Master Mathématiques fondamentales, 2e année *Année 2020/2021*

GÉOMÉTRIE TROPICALE I

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Amibes non archimédiennes (18 février 2021)

Vous pouvez discuter entre vous des exercices mais la rédaction des copies doit être faite individuellement. Celles et ceux qui souhaitent que leur rédaction soit évaluée m'enverront avant le 8 mars 2021 un fichier PDF unique dont le nom aura la forme `<nom>_<prenom>-nonarch.pdf` *comportant la solution de tout ou partie de ces exercices.*

EXERCISE 1

Let K be a field which is complete for some nontrivial absolute value. Let $f \in \mathrm{K}[\mathrm{T}]$ be a nonzero polynomial. The Newton method for solving f starts from $a_0 \in \mathrm{K}$ and defines the sequence (a_n) by the recurrence relation

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)},$$

at least while $f'(a_n) \neq 0$.

1 Let $a \in \mathrm{K}$ be a root of f such that $f'(a) \neq 0$. Prove that there exists $r > 0$ such that for every starting point $a_0 \in \mathrm{K}$ such that $|a_0 - a| < r$, the Newton method converges to a.

In the sequel, we assume that K is nonarchimedean and let R be its valuation ring; we also assume that $f \in \mathrm{R}[\mathrm{T}]$.

2 For $a \in \mathrm{R}$, prove that there is a unique polynomial $h \in \mathrm{R}[\mathrm{T}]$ such that $f(\mathrm{T}) = f(a) + (\mathrm{T} - a)f'(a) + (\mathrm{T} - a)^2 h(\mathrm{T})$.

3 Let $a_0 \in \mathrm{R}$ be such that $|f(a_0)| < |f'(a_0)|^2$. Let $c = |f(a_0)|/|f'(a_0)|^2$.

One considers the outcome (a_n) of the Newton method starting from a_0. Establish the inequalities $|a_1 - a_0| \leqslant c|f'(a_0)|$, $|f'(a_1)| = |f'(a_0)|$ and $|f(a_1)| \leqslant c^2 |f'(a_1)|^2$.

4 Prove that there exists a unique root $a \in \mathrm{K}$ of f $|a - a_0| \leqslant c|f'(a_0)|$ and that the Newton method starting from a_0 converges to it.

EXERCISE 2

Let K be a nonarchimedean valued field and let $\mathrm{R} = \mathrm{K}[\mathrm{T}]$ be the ring of polynomials in one indeterminate with coefficients in K.

1 Let $a \in \mathrm{K}$. Prove that $f \mapsto |f(a)|$ is a multiplicative seminorm p_a on R that defines an element of $\mathscr{M}(\mathrm{R})$. Prove that the map $j \colon a \mapsto p_a$ is a homeomorphism onto its image. Is it surjective ?

2 Assume that K is algebraically closed and that its valuation is nontrivial. Prove that the image of j is dense in $\mathscr{M}(\mathrm{R})$.

3 Assume that $K = \mathbf{Q}_p$ is the field of p-adic numbers. Prove that the image of j is closed in $\mathscr{M}(R)$.

EXERCISE 3

Let $f = 3T_1^2 + 5T_1T_2 - 6T_2^2 + 8T_1 - T_2 + 9 \in \mathbf{Q}[T_1, T_2]$. Fix a prime number p and consider the field of rational numbers endowed with the p-adic valuation v_p.

1 Draw the Newton polytope of f.

2 Suppose $p \geqslant 7$.

a) Determine the initial forms $\mathrm{in}_x(f)$ according to the value of $x \in \mathbf{R}^2$.

b) Determine the (non archimedean) amoeba of f. Make a figure.

3 Redo the preceding question with $p = 3$.

EXERCISE 4

1 Let (a_n) be a sequence of real numbers such that $a_{m+n} \leqslant a_m + a_n$. Prove that the sequence (a_n/n) converge to $\inf_{n\geqslant 1}(a_n/n)$.

Let R be a ring.

2 Let p be a nonarchimedean seminorm on R.

a) Prove that the formula $p^*(x) = \lim_n p(x^n)^{1/n}$ defines a nonarchimedean seminorm on R.

b) Prove that p^* is the largest radical nonarchimedean seminorm on R such that $p^* \leqslant p$.

c) Let S be a multiplicative submonoid of R, let $a \in S$ and let c be a real number such that $p(ax) \geqslant cp(x)$ for all $x \in S$; prove that $p^*(ax) \geqslant cp^*(x)$ for all $x \in S$.

3 Let p be a nonarchimedean radical seminorm on R. Let $a \in R$ be such that $p(a) \neq 0$.

a) Prove that the formula $p_a(x) = \lim_n p(xa^n)/p(a)^n$ defines a radical nonarchimedean seminorm on R such that $p_a \leqslant p$ and $p_a(ax) = p_a(a)p_a(x)$ for all $x \in R$.

b) Let S be a multiplicative submonoid of R, let $a \in S$ and let c be a real number such that $p(ax) \geqslant cp(x)$ for all $x \in S$; prove that $p_a(ax) \geqslant cp_a(x)$ for all $x \in S$.

4 Let p be a nonarchimedean seminorm on R.
There exists a nonarchimedean multiplicative seminorm p^* on R such that

$$p(x) \geqslant p^*(x) \geqslant \inf_{a\in R} \frac{p(ax)}{p(a)}$$

for all $x \in R$.
(*Consider the family of all seminorms on* R *of the form* $p^*_{a_1,a_2,\dots,a_n}$ *and use the fact that the product space* $\prod_{a\in R}[0; p(a)]$ *is compact.*)

5 Let p be a nonarchimedean seminorm on R, let S be a multiplicative submonoid of R. Assume that $p(1) = 1$ and that for every $a, b \in S$, one has $p(ab) = p(a)p(b)$ ("p is multiplicative on S"). Prove that there exists a nonarchimedean multiplicative seminorm p^* on R such that $p^* \leqslant p$ and $p^*(a) = p(a)$ for every $a \in S$.

6 Let p be a nonarchimedean multiplicative seminorm on R, let S be a multiplicative submonoid of R and let I be an ideal of R. One assumes that there does not exist $s \in S$ and $a \in I$ such that $p(a - s) < p(a) = p(s)$. Prove that there exists a nonarchimedean multiplicative seminorm p^* on R such that $p^* \leqslant p$, $p(a) = 0$ for every $a \in I$ and $p^*(a) = p(a)$ for every $a \in S$.