Master 2^e année *Année 2019/2020*

GÉOMÉTRIE TROPICALE, 1

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Exam of Thursday, February 20nd, 2020 (3 hours)

You may write your answers to the three problems in English or in French.

EXERCISE 1

1 Let P be a polyhedron in a finite dimensional $\mathbf{R}$-vector space V. Let $x \in P$.

a) Justify the existence of a unique minimal face F_x of P such that $x \in F_x$.

Let P_x be the set of vectors $v \in V$ for which there exists $\delta > 0$ such that $x + tv \in P$, for any $t \in [0; \delta[$.

b) Prove that P_x is a convex polyhedral cone.

c) Let $V_x = P_x \cap (-P_x)$. Prove that V_x is a vector subspace of V, that $F_x \subset x + V_x$ and that $\dim(F_x) = \dim(V_x)$.

One assumes in the sequel that P is the polytope of bistochastic matrices in $\mathbf{R}^{n^2}$: $n \times n$ real matrices $A = (a_{i,j})$ with positive coefficients and such that the sums $\sum_{j=1}^{n} a_{i,j}$ of each row and the sums $\sum_{i=1}^{n} a_{i,j}$ of each column are all equal to 1.

2 Prove that the vertices of P are the permutation matrices (given, for $\sigma \in \mathfrak{S}_n$, by $a_{i,j} = 1$ if $j = \sigma(i)$, and $a_{i,j} = 0$ otherwise).

3 Let $\sigma \in \mathfrak{S}_n$ be a permutation distinct from id and let x be the midpoint of the two associated permutation matrices A_{id} and A_σ. Compute the space V_x. Prove that the segment $[A_{\mathrm{id}}, A_\sigma]$ is an edge (a 1-dimensional face) of P if and only if σ has only one nontrivial cycle.

EXERCISE 2

Let $f = \sum c_m T^m \in \mathbf{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be a Laurent polynomial. We denote by $\mathscr{V}(f) \subset (\mathbf{C}^*)^n$ the complex hypersurface it defines and by $\mathscr{A}_f$ its amooeba. Let us also denote by NP_f the Newton polytope of f and $V(\mathrm{NP}_f)$ the set of its vertices. One says that the amoeba $\mathscr{A}_f$ is *solid* if the order of each connected component of its complementary subset is a vertex of the Newton polytope NP_f.

1 *a*) Let m be a vertex of the Newton polytope of f and let E be the unique conneceted component of $\mathbf{R}^n - \mathscr{A}_f$ with order m. Prove that the Ronkin function of f is given by $R_f(x) = \log(|c_m|) + \langle m, x\rangle$ on E.

b) Suppose that the amoeba $\mathscr{A}_f$ is solid. Prove that the Passare-Rullgård function coincides with the tropical polynomial, that is,

$$S_f(x) = \sup_{m \in V(\mathrm{NP}_f)} (\log(|c_m|) + \langle m, x\rangle)$$

for every $x \in \mathbf{R}^n$.

Let $(a_{j,k})$ be a real symmetric $n \times n$ matrix whose coefficients satisfy the inequality $0 < |a_{j,k}| < 1$ for every pair (j,k) such that $j \neq k$. Then pose

$$f(T_1, \ldots, T_n) = \sum_{J \subset \{1,\ldots,n\}} \prod_{j \in J} (z_j \prod_{k \notin J} a_{j,k}).$$

2 *a*) Determine the Newton polytope of f. Deduce that the amoeba $\mathscr{A}_f$ is solid.

b) Prove that $f(T_1, \ldots, T_n) = T_1 \ldots T_n f(T_1^{-1}, \ldots, T_n^{-1})$.

c) Prove that the amoeaba of f is symmetric: $\mathscr{A}_f = -\mathscr{A}_f$.

3 *a*) Prove that the origin 0 belongs to the spine of $\mathscr{A}_f$. More precisely, prove that in a neighborhood of 0, the spine of $\mathscr{A}_f$ coincides with the hyperplane with equation $x_1 + \cdots + x_n = 0$.

4 We will prove by induction on n that $\mathbf{R}_+^n \cap \mathscr{A}_f = \{0\}$.

a) Let t be a real number such that $t > 0$. Prove that $(0, \ldots, 0, t) \notin \mathscr{A}_f$.

b) Prove that $\mathbf{R}_+^n \cap \mathscr{A}_f = \{0\} = (-\mathbf{R}_+^n) \cap \mathscr{A}_f = \{0\}$.

c) Deduce that the complex zeroes of the polynomial $f(T, T, \ldots, T)$ in the indeterminate T all have absolute value 1 (*theorem of Lee–Yang*).

EXERCISE 3

Let $f = 3T_1^2 + 5T_1T_2 - 6T_2^2 + 8T_1 - T_2 + 9 \in \mathbf{Q}[T_1, T_2]$. Fix a prime number p and consider the field of rational numbers endowed with the p-adic valuation v_p.

1 Draw the Newton polytope of f.

2 Suppose $p \geqslant 7$.

a) Determine the initial forms $\mathrm{in}_x(f)$ according to the value of $x \in \mathbf{R}^2$.

b) Determine the (non archimedean) amoeab of f. Make a figure.

3 Redo the preceding question with $p = 3$.