GÉOMÉTRIE TROPICALE I

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Vous pouvez discuter entre vous des exercices mais la rédaction des copies doit être faite individuellement. Celles et ceux qui souhaitent que leur rédaction soit évaluée m'enverront avant le 15 février 2021 un fichier PDF unique dont le nom aura la forme `<nom>_<prenom>-archam.pdf` *comportant la solution de tout ou partie de ces exercices.*

EXERCISE 1

1 Let R be a $\mathbf{C}$-algebra which is a field. Let $a \in R \smallsetminus \mathbf{C}$. Prove that the family $((a-z)^{-1})_{z\in\mathbf{C}}$ is linearly independent.

2 Let M be a maximal ideal of the ring $\mathbf{C}[T_1,\dots,T_n]$. Prove that there exists a unique point $a \in \mathbf{C}^n$ such that $M = (T_1 - a_1, \dots, T_n - a_n)$.

If I is an ideal of $\mathbf{C}[T_1,\dots,T_n]$, let $\mathscr{Z}(I)$ be the set of $a \in \mathbf{C}^n$ such that $f(a) = 0$ for all $f \in I$. If Z is a subset of $\mathbf{C}^n$, let $\mathscr{I}(Z)$ be the set of all $f \in \mathbf{C}[T_1,\dots,T_n]$ such that $f(a) = 0$ for all $a \in Z$.

3 Prove that $I \subset \mathscr{I}(\mathscr{Z}(I))$ and $Z \subset \mathscr{Z}(\mathscr{I}(Z))$.

4 Prove that $Z \mapsto \mathscr{I}(Z)$ and $I \mapsto \mathscr{Z}(I)$ induce bijective correspondences, inverse one of the other, between subsets of $\mathbf{C}^n$ of the form $\mathscr{Z}(I)$ and ideals of $\mathbf{C}[T_1,\dots,T_n]$ of the form $\mathscr{I}(Z)$.

5 Let I be an ideal of $\mathbf{C}[T_1,\dots,T_n]$ such that $\mathscr{Z}(I) = \emptyset$. Prove that $I = (1)$.

6 Let I be an ideal of $\mathbf{C}[T_1,\dots,T_n]$ and let $J = \mathscr{I}(\mathscr{Z}(I))$. Let $f \in J$. Considering the ideal $I + (1 - fT)$ of $\mathbf{C}[T_1,\dots,T_n,T]$, prove that there exists $d \in \mathbf{N}$ such that $f^d \in I$. Conclude that $J = \sqrt{I}$.

EXERCISE 2

1 Let $f \in \mathbf{C}[T_1,\dots,T_n]$ be a nonzero polynomial. Prove that the set of $\mathbf{C}^n \smallsetminus \mathscr{Z}(f)$ is dense in $\mathbf{C}^n$.

We have used the following theorem in the proof of the Bieri–Groves theorem : *Let X be a complex irreducible algebraic variety, let Z be a strict Zariski closed subset of X and let $U = X \smallsetminus Z$. Then U is topologically dense in X.*

2 Explain why the first question is a particular instance of this theorem.

3 Look up the proof of this theorem in Mumford's *Red book of varieties of schemes* (theorem 1, p. 58) and, at your will, but using your own words, do *one* of the following :

— Describe a few concepts that you have been led to learn in order to understand this proof;

— Describe one of the concepts that appears in the proof, as well as another appearance of this concept in algebra or geometry ;
— Describe the proof itself.

EXERCISE 3

Describe (qualitatively) the possible amoeabas for polynomials $f \in \mathbf{C}[\mathrm{T}_1, \mathrm{T}_2]$ of degree $\leqslant$ 2 or 3.

EXERCISE 4

1 Let $\varphi \colon \mathbf{Z}^n \to \mathbf{Z}^n$ be an affine map and let $\varphi^* \colon \mathbf{R}^n \to \mathbf{R}^n$ be the transpose of the linear part of φ. For a Laurent polynomial $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$, written as $f = \sum_m c_m \mathrm{T}^m$, one sets $f^\varphi = \sum_m c_m \mathrm{T}^{\varphi(m)}$.
Prove that φ^* maps $\mathscr{A}_{f^\varphi}$ into $\mathscr{A}_f$, and a component E of $\complement \mathscr{A}_{f^\varphi}$ to a component $\varphi^*(\mathrm{E})$ of $\mathscr{A}_f$ whose order satisfies $\nu_f^{\varphi^*(\mathrm{E})} = \varphi(\nu_{f^\varphi}^{\mathrm{E}})$.

2 Prove that the set Γ_f of invertible affine transformations from $\mathbf{Z}^n$ to itself that preserves the Newton polytope of f is a subgroup. Assuming that NP_f has dimension n, identify this group with a subgroup of the (finite) permutation group of the vertices of NP_f.

Let $a \in \mathbf{C}$ and let $f = 1 + \mathrm{T}_1^{n+1} + \dots + \mathrm{T}_n^{n+1} + a\mathrm{T}_1 \dots \mathrm{T}_n$.

3 Describe the Newton polytope of f as well as the possible components of $\complement \mathscr{A}_f$. Which of these possible components are certain to exist ? Draw pictures for $n = 2$ or 3.

4 Identify the group Γ_f as the permutation group $\mathfrak{S}_{n+1}$. Prove that it preserves the amoeba $\mathscr{A}_f$ and permutes the connected components of the complementary subset.

5 Assume that $0 \notin \mathscr{A}_f$ and let E be the connected component of $\complement \mathscr{A}_f$ that contains 0. Prove that E has order $(1, \dots, 1)$.

6 Let E be a a connected component of $\complement \mathscr{A}_f$; assume that E has order $(1, \dots, 1)$. Let $x \in \mathrm{E}$. Prove that $\varphi^*(x) \in \mathrm{E}$ for every affine transformation φ of $\mathbf{Z}^n$ that preserves the Newton polytope of f. Conclude that $0 \in \mathrm{E}$.

7 Describe the set of all $a \in \mathbf{C}$ such that $\complement \mathscr{A}_f$ has a component of order $(1, \dots, 1)$.