TOPICS
IN TROPICAL GEOMETRY

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CHAPTER 6

TROPICAL INTERSECTIONS

6.1. Minkowski weights

All polyhedra are implicitly assumed to be rational.

6.1.1. — Let $L \simeq \mathbf{Z}^n$ be a free finitely generated $\mathbf{Z}$-module and let $V = L_{\mathbf{R}} \simeq \mathbf{R}^n$ be the associated $\mathbf{R}$-vector space.

Let p be an integer such that $0 \leqslant p \leqslant n$. We define as follows the group $F_p(V)$ of p-dimensional *weighted polyhedral subspaces* of V: it is generated by closed polyhedra of dimension $\leqslant p$ in V with the following relations:

(i) $[P] = 0$ for every polyhedron P such that $\dim(P) < p$;

(ii) $[P]+[P\cap H]=[P\cap V_+]+[P\cap V_-]$ whenever P is a p-dimensional polyhedron in V and V_+, V_- are half-spaces such that $V_+\cap V_-$ is a hyperplane H and $V=V_+\cup V_-$.

Note that this second relation is trivial when $P\subset H$; on the other hand, if $P\not\subset H$, then $\dim(P\cap H)<\dim(P)\leqslant p$, so that the first relation implies $[P\cap H]=0$ and that second one relation can be rewritten as $[P]=[P\cap V_+]+[P\cap V_-]$.[1]

The submonoid of $F_p(V)$ generated by the classes [P] of polyhedral subspaces is denoted by $F_p^+(V)$. Its elements are said to be *effective*.

The group $F_0(V)$ identifies with $\mathbf{Z}^{(V)}$, the free abelian group on V. We denote by $\deg\colon F_0(V)\to\mathbf{Z}$ the unique morphism of groups such that $\deg([x])=1$ for every $x\in V$.

6.1.2. — As for any group defined by generators and relations, one defines a morphism λ from $F_p(V)$ to a given abelian group A by prescribing $\lambda(P)$ for every polyhedron P of V such that $\dim(V)\leqslant p$ such that $\lambda(P)=0$ if $\dim(P)<p$ and $\lambda(P)+\lambda(P\cap H)=\lambda(P\cap V_+)+\lambda(P\cap V_-)$ for every hyperplane H of V dividing V into two closed half-spaces V_+ and V_-.

The simplest example of such a morphism is given by the Lebesgue measure μ_W on a subspace W of V such that $\dim(W)=p$. Let indeed C be a compact polyhedron of W; for every polyhedron P of V such that $\dim(P)\leqslant p$, set $\lambda_C(P)=\mu_W(C\cap P)$. If $\dim(P)<p$, then $\dim(C\cap P)<p$ hence $\lambda_C(P)=0$; on the other

[1] Ajouter un dessin avec P, H, V_+, V_-.

hand, if H is a hyperplane of V dividing V into two closed half-spaces V_+ and V_-, then the additivity of measure implies that $\lambda_C(P) + \lambda_C(P \cap H) = \lambda_C(P \cap V_+) + \lambda_C(P \cap V_-)$. Consequently, there exists a unique morphism of abelian groups $\lambda_C : F_p(V) \to \mathbf{R}$ such that $\lambda_C([P]) = \mu_W(P \cap C)$ for every closed polyhedron P of V such that $\dim(P) \leqslant p$.

Observe that $\lambda_C(S) \geqslant 0$ for every effective class $S \in F_p^+(V)$.

6.1.3. — Every closed polyhedral subspace P of V such that $\dim(P) \leqslant p$ has a class [P] in $F_p(V)$: it is the sum of all polyhedra of any polyhedral decomposition of V. This class is effective and vanishes if and only if $\dim(P) < p$.

For every element S of $F_p(V)$, there exists a polyhedral decomposition $\mathscr{C}$ of V and a family $(w_C)_{C\in\mathscr{C}_p}$, where $\mathscr{C}_p$ is the set of all polyhedra $C \in \mathscr{C}$ such that $\dim(C) = p$, such that

$$S = \sum_{C\in\mathscr{C}_p} w_C[C].$$

One then says that $\mathscr{C}$ is *adapted* to S.

Let K be a convex compact polyhedron of dimension p contained in a polyhedron $C \in \mathscr{C}_p$; then one has $\lambda_K(S) = w_C\lambda_K(C \cap K)$. This shows that the family (w_C) is uniquely determined by S and the given polyhedral decomposition. Moreover, S is effective if and only $w_C \geqslant 0$ for every $C \in \mathscr{C}_p$. The element w_C is called the *weight* of C in S.

More generally, if $S' = \sum_{C'\in\mathscr{C}'_p} w'_{C'}[C']$ is another class $S' \in F_p(V)$ adapted to a polyhedral decomposition $\mathscr{C}'$, then the equality $S = S'$ is equivalent to the equalities $w_C = w'_{C'}$ for every pair of polyhedra $(C, C') \in \mathscr{C}_p \times \mathscr{C}'_p$ such that $\dim(C \cap C') = p$.

The union of all polyhedra $C \in \mathscr{C}$ such that $w_C \neq 0$ is called the *support* of S, and is denoted by $|S|$. It is a polyhedral subspace of V, and is everywhere of dimension p.

One has $|S + S'| \subset |S| \cup |S|'$ and $|mS| = |S|$ for every non-zero integer m.

Let A be an abelian group. A similar definition allows to define the group $F_p(V; A)$ of polyhedra with coefficients in A.

6.1.4. — Let us recast the balancing condition in this context. Let $S \in F_p(V)$ be a weighted polyhedral subspace of dimension $\leqslant p$.

Let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S, and let $S = \sum_{C\in\mathscr{C}_p} w_C[C]$.

Let $D \in \mathscr{C}$ be a polyhedron of dimension $p - 1$. Let $\mathscr{C}_D$ be the set of all polyhedra $C \in \mathscr{C}$ of which D is a face and such $\dim(C) = p$.

For every $C \in \mathscr{C}$, let V_C be the lineality space of $\langle C\rangle$; since the polyhedron C is rational, the intersection $L_C = V_C \cap L$ is a free finitely generated submodule of L of rank $\dim(C)$. For every $C \in \mathscr{C}_D$, there exists a vector vector $v_C \in L_C \cap C$ which generates the quotient abelian group L_C/L_D; such a vector is unique modulo L_D. We say that S satisfies the balancing condition along D if one has

$$\sum_{C\in\mathscr{C}_D} w_C v_C \in L_D.$$

We say that S is balanced (in dimension p) if it satisfies the balancing condition along all $(p-1)$-dimensional polyhedra of $\mathscr{C}$.

This condition is independent of the choice of the polyhedral decomposition which is adapted to S.

If $S, S' \in F_p(V)$ are balanced weighted polyhedral subspace, then so are $S + S'$ and mS, for every $m \in \mathbf{Z}$.

6.1.5. — Let $S \in F_p(V)$ and $x \in V$. One says that S is a fan with apex x if there exists a polyhedral decomposition of V adapted to S of which every polyhedron is a cone with apex x.

Let $S \in F_p(V)$ let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S; write $S = \sum w_C[C]$. Let $x \in V$ and let $\mathscr{C}_x$ be the set of polyhedra in $\mathscr{C}$ which contain x; their union is a neighborhood of x in V. For every $C \in \mathscr{C}_x$, let $\lambda_x(C) = \mathbf{R}_+(C - x)$ be the cone with apex x generated by C; the set of all $\lambda_x(C)$, for $C \in \mathscr{C}_x$ is a fan of V. Then $\lambda_x(S) = \sum_{C\in\mathscr{C}_x} w_C[\lambda_x(C)]$ is a fan with apex x.

Moreover, S satisfies the balancing condition along a polyhedron $D \in \mathscr{C}_x$ if and only if $\lambda_x(S)$ satisfies the balancing condition along $\lambda_x(D)$. In particular, if S is balanced, then so is $\lambda_x(S)$.

Definition **(6.1.6)**. — *A balanced p-dimensional weighted polyhedral subspace is called a p-dimensional* Minkowski weight*, or a p-dimensional* tropical cycle.

They form a subgroup $\mathrm{MW}_p(V)$ of $F_p(V)$.
[2]

Example **(6.1.7)**. — Let K be a nonarchimedean valued field, let X be a subvariety of $\mathbf{G}_{\mathrm{m}_K}^n$ and let $p = \dim(X)$. The tropicalization $\mathscr{T}_X$ of X is a polyhedral subspace of $\mathbf{R}^n$ of dimension p. There exists a polyhedral decomposition $\mathscr{C}$ of $\mathbf{R}^n$ such that the set $\mathscr{C}_X$ of all polyhedra in $\mathscr{C}$ that meet $\mathscr{T}_X$ is a polyhedral decomposition of $\mathscr{T}_X$. For $C \in \mathscr{C}_X$ with $\dim(C) = p$, we have defined a multiplicity $\mathrm{mult}_{\mathscr{T}_X}(C)$; Then $S = \sum_{C\in\mathscr{C}_X} \mathrm{mult}_{\mathscr{T}_X}(C)[C]$ is a weighted polyhedral subspace of V of dimension p with support $\mathscr{T}_X$. It satisfies the balancing condition, hence defines a Minkowski weight in $\mathrm{MW}_p(\mathbf{R}^n)$. By abuse of language, this Minkowski weight is still denoted by $\mathscr{T}_X$.

Example **(6.1.8)**. — The Bergman fan $\Sigma(M)$ of a matroid, more generally, the tropical linear space associated with a valuated matroid, is the support of a Minkowski weight (all weights are equal to 1).

Example **(6.1.9)**. — Let $n = \dim(V)$; the class $[V] \in F_n(V)$ is balanced. The morphism $\mathbf{Z} \to \mathrm{MW}_n(V)$ given by $a \mapsto a[V]$ is injective; let us show that it is an isomorphism

Let $S \in \mathrm{MW}_n(V)$ and let $\mathscr{C}$ be polyhedral decomposition of V which is adapted to S; write $S = \sum_C w_C[C]$. Let $D \in \mathscr{C}$ be a polyhedron of dimension $n-1$. There are exactly two polyhedra $C, C' \in \mathscr{C}$ containing D such that $\dim(C) = \dim(C') = n$: the affine space V_D generated by D is a hyperplane that delimits V in

[2]Define $F_p(V; A)$ and $\mathrm{MW}(V; A)$ for any abelian group A?

two half-spaces, one containing C, the other C′. The vectors v_{C} and $v_{\mathrm{C}'}$ that appear in the formulation of the balancing condition can then be chosen opposite, hence $w_{\mathrm{C}} = w_{\mathrm{C}'}$.

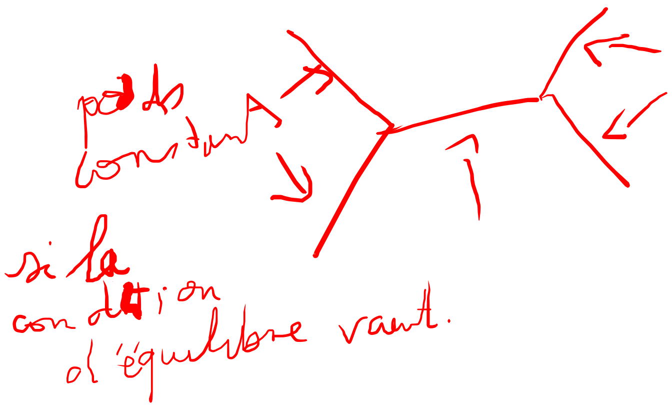
Let then C, C' be arbitrary polyhedra of dimension n in $\mathscr{C}$. There exists a sequence $(\mathrm{C}_0, \dots, \mathrm{C}_m)$ of polyhedra in $\mathscr{C}$ such that $\mathrm{C}_0 = \mathrm{C}$, $\mathrm{C}_m = \mathrm{C}'$, and such that for each $k \in \{1, \dots, m\}$, C_{k-1} and C_k share a face of dimension $n-1$; By what precedes, one then has $w_{\mathrm{C}_{k-1}} = w_{\mathrm{C}_k}$. Consequently, $w_{\mathrm{C}} = w_{\mathrm{C}_0} = w_{\mathrm{C}_1} = \cdots = w_{\mathrm{C}_m} = w_{\mathrm{C}'}$. Let a be this common value.

Finally, one has $\mathrm{S} = \sum_{\mathrm{C}} a[\mathrm{C}] = a[\mathrm{V}]$.

Remark **(6.1.10)**. — One can amplify the previous example for Minkowski weights of arbitrary dimension. Let indeed $\mathrm{S} \in \mathrm{F}_p(\mathrm{V})$ be a weighted polyhedral subspace. The support of S, $|\mathrm{S}|$, is a polyhedral subspace, and the weight of S can be viewed as a function from $|\mathrm{S}|$ to $\mathbf{Z}$ which is defined and locally constant outside of a $(p-1)$-dimensional polyhedral subspace of $|\mathrm{S}|$, the union of the polyhedra of dimension $< p$ contained in $|\mathrm{S}|$ in a polyhedral decomposition of V which is adapted to S.

Let P be a polyhedron of dimension p which is contained in $|\mathrm{S}|$ and such that $|\mathrm{S}|$ is a submanifold at every point of $\mathring{\mathrm{P}}$. In other words, $\mathring{\mathrm{P}}$ is open in $|\mathrm{S}|$.

If S is balanced, then its weight is constant on P.



Example **(6.1.11)**. — Let L, L' be free finitely generated abelian groups, let $V = L_{\mathbf{R}}$ and $V' = L'_{\mathbf{R}}$. There exists a unique bilinear map

$$F_p(V) \times F_q(V') \to F_{p+q}(V \times V')$$

such that $([C],[C']) \to [C \times C']$ for every p-dimensional polyhedron C in V and every q-dimensional polyhedron C' in V'. If $S \in F_p(V)$ and $S' \in F_q(V')$ are weighted polyhedral subspaces, the image of (S, S') is denoted by $S \times S'$.

Choose polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ which are respectively adapted to S and S'. The family $(C \times C')$, for $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$, is a polyhedral decomposition which is adapted to $S \times S'$: one has

$$S \times S' = \sum_{C \in \mathscr{C}_p} \sum_{C' \in \mathscr{C}'_q} w_C w'_{C'} [C \times C']$$

if, for every (C, C'), w_C is the weight of C in S and $w'_{C'}$ is the weight of C' in S'.

If S *and* S' *are balanced, then so is* $S \times S'$. Indeed, let us consider a polyhedron E of dimension $p + q - 1$ belonging to the polyhedral decomposition $\mathscr{C} \times \mathscr{C}'$. Let us write $E = D \times D'$, where $D \in \mathscr{C}$ and $D' \in \mathscr{C}'$.

Let $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$ be polyhedra such that E is a face of $C \times C'$. Then $D \subset C$ and $D' \subset C'$, so that D is a face of C and D' is a face of C'. Since $\dim(D) + \dim(D') = \dim(C) + \dim(C') - 1$, there are two possibilities: either $\dim(D) = \dim(C) - 1$ and $D' = C'$, or $\dim(D') = \dim(C') - 1$ and $D = C$.

This already shows that the balancing condition along E is trivial if $\dim(D) \neq p$ and $\dim(D') \neq q$.

Let us now assume that $\dim(D) = p$ (hence $\dim(D') = q - 1$). By what precedes, the polyhedra of the form $C \times C'$, where $C \in \mathscr{C}_p$ and $C' \in \mathscr{C}'_q$ of which E is a face are of the form $D \times C'$, where $D' \subset C' \in \mathscr{C}'_q$. The balancing condition along E for $S \times S'$ follows from the balancing condition for S' along D'.

Similarly, if $\dim(D') = q$ and $\dim(D) = p - 1$, then the balancing condition along E for $S \times S'$ follows from the balancing condition for S along D.

6.1.12. — A Minkowski weight is said to be *effective* if the corresponding weighted polyhedral subspace is effective. Effective Minkowski weights form a submonoid $MW_p^+(V)$ of $MW_p(V)$.

Proposition **(6.1.13)**. — *Every Minkowski weight is the difference of two effective Minkoswski weights.*

Proof. — Let $S \in MW_p(V)$ be a Minkowski weight and let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S; for $C \in \mathscr{C}_p$, let w_C be the weight of C in S. Let $\mathscr{N}$ be the set of all $C \in \mathscr{C}_p$ such that $w_C < 0$; for $C \in \mathscr{N}$, let $S_C = [\langle C \rangle]$ be the weighted polyhedral subspace associated with the affine space generated by C; it is balanced. Set $S' = \sum_{C\in\mathscr{N}}(-w_C)S_C$; is is an effective Minkowski weight. then one has

$$\begin{aligned} S + S' &= \sum_{C\in\mathscr{C}_p} w_C[C] + \sum_{C\in\mathscr{N}} w_C[\langle C\rangle] \\ &= \sum_{\substack{C\in\mathscr{C}_p\\ w_C>0}} w_C[C] + \sum_{C\in\mathscr{N}} (-w_C)\left([\langle C\rangle] - [C]\right). \end{aligned}$$

Since $C \subset \langle C\rangle$, the weighted polyhedral subspace $[\langle C\rangle] - [C]$ is effective. Consequently, $S + S'$ is effective; it is also balanced. Then $S = (S + S') - S'$ is the difference of two effective Minkowski weights, as was to be shown. □

6.2. Stable intersection

6.2.1. — Let L, L' be free finitely generated abelian groups, let $V = L_{\mathbf{R}}$, $V' = L'_{\mathbf{R}}$ and let $f : V \to V'$ be a linear map such that $f(L) \subset L'$.

There exists a unique linear map $f_* : F_p(V) \to F_p(V')$ satisfying the following properties, for every p-dimensional polyhedron C of V:

(i) If $\dim(f(C)) < p$, then $f_*([C]) = 0$;

(ii) If $\dim(f(\mathrm{C})) = p$, then $f(\mathrm{L_C})$ is subgroup of rank p of $\mathrm{L}_{f(\mathrm{C})}$, so that the index $[\mathrm{L}_{f(\mathrm{C})} : f(\mathrm{L_C})]$ is finite, and $f_*([\mathrm{C}]) = [\mathrm{L}_{f(\mathrm{C})} : f(\mathrm{L_C})][\mathrm{C}]$.

For every $\mathrm{S} \in \mathrm{F}_p(\mathrm{V})$, one has $|f_*(\mathrm{S})| \subset f(|\mathrm{S}|)$.

Proposition **(6.2.2)**. — *If* S *is balanced, then* $f_*(\mathrm{S})$ *is balanced. In other words, one has* $f_*(\mathrm{MW}_p(\mathrm{V})) \subset \mathrm{MW}_p(\mathrm{V}')$.

Proof. — Replacing V′ be its image, we may assume that f is surjective. There is a polyhedral decomposition $\mathscr{C}$ of V such that the polyhedra $f(\mathrm{C})$, for $\mathrm{C} \in \mathscr{C}$, form a polyhedral decomposition $\mathscr{C}'$ of V′ (corollary 1.8.5).

Let D′ be polyhedron of dimension $p-1$ in $\mathscr{C}'$. Let $\mathscr{C}_{\mathrm{D}'}$ be the set of all polyhedra C′ in $\mathscr{C}'$ such that $\dim(\mathrm{C}') = p$ and $\mathrm{D}' \subset \mathrm{C}'$. For $\mathrm{C}' \in \mathscr{C}_{\mathrm{D}'}$, define $v_{\mathrm{C}'/\mathrm{D}'} \in \mathrm{L}'_{\mathrm{C}'}$ which generates $\mathrm{L}'_{\mathrm{C}'}/\mathrm{L}'_{\mathrm{D}'}$ and is such that $x + tv_{\mathrm{C}'} \in \mathrm{C}'$ for every $x \in \mathring{\mathrm{D}}'$ and every small enough positive real number t.

Let $\mathscr{D}_{\mathrm{D}'}$ be the set of all polyhedra D of dimension $p-1$ of $\mathscr{C}$ such that $f(\mathrm{D}) = \mathrm{D}'$. For every $\mathrm{D} \in \mathscr{D}_{\mathrm{D}'}$, let $\mathscr{C}_{\mathrm{D}}$ be the set of all polyhedra $\mathrm{C} \in \mathscr{C}$ such that $\dim(\mathrm{C}) = p$ and $\mathrm{D} \subset \mathrm{C}$. For every $\mathrm{D} \in \mathscr{D}_{\mathrm{D}'}$ and every $\mathrm{C} \in \mathscr{C}_{\mathrm{D}}$, let $v_{\mathrm{C}/\mathrm{D}} \in \mathrm{L_C}$ be a vector that maps to a generator of $\mathrm{L_C}/\mathrm{L_D}$ and is such that $x + tv_{\mathrm{C}} \in \mathrm{C}$ for every $x \in \mathring{\mathrm{D}}$ and every small enough positive real number t. The balancing condition at D for S writes

$$\sum_{\mathrm{D}\in\mathscr{C}_{\mathrm{D}}} w_{\mathrm{C}} v_{\mathrm{C}/\mathrm{D}} \in \mathrm{L_D}.$$

Since $f(\mathrm{C})$ contains $f(\mathrm{D}) = \mathrm{D}'$, the image $f(\mathrm{C})$ of C is either equal to D′, or it belongs to $\mathscr{C}_{\mathrm{D}'}$. In the latter case, set $\mathrm{C}' = f(\mathrm{C})$. There exists $k_{\mathrm{C}} \in \mathbf{N}^*$ such that $f(v_{\mathrm{C}/\mathrm{D}}) = k_{\mathrm{C}} v_{\mathrm{C}'}$; one has

$$k_{\mathrm{C}} = [\mathrm{L}'_{\mathrm{C}'} : (\mathrm{L}'_{\mathrm{D}'} + \mathbf{Z} f(v_{\mathrm{C}/\mathrm{D}}))].$$

Then

$$\begin{aligned} [\mathrm{L}'_{\mathrm{C}'} : f(\mathrm{L}_{\mathrm{C}})] &= [\mathrm{L}'_{\mathrm{C}'} : f(\mathrm{L}_{\mathrm{D}} + \mathbf{Z} v_{\mathrm{C}/\mathrm{D}})] \\ &= [\mathrm{L}'_{\mathrm{C}'} : (f(\mathrm{L}_{\mathrm{D}}) + \mathbf{Z} f(v_{\mathrm{C}/\mathrm{D}}))] \\ &= [\mathrm{L}'_{\mathrm{C}'} : (\mathrm{L}'_{\mathrm{D}'} + \mathbf{Z} f(v_{\mathrm{C}/\mathrm{D}}))]\,[\mathrm{L}'_{\mathrm{D}'} : f(\mathrm{L}_{\mathrm{D}})] \\ &= k_{\mathrm{C}}\,[\mathrm{L}'_{\mathrm{D}'} : f(\mathrm{L}_{\mathrm{D}})], \end{aligned}$$

so that

$$k_{\mathrm{C}} = [\mathrm{L}'_{\mathrm{C}'} : f(\mathrm{L}_{\mathrm{C}})]/[\mathrm{L}'_{\mathrm{D}'} : f(\mathrm{L}_{\mathrm{D}})].$$

Modulo $L'_{D'}$, the vector of L' responsible for the balancing condition along D' is equal to

$$\sum_{C'\in\mathscr{C}_{D'}}\left(\sum_{D\in\mathscr{D}_{D'}}\sum_{\substack{C\in\mathscr{C}_D\\ f(C)=C'}} w_C[L'_{C'}:f(L_C)]\right)v_{C'}$$
$$=\sum_{C'\in\mathscr{C}_{D'}}\left(\sum_{D\in\mathscr{D}_{D'}}\sum_{\substack{C\in\mathscr{C}_D\\ f(C)=C'}} w_C[L'_{D'}:f(L_D)]f(v_{C/D})\right)$$
$$=\sum_{D\in\mathscr{D}_{D'}}[L'_{D'}:L_D]\sum_{\substack{C\in\mathscr{C}_D\\ \dim(f(C))=p}} w_C f(v_{C/D}),$$

hence it belongs to $L'_{D'}$. Indeed, for every $D \in \mathscr{D}_{D'}$, the balancing condition of S along D asserts that $\sum_{C\in\mathscr{C}_D} w_C v_{C/D} \in L_D$; applying f, we get $\sum_{C\in\mathscr{C}_D} w_C f(v_{C/D}) \in L'_{D'}$; on the other hand, if $\dim(f(C)) < p$, then $f(C) \subset D'$ and $f(v_{C/D}) \in L'_{D'}$.

Consequently, $f_*(S)$ is balanced along D', as was to be shown. □

6.2.3. — Let p, q be two integers, let $S \in MW_p(V)$ and $S' \in MW_q(V)$. Choose polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ of V which are respectively adapted to S and S'; write $S = \sum_{C \in \mathscr{C}_p} w_C[C]$ and $S' = \sum_{C \in \mathscr{C}'_q} w'_C[C]$.

Let $C \in \mathscr{C}_p$ and $C' \in \mathscr{C}'_q$ be such that $\dim(C \cap C') = p + q - n$ (in particular, $C \cap C' \neq \varnothing$). This implies $\dim(C + C') = n$.[3]

One says that S and S' *intersect transversally along* $C \cap C'$ if, moreover, $\mathring{C} \cap \mathring{C}' \neq \varnothing$.

For $v \in V$, define

$$\mu(C, C', v) = \sum_{D, D'} w_D w'_{D'} [L : L_D + L'_{D'}],$$

where the sum is over all pairs (D, D') of polyhedra such that $D \in \mathscr{C}_p$, $D' \in \mathscr{C}'_q$, $C \cap C' \subset D \cap D'$, $\dim(D + D') = n$ and $D \cap (v + D') \neq \varnothing$. [4]

This formula implies that for every $x \in (C \cap C')^\circ$, one has $\mu(\mathrm{Star}_x(C), \mathrm{Star}_x(C'), v) = \mu(C, C', v)$. Indeed, the pairs of polyhedra that appear in the formula for $\mu(\mathrm{Star}_x(C), \mathrm{Star}_x(C'), v)$ are precisely of the form $(\mathrm{Star}_x(D), \mathrm{Star}_x(D'))$ where (D, D') appear in the formula for $\mu(C, C', v)$, and the weights are the same.

Lemma **(6.2.4)**. — a) *If* S *and* S' *intersect transversally along* $C \cap C'$, *then* $v \mapsto \mu(C, C', v)$ *is constant in a neighborhood of* 0 *in* V.

[3] Does it?

[4] I'd guess one can/could/should write $C \cap C' = D \cap D'$ here. . .

b) *There exists a strictly positive real number* δ *and a polyhedral subspace* B *of* V *of dimension* $< \dim(V)$, *and an integer* $\mu(C, C')$ *such that* $\mu(C, C', v) = \mu(C, C')$ *for all* $v \in V - B$ *such that* $\|v\| < \delta$.

Proof. — We may assume that $0 \in (C \cap C')^\circ$ and replace S, S′ by the associated conic Minkowski weights with apex at 0. In particular, all polyhedra in $\mathscr{C}$ are cones. Moreover, $C \cap C'$ is a vector subspace, and is contained in the lineality spaces of all cones involved. To check the lemma, we also may mod out by $C \cap C'$, which reduces us to the case where $C \cap C' = \{0\}$.

a) Assume that S and S′ intersect transversally along $C \cap C'$. Since $\mathring{C} \cap \mathring{C}'$ is non-empty, by assumption, it is equal to $(C \cap C')^\circ$, hence it contains 0, so that both C and C′ are linear subspaces.

Let $v \in V$ and (D, D') be a pair of polyhedra that appear in the definition of $\mu(C, C', v)$. Since $0 \in \mathring{C}$, and $0 \in C \cap C' \subset D$, one has $C \subset D$; since $\dim(D) = p$, this implies $D = C$. Similarly, $D' = C'$. Then the sum defining $\mu(C, C', v)$ reduces to $w_C w'_{C'}[L : L_C + L'_{C'}]$; in particular, it is constant.

b) Let $S \times S'$ be the $(p+q)$-dimensional weighted polyhedral subspace of $V \times V$ defined by

$$S \times S' = \sum_{C \in \mathscr{C}_p} \sum_{C' \in \mathscr{C}'_q} w_C w'_{C'} [C \times C'].$$

It is balanced (example 6.1.11).

Let $f : V \times V \to V$ be the linear map given by $f(x, y) = x - y$. Let us consider polyhedral decompositions $\mathscr{C}_1$ of V and $\mathscr{C}_2$ of $V \times V$ that respectively refine $\mathscr{C}$ and $\mathscr{C}'$, and $\mathscr{C} \times \mathscr{C}'$, and such that $f(C \times C')$

is a union of cones in $\mathscr{C}$ for every $C, C' \in \mathscr{C}$ (corollary 1.8.5). The expression $\mu(C, C', v)$ is the coefficient of the cone $[C - C'] = f(C \times C')$ in the Minkowski weight $f_*(S \times S')$. Since this is a Minkowski weight of dimension n, there exists $a \in \mathbf{Z}$ such that $f_*(S \times S') = a[V]$. It follows that $\mu(C, C', v) = a$ for every vector v which does not belong to a polyhedron of $\mathscr{C}$ of dimension $< n$. □

6.2.5. — Let $S \in MW_p(V)$ and $S' \in MW_q(W)$ be Minkowski weights. If $C, C' \in \mathscr{C}$ satisfy $\dim(C) = p$, $\dim(C') = q$ and $\dim(C + C') = n$, let us denote by $\mu(C, C')$ the common value $\mu(C, C', v)$ where $v \in V$ is a generic vector; in this case, one has $\dim(C \cap C') = p + q - n$. Otherwise, let us set $\mu(C, C') = 0$. We thus define an element of $F_{p+q-n}(V)$ by

$$S \cap_{\mathrm{st}} S' = \sum_{C, C'} \mu(C, C')[C \cap C'].$$

In particular, it is 0 if $p + q < n$. Moreover, one has $|S \cap_{\mathrm{st}} S'| \subset |S| \cap |S'|$.

This element is called the *stable intersection* of S and S′. It does not depend on the chosen polyhedral decomposition $\mathscr{C}$ and is bilinear in S and S′.

Since multiplicities $\mu(C, C')$ can be computed after passing to links, one also has $\mathrm{Star}_x(S \cap_{\mathrm{st}} S') = \mathrm{Star}_x(S) \cap_{\mathrm{st}} \mathrm{Star}_x(S')$ for every $x \in V$.

At this point, it is not so clear that $S \cap_{\mathrm{st}} S'$ *belongs to* $F_{p+q-n}(V)$*, because we have not yet proved that the polyhedra* $[C \cap C']$ *involved in its definition have dimension* $p + q - n$*, if* $\mu(C, C') \neq 0$*. (Does it even belong to* $F_*(V)$*?)*

6.2.6. — Let $S \in MW_p(V)$ and $S' \in MW_q(V)$. According to Mikhalkin & Rau (2018), one says that $|S|$ and $|S'|$ *intersect transversally* if $\dim(|S| \cap |S'|) = p + q - n$ and if there exist polyhedral decompositions $\mathscr{C}$ of $|S|$, and $\mathscr{C}'$ of $|S'|$, such that for every polyhedron D satisfying $\dim(D) = p+q-n$ and $D \subset |S| \cap |S'|$, there exists a unique pair (C, C') of polyhedron in $\mathscr{C}$ such that $\dim(C) = p$ and $C \subset |S|$, $\dim(C') = q$ and $C' \subset |S'|$, and $D \subset C \cap C'$.

Proposition **(6.2.7)**. — *If* S *and* S′ *intersect transversally, then* $S \cap_{st} S' \in MW_{p+q-n}(V)$ *and* $|S \cap_{st} S'| = |S| \cap |S'|$.

Proof. — Fix polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ adapted to S and S′ that attest of their transversal intersection; let (w_C), resp. $(w'_{C'})$ be the weights of S, resp. of S′. For every pair (C, C'), where $C \in \mathscr{C}_p$ and $C' \in \mathscr{C}_q$ are such that $w_C \neq 0$, $w'_{C'} \neq 0$ and $C \cap C' \neq \varnothing$, one has $\dim(C \cap C') = p+q-n$, and the definition of $\mu(C, C')$ shows that $\mu(C, C') = w_C w'_{C'}$. In fact, the sum defining $\mu(C, C', v)$ is reduced to (C, C'), for every small enough $v \in V$. This already proves that $S \cap_{st} S'$ belongs to $F_{p+q-n}(V)$ and that $|S \cap_{st} S'| = |S| \cap |S'|$.

Let us prove the balancing condition. By construction, $|S \cap_{st} S'|$ is a union of polyhedra of dimension $p+q-n$ of the form $C \cap C'$, for $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$, and they only meet along faces which are of the form $D \times C'$, or $C \times D'$, where D is a codimension 1 face of C, or D′ is a codimension 1 face of C′. Consequently, the balancing condition needs only be checked along such faces. We thus assume that $E = D \cap C'$, where $D \in \mathscr{C}_{p-1}$ and $C' \in \mathscr{C}_q$, the other case being similar by symmetry. The polyhedra of $S \cap_{st} S'$ that border E are of the form $C \cap C'$, where $C \in \mathscr{C}_p$ contains D.

For every such C, fix a vector $v_{C/D} \in L_C$ which generates L_C/L_D and which is such that $x + tv_{C/D} \in C$ for every $x \in \mathring{D}$ and every small enough positive real number t. The balancing condition for S along D writes $\sum_C w_C v_{C/D} \in L_D$.

Let us fix a normal vector $v'_{C\cap C'/D\cap C'} \in L_{C\cap C'}$ associated with the face $D \times C'$ of $C \times C'$. There exists a unique integer $p_C \in \mathbf{N}^*$ such that $v'_{C\cap C'/D\cap C'} = p_C v_{C/D} \pmod{L}_D$, so that the balancing condition for $S \cap_{st} S'$ along $D \times C'$ writes $\sum_C \mu(C, C') p_C v_{C/D} \in L_D$. To conclude the proof, since $\mu(C, C') = w_C w'_{C'}[L : L_C + L_{C'}]$, it now suffices to prove that $p_C[L_C + L_{C'}]$ is independent of C.

One has

$$L_C \cap L_{C'} = L_{C\cap C'} = L_{D\cap C'} + \mathbf{Z} v_{C\cap C'/D\cap C'},$$

hence

$$[(L_C \cap L_{C'}) + L_D = L_D + \mathbf{Z} v_{C\cap C'/D\cap C'} = L_D + \mathbf{Z} p_C v_{C/D}.$$

Since $L_C = L_D + \mathbf{Z} v_{C/D}$, it follows that

$$p_C = [L_C : (L_C \cap L_{C'}) + L_D] = [L_C + L_{C'} : L_{C'} + L_D]$$

and

$$p_C[L : L_C + L_{C'}] = [L : L_{C'} + L_D].$$

□

Proposition **(6.2.8)**. — a) *There exists a polyhedral subspace* B *of* V *such that* $\dim(\mathrm{B}) < n$ *and such that for every* $v \in \mathrm{V} - \mathrm{B}$*, the Minkowski weights* S *and* $\mathrm{S}' + v$ *intersect transversally.*

b) *If* $n = p + q$*, then* $\deg(\mathrm{S} \cap_{\mathrm{st}} (\mathrm{S}' + v))$ *is independent of* $v \in \mathrm{V} - \mathrm{B}$.

Proof. — We fix polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ of V respectively adapted to S and S′.

Let $\mathscr{I}$ be the set of all pairs $(\mathrm{C}, \mathrm{C}')$ such that $\mathrm{C} \in \mathscr{C}_p$, $\mathrm{C}' \in \mathscr{C}'_q$, $w_{\mathrm{C}} \neq 0$, $w'_{\mathrm{C}'} \neq 0$. Let $(\mathrm{C}, \mathrm{C}') \in \mathscr{I}$. For $v \in \mathrm{V}$, one has $\mathrm{C} \cap (v + \mathrm{C}') \neq \varnothing$ if and only if $v \in \mathrm{C} - \mathrm{C}'$. Let B_1 be the union of all $\partial(\mathrm{C} - \mathrm{C}')$, for $(\mathrm{C}, \mathrm{C}') \in \mathscr{I}$ such that $\dim(\mathrm{C} - \mathrm{C}') < n$. Let $(\mathrm{C}, \mathrm{C}') \in \mathscr{I}$ be such that $\dim(\mathrm{C} - \mathrm{C}') = n$ and let $\partial(\mathrm{C} - \mathrm{C}') = (\mathrm{C} - \mathrm{C}') - (\mathrm{C} - \mathrm{C}')^\circ$; it is a polyhedron of dimension $< n$. If $v \notin (\mathrm{C} - \mathrm{C}')$, then $\mathrm{C} \cap (v + \mathrm{C}') = \varnothing$; if $v \in (\mathrm{C} - \mathrm{C}')^\circ$, then $v \in \mathring{\mathrm{C}} - \mathring{\mathrm{C}}'$, hence $\mathring{\mathrm{C}} \cap (v + \mathring{\mathrm{C}}') \neq \varnothing$. Let B_2 be the union of all $\partial(\mathrm{C} - \mathrm{C}')$, for $(\mathrm{C}, \mathrm{C}') \in \mathscr{I}$ such that $\dim(\mathrm{C} - \mathrm{C}') = n$. Let $\mathrm{B} = \mathrm{B}_1 \cup \mathrm{B}_2$. This is a polyhedral subspace of V of dimension $< n$.

Let $v \in \mathrm{V} - \mathrm{B}$. By construction, S and $\mathrm{S}' + v$ intersect transversally along $\mathrm{C} \cap (\mathrm{C}' + v)$, for every pair $(\mathrm{C}, \mathrm{C}')$ such that $\mathrm{C} \cap (\mathrm{C}' + v) \neq \varnothing$. This proves that S and $\mathrm{S}' + v$ intersect transversally.

Assume that $p + q = n$. Let U be a connected component of $\mathrm{V} - \mathrm{B}$ such that $0 \in \overline{\mathrm{U}}$. Fix $(\mathrm{C}, \mathrm{C}') \in \mathscr{I}$. When $v \in \mathrm{U}$, the pairs $(\mathrm{D}, \mathrm{D}') \in \mathscr{I}$ such that $v \in \mathring{\mathrm{D}} \cap (v + \mathring{\mathrm{D}}')$ remain the same, and in fact, v is their unique point of intersection. This gives

$$\begin{aligned}\deg(\mathrm{S}\cap_{\mathrm{st}}(v+\mathrm{S}')) &= \sum_{(\mathrm{D},\mathrm{D}')} w_{\mathrm{D}}w'_{\mathrm{D}'}[\mathrm{L}:\mathrm{L}_{\mathrm{D}}+\mathrm{L}_{\mathrm{D}'}] \\ &= \sum_{(\mathrm{C},\mathrm{C}')} \sum_{\substack{(\mathrm{D},\mathrm{D}')\\ \mathrm{D}\cap\mathrm{D}'=\mathrm{C}\cap\mathrm{C}'}} w_{\mathrm{D}}w'_{\mathrm{D}'}[\mathrm{L}:\mathrm{L}_{\mathrm{D}}+\mathrm{L}_{\mathrm{D}'}] \\ &= \sum_{(\mathrm{C},\mathrm{C}')} \mu(\mathrm{C},\mathrm{C}') \\ &= \deg(\mathrm{S}\cap_{\mathrm{st}}\mathrm{S}').\end{aligned}$$

This implies the claim. □

Theorem **(6.2.9)**. — *Let p, q be integers such that $p + q \geqslant n$. For any* $\mathrm{S} \in \mathrm{MW}_p(\mathrm{V})$ *and* $\mathrm{S}' \in \mathrm{MW}_q(\mathrm{V})$*, one has* $\mathrm{S}\cap_{\mathrm{st}}\mathrm{S}' \in \mathrm{MW}_{p+q-n}(\mathrm{V})$.

Proof. — Let E be a polyhedron of dimension $p + q - n - 1$ along which we wish to check the balancing condition for $\mathrm{S}\cap_{\mathrm{st}}\mathrm{S}'$. Choosing an origin in $\mathring{\mathrm{E}}$ and replacing S and S′ by the fan-like Minkowski weights, we can assume that there are polyhedral decompositions of V adapted to S and S′, all polyhedra of which are cones. We may also quotient by E and reduce to the case where $\mathrm{E} = \{0\}$; then $p + q = n + 1$.

We will first prove that $S \cap_{st} S' = \operatorname{recc}(S \cap_{st} (v + S'))$ for all $v \in V$. *It suffices to prove this when* S *and* $v + S'$ *intersect transversally.* If C and C' are cones such that $C \cap (C' + v) \neq \varnothing$, then one has $\operatorname{recc}(C \cap (C' + v)) = C \cap C'$. (Let $x \in C \cap (C' + v)$; then for every $u \in C \cap C'$, one has $x + u \in C \cap (C' + v)$. On the other hand, if $x + tu \in C \cap (C' + v)$ for every $t \in \mathbf{R}_+$, then $u \in C \cap C'$, as one sees letting $t \to \infty$.) By transversality, $\dim(C \cap C') = \dim(C \cap (C' + v)) = 1$. Multiplicities add up as well. This implies the equality $\operatorname{recc}(S \cap_{st} (S' + v)) = S \cap_{st} S'$. Since S and $v + S'$ intersect transversally, one has $S \cap_{st} (S' + v) \in MW_1(V)$. To conclude the proof of the theorem, it thus follows to establish the following lemma. □

Lemma **(6.2.10)**. — *Let* $S \in MW_1(V)$. *Then* $\operatorname{recc}(S) \in MW_1(V)$.

Proof. — Let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S; for $C \in \mathscr{C}_1$, let w_C be the weight of C in S.

Let $C \in \mathscr{C}_1$, so that $L_C \simeq \mathbf{Z}$; we fix arbitrarily one generator v_C of L_C. There are three possibilities.

– Either there exist $x_C, y_C \in C$ such that $C = [x_C; y_C]$, chosen such that $y_C \in x_C + \mathbf{R}_+ v_C$. Then its recession cone is 0;

– Or there exists $x \in V$ such that $C = x_C + \mathbf{R}_+ v_C$ or $C = x_C - \mathbf{R}_+ v_C$. Up to changing v_C into $-v_C$, we assume that we are in the former case. Then $\operatorname{recc}(C) = \mathbf{R}_+ v_C$;

– Or there exists $x_C \in V$ such that $C = x_C + \mathbf{R} v_C$; then $\operatorname{recc}(C) = \mathbf{R} v_C$.

Let $\mathscr{C}_1^2, \mathscr{C}_1^1, \mathscr{C}_1^0$ be the corresponding subsets of $\mathscr{C}_1$. The recession fan of S is given by the sum

$$\operatorname{recc}(S) = \sum_{C \in \mathscr{C}_1^1} w_C[\mathbf{R}_+ v_C] + \sum_{C \in \mathscr{C}_1^0} w_C[\mathbf{R} v_C].$$

The balancing condition at the origin for recc(S) is thus the relation

$$\sum_{C \in \mathscr{C}_1^1} w_C v_C = 0.$$

We now write the balancing condition for S at a point $p \in \mathscr{C}_0$. Let $\mathscr{C}_p$ be the set of $C \in \mathscr{C}_1$ such that $p \in C$. If $C \in \mathscr{C}_1^1$, then $p = x_C$; moreover, v_C is an admissible normal vector for (p, C). Otherwise, $C \in \mathscr{C}_1^2$ and there are two possibilities:

- Either $p = x_C$; then v_C is an admissible normal vector for (p, C);
- Or $p = y_C$ and then $-v_C$ is an admissible normal vector for (p, C).

The balancing condition at p thus writes

$$\sum_{\substack{C \in \mathscr{C}_1^1 \\ x_C = p}} w_C v_C + \sum_{\substack{C \in \mathscr{C}_1^2 \\ x_C = p}} w_C v_C - \sum_{\substack{C \in \mathscr{C}_1^2 \\ y_C = p}} w_C v_C = 0.$$

Adding all of these relations, for all $p \in \mathscr{C}_0$, we obtain

$$0 = \sum_{C\in\mathscr{C}_1^1} w_C v_C + \sum_{C\in\mathscr{C}_1^2} w_C v_C - \sum_{C\in\mathscr{C}_1^2} w_C v_C = \sum_{C\in\mathscr{C}_1^1} w_C v_C,$$

as was to be shown. □

Proposition **(6.2.11)**. — *The stable intersection product endowes the abelian group* $\mathrm{MW}(V) = \bigoplus_p \mathrm{MW}_p(V)$ *with a ring structure. The neutral element is* $[V]$.

Proof. — It follows from the definitions that the stable intersection product is commutative and bilinear. It also follows from the definitions that $S \cap [V] = S$.

Let us check associativity. Let S, S', S'' be three Minkowski weights of dimensions p, q, r and let us prove that $(S \cap_{\mathrm{st}} S') \cap_{\mathrm{st}} S'' = S \cap_{\mathrm{st}} (S' \cap_{\mathrm{st}} S'')$. Let us first treat the case where these Minkowski weights intersect transversally, in the sense that $\mathring{C} \cap \mathring{C}'\mathring{C}'' \neq \varnothing$ for every $C \in \mathscr{C}_p$, $C' \in \mathscr{C}'_q$, $C'' \in \mathscr{C}''_r$ such that $w_C, w'_{C'}, w''_{C''} \neq 0$ and $C \cap C' \cap C'' \neq \varnothing$. If this holds, then S' and S'' intersect transversally and

$$S' \cap_{\mathrm{st}} S'' = \sum_{C',C''} w'_{C'} w''_{C''} [L : L_{C'} + L_{C''}][C' \cap C''].$$

$MW_p \times MW_q \to MW_{p+q-n}$

def: $MW^p = MW_{n-p}$ $MW^p \times MW^q \to MW^{p+q}$

Moreover, S and $S' \cap_{st} S''$ intersect transversally and

$$S \cap_{st} (S' \cap_{st} S'') = \sum_{C,C',C''} w_C w'_{C'} w''_{C''} [L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] [C \cap C' \cap C''].$$

By symmetry, one also has

$$(S \cap_{st} S') \cap_{st} S'') = \sum_{C,C',C''} w_C w'_{C'} w''_{C''} [L : L_C + L_{C'}][L : (L_C \cap L_{C'}) + L_{C''}] [C \cap C' \cap C''].$$

It thus suffices to prove the following equality of indices:

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] = [L : L_C + L_{C'}][L : (L_C \cap L_{C'}) + L_{C''}].$$

On the other hand, one has

$$\begin{aligned}[L : L_C + (L_{C'} \cap L_{C''})] &= [L : L_C + L_{C'}][L_C + L_{C'} : L_C + (L_{C'} \cap L_{C''})] \\ &= [L : L_C + L_{C'}][L_{C'} : (L_C \cap L_{C'}) + (L_{C'} \cap L_{C''})],\end{aligned}$$

so that

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})]$$
$$= [L : L_{C'} + L_C]\,[L : L_{C'} + L_{C''}]\,[L_{C'} : (L_C \cap L_{C'}) + (L_{C'} \cap L_{C''})],$$

an expression which is invariant when one exchanges the roles of C and C″. Therefore,

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] = [L : L_{C'} + L_C][L : L_{C''} + (L_{C'} \cap L_C)],$$

as was to be shown.

In the general case, we consider arbitrarily small vectors $v \in V$, $w \in V$ such that S, $S' + v$ and $S'' + w$ intersect transversally. If C, C', C'' are polyhedra of dimensions p, q, r, the multiplicity $\mu(C, C', C'')$ of $[C \cap C' \cap C'']$ in $(S \cap_{st} S') \cap_{st} S''$ is a sum of multiplicities $\mu(D, D', D''; v, w)$, where $C \cap C' \cap C'' = D \cap D' \cap D''$ and $D, D' + v, D'' + w$ intersect transversally, associated with $(S \cap_{st} (S' + v)) \cap_{st} (S'' + w)$. By the case of transverse intersections, they coincide with the multiplicity of $[C \cap C' \cap C'']$ in $S \cap_{st} ((S' + v) \cap_{st} (S'' + w))$. □

Example **(6.2.12)** (Unfinished). — Assume that $L = \mathbf{Z}^n$ and let $(e_1, \dots, e_n)$ be its canonical basis; set also $e_0 = -e_1 - \dots - e_n$. For $I \subsetneq \{0, \dots, n\}$, let C_I be the cone generated by the vectors e_i, for $i \in I$; one has $\dim(C_I) = \mathrm{Card}(I)$. Note that $C_I \cap C_J = C_{I \cap J}$ for $I, J \subsetneq \{0, \dots, n\}$, so that the set of cones $(C_I)_{I \subsetneq \{0,\dots,n\}}$ is a fan in $\mathbf{R}^n$.

so that

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] \\ = [L : L_{C'} + L_C]\,[L : L_{C'} + L_{C''}]\,[L_{C'} : (L_C \cap L_{C'}) + (L_{C'} \cap L_{C''})],$$

an expression which is invariant when one exchanges the roles of C and C″. Therefore,

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] = [L : L_{C'} + L_C][L : L_{C''} + (L_{C'} \cap L_C)],$$

as was to be shown.

In the general case, we consider arbitrarily small vectors $v \in V$, $w \in V$ such that S, $S' + v$ and $S'' + w$ intersect transversally. If C, C', C'' are polyhedra of dimensions p, q, r, the multiplicity $\mu(C, C', C'')$ of $[C \cap C' \cap C'']$ in $(S \cap_{st} S') \cap_{st} S''$ is a sum of multiplicities $\mu(D, D', D''; v, w)$, where $C \cap C' \cap C'' = D \cap D' \cap D''$ and $D, D' + v, D'' + w$ intersect transversally, associated with $(S \cap_{st} (S' + v)) \cap_{st} (S'' + w)$. By the case of transverse intersections, they coincide with the multiplicity of $[C \cap C' \cap C'']$ in $S \cap_{st} ((S' + v) \cap_{st} (S'' + w))$. ☐

Example **(6.2.12)** (Unfinished). — Assume that $L = \mathbf{Z}^n$ and let $(e_1, \dots, e_n)$ be its canonical basis; set also $e_0 = -e_1 - \dots - e_n$. For $I \subsetneq \{0, \dots, n\}$, let C_I be the cone generated by the vectors e_i, for $i \in I$; one has $\dim(C_I) = \mathrm{Card}(I)$. Note that $C_I \cap C_J = C_{I \cap J}$ for $I, J \subsetneq \{0, \dots, n\}$, so that the set of cones $(C_I)_{I \subsetneq \{0, \dots, n\}}$ is a fan in $\mathbf{R}^n$.

For $p \in \{0, \dots, n\}$, we define an effective weighted polyhedral subspace of dimension p by

$$\mathrm{S}_p = \sum_{\mathrm{Card(I)}=p} [\mathrm{C_I}].$$

(This is a tropical linear space of dimension p.) One has $\mathrm{L}_{\mathrm{C_I}} = \sum_{i\in \mathrm{I}} \mathbf{Z} e_i$. It is balanced. The only polyhedra along which the balancing condition is not obvious are of the form $\mathrm{C_J}$, where $\mathrm{Card(J)} = p-1$, and its adjacent polyhedra are of the form $\mathrm{C}_{\mathrm{J}\cup\{i\}}$, for $i \in \{0, \dots, n\} - \mathrm{J}$; one may take e_i as a normal vector. The balancing condition along $\mathrm{C_J}$ then writes

NB S_p est l'esp. linéaire tropical associé au matroïde sur $\{0, \dots, n\}$ $\mathrm{rg}(A) = \inf(p, \mathrm{Card}(A))$

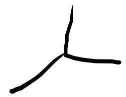
$$\sum_{i\in\{0,\dots,n\}-\mathrm{J}} e_i = \sum_{i\in\{0,\dots,n\}} e_i - \sum_{j\in \mathrm{J}} e_j \in \mathrm{L}_{\mathrm{C_J}}$$

since $\sum_{i=0}^n e_i = 0$.

Let us prove that $\mathrm{S}_p \cap_{\mathrm{st}} \mathrm{S}_q = \mathrm{S}_{p+q-n}$.

$S_1 \cap_{st} S_1 = S_0$

$n = 2$
$p = 1$

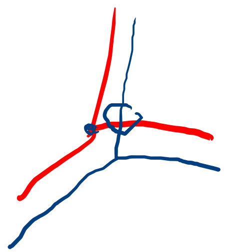


$p = 0$

$p = 2$

Proposition **(6.2.13)**. — *Let* $\mathrm{S} \in \mathrm{MW}_p(\mathrm{V})$ *and let* $\mathrm{S}' \in \mathrm{MW}_q(\mathrm{V})$. *If* $\Delta \in \mathrm{MW}_n(\mathrm{V}\times\mathrm{V})$ *is the diagonal, then one has*

$$\Delta \cap_{\mathrm{st}} (\mathrm{S} \boxtimes \mathrm{S}') = \mathrm{S} \cap_{\mathrm{st}} \mathrm{S}'.$$



6.3. The tropical hypersurface associated with a piecewise linear function

6.3.1. — Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous piecewise affine function and let $\mathscr{C}$ be a polyhedral decomposition of $\mathbf{R}^n$ which is adapted to f. We assume that f *has integral slopes*, in the sense that for every $\mathrm{C} \in \mathscr{C}$, there exists an linear function $\varphi_{\mathrm{C}} \in \mathrm{L}^\vee$ such that $f(y) - f(x) = \varphi_{\mathrm{C}}(y - x)$ for every $x, y \in \mathrm{C}$.

Let $x \in \mathbf{R}^n$ and let D be the unique polyhedron of $\mathscr{C}$ such that $x \in \mathring{\mathrm{D}}$. If $\dim(\mathrm{D}) \neq n - 1$, set $w_f(\mathrm{D}) = 0$. Otherwise, if $\dim(\mathrm{D}) = n - 1$, then D is a face of exactly two n-dimensional polyhedra $\mathrm{C}^+, \mathrm{C}^-$ in $\mathscr{C}$; one has $\mathrm{D} = \mathrm{C}^+ \cap \mathrm{C}^-$.

Fix a point $x \in \mathring{\mathrm{D}}$.

The quotient group $\mathbf{Z}^n/\mathrm{L}_{\mathrm{D}}$ is isomorphic to $\mathbf{Z}$, and it admits a unique generator which is the image of an element v^+ such that $x + tv^+ \in \mathrm{C}^+$ for every small enough $t \in \mathbf{R}_+$.

Define v^- similarly. In fact, one has $v^- = -v^+$.

By assumption, f is affine with integral slopes on C^+; let $\varphi^+ : \mathrm{V} \to \mathbf{R}$ be the unique linear map such that $f(y) - f(x) = \varphi^+(y - x)$ if $x, y \in \mathrm{C}^+$. We define similarly φ^-.

We then set

$$w_{\mathrm{D}} = \varphi^+(v^+) + \varphi^-(v^-).$$

and define

$$\partial(f) = \sum_{\mathrm{D} \in \mathscr{C}_{n-1}} w_{\mathrm{D}}[\mathrm{D}].$$

Proposition **(6.3.2)**. — *Let f be a piecewise linear function f with integral slopes on* V.

a) *The weighted polyhedral subspace $\partial(f)$ is a Minkowski weight of dimension $n-1$ adapted to the polyhedral decomposition $\mathscr{C}$.*

b) *Its support $|\partial(f)|$ is the non-linearity locus of f.*

c) *If f is convex, then $\partial(f)$ is effective.*

Proof. — We have to prove that $\partial(f)$ statisfies the balancing condition.

Let $E \in \mathscr{C}$ be a polyhedron of dimension $n-2$. Fix a point $x \in \mathring{E}$ and consider a 2-dimensional plane through x which is transverse to E. We get a fan in $\mathbf{R}^2$ which reduces the verification of the balancing condition to the case $n=2$, for $E=\{0\}$.

The 1-dimensional polyhedra that contain the origin are (chunks of) rays $D_1 = \mathbf{R}_+ u_1, \dots, D_n = \mathbf{R}_+ u_n$, where $u_1, \dots, u_n \in \mathbf{Z}^2$ are primitive vectors.[5] The balancing condition at 0 is the equation

$$\sum_{j=1}^{n} w_{D_j} u_j = 0.$$

Up to a reordering of the u_j, unique modulo cyclic permutations, the 2-dimensional polyhedra that contain the origin are (chunks) of sectors $C_1 = \operatorname{cone}(u_1, u_2), \dots, C_{n-1} = \operatorname{cone}(u_{n_1}, u_n), C_n = \operatorname{cone}(u_n, u_1)$.

[5]Picture?

Set $\varphi(x) = f(x) - f(0)$; for every j, let φ_j be the linear function on $\mathbf{R}^2$ such that $f(x) = f(0) + \varphi_j(x)$ for every point $x \in \mathrm{C}_j$ which is close to 0.

If ρ is the rotation of angle $\pi/2$, we then may take $\mathrm{D}_j^+ = \mathrm{C}_j$ and $\mathrm{D}_j^- = \mathrm{C}_{j-1}$, $v_j^+ = \rho(u_j)$ and $v_j^- = \rho^{-1}(u_j) = -v_j^+$. Then $w_{\mathrm{D}_j} = \varphi_j(\rho(u_j)) - \varphi_{j-1}(\rho(u_j))$ for all $j \in \{1, \dots, n\}$.

We thus have

$$\begin{aligned}\sum_{j=1}^n w_{\mathrm{D}_j} u_j &= \sum_{j=1}^n \varphi_j(\rho(u_j))u_j - \sum_{j=1}^n \varphi_{j-1}(\rho(u_j))u_j \\ &= \sum_{j=1}^n \varphi_j(\rho(u_j))u_j - \sum_{j=1}^n \varphi_j(\rho(u_{j+1}))u_{j+1}.\end{aligned}$$

The continuity of f along the ray u_j writes $\varphi_j(u_j) = \varphi(u_j) = \varphi_{j-1}(u_j)$. Let $a_j, b_j \in \mathbf{R}$ be such that $\rho(u_j) = a_j u_j + b_j u_{j+1}$. Then

$$\varphi_j(\rho(u_j)) = a_j \varphi_j(u_j) + b_j \varphi_j(u_{j+1}) = a_j \varphi(u_j) + b_j \varphi(u_{j+1}).$$

Similarly, $\rho(u_{j+1}) = a_{j-1} u_{j-1} + b_{j-1} u_j$, hence

$$\varphi_j(\rho(uj+1) = a_{j-1}\varphi_j(u_{j-1}) + b_{j-1}\varphi_j(u_j) = a_{j-1}\varphi(u_{j-1}) + b_{j-1}\varphi(u_j).$$

Finally,

$$\sum_{j=1}^{n} w_{D_j} u_j = \sum_{j=1}^{n} \left(a_j\varphi(u_j) + b_j\varphi(u_{j+1})\right) - \left(a_{j-1}\varphi(u_{j-1}) + b_{j-1}\varphi(u_j)\right) = 0.$$

This proves that ∂f belongs to $\mathrm{MW}_{n-1}(\mathrm{V})$.

By construction, f is locally differentiable on $\mathrm{V} - \bigcup_{\mathrm{D}\in\mathscr{C}_{n-1}} \mathrm{D}$. For $\mathrm{D} \in \mathscr{C}_{n-1}$ and $x \in \mathring{\mathrm{D}}$, observe that f is differentiable on a neighborhood of x if and only if $w_{\mathrm{D}} = 0$. Consequently, the open non-differentiability locus of f is equal to $|\partial(f)|$.

[6]

a) b) With the previously introduced notation, it suffices to prove that $w_{\mathrm{D}} \geqslant 0$ for every $\mathrm{D} \in \mathscr{C}_{n-1}$.

For every positive real number t, one has

$$t w_{\mathrm{D}} = \varphi^{+}(tv^{+}) + \varphi^{-}(tv^{-}) = (f(x + tv^{+}) - f(x)) + (f(x - tv^{+}) - f(x))$$

if t is small enough. By convexity, one has

$$f(x) = \frac{1}{2}\left(f(x + tv^{+}) + f(x - tv^{+})\right),$$

so that $t w_{\mathrm{D}} \geqslant 0$; if $t > 0$, this implies $w_{\mathrm{D}} \geqslant 0$. □

[6] Some points to check. . .

Mikhalkin – Rau
Allerman – Rau . . .

Proposition **(6.3.3)**. — *The map* $f \mapsto \partial(f)$ *from the abelian group* $\mathrm{PL}(\mathrm{V})$ *of piecewise linear functions on* V *with integral slopes to the group* $\mathrm{MW}_{n-1}(\mathrm{V})$ *of* $(n-1)$*-dimensional Minkowski weights is a surjective morphism of groups. Its kernel is the subgroup of affine functions with integral slopes on* V*.*

Proof. — □

Theorem **(6.3.4)**. — *Let* f *be a piecewise linear function with integer slopes and let* $\mathrm{S} \in \mathrm{MW}_p(\mathrm{V})$*. The Minkowski weight* $\partial(f) \cap_{\mathrm{st}} \mathrm{S}$ *can be computed explicitly as follows. Let* $\mathscr{C}$ *be a polyhedral decomposition of* V *which is adapted to* S *and such that* $f|_{\mathrm{C}}$ *is affine, for every* $\mathrm{C} \in \mathscr{C}$*. For every* $\mathrm{D} \in \mathscr{C}_{p-1}$*, let* $\mathscr{C}_{\mathrm{D}}$ *be the set of* $\mathrm{C} \in \mathscr{C}_p$ *such that* $\mathrm{D} \subset \mathrm{C}$*. For* $\mathrm{C} \in \mathscr{C}_{\mathrm{D}}$*, let* $v_{\mathrm{C}/\mathrm{D}} \in \mathrm{L}_{\mathrm{C}}$ *be a vector that generates* $\mathrm{L}_{\mathrm{C}}/\mathrm{L}_{\mathrm{D}}$ *and such that* $x + tv_{\mathrm{C}/\mathrm{D}} \in \mathrm{C}$ *for every* $x \in \mathring{\mathrm{D}}$ *and every small enough positive real number* t*. Set*

$$w'_{\mathrm{D}} = \sum_{\mathrm{C} \in \mathscr{C}_{\mathrm{D}}} w_{\mathrm{C}} \left(\lim_{t \to 0^+} \frac{f(x + tv_{\mathrm{C}/\mathrm{D}}) - f(x)}{t} \right).$$

Then $\partial(f) \cap_{\mathrm{st}} \mathrm{S} = \sum w'_{\mathrm{D}}[\mathrm{D}]$.

Theorem **(6.3.5)** (Projection formula). — *Let* $u \colon \mathrm{L} \to \mathrm{L}'$ *be a morphism of free finitely generated abelian groups, let* $\mathrm{V} = \mathrm{L}_{\mathbf{R}}$ *and* $\mathrm{V}' = \mathrm{L}'_{\mathbf{R}}$*. Still write* u *for* $u_{\mathbf{R}} \colon \mathrm{V} \to \mathrm{V}'$*. Let* S *be a Minkowski weight on* V *and let* f *be a piecewise linear function on* V'*. One has*

$S \in MW_p$

$$u_*(u^*(f) \cap_{\mathrm{st}} \mathrm{S}) = f \cap_{\mathrm{st}} u_*(\mathrm{S}).$$

$u^* f = f \circ u$

$u_*(\partial(u^* f) \cap_{st} S) = \partial(f) \cap_{st} u_* S$ dans MW_{p-1}

Remark **(6.3.6)**. — There should be a projection formula of the form

$$u_*(S \cap_{st} u^*(S')) = u_*(S) \cap_{st} S'$$

if $u \colon L \to L'$ is a morphism of free finitely generated abelian groups.

If u is surjective, then $L \simeq L' \times L''$, and $u^*(S') = S' \boxtimes L''$.

Otherwise, one can/needs to define u^* by stable intersection, say $u^*(S') = p_*(\Gamma_u \cap_{st} (V \boxtimes S'))$, where $\Gamma_u = (\mathrm{id} \times u)_*(V)$ is the graph of u and $p \colon V \times V' \to V$ is the first projection.

6.4. Comparing algebraic and tropical intersections

6.4.1. — Let X and Y be subvarieties of $\mathbf{G}_m{}^n$, respectively defined by ideals I and J of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. Their intersection $X \cap Y$ is the subvariety of $\mathbf{G}_m{}^n$ with ideal $I + J$.

Note that in general, $X \cap Y$ might not be integral. It may have multiple component. It also may be non-reduced, for example if Y is a hyperplane tangent to X at some point a: the tangency will then be reflected by the fact that the local ring $\mathcal{O}_{X\cap Y,a}$ contains non-trivial nilpotent elements.

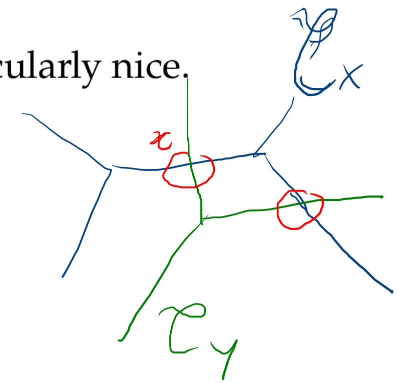
By a general inequality in algebraic geometry, one has

$$\dim_a(X \cap Y) \geqslant \dim_a(X) + \dim_a(Y) - n,$$

for every $a \in X \cap Y$. This inequality is an equality in certain cases, for example when X and Y are smooth at a, and $T_aX + T_aY = T_a\mathbf{G}_m{}^n$ (then, we say that the intersection is *transverse* around a). But the strict inequality may hold, for example in the trivial case where $X = Y$, but also in less obvious cases.

We are interested in computing the tropicalization of $X \cap Y$. How does it compare to the intersection $\mathscr{T}_X \cap \mathscr{T}_Y$, beyond the obvious inclusion? This guess is however often too large, for example if $\mathscr{T}_X = \mathscr{T}_Y$? Then how does it compare to the stable intersection $\mathscr{T}_X \cap_{st} \mathscr{T}_Y$? While that second guess is often too small, it is indubitably better, since we will show that it suffices to translate "generically" Y in $\mathbf{G}_m{}^n$, without changing its tropicalization, to make it correct.

We start with the case of transversal tropical intersections, where the picture is particularly nice.



Lemma **(6.4.2)**. — *Let* X, Y *be subvarieties of* $\mathbf{G}_m{}^n$.

a) *Let* $x \in \mathbf{R}^n$. *If* $\mathscr{T}_X$ *and* $\mathscr{T}_Y$ *meet transversally at* x, *then*

$$\mathrm{Star}_x(\mathscr{T}_{X\cap Y}) = \mathrm{Star}_x(\mathscr{T}_X \cap_{st} \mathscr{T}_Y).$$

b) *If* $\mathscr{T}_X$ *and* $\mathscr{T}_Y$ *intersect transversally everywhere, then*

$$\mathscr{T}_{X\cap Y} = \mathscr{T}_X \cap_{st} \mathscr{T}_Y.$$

Proof. — Let I, J be the ideals of X, Y in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. By assumption, there exists polyhedra C and C′ of the Gröbner polyhedral decompositions of $\mathscr{T}_X$ and $\mathscr{T}_Y$ respectively such that $x \in \mathring{C} \cap \mathring{C}'$; moreover, for every $a \in X \cap Y$. This inequality is an equality in certain cases, for example when X and Y are smooth at a, and $T_a X + T_a Y = T_a \mathbf{G}_m{}^n$ (then, we say that the intersection is *transverse* around a). But the strict inequality may hold, for example in the trivial case where $X = Y$, but also in less obvious cases.

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Aparté Théorie de l'intersection en géométrie algébrique.

X var. lisse de dim n ($X = \mathbb{P}_n$ par ex.)

Cycles de dim p : gr. ab. libre engendré par les ss var irréductibles V de dim p.

$Z_p(X)$

Pas de bonne th. de l'inters au niveau de $Z_p(X)$ à cause des composantes exceptionnelles $[V] \cap [V]$?

Solution relation d'équivalence sur $Z_p(X)$ équivalence rationnelle
(il y en a d'autres : homologique / numérique)

$W \subset X$ $\dim W = p+1$

$f \in K(W)^\times \rightsquigarrow \mathrm{div}_W(f) \in Z_p(W) \subset Z_p(X)$

$$\mathrm{div}_W(f) = \sum_{\substack{\dim(V)=p \\ V \subset W}} \mathrm{ord}_V(f)[V]$$

$\mathrm{ord}_V(f) = \begin{cases} \text{ordre du zéro de } f \text{ le long de } V \\ -\text{ ordre du pôle.} \end{cases}$

les $\mathrm{div}_W(f)$ engendrent $\mathrm{Rat}_p(X)$

$$A_p(X) = Z_p(X) / \mathrm{Rat}_p(X)$$

Il existe une str. d'anneau sur $A(X) = \bigoplus A_p(X)$

$$A_p(X) \times A_q(X) \longrightarrow A_{p+q-n}(X)$$ (Chevalley, Chow, ...)

[Aujourd'hui : Fulton. Intersection theory].

Si $[V] \in Z_p(X)$ et $\dim(V \cap W) = p+q-n$ (inters. propre)
$[W] \in Z_q(X)$

alors $$[V] \cap [W] = \sum_{\substack{S \subset V \cap W \\ \text{comp irréductible}}} m_S \, [S]$$ ← multiplicité calculée par l'alg commutative (ou l'alg homologique Serre.)

(inters. transverse. $m_S = 1$ ...)

En général : déplacement

on prouve $\exists\ z \in Z_p(X)$ $z = \sum n_j [V_j]$
tq $z \sim [V]$
V_j et W s'intersectent proprement

$$[V] \cap [W] \overset{\text{déf}}{=} \sum n_j [V_j] \cap [W]$$

Double difficulté : – prouver que le déplacement est possible
– prouver que la formule est indépendante du choix de Z – dans $A_{p+q-n}(X)$.

L'intersection tropicale décrite dans le début du chapitre est conçue sur ce modèle <u>mais</u> n'a pas besoin de relation d'équivalence / de déplacement.

$A_0(\mathbb{P}_n) = \mathbb{Z}$
"nb. de points"

Comparaison entre les deux théories d'intersection est un peu banale

algébrique

$\cap$ (cycles)

$\cap_{naïve}$

tropical

$\cap_{st}$ int. stable

$\cap$ naïve

déplacement générique

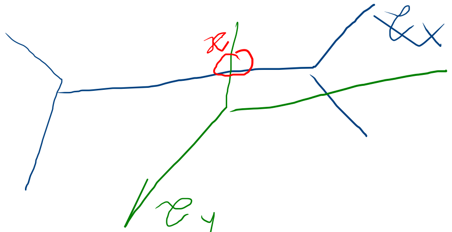
transverse

$Z_0(\mathbb{P}_n) = \mathbb{Z}^{(\mathbb{P}_n(k))}$ (si K est alg clos)

$A_0(\mathbb{P}_n) = \mathbb{Z}$

$A_0(\mathbb{A}^n) = A_0(\mathbb{G}_m^n) = 0$

$A_0(\mathbb{R}^n) = \mathbb{Z}^{(\mathbb{R}^n)}$



$\dim(C + C') = n$. In particular, $W = \mathrm{Star}_x(\mathscr{T}_X)$ and $W' = \mathrm{Star}_y(\mathscr{T}_Y)$ are vector spaces, with a constant multiplicity is constant, and $W + W' = \mathbf{R}^n$. Let $p = \dim(W)$, $q = \dim(W')$; let $W'' = W \cap W'$, so that $r = \dim(W'') = p + q - n$. Choose a rational basis of $\mathbf{R}^n$ as follows, starting from a basis of W'', and extending it to rational bases of W and W'. This shows that there exists a rational isomorphism $\varphi \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\varphi(W'') = \mathbf{R}^r \times \{0\} \times \{0\}$, $\varphi(W) = \mathbf{R}^r \times \mathbf{R}^{p-r} \times \{0\}$ and $\varphi(W') = \mathbf{R}^r \times \{0\} \times \mathbf{R}^{q-r}$. We may also assume that $\varphi(\mathbf{Z}^n) \subset \mathbf{Z}^n$. Let then $f \colon \mathbf{G}_{\mathrm{m}}{}^n \to \mathbf{G}_{\mathrm{m}}{}^n$ be the morphism of tori whose action on cocharacters is given by φ. It is finite and surjective.

Let $X' = f(X)$ and $Y' = f(Y)$; by proposition 3.7.1, one has $\mathscr{T}_{X'} = \varphi_*(\mathscr{T}_X)$, $\mathscr{T}_{Y'} = \varphi_*(\mathscr{T}_Y)$ and $\mathscr{T}_{X' \cap Y'} = \varphi_*(\mathscr{T}_{X\cap Y})$. Since φ_* is a linear isomorphism, we may assume, for proving the lemma, that φ is the identity.[7]

Let $I_x = I \cap k[T_{p+1}^{\pm 1}, \dots, T_n^{\pm 1}]$ and $J_x = J \cap k[T_{r+1}^{\pm 1}, \dots, T_p^{\pm 1}]$. By lemma 3.8.4, one has $I = I_x \cdot k[T_1^{\pm 1}, \dots]$ and $\mathrm{mult}_{\mathscr{T}_X}(C) = \mathrm{codim}(I_x)$; similarly, $J = J_x \cdot k[T_1^{\pm 1}, \dots]$ and $\mathrm{mult}_{\mathscr{T}_Y}(C') = \mathrm{codim}(J_x)$.

We now observe that

$$\mathrm{in}_x(I + J) = \mathrm{in}_x(I) + \mathrm{in}_x(J),$$

and that

$$\mathrm{in}_x(I + J) \cap k[T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1}] = I_x + J_x,$$

[7] Oops! That proposition says nothing about multiplicities. . .

Formule de projection de Sturmfels–Tevelev
(Osserman–Payne, si K n'est pas triv. valué)

so that

$$k[\mathrm{T}_{r+1}^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]/(\mathrm{I}_x+\mathrm{J}_x)\simeq (k[\mathrm{T}_{p+1}^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]/\mathrm{I}_x)\otimes_k (k[\mathrm{T}_{r+1}^{\pm 1},\dots,\mathrm{T}_p^{\pm 1}]/\mathrm{J}_x)$$

has dimension $\mathrm{mult}_{\mathscr{T}_{\mathrm{X}}}(\mathrm{C})\,\mathrm{mult}_{\mathscr{T}_{\mathrm{Y}}}(\mathrm{C}')$. The same result holds for every other point in $\mathring{\mathrm{C}}\cap\mathring{\mathrm{C}}'$. This shows that $\mathrm{C}\cap\mathrm{C}'\subset\mathscr{T}_{\mathrm{X}\cap\mathrm{Y}}$ contains a polyhedron of the Gröbner decomposition of $\mathrm{X}\cap\mathrm{Y}$, and that its multiplicity is the product of the multiplcities of C and C′. This concludes the proof of the first assertion of the lemma, and the second follows directly from it. ☐

6.4.3. — Let K be a valued field. Let $\mathrm{L}=\mathrm{K}(s)$ be the field of rational functions in one indeterminate s with coefficients in K, endowed with the Gauss absolute value. Let $\mathrm{I}\subset\mathrm{L}[\mathrm{T}_1^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]$ be an ideal and let $\mathrm{X}=\mathrm{V}(\mathrm{I})$.

Consider $\mathrm{K}(s)$ as the field of functions of the affine line $\mathbf{A}^1$. The Zariski closure $\mathscr{X}$ of X in $\mathbf{G}_{\mathrm{m}}{}^n_{\mathbf{A}^1}$ is defined by the ideal $\mathscr{I}=\mathrm{K}[s][\mathrm{T}^{\pm 1}]\cap\mathrm{I}$. For every point $a\in\mathrm{K}$, or rather in a valued extension K′ of K, we can then consider the ideal $\mathscr{I}_a$ of $\mathrm{K}'[\mathrm{T}^{\pm 1}]$ deduced from I by setting $s=a$ and the subscheme $\mathscr{X}_a=\mathrm{V}(\mathscr{I}_a)$ of $\mathbf{G}_{\mathrm{m}}{}^n_{\mathrm{K}'}$.

The relations between X and the schemes $\mathscr{X}_a$, its specializations, are well-studied in algebraic geometry. In fact, $\mathscr{X}$ is a flat $\mathbf{A}^1$-scheme, and $\mathscr{X}_a$ is its fiber. In particular, the schemes $\mathscr{X}_a$ are equidimensional if $\mathscr{X}$ is, with the same dimension.

We first prove that, up to finitely many obstructions, the schemes $\mathscr{X}_a$ have the same tropicalization as X provided $v(a)=v(s)=0$.

Pour le cas général, non transverse, il va falloir <u>bouger</u> X ou Y l'un par rapport à l'autre - Facile ! On est dans le <u>groupe</u> $\mathbb{G}_m^n$

$\overline{K}$ $\overline{k}$ — donc on peut regarder X et tY $t \in \mathbb{G}_m^n(K)$ (K alg clos)

corps val K — corps résid k

$\mathcal{C}_X$ $v + \mathcal{C}_Y$ $v = (v(t_1), \ldots, v(t_n))$

On peut choisir v pour que l'intersection tropicale soit transverse (valuation de K <u>non triviale</u>)

On veut permettre <u>$v = 0$</u> (sinon, à quoi bon développer l'int. tropicale !)

Deux voies pour formaliser l'idée du déplacement générique

① $\{ t \in (\overline{K}^\times)^n \mid v(t) = 0,\ \mathcal{C}_{X \cap tY} = \mathcal{C}_X \cap_{st} \mathcal{C}_Y \}$

est « gros » : $\exists\, B \subset \overline{k}^{\times n}$ ouvert Zar dense tq $\{ t \in \ldots \}$ si $\overline{t} \in B$

② $L = K(t_1, \ldots, t_n)$ + norme de Gauss $\quad \|\sum c_m t^m\| = \sup \|c_m\|$

$v(\sum c_m t^m) = \inf v(c_m)$

$\ell = k(t_1, \ldots, t_n)$ $\quad (\|t_i\| = 1)$

$t = (t_1, \ldots, t_n) \in \mathbb{G}_m^n(L)$

$\mathcal{C}(X_L \cap t Y_L) = \mathcal{C}_X \cap_{st} \mathcal{C}_Y$

Pour démontrer ① et ②, le passage par ② est nécessaire,

explicité dans Anders - Yu (2016)

implicite dans Maclagan - Sturmfels (2015)

Je propose de passer par un énoncé intermédiaire, d'intérêt plus large : comparer des tropicalisations

All this should be rewritten replacing $\mathbf{A}^1$ *with* $\mathbf{A}^n$*, possibly even any integral variety.*

Proposition **(6.4.4)**. — *There exists a finite subset* B *of* $\bar{k}$ *such that for every* a *in a valued extension* K′ *of* K *(with residue field* k'*) such that* $v(a) = 0$ *and* $\overline{a} \notin \mathrm{B}$*, one has the equality of initial ideals* $\mathrm{in}_x(\mathrm{I}) \cdot k'(s) = \mathrm{in}_x(\mathscr{I}_a) \cdot k'(s)$*, for every* $x \in \mathbf{R}^n$*. In particular, for all such* a*, one has an equality of tropicalizations* $\mathscr{T}_{\mathrm{X}} = \mathscr{T}_{\mathscr{X}_a}$*.*

Proof. — We start with a few remarks.

Let $(f_1, \dots, f_m)$ be a finite family of elements of $\mathrm{K}(s)[\mathrm{T}^{\pm 1}]$ generating I. Let $h_1 \in \mathrm{K}[s]$ be a non-zero polynomial such that $h_1 f_i \in \mathrm{K}[s][\mathrm{T}^{\pm 1}]$ for every $i \in \{1, \dots, n\}$. Replacing f_i with $h_1 f_i$ for every i, we assume that $f_i \in \mathrm{K}[s][\mathrm{T}^{\pm 1}]$ for every i. Then the ideal $(f_1, \dots, f_m)$ of $\mathrm{K}[s][\mathrm{T}^{\pm 1}]$ is contained in $\mathscr{I}$. Let also $(g_1, \dots, g_p)$ be a generating family of $\mathscr{I}$ in $\mathrm{K}[s][\mathrm{T}^{\pm 1}]$. For every $j \in \{1, \dots, p\}$, there exist Laurent polynomials $k_{j,1}, \dots, k_{j,m} \in \mathrm{K}(s)[\mathrm{T}^{\pm 1}]$ such that $g_j = \sum_{i=1}^m k_{j,i} f_i$. Let $h \in \mathrm{K}[s]$ be a non-zero polynomial such that $h k_{j,i} \in \mathrm{K}[s][\mathrm{T}^{\pm 1}]$ for all i, j. We then obtain inclusions $h\mathscr{I} \subset (f_1, \dots, f_m) \subset \mathscr{I}$ of ideals of $\mathrm{K}[s][\mathrm{T}^{\pm 1}]$. In particular, for every a in a valued extension K′ of K such that $h(a) \neq 0$, the ideal $\mathscr{I}_a$ of $\mathrm{K}'[\mathrm{T}^{\pm 1}]$ coincides with the ideal generated by $f_1(a; \mathrm{T}), \dots, f_m(a; \mathrm{T})$.

Let $f \in \mathrm{K}[s][\mathrm{T}^{\pm 1}]$; write $f = \sum_{m \in \mathrm{S}(f)} f_m(s) c_m \mathrm{T}^m$, where $c_m \in \mathrm{K}^\times$ and $f_m \in \mathrm{K}[s]$ is a polynomial of Gauss-norm 1. The reductions $\overline{f}_m$ of the polynomials f_m are non-zero polynomials in $k[s]$. Let h be their product. By construction, for every a in a valued extension K′ of k such that $v(a) = 0$ and $h(\overline{a}) \neq 0$, one has $v(f_m(a)) = 0$ for all $m \in \mathrm{S}(f)$. It follows that for every such a, one has $\tau_x(f) = \tau_x(f(a;\mathrm{T}))$ and $\mathrm{in}_x(f)(a;\mathrm{T}) = \mathrm{in}_x(f(a;\mathrm{T}))$ for all $x \in \mathbf{R}^n$.

Assume that $(f_1, \dots, f_m)$ contains a basis, a uniform Gröbner basis of I, and a tropical basis. Assume also that the coordinates of x belong to the value group of K. Then the initial ideal $\mathrm{in}_x(\mathrm{I})$ is generated by the initial forms $\mathrm{in}_x(f_i)$, for $i \in \{1, \dots, m\}$. Up to the exceptions described above, the initial ideal $\mathrm{in}_x(\mathrm{I})_{\overline{a}}$ is generated by the initial forms $\mathrm{in}_x(f_i(a;\mathrm{T}))$, hence is contained in the initial ideal $\mathrm{in}_x(\mathscr{I}_a)$. Conversely,. . . □