frac{4}{(1+\tau)^2} + + \sum_{(m,n)\in\mathbf{Z}^2\smallsetminus 0} \left(\frac{1}{(m+n\tau+\frac{1}{2}(1+\tau))^2} - \frac{1}{(m+n\tau)^2} \right)$$

converges to $-2\zeta(2)$ too. This implies that $\Delta(\tau) \to 0$, hence $|j(\tau)| \to \infty$ as before.

9.12. *Surjectivity.* — We now prove that the map j is surjective. It is holomorphic and non constant, so its image is a connected open subset of $\mathbf{C}$. Let us show that $j(\Pi)$ is closed. Since j is invariant under the action of $\mathrm{SL}_2(\mathbf{Z})$, one has $j(\Pi) = j(\mathfrak{F})$. Let (τ_n) be a sequence of elements of $\mathfrak{F}$ such that $j(\tau_n)$ converges to a complex number z; we need to show that z belongs to the image of j. Up to passing to a subsequence, we may assume that either τ_n converges an element $\tau \in \mathfrak{F}$, or that $|\tau_n| \to \infty$. In the first case, we get $j(\tau) = z$, so that $z \in j(\Pi)$. The second case cannot happen, since the preceding lemma shows that $|j(\tau_n)| \to \infty$. Since the complex plane is connected, $j(\Pi) = \mathbf{C}$, as claimed.

It remains to show that the map (g_2, g_3) of Theorem 9.3 is surjective. Let $(a,b) \in \mathbf{C}^2$ be two complex numbers such that $a^3 - 27b^2 \neq 0$. Let $\tau \in \Pi$ be any element such that

$$j(\tau) = 12^3 \frac{g_2(\mathbf{Z}+\mathbf{Z}\tau)^3}{g_2(\mathbf{Z}+\mathbf{Z}\tau)^3 - 27g_3(\mathbf{Z}+\mathbf{Z}\tau)^2} = 12^3 \frac{a^3}{a^3 - 27b^2}.$$

Lemma 9.2 above shows that there exists $u \in \mathbf{C}^*$ such that $g_2(u(\mathbf{Z}+\mathbf{Z}\tau)) = a$ and $g_3(u(\mathbf{Z}+\mathbf{Z}\tau)) = b$. This concludes the proof of Theorems 9.3 and 9.6

9.13. To conclude this section, let us give some examples of computations of j-invariants.

For $\tau = i$, $\Lambda = \mathbf{Z} + i\mathbf{Z}$, one has $i\Lambda = \Lambda$, so that $g_3(\Lambda) = i^{-6}g_3(\Lambda) = -g_3(\Lambda)$. Consequently, $g_3(\Lambda) = 0$ and $j(i) = 12^3 = 1728$.

Let $\rho = \exp(i\pi/3) = (1+i\sqrt{3})/2$; since $\rho^2 = (-1+i\sqrt{3})/2 = \rho - 1$, one has $\rho\Lambda = \Lambda$. This implies that $g_2(\Lambda) = \zeta^{-4}g_2(\Lambda)$, hence $g_2(\Lambda) = 0$ since $\zeta^4 = -\zeta \neq 1$. Finally, $j(\rho) = 0$.

For $\lambda \in \mathbf{C} \smallsetminus \{0,1\}$, let E_λ be the elliptic curve with affine equation $y^2 = x(x-1)(x-\lambda)$. It is called the *Legendre elliptic curve* with parameter λ. Let Λ be the lattice corresponding to the curve E_λ. With the notation of (8.4), $\sigma_1 = 1+\lambda$ and $\sigma_2 = \lambda$. As we have shown,

(9.13.1)
$$g_2(\Lambda) = -4\lambda + \frac{4}{3}(1+\lambda)^2 = \frac{4}{3}(1-\lambda+\lambda^2)$$

and

(9.13.2)
$$\Delta(\Lambda) = 16\lambda^2(1-\lambda)^2.$$

Consequently,

(9.13.3)
$$j(\Lambda) = 2^8 \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda^2)}.$$

§ 10. ELLIPTIC CURVES WITH A STRUCTURE OF LEVEL 2

10.1. Let E be an elliptic curve. Since it is an Abelian group, multiplication by 2 is an endomorphism; let E_2 be its kernel. If Λ is a lattice in $\mathbf{C}$ and $E = C/\Lambda$, then $E_2 = (\frac{1}{2}\Lambda)/\Lambda$. Let (ω_1, ω_2) be a basis of E; then $E_2 = \{0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2\}$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$.

Definition **(10.2)**. — *A 2-marking of an elliptic curve E is an isomorphism of groups $\eta\colon (\mathbf{Z}/2\mathbf{Z})^2 \to E_2$.*

We shall sometimes identify a 2-marking with the two points of order 2, $\eta(1,0)$ and $\eta(0,1)$, of E. Conversely any ordered pair of distinct points E_2 defines a marking. Since there are three points of order 2, it follows that an elliptic curve admits exactly 6 distinct 2-markings. More precisely, the group $\mathrm{SL}_2(\mathbf{F}_2) = \mathrm{Aut}((\mathbf{Z}/2\mathbf{Z})^2)$ acts (by composition on the left, hence on the right) on the set of all 2-markings and makes it a principal homogeneous space.

Lemma **(10.3)**. — *Let $(E,(P_1,P_2))$ be an elliptic curve with a 2-marking. An automorphism φ of E respects the 2-marking if and only if $\varphi = \mathrm{Id}$ or $\varphi = -\mathrm{Id}$.*

Proof. — Let φ be an automorphism of E such that $\varphi(P_i) = P_i$ for every i. We may assume that $E = \mathbf{C}/\Lambda$; then Λ has a basis (ω_1,ω_2) such that $P_1 = \omega_1/2 \pmod{\Lambda}$ and $P_2 = \omega_2/2 \pmod{\Lambda}$. There exists a complex number $u \in \mathbf{C}^*$ such that $u\Lambda = \Lambda$ and such that the automorphism φ is given by $\varphi(z) = uz \pmod{\Lambda}$. Necessarily, $|u| = 1$.

Assume that $u \neq \pm 1$ and let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ be such that

$$u(\omega_1,\omega_2) = g\cdot(\omega_1,\omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2).$$

We see that u is an eigenvalue of the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, with eigenvector $\left(\begin{smallmatrix} \omega_1 \\ \omega_2 \end{smallmatrix}\right)$, the other eigenvalue being u^{-1} since $g \in \mathrm{SL}_2(\mathbf{Z})$. Consequently, $|a+d| \leq |u+u^{-1}| \leq 2$; since $u \notin \mathbf{R}$, the last inequality is strict, so that $|a+d| < 2$.

Since $\varphi(P_1) = P_1$, one has $u\omega_1/2 \equiv \omega_1/2 \pmod{\Lambda}$ and $u\omega_2/2 \equiv \omega_2/2 \pmod{\Lambda}$. Consequently, $u\omega_i \equiv \omega_i \pmod{2\Lambda}$. This implies that $a, d \equiv 1 \pmod{2}$ and $b, c \equiv 0 \pmod{2}$. On particular, $a+d \equiv 2 \pmod{4}$. Combined with the inequality $|a+d| < 2$, we obtain $a+d = 0$. Since $ad - bc = 1$, we get $a^2 = -1 - bc \equiv -1 \pmod{4}$, which is absurd. So $u \in \{\pm 1\}$, and $\varphi \in \{\pm \mathrm{Id}\}$. ☐

10.4. Let $\Gamma = \mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\{\pm\mathrm{Id}\}$. Since $-\mathrm{Id} \equiv \mathrm{Id} \pmod{2}$, there exists a morphism of groups $\Gamma \to \mathrm{SL}_2(\mathbf{F}_2)$ which maps the class of a matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ to its reduction modulo 2. The kernel of this morphism is denoted $\Gamma(2)$.

Lemma **(10.5)**. — *Let $\tau, \tau' \in \Pi$. The 2-marked elliptic curves $(\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z}), 1/2, \tau/2)$ and $(\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z}), 1/2, \tau/2)$ are isomorphic if and only if $\tau' \in \Gamma(2)\cdot\tau$. If this holds, there exists a unique $\gamma \in \Gamma(2)$ such that $\tau' = \gamma\cdot\tau$.*

Proof. — Let Λ and Λ' be the lattices $\mathbf{Z} + \mathbf{Z}\tau$ and $\mathbf{Z} + \mathbf{Z}\tau'$. Let φ be an isomorphism from $\mathbf{C}/\Lambda$ to $\mathbf{C}/\Lambda'$. There exists $u \in \mathbf{C}^*$ such that $u\Lambda = \Lambda'$ and such that $\varphi(z \pmod{\Lambda}) = uz \pmod{\Lambda'}$ for any $z \in \mathbf{C}$. For such an isomorphism to exist, it is necessary and sufficient that there exists $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$, such that $u = a + b\tau'$ and $u\tau = c + d\tau'$. In that case, one has $\tau = \frac{c+d\tau'}{a+b\tau'} = g\cdot\tau'$. Moreover, such an isomorphism respects the 2-markings if and only if $u = 1 \pmod{2\Lambda'}$ and $u\tau = \tau' \pmod{2\Lambda'}$. These congruences are equivalent to the fact $a, d \equiv 1 \pmod 2$ and $b, c \equiv 0 \pmod 2$, or that $g \in \Gamma(2)$.

To show the uniqueness, it suffices to prove that the action of $\Gamma(2)$ on Π is free. So let $\tau \in \Pi$, let $\Lambda = \mathbf{Z} + \mathbf{Z}\tau$, and let $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ a matrix in $\mathrm{SL}_2(\mathbf{Z})$ which is congruent to the identity modulo 2 and such that $\tau = g\cdot\tau$.

As above, g defines an automorphism $\varphi\colon z \mapsto (a+b\tau)z \pmod\Lambda$ of $\mathbf{C}/\Lambda$, and this automorphism respects the 2-marking $(1/2, \tau/2)$. By Lemma 10.3, $\varphi = \mathrm{Id}$ or $\varphi = -\mathrm{Id}$, that is, $a + b\tau \in \{1, -1\}$. This implies that $a \in \{\pm 1\}$ and $b = 0$. Finally, $\tau = g\cdot\tau = (c+d\tau)/a$, so that $d = a$ and $c = 0$. Therefore, $g = a\,\mathrm{Id}$ and its class in $\Gamma(2)$ is the neutral element. ☐

10.6. Let $\lambda \in \mathbf{C} \smallsetminus \{0,1\}$, and let E_λ be the Legendre elliptic curve with parameter λ. This is the curve with affine equation $y^2 = x(x-1)(x-\lambda)$, whose points of order 2 are the three points $[1:0:0]$, $[1:1:0]$ and $[1:\lambda:0]$ of $\mathbf{P}^2(\mathbf{C})$. We shall write $\tilde{E}_\lambda$ for the elliptic curve E_λ endowed with the 2-marking $([1:1:0],[1:0:0])$.

Proposition (**10.7**). — *For any elliptic curve E with a 2-marking (p,q), there exists a unique $\lambda \in \mathbf{C}\smallsetminus\{0,1\}$ such that (E,p,q) is isomorphic to $\tilde{E}_\lambda$.*

Proof. — Let Λ be the lattice such that $E \simeq \mathbf{C}/\Lambda$. There exists a unique homography φ fixing ∞ such that $\varphi(\wp_\Lambda(p)) = 1$ and $\varphi(\wp_\Lambda(q)) = 0$; it is given by

$$\varphi(t) = \frac{t - wp_\Lambda(q)}{\wp_\Lambda(p) - \wp_\Lambda(q)}.$$

Besides p and q, the third point of order 2 of E is $p+q$; set

(10.7.1) $$\mathscr{L}(E,p,q) = \varphi(\wp_\Lambda(p+q)) = \frac{\wp_\Lambda(p+q) - \wp_\Lambda(q)}{\wp_\Lambda(p) - \wp_\Lambda(q)}.$$

I claim that $\mathscr{L}(E,p,q)$ is the unique complex number $\lambda \in \mathbf{C}\smallsetminus\{0,1\}$ such that (E,p,q) is isomorphic to $\tilde{E}_\lambda$. ☐

10.8. Let $\lambda\colon \Pi \to \mathbf{C}\smallsetminus\{0,1\}$ be the map given by

(10.8.1) $$\lambda(\tau) = \mathscr{L}(\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau); 1/2, \tau/2) = \frac{\wp_{\mathbf{Z}+\mathbf{Z}\tau}(\frac{1+\tau}{2}) - \wp_{\mathbf{Z}+\mathbf{Z}\tau}(\frac{\tau}{2})}{\wp_{\mathbf{Z}+\mathbf{Z}\tau}(\frac{1}{2}) - \wp_{\mathbf{Z}+\mathbf{Z}\tau}(\frac{\tau}{2})}.$$

It is holomorphic, surjective, and invariant under the action of $\Gamma(2)$ on Π. Moreover, the induced map $\lambda\colon \Gamma(2)\backslash\Pi \to \mathbf{C}\smallsetminus\{0,1\}$ is bijective.

Observe also that the quotient map $\Pi \to \Gamma(2)\backslash\Pi$ is a covering, because $\Gamma(2)$ acts freely on Π. There exists a unique structure of a Riemann surface on $\Gamma(2)\backslash\Pi$ such that this quotient map is a local biholomorphic diffeomorphism. Then, the map $\lambda\colon\Gamma(2)\backslash\Pi \to \mathbf{C}\smallsetminus\{0,1\}$ is an isomorphism of Riemann surfaces.

Lemma (**10.9**). — *For any $\tau \in \Pi$, one has*

(10.9.1) $$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \quad \lambda(-\frac{1}{\tau}) = 1 - \lambda(\tau).$$

Proof. — By definition, $\lambda(\tau)$ is the Legendre invariant $\mathscr{L}(\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau); 1/2, \tau/2)$ of the elliptic curve $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$, endowed with the 2-marking $(1/2, \tau/2)$. Let $\Lambda = \mathbf{Z}+\mathbf{Z}\tau$ and write $e_1 = \wp_\Lambda(1/2)$, $e_2 = \wp_\Lambda(\tau/2)$ and $e_3 = \wp_\Lambda((1+\tau)/2)$. According to Equation (10.7.1),

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2},$$

so that

$$e_3 = e_2 + (e_1 - e_2)\lambda(\tau).$$

Since $\mathbf{Z} + \mathbf{Z}(\tau+1) = \Lambda$, $\lambda(\tau+1)$ is the Legendre invariant of the same elliptic curve, but with the 2-marking $(1/2, (1+\tau)/2)$. By Equation (10.7.1),

$$\lambda(\tau+1) = \frac{e_2 - e_3}{e_1 - e_3} = \frac{-(e_1 - e_2)\lambda(\tau)}{(e_1 - e_2) - (e_1 - e_2)\lambda(\tau)} = \frac{-\lambda(\tau)}{1 - \lambda(\tau)}.$$

One has $\mathbf{Z} + (-1/\tau)\mathbf{Z} = \frac{1}{\tau}\Lambda$, so that $\lambda(-1/\tau)$ is the Legendre invariant of the elliptic curve $\mathbf{C}/\tau^{-1}\Lambda$ with the 2-marking $(1/2, -1/2\tau)$, which is isomorphic to the elliptic curve $\mathbf{C}/\Lambda$ with the 2-marking $(\tau/2, -1/2)$. Therefore,

$$\lambda(-1/\tau) = \frac{e_3 - e_1}{e_2 - e_1} = \frac{(e_2 - e_1) + (e_1 - e_2)\lambda(\tau)}{e_2 - e_1} = 1 - \lambda(\tau).$$

□

10.10. Let $\mathrm{PGL}_2(\mathbf{C})$ act on $\mathbf{P}^1(\mathbf{C})$ by homographies and let G be the stabilizer of $\{0, 1, \infty\}$. It follows from the preceding lemma that there is a unique isomorphism $\iota\colon \Gamma/\Gamma(2) \simeq G$ such that, for any $\tau \in \Pi$ and any $\gamma \in \Gamma$,

$$\lambda(\gamma(\tau)) = \iota(\gamma)(\lambda(\tau)).$$

In fact, for $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, one has $\iota(\gamma)(z) = z/(z-1)$, while for $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, one has $\iota(\gamma)(z) = 1-z$.

Lemma **(10.11)**. — *One has*

(10.11.1)
$$\lim_{\Im(\tau)\to+\infty} \lambda(\tau) = 0.$$

Proof. — We have proved that $|j(\tau)| \to \infty$ when $\Im(\tau) \to +\infty$. According to Equation (9.13.3), the only limit values of $\lambda(\tau)$ belong to $\{0, 1, \infty\}$. a connectedness argument would then show that $\lambda(\tau)$ has a limit for $\Im(\tau) \to +\infty$. To determine the value of this limit, let us return to the notation of Section (9.11). The computations there show that when $\Im(\tau) \to \infty$, while $\tau \in \mathfrak{F}$, then e_1, e_2, e_3 have finite limits, and that the limits of e_2 and e_3 coincide. This implies that $\lambda(\tau) \to 0$, when $\tau \in \mathfrak{F}$ and $\Im(\tau) \to +\infty$. The lemma follows from that, using that $\lambda(\tau+1) = -\lambda(\tau)/(1-\lambda(\tau))$, and $\lambda(\tau+2) = \lambda(\tau)$ for any $\tau \in \Pi$. □

§ 11. THE THEOREMS OF PICARD

Theorem **(11.1)** ("Little Picard Theorem"). — *Let $f\colon \mathbf{C} \to \mathbf{C}$ be an entire function. If f is not constant, then f omits at most one value.*

Proof. — Assume that f omits two values; replacing f by $af + b$ for suitable complex numbers a and b, we may suppose that these two omitted values are 0 and 1. The map $\lambda\colon \Pi \to \mathbf{C} \smallsetminus \{0,1\}$ is a holomorphic covering map. Since $\mathbf{C}$ is simply connected, there exists a holomorphic map $\tilde{f}\colon \mathbf{C} \to \Pi$ such that $f = \lambda \circ \tilde{f}$. However, Π is biholomorphic to the unit disk $\mathbf{D}$, for example through the map $b\colon \tau \mapsto \frac{\tau - i}{\tau + i}$. Then, $b \circ \tilde{f}$ is entire and bounded; by Liouville's Theorem, it is constant. It follows that f is constant. □

Theorem **(11.2)** ("Great Picard Theorem"). — *Let* $\dot{\mathbf{D}}(0,1)$ *be the unit disk deprived of* 0 *and let* $f\colon \dot{\mathbf{D}}(0,1) \to \mathbf{C}$ *be a holomorphic function. If f has an essential singularity, then f omits at most one value.*

11.3. Let $\mathrm{e}\colon \Pi \to \dot{\mathbf{D}}(0,1)$ be the map $\tau \mapsto \exp(2\pi i \tau)$. It is a universal covering map, its group of deck transformations is $\mathbf{Z}$, given by the translations by integers in Π. As in the proof of the Little Picard Theorem, there exists a holomorphic map $\tilde{f}\colon \Pi \to \Pi$ such that $\lambda \circ \tilde{f}(\tau) = f(\mathrm{e}(\tau))$ for any $\tau \in \Pi$.

Observe that $\tilde{f}(i+1)$ and $\tilde{f}(i)$ have the same image by λ, namely $f(\mathrm{e}(i))$. Consequently, there is a unique element $\gamma \in \Gamma(2)$ such that $\gamma \cdot \tilde{f}(i+1) = \tilde{f}(i)$. The map $\tau \mapsto \gamma^{-1} \cdot \tilde{f}(\tau+1)$ is another lift of $f \circ \mathrm{e}$ to Π which coincides with $\tilde{f}$ at $\tau = i$. Since λ is a covering map, it must coincide everywhere, so that

(11.3.1) $$\tilde{f}(\tau+1) = \gamma \cdot \tilde{f}(\tau), \quad \text{for any } \tau \in \Pi.$$

Let $\theta\colon \Pi \to \langle\gamma\rangle\backslash\Pi$ be the quotient of Π by the subgroup generated by γ. The holomorphic map λ descends to a holomorphic map $\lambda_\gamma\colon \langle\gamma\rangle\backslash\Pi \to \mathbf{C} \smallsetminus \{0,1\}$ and there exists a holomorphic map $h\colon \dot{\mathbf{D}}(0,1) \to \langle\gamma\rangle\backslash\Pi$ such that $f = \lambda_\gamma \circ h$.

Lemma **(11.4)**. — *Let* $g \in \mathrm{SL}_2(\mathbf{Z})$ *be any element lifting* γ.

If $\gamma = \mathrm{Id}$, *then* $\langle\gamma\rangle\backslash\Pi \simeq \Pi$.

If $|\mathrm{Tr}(g)| > 2$, *then* $\langle\gamma\rangle\backslash\Pi$ *is isomorphic to an open annulus with positive inner radius and finite outer radius.*

Otherwise, $|\mathrm{Tr}(g)| = 2$. *Then, there exists an isomorphism b from* $\langle\gamma\rangle\backslash\Pi$ *to* $\dot{\mathbf{D}}(0,1)$ *such that* $\lambda_\gamma \circ b^{-1}\colon \dot{\mathbf{D}}(0,1) \to \mathbf{C}$ *is meromorphic at* 0.

11.5. From this lemma, we can finish the proof of the Great Picard Theorem.

If $\gamma = \mathrm{Id}$, then composing g with a biholomorphism $b\colon \Pi \to \mathbf{D}(0,1)$, we obtain a meromorphic map $b \circ g\colon \dot{\mathbf{D}}(0,1) \to \mathbf{D}(0,1)$. Necessarily, $b \circ h$ extends to a holomorphic map from $\mathbf{D}(0,1)$ to $\overline{\mathbf{D}(0,1)}$. Moreover, the maximum principle implies that unless $b \circ h$ is constant, $|b(h(0))| < \sup_{\dot{\mathbf{D}}(0,1)} |b \circ h| \leq 1$. In any case, $h(0) \in \mathbf{D}(0,1)$ and f itself extends holomorphically at 0.

If $|\mathrm{Tr}(g)| > 2$, the lemma shows that there exists a biholomorphism $b\colon \langle\gamma\rangle\backslash\Pi \to \mathrm{C}(r,R)$. In particular, $b \circ h$ extends to a holomorphic map from $\mathbf{D}(0,1)$ to the closed annulus $\overline{\mathrm{C}(r,R)}$, for some real numbers r and R such that $R > r > 0$. As above, it follows from the maximum principle that $|h(0)| < R$; by the maximum principle applied to the map $z \mapsto 1/b(h(z))$, we also have $|h(0)| < r$. This shows that h extends to a holomorphic map from $\mathbf{D}(0,1)$ to $\gamma\rangle\backslash\Pi$, hence f extends holomorphically at 0.

In the remaining case, let us show that f is meromorphic. As in the first two cases, the holomorphic map $b \circ h \colon \dot{\mathbf{D}}(0,1) \colon \dot{\mathbf{D}}(0,1) \to \dot{\mathbf{D}}(0,1)$ extends to a holomorphic map from $\mathbf{D}(0,1)$ to $\mathbf{D}(0,1)$. Since $\lambda_\gamma \circ b^{-1}$ is meromorphic at 0, the formula

$$f = \lambda_\gamma \circ h = (\lambda_\gamma \circ b^{-1}) \circ (b \circ h)$$

implies that f is meromorphic, as claimed.

This concludes the proof of the Great Picard Theorem, but it remains to prove Lemma 11.4.

11.6. We first recall the classification of matrices in $\mathrm{SL}_2(\mathbf{R})$, and of the corresponding transformations of the upper half plane. Let $g \in \mathrm{SL}_2(\mathbf{R})$ and let u, v be its eigenvalues; one has $uv = 1$ and $u + v = \mathrm{Tr}(g)$ is a real number. Let γ be the image of g in $\mathrm{PSL}_2(\mathbf{R})$.

If u and v are real and distinct, then g is conjugate to the diagonal matrix $\left(\begin{smallmatrix} 1/u & 0 \\ 0 & u \end{smallmatrix}\right)$, so that the homography of Π it defines is conjugate to the map $\tau \mapsto u^2\tau$. Moreover, since u and $1/u$ have the same sign, $|\mathrm{Tr}(g)| = |u| + |u|^{-1} > 2$. One says that g, or γ, is *hyperbolic*.

Assume that $u = v$. Since $u + v$ is a real number and $uv = 1$, if comes $u = v = \pm 1$ and $|\mathrm{Tr}(g)| = 2$. Observe that g is semisimple, if and only if $g = \pm\,\mathrm{Id}$, if and only if the corresponding homography is the identity. Otherwise, g is conjugate to a matrix $\pm\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and the corresponding homography is conjugate to the map $\tau \mapsto \tau + 1$. One says that g (or γ) is *parabolic*.

Finally, assume that u and v are non-real, and distinct. Then $v = \overline{u} = u^{-1}$, so that $|u| = |v| = 1$. Then, $|\mathrm{Tr}(g)| = |u + v| < 2$ and one says that g and γ are *elliptic*. In fact, this case will not happen in the discussion below. Indeed, if one assumes, moreover, that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of $\mathrm{SL}_2(\mathbf{Z})$, congruent the identity modulo 2, then $a + d$ is an even integer, so that $a = -d$. It follows that $1 = ad - bc \equiv -a^2 \pmod 4$, a contradiction.

11.7. We now prove Lemma 11.4. Since $\gamma \in \Gamma(2)$, the classification of elements in $\mathrm{SL}_2(\mathbf{R})$ shows that we need to treat the cases where $\gamma = \pm\,\mathrm{Id}$, γ is hyperbolic, or γ is parabolic. There is nothing to do if $\gamma = \pm\,\mathrm{Id}$, so assume that γ is hyperbolic. Up to conjugation, we may suppose that γ induces the automorphism $\tau \mapsto u\tau$ of Π, for some real number $u > 1$. Let $\log \colon \mathbf{C} \smallsetminus \mathbf{R}_{\leq 0} \to \mathbf{C}$ be the principal determination of logarithm. It induces a biholomorphic map, still denoted log, from Π to the band $\mathrm{B} = \{0 < \mathfrak{I}(z) < \pi\}$ in $\mathbf{C}$. Moreover, since $\log(u\tau) = \log(u) + \log(\tau)$, the action of γ on Π is conjugate to the translation t by $\log(u)$ on B, so that $\mathrm{B}/\langle t\rangle$ is biholomorphic with $\langle\gamma\rangle\backslash\Pi$. The map $z \mapsto \exp(2\pi i z/u)$ is an injective holomorphic from $\mathrm{B}/\langle t\rangle$ to $\mathbf{C}$, whose image is the annulus $\mathrm{C}(\exp(-2\pi^2/u), 1)$.

11.8. Finally, let us treat the case where γ is parabolic. We have seen that g is conjugate to a matrix $\pm\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$. However, since $g \in \mathrm{SL}_2(\mathbf{Z})$, $g \neq \pm\,\mathrm{Id}$, there exists $g_0 \in \mathrm{SL}_2(\mathbf{Z})$ and a positive integer n such that $g = \pm g_0 \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) g_0^{-1}$, so that γ acts by $\tau \mapsto g_0 \cdot (g_0^{-1} \cdot \tau + n)$. Therefore, the map $b \colon \tau \mapsto \exp(2\pi i (g_0^{-1} \cdot \tau)/n)$ from Π to $\mathbf{C}^*$ induces a biholomorphic map b_γ from $\langle\gamma\rangle\backslash\Pi$ to $\dot{\mathbf{D}}(0,1)$.

Then, there is a unique holomorphic map $v \colon \dot{\mathbf{D}}(0,1) \to \mathbf{C} \smallsetminus \{0,1\}$ such that $\lambda(\tau) = v \circ b(\tau)$ for any $\tau \in \Pi$. We need to show that v is meromorphic at 0.

We have seen that there is a homography $\varphi_0 \in \mathrm{PGL}_2(\mathbf{C})$ such that $\varphi_0(\{0,1,\infty\}) = \varphi_0(\{0,1,1\infty\})$ and $\lambda(g_0 \cdot \tau) = \varphi_0(\lambda(\tau))$.

Let $q \in \dot{\mathbf{D}}(0,1)$; write $q = b(g_0 \cdot \tau)$, for $\tau \in \Pi$; one has $|q| = \exp(-2\pi\Im(\tau)/n)$ and

$$\nu(q) = \lambda(g_0^{-1} \cdot \tau) = \varphi_0(\lambda(\tau)).$$

When $q \to 0$, $\Im(\tau) \to +\infty$, hence $\lambda(\tau) \to 0$. Since φ_0 is a rational function such that $\varphi_0(0) \in \{0, 1, \infty\}$, $\nu(q)$ converges to $\varphi_0(0)$. In particular, ν is meromorphic at the origin. The precise nature of ν, whether it extends holomorphically, or whether it has a pole, depends on the actual value of $\varphi_0(0)$.

CHAPTER 2

PRELIMINARIES FROM COMPLEX GEOMETRY

This chapter gathers material from differential geometry and complex analysis that will be used in the sequel. The reader may either read it first, or only when needed.

§ 1. DIFFERENTIAL CALCULUS ON COMPLEX MANIFOLDS

1.1. As usual, a complex number $z = x + iy$ can be viewed as the pair (x, y) consisting of its real and its imaginary parts, giving an identification of the complex line **C** with the real plane $\mathbf{R}^2$. This furnishes equalities of differential forms on **C**:

(1.1.1) $$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y, \quad \mathrm{d}\overline{z} = \mathrm{d}x - i\mathrm{d}y,$$

which compares the two bases $(\mathrm{d}z, \mathrm{d}\overline{z})$ and $(\mathrm{d}x, \mathrm{d}y)$ of the complex 2-dimensional vector space $\mathrm{Hom}_{\mathbf{R}}(\mathbf{C}, \mathbf{C})$.

For any complex valued differentiable function f on Let Ω be an open subset Ω of **C** and let $f\colon \Omega \to \mathbf{C}$ be a complex-valued differentiable function. For any $a \in \Omega$, the differential $\mathrm{d}f_a$ is a **R**-linear map $\mathbf{C} \to \mathbf{C}$ given in the two given bases by:

$$\mathrm{d}f_a = \frac{\partial}{\partial z} f(a)\mathrm{d}z + \frac{\partial}{\partial \overline{z}} f(a)\mathrm{d}\overline{z} = \frac{\partial}{\partial x} f(a)\mathrm{d}x + \frac{\partial}{\partial y} f(a)\mathrm{d}y,$$

so that

(1.1.2) $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If one defines

(1.1.3) $$\partial f = \frac{\partial}{\partial z} f(a)\mathrm{d}z, \quad \overline{\partial} f = \frac{\partial}{\partial \overline{z}} f(a)\mathrm{d}\overline{z},$$

then

(1.1.4) $$\mathrm{d}f_a = \partial f + \overline{\partial} f$$

is the decomposition of the **R**-linear map $\mathrm{d}f_a \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{C},\mathbf{C})$ into a **C**-linear part ∂f and a **C**-antilinear part $\overline{\partial} f$. Observe also that

(1.1.5) $$\partial\overline{f} = \overline{\overline{\partial} f}, \quad \overline{\partial f} = \overline{\overline{f}}.$$

Moreover, one has the following formula for the 2-form giving the area element:

(1.1.6) $$\mathrm{d}x \wedge \mathrm{d}y = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

1.2. These decompositions extend to the higher dimensional case. Let d be a positive integer. We can identify a point $(z_1,\dots,z_d)\mathbf{C}^d$ with the point $(x_1,y_1,\dots,x_d,y_d) \in (\mathbf{R}^2)^d = \mathbf{R}^{2d}$, where $z_k = x_k + ix_k$ for every $k \in \{1,\dots,d\}$.

1.3. Let X be a complex manifold of dimension d. Let $\mathscr{A}^n(X)$ be the space of complex-valued smooth differential forms of degree n on X, let $\mathscr{A}^n_{\mathrm{c}}(X)$ be the subspace of forms with compact support.

Let α be a differential form of degree 1 on X. For any point a on X, α is a **R**-linear map from the tangent space T_aX to **C**. Since X is a complex manifold, T_aX is a complex vector space and α can be decomposed canonically as the sum of a **C**-linear part, and of **C**-antilinear part. This decomposes the space $\mathscr{A}^1(X)$ as the direct sum of two subspaces $\mathscr{A}^{1,0}(X)$ and $A^{0,1}(X)$ consisting of forms which are respectively **C**-linear and **C**-antilinear at each point of X.

For any integer n, the space $\mathscr{A}^n(X)$ has a similar decomposition

(1.3.1) $$\mathscr{A}^n(X) = \bigoplus_{p+q=n} A^{p,q}(X),$$

where a form $\alpha \in \mathscr{A}^n(X)$ belongs to $A^{p,q}(X)$ if and only if it is p-times **C**-linear, and q-times **C**-antilinear, meaning: $\alpha_a(\lambda v) = \lambda^p\overline{\lambda}^q\alpha_a(v)$ for any $v \in T_a(X)$ and any $\lambda \in \mathbf{C}$.

There are differential operators

(1.3.2) $$\partial\colon \mathscr{A}^{p,q}(X) \to A^{p+1,q}(X), \quad \overline{\partial}\colon \mathscr{A}^{p,q}(X) \to A^{p,q+1}(X)$$

such that

(1.3.3) $$\mathrm{d} = \partial + \overline{\partial}.$$

From the relation $\mathrm{d}\circ\mathrm{d} = 0$ and looking at the possible degrees, one gets

(1.3.4) $$\partial\circ\partial = 0, \quad \overline{\partial}\circ\overline{\partial} = 0, \quad \partial\circ\overline{\partial} + \overline{\partial}\circ\partial = 0.$$

Theses decompositions respect the support, hence similar decomposition of $\mathscr{A}^n_{\mathrm{c}}(X)$.

(1.3.1) One defines a differential operator $\mathrm{d^c}$ by the formula

(1.3.5) $$\mathrm{d^c} = \frac{1}{2\pi i}(\partial - \overline{\partial}).$$

Observe that this is a real operator, namely: $\mathrm{d^c}\overline{f} = \overline{\mathrm{d^c} f}$. Moreover, if $f = \Re(u)$ is the real part of a holomorphic function u,

$$\mathrm{d^c} f = \mathrm{d^c}(\Re(u)) = \frac{1}{4\pi i}(\partial - \overline{\partial})(u + \overline{u}) = \frac{1}{4\pi i}(\partial u - \overline{\partial}\overline{u}) = \frac{1}{2\pi}\Im(\partial u)$$

since $\bar\partial u = \partial \bar u = 0$.

The composition $\mathrm{d}\circ\mathrm{d}^{\mathrm{c}}$ of the operators d and d^{c} is of tremendous importance of complex analysis; it satisfies

$$\mathrm{dd}^{\mathrm{c}} = \frac{1}{2\pi i}(\partial+\bar\partial)(\partial-\bar\partial) = \frac{i}{\pi}\partial\bar\partial = -\frac{i}{\pi}\bar\partial\partial. \tag{1.3.6}$$

Proposition **(1.4)**. — *Let X be a complex manifold and let f be an invertible holomorphic function on X. Then* $\mathrm{dd}^{\mathrm{c}}\log|f| = 0$.

Proof. — It suffices to prove the result locally, hence we may assume that X is simply connected. Since the exponential $\mathbf{C}\to\mathbf{C}^*$ is a covering, there exists a holomorphic function h on X such that $f = e^h$. Then,

$$\mathrm{dd}^{\mathrm{c}}\log|f| = \mathrm{dd}^{\mathrm{c}}\Re(h) == \frac{i}{2\pi}\left(\partial\bar\partial h + \partial\bar\partial \bar h\right)\frac{i}{2\pi}\left(\partial\bar\partial h - \bar\partial\partial \bar h\right) = 0.$$

Indeed, h being holomorphic, $\bar\partial h = 0$, and $\partial\bar h = \overline{\bar\partial h} = 0$. □

1.5. Assume that X is an open subset of $\mathbf{C}$. For any $\mathscr{C}^\infty$-function on X, one can write

$$\mathrm{d}^{\mathrm{c}}u = \frac{1}{2\pi}\left(-\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y\right) \tag{1.5.1}$$

so that

$$\mathrm{dd}^{\mathrm{c}}u = \frac{1}{2\pi}\Delta u\,\mathrm{d}x\wedge\mathrm{d}y, \tag{1.5.2}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is the Laplace operator applied to u. In particular, this shows directly that harmonic functions are preserved by a holomorphic changes of variables.

Let u, v be $\mathscr{C}^\infty$-functions on X. Since non-zero 2-forms on X have bidegree $(1,1)$, $\partial u\wedge\partial v = \bar\partial u\wedge\bar\partial v = 0$ and

$$\mathrm{d}u\wedge\mathrm{d}^{\mathrm{c}}v = \frac{i}{2\pi}\left(\partial u\wedge\bar\partial v + \partial v\wedge\bar\partial u\right) \tag{1.5.3}$$

$$= \frac{1}{2\pi}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\mathrm{d}x\wedge\mathrm{d}y \tag{1.5.4}$$

$$= \mathrm{d}v\wedge\mathrm{d}^{\mathrm{c}}u. \tag{1.5.5}$$

In particular, this is a symmetric expression in u and v.

Proposition **(1.6)** (Green formula). — *Let X be a Riemann surface with boundary, let u and v be $\mathscr{C}^2$-functions on X such that* $\operatorname{supp}(u)\cap\operatorname{supp}(v)$ *is compact. Then,*

$$\int_X (u\mathrm{dd}^{\mathrm{c}}v - v\mathrm{dd}^{\mathrm{c}}u) = \int_{\partial X}(u\mathrm{d}^{\mathrm{c}}v - v\mathrm{d}^{\mathrm{c}}u)\,. \tag{1.6.1}$$

Proof. — Since
$$\mathrm{d}u \wedge \mathrm{d^c}v = \mathrm{d}(u\mathrm{d^c}v) - u\mathrm{dd^c}v,$$
Stokes formula
$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega,$$
valid for any $\mathscr{C}^1$-differential form of degree 1 on X, implies
$$\int_X \mathrm{d}u \wedge \mathrm{d^c}v = \int_{\partial X} u\mathrm{d^c}v - \int_X u\mathrm{dd^c}v.$$
Green formula follows by symmetry. □

1.7. It is occasionally useful to express the operators d, $\mathrm{d^c}$, $\mathrm{dd^c}$ in polar coordinates. If $z = x + iy = re^{i\theta}$ we have

(1.7.1) $$\mathrm{d}z = e^{i\theta}(\mathrm{d}r + ir\mathrm{d}\theta), \quad \mathrm{d}\bar{z} = e^{-i\theta}(\mathrm{d}r - ir\mathrm{d}\theta),$$

(1.7.2) $$\mathrm{d}r = \frac{1}{2}(e^{-i\theta}\mathrm{d}z + e^{i\theta}\mathrm{d}\bar{z}), \quad \mathrm{d}\theta = \frac{1}{2ir}(e^{-i\theta}\mathrm{d}z - e^{i\theta}\mathrm{d}\bar{z}), \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z} = r\mathrm{d}r\mathrm{d}\theta.$$

Then, for any differentiable function u on an open subset of $\mathbf{C}^*$,
$$\begin{aligned}\mathrm{d}u &= \frac{\partial u}{\partial r}\mathrm{d}r + \frac{\partial u}{\partial \theta}\mathrm{d}\theta \\ &= \frac{e^{-i\theta}}{2}\left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}+\right)\mathrm{d}z + \frac{e^{i\theta}}{2}\left(\frac{\partial u}{\partial r} + \frac{i}{r}\frac{\partial u}{\partial \theta}\right)\mathrm{d}\theta\end{aligned}$$
so that

(1.7.3) $$\partial u = \frac{1}{2}\left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}+\right)(\mathrm{d}r + ir\mathrm{d}\theta), \quad \bar{\partial} u = \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{i}{r}\frac{\partial u}{\partial \theta}\right)(\mathrm{d}r - ir\mathrm{d}\theta).$$

Finally,

(1.7.4) $$\mathrm{d^c}u = \frac{1}{2\pi i}(\partial u - \bar{\partial} u) = \frac{1}{2\pi}\left(r\frac{\partial u}{\partial r}\mathrm{d}\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\mathrm{d}r\right).$$

and, if u is twice differentiable,

(1.7.5) $$\mathrm{dd^c}u = \frac{i}{\pi}\partial\bar{\partial}u = \frac{1}{2\pi}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)r\mathrm{d}r\mathrm{d}\theta.$$

§ 2. CURRENTS

2.1. Let X be an oriented manifold (everywhere) of dimension d. Currents of degree p on X are continuous linear forms on $\mathscr{A}_{\mathrm{c}}^{d-p}(X)$, where the continuity condition comes from distribution theory: one says that a linear form T on $\mathscr{A}_{\mathrm{c}}^{d-p}(X)$ is continuous if for any compact subset K contained in a coordinate open Ω set of X, there exists an integer k such that for any sequence (α_n) of forms in $\mathscr{A}_{\mathrm{c}}^{d-p}(X)$ whose support is contained in K whose coefficients converge uniformly to 0, as well as their partial derivatives up to order k, $T(\alpha_n)$ converges to 0. (One then says that T has order $\leq k$ on K.)

The space of currents of degree p is denoted $\mathscr{D}^p(X)$.

2.2. *Currents and integration theory.* — Since X is oriented, one can integrate any differential form with compact support of degree d. The map $\alpha \mapsto \int_X \alpha$ is linear, continuous, hence defines a current $[X]$ (of order 0) of degree 0 on X.

More generally, a locally integrable function f on X defines a current $[f]$ of degree 0 on X , by the formula

(2.2.1)
$$[f](\alpha) = \int_X f\alpha, \quad f \in \mathrm{L}^1_{\mathrm{loc}}(X), \quad \alpha \in \mathscr{A}^d_{\mathrm{c}}(X).$$

This gives an injection of $\mathrm{L}^1_{\mathrm{loc}}(X)$ in $\mathscr{D}^0(X)$.

Let μ be a Radon measure on X. By definition, μ is a continuous linear form on the space $\mathscr{C}^0_{\mathrm{c}}(X)$ of continuous functions with compact support on X. It defines naturally a current of degree d. The map $\mathscr{M}(X) \to \mathscr{D}^d(X)$ so defined is injective.

2.3. Let $T \in \mathscr{D}^p(X)$ be a current of degree p on X and let ω be a differential form of degree q. One defines a current $T \wedge \omega$ on X of degree $p+q$ by the formula:

(2.3.1)
$$T \wedge \omega(\alpha) = T(\omega \wedge \alpha), \quad \alpha \in \mathscr{A}^{d-p-q}_{\mathrm{c}}(X).$$

In particular, the map $\alpha \mapsto [\alpha] = [X] \wedge \alpha$ associates to every differential form of degree q a current of degree q. This map is injective.

2.4. Let U be an open subset of X. Any form $\alpha \in \mathscr{A}^{d-p}_{\mathrm{c}}(U)$ can be viewed as a form with compact support on X (whose support is actually contained in U). Therefore, any current T of degree p on X defines a current $T|_U$ of degree p on X, obtained by evaluating T on forms with compact support on U.

As shown be the following lemma, the restriction maps $\mathscr{D}^p(X) \to \mathscr{D}^p(U)$ define a *sheaf* of vector spaces on X.

Lemma (**2.5**). — *Let $(U_i)_{i\in I}$ be an open cover of X; for every $i \in I$, let $T_i \in \mathscr{D}(U_i)$ be a current of degree p on U_i. Assume that for every $i, j \in I$, the currents $T_i|_{U_i\cap U_j}$ and $T_j|_{U_i\cap U_j}$ on $U_i \cap U_j$ coincide. Then, there exists a unique current $T \in \mathscr{D}^p(X)$ such that $T_i = T|_{U_i}$ for every $i \in I$.*

Proof. — Manifolds are assumed to be paracompact, and possess smooth partitions of unity. In other words, there is a family $(\lambda_j)_{j\in J}$ of nonnegative $\mathscr{C}^\infty$-functions on X, and a map $i\colon J \to I$ such that for every $j \in J$, $\operatorname{supp}(\lambda_j) \subset U_{i(j)}$ and such that $\sum_{j\in J} \lambda_j(x) = 1$, the sum being locally finite.

Let $\alpha \in \mathscr{A}^{d-p}_{\mathrm{c}}(X)$. For every $j \in J$, the form $\lambda_j\alpha$ on X is supported by $U_{i(j)}$ and one has $\alpha = \sum_{j\in J} \lambda_j\alpha$; the sum is locally finite but since α has compact support, only finitely many terms are nonzero. Define
$$T(\alpha) = \sum_{j\in J} T_{i(j)}(\lambda_j\alpha).$$
I claim that this is the unique current on X which statisfies the required conditions.

Let us first check uniqueness: if S is any current such that $T_i = S|_{U_i}$ for every $i \in I$, then $S(\lambda_j\alpha) = T_{i(j)}(\alpha)$, since $\operatorname{supp}(\lambda_j\alpha) \subset T_{i(j)}$. Consequently, $S(\alpha) = S(\sum \lambda_j\alpha) = \sum T_{i(j)}(\lambda_j\alpha)$.

One leaves to the reader to check that T is continuous, hence really is a current. Let $i \in I$ and assume that $\operatorname{supp}(\alpha) \subset U_i$. Then, for every $j \in J$, $\operatorname{supp}(\lambda_j \alpha) \subset U_i \cap U_{i(j)}$, so that

$$T_{i(j)}(\lambda_j \alpha) = T_i(\lambda_j \alpha).$$

Consequently,

$$T(\alpha) = \sum_{j \in J} T_{i(j)}(\lambda_j \alpha) = \sum_{j \in J} T_i(\lambda_j \alpha) = T_i(\sum_{j \in J} \lambda_j \alpha) = T_i(\alpha).$$

In other words, $T|_{U_i} = T_i$. □

2.6. Although the exterior product of differential forms is well defined, there is no exterior product of currents in general.

Let T be a current of degree p on X. There is a largest open subset U of X such that $T|_U$ is of the form $[\alpha_U]$, for some form $\alpha_U \in \mathscr{A}^p(U)$. The complement of U is the *singular support* of T; it is closed subset of X, and is denoted $\operatorname{sing\,supp}(T)$.

Let S and T be two currents of degrees p and q on X such that $\operatorname{sing\,supp}(S) \cap \operatorname{sing\,supp}(T) = \varnothing$. Then, one can define the current $S \wedge T$ as follows: let α be a form on $U = X \smallsetminus \operatorname{sing\,supp}(S)$ such that $S|_U = [\alpha]$, let β be a form on $V = X \smallsetminus \operatorname{sing\,supp}(T)$ such that $T|_U = [\beta]$; then $S \wedge T$ is the unique current on X such that

$$(S \wedge T)|_U = [\alpha] \wedge T, \quad (S \wedge T)|_V = S \wedge [\beta].$$

To see that it exists, observe that $U \cup V = X$, and that, denoting $W = U \cap V$, $([\alpha] \wedge T)|_W = [\alpha|_W] \wedge [\beta|_W] = (S \wedge [\beta])|_W$.

2.7. Functoriality f_*, f^*, projection formula.

2.8. *Differential calculus for currents.* — Let T be a current of degree p on X; one defines a current $\mathrm{d}T$ of degree $p+1$ by the formula

(2.8.1) $$\mathrm{d}T(\beta) = (-1)^{p+1} T(\mathrm{d}\beta), \quad \beta \in \mathscr{A}_{\mathrm{c}}^{d-p-1}(X).$$

This definition is compatible with the injection of $\mathscr{A}^p(X)$ in $\mathscr{D}^p(X)$. Indeed, for any $\alpha \in \mathscr{A}^p(X)$ and any $\beta \in \mathscr{A}_{\mathrm{c}}^{d-p-1}(X)$, Stokes's formula implies that

$$\int_X \mathrm{d}(\alpha \wedge \beta) = 0.$$

Since

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta,$$

it comes:

$$\begin{aligned} \mathrm{d}[\alpha](\beta) &= (-1)^{p+1}[\alpha](\mathrm{d}\beta) \\ &= (-1)^{p+1} \int_X \alpha \wedge \mathrm{d}\beta \\ &= -\int_X (\mathrm{d}(\alpha \wedge \beta) - \mathrm{d}\alpha \wedge \beta) \\ &= \int_X \mathrm{d}\alpha \wedge \beta \\ &= [\mathrm{d}\alpha](\beta). \end{aligned}$$

The following formulae follow from the definition and their counterparts for differential forms: for any current T, and any differential form α,

$$\mathrm{d}\circ\mathrm{d}T = 0 \tag{2.8.2}$$

and

$$\mathrm{d}(S\wedge\alpha) = \mathrm{d}S\wedge\alpha + (-1)^{\deg(S)}S\wedge\mathrm{d}\alpha. \tag{2.8.3}$$

Indeed, for any form $\beta\in\mathscr{A}_{\mathrm{c}}^{d-p}(X)$,

$$(\mathrm{d}\circ\mathrm{d}T)(\beta) = (-1)^{\deg(T)}\mathrm{d}T(\beta) = -T(\mathrm{d}\circ\mathrm{d}\beta) = 0,$$

while

$$\begin{aligned}
\mathrm{d}(S\wedge\alpha)(\beta) &= (-1)^{\deg(S)+\deg(\alpha)+1}S\wedge\alpha(\mathrm{d}\beta)\\
&= (-1)^{\deg(S)+\deg(\alpha)+1}S(\alpha\wedge\mathrm{d}\beta)\\
&= (-1)^{\deg(S)+\deg(\alpha)+1}S\big((-1)^{\deg(\alpha)}\mathrm{d}(\alpha\wedge\beta) - (-1)^{\deg(\alpha)}\mathrm{d}\alpha\wedge\beta\big)\\
&= (-1)^{\deg(S)+1}(\mathrm{d}(\alpha\wedge\beta)) + (-1)^{\deg(S)}S(\mathrm{d}\alpha\wedge\beta)\\
&= \mathrm{d}S(\alpha\wedge\beta) + (-1)^{\deg(S)}S(\wedge\mathrm{d}\alpha)(\beta)\\
&= \big(\mathrm{d}S\wedge\alpha + (-1)^{\deg(S)}S\wedge\mathrm{d}\alpha\big)(\beta).
\end{aligned}$$

This formula extends to the product of two currents whose singular supports do not meet,

$$\mathrm{d}(S\wedge T) = \mathrm{d}S\wedge T + (-1)^{\deg(S)}S\wedge\mathrm{d}T. \tag{2.8.4}$$

Indeed, the latter formula needs only to be shown after restricting to an open subset of X whese S (resp. T) is given by a differential form, in which case it reduces to Equation (2.8.3).

2.9. On complex manifolds, bigraduation of currents, operators ∂, $\overline{\partial}$, $\mathrm{d^c}$.

§ 3. THE POINCARÉ-LELONG FORMULA (IN DIMENSION 1)

3.1. Let X be a connected Riemann surface. A *divisor* on X is a function from X to $\mathbf{Z}$ whose support is discrete — this means that for any compact subset K of X, there are only finitely many points of K which are mapped to a nonzero value. In fact, whatever the technical definition, one does not consider a divisor $D\colon p\mapsto n_p$ as a function but as a linear combination $D=\sum n_p p$ of points. The support $|D|$ of this divisor is then the set of all points $p\in X$ such that $n_p\neq 0$.

The set $\mathrm{Div}(X)$ of divisors on X is an Abelian group.

3.2. Let $D=\sum n_p p$ be a divisor on X; the current $\delta_D=\sum n_p\delta_p$ is defined by the formula

$$\delta_D(f) = \sum_p n_p f(p), \quad \text{for } f\in\mathscr{A}_{\mathrm{c}}^{0}(X). \tag{3.2.1}$$

This is a finite sum.

3.3. Let $\varphi \in \mathscr{M}(X)^*$ be a meromorphic function on X. We shall say that φ is "*regular*" (1) if it does not vanish identically on any non-empty open subset of X.

If φ is "regular", its *divisor* $\operatorname{div}(\varphi)$ is defined by:

$$\operatorname{div}(\varphi) = \sum_{p \in X} v_p(\varphi) p. \tag{3.3.1}$$

It is a divisor; indeed, the set of zeroes and poles of any "regular" meromorphic function on X is discrete, and its order of vanishing is well-defined.

Proposition **(3.4)** (Poincaré–Lelong). — *Let φ be a "regular" meromorphic function on X. The function* $\log|\varphi|$ *on X is locally integrable and*

$$\mathrm{dd^c}[\log|\varphi|] = \delta_{\operatorname{div}(\varphi)}. \tag{3.4.1}$$

Proof. — Let $p \in X$; there exists a neighborhood U of P and an isomorphism $z\colon U \to \mathbf{D}(0,1)$ such that $z(P) = 0$. One can choose U small enough so that, besides possibly p, φ has no zero and no pole on U. One can then write $\varphi = (z - z(P))^{v_p(\varphi)} \tilde{\varphi}$, where $\tilde{\varphi}$ is a holomorphic function on U, without zero nor pole. Then,

$$\log|\varphi| = -v_p(\varphi) \log|z - z(P)|^{-1} + \log|\tilde{\varphi}|.$$

The first term, $\log|z - z(P)|^{-1}$, is integrable on U since it is nonnegative and

$$\int_U \log|z - z(P)|^{-1} \frac{i}{2} \mathrm{d}z \wedge d\bar{z} = \int_{D(0,1)} \log|z|^{-1} \frac{i}{2} \mathrm{d}z \wedge d\bar{z} = \int_0^1 \int_0^{2\pi} \log(1/r)\, r \,\mathrm{d}r \,\mathrm{d}\theta$$

and the integral $\int_0^1 r \log(1/r)\,\mathrm{d}r$ converges absolutely. The second term is continuous, hence is locally integrable. This shows that $\log|\varphi|$ is locally integrable on X, hence defines a current.

Let us show the given equality of currents. Let $u \in \mathscr{A}_c^0(X)$ be a smooth function with compact support on X. Let I be the set of points $p \in \operatorname{supp}(u)$ such that $v_p(\varphi) \neq 0$; since $\operatorname{supp}(u)$ is compact, it is a finite set. Any point $p \in I$ has an open neighborhood U that is isomorphic to a unit disk via an isomorphism $z_p\colon U_p \to \mathbf{D}(0,1)$ such that $z_p(p) = 0$. We may moreover assume that the closures of the open sets U_p, for $p \in I$, are pairwise disjoint.

For $r \in (0,1)$ and any $p \in I$, let $D(p,r)$ be the set of points $q \in U_p$ such that $|z_p(p)| < r$. Let X_r be the complement of $|\operatorname{div}(\varphi)| \cup \bigcup_{p \in I} D(p,r)$ in X; this is a Riemann surface with boundary. The function φ is holomorphic on X_r and has neither zeroes, nor poles, so that $\log|\varphi|$ is a $\mathscr{C}^\infty$-function on X_r; the function u has compact support on X_r. One has

$$[\mathrm{dd^c} \log|\varphi|](u) = \int_X \log|\varphi|\, \mathrm{dd^c} u = \lim_{r \to 1} \int_{X_r} \log|\varphi|\, \mathrm{dd^c} u.$$

Moreover, for any $r \in (0,1)$, the Green formula asserts that

$$\int_{X_r} \log|\varphi|\, \mathrm{dd^c} u = \int_{X_r} u \mathrm{dd^c} \log|\varphi| + \int_{\partial X_r} \left(\log|\varphi|\, \mathrm{d^c} u - u \mathrm{d^c} \log|\varphi|\right).$$

(1) The terminology, borrowed from Grothendieck's Éléments de géométrie algébrique, refers to the fact that regular elements, *e.g.*, non-zero-divisors, of the ring of meromorphic functions are precisely such meromorphic functions which are not identically zero on any non-empty open subsets; the quotes are there to prevent any confusion: regular meromorphic functions may have poles!

Since φ is holomorphic on X_r and has no zeroes and no poles, $\mathrm{dd^c}\log|\varphi| = 0$ on $\mathring{X}_r$. Moreover, the boundary of X_r is the union, for $p \in I$, of the boundaries of the disks $D_p(r)$ with the clockwise orientation. First of all, for any $p \in I$, $\log|\varphi| = \mathrm{O}(\log r^{-1})$ on $\partial D_p(r)$, and the length of $\partial D_p(r) = \mathrm{O}(r)$. Consequently,

$$\int_{\partial D_p(r)} \log|\varphi| \,\mathrm{d^c}u = \mathrm{O}(r\log(r^{-1}))$$

converges to 0 when $r \to 0$. Then we analyse the terms $\int_{\partial D_p(r)} u\mathrm{d^c}\log|\varphi|$. On U_p, we can write $\varphi = z_p^{n_p}\tilde\varphi$, where $n_p = v_p(\varphi)$ and $\tilde\varphi$ is a holomorphic function without zeroes nor poles. Then,

$$\mathrm{d^c}\log|\varphi| = \mathrm{d^c}\log|\tilde\varphi| + n_p\mathrm{d^c}\log|z_p|.$$

The integrals $\int_{\partial D_p(r)} u\mathrm{d^c}\log|\tilde\varphi|$ converge to 0 when $r \to 0$. Passing in polar coordinates and writing $z = re^{i\theta}$, we have (see §1.7)

$$\mathrm{d^c}\log|z| = \frac{1}{2\pi}\,\mathrm{d}\theta$$

(we could also have used $\mathrm{d^c}\log|z| = \mathrm{d^c}\Re(\log z) = \frac{1}{2\pi}\Im(\mathrm{d}z/z)$), so that

$$\int_{\partial D_p(r)} u\,\mathrm{d^c}\log|z_p| = \int_0^{2\pi} u(z_p^{-1}(re^{i\theta}))\,\frac{1}{2\pi}\,\mathrm{d}\theta$$

converges to $u(z_p^{-1}(0)) = u(p)$ when $r \to 0$. Finally,

$$\int_X u\log|\varphi| = \lim_{r\to 0}\sup_{p\in I}\int_{\partial D_p(r)} u\mathrm{d^c}\log|\varphi| = \sum_{p\in I} n_p\lim_{r\to 0}\int_{\partial D_p(r)} u\mathrm{d^c}\log|z_p| = \sum_{p\in I} n_p u(p).$$

This concludes the proof of the proposition. □

Corollary **(3.5)**. — *Let X be a Riemann surface and let S, T be currents of order* 0 *on X. Let* Ω *be a relatively compact open subset of X whose boundary* $\partial\Omega$ *is* $\mathscr{C}^1$. *Assume that* $\operatorname{sing\,supp}(S)$, $\operatorname{sing\,supp}(T)$ *and* $\partial\Omega$ *are pairwise disjoint. Then,*

(3.5.1)
$$\int_\Omega (S\mathrm{dd^c}T - T\mathrm{dd^c}S) = \int_{\partial\Omega}(S\mathrm{d^c}T - T\mathrm{d^c}S).$$

Proof. — First observe that every term of this formula is well defined. Since the singular supports of the currents S and T do not meet, the products $S\mathrm{dd^c}T$ and $T\mathrm{dd^c}S$ are defined as in §2.6. Moreover, S and T are given by smooth forms in a neighborhood of $\partial\Omega$.

Let a, b, c be $\mathscr{C}^\infty$-functions on X such that $a + b + c = 1$, $a \equiv 0$ in a neighborhood of $\operatorname{sing\,supp}(T) \cup \partial\Omega$, $b \equiv 0$ in a neighborhood of $\operatorname{sing\,supp}(S) \cup \partial\Omega$ and $c \equiv 0$ in a neighborhood of $\operatorname{sing\,supp}(S) \cup \operatorname{sing\,supp}(T)$.

After multiplying everything by a, the formula reduces to Green formula in the case where T is a differential form with compact support contained in Ω. Then the right hand side vanishes and the asserted formula is the definition of the current $\mathrm{dd^c}S$.

After multiplying everything by b, one gets the analogous situation where the roles of S and T are exchanged.

After multiplying by c, one is reduced to the case where S and T are smooth differential forms. The formula is then nothing but Green formula for the Riemann surface with boundary Ω.

Since $a + b + c = 1$, the result follows. □

Let us now derive a few consequences.

Proposition **(3.6)** (Jensen's formula). — *Let r be a positive real number, let f be a meromorphic function on a neighborhood of $\overline{D(0,r)}$ such that $f(0) \in \mathbf{C}^*$. Then,*

$$\log|f(0)| = \sum_{z\in D(0,r)} v_p(f)\log\frac{|z|}{r} + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right| \mathrm{d}\theta. \tag{3.6.1}$$

Proof. — We define currents $S = \log|f|$ and $T = \log|z/r|$ on a neighborhood of $\overline{D(0,r)}$. Their singular supports are $|\mathrm{div}(f)|$ and $\{0\}$. First assume that $|\mathrm{div}(f)|$ does not meet the boundary $\partial D(0,r)$. Then we can apply the Green formula for currents on the open set $D(0,r)$. Since $\mathrm{dd^c} S = \delta_{\mathrm{div}(f)}$ and $\mathrm{dd^c} T = \delta_0$,

$$\int_{D(0,r)}(S\mathrm{dd^c}T - T\mathrm{dd^c}S) = \log|f(0)| - \sum_{z\in D(0,r)}\log|z/r|.$$

Moreover, in a neighborhood of $\partial D(0,r)$ which does not contain any zero or pole of f, nor the origin,

$$S\mathrm{d^c}T = \log|f(z)|^2\,\frac{1}{2\pi}\,\mathrm{d}\theta,$$

and $T\mathrm{d^c}S$ vanishes on $\partial D(0,r)$. Consequently,

$$\int_{\partial D(0,r)}(S\mathrm{d^c}T - T\mathrm{d^c}S) = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|^2 \mathrm{d}\theta,$$

and Jensen's formula follows from Green's one.

In the general case, f may have zeroes or poles on the boundary $\partial D(0,r)$, and we apply Jensen's formula for $s < r$, and let s converge to r. It thus suffices to show that

$$\int_0^{2\pi}\log\left|f(se^{i\theta})\right| \mathrm{d}\theta \to \int_0^{2\pi}\log\left|f(re^{i\theta})\right| \mathrm{d}\theta.$$

The convergence is pointwise, and is locally uniform around any θ such that $re^{i\theta}$ is neither a zero nor a pole of f. We shall prove that the convergence is dominated in a neighborhood of these points. Assume indeed that f has a zero of order n at $re^{i\varphi}$; then we can estimate $f(z)$ for z close to $re^{i\varphi}$ by

$$c_1\left|re^{i\varphi} - z\right|^n \leq |f(z)| \leq c_2,$$

where c_1 and c_2 are positive real numbers. If f has a pole of order n, then a similar inequality holds for $|f(z)|^{-1}$. Since $|re^{i\varphi} - se^{i\theta}| \geq s\,|1 - e^{i(\theta-\varphi)}|$, we get in both cases that

$$\left|\log\left|f(se^{i\theta})\right|\right| \leq \mathrm{O}(\log\left|1 - e^{i(\theta-\varphi)}\right|^{-1}) \leq \mathrm{O}(\log|\theta-\varphi|^{-1})$$

Since the function $t \mapsto \log|t|^{-1}$, is integrable in a neighborhood of the origin, the convergence is dominated, whence the result by Lebesgue's theorem. □

3.7. One of the ideas behind value distribution theory is that there is a relation between the zeroes of a meromorphic function and its growth. In the case of entire functions, Jensen's formula allows to turn this idea into a precise estimate.

Let thus $f \in \mathscr{O}(\mathbf{C})$ be a holomorphic function on $\mathbf{C}$. Assume for simplicity that $f(0) \neq 0$. For any positive real number r, Jensen's formula asserts that

$$\sum_{z\in\mathbf{D}(0,r)} v_z(f)\log\frac{r}{|z|} = \frac{1}{2\pi}\log\left|f(re^{i\theta})\right| - \log|f(0)|.$$

Let $\varepsilon > 0$. Applying Jensen's formula again on the disk of radius $r(1+\varepsilon)$ and neglecting zeroes of f in the annulus $\mathrm{C}(r, r(1+\varepsilon))$, we obtain

$$\sum_{z\in\mathbf{D}(0,r)} v_z(f)\log\frac{r(1+\varepsilon)}{|z|} = \frac{1}{2\pi}\log\left|f(r(1+\varepsilon)e^{i\theta})\right| - \log|f(0)|,$$

so that

$$\sum_{z\in\mathbf{D}(0,r)} v_z(f) \leq c(\varepsilon)\log\|f\|_{\mathrm{L}^\infty(\mathbf{D}(0,r(1+\varepsilon)))}.$$

Proposition **(3.8)** (Formula of "Poisson-Jensen"). — *Let r be a positive real number, let f be a meromorphic function on a neighborhood of $\overline{D(0,r)}$. Let $w \in D(0,r)$ be any point such that $w \notin |\mathrm{div}(f)|$, write $w = \rho e^{i\varphi}$. Then*

(3.8.1)
$$\log|f(w)| = \sum_{z\in D(0,r)} v_z(f)\log\left|\frac{r^2-\bar{z}w}{r(z-w)}\right| + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\frac{r^2-\rho^2}{r^2-2r\rho\cos(\theta-\varphi)+\rho^2}\,\mathrm{d}\theta.$$

Proof. — We apply Green's formula to the currents $S = [\log|f|]$ and $T(z) = [\log\left|\frac{r^2-\bar{z}w}{r(z-w)}\right|]$ on $\Omega = D(0,r)$. One has $\mathrm{dd^c}S = \delta_{\mathrm{div}(f)}$ and $\mathrm{dd^c}T = -\delta_w$. We assume that f has no zeroes and no poles on the boundary $\partial D(0,r)$, the general case can be treated by a similar argument as in the proof of Jensen's formula. Then, Green's formula implies

$$-\log|f(w)| - \sum_{z\in D(0,r)} v_z(f)\log\left|\frac{r^2-\bar{z}w}{r(z-w)}\right| = \int_{\partial D(0,r)}(\log|f|\,\mathrm{d^c}T - T\mathrm{d^c}\log|f|).$$

In a neighborhood of $\partial D(0,r)$, the current T is given by a $\mathscr{C}^\infty$-function which vanishes identically on $\partial D(0,r)$, so that $\int_{\partial D(0,r)} T\mathrm{d^c}\log|f|^2 = 0$. On the other hand,

$$\mathrm{d^c}T = \frac{1}{2\pi}\mathfrak{I}\Big(\frac{\mathrm{d}(r^2-\bar{z}w)}{r^2-\bar{z}w} - \frac{\mathrm{d}(z-w)}{z-w}\Big).$$

We express the integral on $\partial D(0,r)$ in polar coordinates and set $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$; then $\mathrm{d}z = iz\mathrm{d}\theta$ and $\mathrm{d}\bar z = -i\bar z\mathrm{d}\theta$. Moreover, $\bar z = r^2/r$, so that, on $\partial D(0,r)$,

$$\begin{aligned}
\mathrm{d}^c T &= -\frac{1}{2\pi}\Re\left(\frac{\bar z w}{r^2-\bar z w} + \frac{z}{z-w}\right)\mathrm{d}\theta \\
&= -\frac{1}{2\pi}\Re\left(\frac{z+w}{z-w}\right)\mathrm{d}\theta \\
&= -\frac{1}{2\pi}\Re\left(\frac{r^2 - 2ir\rho\sin(\varphi-\theta) - \rho^2}{r^2 - 2r\rho\cos(\varphi-\theta)+\rho^2}\right)\mathrm{d}\theta \\
&= -\frac{1}{4\pi}\frac{r^2-\rho^2}{r^2-2r\rho\cos(\varphi-\theta)+\rho^2}\mathrm{d}\theta.
\end{aligned}$$

Therefore,

$$\int_{\partial \mathbf{D}(0,r)} \log|f|\,\mathrm{d}^c T = -\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\frac{r^2-\rho^2}{r^2-2r\rho\cos(\theta-\varphi)+\rho^2}\,\mathrm{d}\theta.$$

Combining these equalities, we get the asserted formula. ☐

Proposition **(3.9)** (Weil). — *Let X be a compact connected Riemann surface, let f, g be nonzero meromorphic functions on X such that* $|\mathrm{div}(f)| \cap |\mathrm{div}(g)| = \varnothing$. *Then,*

$$\prod_{z\in X}|g(z)|^{v_z(f)} = \prod_{z\in X}|f(z)|^{v_z(g)}.$$

Proof. — It follows directly from Green's formula, as applied to the currents $S = [\log|f|]$, $T = [\log|g|]$ on X. ☐

Exercise **(3.10)**. — Prove that the formula holds without the absolute values, namely

$$\prod_{z\in X} g(z)^{v_z(f)} = \prod_{z\in X} f(z)^{v_z(g)}.$$

(First treat the case where $X = \mathbf{P}^1(\mathbf{C})$, identifying f and g with rational functions. In the general case, view f as a morphism from X to $\mathbf{P}^1(\mathbf{C})$.)

3.11. Let Ω be as in Green's formula. For any point $w \in \Omega$, there exists a unique function $g_{w,\Omega}$ on $\Omega \setminus \{w\}$ satisfying the following properties:

- it is harmonic on $\Omega \setminus \{w\}$;
- it has a logarithmic singularity at w: if z is a local holomorphic coordinate in a neighborhood of w, then $g_{w,\Omega} - \log|z - z(w)|^2$ extends to a harmonic function near w;
- it extends to a harmonic function in a neighborhood of $\partial\Omega$, vanishing identically on $\partial\Omega$.

Such a function can be defined as follows. First glue two copies of the compact set $\overline{\Omega}$ along the boundary via the complex conjugation; this furnishes a compact Riemann surface Y with an antiholomorphic automorphism c. There exists a harmonic function u on $Y \setminus \{w, c(w)\}$ with the prescribed holomorphic singularity at w, the opposite one at $c(w)$ which is changed into its opposite by c. It remains to identify a neighborhood of $\overline{\Omega}$ in X to a neighborhood of $\overline{\Omega}$ in Y, and to consider the restriction of u. Since $c(z) = z$ for any $z \in \partial\Omega$, $u \equiv 0$ on $\partial\Omega$.

Then $g_{w,\Omega}$ is locally integrable, hence defines a current on a neighborhood of $\overline{\Omega}$ and

$$\mathrm{dd^c}[g_{w,\Omega}] = \delta_w \quad \text{on a neighborhood of } \overline{\Omega}. \tag{3.11.1}$$

Moreover, for any continuous function on $\overline{\Omega}$ which is harmonic function on Ω, one has

$$h(w) = \int_{\partial\Omega} h\mathrm{d^c} g_{w,\Omega}. \tag{3.11.2}$$

§ 4. GEOMETRY OF THE RIEMANN SPHERE

4.1. The projective line $\mathbf{P}^1(\mathbf{C})$ is the set of lines of $\mathbf{C}^2$ passing through the origin. For $(x_0, x_1) \in \mathbf{C}^2 \setminus \{0\}$, write $[x_0 : x_1]$ for the line $l = \mathbf{C}(x_0, x_1)$. One says that x_0 and x_1 are the homogeneous coordinates of l; they are well-defined up to a common multiplicative constant. Let $p\colon \mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1(\mathbf{C})$ be the natural projection.

The map $z \mapsto [1 : z]$ from $\mathbf{C}$ to $\mathbf{P}^1(\mathbf{C})$ identifies the complex line with the complement of the *point at infinity* $\infty = [0 : 1]$ of $\mathbf{P}^1(\mathbf{C})$.

4.2. Let $\mathbf{S}_2$ be the unit sphere in $\mathbf{R}^3$. The *stereographic projection* is the map

$$s\colon \mathbf{S}_2 \to \mathbf{P}^1(\mathbf{C}), \quad (x, y, z) \mapsto [1 - z : x + iy]. \tag{4.2.1}$$

In other words, $s(x, y, z) = \frac{x+iy}{1-z}$ if $z \neq 1$, while the north pole $N = (0, 0, 1)$ is mapped to ∞. It is a bijection, the inverse of $w \in \mathbf{C}$ is given by the formulae

$$x = \Re\left(\frac{2w}{1 + |w|^2}\right), \quad y = \Im\left(\frac{2w}{1 + |w|^2}\right), \quad z = \frac{|w|^2 - 1}{|w|^2 + 1}. \tag{4.2.2}$$

The groups $\mathrm{SO}(3)$ and $\mathrm{SU}(2)/\{\pm 1\}$ act transitively on the sphere $\mathbf{S}_2$ and on $\mathbf{P}^1(\mathbf{C})$ respectively; There is a unique group isomorphism $\rho\colon \mathrm{SO}(3) \to \mathrm{SU}(2)/\{\pm 1\}$ such that $s(g \cdot p) = \rho(g) \cdot s(p)$ for any $p \in \mathbf{S}_2$ and any $g \in \mathrm{SO}(3)$.

4.3. Let $v_0, v_1 \in \mathbf{C}^2 \setminus \{0\}$; write $[v_0] = p(v_0)$ and $[v_1] = p(v_1)$ in $\mathbf{P}^1(\mathbf{C})$. One defines the *chordal distance* between $[v_0]$ and $[v_1]$ by the formula

$$\|[v_0], [v_1]\| = \frac{\|v_0 \wedge v_1\|}{\|v_0\| \, \|v_1\|}. \tag{4.3.1}$$

Observe that for α and $\beta \in \mathbf{C}^*$,

$$\frac{\|\alpha v_0 \wedge \beta v_1\|}{\|\alpha v_0\| \, \|\beta v_1\|} = \frac{\|v_0 \wedge v_1\|}{\|v_0\| \, \|v_1\|},$$

so that the chordal distance between two points p_0 and p_1 of $\mathbf{P}^1(\mathbf{C})$ is well-defined, independently of the choice of homogeneous coordinates needed for its computation.

Moreover, $\|[v_0], [v_1]\|$ is an element of $[0, 1]$, and it vanishes if and only if v_0 and v_1 are collinear, i.e., if $[v_0] = [v_1]$. It is also symmetric in $[v_0]$ and $[v_1]$.

Let w_1 and $w_2 \in \mathbf{C}$, write $v_1 = (1, w_1)$ and $v_2 = (1, w_2)$ so that $p(v_1) = w_1$ and $p(v_2) = w_2$. Then,

(4.3.2)
$$\|w_1, w_2\|^2 = \frac{|w_1 - w_2|^2}{(1 + |w_1|^2)(1 + |w_2|^2)}.$$

Moreover,

(4.3.3)
$$\|w_1, \infty\|^2 = \frac{1}{1 + |w_1|^2}.$$

Let also $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ be two points of the sphere $\mathbf{S}_2$. Then,

(4.3.4)
$$\|s(p_1), s(p_2)\| = \frac{1}{2} \|p_1 - p_2\|,$$

where the norm in $\mathbf{R}^3$ is the Euclidean one. This formula explains the terminology chosen: up to the normalization factor $\frac{1}{2}$, the chordal distance between two points of $\mathbf{P}^1(\mathbf{C})$ is the length of the chord that joins the corresponding points of the unit sphere.

Indeed, in the case where p_2 is the North pole and $p_1 \neq p_2$, one has $s(p_1) = [1 - z_1 : x_1 + iy_1]$ and $s(p_2) = [0 : 1]$, so that

$$\|s(p_1), s(p_2)\|^2 = \frac{(1 - z_1)^2}{(1 - z_1)^2 + x_1^2 + y_1^2} = \frac{4(1 - z_1)^2}{1 - 2z_1 + x_1^2 + y_1^2 + z_1^2} = 2(1 - z_1)$$

using that $p_1 \in \mathbf{S}_2$ hence $x_1^2 + y_1^2 + z_1^2 = 1$. On the other hand,

$$\|p_1 - p_2\| = x_1^2 + y_1^2 + (z_1 - 1)^2 = 2(1 - z_1).$$

This implies the given formula in this particular case. The general case follows from the particular one and the fact that both sides of the formula are unchanged under the actions of $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$.

Anyway, we can also make the computation. In the remaining case where p_1 and p_2 are both distinct from the North pole, one has $s(p_1) = w_1 = (x_1 + iy_1)/(1 - z_1)$ and $s(p_2) = w_2 = (x_2 + iy_2)/(1 - z_2)$, so that

$$\begin{aligned}
\|s(p_1), s(p_2)\|^2 &= \frac{(x_1(1 - z_2) - x_2(1 - z_1))^2 + (y_1(1 - z_2) - y_2(1 - z_1))^2}{\left((1 - z_1)^2 + x_1^2 + y_1^2\right)\left((1 - z_2)^2 + x_2^2 + y_2^2\right)} \\
&= \frac{(x_1^2 + y_1^2)(1 - z_2)^2 + (x_2^2 + y_2^2)(1 - z_1)^2 - 2(x_1x_2 + y_1y_2)(1 - z_1)(1 - z_2)}{\left(1 - 2z_1 + x_1^2 + y_1^2 + z_1^2\right)\left(1 - 2z_2 + x_2^2 + y_2^2 + z_2^2\right)}.
\end{aligned}$$

Since p_1 and p_2 belong to $\mathbf{S}_2$, it comes

$$x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2 = 1,$$

hence

$$\begin{aligned}\|s(p_1), s(p_2)\|^2 &= \frac{(1-z_1^2)(1-z_2)^2 + (1-z_2^2)(1-z_1)^2 - 2(1-z_1)(1-z_2)(x_1x_2+y_1y_2)}{4(1-z_1)(1-z_2)} \\ &= \frac{1}{2}\big((1+z_1)(1-z_2) + (1+z_2)(1-z_1) - 2(x_1x_2+y_1y_2)\big) \\ &= \frac{1}{2}(1-2p_1\cdot p_2) \\ &= \frac{1}{2}\|p_1-p_2\|^2 .\end{aligned}$$

Proposition **(4.4)**. — *There exists a unique differential form ω on $\mathbf{P}^1(\mathbf{C})$ such that*

$$p^*\omega = \frac{1}{2}\mathrm{dd^c}\log\left(|x_0|^2+|x_1|^2\right) \quad \text{on } \mathbf{C}^2 \smallsetminus \{0\}. \tag{4.4.1}$$

Proof. — Write $\tilde\omega = \mathrm{dd^c}\log\left(|x_0|^2+|x_1|^2\right)$. Since the map p is a holomorphic submersion, it induces an injection on differential forms, so that there exists at most one such ω. Moreover, since $\tilde\omega$ is of type $(1,1)$, so will be ω.

To prove the existence of ω, we show that for any open set $U \subset \mathbf{P}^1(\mathbf{C})$ and any holomorphic section s of p, $\tilde\omega = p^*s^*\tilde\omega$ on $p^{-1}(U)$. By definition, $s\circ p(x_0,x_1)$ is a nonzero point of the line $\mathbf{C}(x_0 : x_1)$, for any $(x_0,x_1)\in p^{-1}(U)$. Consequently, there exists an holomorphic function $\lambda\colon p^{-1}(U)\to\mathbf{C}^*$ such that

$$s\circ p(x_0,x_1) = \lambda(x_0,x_1)\,(x_0,x_1).$$

Then,

$$\begin{aligned} p^*s^*\tilde\omega &= \frac{1}{2}\mathrm{dd^c}\log\|s\circ p\|^2 \\ &= \frac{1}{2}\mathrm{dd^c}\log\left(|\lambda(x_0,x_1)|^2\ (|x_0|^2+|x_1|^2)\right) \\ &= \mathrm{dd^c}\log|\lambda| + \tilde\omega \\ &= \tilde\omega, \end{aligned}$$

since λ is holomorphic and invertible.

The projective line $\mathbf{P}^1(\mathbf{C})$ can be covered by open sets over which the map p admits holomorphic sections, *e.g.*, the section $s_0\colon[1 : w]\mapsto(1,w)$ on $U_0 = \mathbf{P}^1(\mathbf{C})\smallsetminus\{\infty\}$ and the section $s_1\colon[z : 1]\mapsto(z,1)$ on $U_1 = \mathbf{P}^1(\mathbf{C})\smallsetminus\{[1 : 0]\}$. The restrictions to $U_0\cap U_1$ of the forms $\omega_0 = s_0^*\tilde\omega$ on U_0 and $\omega_1 = s_1^*\tilde\omega$ on U_1 coincide, since they both pull-back to $\tilde\omega$ on $p^{-1}(U_0\cap U_1)$ by the submersion p. Consequently, they glue to a global differential form ω on $U_0\cup U_1 = \mathbf{P}^1(\mathbf{C})$. □

Lemma **(4.5)**. — *Let $a\in\mathbf{P}^1(\mathbf{C})$. One has the following equality of currents on $\mathbf{P}^1(\mathbf{C})$:*

$$\omega = -\mathrm{dd^c}\log\|a,\cdot\| + \delta_a. \tag{4.5.1}$$

Proof. — First assume that $a\neq\infty$. Then, for $z\in\mathbf{P}^1(\mathbf{C})\smallsetminus\{\infty\}$,

$$\|a,z\|^2 = \frac{|a-z|^2}{(1+|a|^2)(1+|z|^2)},$$

so that, as currents on $\mathbf{P}^1(\mathbf{C}) \smallsetminus \{\infty\}$,

$$\mathrm{dd^c}\log\|a,z\| = \mathrm{dd^c}\log|a-z| - \frac{1}{2}\mathrm{dd^c}\log\left(1+|z|^2\right) = \delta_a - \omega.$$

Since the function $\log\|a,\cdot\|$ is $\mathscr{C}^\infty$ on $\mathbf{P}^1(\mathbf{C}) \smallsetminus \{a\}$, the singular support of the current $\mathrm{dd^c}\log\|a,\cdot\| - \delta_a$ is contained in $\{a\}$. Since it equals the restriction of the smooth differential form ω on this open set, one must have $\mathrm{dd^c}\log\|a,\cdot\| - \delta_a = \omega$ on $\mathbf{P}^1(\mathbf{C})$, hence the lemma when $a \neq \infty$.

When $a = \infty = [0:1]$, one identifies $U_1 = \mathbf{P}^1(\mathbf{C}) \smallsetminus \{[1:0]\}$ with $\mathbf{C}$ via the map $w \mapsto [w:1]$ and one writes

$$\|\infty, [w:1]\|^2 = \frac{|w|^2}{1+|w|^2}$$

so that

$$\mathrm{dd^c}\log\|\infty,[w:1]\| = \mathrm{dd^c}\log|w| - \omega = \delta_\infty - \omega.$$

□

4.6. Let us compute explicitly the form ω on the open set $\mathbf{C} = \mathbf{P}^1(\mathbf{C}) \smallsetminus \{\infty\}$. We have

$$\begin{aligned}
\omega &= \frac{1}{2}\mathrm{dd^c}\log\left(1+|z|^2\right) \\
&= \frac{i}{2\pi}\partial\bar\partial\log\left(1+z\bar z\right) \\
&= \frac{i}{2\pi}\partial\frac{z}{1+z\bar z}\,\mathrm{d}\bar z \\
&= \frac{i}{2\pi}\left(\frac{1}{1+z\bar z} - \frac{i}{2\pi}\frac{z\bar z}{(1+z\bar z)^2}\right)\mathrm{d}z\wedge\mathrm{d}\bar z \\
&= \frac{i}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{(1+|z|^2)^2}. \qquad \textbf{(4.6.1)}
\end{aligned}$$

In particular,

$$\begin{aligned}
\int_{\mathbf{P}^1(\mathbf{C})}\omega = \int_{\mathbf{C}}\omega &= \frac{1}{\pi}\int_{\mathbf{R}^2}\frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2} \\
&= 2\int_0^\infty \frac{r\mathrm{d}r}{(1+r^2)^2} = \int_0^\infty\frac{\mathrm{d}r^2}{(1+r^2)^2} = 1, \qquad \textbf{(4.6.2)}
\end{aligned}$$

a formula which justifies various normalizations. On the other hand, we could have also used the equations of currents given by lemma 4.5 and written, for some $a \in \mathbf{P}^1(\mathbf{C})$,

$$\int_{\mathbf{P}^1(\mathbf{C})}\omega = \int_{\mathbf{P}^1(\mathbf{C})}(\mathrm{dd^c}\log\|a,\cdot\| + \delta_a) = 1,$$

since, for any current T of degree 0,

$$\int_{\mathbf{P}^1(\mathbf{C})}\mathrm{dd^c}T = \mathrm{dd^c}T(1) = T(\mathrm{dd^c}1) = 0.$$

4.7. The action of $\mathrm{SU}(2)$ on $\mathbf{P}^1(\mathbf{C})$ is transitive, and lets invariant the form ω. Consequently, the normalization condition $\int_{\mathbf{P}^1(\mathbf{C})}\omega = 1$ characterizes the form ω among invariant forms.

§ 5. LINE BUNDLES

5.1. Let X be a complex manifold. From the point of view of differential geometry, a *holomorphic line bundle* on X is a complex manifold L together with a holomorphic map $p\colon L \to X$ which is a locally trivial fibration with fiber $\mathbf{C}$ and structure group $\mathbf{C}^*$. In other words, the space X is covered by open subsets U_i on which there is an isomorphism $f_i = (p|_{U_i}, g_i)\colon p^{-1}(U_i) \simeq U_i \times \mathbf{C}$ such that for any i and j, there is a holomorphic function $f_{ij}\colon U_i \cap U_j \to \mathbf{C}^*$ such that $g_i(y) = f_{ij}(p(x))g_j(y)$ for any $y \in p^{-1}(U_i \cap U_j)$. The maps f_i are called the *trivializations* of L, and the f_{ij} the *transition maps*.

These data furnish a structure of $\mathbf{C}$-vector space (of dimension 1) on the fibers of p, given by

$$(x, g_i(y + y')) = (x, g_i(y) + g_i(y')), \quad \lambda \cdot (x, g_i(y)) = (x, \lambda g_i(y)),$$

for $x \in U_i$, $y, y' \in p^{-1}(x)$ and $\lambda \in \mathbf{C}$, because this structure is independent on the choice of i such that $x \in U_i$.

If M is a second line bundle on X, a morphism of line bundles $\varphi\colon L \to M$ is a holomorphic map which is $\mathbf{C}$-linear on each fiber of p.

5.2. A holomorphic section of p on an open set U of X is a holomorphic map $s\colon U \to p^{-1}(U)$ such that $p \circ s = \mathrm{id}_U$. The addition on the fibers of p varies "varies holomorphically"with the point x, in the following sense: if s and s' are holomorphic sections of p over an open subset U of X, then $x \mapsto s(x) + s'(x)$ is again holomorphic. Indeed, on $U \cap U_i$, one can write

$$s(x) + s'(x) = f_i^{-1}(x, g_i(s(x)) + g_i(s'(x))).$$

Similarly, if $\lambda\colon U \to \mathbf{C}$ is holomorphic and s is a holomorphic section of p, then $x \mapsto \lambda(x) \cdot s(x)$ is a holomorphic section of p.

Let s be a section of p on an open set U of X. On $U \cap U_i$, this section induces a holomorphic map $g_i \circ s\colon U \cap U_i \to \mathbf{C}$. Conversely, if $h_i\colon U \cap U_i \to \mathbf{C}$ is holomorphic, there exists a unique section s on $U \cap U_i$ such that $g_i \circ s = h_i$; it is given by $s(x) = f_i^{-1}(x, h_i(x))$ for $x \in U \cap U_i$. Moreover, for $x \in U \cap U_i \cap U_j$, one has the relation

$$h_i(x) = f_{ij}(x)h_j(x).$$

Conversely, for any family (g_i), where g_i is a holomorphic function $U \cap U_i \to \mathbf{C}$, satisfying these relations, there exists a unique section s on U such that $h_i \circ s = h_i$ for every i.

For each open subset U of X, let $\mathscr{L}(U)$ be the set of holomorphic sections of p on U. It is naturally a module over the ring $\mathscr{O}_X(U)$ of holomorphic functions on U. Moreover, $\mathscr{L}$ and $\mathscr{O}_X$ satisfy the axioms of sheaves, so that $\mathscr{L}$ is a sheaf of $\mathscr{O}_X$-modules. Moreover, on any open set $U \subset U_i$, we have described a bijection $s \mapsto h = g_i \circ s$ between sections $s \in \mathscr{L}(U)$ and holomorphic functions $h \in \mathscr{O}_X(U)$. This bijection is compatible with restriction to open subsets and to the module structure. Consequently, it defines an isomorphism of sheaves from the restriction to U of the sheaf $\mathscr{L}$ to the restriction to U of the sheaf $\mathscr{O}_X$. In particular, the sheaf $\mathscr{L}$ is locally isomorphic to $\mathscr{O}_X$; we shall sum up this property by saying that $\mathscr{L}$ is a *line sheaf*.

Let M is a second line bundle on X and let $\mathscr{M}$ be its sheaf of sections. Let $\varphi\colon L \to M$ be a morphism. For each open set $U \subset X$ and each section $s \in \mathscr{L}(U)$, $x \mapsto \varphi(s(x))$ is a section of M on U, so is an element of $\mathscr{M}(U)$. This defines a map $\varphi_U: \mathscr{L}(U) \to \mathscr{M}(U)$; it is a morphism of $\mathscr{O}_X(U)$-modules. Moreover, when U varies, these maps define a morphism of sheaves Φ from $\mathscr{L}$ to $\mathscr{M}$.

5.3. By associating to a holomorphic line bundle its sheaf of sections, we thus have defined a functor from the category of holomorphic line bundles on X to the category of *line sheaves,* that is, the category of sheaves of $\mathscr{O}_X$-modules which are locally isomorphic to $\mathscr{O}_X$. Let us show that this functor is an equivalence of categories.

We first prove that it is fully faithful, namely: *if L and M are holomorphic line bundles on X and Φ is a morphism of sheaves from $\mathscr{L}$ to $\mathscr{M}$, there exists a unique morphism of holomorphic line bundles $\varphi: L \to M$ giving rise to Φ.*

It is necessarily given by the formula $\Phi(s)(x) = \varphi(s(x))$ for any open set $U \subset X$, any section $s\mathscr{L}(U)$ and any point $x \in U$. Conversely, it suffices to check that this formula actually defines a morphism of holomorphic line bundles.

Then, we show that this functor is essentially surjective: *any line sheaf $\mathscr{L}$ is associated to a holomorphic line bundle on X.*

Let (U_i) an open cover of X and, for every i, let φ_i be an isomorphism of $\mathscr{O}_{U_i}$ to $\mathscr{L}|_{U_i}$; let $s_i \in \mathscr{L}(U_i)$ be the section $\varphi_i(1)$. Let $\varphi_{ij} = \varphi_i|_{U_i \cap U_j} \circ (\varphi_j|_{U_i \cap U_j})^{-1}$; it is an automorphism of the sheaf $\mathscr{O}_{U_i \cap U_j}$ (as a sheaf of modules over itself); let $f_{ij} = \varphi_{ij}(1)$. By the sheaf property, one has $\varphi_{ij}(h) = f_{ij}|_U h$ for any open set $U \subset U_i \cap U_j$ and any section $h \in \mathscr{O}_X(U)$. The function f_{ij} is invertible because φ_{ij} is an automorphism. Moreover, on $U_i \cap U_j \cap U_k$, we have the *cocycle relation:*

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}.$$

We can now define a holomorphic line bundle L on X as follows. On the disjoint union of the spaces $U_i \times \mathbf{C}$, we define a binary relation $\sim$ by

$$(x, t)_i \sim (x, f_{ij}(x)t)_j, \quad \text{for } x \in U_i \cap U_j,\ t \in \mathbf{C}.$$

The cocycle relation shows indeed that it is an equivalence relation. The quotient space is the desired holomorphic line bundle.

If $\mathscr{L}$ is a line sheaf on X, a *basis* of $\mathscr{L}$ over an open subset U is a section $s \in \mathscr{L}(U)$ such that $s|_V$ is a basis of $\mathscr{L}(V)$ as a $\mathscr{O}_X(V)$-module for any open subset V of U—in other words, the morphism of sheaves from $\mathscr{O}_U$ to $\mathscr{L}|_U$ given by $f \mapsto fs$ is an isomorphism.

5.4. *Divisors and line bundles on Riemann surfaces.* — Let X be a Riemann surface and let D be a divisor on X. Let $\mathscr{O}_X(D)$ be the subsheaf of the sheaf of meromorphic functions on X defined as follows: a section of $\mathscr{O}_X(D)$ on an open subset U of X is a meromorphic function f on U such that $\operatorname{div}(f) + D|_U \geq 0$. Let us detail this condition. Recall that a divisor on U is a function from X to $\mathbf{Z}$ whose support is locally finite. The divisor $D|_U$ is the restriction to U of the divisor D; it is a divisor on U. The inequality $\operatorname{div}(f) + D|_U \geq 0$ means that the divisor $\operatorname{div}(f) + D_U$ takes only nonnegative values; it is rephrased by saying that this divisor is *effective*.

Let us show that $\mathscr{O}_X(D)$ is line sheaf on X. First of all, for any open subset U of X, any meromorphic function f on U such that $\operatorname{div}(f) + D|_U \geq 0$ and any holomorphic function g

on U, one has $\operatorname{div}(g) \geq 0$, hence

$$\operatorname{div}(gf) + D|_U = \operatorname{div}(g) + \operatorname{div}(f) + D|_U \geq 0.$$

This endowes $\mathscr{O}_X(D)(U)$ with a structure of a $\mathscr{O}_X(U)$-module, and this structure is compatible with restriction to open subsets of U, so that $\mathscr{O}_X(D)$ is a sheaf of $\mathscr{O}_X$-modules.

Let $p \in X$, let U be an open neighborhood of p in X which is isomorphic to the unit disk; fix such an isomorphism $z\colon U \simeq D(0,1)$. We may also assume that $U \smallsetminus \{p\}$ does not contain any point of the support $|D|$ of D. Let n_p be the coefficient of p in D, so that $D|_U = n_p p$. Since $\operatorname{div}(z - z(p)) = p$, a meromorphic function f on U belongs to $\mathscr{O}_X(D)(U)$ if and only if $\operatorname{div}((z - z(p)^{n_p} f)) \geq 0$, that is, if and only if $(z - z(p))^{n_p} f$ is holomorphic on U. Since $z - z(p)$ is invertible as a meromorphic function, $\mathscr{O}_X(D)(U)$ is a free $\mathscr{O}_X(U)$-module with basis $z - z(p)$. This implies the result.

5.5. Let $\mathscr{L}$ be a line sheaf on a Riemann surface X. Let U be an open subset of X. A *meromorphic section* of $\mathscr{L}$ on U is the datum of a discrete subset A of U and of a section $s \in \mathscr{L}(U \smallsetminus A)$ satisfying the following condition: for any point $p \in U$, there exists an open neighborhood V of p, a basis s_0 of $\mathscr{L}$ on V, such that the unique holomorphic function f $V \smallsetminus (A \cap V)$ such that $s = f s_0$ is meromorphic on V. We identify two meromorphic sections $s \in \mathscr{L}(U \smallsetminus A)$ and $s \in \mathscr{L}(U \smallsetminus A')$ if they coincide on $U \smallsetminus (A \cup A')$.

Let s be a meromorphic section of $\mathscr{L}$ on X, holomorphic on $X \smallsetminus A$. In analogy with the corresponding definition for meromorphic functions, we shall say that s is "regular" if it does not vanish identically on any non-empty open subset of $X \smallsetminus A$. If this holds, there is a unique divisor D on X such that, for any open subset U of X, any basis s_0 of $\mathscr{L}$ on U, $D|_U = \operatorname{div}(f)$, where f is the unique meromorphic function on U such that $s = f s_0$ on $U \smallsetminus (A \cap U)$. This divisor is denoted $\operatorname{div}(s)$, and referred to as the divisor of the meromorphic section s. Observe that $\operatorname{div}(s)$ is effective if and only s extends to a holomorphic section of $\mathscr{L}$ on X.

For example, a *meromorphic* section of $\mathscr{O}_X(D)$ on an open set U of X is nothing but a meromorphic function f on U. If moreover f is "regular", then its divisor (as a section of $\mathscr{O}_X(D)$) is given by $\operatorname{div}(f) + D$.

Lemma (**5.6**). — *Let X be a Riemann surface. The map that associates to a pair $(\mathscr{L}, s)$ consisting of a line sheaf $\mathscr{L}$ on X and of a regular meromorphic section s of $\mathscr{L}$ its divisor* $\operatorname{div}(s)$ *induces a bijection from the set of isomorphism classes of such pairs to the set of divisors on X.*

Proof. — Let $(\mathscr{L}, s)$ and $(\mathscr{L}', s')$ be such pairs, and let $\varphi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ be an isomorphism of line sheaves such that $\varphi(s) = s'$. By the very definition of these divisors, $\operatorname{div}(s) = \operatorname{div}(s')$.

Conversely, let $D = \operatorname{div}(s)$; let us show that there is an isomorphism of $\mathscr{O}_X(D)$ to $\mathscr{L}$ that maps the meromorphic section 1 to the meromorphic section s. It suffices to observe that for any meromorphic function $f \in \mathscr{O}_X(D)(U)$ on an open subset U of X, the meromorphic section fs of $\mathscr{L}|_U$ is in fact a holomorphic section. Moreover, if f is a basis of $\mathscr{O}_X(D)|_U$, namely, if $\operatorname{div}(f) + D|_U = 0$, then $\operatorname{div}_{\mathscr{L}}(fs) = 0$ and fs is a basis of $\mathscr{L}|_U$. □

Corollary (**5.7**). — *Let $(\mathscr{L}, s)$ and $(\mathscr{M}, t)$ be line sheaves on a Riemann surface X both equipped with a "regular" meromorphic section. One has canonical isomorphisms*

$$\operatorname{Hom}(\mathscr{L}, \mathscr{M}) \simeq \mathscr{O}_X(\operatorname{div}(t) - \operatorname{div}(s))(X).$$

Similarly, if D_1 and D_2 are divisors on X, then

$$\operatorname{Hom}(\mathcal{O}_X(D_1), \mathcal{O}_X(D_2)) \simeq \mathcal{O}_X(D_2 - D_1)(X).$$

Proof. — By the preceding lemma, it suffices to show the second assertion. A morphism from $\mathcal{O}_X(D_1)$ to $\mathcal{O}_X(D_2)$ is characterized by the image f of the meromorphic section 1 of $\mathcal{O}_X(D_2)$; and there is such a morphism if and only if fg is a holomorphic section of $\mathcal{O}_X(D_2)$ whenever g is a holomorphic section of $\mathcal{O}_X(D_1)$. This condition means that $\operatorname{div}(fg)+D_2 \geq 0$ whenever $\operatorname{div}_1 \geq 0$. It certainly suffices that $\operatorname{div}(f)+D_2-D_1 \neq 0$. Conversely, any point of X has a neighborhood U on which there is a meromorphic function g with $\operatorname{div}(g) + D_1|_U = 0$; then $\operatorname{div}(fg) + D_2|_U = \operatorname{div}(f)|_U + D_2 - D_1|_U$, so that the condition $\operatorname{div}(f) + D_2 - D_1 \geq 0$ is necessary. This concludes the proof of the corollary. □

5.8. *Operations on line bundles and line sheaves*

(5.8.1) *Tensor product.* — Let L_1 and L_2 be line bundles on a complex space X, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the corresponding line sheaves. There is a unique line bundle L on X whose fiber over a point $x \in X$ is the tensor product $L_{1,x} \otimes L_{2,x}$ such that, whenever s_1 and s_2 are nonvanishing sections of L_1 and L_2 on an open subset U of X, the section $x \mapsto s_1(x) \otimes s_2(x)$ is a nonvanishing section of L on U.

Let (U_i) be an open cover of X such that L_1 is defined by a cocycle (f_{ij}) and L_2 is defined by a cocycle (g_{ij}). Then L can be defined by the cocycle $(h_{ij}) = (f_{ij}g_{ij})$.

This line bundle L is called the tensor product of L_1 and L_2 and denotes $L_1 \otimes L_2$. Its sheaf of sections is called the tensor product of the line sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ and denotes $\mathcal{L}_1 \otimes \mathcal{L}_2$. In fact, $\mathcal{L}$ is the sheaf of $\mathcal{O}_X$-modules associated to the presheaf $U \mapsto \mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U)$.

When X is a Riemann surface and D_1, D_2 are divisors on X, $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ is isomorphic to $\mathcal{O}_X(D_1 + D_2)$.

(5.8.2) *Dual.* — Let L be a line bundle on a complex space X, let $\mathcal{L}$ be its sheaf of sections. There is a unique line bundle $L^\vee$ on X such that the fiber of $L^\vee$ at any point $x \in X$ is the dual $(L_x)^\vee$ of L_x, and such that for any local nonvanishing section s of L, the map $x \mapsto \varphi_s(x)$ is a local nonvanishing section of $L^\vee$, where, for any $x \in U$, $\varphi_s(x)$ is the unique linear form on L_x that maps s to 1. Its sheaf of sections is the sheaf $\mathcal{L}^\vee$ associated to the the presheaf $U \mapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{L}(U); \mathbf{C})$.

Let (U_i) be an open cover of X such that L is defined by a cocycle (f_{ij}); then $L^\vee$ can be defined by the cocycle (f_{ij}^{-1}).

When X is a Riemann surface and D is a divisor on X, $\mathcal{O}_X(D)^\vee$ is isomorphic to $\mathcal{O}_X(-D)$.

There are isomorphisms

$$L \otimes L^\vee \simeq \mathbf{C} \times X, \quad s \otimes \varphi \mapsto (\varphi(s), x),$$

whenever $x \in X$, $s \in L_x$ and $\varphi \in L_x^\vee = \operatorname{Hom}(L_x; \mathbf{C})$, and

$$(L_1 \otimes L_2)^\vee simeq L_1^\vee \otimes L_2^\vee,$$

where L_1 and L_2 are line bundles on X.

(5.8.3) *Inverse image.* — Let $f\colon X \to Y$ be a holomorphic map of complex spaces, let L be a line bundle on Y. Then, the space $f^*L = L \times_Y X$ consisting of pairs $(v, x) \in L \times X$ such that $v \in L_{f(x)}$, together with the natural projection to X, is a line bundle on X. The fiber f^*L_x of f^*L over a point $x \in X$ is identified with the fiber $L_{f(x)}$ of L over $f(x)$, by the map $(v, x) \mapsto v$, and the structure of a complex line on f^*L_x is deduced from this identification.

Let s be a nonvanishing section of L over an open subset U of Y. Then, $f^*s\colon x \mapsto (s(f(x)), x)$ is a nonvanishing section of f^*L over $f^{-1}(U)$.

Assume that X and Y are Riemann surfaces. Let s be a meromorphic section of L on Y, holomorphic on the complement $U = Y \smallsetminus B$ to a discrete subset B of Y. If no connected component of X is sent identically to a point, then $f^{-1}(B)$ is discrete in X and the section f^*s of f^*L over $f^{-1}(U)$ is actually a meromorphic section of f^*L on X. Its divisor $\operatorname{div}_{f^*L}(f^*s)$ and the divisor $\operatorname{div}_L(s)$ of s satisfy the following relation:

$$\operatorname{div}_{f^*L}(f^*s) = f^*(\operatorname{div}_L(s)).$$

In the right hand side, $f^*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ is the linear map that sends a divisor p (namely, the divisor sending p to 1 and any other point to 0) to the divisor $\sum_{q\in f^{-1}(p)} m_{f,q} q$, for adequate multiplicities $m_{f,q}$ defined as follows. Let $q \in X$ and $p = f(x) \in Y$; let U be an open neighborhood of p which admits an isomorphism $z\colon U \to \mathbf{D}(0,1)$ such that $z(p) = 0$; let V be an open neighborhood of q such that $f(V) \subset U$ and which admits an isomorphism $w\colon V \to \mathbf{D}(0,1)$ such that $w(q) = 0$. Let $\varphi\colon \mathbf{D}(0,1) \to \mathbf{D}(0,1)$ be the unique holomorphic map such that $z \circ f = \varphi \circ w$ on V. It satisfies $\varphi(0) = 0$ and is not constant; one has $m_{f,p} = v_0(\varphi)$.

One has the following compatibilities between tensor products, duals and inverse image:

(5.8.1) $$f^*L_1 \otimes f^*L_2 \simeq f^*(L_1 \otimes L_2),$$

(5.8.2) $$(f^*L)^\vee \simeq f^*(L^\vee),$$

both defined in the obvious way fiberwise.

5.9. *The tautological line bundle on the Riemann sphere.* — We now discuss a fundamental example. We consider the product $\mathbf{P}^1(\mathbf{C}) \times \mathbf{C}^2$ over the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, which we view as a family of two-dimensional vector spaces $\mathbf{C}^2$ (the trivial vector bundle of rank two). Let $\mathscr{O}(-1)$ be the subspace of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{C}^2$ consisting of pairs (x, v) such that v belongs to the line L_x corresponding to x — we call it the *tautological line bundle.* It is indeed a line bundle. On the complement U_0 to the point at infinity $[0 : 1]$ of $\mathbf{P}^1(\mathbf{C})$, we have the isomorphism

(5.9.1) $$\varphi_0\colon U_0 \times \mathbf{C} \simeq \mathscr{O}(-1)|_{U_0}, \qquad ([1 : z], t) \mapsto ([1 : z], (t, tz)),$$

while on the complement U_1 to the point $[1 : 0]$, we have an isomorphism

(5.9.2) $$\varphi_1\colon U_1 \times \mathbf{C} \simeq \mathscr{O}(-1)|_{U_1}, \qquad ([z : 1], u) \mapsto ([z : 1], (uz, u)).$$

On the intersection $U_0 \cap U_1$, any point can be written as $[1 : z]$ or as $[1/z : 1]$ and these two isomorphisms differ by the composition by the automorphism

(5.9.3) $$(U_0 \cap U_1) \times \mathbf{C} \simeq (U_0 \cap U_1) \times \mathbf{C}, \qquad ([1 : z], t) \mapsto ([1/z : 1], tz),$$

so that $\mathscr{O}(-1)$ is the line bundle on $\mathbf{P}^1(\mathbf{C})$ associated to the cocycle $(f_{01}\colon z \mapsto z)$ relative to the open covering (U_0, U_1).

In other words, the line bundle $\mathscr{O}(-1)$ has trivializing sections s_0 on U_0 and s_1 on U_1 given by

(5.9.4) $$s_0([1:z]) = (1,z), \quad s_1([z:1]) = (z,1).$$

The line bundle $\mathscr{O}(1)$ is defined as the *dual* of the line bundle $\mathscr{O}(-1)$. Its fiber over a point $x \in \mathbf{P}^1(\mathbf{C})$ is the complex space of linear forms on the line L_x. In particular, the linear forms on $\mathbf{C}^2$ induce global sections of $\mathscr{O}(1)$, hence a morphism of complex vector spaces

$$(\mathbf{C}^2)^\vee \to \Gamma(\mathbf{P}^1(\mathbf{C}), \mathscr{O}(1)), \quad \varphi \mapsto (x \mapsto \varphi|_{L_x}).$$

Let X_0 and X_1 be the images in $\Gamma(\mathbf{P}^1(\mathbf{C}), \mathscr{O}(1))$ of the two coordinates on $\mathbf{C}^2$.

Let us compute their divisors. By definition, $\operatorname{div}_{\mathscr{O}(1)}(X_0)$ is the divisor on $\mathbf{P}^1(\mathbf{C})$ which coincides with the divisor $\operatorname{div}(X_0(s_0))$ on U_0 and with the divisor $\operatorname{div}(X_0(s_1))$ on U_1. Returning to the definitions, $X_0(s_0)$ is the holomorphic function on U_0 given by $[1:z] \mapsto 1$, and $X_0(s_1)$ is the holomorphic function on U_1 given by $[z:1] \mapsto z$. Consequently,

(5.9.5) $$\operatorname{div}_{\mathscr{O}(1)}(X_0) = [0:1].$$

Similarly,

(5.9.6) $$\operatorname{div}_{\mathscr{O}(1)}(X_1) = [1:0].$$

and, more generally, for any $(a,b) \in \mathbf{C}^2 \smallsetminus \{(0,0)\}$,

(5.9.7) $$\operatorname{div}_{\mathscr{O}(1)}(aX_0 + bX_1) = [-b:a].$$

§ 6. HERMITIAN LINE BUNDLES

Definition **(6.1)**. — *Let $L \to X$ be a holomorphic line bundle on a complex space X. A* hermitian metric *on L is the data of a map $\|\cdot\|_L : L \to \mathbf{R}_+$ such that:*

- *For any $x \in X$, the restriction of $\|\cdot\|_L$ to the complex line L_x is a hermitian norm;*
- *For any non-vanishing section s of L on an open subset U of X, the map $U \to \mathbf{R}_+^*$ given by $x \mapsto \|s(x)\|_L$ is $\mathscr{C}^\infty$.*

The last condition amounts to saying that the restriction of $\|\cdot\|_L$ to the complement of the zero section in L is $\mathscr{C}^\infty$. There is an analogous notion of a continuous metric, where this map is only assumed to be continuous.

A *hermitian line bundle* $(L, \|\cdot\|_L)$ is a holomorphic line bundle equiped with a hermitian metric. We will often write $\overline{L}$ to indicate that we are talking of a hermitian line bundle L.

6.2. Let X be a complex space which is paracompact (this assumption is automatic if X is a Riemann surface) and let L be a holomorphic line bundle on X. Then, L admits hermitian metrics.

Indeed, let $(U_i)_{i\in I}$ be an open cover of X and, for every i, let s_i be a non-vanishing section of L on U_i. Since X is paracompact, we may consider a partition of unity relative to the open cover (U_i) : this is a family $(\lambda_j)_{j\in J}$ of nonnegative $\mathscr{C}^\infty$-functions on X satisfying the following properties:

– It is locally finite (every $p \in X$ has a neighborhood U such that the set of indices j such that $\operatorname{supp}(\lambda_j)$ meets U is finite);
– For every j, there exists an index $i = i(j)$ such that $\operatorname{supp}(\lambda_j) \subset U_i$;
– One has $\sum_j \lambda_j = 1$.

Let $x \in X$ and let $v \in L_x$. Let J_x be the set of indices j such that $x \in \operatorname{supp}(\lambda_j)$; for any such j, $x \in U_{i(j)}$ by assumption, so that $s_{i(j)}(x) \neq 0$. In particular, there is a complex number a_j such that $v = a_j s_{i(j)}$ and we set

$$\|v\|^2 = \sum_{j \in J_x} \lambda_j(x) \left|a_j\right|^2 .$$

I claim that this defines a hermitian metric on the line bundle L.

On each complex line L_x, the right hand side is a sum of hermitian forms with nonnegative coefficients, so it is a nonnegative hermitian form. If $v \neq 0$, then $a_j \neq 0$ for any $j \in J_x$; since $\sum \lambda_j = 1$, one has $\|v\| \neq 0$, hence this hermitian form is positive definite.

Let us show that for any non-vanishing section s of L on an open set U of X, the function $x \mapsto \|s(x)\|$ on U is $\mathscr{C}^\infty$.

Let $x \in U$; since the family (λ_j) is locally finite, there exists a neighborhood V of x contained in U and a finite subset J_0 of J such that $J_y \subset J_0$ for any point $y \in V$. Moreover, we may replace V by its intersection with the finitely many open sets $U_{i(j)}$, for $j \in J_0$. This allows us to assume that $V \subset U_{i(j)}$ for any $j \in J_0$. Then, for any $j \in J_0$, there exists a $\mathscr{C}^\infty$-function a_j on V such that $s|_V = a_j s_{i(j)}|_V$. Consequently, for any $y \in V$,

$$\|s(y)\|^2 = \sum_{j \in J_y} \lambda_j(y) \left|a_j(y)\right|^2 = \sum_{j \in J_0} \lambda_j(y) \left|a_j(y)\right|^2 .$$

The right hand side is a finite sum of $\mathscr{C}^\infty$-functions on V, hence it defines a $\mathscr{C}^\infty$-function on V.

6.3. *Operations on hermitian line bundles.* — Tensor product, duals, inverse images of line bundles have a counterpart for hermitian metrics. The formulae are as follows.

Let $\overline{L_1}$ and $\overline{L_2}$ be hermitian line bundles on X; let s_1 and s_2 be sections of L_1 and L_2 on an open subset U of X. The hermitian metric on the tensor product $L_1 \otimes L_2$ is given by

(6.3.1) $$\|s_1 \otimes s_2(x)\|_{L_1 \otimes L_2} = \|s_1(x)\|_{L_1} \|s_2(x)\|_{L_2}$$

for any $x \in U$.

Let $\overline{L}$ be a hermitian line bundle on X, let s be a non-vanishing section of L on an open subset U of X; let φ be the section of the dual line bundle $L^\vee$ which maps s to 1; then, for any $x \in U$,

(6.3.2) $$\|\varphi_s(x)\|_{L^\vee} = \|s(x)\|_L^{-1} .$$

Finally, let $\overline{L}$ be a hermitian line bundle on a complex space Y, let $f\colon X \to Y$ be a morphism of complex spaces; the hermitian metric $\|\cdot\|_{f^*L} : f^*L \to \mathbf{R}_+$ on the inverse image f^*L is just the composition of the projection $f^*L \to L$ with the hermitian metric $\|\cdot\|_L$ on L. For any section s of L on an open subset U of Y, one has

(6.3.3) $$\|f^* s(x)\|_{f^*L} = \|s(f(x))\|_L$$

for any $x \in f^{-1}(U)$.

Observe that with these definitions, all canonical isomorphisms described earlier become isometries.

Proposition **(6.4)**. — *Let $\overline{L}$ be a hermitian line bundle on a complex space X. There exists a unique real differential form $\alpha \in \mathscr{A}^{1,1}(X)$ such that, for any non-vanishing section s of L on an open subset U of X,*

(6.4.1)
$$\alpha|_U = -\mathrm{dd^c} \log \|s\| = \frac{i}{\pi} \bar\partial \partial \log \|s\| .$$

This form is denoted $c_1(\overline{L})$ and is called the *curvature form* of the hermitian line bundle $\overline{L}$.

Proof. — We first prove that whenever s and t are non-vanishing sections of L on an open subset U of X, one has the following equality

$$\mathrm{dd^c} \log \|s\| = \mathrm{dd^c} \log \|t\|$$

of differential forms of degree 2 on U. Indeed, there exists a holomorphic function f on U such that $t = fs$, and f is invertible. By the definition of a hermitian metric

$$\|t(x)\| = \|f(x)s(x)\| = |f(x)| \, \|s(x)\| ,$$

hence

$$\log \|s(x)\| = \log |f(x)| + \log \|s(x)\| .$$

By Proposition 1.4, the image of $\log |f|$ under the operator $\mathrm{dd^c}$ vanishes, hence the desired formula.

Let us cover X by open subsets U_i such that for each i L admits a non-vanishing section s_i on U_i. Let α_i be the differential form $-\mathrm{dd^c} \log \|s_i\|$ on U_i. Applying our first observation to the restrictions to $U_i \cap U_j$ of the sections s_i and s_j, we see that α_i and α_j coincide on $U_i \cap U_j$. Consequently, there exists a unique differential form α on X such that $\alpha|_{U_i} = \alpha_i$.

Let U be any open subset of X and s be a non-vanishing section of L on U. We apply the observation to the restrictions to $U_i \cap U$ of s and s_i; this says that the restriction of $\mathrm{dd^c} \log \|s\|^{-1}$ to $U \cap U_i$ coincides with the form $\alpha_i|_{U \cap U_i} = \alpha|_{U \cap U_i}$. Consequently, $-\mathrm{dd^c} \log \|s\| = \alpha|_U$.

The proposition now follows from the fact that the operator $\mathrm{dd^c}$ maps real functions to real differential forms of bidegree $(1,1)$. ☐

6.5. *Examples.* — The trivial line bundle $\mathbf{C} \times X$ on X has a hermitian metric such that $\|(a,x)\| = |a|$ for any $(a,x) \in \mathbf{C} \times X$. This line bundle admits a global section $\mathbf{1}$, given by $x \mapsto (1,x)$, and $\log \|1\| = 0$. Consequently, its curvature form vanishes.

Let $\overline{L}$ and $\overline{M}$ be hermitian line bundles on X. Then, when the line bundle $L \otimes M$ is equiped with the tensor product of the hermitian metrics on L and M, we have

(6.5.1)
$$c_1(\overline{L \otimes M}) = c_1(\overline{L}) + c_1(\overline{M}).$$

Indeed, this follows from the definition applied to sections s of L, t of M and $s \otimes t$ of $L \otimes M$.

Similarly, the curvature form of the dual hermitial line bundle $\overline{L^\vee}$ satisfies

(6.5.2)
$$c_1(\overline{L^\vee}) = -c_1(\overline{L}).$$

Finally, let $f\colon X \to Y$ be a morphism of complex spaces and let $\overline{L}$ be a hermitian line bundle on Y. Then,

(6.5.3) $$c_1(f^*\overline{L}) = f^* c_1(\overline{L}).$$

Proposition **(6.6)** (Formula of Poincaré–Lelong). — *Let X be a Riemann surface, let $\overline{L}$ be a hermitian line bundle on X, and let s be a "regular" meromorphic section of L on X.*

Then, the function $\log \|s\|_L$ *on X is locally integrable and the current it defines satisfies the equation:*

(6.6.1) $$\mathrm{dd^c}[\log \|s\|_L] = \delta_{\mathrm{div}_L(s)} - c_1(\overline{L}).$$

Proof. — This proposition can be proved locally, hence we may assume that the line bundle L admits a non-vanishing section s_0. Then, there exists a meromorphic function f on X such that $s = f s_0$. By definition,

$$\log \|s\|_L = \log |f| + \log \|s_0\|_L .$$

Moreover, $\log \|s_0\|_L$ is $\mathscr{C}^\infty$ on X, hence locally integrable. As we saw earlier (Proposition 3.4), $\log |f|$ is locally integrable and $\mathrm{dd^c}[\log |f|] = \delta_{\mathrm{div}(f)}$; moreover, $\mathrm{div}_L(s) = \mathrm{div}(f)$. Finally, $-\mathrm{dd^c} \log \|s_0\|_L = c_1(\overline{L})$, by definition of the curvature of a hermitian line bundle. Consequently,

$$\mathrm{dd^c}[\log \|s\|_L] = \mathrm{dd^c}[\log |f|] + \mathrm{dd^c} \log \|s_0\|_L = \delta_{\mathrm{div}_L(s)} - c_1(\overline{L}),$$

as was to be shown. □

6.7. Let X be a compact Riemann surface and let D be a divisor on X. Write $D = \sum n_p p$. Since D is locally finite and X is compact, there are only finitely many points $p \in X$ such that $n_p \neq 0$. The sum $\sum_{p\in X} n_p$ is called the *degree*of D and denoted $\deg(D)$.

If D_1 and D_2 are divisors on X, one has $\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2)$.

Corollary **(6.8)**. — *Let L be a holomorphic line bundle on a compact Riemann surface X. For any hermitian metric $\|\cdot\|_L$ on L and any "regular" meromorphic section s of L, one has*

$$\deg(\mathrm{div}_L(s)) = \int_X c_1(\overline{L}).$$

Proof. — Since X is compact, we may integrate the Poincaré–Lelong equation on X; the right hand side gives $\deg(\mathrm{div}_L(s)) - \int_X c_1(\overline{L})$. By the definition of the operator $\mathrm{dd^c}$ for currents, the left hand side is equal to

$$\int_X [\mathrm{dd^c} \log \|s\|_L] 1 = \int_X [\log \|s\|_L] \mathrm{dd^c}(1) = 0.$$

□

6.9. In particular, the degree of the divisor of a regular meromorphic section of L is independent on the choice of that section. It is called the *degree of the line bundle L*. Two isomorphic line bundles have the same degree. Moreover, for any two line bundles L_1 and L_2 on X, one has

$$\deg(L_1 \otimes L_2) = \deg(L_1) + \deg(L_2).$$

6.10. Let X be a compact *connected* (non-empty) Riemann surface. One has the following exact sequence of complex vector spaces:

$$0 \to \mathbf{C} \to \mathscr{A}^0(X) \xrightarrow{\mathrm{dd^c}} \mathscr{A}^{1,1}(X) \xrightarrow{\int_X} \mathbf{C} \to 0. \tag{6.10.1}$$

The map $\mathbf{C} \to \mathscr{A}^0(X)$ associates to any complex number a the constant function with value a; it is injective. The image by $\mathrm{dd^c}$ of a constant function is zero, so that the composition of the first two maps is zero.

Conversely, let $\alpha \in \mathscr{A}^0(X)$ be any function such that $\mathrm{dd^c}(\alpha) = 0$. Since X is compact, there exists a point $p \in X$ such that $\alpha(p) = \sup_X \alpha$. We will prove that α is constant, with value $\alpha(p)$. The set $\alpha^{-1}(\alpha(p))$ is closed in X. Let U be an open neighborhood of p which is isomorphic to the unit disk and let $z \colon U \to \mathbf{D}(0,1)$ be such an isomorphism; let f be the $\mathscr{C}^\infty$-function on $\mathbf{D}(0,1)$ such that $\alpha(q) = f(z(q))$ for any $q \in U$. One has $\mathrm{dd^c} f = 0$, hence $\Delta(f) = 0$. By the maximum principle for harmonic functions, f is constant, equal to $\alpha(p)$, so that α is constant in a neighborhood of p. Applying this argument to any point of $\alpha^{-1}(\alpha(p))$ implies that this set is open. Since X is connected, this implies that $\alpha^{-1}(\alpha(p)) = X$, hence α is constant.

Let again $\alpha \in \mathscr{A}^0(X)$. By the Green formula, One has $\int_X \mathrm{dd^c}\alpha = 0$. The converse holds: if $\omega \in \mathscr{A}^{1,1}(X)$ is a differential form of bidegree $(1,1)$ such that $\int_X \omega = 0$, then there exists $\alpha \in \mathscr{A}^0(X)$ such that $\omega = \mathrm{dd^c}\alpha$. This follows from the analysis of the Laplace operator on compact Riemann surfaces, but we shall not prove it here.

6.11. Let X be a compact connected (non-empty) Riemann surface and let L be a holomorphic line bundle on X. Let α be a real form of bidegree $(1,1)$ such that $\int_X \alpha = \deg(L)$. Then, there exists a hermitian metric $\|\cdot\|_L$ on L such that $\alpha = c_1(\overline{L})$. Two such metrics are proportional: if $\|\cdot\|_1$ and $\|\cdot\|_2$ are hermitian metrics on L such that $\alpha = c_1(L, \|\cdot\|_1) = c_1(L, \|\cdot\|_2)$, there exists a real number c such that $\|\cdot\|_2 = e^{-c}\,\|\cdot\|_1$.

Indeed, let us choose an arbitrary hermitian metric $\|\cdot\|_0$ on L. If $\|\cdot\|_L$ is any other metric on L, there exists a (unique) real valued $\mathscr{C}^\infty$-function φ on X such that $\|v\|_L = e^{-\varphi(x)}\,\|v\|_0$ for any $x \in X$ and any $v \in L_x$. Conversely, any real-valued $\mathscr{C}^\infty$-function φ gives rise to a metric on L. Then, the definition of the curvature form gives

$$c_1(\overline{L}) = c_1(L, \|\cdot\|_0) + \mathrm{dd^c}\varphi,$$

so that the equation $\alpha = c_1(\overline{L})$ is equivalent to the condition

$$\mathrm{dd^c}\varphi = \alpha - c_1(L, \|\cdot\|_0).$$

Since $\int_X \alpha = \deg(L) = \int_X c_1(L, \|\cdot\|_0)$, the integral of the right hand side is zero. By the preceding paragraph, there exists a $\mathscr{C}^\infty$-function φ on X satisfying this equation. Since α is real, its real part is again a solution and the associated hermitian metric is a solution to our problem.

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two hermitian metrics on L with curvature form α, the corresponding functions φ_1 and φ_2 satisfy $\mathrm{dd^c}(\varphi_1) = \mathrm{dd^c}(\varphi_2)$, hence $\mathrm{dd^c}(\varphi_1 - \varphi_2) = 0$. By the results of the preceding paragraph, $\varphi_1 - \varphi_2$ is constant (and real) on X, hence the claim.

6.12. *The tautological line bundle on the Riemann sphere.* — We now discuss a hermitian metric on the tautological line bundles $\mathscr{O}(1)$ and $\mathscr{O}(-1)$ on $\mathbf{P}^1(\mathbf{C})$ that we had defined in §5.9. Recall that $\mathscr{O}(-1)$ had been defined as a subspace (a sub-bundle, in fact) of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{C}^2$. On this larger space, we can consider the function induced by the hermitian norm on the factor $\mathbf{C}^2$. It induces an hermitian metric on $\mathscr{O}(-1)$. In fact, for any $x \in \mathbf{P}^1(\mathbf{C})$, and any vector $v = (v_0, v_1)$ in the line L_x, $\|v\| = (|v_0|^2 + |v_1|^2)^{1/2}$.

On the dual line bundle $\mathscr{O}(-1)$, this gives a dual hermitian metric: if $x \in \mathbf{P}^1(\mathbf{C})$ and φ is a linear form on the line L_x, then

(6.12.1) $$\|\varphi\| = \frac{|\varphi(v)|}{\|v\|} \quad \text{for any vector } v \in L_x.$$

Explicitly,

(6.12.2) $$\|aX_0 + bX_1\|\,([x_0 : x_1]) = \frac{|ax_0 + bx_1|}{(|x_0|^2 + |x_1|^2)^{1/2}}.$$

Proposition **(6.13)**. — *With this metric, the curvature of $\mathscr{O}(1)$ is equal to the form ω.*

Proof. — The section X_0 of $\mathscr{O}(1)$ does not vanish on the open set U_0 of $\mathbf{P}^1(\mathbf{C})$ consisting of points of the form $[1 : z]$, with $z \in \mathbf{C}$ and

$$\|X_0\|\,([1 : z]) = \frac{1}{(1 + |z|^2)^{1/2}}.$$

Consequently,

$$c_1(\overline{\mathscr{O}(1)})|_{U_0} = \mathrm{dd^c} \log \|X_0\|^{-1}|_{U_0} = \frac{1}{2}\mathrm{dd^c} \log(1 + |z|^2) = \omega|_{U_0}.$$

The computation on the open set U_1 (consisting of points of the form $[z : 1]$, with $z \in \mathbf{C}$) is similar. Anyway, the differential forms $c_1(\overline{\mathscr{O}(1)})$ and ω agree on the dense open subset U_0, so must be equal. ☐

§ 7. RIEMANNIAN METRICS AND CURVATURE ON RIEMANN SURFACES

7.1. Let X be a Riemann surface. Let T_X denote its *holomorphic tangent bundle.* For any point $p \in X$, $T_{X,p}$ is the tangent space to X at p, endowed with its natural structure of a complex vector space.

When X is an open subset of $\mathbf{C}$, T_X is a trivial line bundle. Indeed, let $p \in X$. A basis of $T_{X,p}$ as a real vector space is $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Its complex structure is defined by the relation $i\frac{\partial}{\partial x} = -\frac{\partial}{\partial y}$, so that $\frac{\partial}{\partial x}$ is a basis of $T_{X,p}$ as a complex vector space. This gives an identification of T_X with $X \times \mathbf{C}$.

As a complex vector space of dimension 2, $T_{X,p} \otimes_{\mathbf{R}} \mathbf{C}$ admits the basis $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z})$ and the natural injection $T_{X,p} \to T_{X,p} \otimes_{\mathbf{R}} \mathbf{C}$ maps $\frac{\partial}{\partial x}$ to $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial y}$ to $i\frac{\partial}{\partial z}$. Therefore, we can view $T_{X,p}$ as a complex line in the complex vector space $T_{X,p} \otimes \mathbf{C}$ (of dimension 2). In this identification, T_X is a subbundle of a trivial vector bundle of rank 2.

Let us return to the case of a general Riemann surface X. Let U be an open subset of X together with a holomorphic bijection $\varphi\colon U \to \Omega$ to an open subset Ω of $\mathbf{C}$. Then the differential operator $\partial = \varphi^{-1}(\frac{\partial}{\partial z})$ is a non-vanishing section of the restriction to U of T_X, so that we identify T_U with $U \times \mathbf{C}$.

7.2. The dual line bundle $\Omega^1_X = T_X^\vee$ is the holomorphic cotangent bundle.

7.3. The data of a hermitian metric on T_X is equivalent to various other data.

Let U be an open subset of X together with a holomorphic local chart $z\colon U \to \mathbf{C}$. This furnishes a non-vanishing section $\frac{\partial}{\partial z}$ on U. Consequently, there exists a positive $\mathscr{C}^\infty$-function λ on U such that

(7.3.1)
$$\left\| \frac{\partial}{\partial z} \right\|_{T_X} = \lambda.$$

By duality, the cotangent bundle Ω^1_X admits a non-vanishing section dz on U, and its norm satisfies the equation

(7.3.2)
$$\|\mathrm{d}z\| = \lambda^{-1}.$$

For any point $p \in X$, we have identified the (complex) tangent line of X at p with the (real) tangent plane of X at p. Consequently, the hermitian metric on T_X also furnishes a Riemannian metric on X, given by the length element

(7.3.3)
$$\mathrm{d}s^2 = \left\| \frac{\partial}{\partial z} \right\|^2 = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2.$$

Observe that this Riemannian metric is a multiple of the euclidean metric $\mathrm{d}x^2 + \mathrm{d}y^2$; we say that it is *conformal*. In the coordinates (x, y), angles are the same whether they are computed with the Riemannian metric or with the euclidean metric.

Finally, we also get a positive area form on X, given locally by

(7.3.4)
$$\mathrm{d}A = \lambda^2 \mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2\,\|\alpha\|^2} \alpha \wedge \overline{\alpha}$$

for any local non-vanishing section α of Ω^1_X.

We see from the formulae that any of these data allows to recover the function λ, so that they are all equivalent.

7.4. Let us endow T_X with a Hermitian metric. The Gauß curvature R of the associated Riemannian metric is related to the curvature form of the hermitian line bundle $\overline{T_X}$ by the following equation

(7.4.1)
$$c_1(\overline{T_X}) = \frac{1}{2\pi} R\,\mathrm{d}A.$$

Assuming that X is connected, compact and non-empty, and integrating over X, we deduce the *Gauß–Bonnet theorem*:

(7.4.2)
$$\chi(X) = \deg(T_X) = \int_X c_1(\overline{T_X}) = \frac{1}{2\pi} \int_X R\,\mathrm{d}A.$$

In this formula, $\chi(X)$ is the Euler characteristic of the Riemann surface X. The equality between $\chi(X)$ and $\deg(T_X)$ reflects the computation of $\chi(X)$ via indices of vector fields. The rest of the equation is a consequence of the Poincaré–Lelong equation.

7.5. Let $f\colon X \to Y$ be a nowhere locally constant morphism between two Riemann surfaces.

For each point x of X, the derivative of f at x is a $\mathbf{C}$-linear map $Df(x)\colon T_{X,x} \to T_{Y,f(x)}$, hence an element of $T_{X,x}^\vee \otimes T_{Y,f(x)}$. Consequently, the derivative of f is a section Df of the line bundle $T_X^\vee \otimes f^* T_Y$. This section is holomorphic, and nowhere locally zero. Its divisor $\operatorname{div}(Df)$ is called the *ramification divisor* of f, and denoted $\operatorname{Ram}(f)$. This is an effective divisor (each point of X comes with a nonnegative coefficient) and its support is the set of points of X at which f is not a local diffeomorphism.

Endow the line bundles T_X and T_Y with hermitian metrics. Then, $T_X^\vee \otimes f^* T_Y$ has a natural hermitian metric too and

$$\text{(7.5.1)} \qquad \mathrm{dd^c} \log \|Df\| = \delta_{\operatorname{Ram}(f)} - c_1(\overline{T_X}^\vee \otimes f^*\overline{T_Y}) = \delta_{\operatorname{Ram}(f)} + c_1(\overline{T_X}) - f^* c_1(\overline{T_Y}).$$

In the case where both X and Y are connected, we may integrate this relation on X. Since

$$\int_X f^* c_1(\overline{T_Y}) = \int_Y f_* 1\, c_1(\overline{T}_Y) = \deg(f)\chi(Y),$$

we obtain the Riemann–Hurwitz formula:

$$\text{(7.5.2)} \qquad \deg(\operatorname{Ram}(f)) + \chi(X) = \deg(f)\chi(Y).$$

7.6. *The sphere.* — Let $s\colon \mathbb{S}_2 \to \mathbf{P}^1(\mathbf{C})$ be the stereographic projection and let us define a hermitian metric on $T_{\mathbf{P}^1(\mathbf{C})}$ by the formula

$$\text{(7.6.1)} \qquad \|Ds(v)\|_{T_{\mathbf{P}^1(\mathbf{C})}} = \frac{1}{2}\|v\|$$

for any vector $v \in \mathbf{R}^3$. The stereographic projection maps the complement to the North pole to the complement of the point at infinity in $\mathbf{P}^1(\mathbf{C})$, which is identified to $\mathbf{C}$; the formula is

$$s(x,y,z) = \frac{x+iy}{1-z}, \qquad \text{for } (x,y,z) \in \mathbb{S}_2 \setminus \{N\}.$$

Its inverse is given by

$$s^{-1}(w) = \frac{1}{1+|w|^2}(\Re(2w), \Im(2w), 1-|w|^2) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, 1-\frac{2}{1+u^2+v^2}\right),$$

where $w = u + iv$, so that the vector field $\frac{\partial}{\partial u}$ is the image of the vector field

$$\left(\frac{2}{1+u^2+v^2} - \frac{4u^2}{(1+u^2+v^2)^2}, -\frac{4uv}{(1+u^2+v^2)^2}, -\frac{4u}{(1+u^2+v^2)^2}\right) = \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2, -2uv, -2u).$$

Consequently,

$$\begin{aligned}\left\|\frac{\partial}{\partial}u\right\| &= \frac{1}{(1+|w|^2)^4}\left(1+u^4+v^4-2u^2+2v^2-2u^2v^2+4u^2v^2+4u^2\right)\\ &= \frac{1}{(1+|w|^2)^4}\left(1+u^4+v^4+2u^2+2v^2+2u^2v^2\right)\\ &= \frac{1}{(1+|w|^2)^2}.\end{aligned}$$

Moreover, the identification of the holomorphic tangent space to a Riemann surface at a point (a complex line) with its real tangent space at that point (a real plane) maps $\frac{\partial}{\partial w}$ to $\frac{\partial}{\partial u}$, where w is a local holomorphic coordinate and (u, v) are its real and imaginary parts. It follows that on $T_{\mathbf{P}^1(\mathbf{C})}$,

(7.6.2)
$$\left\|\frac{\partial}{\partial w}\right\| = \left\|\frac{\partial}{\partial u}\right\| = \left\|\frac{\partial}{\partial u}\right\| = \frac{1}{(1+|w|^2)^2}.$$

§ 8. JENSEN'S INEQUALITY

Theorem **(8.1)**. — *Let $\varphi\colon (a,b) \to \mathbf{R}$ be a convex function. Let (X,μ) be a probability space and let $f\colon X \to (a,b)$ be a μ-integrable function. The following properties hold:*

a) *One has $\int_X f\,\mathrm{d}\mu \in (a,b)$;*

b) *The function $\varphi\circ f$ is μ-measurable, and is bounded from below by some μ-integrable function. In particular, $\int_X(\varphi\circ f)\,\mathrm{d}\mu$ is a well defined element of $\mathbf{R}\cup\{+\infty\}$;*

c) *One has*

(8.1.1)
$$\varphi\Big(\int_X f\,\mathrm{d}\mu\Big) \leq \int_X (\varphi\circ f)\,\mathrm{d}\mu.$$

Proof. — Since φ is convex on the open interval (a,b) of $\mathbf{R}$, it is continuous, and admits left and right derivatives, $\varphi'_{\mathrm{l}}(t)$ and $\varphi'_{\mathrm{r}}(t)$ at any point $t\in(a,b)$. Moreover, $\varphi'_{\mathrm{l}}(t)\leq\varphi'_{\mathrm{r}}(t)$ and for any real number λ such that $\varphi'_{\mathrm{l}}(t)\leq\lambda\leq\varphi'_{\mathrm{r}}(t)$, and any $s,t\in(a,b)$, one has

$$\varphi(s)\geq\varphi(t)+\lambda(s-t).$$

Consequently, for any $x\in X$,

$$\varphi(f(x))\geq\varphi(t)+\lambda(f(x)-t).$$

This shows that $\varphi\circ f$ is bounded from below by a μ-integrable function. Let us integrate this inequality on X; we get

$$\int_X(\varphi\circ f)\,\mathrm{d}\mu\geq\varphi(t)+\lambda\left(\int_X f\,\mathrm{d}\mu - t\right).$$

Set $t=\int_X f\,\mathrm{d}\mu$; one has $t\in(a,b)$. Taking any $\lambda\in[\varphi'_{\mathrm{l}}(t),\varphi'_{\mathrm{r}}(t)]$, we obtain the desired inequality. ☐

CHAPTER 3

NEVANLINNA THEORY FOR MEROMORPHIC FUNCTIONS IN ONE VARIABLE

Let r_0 be a positive real number. Let Ω be an open subset of $\mathbf{C}$ containing the complement $\overline{\mathrm{C}(r_0,\infty)}$ of the disk $\mathbf{D}(0,r_0)$ in $\mathbf{C}$ and let f be a non constant meromorphic function on Ω. Since the inversion $z\mapsto 1/z$ identifies $\overline{\mathrm{C}(r_0,\infty)}$ with the complement $\dot{\mathbf{D}}(0,r_0)$ of the origin in the disk $\mathbf{D}(0,r_0)$, we really are in the situation of the Great Picard Theorem.

The point of Nevanlinna theory is to consider f as a *holomorphic* function from Ω with values in the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, without (almost) any reference to holomorphic functions on Ω. Let ω be the canonical differential form of degree 2 on $\mathbf{P}^1(\mathbf{C})$.

§ 1. THE CHARACTERISTIC FUNCTION

1.1. Let $r\in[r_0,+\infty)$. One defines

(1.1.1) $$A(f,r_0;r)=\int_{\mathrm{C}(r_0,r)} f^*\omega=\frac{1}{\pi}\int_{\mathrm{C}(r_0,r)}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,\mathrm{d}x\mathrm{d}y.$$

This is the area of $f(\mathrm{C}(r_0,r))$ on the sphere with respect to the measure ω, taking multiplicities into account.

Definition **(1.2)**. — *The* characteristic function *is defined by*

(1.2.1) $$T(f,r_0;r)=\int_{r_0}^{r} A(f,r_0;t)\,\frac{\mathrm{d}t}{t}=\int_{\mathrm{C}(r_0,\infty)}\log^+\frac{r}{|z|}f^*\omega.$$

As for the function $A(f,r_0;\cdot)$, the characteristic function measures the "growth" of f, that is, the growth of the area of the Riemann sphere it covers.

The equality of the two formulae in the definition is a consequence of the following lemma.

Lemma **(1.3)**. — *Let α be a positive Borel measure on Ω. For any real number $r\in[r_0,\infty)$, one has*

$$\int_{\mathrm{C}(r_0,r)}\log^+\frac{r}{|z|}\,\mathrm{d}\alpha(z)=\int_{r_0}^{r}\alpha(\mathrm{C}(r_0,t))\,\frac{\mathrm{d}t}{t},$$

an equality of elements of $[0,+\infty]$.

Proof. — Indeed,

$$\begin{aligned}\int_{r_0}^{r} \alpha(\mathrm{C}(r_0,t))\,\frac{\mathrm{d}t}{t} &= \int_{r_0}^{r}\Big(\int_{\mathrm{C}(r_0,t)} \mathrm{d}\alpha(z)\Big)\,\frac{\mathrm{d}t}{t} \\ &= \int_{r_0}^{\infty}\Big(\int_{\Omega} \mathbf{1}_{[r_0,t]}(|z|)\,\mathrm{d}\alpha(z)\Big)\,\mathbf{1}_{t<r}\frac{\mathrm{d}t}{t} \\ &= \int_{\Omega}\Big(\int_{r_0}^{\infty}\mathbf{1}_{t<r}\mathbf{1}_{r_0\le|z|\le t}\frac{\mathrm{d}t}{t}\Big)\,\mathrm{d}\alpha(z)\end{aligned}$$

by Fubini's theorem. Moreover, the expression within parentheses is an integral on the interval given by $|z|\le t<r$; it vanishes if $r\ge|z|$ and equals

$$\int_{|z|}^{r}\frac{\mathrm{d}t}{t} = \log\frac{r}{|z|}$$

if $|z|\ge r$. This implies the lemma. □

Proposition **(1.4)**. — *The function* $T(f,r_0;\cdot)$ *is continuous on* $[r_0,+\infty)$ *with values in* $\mathbf{R}_+$, *vanishing at* r_0. *It is a strictly convex function of* $\log(r)$.

Proof. — The function $r\mapsto\log^+\frac{r}{|z|}=\max(0,\log(r)-\log|z|)$ is a convex, continuous, nondecreasing function of $\log(r)$. Since ω is a positive measure, this implies that $T(f,r_0;r)$ is a continuous nondecreasing, and convex function of $\log(r)$ as well, while it is obvious that it vanishes at r_0. In fact, one has

$$\frac{\mathrm{d}}{\mathrm{d}\log(r)}T(f,r_0;r) = rT'(f,r_0;r) = A(f,r_0;r),$$

a strictly increasing and positive function of $\log(r)$, so that $T(f,r_0;r)$ is a strictly convex function of $\log(r)$. □

Corollary **(1.5)**. — *When* $r\to+\infty$, $T(f,r_0;r)$ *converges to* $+\infty$; *more precisely,*

(1.5.1)
$$\lim_{r\to+\infty}\frac{T(f,r_0;r)}{\log(r)}>0.$$

Proof. — This is a general fact: for any convex, strictly increasing function $\varphi\colon[u_0,+\infty)\to\mathbf{R}$, $\varphi(u)/u$ has a positive limit when $u\to\infty$. Indeed, for any $u>u_0$, one has the following equality

$$\frac{\varphi(u)-\varphi(u_0+0)}{u-u_0} = \int_{u_0}^{u}\varphi'_{\mathrm{r}}(u_0+t(u-u_0)).$$

Since φ'_{r} is increasing, this shows that $(\varphi(u)-\varphi(u_0+0))/(u-u_0)$ is increasing and positive for $u>u_0$, hence has a positive limit when $u\to\infty$. The claim follows. □

1.6. Let Ω' be an open neighborhood of $\overline{C(r_0, r)}$ on which $f\colon \Omega \to \mathbf{P}^1(\mathbf{C})$ lifts to a map $\tilde f\colon \Omega \to \mathbf{C}^2 \smallsetminus \{0\}$. For example, f has finitely many poles on $\overline{C(r_0, r)}$, so that there exists a nonzero polynomial $P \in \mathbf{C}[z]$ such that $P(z)f(z)$ is holomorphic in a neighborhood Ω' of this compact set; we can then set $\tilde f(z) = (P(z), P(z)f(z))$. Then, one has the following equality

$$f^*\omega = \frac{1}{2}\tilde f^* \mathrm{dd^c} \log\left(|x_0|^2 + |x_1|^2\right) = \mathrm{dd^c}\log\|\tilde f\|$$

of differential forms on Ω'. By Green's formula,

$$T(f, r_0; r) = \frac{1}{2\pi}\int_{\partial D(0,r)-\partial D(0,r_0)} \log\|\tilde f\|\,\mathrm{d}\theta - \log\frac{r}{r_0}\int_{\partial D(0,r_0)} \mathrm{d^c}\log\|\tilde f\|.$$

In the particular case where f is holomorphic, we may take $\tilde f(z) = (1, f(z))$. It follows that there exist real numbers A and B such that

(1.6.1) $$T(f, r_0; r) = \frac{1}{4\pi}\int_{\partial D(0,r)} \log\left(1 + |f(re^{i\theta})|^2\right)\mathrm{d}\theta + A\log(r) + B.$$

Moreover, using the inequality

(1.6.2) $$\max(1, |u|)^2 \le 1 + |u|^2 \le 2\max(1, |u|^2)$$

we deduce that

(1.6.3) $$T(f, r_0; r) = \frac{1}{2\pi}\int_0^{2\pi} \log\max(1, |f(re^{i\theta})|)\,\mathrm{d}\theta + A\log(r) + O(1).$$

Consequently, the characteristic function $T(f, r_0; \cdot)$ is a measure of the growth of f.

§ 2. THE COUNTING FUNCTION

2.1. Let $a \in \mathbf{P}^1(\mathbf{C})$. Let $f^*(a)$ be the divisor on Ω, inverse image by f of the divisor a. One has

(2.1.1) $$f^*(a) = \operatorname{div}^+(f - a) = \sum_{z\in\Omega} \max(0, v_z(f - a))z, \qquad \text{if } a \neq \infty;$$

(2.1.2) $$f^*(\infty) = \operatorname{div}^+(1/f) = \sum_{z\in\Omega} \max(0, -v_z(f))z.$$

For any $r \in [r_0, +\infty)$, let $n(f, r_0; r)$ be the number of solutions of $f(z) = a$ in $C(r_0, r)$, counted with multiplicities. Thus,

(2.1.3) $$n(f, r_0; r, a) = \int_{C(r_0,r)} \delta_{f^*(a)}.$$

Definition **(2.2)**. — *The* counting function *is defined by*

(2.2.1) $$N(f, r_0; r) = \int_{r_0}^{r} n(f, r_0; t, a)\frac{\mathrm{d}t}{t} = \int_{C(r_0,\infty)} \log^+\frac{r}{|z|}\delta_{f^*(a)}.$$

It measures the incidence of f on a, *i.e.*, how many times f takes the value a, with a logarithmic weight.

The second equality follows from Lemma 1.3.

Proposition (**2.3**). — *The function $r \mapsto N(f, r_0; r, a)$ is continuous, nondecreasing, and vanishes at r_0; it is a convex function of* $\log(r)$.

Proof. — It is similar to that of Proposition 1.4, up to replacing $f^*\omega$ by $\delta_{f^*(a)}$. □

Proposition (**2.4**). — [1] *The function $a \mapsto N(f, r_0; r, a)$ is upper semicontinuous, locally uniformly in r.*

§ 3. THE PROXIMITY FUNCTION

Definition (**3.1**). — *Let $a \in \mathbf{P}^1(\mathbf{C})$. The* proximity function *is defined, for $r \in [r_0, \infty)$, by*

(3.1.1)
$$m(f; r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log \left\| f(re^{i\theta}), a \right\|^{-1} \mathrm{d}\theta.$$

The proximity function measures how much f is close to a on a circle of radius r. When $a = \infty$, this is a measure of the size of f on the circle of radius r; other definitions can be found in the litterature. For example, since $\|u, \infty\| = 1/\sqrt{1 + |u|^2}$ for any $u \in \mathbf{C}$, it follows from Equation 1.6.2 that

(3.1.2)
$$m(f; r, \infty) = \frac{1}{2\pi} \log^+ \left| f(re^{i\theta}) \right| \mathrm{d}\theta + \mathrm{O}(1).$$

Proposition (**3.2**). — *The proximity function is continuous and takes nonnegative values.*

Proof. — Since the chordal distance $\|a, b\|$ of any two points on $\mathbf{P}^1(\mathbf{C})$ belongs to $[0, 1]$, we have $m(f; r, a) \geq 0$ for any r. Let us show the continuity assertion. By Lebesgue's dominated convergence theorem, it suffices to show that for any $\varphi \in [0, 2\pi]$, there exists a neighborhood U of φ in $[0, 2\pi]$ and a neighborhood V of r such that $\sup_{t \in V} \log \left\| f(te^{i\theta}), a \right\|^{-1}$ is integrable on U. This is clear if $f(re^{i\varphi}) \neq a$, for then the function $z \mapsto \log \| f(z, a) \|$ is uniformly continuous in a neighborhood of $re^{i\varphi}$. In general, letting n be the coefficient of $re^{i\varphi}$ in $f^*(a)$, there exists a positive real number c and an open neighborhood B of $re^{i\varphi}$ such that

$$\| f(z); a \| \geq c \left| z - re^{i\varphi} \right|^n, \quad \text{for any } z \in B.$$

[1] Vérifier si c'est vrai, et si c'est utile. En fait, oui : c'est l'endroit pour discuter la propriété de semicontinuité de $n(f, r_0, r)$ en lien avec le théorème de Rouché.

By continuity of the exponential function, there are open neighborhoods U of φ and V of r such that $te^{i\theta} \in B$ for any $\theta \in U$. and any $t \in V$. Then, for any $\theta \in U$,

$$\begin{aligned}\sup_{t\in V} \log \left\| f(te^{i\theta}); a\right\|^{-1} &\leq \log c^{-1} + n \log \left| te^{i\theta} - re^{i\varphi}\right|^{-1} \\ &\leq \log c^{-1} + n \log \left((t - r\cos(\theta-\varphi))^2 + r^2 \sin^2(\theta-\varphi)\right)^{-1} \\ &\leq \mathrm{O}(\log(\theta-\varphi)^{-1}).\end{aligned}$$

The result follows, since the function $u \mapsto \log |u|^{-1}$ is locally integrable on $\mathbf{R}$. □

3.3. For later reference, we observe the following estimate

$$m(f; r, \infty) = \frac{1}{2\pi} \log \max(1, \left|f(re^{i\theta})\right|)\, \mathrm{d}\theta + \mathrm{O}(1), \tag{3.3.1}$$

which follows from Equation 1.6.2.

§ 4. NEVANLINNA'S FIRST THEOREM

Theorem **(4.1)** (Nevanlinna's first theorem). — *Let $a \in \mathbf{P}^1(\mathbf{C})$ and any $r \in [r_0, +\infty)$. If $a \notin f(\partial\mathbf{D}(0, r_0))$, then*

(4.1.1)

$$T(f, r_0; r) = N(f, r_0; r, a) + m(f; r, a) - m(f; r_0, a) + \log \frac{r}{r_0} \int_{\partial D(0,r_0)} \mathrm{d}^c \log \|f, a\|^{-1}.$$

Moreover, for $r \to +\infty$,

$$T(f, r_0; r) = N(f, r_0; r, a) + m(f; r, a) + \mathrm{O}(\log(r)), \tag{4.1.2}$$

where the constant underlying the O *is independent of $a \in \mathbf{P}^1(\mathbf{C})$.*

Lemma **(4.2)**. — *For any $a \in \mathbf{P}^1(\mathbf{C})$, one has the following equality of currents on Ω:*

$$f^*\omega = \delta_{f^*(a)} - \mathrm{dd^c}[\log \|a, f\|]. \tag{4.2.1}$$

Proof. — We know the equality

$$\omega = \delta_a - \mathrm{dd^c}[\log \|a, \cdot\|] \tag{4.2.2}$$

in $\mathscr{D}^2(\Omega)$ and, in principle, we would want to pull back this relation by f. However, the inverse image of a current is not defined in general, so we need a computation.

First of all, it suffices to prove the equality locally, in the neighborhood of any point b of Ω. There exists an open neighborhood V of $f(b)$ in $\mathbf{P}^1(\mathbf{C})$ and a section s of the projection $p\colon \mathbf{C}^2 \smallsetminus \{0\} \to \mathbf{P}^1(\mathbf{C})$ defined over V. Let U be an open neighborhood of b such that $f(U) \subset V$ and set $\tilde f = s \circ f$. The map $\tilde f\colon U \to \mathbf{C}^2 \smallsetminus \{0\}$ is holomorphic and satisfies $p \circ \tilde f = f$. Fix a point $\tilde a$ in $p^{-1}(a)$. Then one has, for any $z \in U$,

$$\log \|a, f(z)\| = \frac{\left|\tilde a \wedge \tilde f(z)\right|}{\|\tilde a\| \left\|\tilde f(z)\right\|}.$$

The function $\varphi = \tilde{a} \wedge \tilde{f}$ on U is holomorphic and the construction of the divisor $f^*(a)$ gives

$$(f|_U)^*(a) = \sum_{z\in U} v_z(\varphi)z.$$

Moreover, by the definition of the form ω, we have

$$(f|U)^*\omega = (f|U)s^*\mathrm{dd^c}[\log\|\cdot\|] = \mathrm{dd^c}[\log\|\tilde{f}\|].$$

Consequently, one gets the following equality of currents on U,

$$\mathrm{dd^c}[\log\|a,f\|] = \mathrm{dd^c}[\log|\varphi|] - \mathrm{dd^c}[\log\|\tilde{f}\|] = \delta_{f^*(a)} - \omega|_U.$$

This implies the lemma. □

4.3. *Proof of Nevanlinna's first theorem.* — Among the three currents in the equality of the previous lemma, two of them are measures on Ω, hence so is the third. In particular, we may multiply this relation by the function $\log(r/|z|)$ and integrate it on the annulus $\mathrm{C}(r_0, r)$. This gives

$$\begin{aligned} T(f, r_0; r) &= \int_{\mathrm{C}(r_0,r)} \log\frac{r}{|z|} f^*\omega \\ &= \int_{\mathrm{C}(r_0,r)} \log\frac{r}{|z|}\delta_{f^*(a)} - \int_{\mathrm{C}(r_0,r)} \mathrm{dd^c}\log\|a,f\| \\ &= N(f, r_0; r, a) - \int_{\mathrm{C}(r_0,r)} \mathrm{dd^c}\log\|a,f\|. \end{aligned}$$

Let us apply Green's formula to the currents $S = [\log\frac{r}{|z|}]$ and $T = [\log\|a,f\|]$ on the closed annulus $\overline{\mathrm{C}(r_0, r)}$. It follows that if $f(z)$ does not take the value a for $|z| = r_0$ or $|z| = r$,

$$\begin{aligned} \int_{\mathrm{C}(r_0,r)} \mathrm{dd^c}\log\|a,f\| &= -\int_{\partial\mathbf{D}(0,r)} \left(\log\frac{r}{|z|}\mathrm{d^c}\log\|a,f\| - \log\|a,f\|\,\mathrm{d^c}\log\frac{r}{|z|}\right) \\ &\quad + \int_{\partial\mathbf{D}(0,r_0)} \left(\log\frac{r}{|z|}\mathrm{d^c}\log\|a,f\| - \log\|a,f\|\,\mathrm{d^c}\log\frac{r}{|z|}\right). \end{aligned}$$

Observe that $\log(r/|z|)$ is identically 0 on $\partial\mathbf{D}(0,r)$, $\mathrm{d^c}\log|z| = \frac{1}{2\pi}\mathrm{d}\theta$; similarly, $\log(r/|z|)$ is identically $\log(r/r_0)$ on $\partial\mathbf{D}(0,r)$, $\mathrm{d^c}\log|z| = \frac{1}{2\pi}\mathrm{d}\theta$. Since $\log\|a,f\|$ is $\mathscr{C}^\infty$ in a neighborhood of these circles, the definition of the proximity function implies that

$$\int_{\mathrm{C}(r_0,r)} \mathrm{dd^c}\log\|a,f\| = -m(f;r,a) + m(f;r_0,a) + \log\frac{r}{r_0}\int_{\partial\mathbf{D}(0,r_0)} \mathrm{d^c}\log\|a,f\|.$$

This gives the first asserted formula, namely

$$T(f,r_0;r) = N(f,r_0;r,a) + m(f;r,a) - m(f;r_0,a) - \log\frac{r}{r_0}\int_{\partial\mathbf{D}(0,r_0)} \mathrm{d^c}\log\|a,f\|,$$

under the stronger assumption that f does not take the value a on $\partial\mathbf{D}(0,r)$ or on $\partial\mathbf{D}(0,r_0)$. In fact, all terms of this equation define continuous functions of r, so that we only need that a does not belong to $f(\partial\mathbf{D}(0,r_0))$.

Turning our interest to order of growth, the relation obviously implies that for $r \to \infty$,

$$T(f,r_0;r) = N(f,r_0;r,a) + m(f;r,a) + \mathrm{O}(\log(r)),$$

uniformly for a in any compact subset of $\mathbf{P}^1(\mathbf{C}) \smallsetminus f(\partial\mathbf{D}(0, r_0))$. However, considering another radius r_1, we have

$$T(f, r_0; r) = T(f, r_1; r) + \mathrm{O}(\log(r)), \quad N(f, r_0; r, a) = N(f, r_1; r, a) + \mathrm{O}(\log(r)),$$

so that the desired asymptotic behavior also holds if a does not belong to $f(\partial\mathbf{D}(0, r_1))$. Since

$$f(\partial\mathbf{D}(0, r_1)) \cap f(\partial\mathbf{D}(0, r_2))$$

is finite, for any pair (r_1, r_2) of distinct real numbers such that $\overline{\mathrm{C}(r_2, r_2)} \subset \Omega$, we may find tow real numbers $r_0 < r_1 < r_2$ such that

$$f(\partial\mathbf{D}(0, r_0)) \cap f(\partial\mathbf{D}(0, r_1)) \cap f(\partial\mathbf{D}(0, r_2)) = \varnothing,$$

and three compact subsets U_0, U_1, U_2 covering $\mathbf{P}^1(\mathbf{C})$ such that U_j does not meet $f(\partial\mathbf{D}(0, r_j))$ for any $j \in \{0, 1, 2\}$. The theorem follows from that.

Theorem **(4.4)** (Mean theorem). — *For any $r \in [r_0, +\infty)$, $a \mapsto N(f, r_0; r, a)$ is a nonnegative bounded Borel function of $a \in \mathbf{P}^1(\mathbf{C})$ and*

(4.4.1)
$$T(f, r_0; r) = \int_{\mathbf{P}^1(\mathbf{C})} N(f, r_0; r, a)\, \omega(a).$$

Lemma **(4.5)**. — *Let X and Y be Riemann surfaces and let $f \colon X \to Y$ be a holomorphic map. Let $\varphi \colon X \to \mathbf{C}$ be a Borel function on X; assume that φ either takes nonnegative real values, or has compact support. For any $y \in Y$, define*

$$f_*(\varphi)(y) = \int_X \varphi \delta_{\varphi^*(y)} = \sum_{x \in X} n_x \varphi(x),$$

with $f^*(y) = \sum_{x \in X} n_x x$.

Assume that φ is continuous and compactly supported (resp. that φ is Borel and takes nonnegative values). Then so is $f_*\varphi$ and

$$\int_Y f_*(\varphi)\, \alpha = \int_X \varphi f^* \alpha$$

for any differential form (resp. for any positive differential form) of degree 2 on Y.

Proof. — We shall use the Theorem of Rouché in the following form: for any $x \in X$, there exists a neighborhood U of x and a neighborhood V of $f(x)$ such that, for any $y \in V \smallsetminus \{f(x)\}$, the equation $f = f(y)$ admits exactly n_x distinct roots in U, all with multiplicity 1. Moreover, U and V can be chosen to be contained in any prescribed neighborhoods of x and $f(x)$ respectively.

Assume that φ is continuous and that its support K, is compact. Then, the function $f_*(\varphi)$ vanishes outside of the compact subset $f(K)$ of Y. It suffices to show its continuity at every point of $f(K)$. Let $b \in f(K)$, let $(a_1, \dots, a_m)$ be the family of preimages of y in K, write n_i for the multiplicity of a_i as a root of $f(x) = f(b)$. For each i, let U_i be an open neighborhood of a_i as above, and let $V_i = f(U_i)$. We may assume that $|\varphi(x) - \varphi(a_i)| \leq \varepsilon$ for any $x \in U_i$. Replacing V_i by the intersection $V = V_1 \cap \dots \cap V_m$ and U_i by $U_i \cap f^{-1}(V)$, we may also assume that all V_i are equal to a common neighborhood V.

I claim that there exists a neighborhood W of b, contained in V, such that for any $y \in W$ and any $x \in f^{-1}(y)$, either x belongs to $U_1 \cup \cdots \cup U_m$, or $\varphi(x) = 0$. Otherwise, there would exist a sequence (y_n) of elements of W converging to b, and for each n, an element $x_n \in K \cap f^{-1}(y_n)$ which does not belong to $U_1 \cup \cdots \cup U_m$. By compactness of K, we may replace the sequence (x_n) by a subsequence and assume that the sequence (x_n) converges to some point $x \in K$. Then, $f(x) = b$ so that $x \in f^{-1}(b) \cap K$. It follows that $x \in \{a_1, \ldots, a_m\}$. If $x = a_i$, then $x_n \in U_i$ for n large enough, contradiction.

For any $y \in W \setminus \{b\}$, we thus have

$$(f_*\varphi)(y) - (f_*\varphi)(b) = \sum_{i=1}^{m} \left(\sum_{x \in U_i \cap f^{-1}(y)} \varphi(x) \right) - \sum_{i=1}^{m} n_i \varphi(a_i)$$
$$= \sum_{i=1}^{m} \sum_{x \in U_i \cap f^{-1}(y)} (\varphi(x) - \varphi(a_i)),$$

so that

$$|(f_*\varphi)(y) - (f_*\varphi)(b)| \leq \left(\sum_{i=1}^{m} n_i \right) \varepsilon.$$

This shows that $f_*\varphi$ is continuous at b.

When φ is lower semi-continuous and nonnegative, the same argument shows that $f_*\varphi$ is lower semi-continuous and nonnegative at each point where it is finite. Indeed, if $f_*\varphi(b) < \infty$, we first choose finitely many elements $a_i \in f^{-1}(b)$ such that $(f_*\varphi)(b) \leq \sum_{i=1}^{m} n_i \varphi(a_i) + \varepsilon$, where n_i is the multiplicity of a_i as a root of $f(x) = f(b)$. For each i, choose also an open neighborhood U_i of a_i such that $\varphi(x) \geq \varphi(a_i) - \varepsilon$ for any $x \in U_i$. Since φ is nonnegative, we then have

$$f_*\varphi(y) \geq \sum_{i=1}^{m} m_i \varphi(a_i) - (\sum_{i=1}^{m} n_i)\varepsilon \geq (f_*\varphi)(b) - \big(1 + \sum_{i=1}^{m} n_i\big)\varepsilon.$$

If φ is measurable and nonnegative, then so is $f_*\varphi$. [2]

We now show that the two indicated integrals coincide. This is exactly the change of variables formula in the case where f is a diffeomorphism of X onto its image. Let us explain how to reduce to this case.

Let $Z \subset Y$ be the set of critical values of f. This is a countable subset of Y and f is a local diffeomorphism at any point of $X \setminus f^{-1}(Z)$. Let (λ_i) be a partition of unity in $X \setminus f^{-1}(Z)$ subordinate to an open covering (U_i) of $X \setminus f^{-1}(Z)$ such that f induces a diffeomorphism

(2) Prouver la mesurabilité...

of U_i onto its image $f(U_i)$ in Y. Then,

$$\begin{aligned}\int_X \varphi f^*\alpha &= \int_{X\smallsetminus f^{-1}(Z)} \varphi f^*\alpha \\ &= \sum_i \int_{U_i} \lambda_i \varphi f^*\alpha \\ &= \sum_i \int_{f(U_i)} f_*(\lambda_i\varphi)\alpha \\ &= \sum_i \int_Y f_*(\lambda_i\varphi)\alpha,\end{aligned}$$

since $f_*(\lambda_i\varphi)$ vanishes outside of $f(U_i)$. Now, for any $y \in Y \smallsetminus Z$,

$$\sum_i f_*(\lambda_i\varphi_i)(y) = \sum_i \sum_{x\in f^{-1}(y)} \lambda_i(x)\varphi(x) = \sum_{x\in f^{-1}(y)} \varphi(x) = f_*\varphi(y),$$

so that $\int_X \varphi f^*\alpha = \int_Y (f_*\varphi)\alpha$, as claimed. □

4.6. *Proof of the Mean theorem.* — Let $\varphi = \mathbf{1}_{C(r_0,r)} \log \frac{r}{|z|}$; For any $a \in \mathbf{P}^1(\mathbf{C})$, one has

$$N(f, r_0; r, a) = \int_\Omega \varphi \delta_{f^*(a)} = f_*\varphi(a).$$

By Lemma 4.5,

$$\int_{\mathbf{P}^1(\mathbf{C})} N(f, r_0; r, a) = \int_{\mathbf{P}^1(\mathbf{C})} \varphi f^*\omega = \int_{C(r_0,r)} \log \frac{r}{|z|} f^*\omega = T(f, r_0; r).$$

Corollary (**4.7**). — *For any $a \in \mathbf{P}^1(\mathbf{C})$ and any $r \in [r_0, +\infty)$,*

(4.7.1) $$N(f, r_0; r, a) \le T(f, r_0; r) + \mathrm{O}(\log(r)).$$

Proof. — Indeed, $m(f; r, a) \ge 0$. □

Corollary (**4.8**). — *The function f is meromorphic at infinity (equivalently, $f(1/z)$ does not have en essential singularity at 0) if and only if*

$$\lim_{r\to\infty} \frac{T(f, r_0; r)}{\log(r)} < \infty.$$

(Recall that, according to Corollary 1.5, the limit exists.)

Proof. — If f is meromorphic at infinity, it extends to a holomorphic function $\tilde f$ from $\mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{D}(0, r_0)$ to $\mathbf{P}^1(\mathbf{C})$. Then, $\tilde f^*\omega$ is a $\mathscr{C}^\infty$ differential form on $\mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{D}(0, r_0)$. Since ω is positive on $\mathbf{P}^1(\mathbf{C})$, there exists a $\mathscr{C}^\infty$ function φ on $\mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{D}(0, r_0)$ such that $f^*\omega = \varphi\omega$. Then φ is bounded, because $\mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{D}(0, r_0)$ is compact, so that

$$T(f, r_0; r) = \int_{r_0}^{r} \log \frac{r}{|z|} f^*\omega \ll \int_{r_0}^{r} \log \frac{r}{|z|} \omega \ll \log \frac{r}{r_0}.$$

Conversely, if $T(f, r_0; r) = \mathrm{O}(\log(r))$, then

$$N(f, r_0; r, a) \le T(f, r_0; r) + \mathrm{O}(\log(r)) \le \mathrm{O}(\log(r)),$$

as well. However, if $m = n(f, r_0; r_1)$, one has, for $r \geq r_1$,

$$N(f, r_0; r, a) = \int_{r_0}^{r} n(f, r_0; t) \frac{\mathrm{d}t}{t} \geq n(f, r_0; r_1, a) \int_{r_1}^{r} \frac{\mathrm{d}t}{t} = n(f, r_0; r_1, a) \log \frac{r}{r_1}.$$

Consequently, $n(f, r_0; r_1, a)$ is uniformly bounded from above, when $r_1 \geq r_0$ and $a \in \mathbf{P}^1(\mathbf{C})$. In other words, f takes each value at most finitely many times.

For any $n \in \mathbf{N}$, let $F_n \subset \mathbf{P}^1(\mathbf{C})$ be the set of points a such that f takes the value a at most n times, counted with multiplicities. Rouché's theorem asserts that if f takes the value a at $w \in \Omega$, with multiplicity m, there exists an open neighborhood V of a in $\mathbf{P}^1(\mathbf{C})$ and an open neighborhood U of w in Ω such that for any $b \in V$, the function f takes m times the value b on V, again counted with multiplicities. This implies that the complement to F_n in $\mathbf{P}^1(\mathbf{C})$, the set of values taken at least $n+1$ times, is open, so that F_n is closed in $\mathbf{P}^1(\mathbf{C})$.

Moreover, $\bigcup_{n\geq 1} F_n = \mathbf{P}^1(\mathbf{C})$. By Baire's theorem, one of these sets, say F_n, has a non-empty interior. For any element in $\mathring{F}_n$, the number of solutions of the equation $f(z) = a$, counted with multiplicities, is at most n. Let a be such an element where this number is maximal, say m, so that $a \in \mathring{F}_m$; let r_a be the largest absolute value of an element in $f^{-1}(a)$. By Rouché's theorem again, there is an open neighborhood U of a in $\mathbf{P}^1(\mathbf{C})$, contained in F_m, such that for any $b \in U$, the equation $f(z) = w$ has at least m roots in $\overline{\mathrm{C}(r_0, r_a + 1)}$, counted with multiplicities.

Necessarily f omits every value in U on $\mathrm{C}(r_a + 1, \infty)$: indeed, such a value is taken at most m times on $\mathrm{C}(r_0, \infty)$, and at least m times on $\overline{\mathrm{C}(r_0, r_a + 1)}$. It then follows from the theorem of Casorati-Weierstrass (see §1.4) that f is meromorphic at infinity. □

4.9. *Assume that f has an essential singularity at infinity.* By the preceding corollary,

$$\lim_{r\to+\infty} \frac{T(f, r_0; r)}{\log(r)} = +\infty.$$

For $a \in \mathbf{P}^1(\mathbf{C})$, the *defect* of f at a is defined by

(4.9.1)
$$\delta(f, a) = \underline{\lim}_{r\to\infty} \frac{m(f; r, a)}{T(f, r_0; r)}.$$

Since

$$T(f, r_0; r) = N(f, r_0; r, a) + m(f; r, a) + \mathrm{O}(\log(r)),$$

we also have

(4.9.2)
$$\delta(f, a) = 1 - \overline{\lim}_{r\to\infty} \frac{N(f, r_0; r, a)}{T(f, r_0; r)}.$$

From these two formulae, we see in particular that

(4.9.3)
$$\delta(f, a) \in [0, 1] \quad \text{for every } a \in \mathbf{P}^1(\mathbf{C}).$$

One has $\delta(f, a) = 1$ if f omits the value a. In general, the defect $\delta(f, a)$ measures in what respect the function f does not take the value a as much as is allowed by its growth.

By the Mean theorem,

$$\int_{\mathbf{P}^1(\mathbf{C})} \left(1 - \frac{N(f, r_0; r, a)}{T(f, r_0; r)}\right) \omega(a) = 0.$$

When $r \to +\infty$, the infimum limit of the term within the parentheses is precisely $\delta(f,a)$. It then follows from Fatou's lemma that

$$\int_{\mathbf{P}^1(\mathbf{C})} \delta(f,a)\omega(a) \leq \varliminf_{r\to\infty} \int_{\mathbf{P}^1(\mathbf{C})} \left(1 - \frac{N(f,r_0;r,a)}{T(f,r_0;r)}\right) \omega(a) = 0.$$

Since ω is a positive differential form, this implies that

(4.9.4) $$\delta(f,a) = 0 \quad \text{for almost every } a \in \mathbf{P}^1(\mathbf{C}).$$

In other words, the set of points a such that $\delta(f,a) > 0$ is Lebesgue negligible in $\mathbf{P}^1(\mathbf{C})$. In particular, the set of omitted values (at which the defect equals 1) is negligible, a much stronger property than the one asserted by theorem of Casorati-Weierstraß.

We shall prove later, this is the content of Nevanlinna's Second theorem, that

$$\sum_{a\in\mathbf{P}^1(\mathbf{C})} \delta(f,a) \leq 2.$$

In particular, the set of points a such that $\delta(f,a) > 0$ is at most countable, and the set of omitted values has cardinality at most 2 — we will thus recover the Great Picard theorem!

Example **(4.10)**. — Let us assume that $f(z) = e^z$. Then, $|f(re^{i\theta})| = e^{r\cos(\theta)}$ so that

$$\log\max(1, |f(re^{i\theta})|) = \max(0, r\cos(\theta)).$$

One has

$$\frac{1}{2\pi}\int_0^{2\pi} \log\max(1,|f(re^{i\theta})|)\mathrm{d}\theta = \frac{1}{\pi}\int_0 \pi/2 r\cos(\theta)\,\mathrm{d}\theta = \frac{r}{\pi}.$$

Given Equation (1.6.3), this implies

(4.10.1) $$T(f,r_0;r) = \frac{1}{\pi}r + \mathrm{O}(\log(r)).$$

Since f is holomorphic,

(4.10.2) $$N(f,r_0;r,\infty) = n(f,r_0;r,\infty) = 0$$

and $\delta(f,\infty) = 1$. Similarly,

(4.10.3) $$N(f,r_0;r,\infty) = n(f,r_0;r,\infty) = 0, \quad \delta(f,0) = 1.$$

Let now $a \in \mathbf{C}^*$. The roots of the equation $f(z) = a$ are $z = \log(|a|) + i(\arg(a) + 2k\pi)$, for $k \in \mathbf{Z}$, and all have multiplicity 1. Consequently,

(4.10.4) $$n(f,r_0;r,a) = \frac{1}{\pi}r + \mathrm{O}(1),$$

so that

(4.10.5) $$N(f,r_0;r,a) = \frac{1}{\pi}r + \mathrm{O}(\log(r))$$

and $\delta(f,a) = 0$. By Nevanlinna's first theorem, it follows that

(4.10.6) $$m(f;r,a) = \mathrm{o}(\log(r)) \quad \text{for any } a \in \mathbf{C}^*.$$

Example **(4.11)**. — Let λ be a real number such that $0 < \lambda < 1$ and let $g(z) = e^z + e^{\lambda z}$. In this case, we have $|g(z)| \leq 2$ if $\Re(z) \leq 0$ while, for $\theta \in [-\pi/2, \pi/2]$,

$$\left|g(re^{i\theta})\right| = e^{r\cos\theta}\left|1 + e^{-(1-\lambda)re^{i\theta}}\right| = e^{r\cos\theta}(1 + \mathrm{O}(e^{-(1-\lambda)r\cos\theta})).$$

Consequently,

(4.11.1) $$T(g, r_0; r) = \frac{1}{\pi}r + \mathrm{O}(\log(r)).$$

Since g is holomorphic, one has $\delta(g, \infty) = 0$. The equation $g(z) = 0$ is equivalent to $e^{(1-\lambda)z} = -1$ and is roots are $(2k+1)i\pi/(1-\lambda)$, for $k \in \mathbf{Z}$, each with multiplicity 1. Therefore,

(4.11.2) $$n(g, r_0; r, 0) = \frac{1-\lambda}{\pi}r + \mathrm{O}(1), N(g, r_0; r, 0) = \frac{1-\lambda}{\pi}r + \mathrm{O}(\log(r)), \delta(g, 0) = \lambda.$$

Moreover, one can prove that [3]

(4.11.3) $$\delta(g, a) = 0 \quad \text{for any } a \in \mathbf{C}^*.$$

§ 5. A VARIANT OF NEVANLINNA'S FIRST THEOREM

5.1. Let μ be a Borel probability measure on $\mathbf{P}^1(\mathbf{C})$. We define analogues of the counting, and proximity functions by averaging their values with respect to μ. Namely, we let, for any $r > r_0$,

(5.1.1) $$N(f, r_0; r, \mu) = \int_{\mathbf{P}^1(\mathbf{C})} N(f, r_0; r, a)\,\mathrm{d}\mu(a)$$

and

(5.1.2) $$m(f; r, \mu) = \int_{\mathbf{P}^1(\mathbf{C})} m(f; r, a)\,\mathrm{d}\mu(a).$$

By Nevanlinna's first theorem, especially its uniformity in $a \in \mathbf{P}^1(\mathbf{C})$, one has

(5.1.3) $$T(f, r_0; r) = N(f, r_0; r, \mu) + m(f; r, \mu) + \mathrm{O}(\log(r)).$$

The function $N(f, r_0; \cdot, \mu)$ is an increasing, convex function of $\log(r)$.
The function $m(f; \cdot, \mu)$ is nonnegative and continuous. [4]

Lemma **(5.2)**. — *Let g_μ be the function on $\mathbf{P}^1(\mathbf{C})$ given by*

(5.2.1) $$g_\mu(x) = \int_{\mathbf{P}^1(\mathbf{C})} \log \|x, a\|^{-1}\,\mathrm{d}\mu(a).$$

a) *The function g_μ is lower semi-continuous: for any real number t, the set of all x such that $g_\mu(x) > t$ is open in $\mathbf{P}^1(\mathbf{C})$.*

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b) *The function g_μ is integrable on $\mathbf{P}^1(\mathbf{C})$ and*

(5.2.2)
$$\int_{\mathbf{P}^1(\mathbf{C})} g_\mu(x)\,\omega(x) = \frac{1}{2}.$$

c) *One has*

(5.2.3)
$$\mathrm{dd^c}[g_\mu] = \omega - \mu.$$

Proof. — a) For any positive integer n and any $x \in \mathbf{P}^1(\mathbf{C})$, define
$$g_{\mu,n}(x) = \int_{\mathbf{P}^1(\mathbf{C})} \min(n, \log \|x,a\|^{-1})\,\mathrm{d}\mu(a).$$
By Lebesgue's dominated convergence theorem, $g_{\mu,n}$ is a continuous function on $\mathbf{P}^1(\mathbf{C})$. Since the sequence $(g_{\mu,n})$ is nondecreasing, the monotone convergence theorem implies
$$g_\mu(x) = \lim_{n\to\infty} \int_{\mathbf{P}^1(\mathbf{C})} \min(n, \log \|x,a\|^{-1})\,\mathrm{d}\mu(a)$$
for any $x \in \mathbf{P}^1(\mathbf{C})$. In particular, g_μ is a non-decreasing limit of continuous functions. This implies the claim. Indeed, let $t \in \mathbf{R}$ and let $x \in \mathbf{P}^1(\mathbf{C})$ such that $g_\mu(x) > t$. Let n be an integer such that $g_{\mu,n}(x) > t$. Since $g_{\mu,n}$ is continuous, the point x has an open neighborhood U such that $g_{\mu,n}(y) > t$ for any $y \in U$. In particular, $g_\mu(y) > t$ for any $y \in U$. We have shown that $g_\mu^{-1}((t,\infty))$ is open.

b) Let us apply the theorem of Fubini for the nonnegative function $\log\|x,a\|^{-1}$ on $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ endowed with the measure $\mathrm{d}\mu(a) \otimes \mathrm{d}\omega(x)$. This implies
$$\begin{aligned}\int_{\mathbf{P}^1(\mathbf{C})} g_\mu(x)\,\mathrm{d}\omega(x) &= \int_{\mathbf{P}^1(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \log\|x,a\|^{-1}\,\mathrm{d}\mu(a)\mathrm{d}\omega(x)\\ &= \int_{\mathbf{P}^1(\mathbf{C})} \Big(\int_{\mathbf{P}^1(\mathbf{C})} \log\|x,a\|^{-1}\,\mathrm{d}\omega(x)\Big)\,\mathrm{d}\mu(a).\end{aligned}$$
Since the chordal distance and the measure $\mathrm{d}\omega(x)$ on $\mathbf{P}^1(\mathbf{C})$ are invariant under the action of the group $\mathrm{SU}(2)$,
$$\lambda(a) = \int_{\mathbf{P}^1(\mathbf{C})} \log\|x,a\|^{-1}\,\mathrm{d}\omega(x)$$
is independent of a. Let us compute it when $a = \infty$. For $z \in \mathbf{C}$, we know that $\|z,\infty\| = (1+|z|^2)^{-1/2}$ and $\omega = \frac{1}{\pi}(1+|z|^2)^{-2}\,\mathrm{d}x\mathrm{d}y$, hence
$$\begin{aligned}\lambda(\infty) &= \frac{1}{2\pi}\int_{\mathbf{C}} \frac{\log(1+|z|^2)}{(1+|z|^2)^2}\,\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \frac{\log(1+r^2)}{(1+r^2)^2}\,r\mathrm{d}r\mathrm{d}\theta\\ &= \frac{1}{2}\int_0^\infty \frac{\log(1+s)}{(1+s)^2}\,\mathrm{d}s\end{aligned}$$
by the change of variables $s = r^2$. Integrating by parts, we have
$$\lambda(\infty) = \frac{1}{2}\left[-\frac{\log(1+s)}{1+s}\right]_0^\infty + \frac{1}{2}\int_0^\infty \frac{1}{(1+s)^2}\,\mathrm{d}s = \frac{1}{2}.$$

Since $\lambda(\infty) < \infty$, we conclude that g_μ is integrable on $\mathbf{P}^1(\mathbf{C})$ for the measure $\mathrm{d}\omega$, and that $\int_{\mathbf{P}^1(\mathbf{C})} g_\mu(x)\,\mathrm{d}\omega(x) = 1/2$, as claimed.

c) Let $\varphi \in \mathscr{A}^0(\mathbf{P}^1(\mathbf{C}))$. By definition of the current $\mathrm{dd^c}[g_\mu]$ and Fubini's theorem, we have

$$\begin{aligned}\mathrm{dd^c}[g_\mu](\varphi) &= \int_{\mathbf{P}^1(\mathbf{C})} g_\mu(x)\mathrm{dd^c}\varphi(x)\\ &= \int_{\mathbf{P}^1(\mathbf{C})}\int_{\mathbf{P}^1(\mathbf{C})} \log\|x,a\|^{-1}\,\mathrm{d}\mu(a)\mathrm{dd^c}\varphi(x)\\ &= \int_{\mathbf{P}^1(\mathbf{C})}\left(\int_{\mathbf{P}^1(\mathbf{C})} \log\|x,a\|^{-1}\,\mathrm{dd^c}\varphi(x)\right)\mathrm{d}\mu(a).\end{aligned}$$

By Lemma 4.5,

$$\begin{aligned}\int_{\mathbf{P}^1(\mathbf{C})} \log\|x,a\|^{-1}\,\mathrm{dd^c}\varphi(x) &= -\mathrm{dd^c}[\log\|x,a\|](\varphi) = (\omega-\delta_a)(\varphi)\\ &= \int_{\mathbf{P}^1(\mathbf{C})}\varphi(x)\omega(x) - \varphi(a).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{dd^c}[g_\mu](\varphi) &= \int_{\mathbf{P}^1(\mathbf{C})}\int_{\mathbf{P}^1(\mathbf{C})}\varphi(x)\omega(x)\,\mathrm{d}\mu(a) - \int_{\mathbf{P}^1(\mathbf{C})}\varphi(a)\,\mathrm{d}\mu(a)\\ &= \int_{\mathbf{P}^1(\mathbf{C})}\varphi(x)\omega(x) - \int_{\mathbf{P}^1(\mathbf{C})}\varphi(a)\,\mathrm{d}\mu(a)\\ &= (\omega-\mu)(\varphi),\end{aligned}$$

as was to be shown. □

Proposition **(5.3)**. — *For any real number $r > r_0$,*

(5.3.1)
$$m(f;r,\mu) = \frac{1}{2\pi}\int_0^{2\pi} g_\mu(f(re^{i\theta}))\mathrm{d}\theta.$$

Proof. — This is a simple application of the theorem of Fubini for nonnegative functions. Indeed,

$$\begin{aligned}m(f;r,\mu) &= \int_{\mathbf{P}^1(\mathbf{C})} m(f;r,a)\,\mathrm{d}\mu(a)\\ &= \int_{\mathbf{P}^1(\mathbf{C})}\frac{1}{2\pi}\int_0^{2\pi}\log\left\|f(re^{i\theta}),a\right\|^{-1}\mathrm{d}\theta\mathrm{d}\mu(a)\\ &= \frac{1}{2\pi}\int_{\mathbf{P}^1(\mathbf{C})}\log\left\|f(re^{i\theta}),a\right\|^{-1}\mathrm{d}\mu(a)\mathrm{d}\theta\\ &= \frac{1}{2\pi}g_\mu(f(re^{i\theta}))\,\mathrm{d}\theta.\end{aligned}$$
□

5.4. Let us assume that $\mu = \omega$. The computation done in Part b) of the proof of the preceding lemma shows that g_ω is the constant function $1/2$. Then $m(f;r,\omega) = 1/2$. We recover the Mean Theorem. (5)

(5) Ah bon...

Proposition (**5.5**). — *Let us assume that g_μ is bounded on $\mathbf{P}^1(\mathbf{C})$. If f has an essential singularity at infinity, then $\delta(f,a)=0$ for μ-almost every $a\in\mathbf{P}^1(\mathbf{C})$.*

Proof. — By Proposition 5.3, $m(f;r,\mu)$ is a bounded function of r. Then, Equation (5.1.3) implies that for $r\to\infty$,

$$T(f,r_0;r)=N(f,r_0;r,\mu)+\mathrm{O}(\log(r))=\int_{\mathbf{P}^1(\mathbf{C})}N(f,r_0;r,a)\,\mathrm{d}\mu(a)+\mathrm{O}(\log(r)).$$

Then we write

$$\int_{\mathbf{P}^1(\mathbf{C})}\left(1-\frac{N(f,r_0;r,a)}{T(f,r_0;r)}\right)\mathrm{d}\mu(a)=\mathrm{O}\Big(\frac{\log(r)}{T(f,r_0;r)}\Big)=\mathrm{o}(1).$$

Arguing as in §4.9, it follows from Fatou's lemma that

$$\int_{\mathbf{P}^1(\mathbf{C})}\delta(f,a)\,\mathrm{d}\mu(a)=0,$$

hence the proposition. □

Remark (**5.6**). — This is a remarkable strengthening of the property that the defects $\delta(f,a)$ vanish for almost every $a\in\mathbf{P}^1(\mathbf{C})$. Indeed, it is possible to construct measures μ for which g_μ is bounded and whose supports have ω-measure zero in $\mathbf{P}^1(\mathbf{C})$.

For example, let $\gamma\colon[0,1]\to\mathbf{P}^1(\mathbf{C})$ be a parameterized $\mathscr{C}^1$-curve with nonzero derivative. Let $\mu=\gamma_*\mathrm{d}t$ be the image of Lebesgue measure on $[0,1]$. Then, for any $x\in\mathbf{P}^1(\mathbf{C})$, one has

$$g_\mu(x)=\int_0^1\log\|x,\gamma(t)\|^{-1}\,\mathrm{d}t.$$

By Jensen's formula, for any $\alpha>0$, one has

$$g_\mu(x)=\frac{1}{\alpha}\int_0^1\log\|x,\gamma(t)\|^{-\alpha}\,\mathrm{d}t\le\frac{1}{\alpha}\log\int_0^1\|x,\gamma(t)\|^{-\alpha}\,\mathrm{d}t.$$

Up to decomposing the curve γ in finitely many part, small enough so as to be contained in open charts, we may assume that the image of γ is contained in the domain U of a chart $\varphi\colon U\to\mathbf{C}$. Up to a diffeomorphism, we may assume that the image of φ is the disk $D(0,2)$ and that $\gamma(t)=t$ for $t\in[0,1]$. It suffices to prove that g_μ is bounded on some neighborhood of $\gamma([0,1])$.

$m=(x,y)$; $d((x,y),(0,t))=|x|+|y-t|$;

$$\int_0^1\log|y-t|^{-1}\,\mathrm{d}t\le 2\int_0^1\log|u|^{-1}\,\mathrm{d}t=2.$$

§ 6. NEVANLINNA'S SECOND THEOREM

6.1. If we view f as a holomorphic map from Ω to $\mathbf{P}^1(\mathbf{C})$, its derivative is a holomorphic map from the tangent bundle T_Ω of Ω to the tangent bundle of $\mathbf{P}^1(\mathbf{C})$. We can view it as a section on Ω of the line bundle $T_\Omega^\vee\otimes f^*T_{\mathbf{P}^1(\mathbf{C})}$. For any $z\in\Omega$, the order of vanishing

$v_z(Df)$ at z of Df is given in terms of f, now viewed as a meromorphic function to $\mathbf{C}$, by the formulae:

(6.1.1)
$$v_z(Df) = \begin{cases} v_z(f') & \text{if } f(z) \neq \infty; \\ v_z(f'/f^2) & \text{if } f(z) = \infty. \end{cases}$$

The *ramification divisor* of f, $\mathrm{Ram}(f)$, is the divisor on Ω given by

(6.1.2)
$$\mathrm{Ram}(f) = \sum_{z\in\Omega} v_z(Df)z.$$

Its support is the set of points at which the holomorphic map $f\colon \Omega \to \mathbf{P}^1(\mathbf{C})$ is not a local biholomorphism. For any $a \in \mathbf{P}^1(\mathbf{C})$, one defines $\mathrm{Ram}(f,a)$ as

(6.1.3)
$$\mathrm{Ram}(f,a) = \sum_{z\in f^{-1}(a)} v_z(Df)z.$$

For any divisor D, one can define the corresponding reduced divisor D_{red} by

$$D_{\mathrm{red}}(z) = \begin{cases} 1 & \text{if } D(z) > 0; \\ 0 & \text{if } D(z) = 0; \\ -1 & \text{if } D(z) < 0. \end{cases}$$

with this notation,

(6.1.4)
$$\mathrm{Ram}(f,a) = f^*(a) - f^*(a)_{\mathrm{red}},$$

and

(6.1.5)
$$\mathrm{Ram}(f) = \sum_{a\in\mathbf{P}^1(\mathbf{C})} \mathrm{Ram}(f,a).$$

For any effective divisor D, one defines naturally the counting function with respect to D by the formula

(6.1.6)
$$N(D,r_0;r) = \int_{C(r_0,+\infty)} \log^+ \frac{r}{|z|}\delta_D.$$

For $a \in \mathbf{P}^1(\mathbf{C})$, the *ramification excess* of f at the point a is then defined by

(6.1.7)
$$\varepsilon(f,a) = \liminf_{r\to\infty} \frac{N(\mathrm{Ram}(f,a),r_0;r)}{T(f,r_0;r)}.$$

Since $\mathrm{Ram}(f,a) \leq f^*(a)$, it follows from Nevanlinna's First Theorem that $\varepsilon(f,a)$ is a nonnegative real number.

Theorem **(6.2)** (Nevanlinna's Defects Relation). — *Assume that f has an essential singularity at infinity. Then,*

(6.2.1)
$$\sum_{a\in\mathbf{P}^1(\mathbf{C})} \delta(f,a) + \varepsilon(f,a) \leq 2.$$

Corollary **(6.3)**. — *The set of points $a \in \mathbf{P}^1(\mathbf{C})$ such that $\delta(f,a) \neq 0$ or $\varepsilon(f,a) \neq 0$ is countable.*

Corollary **(6.4)** (Picard's Great Theorem). — *If f omits at least three values, then f is meromorphic at infinity.*

Proof. — Assume that f has an essential singularity at infinity. If f omits the value a, then the defect $\delta(f,a) = 1$. All other terms in Nevanlinna's Second Theorem being nonnegative, we obtain that f omits at most two values. □

We now prove Nevanlinna's Defects Relation, beginning with a technical proposition whose interest will appear later.

Lemma **(6.5)** (Émile Borel). — *Let u_0 be a real number, let I be an interval in $\mathbf{R}$ let $\varphi\colon [u_0,+\infty) \to I$ be an increasing $\mathscr{C}^1$-function.*

a) *For any Borel function $\alpha\colon I \to \mathbf{R}_+^*$ there exists a Borel subset E of $[u_0,+\infty)$ of measure at most $\int_I \mathrm{d}t/\alpha(t)$ such that $\varphi'(u) < \alpha(\varphi(u))$ for every $u \in [u_0,+\infty)$ such that $u \notin E$.*

b) *Assume that* $\inf(I) > 0$. *For any $c > 1$, there exists a Borel subset E of $[u_0,+\infty)$ of finite Lebesgue measure such that $\varphi'(u) < \varphi(u)^c$ for every $u \in [u_0,+\infty)$ such that $u \notin E$.*

Proof. — a) Let E be the set of real numbers $t \in [u_0,+\infty)$ such that $\varphi'(u) \geq \alpha(\varphi(u))$. Then, the measure of E can be estimated as follows:

$$\int_E \mathrm{d}u \leq \int_E \frac{\varphi'(u)}{\alpha(\varphi(u))}\mathrm{d}u \leq \int_{\varphi(E)} \frac{\mathrm{d}t}{\alpha(t)} \leq \int_I \frac{\mathrm{d}t}{\alpha(t)}.$$

b) Since $\inf(I) > 0$ and $c > 1$, the function $\alpha\colon t \mapsto t^{-c}$ is integrable on I. The first part of the lemma shows that there exists a Borel subset of finite Lebesgue measure such that $\varphi'(u) < \varphi(u)^c$ for any $u \in [u_0,+\infty)$ such that $u \notin E$. □

Corollary **(6.6)**. — *Let u_0 be a real number, let $\Lambda\colon [u_0,+\infty) \to \mathbf{R}_+$ be a nonnegative continuous function. For any $u \in [u_0,+\infty)$,*

$$\Theta(u) = \int_{u_0}^{+\infty} \max(u-t,0)\Lambda(t)\,\mathrm{d}t.$$

The function Θ is $\mathscr{C}^2$ and satisfies $\Theta(u_0) = \Theta'(u_0) = 0$, and $\Theta'' = \Lambda$. Moreover, for any $c > 1$, there exists an open subset E of $\mathbf{R}$ of finite Lebesgue measure surch that $\Lambda(u) \leq \Theta(u)^c$ for every $u \in [u_0,+\infty)$ such that $u \notin E$.

Proof. — For every $u \in [u_0,+\infty)$, one has

$$\Theta(u) = \int_{u_0}^{u} (u-t)\Lambda(t)\mathrm{d}t = u\int_{u_0}^{u} \Lambda(t)\mathrm{d}t - \int_{u_0}^{u} t\Lambda(t)\mathrm{d}t$$

so that Θ is $\mathscr{C}^1$ and

$$\Theta'(u) = u\Lambda(u) + \int_{u_0}^{u} \Lambda(t)\mathrm{d}t - u\Lambda(u) = \int_{u_0}^{u} \Lambda(t)\mathrm{d}t$$

for every $u \in [u_0,+\infty)$. It follows that Θ' is $\mathscr{C}^1$ and that $\Theta'' = \Lambda$. In particular, Θ is convex and increasing.

There is nothing to show if $\Lambda \equiv 0$. Otherwise, there exists $u_1 > 0$ such that $\Lambda(u_1) > 0$, so that $\Theta(u_1) > 0$ and $\Theta'(u_1) > 0$. Let $b = \sqrt{c}$. By the preceding lemma, applied to the function Θ', the interval $I = [\Theta'(u_1),+\infty)$ and the real number b, there exists a subset $E' \subset [u_1,+\infty)$ of finite Lebesgue measure such that $\Theta''(u) \leq \Theta'(u)^b$ for every $u \in [u_1,+\infty) \setminus E'$. Applying the lemma once again, to the function Θ, the interval $[\Theta(u_1,+\infty)$ and the real number b, there exists a subset $E \subset [u_1, +\infty)$ of finite Lebesgue measure such that $\Theta'(u) \leq \Theta(u)^b$ for every $u \in [u_1, +\infty) \setminus E$.

Consequently, for every $u \in [u_0, +\infty)$ such that $u \notin [u_0, u_1] \cup E \cup E'$, one has

$$\Lambda(u) = \Theta''(u) \leq \Theta'(u)^b \leq \Theta(u)^{b^2} = \Theta(u)^c,$$

whence the corollary. □

6.7. Let q be a nonnegative integer and let $a_1, \dots, a_q$ be distinct elements of $\mathbf{P}^1(\mathbf{C})$. To prove Theorem 6.2, it suffices to show the inequality

$$\sum_{n=1}^{q} \delta(f, a_n) + \varepsilon(f, a_n) \leq 2. \tag{6.7.1}$$

For the proof, we shall consider the section Df of the line bundle $T_X^\vee \otimes f^* T_Y$, and its norm $\|Df\|$ for suitable hermitian metrics on T_X and T_Y.

The chosen metric on T_X is the one for which

$$\left\| z \frac{\partial}{\partial z} \right\| = 1. \tag{6.7.2}$$

The metric on T_Y takes the points a_n into account, and is given by

$$\|\cdot\|_\varphi = \|\cdot\|_{\mathrm{FS}}\, e^\varphi, \tag{6.7.3}$$

where the function φ is defined by

$$\varphi(x) = \sum_{n=1}^{q} \log \frac{1}{\|a_n, x\| \log(e \, \|a_n, x\|^{-1})} + c, \tag{6.7.4}$$

for some real number c. We then get a new measure

$$\omega_\varphi = e^{2\varphi} \omega \tag{6.7.5}$$

on $\mathbf{P}^1(\mathbf{C})$.

Lemma **(6.8)**. — *For any $a \in \mathbf{P}^1(\mathbf{C})$, let φ_a be the function $x \mapsto \log\big(\|a, x\| \log(e/\|a, x\|)\big)^{-1}$ on $\mathbf{P}^1(\mathbf{C})$.*

a) *The function φ_a is smooth outside of a, the function $e^{2\varphi_a}$ is integrable on $\mathbf{P}^1(\mathbf{C})$ with respect to the measure given by the canonical 2-form ω.*

b) *There is a unique real number c such that ω_φ is a probability measure on $\mathbf{P}^1(\mathbf{C})$.*

Proof. — a) For $x \in \mathbf{P}^1(\mathbf{C})$, $e/\|a, x\| \geq e$, hence $\log(e/\|a, x\|) \geq 1$, so that the function φ_a is well-defined and nonnegative. By its definition, we see that it is smooth outside of a. It remains to show that $\int e^{2\varphi_a} \omega$ is finite. Since the form ω is invariant under the (transitive) action of the group $\mathrm{SU}(2,\mathbf{C})$, it suffices to treat the case $a=\infty$. Then, for $z\in\mathbf{C}$,

$$\begin{aligned} e^{2\varphi_\infty(z)}\omega(z) &= \frac{1}{\|\infty,z\|^2\left(\log(e\,\|\infty,z\|^{-1})\right)^2}\omega(z) \\ &= \frac{1}{2\pi}\frac{1+|z|^2}{\left(\log(e\sqrt{1+|z|^2})\right)^2}\frac{i\,\mathrm{d}z\mathrm{d}\bar z}{(1+|z|^2)^2} \\ &= \frac{1}{\pi}\frac{r\mathrm{d}r\mathrm{d}\theta}{(1+r^2)(1+\frac12\log(1+r^2))^2}, \end{aligned}$$

using polar coordinates. Since $1/r(\log(r))^2$ is integrable around ∞, we obtain the required integrability.

b) Since the points a_n are pairwise distinct, it follows from part a) that e^φ is smooth outside of $\{a_1,\dots,a_q\}$ and that it is bounded by a constant multiple of $e^{\varphi_{a_n}}$ in a neighborhood of a_n, for any $n\in\{1,\dots,q\}$. Consequently, $e^{2\varphi}$ is integrable on $\mathbf{P}^1(\mathbf{C})$ with respect to ω. It is then clear that there exists a unique real number c such that $e^{2\varphi}\omega$ is a probability measure. □

6.9. Since Df sends the tangent vector $\partial/\partial z$ at $z\in\Omega$ to the tangent vector $f'(z)\partial/\partial w$ at $w=f(z)$, one has

$$\|Df\|_\varphi\left\|\frac{\partial}{\partial z}\right\| = |f'(z)|\left\|\frac{\partial}{\partial w}\right\|_\varphi,$$

so that

(6.9.1)
$$\|Df(z)\|_\varphi = |z|\,e^{\varphi(f(z))}\frac{|f'(z)|}{1+|f(z)|^2}$$

Consequently,

(6.9.2)
$$f^*\omega_\varphi = e^{2\varphi(z)}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\frac{i\mathrm{d}z\mathrm{d}\bar z}{2\pi} = \|Df(z)\|_\varphi^2\frac{i\mathrm{d}z\mathrm{d}\bar z}{2\pi|z|^2}.$$

We then consider the counting function with respect to the probability measure ω_φ. As in the proof of the Mean theorem (Theorem 4.4), one deduces from Lemma 4.5 that

(6.9.3)
$$N(f,r_0;r,\omega_\varphi) = \int_{C(r_0,+\infty)}\log^+\frac{r}{|z|}f^*\omega_\varphi.$$

Replacing $f^*\omega_\varphi$ by its expression (6.9.2) in terms of $\|Df(z)\|$ and passing in polar coordinates, we obtain

(6.9.4)
$$N(f,r_0;r,\omega_\varphi) = \frac{1}{\pi}\int_{C(r_0,+\infty)}\max(\log(r)-\log(|z|),0)\,\|Df(z)\|_\varphi^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\theta.$$

We make the change of variables $u=\log(r)$, and $u_0=\log(r_0)$. For any $r>r_0$, set

(6.9.5)
$$\Lambda(r) = \frac{1}{\pi}\int_0^{2\pi}\left\|Df(re^{i\theta})\right\|_\varphi^2\,\mathrm{d}\theta.$$

This gives

(6.9.6)
$$N(f,r_0;r,\omega_\varphi) = \int_{u_0}^{\infty}\max(\log(r)-\log(t),0)\Lambda(t)\,\frac{\mathrm{d}t}{t}.$$

Let $b > 1$. By Corollary 6.6, there exists a set E of finite Lebesgue measure in $[u_0, +\infty)$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \left\|Df(re^{i\theta})\right\|_\varphi^2 \mathrm{d}\theta \leq \frac{1}{2}N(f, r_0; r, \omega_\varphi)^b$$

for any real number $r > r_0$ such that $\log(r) \notin E$. By Equation (5.1.3), we also have

$$N(f, r_0; r, \omega_\varphi) = T(f, r_0; r) - m(f; r, \omega_\varphi) + \mathrm{O}(\log(r)),$$

and $m(f; r, \omega_\varphi)$ is nonnegative. Therefore,

$$\textbf{(6.9.7)} \qquad \log\left(\frac{1}{2\pi}\int_0^{2\pi} \left\|Df(re^{i\theta})\right\|_\varphi \mathrm{d}\theta\right) \leq \mathrm{O}(\log(T(f, r_0; r))) + \mathrm{O}(\log(r)),$$

for any real number $r > r_0$ such that $\log(r) \notin E$.

Applying Jensen's inequality, it follows that

$$\textbf{(6.9.8)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \log\left\|Df(re^{i\theta})\right\|_\varphi^2 \mathrm{d}\theta \leq \mathrm{O}(\log(T(f, r_0; r))) + \mathrm{O}(\log(r))$$

for any $r > r_0$ such that $\log(r) \notin E$.

On the other hand, we have the following estimate:

Proposition **(6.10)**. — *For any $r > r_0$, one has*

$$\textbf{(6.10.1)} \quad \frac{1}{2\pi}\int_0^{2\pi} \log\left\|Df(re^{i\theta})\right\|_\varphi \mathrm{d}\theta$$
$$= N(\mathrm{Ram}(f), r_0; r) - 2T(f, r_0; r) + \sum_{n=1}^{q} m(f; r, a_n)$$
$$+ \mathrm{O}(\log(r)) + \mathrm{O}(\log T(f; r_0, r)).$$

Proof. — Recall that $\|Df(z)\|_\varphi = e^{\varphi(z)} \|Df(z)\|$, so that

(6.10.2)

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left\|Df(re^{i\theta})\right\|_\varphi \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\left\|Df(re^{i\theta})\right\| \mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \varphi(f(re^{i\theta}))\,\mathrm{d}\theta$$

The proposition obviously follows from the two following lemmas. □

Lemma **(6.11)**. — *For any $r > r_0$, one has*

$$\textbf{(6.11.1)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \log\left\|Df(re^{i\theta})\right\| \mathrm{d}\theta = N(\mathrm{Ram}(f), r_0; r) - 2T(f, r_0; r) + \mathrm{O}(\log(r)).$$

Proof. — Observe that $c_1(\overline{T_X}) = 0$ and $c_1(\overline{T_Y}) = 2\omega$. Consequently,

$$\mathrm{dd^c}\log\|Df\| = \delta_{\mathrm{Ram}(f)} - 2f^*\omega.$$

This also follows from the formula:

$$\|Df\| = |z|\,\frac{|f'(z)|}{1 + |f(z)|^2}.$$

Indeed, this expression, together with the definition of the form ω, implies the desired relation in a neighrborhood of any point $z \in \Omega$ such that $f(z) \neq \infty$, since $v_z(Df) = v_z(f')$ in that case. On the other hand, if $f(z) = \infty$, we write

$$\|Df\| = |z| \frac{|f'(z)|}{f^2(z)} \frac{1}{1 + |1/f(z)|^2},$$

and we get the desired formula, since $v_z(Df) = v_z(f'/f^2)$. Since $\mathrm{d^c} \log |z| = \mathrm{d}\theta/2\pi$, Green's formula implies that

$$\begin{aligned} N(\mathrm{Ram}(f), r_0; r) - 2T(f, r_0; r) &= \int_{C(r_0;r)} \log \frac{r}{|z|} (\delta_{\mathrm{Ram}(f)} - 2f^*\omega) \\ &= \int_{C(r_0;r)} \log \frac{r}{|z|} \mathrm{dd^c} \log \|Df\| \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left\|Df(re^{i\theta})\right\| \mathrm{d}\theta - \frac{1}{2\pi} \log \frac{r}{r_0} \int_0^{2\pi} \log \left\|Df(r_0 e^{i\theta})\right\| \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left\|Df(re^{i\theta})\right\| \mathrm{d}\theta + \mathrm{O}(\log(r)). \end{aligned}$$

□

Lemma **(6.12)**. — *Let* $a \in \mathbf{P}^1(\mathbf{C})$ *and let* $\varphi_a(x) = \log\left(\|a, x\| \log\left(e \|a, x\|^{-1}\right)\right)^{-1}$. *For any* $r > r_0$, *one has*

(6.12.1) $$\frac{1}{2\pi} \int_0^{2\pi} \varphi_a(f(re^{i\theta}))\mathrm{d}\theta = m(f; r, a) + \mathrm{O}(\log(T(f, r_0; r))).$$

Proof. — By definition of the proximity function,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \varphi_a(f(re^{i\theta}))\mathrm{d}\theta &= \frac{1}{2\pi} \int_0^{2\pi} \log\left(\left\|a, f(re^{i\theta})\right\|^{-1} \right) \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \log\left(\log(e \|a, x\|^{-1})^{-1} \right)\mathrm{d}\theta \\ &= m(f; r, a) - \frac{1}{2\pi} \int_0^{2\pi} \log\left(\log e \|a, x\|^{-1} \right)\mathrm{d}\theta. \end{aligned}$$

By Jensen's formula, the second term satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left(\log(e \|a, x\|^{-1}) \right)\mathrm{d}\theta \leq \log\left(\frac{1}{2\pi} \int_0^{2\pi} \log(e \|a, x\|^{-1})\mathrm{d}\theta \right) = \log\left(1 + m(f; r, a)\right).$$

By Nevanlinna's First Theorem,

$$1 + m(f; r, a) \leq T(f, r_0; r) + \mathrm{O}(\log(r)) = \mathrm{O}(T(f, r_0; r)).$$

This concludes the proof of the lemma. □

At this point, we have proved the following theorem.

Theorem **(6.13)** (Nevanlinna's Second Theorem). — *There exists a set E of finite Lebesgue measure such that, for any $r > r_0$ such that* $\log(r) \notin E$,

(6.13.1)

$$\sum_{n=1}^{q} m(f;r,a_n) + N(\mathrm{Ram}(f),r_0;r) - 2T(f,r_0;r) \le \mathrm{O}(\log(r)) + \mathrm{O}(\log(T(f,r_0;r))).$$

6.14. We may now complete (at last!) the proof of Nevanlinna's Defect Relation. Since $\mathrm{Ram}(f)$ is the sum of the effective divisors $\mathrm{Ram}(f,a)$, for $a \in \mathbf{P}^1(\mathbf{C})$, we also have

$$\sum_{n=1}^{q} N(\mathrm{Ram}(f,a_n),r_0;r) \le N(\mathrm{Ram}(f),r_0;r)$$

for any $r > r_0$. Consequently, if $\log(r) \notin E$,

(6.14.1)

$$\sum_{n=1}^{q} \frac{m(f;r,a_n)}{T(f,r_0;r)} + \frac{N(\mathrm{Ram}(f,a_n),r_0;r)}{T(f,r_0;r)} \le 2 + \mathrm{O}\Big(\frac{\log(r)}{T(f,r_0;r)}\Big) + \mathrm{O}\Big(\frac{\log(T(f,r_0;r))}{T(f,r_0;r)}\Big).$$

Assume that f has an essential singularity at infinity. By Corollary 4.8

$$\lim_{r\to+\infty} \frac{T(f,r_0;r)}{\log(r)} = +\infty.$$

If we let r converge to infinity within the set of real numbers such that $\log(r) \notin E$, we thus obtain the inequality

$$\sum_{n=1}^{q} \underline{\lim} \frac{m(f;r,a_n)}{T(f,r_0;r)} + \underline{\lim} \frac{N(\mathrm{Ram}(f,a_n),r_0;r)}{T(f,r_0;r)} \le 2.$$

In other words,

(6.14.2)
$$\sum_{n=1}^{q} \delta(f,a_n) + \varepsilon(f,a_n) \le 2.$$

This concludes the proof of Nevanlinna's Defects Relation. Considering the particular case where we have only two points, a similar analysis will show the following theorem.

Theorem **(6.15)** (Theorem of the logarithmic derivative). — *There exists a subset E of finite Lebesgue measure in* $\mathbf{R}$ *such that*

(6.15.1)
$$m(f'/f;r,\infty) \le \mathrm{O}(\log(T(f,r_0;r))).$$

for any real number $r > r_0$ such that $\log(r) \notin E$.

Proof. — Let us consider the particular case where we have only two points, taken to 0 and ∞ (in other words, $q = 2$, $a_1 = 0$, $a_2 = \infty$). Then,

$$\begin{aligned}\|Df(z)\|_\varphi &= \frac{|f'(z)|}{1+|f(z)|^2} |z|\, e^{\varphi_0(z)} e^{\varphi_\infty(z)} \\ &= \left|\frac{f'}{f}(z)\right| |z| \frac{1}{\log(e\,\|f(z),0\|^{-1})} \frac{1}{\log(e\,\|f(z),\infty\|^{-1})}.\end{aligned}$$

This will allow to estimate $m(f'/f;r,\infty)$ in terms of $\|Df\|_\varphi$. Indeed, for $|z| \geq 1$, we have

$$\begin{aligned}\log\left|\frac{f'}{f}(z)\right| &= \log\|Df(z)\|_\varphi - \log|z| + \log\log(e\,\|f(z),0\|^{-1}) + \log\log(e\,\|f(z),\infty\|^{-1})\\ &\leq \frac{1}{2}\log(1+\|Df(z)\|_\varphi^2)\\ &\qquad + \log\log(e\,\|f(z),0\|^{-1}) + \log\log(e\,\|f(z),\infty\|^{-1}).\end{aligned}$$

Since the right hand side is nonnegative, it follows that for $|z| \geq 1$,

$$\begin{aligned}\log\max(|f'/f(z)|,0) &\leq \frac{1}{2}\log(1+\|Df(z)\|_\varphi^2)\\ &\qquad + \log\log(e\,\|f(z),0\|^{-1}) + \log\log(e\,\|f(z),\infty\|^{-1}).\end{aligned}$$

Let us integrate this relation on $\partial D(r)$, for $r \geq 1$. Recalling Equation 3.1.2 and applying Jensen's inequality, this gives

$$\begin{aligned}m(f'/f;r,\infty) &\leq \frac{1}{2}\log\left(\frac{1}{2\pi}\int_0^{2\pi}(1+\left\|Df(re^{i\theta})\right\|_\varphi^2)\mathrm{d}\theta\right)\\ &\qquad + \log\left(\frac{1}{2\pi}\int_0^{2\pi}\log(e\left\|f(re^{i\theta}),0\right\|^{-1})\mathrm{d}\theta\right)\\ &\qquad + \log\left(\frac{1}{2\pi}\int_0^{2\pi}\log(e\left\|f(re^{i\theta}),\infty\right\|^{-1})\mathrm{d}\theta\right)\\ &\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi}\int_0^{2\pi}\left\|Df(re^{i\theta})\right\|_\varphi^2\mathrm{d}\theta\right)\\ &\qquad + \log\left(1+m(f;r,0)\right) + \log\left(1+m(f;r,\infty)\right).\end{aligned}$$

For any $a \in \mathbf{P}^1(\mathbf{C})$, we have

$$m(f;r,a) = T(f,r_0;r) - N(f,r_0;r,a) + \mathrm{O}(\log(r)) \leq \mathrm{O}(T(f,r_0;r))$$

since $N(f,r_0;r,a) \geq 0$ and $\log(r) = \mathrm{O}(T(f,r_0;r))$. By Equation (6.9.7), there exists a set $E \subset \mathbf{R}$ of finite Lebesgue measure such that

$$\log\left(\frac{1}{2\pi}\int_0^{2\pi}\left\|Df(re^{i\theta})\right\|_\varphi^2\mathrm{d}\theta\right) \leq \mathrm{O}\big(\log(T(f,r_0;r))\big),$$

for any real number $r > r_0$ such that $\log(r) \notin E$. This concludes the proof of the theorem of the logarithmic derivative. □

CHAPTER 4

ANALYTIC CURVES IN PROJECTIVE VARIETIES

§ 1. GEOMETRY OF THE PROJECTIVE SPACE

1.1. *The projective space.* — We will consider holomorphic functions with values into the projective space $\mathbf{P}^n(\mathbf{C})$. Recall that it is the space of lines in $\mathbf{C}^{n+1}$, written as the quotient of $\mathbf{C}^{n+1} \setminus \{0\}$ by the action of $\mathbf{C}^*$ acting by multiplication coordinatewise. For $(z_0, \dots, z_n) \in \mathbf{C}^{n+1} \setminus \{0\}$, the line $\mathbf{C}(z_0, \dots, z_n)$ will be written $[z_0 : \dots : z_n]$; the complex numbers $(z_0, \dots, z_n)$ will be called the homogeneous coordinates of $[z_0 : \dots : z_n]$.

The projective space has a natural structure of a complex manifold. Let $j \in \{0, \dots, n\}$ and let U_j be the open subset of $\mathbf{P}^n(\mathbf{C})$ consisting of points $[z_0 : \dots : z_n]$ with $z_j \neq 0$. On U_j, one may choose homogeneous coordinates so that the one with index j is equal to 1; this identifies U_j with the affine hyperplane of $\mathbf{C}^{n+1}$ with equation $z_j = 1$ or, forgetting this coordinate the coordinate with index j, with the affine space $\mathbf{C}^n$.

1.2. *The tautological line bundle on* $\mathbf{P}^n(\mathbf{C})$. — Let $\mathscr{O}(-1)$ be the subspace of $\mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{n+1}$ consisting of pairs (p, z) such that $z \in p$. Together with the first projection $\pi\colon \mathscr{O}(-1) \to \mathbf{P}^n(\mathbf{C})$, it is a line bundle, the structure of a vector space on the fibers is induced by the corresponding structure of $\mathbf{C}^{n+1}$.

The restriction to U_j of the line bundle $\mathscr{O}(-1)$ is trivial, for $\mathscr{O}(-1)|_{U_j}$ possesses a nonvanishing section ε_j, associating to a point $p \in U_j$ the unique pair (p, z) such that $z \in p$ and $z_j = 1$. For $p = [z_0 : \dots : z_n] \in U_i \cap U_j$, one has

$$z_i \varepsilon_i(p) = (z_0, \dots, z_n) = z_j \varepsilon_j(p).$$

The line bundle $\mathscr{O}(1)$ is defined as the dual of $\mathscr{O}(-1)$. Any linear form ξ on $\mathbf{C}^{n+1}$ gives rise to a section s_ξ of $\mathscr{O}(1)$. In this way, we get a morphism of vector spaces $(\mathbf{C}^{n+1})^\vee \to \Gamma(\mathbf{P}^n(\mathbf{C}), \mathscr{O}(1))$. This morphism is an isomorphism. First of all, it is injective: for $\xi = (\xi_0, \dots, \xi_n) \in (\mathbf{C}^{n+1})^\vee$, $p = [z_0 : \dots : z_n] \in U_j$, we have

$$z_j \varepsilon_j(p) = (z_0, \dots, z_n), \quad z_j s_\xi(\varepsilon_j(p)) = \xi_0 z_0 + \dots + \xi_n z_n.$$

If $s_\xi = 0$, then $\xi_0 z_0 + \dots + \xi_n z_n = 0$ for any $(z_0, \dots, z_n) \in \mathbf{C}^{n+1} \setminus \{0\}$, hence $\xi = 0$. It is also surjective.

1.3. Endow $\mathbf{C}^{n+1}$ with its natural structure of a hermitian space, given by $\|(z_0,\dots,z_n)\|^2 = \sum_{j=0}^n |z_j|^2$. This induces a hermitian metric on the line bundle $\mathscr{O}(-1)$, as well as on its dual $\mathscr{O}(1)$. Let $p \in \mathbf{P}^n(\mathbf{C})$ be a point of the projective space, let $L_p \subset \mathbf{C}^{n+1}$ be the corresponding line, let $z = (z_0,\dots,z_n) \in L_p$ and let ξ be a linear form on L_p. So

$$\|z\|^2 = |z_0|^2 + \cdots + |z_n|^2, \qquad \|\xi\| = \frac{\xi(z)}{\|z\|}.$$

In particular, let $(\xi_0,\dots,\xi_n) \in \mathbf{C}^{n+1}$ considered as the linear form ξ on $\mathbf{C}^{n+1}$ such that $\xi(z_0,\dots,z_n) = \sum \xi_j z_J$. Then,

$$\|s_\xi(p)\| = \frac{|\xi_0 z_0 + \cdots + \xi_n z_n|}{\left(|z_0|^2 + \cdots + |z_n|^2\right)^{1/2}}.$$

We will write $\overline{\mathscr{O}(1)}$ to denote the line bundle $\mathscr{O}(1)$ together with this hermitian metric.

1.4. *The canonical Fubiny-Study form on the projective space.* — The Fubini-Study form ω on $\mathbf{P}^n(\mathbf{C})$ is defined as the curvature of the metrized line bundle $\overline{\mathscr{O}(1)}$. It is also the unique differential form on $\mathbf{P}^n(\mathbf{C})$ such that

$$\pi^*\omega = \frac{1}{2}\mathrm{dd^c}\log\left(\sum_{j=0}^n |z_j|^2\right),$$

where $\pi\colon \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n(\mathbf{C})$ is the natural projection.

1.5. A (projective) hyperplane of $\mathbf{P}^n(\mathbf{C})$ is the image by the projection $\pi\colon \mathbf{C}^{n+1} \setminus \{0\}$ of a hyperplane of $\mathbf{C}^{n+1}$. In other words, a hyperplane H of $\mathbf{P}^n(\mathbf{C})$ is the set of points $p = [z_0 : \dots : z_n]$ whose homogeneous coordinates satisfy some linear equation $\xi_0 z_0 + \cdots + \xi_n z_n$.

For $p = [z_0 : \dots : z_n] \in \mathbf{P}^n(\mathbf{C})$, one defines

$$d(p,H) = \frac{|\xi_0 z_0 + \cdots + \xi_n z_n|}{\left(|\xi_0|^2 + \cdots + |\xi_n|^2\right)^{1/2}}.$$

This is the *distance* of p to the hyperplane H. It vanishes if and only if $p \in H$. If one writes $\xi = (\xi_0,\dots,\xi_n)$, it follows that

$$d(p,H) = \|s_\xi(p)\|\,\|\xi\|.$$

§ 2. CHARACTERISTIC, COUNTING AND PROXIMITY FUNCTIONS

Definition (**2.1**). — *The characteristic function of f is defined by*

$$T(f,r_0,r) = \int_{C(r_0,+\infty)} \log^+\left|\frac{r}{z}\right| f^*\omega.$$

Proposition (**2.2**). — *The function $T(f,r_0;r)$ is increasing, and a convex function in* $\log(r)$. *In particular, $T(f,r_0;r)/\log(r)$ has a limit when $r \to +\infty$.*

Definition **(2.3)**. — *Let H be a hyperplane of $\mathbf{P}^n(\mathbf{C})$, with equation $\xi_0 z_0 + \dots + \xi_n z_n = 0$; let $\xi = (\xi_0, \dots, \xi_n)$.*

If $f(\Omega) \not\subset H$, then, the proximity function is defined by

$$m(f;r,H) = \int_{C(r)} \log d(f,H)^{-1} = \int_0^{2\pi} \log d(f(re^{i\theta}),H)^{-1} \frac{d\theta}{2\pi}.$$

Proposition **(2.4)**. — *The proximity function is a continuous and nonnegative function of r.*

2.5. Let H be a hyperplane of $\mathbf{P}^n(\mathbf{C})$, with equation $\xi_0 z_0 + \dots + \xi_n z_n = 0$; let $\xi = (\xi_0, \dots, \xi_n)$. Then, $f^* s_\xi$ is a section of $f^*\mathscr{O}(1)$ on Ω which vanishes at points $z \in \Omega$ such that $f(z) \in H$. In particular, it is not identically 0 if $f(\Omega) \not\subset H$; in this case, we will write $f^* H$ for its divisor.

Lemma **(2.6)**. — *Assume that $f(\Omega) \not\subset H$. Then, $f^* \log \|s_\xi\|$ is locally integrable on Ω and*

$$f^*\omega = \delta_{f^*H} - dd^c[f^* \log s_\xi].$$

Definition **(2.7)**. — *The counting function of f with respect to H is defined by*

$$N(f,r_0;r,H) = \int_{C(r_0,\infty)} \log^+ \left|\frac{r}{z}\right| \delta_{f^*H}.$$

Theorem **(2.8)** (First main theorem). — *For any hyperplane H such that $f(\Omega) \not\subset H$, one has*

$$T(f,r_0;r) = N(f,r_0;r,H) + m(f;r,H) + O(\log(r)).$$

Proposition **(2.9)**. — *Embed $\mathbf{P}^n(\mathbf{C})$ into $\mathbf{P}^{n+1}(\mathbf{C})$ by the map $i\colon [z_0 : \dots : z_n] \mapsto [z_0 : \dots : z_n : 0]$. Then $T(f,r_0;r) = T(i \circ f, r_0; r)$ for any $r > r_0$.*

Corollary **(2.10)**. — *When $r \to +\infty$, $T(f,r_0;r)/\log(r)$ has a finite limit if and only if f is holomorphic at infinity,* i.e., *extends to a holomorphic function from $\Omega \cup \{\infty\}$ to $\mathbf{P}^n(\mathbf{C})$.*

§ 3. WRONSKIAN

Definition **(3.1)**. — *Let U be an open subset of $\mathbf{C}$ and let $F = (f_0, \dots, f_n)$ be a holomorphic function from U to $\mathbf{C}^{n+1}$. The* Wronskian *of F is the holomorphic function on U given by*

$$W(F) = \det \begin{pmatrix} f_0 & f_0' & \dots & f_0^{(n)} \\ f_1 & f_1' & \dots & f_1^{(n)} \\ \vdots & \vdots & & \vdots \\ f_n & f_n' & \dots & f_n^{(n)} \end{pmatrix}.$$

Lemma **(3.2)**. — *Let U be an open subset of $\mathbf{C}$, let $F\colon U \to \mathbf{C}^{n+1}$ be a holomorphic function from U to C^{n+1}.*

a) *For any matrix $A \in M_{n+1}(\mathbf{C})$, one has*

$$W(A \cdot F) = \det(A)\, W(F).$$

b) *For any holomorphic function φ on U, one has $W(\varphi F) = \varphi^{n+1} W(F)$.*

Proof. — *a*) Write $A = (a_{ij})$, $F = (f_0, \dots, f_n)$ and $G = A \cdot F = (g_0, \dots, g_n)$. For any i and $j \in \{0, \dots, n\}$,

$$g_i = \sum_{k=0}^{n} a_{ik} f_k,$$

hence

$$g_i^{(j)} = \sum_{k=0}^{n} a_{ik} f_k^{(j)}.$$

This implies the following equality of matrices

$$\left(g_i^{(j)}\right) = A \cdot \left(f_i^{(j)}\right),$$

hence $\mathrm{W}(G) = \det(A)\,\mathrm{W}(F)$.

b) Set $g_i = \varphi f_i$. By the Leibniz rule for derivation of products, one has

$$g_i^{(j)} = \sum_{k=0}^{j} \binom{j}{k} \varphi^{(j-k)} f_i^{(k)},$$

for any i and $j \in \{0, \dots, n\}$. Let $\Phi = (\varphi_{kj})$ be the matrix with (k, j)-entry given by

$$\varphi_{kj} = \begin{cases} \binom{j}{k} \varphi^{(j-k)} & \text{if } j \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

One has the equality of matrices

$$\left(g_i^{(j)}\right) = \left(f_i^{(j)}\right) \cdot \Phi.$$

The matrix Φ is upper-triangular, and all diagonal entries are equal to φ, so that $\det(\Phi) = \varphi^{n+1}$. Consequently, $\mathrm{W}(\varphi F) = \varphi^{n+1}\,\mathrm{W}(F)$, as was to be shown. □

Proposition **(3.3)**. — *Let U be a connected open subset of $\mathbf{C}$, let $F\colon U \to \mathbf{C}^{n+1}$ be a holomorphic function from U to $\mathbf{C}^{n+1}$. Then the following properties are equivalent:*

a) *The Wronskian $\mathrm{W}(F)$ vanishes identically on U;*

b) *There exists an hyperplane of $\mathbf{C}^{n+1}$ which contains the image of F;*

c) *There exists a nonzero linear form φ on $\mathbf{C}^{n+1}$ such that $\langle \varphi, F(z) \rangle = 0$ for every $z \in U$.*

Proof. — The last two properties are obviously equivalent; assume they hold and let $(a_0, \dots, a_n)$ be a nonzero vector in $\mathbf{C}^{n+1}$ such that $a_0 f_0 + \dots + a_n f_n$ vanishes identically on U. Consequently,

$$a_0 f_0^{(j)} + \dots + a_n f_n^{(j)} \equiv 0$$

for each $j \in \{0, \dots, n\}$, so that the columns of the Wronskian matrix of F are linearly dependent. This implies $\mathrm{W}(F) = 0$.

Let us prove the result in the other direction. For $n = 0$, $F = f_0 = \mathrm{W}(F)$, hence the result in that case. We prove the result by induction on n, assuming that $n \geq 1$ and that the result holds for $n - 1$. Let $F\colon U \to \mathbf{C}^{n+1}$ be a holomorphic map such that $\mathrm{W}(F) = 0$. If f_0 is identically 0, then the image of F is contained in the hyperplane with equation $x_0 = 0$. Otherwise, there exists a non-empty connected open subset V of U such that f_0 is invertible on V. Set $g_i = f_i / f_0$ for $i \in \{0, \dots, n\}$ and let $G\colon V \to \mathbf{C}^{n+1}$ be the holomorphic map given by $(g_0, \dots, g_n)$. One has $G = f_0^{-1}F$, hence $\mathrm{W}(G) = f_0^{-n-1}\mathrm{W}(F) = 0$. On the other hand, since $g_0 \equiv 1$, one has

$$\mathrm{W}(G) = \det\begin{pmatrix} 1 & 0 & \dots & 0 \\ g_1 & g_1' & \dots & g_1^{(n)} \\ \vdots & \vdots & & \vdots \\ g_n & g_n' & \dots & g_n^{(n)} \end{pmatrix} = \det\begin{pmatrix} g_1' & \dots & g_1^{(n)} \\ \vdots & & \vdots \\ g_n' & \dots & g_n^{(n)} \end{pmatrix}.$$

By induction, $g_1', \dots, g_n'$ are linearly dependent and there exist a nonzero family $(a_1, \dots, a_n)$ of complex numbers such that $a_1 g_1' + \dots + a_n g_n'$ vanishes identically on V. Since V is connected, $a_1 g_1 + \dots + a_n g_n$ takes a constant value on V, say $-a_0$, so that $a_0 + a_1 g_1 + \dots + a_n g_n$ vanishes identically on V. Multiplying by f_0, we see that $a_0 f_0 + a_1 f_1 + \dots + a_n f_n$ vanishes identically on V. Since U is connected, the principle of analytic extension implies that this identity holds on the whole of U, as was to be shown. ☐

§ 4. THE THEOREM OF CARTAN

4.1. Let Ω be a connected open neighborhood of ∞ in $\mathbf{C}$; let $r_0 > 0$ be such that $\Omega \supset \mathrm{C}(r_0, +\infty)$. Let $f\colon \Omega \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic map. One says that f is *nondegenerate* if its image $f(\Omega)$ is not contained in a hyperplane of $\mathbf{P}^n(\mathbf{C})$. As we have see, this is equivalent to the fact that the Wronskian $\mathrm{w}(f) \in \Gamma(\Omega, f^*\mathscr{O}(n+1))$ of f is nonzero. Its divisor $\mathrm{div}(\mathrm{w}(f))$ is called the *ramification divisor.*

4.2. Let $H_1, \dots, H_q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$. For every $j \in \{1, \dots, q\}$, let ξ_j be a nonzero linear form on $\mathbf{C}^{n+1}$ defining H_j. One says that $(H_1, \dots, H_q)$ are in general position if $(\xi_{j_1}, \dots, \xi_{j_p})$ is free, for any integer $p \in \{1, \dots, n+1\}$ and any sequence $(j_1, \dots, j_p)$ of distinct integers in $\{1, \dots, q\}$.

Theorem **(4.3)** (Cartan). — *Let $f\colon \Omega \to \mathbf{P}^n(\mathbf{C})$ be a nondegenerate holomorphic map. Let $(H_1, \dots, H_q)$ be a family of hyperplanes in general position. Then, there exists a subset E of* $\mathbf{R}$ *such that* $\log(E)$ *has finite measure and such that*

$$(q-n-1)T(f, r_0; r) \le \sum_{j=1}^{q} N(f, r_0; r, H_j) - N(r_0; r, \mathrm{Ram}(f)) + \mathrm{O}(\log(T(f, r_0; r))) + \mathrm{O}(\log(r)),$$

for any $r \in (r_0, +\infty) \setminus E$.

Remark **(4.4)**. — The case where $n = 1$ is exactly Nevanlinna's second theorem.

Corollary **(4.5)**. — *Let $f\colon \Omega \to \mathbf{P}^n$ be a non degenerate holomorphic map. Let $q \ge n+2$ and let $H_1, \dots, H_q$ be hyperplanes in general positition. Assume that $f(\Omega) \subset \complement \bigcup_{j=1}^{q} H_j$. Then f is holomorphic at infinity.*

Proof. — Indeed, $N(f, r_0; r, H_j) = 0$ for every j and every $r > r_0$. Since $q \ge n+2$ and $N(r_0; r, \mathrm{Ram}(f)) \ge 0$, we get

$$T(f, r_0; r) \le \mathrm{O}(\log(T(f, r_0; r))) + \mathrm{O}(\log(r)),$$

for every $r > r_0$ such that $r \notin E$. As a consequence, $T(f, r_0; r) \leq O(\log(r))$, hence f is holomorphic at infinity. □

Corollary **(4.6)**. — *Let $f\colon \mathbf{C} \to \mathbf{P}^n$ be a holomorphic map, let $q \geq n+2$ and let $H_1, \dots, H_q$ be hyperplanes in general position. If $f(\mathbf{C}) \subset \complement \bigcup H_j$, then f is degenerate.*

Proof. — By the preceding corollary, f extends to a holomorphic map from $\mathbf{P}^1$ to $\mathbf{P}^n$. Since $q \geq n+2 \geq 2$, there exists $j \in \{1, \dots, n+2\}$ such that $f(\infty) \notin H_j$, so that f is a holomorphic map from $\mathbf{P}^1(\mathbf{C})$ to $\mathbf{P}^n(\mathbf{C}) \smallsetminus H_j$. However, $\mathbf{P}^n(\mathbf{C}) \smallsetminus H_j$ is biholomorphically isomorphic to $\mathbf{C}^n$. This identifies f with a holomorphic map from $\mathbf{P}^1(\mathbf{C})$ to $\mathbf{C}^n$. It must be bounded, hence constant. □

Corollary **(4.7)** (Borel). — *Let $f_0, \dots, f_n$ be holomorphic nonvanishing functions on $\mathbf{C}$ such that $f_0 + \dots + f_n = 1$. Then there exists a finite family $(g_m)_{m\in M}$ of holomorphic nonvanishing functions on $\mathbf{C}$, a surjective map $\varphi\colon \{0, \dots, n\} \to M$, as well as complex numbers $(a_i)_{0\leq i\leq n}$ such that*

- *for $m \neq m'$, g_m and $g_{m'}$ are not proportional;*
- *for any $i \in \{0, \dots, n\}$, $f_i = a_i g_{\varphi(i)}$;*
- *for any $m \in M$ such that g_m is nonconstant, then $\sum_{i\in\varphi^{-1}(m)} a_i = 0$.*

Say two functions are equivalent if their quotient is constant. In the sum $f_0 + \dots + f_n$, we can trivially combine all terms from a given equivalence class; we get either the function 0, or an invertible function of the same class. The meaning of the corollary is the following: the sum of all functions f_j in a given equivalence class is equal to 0, unless this is the class of the constant function.

Proof. — We may also assume that the relation $f_0 + \dots + f_n$ is minimal, in the sense that no two functions $f_0, \dots, f_n$ are proportional; we then need to prove that $n = 0$. We shall therefore prove by induction on n the following statement: there is no minimal relation if $n \geq 1$.

Argue by contradiction, and let us consider a minimal relation $f_0 + \dots + f_n$. Let us consider the map $f\colon \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ given by $f(z) = [f_0(z) : \ldots : f_n(z)]$. For $j \in \{0, \dots, n\}$, let H_j be the hyperplane with equation $z_j = 0$; let H_{n+1} be the hyperplane with equation $z_0 + \dots + z_n = 0$. By assumption, $f(\mathbf{C}) \subset \complement \bigcup H_j$. By the preceding corollary, the map f is degenerate, hence there exist complex numbers $a_0, \dots, a_n$, not all 0, such that $a_0 f_0 + \dots + a_n f_n = 0$. In other words, $(f_0, \dots, f_n)$ are linearly dependent.

To fix the ideas, suppose that $a_n \neq 0$. Then, by subtraction, we get a relation

$$(1 - \frac{a_0}{a_n}) f_0 + \dots + (1 - \frac{a_{n-1}}{a_n}) f_{n-1} = 1.$$

Eliminating the terms with $a_j = a_n$, we obtain a relation of the same form, but with $\leq n$ terms. By induction and the assumption that no two f_j are proportional, this relation must be trivial. In particular, up to renumbering the indices, we have $a_1 = \dots = a_{n-1} = a_n$, $a_0 \neq a_n$, and f_0 is constant. It follows that

$$f_1 + \dots + f_n = \frac{1}{a_n}(a_0 f_0 + \dots + a_n f_n) - \frac{a_0}{a_n} f_0 = -\frac{a_0}{a_n} f_0.$$

If $a_0 \neq 0$, the right hand side is a nonzero constant; dividing by $-a_0 f_0/a_n$, we obtain a minimal relation with n terms. This implies that $n = 1$ and that f_1 is constant, contradicting the minimality of the original relation $f_0 + \cdots + f_n = 1$.

Therefore, $a_0 = 0$ and $f_1 + \cdots + f_n = 0$. We now write the latter relation as

$$(-f_2/f_1) + \cdots + (-f_n/f_1) = 1.$$

This is a minimal relation of the same form with $n - 1$ terms. Therefore, $n = 2$ and $-f_2/f_1 = 1$, so that f_1 and f_2 are proportional, contradicting again the minimality of the original relation. □

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