Master Mathématiques fondamentales, 2^e^ année *Année 2020/2021*

GÉOMÉTRIE TROPICALE I

Antoine Chambert-Loir

Polyèdres et polytopes (11 janvier 2021)

Vous pouvez discuter entre vous des exercices mais la rédaction des copies doit être faite individuellement. Celles et ceux qui souhaitent que leur rédaction soit évaluée m'enverront avant le 25 janvier 2021 un fichier PDF unique dont le nom aura la forme `<nom>_<prenom>-convex.pdf` *comportant la solution de tout ou partie de ces exercices.*

EXERCISE 1

1 Let P be the polyhedron in $\mathbf{R}^2$ defined by the inequalities $x, y \geqslant -1$, $y \leqslant x+1$, $x \leqslant 2y+3$. Compute its face lattice and its recession cone. Draw a picture.

2 Same question for the polyhedron Q obtained by adding to this system of inequalities the inequality $x + y \leqslant 2$.

Let V *be a finite dimensional real vector space. If* P *is a polyhedron in* V*, one lets* P° *be the subset of* V^* *consisting of all linear forms* $f \in V^*$ *such that* $f(x) \leqslant 1$ *for all* $x \in P$.

3 Compute P° and Q° in the case of the given examples.
In the remainder of this exercise, P *is a general polyhedron in* V*. Illustrate your solutions to the following questions on the given examples of questions 1 and 2.*

4 Prove that P° is a polyhedron containing 0.

5 If P is a convex cone, then P° is the set of all $f \in V^*$ such that $f(x) \leqslant 0$ for all $x \in P$; it is a convex cone.
From now on, one assumes that $0 \in P$.

6 Identifying V with V^{**}, prove that $P = P^{\circ\circ}$.

7 Prove that $\dim(P) = \operatorname{codim}_V(\operatorname{linsp}(P^\circ))$.

8 Prove that P° is a polytope of and only of 0 is an interior point of P.
From now on, one assumes that P *is a polytope of which* 0 *is an interior point.*

9 Let F be a face of P. Let $F^\diamond$ be the subset of all $f \in P^\circ$ such that $f|_F \equiv 1$. Prove that $F^\diamond$ is a face of P°; compute its dimension in terms of that of F.

10 Compare the face lattices of P and P°.

EXERCISE 2

Let n be an integer $\geqslant 1$ and let C be the curve in $\mathbf{R}^n$ parameterized by $c\colon t \mapsto (t, t^2, \dots, t^n)$.

1 Let $(t_0, \dots, t_n)$ be a strictly increasing sequence of real numbers. Prove that the points $c(t_0), \dots, c(t_n)$ of $\mathbf{R}^n$ are affinely independent (*ie*, not all contained in any affine hyperplane).

Let v be an integer $\geqslant n$ and fix a strictly increasing sequence $\tau = (t_0, \dots, t_v)$ of real numbers. Let P *be the convex hull of the points $c(t_0), \dots, c(t_v)$. More generally, for every subset* I *of $\{0, \dots, v\}$, let P_I be the convex hull of the points $c(t_i)$, for $i \in \mathrm{I}$.*

2 Prove that P is a polytope all of which faces are simplices.

3 Let p be an integer such that $1 \leqslant 2p \leqslant n$ and let I be a subset of $\{0, \dots, v\}$ with cardinality p. Prove that P_I is a face of P of dimension $p - 1$.

4 Let I be a subset of $\{0, \dots, v\}$ with cardinality n. Considering the polynom $f = \prod_{i \in \mathrm{I}}(\mathrm{T} - t_i)$, prove that P_I is a facet of P if and only if, for all $i, j \in \{0, \dots, v\} - \mathrm{I}$ such that $i < j$, the cardinality of $\mathrm{I} \cap [i; j]$ is even. (In words, any two elements i, j not in I are separated by an even number of elements of I.)

5 Prove that the face lattice of P does not depend on the choice of the sequence τ.

EXERCISE 3

Let P and Q be polytopes in a finite dimensional real vector space V. Let ξ be a vertex of Q and assume that $\mathrm{Q} = \mathrm{conv}(\mathrm{P} \cup \{\xi\})$.

One says that a facet G of a polytope P separates ξ from P if there exists an affine hyperplane H such that $\mathrm{H} \cap \mathrm{P} = \mathrm{G}$ and such that ξ does not belong to the half-space of V delimited by H which contains P.

1 Let F be a face of P. Prove that F is a face of Q if and only if there exists a facet of F containing F which *does not* separate ξ from P.

2 Let F be a face of P and let $\mathrm{F}^* = \mathrm{conv}(\mathrm{F} \cup \{\xi\})$. Prove that F^* is a face of Q if and only if, either $\xi \in \mathrm{affsp}(\mathrm{F})$, or among all facets of P *containing* F, there exists one which separates ξ from P and one which doesn't.

3 Prove that each face of Q is of one of the preceding forms, exclusively.

EXERCISE 4

Using any source of information of your choice, describe *in your own words* one open problem regarding polytopes, which you find interesting, and explain why you find it interesting. Your description shall rely on references to the *academic* mathematical litterature. (Despite its importance and usefulness, Wikipedia does not count as an academic reference.)