*Remark* **(6.3.6)**. — There should be a projection formula of the form

$$u_*(S \cap_{st} u^*(S')) = u_*(S) \cap_{st} S'$$

if $u : L \to L'$ is a morphism of free finitely generated abelian groups.

If $u$ is surjective, then $L \simeq L' \times L''$, and $u^*(S') = S' \boxtimes L''$.

Otherwise, one can/needs to define $u^*$ by stable intersection, say $u^*(S') = p_*(\Gamma_u \cap_{st} (L \boxtimes S'))$, where $\Gamma_u = (\mathrm{id} \times u)_*(V)$ is the graph of $u$ and $p : V \times V' \to V$ is the first projection.

## 6.4. Comparing algebraic and tropical intersections

**6.4.1.** — Let X and Y be subvarieties of $\mathbf{G}_m{}^n$, respectively defined by ideals I and J of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}$. Their intersection $X \cap Y$ is the subvariety of $\mathbf{G}_m{}^n$ with ideal $I + J$.

Note that in general, $X \cap Y$ might not be integral. It may have multiple component. It also may be non-reduced, for example if Y is a hyperplane tangent to X at some point $a$: the tangency will then be reflected by the fact that the local ring $\mathscr{O}_{X \cap Y, a}$ contains non-trivial nilpotent elements.

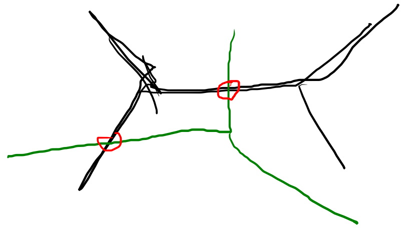
By a general inequality in algebraic geometry, one has

$$\dim_a(X \cap Y) \geqslant \dim_a(X) + \dim_a(Y) - n,$$

for every $a \in X \cap Y$. This inequality is an equality in certain cases, for example when X and Y are smooth at $a$, and $T_aX + T_aY = T_a\mathbf{G}_m{}^n$ (then, we say that the intersection is *transverse* around $a$). But the strict inequality may hold, for example in the trivial case where $X = Y$, but also in less obvious cases.

We are interested in computing the tropicalization of $X \cap Y$. How does it compare to the intersection $\mathscr{T}_X \cap \mathscr{T}_Y$, beyond the obvious inclusion? This guess is however often too large, for example if $\mathscr{T}_X = \mathscr{T}_Y$? Then how does it compare to the stable intersection $\mathscr{T}_X \cap_{st} \mathscr{T}_Y$? While that second guess is often too small, it is indubitably better, since we will show that it suffices to translate "generically" Y in $\mathbf{G}_m{}^n$, without changing its tropicalization, to make it correct.

We start with the case of transversal tropical intersections, where the picture is particularly nice.



*Lemma* **(6.4.2)**. — *Let* X, Y *be subvarieties of* $\mathbf{G}_m{}^n$.

a) *Let* $x \in \mathbf{R}^n$. *If* $\mathscr{T}_X$ *and* $\mathscr{T}_Y$ *meet transversally at* $x$, *then*

$$\mathrm{Star}_x(\mathscr{T}_{X\cap Y}) = \mathrm{Star}_x(\mathscr{T}_X \cap_{st} \mathscr{T}_Y).$$

b) *If* $\mathscr{T}_X$ *and* $\mathscr{T}_Y$ *intersect transversally everywhere, then*

$$\mathscr{T}_{X\cap Y} = \mathscr{T}_X \cap_{st} \mathscr{T}_Y.$$

(ensemblistement, $\mathscr{C}_X \cap \mathscr{C}_Y$ multiplicités produits)

*Proof*. — Let I, J be the ideals of X, Y in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. By assumption, there exists polyhedra C and C′ of the Gröbner polyhedral decompositions of $\mathscr{T}_X$ and $\mathscr{T}_Y$ respectively such that $x \in \mathring{C} \cap \mathring{C}'$; moreover,

$\dim(C + C') = n$. In particular, $W = \mathrm{Star}_x(\mathcal{T}_X)$ and $W' = \mathrm{Star}_y(\mathcal{T}_Y)$ are vector spaces, with a constant multiplicity is constant, and $W + W' = \mathbf{R}^n$. Let $p = \dim(W)$, $q = \dim(W')$; let $W'' = W \cap W'$, so that $r = \dim(W'') = p + q - n$. Choose a rational basis of $\mathbf{R}^n$ as follows, starting from a basis of $W''$, and extending it to rational bases of $W$ and $W'$. This shows that there exists a rational isomorphism $\varphi \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\varphi(W'') = \mathbf{R}^r \times \{0\} \times \{0\}$, $\varphi(W) = \mathbf{R}^r \times \mathbf{R}^{p-r} \times \{0\}$ and $\varphi(W') = \mathbf{R}^r \times \{0\} \times \mathbf{R}^{q-r}$. We may also assume that $\varphi(\mathbf{Z}^n) \subset \mathbf{Z}^n$. Let then $f \colon \mathbf{G_m}^n \to \mathbf{G_m}^n$ be the morphism of tori whose action on cocharacters is given by $\varphi$. It is finite and surjective.

Let $X' = f(X)$ and $Y' = f(Y)$; by proposition 3.7.1, one has $\boxed{\mathcal{T}_{X'} = \varphi_*(\mathcal{T}_X)}$, $\mathcal{T}_{Y'} = \varphi_*(\mathcal{T}_Y)$ and $\mathcal{T}_{X' \cap Y'} = \varphi_*(\mathcal{T}_{X\cap Y})$. Since $\varphi_*$ is a linear isomorphism, we may assume, for proving the lemma, that $\varphi$ is the identity. [5]

Let $I_x = I \cap k[T_{p+1}^{\pm 1}, \dots, T_n^{\pm 1}]$ and $J_x = J \cap k[T_{r+1}^{\pm 1}, \dots, T_p^{\pm 1}]$. By lemma 3.8.4, one has $I = I_j \cdot k[T_1^{\pm 1}, \dots]$ and $\mathrm{mult}_{\mathcal{T}_Y}(C') = \mathrm{codim}(J_x)$; similarly, $J = J_x \cdot k[T_1^{\pm 1}, \dots]$ and $\mathrm{mult}_{\mathcal{T}_Y}(C') = \mathrm{codim}(J_x)$.

We now observe that

$$\mathrm{in}_x(I + J) = \mathrm{in}_x(I) + \mathrm{in}_x(J),$$

and that

$$\mathrm{in}_x(I + J) \cap k[T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1}] = I_x + J_x,$$

[5] Oops! That proposition says nothing about multiplicities. . .

$T = 1$

$(sT = s)$

$s = 0$

$0 = 0$

so that

$$k[T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1}]/(I_x + J_x) \simeq (k[T_{p+1}^{\pm 1}, \dots, T_n^{\pm 1}]/I_x) \otimes_k (k[T_{r+1}^{\pm 1}, \dots, T_p^{\pm 1}]/J_x)$$

has dimension $\operatorname{mult}_{\mathscr{T}_X}(C)\operatorname{mult}_{\mathscr{T}_Y}(C')$. The same result holds for every other point in $\mathring{C} \cap \mathring{C}'$. This shows that $C \cap C' \subset \mathscr{T}_{X\cap Y}$ contains a polyhedron of the Gröbner decomposition of $X \cap Y$, and that its multiplicity is the product of the multiplcities of C and C′. This concludes the proof of the first assertion of the lemma, and the second follows directly from it. □

**6.4.3.** — Let K be a valued field. Let $L = K(s)$ be the field of rational functions in one indeterminate $s$ with coefficients in K, endowed with the Gauss absolute value. Let $I \subset L[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be an ideal and let $X = V(I)$. Assume that X is equidimensional and let $d = \dim(X)$.

$\mathbb{G}_m^n, L$ ?

Consider $K(s)$ as the field of functions of the affine line $\mathbf{A}^1$. The Zariski closure $\mathscr{X}$ of X in $\mathbf{G}_{m\,\mathbf{A}^1}^{\,n}$ is defined by the ideal $\mathscr{I} = K[s][T^{\pm 1}] \cap I$. For every point $a \in K$, or rather in a valued extension K′ of K, we can then consider the ideal $\mathscr{I}_a$ of $K'[T^{\pm 1}]$ deduced from I by setting $s = a$ and the subscheme $\mathscr{X}_a = V(\mathscr{I}_a)$ of $\mathbf{G}_{m\,K'}^{\,n}$.

The relations between X and the schemes $\mathscr{X}_a$, its specializations, are well-studied in algebraic geometry. In fact, $\mathscr{X}$ is a flat $\mathbf{A}^1$-scheme, and $\mathscr{X}_a$ is its fiber. In particular, the schemes $\mathscr{X}_a$ are equidimensional if $\mathscr{X}$ is, with the same dimension.

We first prove that, up to finitely many obstructions, the schemes $\mathscr{X}_a$ have the same tropicalization as X provided $v(a) = v(s) = 0$.

$\mathscr{C}_X$ $\quad$ $X \subset \mathbb{G}^n_{m,K(s)}$ $\quad$ $\overline{X}^{Zar} = \mathscr{X} \subset \mathbb{A}^1_s \times \mathbb{G}^n_{m,K}$ $\quad$ $\mathscr{C}_{\mathscr{X}_a}$ $\quad$ $\mathscr{X}_a$

$\|\ \|_{K(s)}$ $\quad$ Spec(K(s)) $\quad$ plat $\quad$ Spec(K[s]) = $\mathbb{A}^1_s$ $\quad$ $|\ |_{K'}$ $\quad$ $K' \ni a$

$\|\sum_m c_m s^m\|_{K(s)} = \sup_m |c_m|$

$\|s\| = 1$

$|a| = 1 = \|s\|$

revenir plus tard

*All this should be rewritten replacing* $\mathbf{A}^1_s$ *with* $\mathbf{A}^n$*, possibly even any integral variety* V*; the outcome is an analytic domain containing a given Zariski-dense point of* $\mathrm{V}^{\mathrm{an}}$.

$\mathrm{K}' \to k'$

$\mid \quad \mid$

$\mathrm{K} \to k$

*Proposition* **(6.4.4)**. — *There exists a finite subset* B *of* $\bar{\mathrm{K}}$*, a finite subset* C *of* $\bar{k}$ *such that for every* $a$ *in a valued extension* K′ *of* K *(with residue field* $k'$*) such that* $a \notin \mathrm{B}$*,* $v(a) = 0$*, and* $\bar{a} \notin \mathrm{C}$*, the variety* $\mathscr{X}_a$ *has the same tropicalization than* X*: one has an equality of weighted polyhedra* $\mathscr{T}_{\mathrm{X}} = \mathscr{T}_{\mathscr{X}_a}$.

$\mathrm{in}_x(\mathrm{I}^h) = (\mathrm{in}_x(f_1), \dots, \mathrm{in}_x(f_m))$

*Proof.* — Let us consider the homogeneization $\mathrm{I}^{\mathrm{h}} \subset \mathrm{K}(s)[\mathrm{T}_0, \dots, \mathrm{T}_n]$ of I. Let $(f_1, \dots, f_m)$ be a finite set of homogeneous polynomials in $\mathrm{I}^{\mathrm{h}}$ which is a universal Gröbner basis, *i.e.*, at every $x \in \mathbf{R}^{n+1}$; we may assume that it is contained in $\mathscr{I}^{\mathrm{h}} = \mathrm{K}[s][\mathrm{T}_0, \dots, \mathrm{T}_n] \cap \mathrm{I}^{\mathrm{h}}$.

a) The family $(f_1, \dots, f_m)$ generates the ideal $\mathrm{I}^{\mathrm{h}}$ in $\mathrm{K}(s)[\mathrm{T}_0, \dots, \mathrm{T}_n]$. Since $\mathrm{K}[s][\mathrm{T}_0, \dots, \mathrm{T}_n]$ is a noetherian ring, the homogeneous ideal $\mathscr{I}^{\mathrm{h}}$ has a finite basis $(g_1, \dots, g_p)$ consisting of homogeneous polynomials. For every $j \in \{1, \dots, p\}$, there exist homogeneous polynomials $k_{j,1}, \dots, k_{j,m} \in \mathrm{K}(s)[\mathrm{T}_0, \dots, \mathrm{T}_n]$ such that $g_j = \sum_{i=1}^m k_{j,i} f_i$. Let $h \in \mathrm{K}[s]$ be a non-zero polynomial such that $h k_{j,i} \in \mathrm{K}[s][\mathrm{T}_0, \dots, \mathrm{T}_n]$ for all $i, j$. We then obtain inclusions $h\mathscr{I}^{\mathrm{h}} \subset (f_1, \dots, f_m) \subset \mathscr{I}^{\mathrm{h}}$ of homogeneous ideals of $\mathrm{K}[s][\mathrm{T}_0, \dots, \mathrm{T}_n]$. In particular, for every $a$ in a valued extension K′ of K such that $h(a) \neq 0$, the ideal $\mathscr{I}_a$ of $\mathrm{K}'[\mathrm{T}_0, \dots, \mathrm{T}_n]$ coincides with the ideal generated by $f_1(a; \mathrm{T}), \dots, f_m(a; \mathrm{T})$. We define $\mathrm{B} \subset \bar{\mathrm{K}}$ as the set of roots of $h$.

b) Let $f \in \mathrm{K}[s][\mathrm{T}_0, \dots, \mathrm{T}_n]$; write $f = \sum_{m \in \mathrm{S}(f)} f_m(s) c_m \mathrm{T}^m$, where $c_m \in \mathrm{K}^\times$ and $f_m \in \mathrm{K}[s]$ is a polynomial of Gauss-norm 1. The reductions $\overline{f}_m$ of the polynomials $f_m$ are non-zero polynomials in $k[s]$. Let $h_f$ be

$\tau_x(f) = \inf_m v(f_m(s) c_m) + \langle m, x \rangle$

$= \inf_m v(c_m) + \langle m, x \rangle$

$v(f_m(a)) \geqslant 0$ $\quad \bar{a} \in k'$ $\quad \|f_m\| = 1$

$k \ni \overline{f_m(a)} = \bar{f}_m(\bar{a})$

$\neq 0$

$f(a; \mathrm{T}) = \sum f_m(a) c_m \mathrm{T}^m$

si $= 0$ ?

$\tau_x(f(a; \mathrm{T})) = \inf_m v(f_m(a)) + v(c_m) + \langle m, x \rangle$

$\mathrm{in}_x(f) = \sum \bar{f}_m(s) \tilde{c}_m T^m$
$m \in S_x(f)$

their product. By construction, for every $a$ in a valued extension K′ of $k$ such that $v(a) = 0$ and $h_f(\overline{a}) \neq 0$, one has $v(f_m(a)) = 0$ for all $m \in \mathrm{S}(f)$. It follows that for every such $a$, one has $\tau_x(f) = \tau_x(f(a;\mathrm{T}))$ and $\mathrm{in}_x(f)(\bar{a};\mathrm{T}) = \mathrm{in}_x(f(a;\mathrm{T}))$ for all $x \in \mathbf{R}^{n+1}$.

c) Let $h'$ be the product of the polynomials $h_{f_j}$ and let $\mathrm{C}_1 \subset \bar{k}$ be the set of roots of $h'$.

For $x \in \mathbf{R}^{n+1}$, set $\mathrm{J}_x = \mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$; note that there are only finitely many ideals of the form $\mathrm{J}_x$, when $x \in \mathbf{R}^{n+1}$. Let $\mathscr{J}_x = \mathrm{J}_x \cap k[s][\mathrm{T}_0, \dots, \mathrm{T}_n]$; for $b$ in an extension $k'$ of $k$, let $\mathscr{J}_{x,b}$ be the image of $\mathscr{J}_x$ in $k'[\mathrm{T}_0, \dots, \mathrm{T}_n]$.

$\mathrm{in}_x(f(a;T))$
$= \sum \bar{f}_m(\bar{a}) \tilde{c}_m T^m$
$m \in S_x(f(a;T))$

For $x \in \mathbf{R}^{n+1}$, the ideal $\mathrm{J}_x$ is generated by $(\mathrm{in}_x(f_1), \dots, \mathrm{in}_x(f_m))$, by definition of a universal Gröbner basis. It follows that there exists a finite subset $\mathrm{C}_2$ of $\bar{k}$ such that for every $b$ in an extension $k'$ of $k$ such that $b \notin \mathrm{C}_2$, one has $\mathscr{J}_{x,b} = (\mathrm{in}_x(f_1)(b;\mathrm{T}), \dots, \mathrm{in}_x(f_m)(b;\mathrm{T}))$.

Let $a$ be an element of a valued extension K′ of K such that $a \notin \mathrm{B}$, $v(a) = 0$ and $\bar{a} \notin \mathrm{C}_1 \cup \mathrm{C}_2$. Then one has $\mathrm{in}_x(f_j(a;\mathrm{T})) = \mathrm{in}_x(f_j)(\bar{a};\mathrm{T})$, so that $\mathscr{J}_{x,\bar{a}} = (\mathrm{in}_x(f_1)(\bar{a};\mathrm{T}), \dots, \mathrm{in}_x(f_m)(\bar{a};\mathrm{T})) \subset \mathrm{in}_x(\mathscr{I}_a)$.

$I \to J \to J_a$
$\mathrm{in}_x(J_a)$
$\mathrm{in}_x(I) = J \to J \to J_{\bar{a}}$

By flatness of $\mathrm{K}[s][\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathscr{I}^h$ over $\mathrm{K}[s]$, the homogeneous ideals $\mathscr{I}_a^h$ and $\mathrm{I}^h$ have the same Hilbert function. Similarly, the homogeneous ideals $\mathscr{J}_{x,\bar{a}}$ and $\mathrm{J}_x$ have the same Hilbert function. Moreover, by theorem 3.4.12, the homogeneous ideals $\mathrm{I}^{\mathrm{h}} \subset \mathrm{K}(s)[\mathrm{T}_0, \dots, \mathrm{T}_n]$ and $\mathrm{J}_x = \mathrm{in}_x(\mathrm{I}^{\mathrm{h}}) \subset k(s)[\mathrm{T}_0, \dots, \mathrm{T}_n]$ have the same Hilbert function; similarly, the homogenous ideals $\mathscr{I}_a^{\mathrm{h}} \subset \mathrm{K}'[\mathrm{T}_0, \dots, \mathrm{T}_n]$ and $\mathrm{in}_x(\mathscr{I}_a^{\mathrm{h}}) \subset k'[\mathrm{T}_0, \dots, \mathrm{T}_n]$ have the same Hilbert function. It follows that the inclusion $\mathscr{J}_{x,\bar{a}} \subset \mathrm{in}_x(\mathscr{I}_a^{\mathrm{h}})$ is an equality: $\boxed{\mathscr{J}_{x,\bar{a}} = \mathrm{in}_x(\mathscr{I}_a^{\mathrm{h}})}$.

$S_x(f) = S_x(f(a;T))$

d) These equalities imply that the Gröbner decompositions of $\mathbf{R}^{n+1}$ associated with the homogeneous ideals $\mathrm{I}^{\mathrm{h}}$ and $\mathscr{I}_a^{\mathrm{h}}$ coincide, for every such $a$. Let $x \in \mathbf{R}^n$ and let $x' = (0, x) \in \mathbf{R}^{n+1}$; we know that $x \in \mathscr{T}_{\mathrm{X}}$

if and only if $\mathrm{in}_x(\mathrm{I}) \neq (1)$, if and only if $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ contains no monomials. Similarly, $x \in \mathscr{T}_{\mathscr{X}_a}$ if and only if $\mathrm{in}_x(\mathscr{I}_a) \neq (1)$, if and only if $\mathrm{in}_x(\mathscr{I}_a^{\mathrm{h}})$ contains no monomial.

For good $a$ as above, this already implies that $\mathscr{T}_{\mathscr{X}_a} \subset \mathscr{T}_{\mathrm{X}}$. Let indeed $x \in \mathbf{R}^n - \mathscr{T}_{\mathrm{X}}$ and let $x' = (0, x)$. Then $\mathrm{J}_{x'} = \mathrm{in}_{x'}(\mathrm{I}^{\mathrm{h}})$ contains a monomial; it then belongs to $\mathscr{J}_{x'}$, so that $\mathrm{in}_{x'}(\mathscr{I}_a^{\mathrm{h}}) = \mathscr{J}_{x',\bar{a}}$ contains a monomial as well. Consequently, $x \notin \mathscr{T}_{\mathscr{X}_a}$.

The converse inclusion will require to put an additional restriction on the set of good $a$. Let $\mathscr{Y}_x \subset \mathbf{G}_{\mathrm{m}}{}^n_{k[s]}$ be the closed subscheme defined by the ideal $\mathscr{J}_x$. Its image $\mathrm{V}_x$ in $\mathbf{A}^1_k = \mathrm{Spec}(k[s])$ is the set of points $\alpha$ of $\mathbf{A}^1$ such that $\mathscr{Y}_{x,\alpha} \neq (1)$. By a theorem of Chevalley, it is a constructible subset of $\mathbf{A}^1_k$. Since $\mathbf{A}^1_k$ has dimension 1, there are only two possibilities: either $\mathrm{V}_x$ is a strict closed subset, or $\mathrm{V}_x$ is a dense open subset and its complement is finite. The first case happens if and only if the generic point of $\mathbf{A}^1_k$ does not belong to $\mathrm{V}_x$, *i.e.*, if $\mathrm{J}_x$ contains 1, that is, if and only if $x \notin \mathscr{T}_{\mathrm{X}}$. Let $\mathrm{C}_3$ be the set of points in $\bar{k}$ which do not belong to those $\mathrm{V}_x$, for $x \in \mathscr{T}_{\mathrm{X}}$. Since there are only finitely many ideals of the form $\mathrm{J}_x$, the set $\mathrm{C}_3$ is finite.

Let $a$ be an element of a valued extension K′ of K such that $a \notin \mathrm{B}$, $v(a) = 0$ and $\bar{a} \notin \mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{C}_3$. By construction, if a point $x \in \mathbf{R}^n$ belongs to $\mathscr{T}_{\mathrm{X}}$, then $\mathscr{Y}_{x,\bar{a}} \neq \varnothing$, hence $\mathrm{in}_x(\mathscr{I}_a) \neq (1)$ and $x \in \mathscr{T}_{\mathscr{X}_a}$.

This proves the equality $\mathscr{T}_{\mathrm{X}} = \mathscr{T}_{\mathscr{X}_a}$ for all such $a$. We also saw above the coincidence of the Gröbner polyhedral decompositions of this polyhedral subset of $\mathbf{R}^n$ respectively associated with the ideals I and $\mathscr{I}_a$.

e) It remains to prove the equality of multiplicities. Let $x \in \mathbf{R}^n$ and let C be a polyhedron of these Gröbner decompositions. Up to a monomial change of variable, we may assume that the affine span of C

$\mathcal{C}_X \subset \mathcal{C}_{X_a}$

$x \notin \mathcal{C}_{X_a}$

$\Rightarrow x \notin \mathcal{C}_X$

$\mathcal{J}$

$X_x = V(\mathrm{in}_x(I))$

$Y_x \xrightarrow{p} \mathbb{A}^1_s$

$x \notin \mathcal{C}_{X_a}$

$\Leftrightarrow Y_{x,\bar a} \neq \emptyset$

$\Leftrightarrow p(Y_x) \not\ni \bar a$

c.à.d. $p(Y_x)$ contient le point générique $\Leftrightarrow$ $\mathrm{in}_x(I) \neq 1$ $\Leftrightarrow$ $x \in \mathcal{C}_X$.

# Théorème de constructibilité de Chevalley.

$k$ corps (alg. clos si vous voulez)

$X$ var. alg $/k$ (schéma t.f.)

Parties constructibles de $X$ : plus petite sous alg. de Boole $\mathcal{C}_X$ de $\mathcal{P}(X)$

qui contient les ouverts

$\to$ contient les fermés $\quad F = \complement U$

$\to$ ——— les $U \cap F \quad U$ ouvert, $F$ fermé

$\to$ ——— $\bigcup_{i=1}^{n} U_i \cap F_i$

$$\mathcal{C}_X = \left\{ \bigcup_{i=1}^{n} U_i \cap F_i \ , \ \begin{array}{l} U_1, \dots, U_n \subset X \text{ ouverts}, \\ F_1, \dots, F_n \subset X \text{ fermés} \end{array} \right\}.$$

**Th.** $f : X \to Y$ morphisme de var. alg.

$\Rightarrow f(X)$ est constructible dans $Y$.

$f_*(\mathcal{C}_X) \subset \mathcal{C}_Y$

Tout repose sur un résultat d'algèbre commutative, plus ~~ou~~ moins équivalent au théorème des zéros de Hilbert :

Th. $\varphi : A \to B$ un morphisme injectif de $k$-alg. t.f. intègres

Pour tout $b \in B - \{0\}$, il existe $a \in A - \{0\}$ tel que : si $f : A \to K$ est un morphisme à valeurs dans une extension $K$ de $k$ (alg. close) tel que $f(a) \neq 0$, il existe $g : B \to K$ tel que $\begin{cases} g|_A = f \\ g(b) \neq 0. \end{cases}$

$B = A[b_1, \dots, b_m]$

récurrence sur $m$

$B = A[b]$

$\to$ $b$ transcendant

$\to$ $b$ algébrique

$P = a_n T^n + \dots + a_0$

imposer $f(a_n) \neq 0$

${}^a\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$
$\qquad = X \qquad\qquad\qquad = Y$

$\varphi$ injectif : ${}^a\varphi(X)$ est dense dans $Y$.

Soit $a$ tq $f : A \to K$ se prolonge à $B$ dès que $f(a) \neq 0$

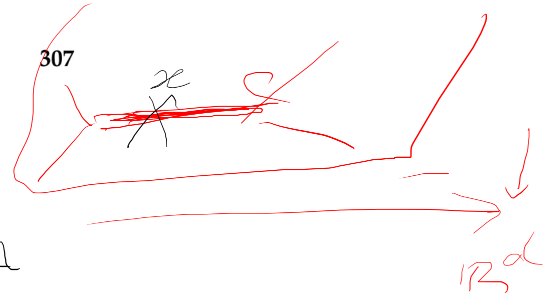
$y \in Y(K)$ $\qquad x \in X(K)$ $\qquad {}^a\varphi(x) = y$.

$\Rightarrow$ ${}^a\varphi(X) \supset D(a)$

ouvert dense.

$Y' = V(a) = Y - D(a)$ $\qquad \cancel{X'} = ({}^a\varphi)^{-1}(Y') \longrightarrow Y'$

$\mathcal{J}_x = \mathrm{in}_x(I)$



is $x + \mathbf{R}^d \times \{0\}$. Then one has

$$\mathrm{mult}_{\mathcal{T}_X}(C) = \dim(k(s)[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]/J_x \cap k(s)[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}])$$

$\mathcal{A} \otimes_{k[s]} k(s)$

and

$$\mathrm{mult}_{\mathcal{T}_{X_a}}(C) = \dim(k'[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]/\mathcal{J}_{x,\bar a} \cap k'[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]).$$

$\mathcal{A}$ $\quad \mathcal{A} \otimes_{k(s)} k'$

Let $\mathcal{A}$ be the finitely generated $k[s]$-algebra $k[s][T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]/\mathcal{J}_x \cap k[s][T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]$. It is flat, by construction, and its generic fiber $\mathcal{A} \otimes_{k[s]} k(s)$ is a finite $k(s)$-algebra of rank $\mathrm{mult}_{\mathcal{T}_X}(C)$. Consequently, $\mathcal{A}$ is finite over $k[s]$, of constant rank. In particular,

si $a \notin C_h$

$$\mathrm{mult}_{\mathcal{T}_{X_a}}(C) = \dim_{k'}(\mathcal{A} \otimes_{k[s]} k') = \mathrm{mult}_{\mathcal{T}_X}(C).$$

This concludes the proof. □

$\mathcal{A}$ fini qu'un ouv dense

$\mathcal{A}'_s$

*Lemma* **(6.4.5)**. — *Let* $I \subset K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *and let* $x \in \mathbf{R}^{n-1} \times \{0\}$*. One has the following equality of ideals in* $k(s)[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$:

$$\mathrm{in}_x(I_{K(s)} + (T_n - s)) = \mathrm{in}_x(I)_{k(s)} + (T_n - s).$$

$J$

$k[s, \frac{1}{h}]$

Recall that the field $K(s)$ is endowed with the Gauss absolute value; in particular, $v(s) = 0$.

$h \in k[s] - \{0\}$

*Proof*. — One has $\mathrm{in}_x(I)_{k(s)} = \mathrm{in}_x(I_{K(s)})$ and $\mathrm{in}_x(T_n - s) = T_n - s$ since $x_n = 0$. This implies the inclusion

$C_h =$ zéros de $h$

$$\mathrm{in}_x(I)_{k(s)} + (T_n - s) \subset \mathrm{in}_x(I_{K(s)} + (T_n - s)).$$

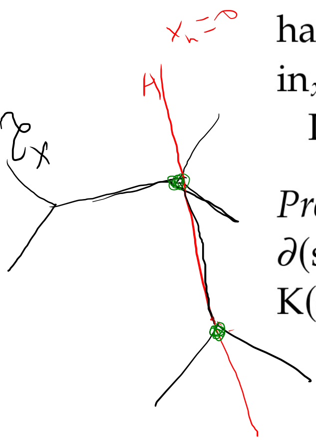
Conversely, let $h \in \mathrm{I}_{\mathrm{K}(s)} + (\mathrm{T}_n - s)$ and let us prove that $\mathrm{in}_x(h) \in \mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s)$. Up to multiplying $h$ by a non-zero element of $\mathrm{K}[s]$, we may assume that there exist $p \in \mathrm{K}[s]$, $f \in \mathrm{I}$ and $g \in \mathrm{K}[s][\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ such that $h = pf + (\mathrm{T}_n - s)g$. Writing $s = \mathrm{T}_n - (\mathrm{T}_n - s)$, there exists a polynomial $q \in k[s][\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ such that $p = p(\mathrm{T}_n) + (\mathrm{T}_n - s)q$. We then write $h = pf + (\mathrm{T}_n - s)g = p(\mathrm{T}_n)f + (\mathrm{T}_n - s)(g + q)$. This allows to assume that $p = 1$.

Observe that $\tau_x((\mathrm{T}_n - s)g) = \tau_x(\mathrm{T}_n - s) + \tau_x(g) = \tau_x(g)$ since $x_n = 0$ and $v(s) = 0$; moreover, $\mathrm{in}_x((\mathrm{T}_n - s)g) = (\mathrm{T}_n - s)\mathrm{in}_x(g)$.

If $\tau_x(f) < \tau_x((\mathrm{T}_n - s)g)$, then $\tau_x(h) = \tau_x(f + (\mathrm{T}_n - s)g) = \tau_x(f)$ and $\mathrm{in}_x(h) = \mathrm{in}_x(f)$.

Similarly, if $\tau_x(f) > \tau_x((\mathrm{T}_n - s)g)$, then $\tau_x(h) = \tau_x((\mathrm{T}_n - s)g) = \tau_x(g)$ and $\mathrm{in}_x(h) = \mathrm{in}_x((\mathrm{T}_n - s)g) = (\mathrm{T}_n - s)\mathrm{in}_x(g)$.

Assume finally that $\tau_x(f) = \tau_x((\mathrm{T}_n - s)g)$. Since $\deg_s(\mathrm{in}_x(f)) = 0$ and $\deg_s(\mathrm{in}_x((\mathrm{T}_n - s)g)) \geqslant 1$, one has $\mathrm{in}_x(f) + \mathrm{in}_x((\mathrm{T}_n - s)g) \neq 0$. Consequently, $\tau_x(h) = \tau_x(f)$ and $\mathrm{in}_x(h) = \mathrm{in}_x(f) + \mathrm{in}_x((\mathrm{T}_n - s)g) = \mathrm{in}_x(f) + (\mathrm{T}_n - s)\mathrm{in}_x(g)$.




In these three cases, this proves that $\mathrm{in}_x(h) \in \mathrm{I}_{k(s)} + (\mathrm{T}_n - s)$. This concludes the proof of the lemma. ☐

*Proposition* **(6.4.6)** (Jensen & Yu (2016)). — *Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$, *let* $\mathrm{X} = \mathrm{V}(\mathrm{I})$. *Let* $\mathrm{H} = \partial(\sup(x_n, 0)) \subset \mathbf{R}^n$ — *the hyperplane defined by* $x_n = 0$ *with multiplicity* 1. *Let* $\mathrm{J} = \mathrm{I}_{\mathrm{K}(s)} + (\mathrm{T}_n - s) \subset \mathrm{K}(s)[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* $\mathrm{Y} = \mathrm{V}(\mathrm{J})$. *One has the equality of* ~~*tropicalizations*~~ variétés tropicales.

$$\mathscr{T}_{\mathrm{Y}} = \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}.$$

$H = \mathscr{T}_{V(T_n - 1)}$ ?

$\mathscr{T}_Y$

$Y_1 = X \cap V(T_n - 1)$ $\neq \emptyset$

$J_0 = I$

*Proof*. — Let us prove that the following five assertions, for $x \in \mathbf{R}^n$, are equivalent.

(i) One has $x \in \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}$;

(ii) One has $\mathscr{T}_{\mathrm{in}_x(\mathrm{I})} \not\subset \mathrm{H}$;

(iii) One has $\mathrm{in}_x(\mathrm{I}) \cap k[\mathrm{T}_n, \mathrm{T}_n^{-1}] = (0)$;

(iv) One has $\mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s) \neq (1)$;

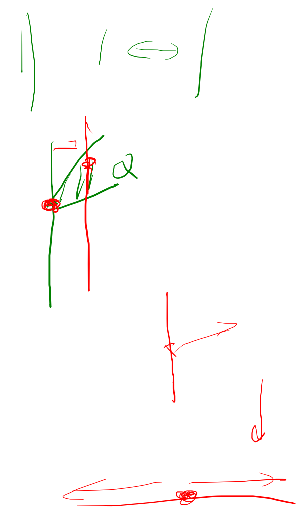
(v) One has $x \in \mathscr{T}_{\mathrm{Y}}$.

(i)$\Leftrightarrow$(ii). One has $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) = \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$.

If $\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})} \subset \mathrm{H}$, then a generic deplacement by a vector $v$ such that $v_n \neq 0$ shows that the stable intersection is empty; in particular, $x \notin \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})} \cap_{\mathrm{st}} \mathrm{H}$, hence $x \notin \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}$.

Otherwise, there exists a polyhedral convex cone $\mathrm{Q} \subset \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})}$ such that that $x \in \mathrm{Q}$ and $\mathrm{Q} \not\subset \mathrm{H}$. The polyhedral convex cone $\mathrm{Q}+\mathrm{H}$ has dimension $n$. If we perform a generic deplacement by a vector $v \in \mathring{\mathrm{Q}}+\mathrm{H}$ such that $v_n > 0$, we obtain a strictly positive contribution of $(\mathrm{Q}, \mathrm{H})$ to the intersection $\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})} \cap_{\mathrm{st}} \mathrm{H}$. In particular, $x \in \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}$.

(ii)$\Leftrightarrow$(iii). Let $p : \mathbf{G}_{\mathrm{m}}{}^n \to \mathbf{G}_{\mathrm{m}}$ be the projection to the last factor; similarly, let $\pi : \mathbf{R}^n \to \mathbf{R}$ be the projection to the last factor. One has $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) = \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$ and $\pi(\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}})) = \mathscr{T}_{\mathrm{V}(\mathrm{I}_n)}$, where $\mathrm{I}_n = \mathrm{in}_x(\mathrm{I}) \cap \mathrm{K}[\mathrm{T}_n^{\pm 1}]$, since $\overline{p(\mathrm{V}(\mathrm{in}_x(\mathrm{I})))} = \mathrm{V}(\mathrm{I}_n)$. The inclusion $\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))} \subset \mathrm{H}$ is equivalent to $\pi(\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}) = \{0\}$, hence to $\mathscr{T}_{\mathrm{V}(\mathrm{I}_n)} = \{0\}$. It implies that $\mathrm{I}_n \neq (0)$ (otherwise, $\mathrm{V}(\mathrm{I}_n) = \mathbf{G}_{\mathrm{m}k}$ and $\mathscr{T}_{\mathrm{V}(\mathrm{I}_n)} = \mathbf{R}$). Conversely, if $\mathrm{I}_n \neq (0)$, then $\mathrm{V}(\mathrm{I}_n)$ is a finite subscheme of $\mathbf{G}_{\mathrm{m}}$, $\pi(\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}))$ is finite; since it is a fan, it is then reduced to 0.



$f(s) \equiv f \mod (T_n - s)$

$\neg$(iii) $\Rightarrow \neg$(iv)

(iii)$\Leftrightarrow$(iv). — Let $f \in k[T_n^{\pm 1}]$ be a non-zero Laurent polynomial. Since $s$ is transcendental, one has $f(s) \neq 0$ and the ideal $(f, T_n - s)$ of $k(s)[T_n^{\pm 1}]$ contains 1. If, moreover, $f \in \mathrm{in}_x(I)$, this implies that $\mathrm{in}_x(I)_{k(s)} + (T_n - s) = (1)$. Assume conversely that $\mathrm{in}_x(I)_{k(s)} + (T_n) = (1)$ and let us consider a relation of the form $1 = \sum g_j \mathrm{in}_x(f_j) + (T_n - s)h$, where $f_j \in I$, $g_j \in k(s)$ and $h \in k(s)[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. Let $p \in k[s]$ be a non-zero polynomial such that $pg_j \in k[s]$ for all $j$, and $ph \in k[s][T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. In the relation $p = \sum pg_j \mathrm{in}_x(f_j) + (T_n - s)ph$ we substitute $T_n$ to $s$. We obtain $p(T_n) = \underbrace{\sum_j (pg_j)(T_n)\mathrm{in}_x(f_j)}_{\in \mathrm{in}_x(I)}$, which proves that $p(T_n) \in \mathrm{in}_x(I) \cap k[T_n^{\pm 1}]$.

$\neg$(iv) $\Rightarrow \neg$(iii)

$k(s)(T_n^{\pm})$

The equivalence (iv)$\Leftrightarrow$(v) follows from the preceding lemma. Indeed, $x \in \mathscr{T}_Y$ if and only if $\mathrm{in}_x(J) \neq (1)$, which is then equivalent to $\mathrm{in}_x(I)_{k(s)} + (T_n - s) \neq (1)$.

$\mathrm{in}_x(I_{k(s)}) = \mathrm{in}_x(I)_{k(s)}$

*It remains to explain compare the multiplicities.* □

## 6.5. A tropical version of Bernstein's theorem

$f_1, \dots, f_n \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ $P_i = NP(f_i)$ polytope de Newton

$X_i = V(f_i)$ $Z = X_1 \cap \dots \cap X_n \subset \mathbb{G}_m^n$

Th (Bernstein) : si les coeff. de $f_i$ sont "génériques" (à pol. de Newton fixé)

alors $Z$ est fini et $\mathrm{Card}(Z) = V(P_1, \dots, P_n) \cdot n!$

← volume mixte, coeff. de $a_1 \cdots a_n$

dans $\mathrm{Vol}(\sum a_i P_i)$ = pol. en $a_1, \dots, a_n$ ~~de degré~~ homogène de degré $n$

Version tropicale (Jensen-Yu, Maclagan-Sturmfels, Katz Oosterman-Payne, ...)

si les coeff. sont génériques — sans changer les polytopes de Newton, ni les valeurs absolues des coeff.,

$$\mathcal{C}_Z = \underbrace{\mathcal{C}_{X_1} \cap_{st} \cdots \cap_{st} \mathcal{C}_{X_n}}_{\text{ens fini de points avec multiplicités}} .$$

degré total $= V(P_1, \dots, P_n)$.

On retrouve en plus du th. de Bernstein une information sur la valeur absolue des points de $Z$.