TOPICS
IN TROPICAL GEOMETRY

CONTENTS

- $(\mathbb{C}^*)^n \xrightarrow{\lambda} \mathbb{R}^n$

  $(z_1 \ldots z_n) \longmapsto (\log|z_1|, \ldots\ )$

- $V \subset (\mathbb{C}^*)^n$

  $\lambda(V)$ amibe de V

zéros de f

$\lambda(V(f)) = \mathcal{A}_f \qquad f \in \mathbb{C}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$

les c.c. de $\mathbb{R}^n - \mathcal{A}_f$ sont convexes paramétrées par un sous ensemble de $\underbrace{NP_f \cap \mathbb{Z}^n}_{\nu \in \mathbb{C}}$

$\text{ord}(E) = N_{\nu_E}(NP_f)$

$R : \mathbb{R}^n \to \mathbb{R}$ fonction de Ronkin

Conversely, let $x \in \mathrm{E}$ and let $\xi \in \mathbf{R}^n$ be such that $x + \mathbf{R}_+\xi \subset \mathrm{E}$. Then $x + \mathbf{R}_+\xi \subset \mathrm{E}'$, with the notation of the proof, and we have seen how this implies that $\xi \in \mathrm{N}_{\nu_f^{\mathrm{E}}}(\mathrm{NP}_f)$. This concludes the proof. □

2.6. The logarithmic limit set of a variety

Definition **(2.6.1)**. — *Let* V *be an algebraic subvariety of* $(\mathbf{C}^*)^n$. *The* logarithmic limit set *of* V *is the set of points* $x \in \mathbf{R}^n$ *such that there exists sequences* $(x_k) \in \lambda(\mathrm{V})$ *and* $(h_k) \in \mathbf{R}_+^*$ *such that* $h_k \to 0$ *and* $h_k x_k \to x$. *We denote it by* $\lambda_\infty(\mathrm{V})$.

This set has been introduced by Bergman (1971) who gave a description of the set when V is a hypersurface. His work has been completed by Bieri & Groves (1984).

It is also called the *asymptotic cone* of $\lambda(\mathrm{C})$, and can be defined as the limit of the closed subsets $h\lambda(\mathrm{V})$, when $h \to 0$ (restricted to $h > 0$) for the topology defined by the Hausdorff distance on compact sets.

In this section, we describe $\lambda_\infty(\mathrm{V})$ when $\mathrm{V} = \mathscr{V}(f)$ is defined by a nonzero Laurent polynomial in $\mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$.

Lemma **(2.6.2)**. — *Let* V *be a nonempty closed algebraic subvariety of* $(\mathbf{C}^*)^n$. *Then its logarithmic limit set* $\lambda_\infty(\mathrm{V})$ *is a closed conic subset of* $\mathbf{R}^n$.

Proof. — Since V is nonempty, one has $\lambda(\mathrm{V}) \neq \varnothing$; one then may choose x_k to be equal to a given element of $\lambda(\mathrm{V})$ and $h_k = 1/k$; this shows that $0 \in \lambda_\infty(\mathrm{V})$.

Let $x \in \lambda_\infty(\mathrm{V})$; write $x = \lim h_k x_k$, with $x_k \in \lambda(\mathrm{V})$ and $(h_k) \to 0$. For every $t > 0$, one has $tx = \lim (th_k)x_k$, and $th_k \to 0$, so that $tx \in \lambda_\infty(\mathrm{V})$. This proves that $\lambda_\infty(\mathrm{V})$ is a cone. Let us prove that it is closed.

Let $(x^{(m)})$ be a sequence of points of $\lambda_\infty(\mathrm{V})$ that converges to a point $x \in \mathbf{R}^n$ and let us prove that $x \in \lambda_\infty(\mathrm{V})$. For every m, choose a point $x_m \in \lambda(\mathrm{V})$ and a real number h_m such that $0 < h_m < 1/m$ and $\left\| x^{(m)} - h_m x_m \right\| < 1/m$. Then $\| x - h_m x_m \| < \left\| x - x^{(m)} \right\| + 1/m$, so that $x = \lim h_m x_m$, hence $x \in \lambda_\infty(\mathrm{V})$. This proves that $\lambda_\infty(\mathrm{V})$ is closed. □

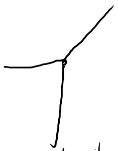
Definition **(2.6.3)**. — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a nonzero Laurent polynomial and let* $\mathrm{S} \subset \mathbf{Z}^n$ *be its support. The* tropical variety *defined by* f *is the*

espace métrique (X, d)

→ cône asymptotique

« $\lim_{\varepsilon \to 0} (X, \varepsilon d)$ »

Exemple $f = T_1 + T_2 - 1$



3 demi droites

ensemble limite logarithmique

vision à grande distance, en papier log log, de $V \subset (\mathbb{C}^*)^n$

$\lambda(V)$

$\lambda_\infty(V) = \{ \lim_{k \to \infty} h_k x_k,\ (x_k) \text{ suite dans } \lambda(V),\ 0 < h_k \to 0 \}$

$V \subset (\mathbb{C}^\times)^n$

• idéal de $V = \mathcal{I}(V) = \{ f \in \mathbb{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \,/\, f|_V \equiv 0 \}$

• $I \subset \mathbb{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$\mathcal{V}(I)$ variété de I $= \{ z \mid f(z) = 0 \ \forall z \in I \}$

Th. des zéros de Hilbert (Nullstellensatz)

• $\mathcal{V}(\mathcal{I}(V)) = V$

• $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I} = \{ f \mid \exists m \geq 1,\ f^m \in I \}$

set of all points $x \in \mathbf{R}^n$ such that $\sup_{m \in S} \langle x, m \rangle$ is attained for at least two values of $m \in V$. We denote it by $\mathcal{T}_f$.

It follows from the definition of $\mathcal{T}_f$ that it is a closed **Q**-rational cone (non convex, in general).

In general, if V is a closed subvariety of $(\mathbf{C}^*)^n$, one defines its *tropical variety* $\mathcal{T}_V$ as the intersection of all $\mathcal{T}_f$, for $f \in \mathcal{I}(V) - \{0\}$, where $\mathcal{I}(V)$ is the *ideal* of V, namely the ideal of all Laurent polynomials $f \in \mathbf{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ such that $f|_V \equiv 0$.

If $V \subset W$, one has $\mathcal{I}(W) \subset \mathcal{I}(V)$, hence $\mathcal{T}_V \subset \mathcal{T}_W$.

The tropical variety $\mathcal{T}_V$ is a closed conic subset of $\mathbf{R}^n$, as an intersection of a family of such subsets.

Lemma **(2.6.4)**. — *Assume that $V = \mathcal{V}(f)$ is a hypersurface defined by a nonzero Laurent polynomial $f \in \mathbf{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. Then $\mathcal{T}_V = \mathcal{T}_f$. In particular, $\mathcal{T}_V$ is a **Q**-rational polyhedral set.*

Proof. — It suffices to prove that $\mathcal{T}_f \subset \mathcal{T}_{fg}$ for every nonzero Laurent polynomial g. One has $\mathrm{NP}_{fg} = \mathrm{NP}_f + \mathrm{NP}_g$; indeed, if $m \in \mathbf{Z}^n$ is a vertex of NP_{fg}, it must be a vertex of both NP_f and NP_g. In other words, if a linear form defines a nonpunctual face of NP_f, then it defines a nonpunctual face of NP_{fg}; this means exactly that $\mathcal{T}_f \subset \mathcal{T}_{fg}$. □

Using Gröbner bases and the notion of nonarchimedean amoebas, we shall prove in the next chapter (remark 3.6.7) a conjecture put forward by Bergman (1971) and proved by Bieri & Groves (1984) that there is a finite family (f_i) of Laurent polynomials such that $\mathcal{T}_V = \bigcap_i \mathcal{T}_{f_i}$. In particular, $\mathcal{T}_V$ is a **Q**-rational polyhedral set. The motivation for the work of Bieri & Groves (1984) came from the following consequence regarding the logarithmic limit set of an algebraic variety.

$f_i \in \mathcal{I}(V)$

Theorem **(2.6.5)** (Bieri & Groves, 1984). — *For every closed subvariety V of $(\mathbf{C}^*)^n$, the tropical variety of V coincides with its logarithmic limit set: $\mathcal{T}_V = \lambda_\infty(V)$.*

combinatoire

topologie même

For the moment, we need to content ourselves with the weakest result.

$f = \sum_{m \in \mathbb{Z}^n} c_m T^m = \sum_{m \in S} c_m T^m$

$S = \mathrm{supp}(f) = \{ m \mid c_m \neq 0 \}$

polynôme tropical associé à f

$\times \rightsquigarrow +$

$+ \rightsquigarrow \sup$ « algèbre $(\max, +)$ »

$\prod T_i^{m_i} = T^m \rightsquigarrow \langle x, m \rangle = \sum m_i x_i$

$$\boxed{\tau_f(x) = \sup_{m \in S} \langle x, m \rangle}$$

fonction affine par morceaux sur $\mathbb{R}^n$, convexe.

[cf déf. de l'épine]

$\mathcal{T}_f = \{ x \mid \tau_f$ n'est pas affine au vois. de $x \}$

$= \{ x \mid \exists m' \neq m'' \in S,\ \tau_f(x) = \langle x, m' \rangle = \langle x, m'' \rangle \}$

hypersurface tropicale

($\mathbb{Q}$-rationnelle)

Lemma **(2.6.4)**. — *Assume that* $\mathrm{V} = \mathscr{V}(f)$ *is a hypersurface defined by a nonzero Laurent polynomial* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. *Then* $\mathscr{T}_{\mathrm{V}} = \mathscr{T}_f$. *In particular,* $\mathscr{T}_{\mathrm{V}}$ *is a* **Q***-rational polyhedral set.*

réunion finie de polyèdres.

$$\mathscr{T}_V = \bigcap_{h \in \mathcal{I}(V)} \mathscr{T}_h = \bigcap_{g \in \mathbb{C}[T_n^{\pm 1}]} \mathscr{T}_{fg}$$

$$\mathcal{I}(V) = \sqrt{(f)} = \{ h \mid \exists m \geq 1 \ \exists g \quad h^m = fg \}.$$

Proof. — It suffices to prove that $\mathscr{T}_f \subset \mathscr{T}_{fg}$ for every nonzero Laurent polynomial g. One has $\boxed{\mathrm{NP}_{fg} = \mathrm{NP}_f + \mathrm{NP}_g}$; indeed, ~~if $m \in \mathbf{Z}^n$ is a vertex of NP_{fg}, it must be a vertex of both NP_f and~~ NP_g. In other words, if a linear form defines a nonpunctual face of NP_f, then it defines a nonpunctual face of NP_{fg}; this means exactly that $\mathscr{T}_f \subset \mathscr{T}_{fg}$. □

$$\mathrm{NP}_{fg} = \mathrm{NP}_f + \mathrm{NP}_g.$$

$$f = \sum a_p T^p \qquad g = \sum b_q T^q \qquad fg = \sum c_m T^m$$

$$c_m = \sum_{p+q=m} a_p b_q$$

$$\mathscr{T}_f \subset \mathscr{T}_{fg} = \mathscr{T}_{h^m}$$

$$\mathscr{T}_{h^m} = \mathscr{T}_h$$

si m est un sommet de NP_{fg}

$h \in (\mathbb{R}^n)^*$ extrémal en m

$\langle h, m \rangle = \langle h, p \rangle + \langle h, q \rangle$

h doit être extrémale en p et q sur NP_f sur NP_g.

$\leadsto$

$$\mathrm{Supp}(fg) \subset \mathrm{Supp}(f) + \mathrm{Supp}(g) \subset \mathrm{NP}_f + \mathrm{NP}_g$$

$$\mathrm{NP}_{fg} \subset \mathrm{NP}_f + \mathrm{NP}_g.$$

Au moins dans le cas $V = \mathcal{V}(f)$

$\mathscr{T}_V = \mathscr{T}_f$ est une hypersurface tropicale

— une intersection finie d' —

Theorem **(2.6.6)** (Bergman, 1971). — *Let* V *be a closed subvariety such that* $\mathscr{T}_{\mathrm{V}}$ *is a* **Q**-*rational polyhedral set. Then* $\mathscr{T}_{\mathrm{V}} = \lambda_\infty(\mathrm{V})$. *In particular, for every non zero Laurent polynomial* $f \in \mathbf{C}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$, *one has* $\mathscr{T}_f = \lambda_\infty(\mathscr{V}(f))$.

We split the proof of this equality as two inclusions. The proof of the first one is relatively elementary, the second will require a bit of algebraic geometry.

Proposition **(2.6.7)**. — *One has* $\lambda_\infty(\mathrm{V}) \subset \mathscr{T}_{\mathrm{V}}$.

Proof. — It suffices to prove that $\boxed{\lambda_\infty(\mathscr{V}(f)) \subset \mathscr{T}_f}$ for every non zero Laurent polynomial f. Fix $x \in \mathbf{R}^n$. Let S be the support of f and write $f = \sum_{m\in\mathrm{S}} c_m \mathrm{T}^m$; let S_x be the set of $m \in \mathrm{S}$ such that $\langle x, m\rangle = \sup_{m\in\mathrm{S}} \langle x, m\rangle$. By definition, $x \in \mathscr{T}_f$ if and only if $\mathrm{Card}(\mathrm{S}_x) \geqslant 2$. Let us assume that $x \notin \mathscr{T}_f$, that is, $\mathrm{Card}(\mathrm{S}_x) = 1$, and let us prove that $x \notin \lambda_\infty(\mathscr{V}(f))$. We argue by contradiction, assuming that there is a sequence (z_k) in $\mathscr{V}(f)$ and a sequence (h_k) of strictly positive real numbers such that $h_k \to 0$ and $h_k\lambda(z_k) \to x$. Let $\mu \in \mathrm{S}$ be the unique element such that $\mathrm{S}_x = \{\mu\}$. By assumption, one has $\langle x, m\rangle < \langle x, \mu\rangle$ for every $m \in \mathrm{S} - \{\mu\}$. Let $\varepsilon > 0$ be such that $\langle x, m\rangle < \langle x, \mu\rangle - \varepsilon$ for every $m \in \mathrm{S} - \{\mu\}$; by continuity, this inequality holds in a neighborhood U of x. For k large enough such that $h_k\lambda(z_k) \in \mathrm{U}$, one then has

$$\log(z_k^{m-\mu}) = \langle \lambda(z_k), m - \mu\rangle = h_k^{-1}\langle h_k\lambda(z_k), m - \mu\rangle \leqslant -h_k^{-1}\varepsilon$$

for all $m \in \mathrm{S} - \{\mu\}$. Since h_k tends to 0, this shows that $\log(|z_k^{m-\mu}|$ converges to $-\infty$, hence $|z_k^{m-\mu}|$ converges to 0. From the equality $f(z_k) = 0$, we deduce that

$$1 = -\sum_{m\in\mathrm{S}-\{\mu\}} \frac{c_m}{c_\mu} z_k^{m-\mu}.$$

By the preceding estimate, the right hand side of the previous equality converges to 0, whence the desired contradiction. □

Lemma **(2.6.8)**. — *Let* $t \in \mathbf{R}_+$ *and let* $x = (0, \dots, 0, -t)$; *if* $x \in \mathscr{T}_{\mathrm{V}}$, *then* $x \in \lambda_\infty(\mathrm{V})$.

Proof. — The result is obvious if $x = 0$. Since both $\mathscr{T}_{\mathrm{V}}$ and $\lambda_\infty(\mathrm{V})$ are invariant by multiplication by a positive real number, we may assume that $x = (0, \dots, 0, -1)$.

$f = \sum_{m\in S} c_m T^m$ $\quad S = \mathrm{supp}(f)$

$\boxed{x \notin \mathscr{T}_f :}$ $\sup_{m\in S} \langle x, m\rangle$

est atteint en un seul point $\mu \in S$

$m \in S - \{\mu\}$

$\langle x, m\rangle < \langle x, \mu\rangle$.

reste vrai au voisinage de x avec une marge d'erreur ε.

$x \in U$ ouvert

tel que $\langle y, m\rangle < \langle y, \mu\rangle - \varepsilon$

$\forall y \in U$

$\boxed{x \in \lambda_\infty(V(f))}$

$x = \lim h_k \lambda(z_k)$

$0 < h_k \to 0$ $\quad z_k \in (\mathbf{C}^*)^n$

$f(z_k) = 0$

. $\lambda_\infty(V) \subset \mathscr{T}_V$ assez élémentaire

. $\mathscr{T}_V \subset \lambda_\infty(V)$ demande un peu de géométrie algébrique.

$\mathscr{T}_V = \bigcap_{f\in \mathscr{I}(V)} \mathscr{T}_f$

? $\lambda_\infty(V) \subset \mathscr{T}_f$ si $f \in \mathscr{I}(V)$

$V \subset \mathscr{V}(f)$

$\lambda_\infty(V) \subset \lambda_\infty(\mathscr{V}(f))$

$x \in \mathbf{R}^n$ on raisonne par l'absurde : $x \notin \mathscr{T}_f$

et $x \in \lambda_\infty(\mathscr{V}(f))$

this inequality holds in a neighborhood U of x. For k large enough such that $h_k\lambda(z_k) \in \mathrm{U}$, one then has

$$\log(z_k^{m-\mu}) = \langle \lambda(z_k), m-\mu\rangle = h_k^{-1}\langle h_k\lambda(z_k), m-\mu\rangle \leqslant -h_k^{-1}\varepsilon$$

for all $m \in \mathrm{S} - \{\mu\}$. Since h_k tends to 0, this shows that $\log(|z_k^{m-\mu}|$ converges to $-\infty$, hence $|z_k^{m-\mu}|$ converges to 0. From the equality $f(z_k) = 0$, we deduce that

$$1 = -\sum_{m\in \mathrm{S}-\{\mu\}} \frac{c_m}{c_\mu} z_k^{m-\mu}.$$

By the preceding estimate, the right hand side of the previous equality converges to 0, whence the desired contradiction. □

$x = \lim h_k \lambda(z_k)$

donc $h_k\lambda(z_k) \in U$ pour $k >> 0$.

$\langle h_k\lambda(z_k), m\rangle < \langle h_k\lambda(z_k), \mu\rangle - \varepsilon$

si $m \in S - \{\mu\}$

$|z_k^m| < |z_k^\mu| \cdot \underbrace{e^{-\varepsilon/h_k}}_{<<1}$

$f(z_k) = 0$ $\quad \sum_{m\neq\mu} c_m z_k^m = -c_\mu z_k^\mu$

$\sum_{m\neq\mu} \underbrace{\frac{c_m}{c_\mu}}_{\text{constant}} \underbrace{z_k^{m-\mu}}_{\to 0} = -1$ contradiction.

$V \subset (\mathbb{C}^*)^n \rightsquigarrow V' \subset (\mathbb{C}^*)^{n-1} \times \mathbb{C}$

adhérence pour la topologie de Zariski

$\overset{R}{\mathbb{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]} \supset \overset{R'}{\mathbb{C}[T_1^{\pm 1}, \dots, T_{n-1}^{\pm 1}, T_n]}$

$\mathcal{J}(V) = I \qquad I' = I \cap R'$

$V' = \mathcal{V}(I')$

$\rightarrow V_0 \subset (\mathbb{C}^*)^{n-1} \times \{0\}$

$\quad \| $

$V' \cap (\mathbb{C}^*)^{n-1} \times \{0\}.$

$R_0 = \mathbb{C}[T_1^{\pm 1}, \dots, T_{n-1}^{\pm 1}]$

$\varphi : R' \twoheadrightarrow R_0$

$T_n \longmapsto 0$

$I_0 = \varphi(I')$

Let $\mathrm{R}_0 = \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_{n-1}^{\pm 1}]$, let $\mathrm{R} = \mathrm{R}_0[\mathrm{T}_n^{\pm 1}]$ and $\mathrm{R}' = \mathrm{R}_0[\mathrm{T}_n]$; let $\varphi : \mathrm{R}' \to \mathrm{R}_0$ be the unique morphism of R_0-algebras such that $\varphi(\mathrm{T}_n) = 0$. These rings R, R′ and R_0 are respectively viewed as the rings of functions on the algebraic varieties $(\mathbf{C}^*)^n$, $(\mathbf{C}^*)^{n-1} \times \mathbf{C}$ and $(\mathbf{C}^*)^{n-1} \times \{0\}$. Let $\mathrm{I} = \mathscr{I}(\mathrm{V})$ be the ideal of V in R; let $\mathrm{I}' = \mathrm{I} \cap \mathrm{R}'$ and let $\mathrm{I}_0 = \varphi(\mathrm{I}')$. Geometrically, I′ is the ideal of the Zariski closure V′ of V in $(\mathbf{C}^*)^{n-1} \times \mathbf{C}$, and I_0 is the ideal of $\mathrm{V}_0 = \mathrm{V}' \cap (\mathbf{C}^*)^{n-1} \times \{0\}$.

Let us prove that $\mathrm{I}_0 \neq (1)$. Otherwise, there exists $f \in \mathrm{I}' = \mathrm{I} \cap \mathrm{R}'$ such that $\varphi(f) = 1$; let S be the support of f and write $f = \sum_{m \in \mathrm{S}} c_m \mathrm{T}^m$, so that

$$\varphi(f) = \sum_{\substack{m \in \mathrm{S} \\ m_n = 0}} c_m \mathrm{T}_1^{m_1} \dots \mathrm{T}_{n-1}^{m_{n-1}}.$$

Since $f \in \mathrm{I}'$, one has $\mathrm{S} \subset \mathbf{Z}^{n-1} \times \mathbf{N}$, so that $\langle x, m \rangle = -m_n \leqslant 0$ for all $m \in \mathrm{S}$. From the equality $\varphi(f) = 1$, we see that there exists $m \in \mathrm{S}$ such that $m_n = 0$ and $(m_1, \dots, m_{n-1}) = 0$, that is, $0 \in \mathrm{S}$. In particular, $\sup_{m \in \mathrm{S}} \langle x, m \rangle = 0$.

Since $x \in \mathscr{T}_f$, there are at least two distinct elements $m, m' \in \mathrm{S}$ such that $0 = \langle x, m \rangle = \langle x, m' \rangle$, that is, $m_n = m'_n = 0$. Then $(m_1, \dots, m_{n-1}) \neq (m'_1, \dots, m'_{n-1})$, hence $\varphi(f)$ is not a monomial, contrary to the hypothesis $\varphi(f) = 1$. Consequently, $\mathrm{V}_0 \neq \varnothing$. Let $z \in (\mathbf{C}^*)^{n-1}$ be a point such that $(z, 0) \in \mathrm{V}_0$.

By definition, V is a dense open subset of V′ for the Zariski topology. It is therefore an open subset of V′ for the classical topology. Moreover, a basic but nontrivial result of algebraic geometry asserts it is also dense; see, for example, (Mumford, 1994), p. 58, theorem 1. Consequently, there is a sequence (z'_k) of points of V such that $z_k \to (z, 0)$. If one writes $z'_k = (z_k, u_k)$, with $z_k \in (\mathbf{C}^*)^{n-1}$ and $u_k \in \mathbf{C}^*$, this means that $z_k \to z$ and $u_k \to 0$. In particular, $\lambda(z_k) \to \lambda(z)$ and $\lambda(u_k) \to -\infty$; For k large enough, one thus has $\log(u_k) < 0$; removing a few terms, we assume that $\log(u_k) < 0$ for all k; setting $h_k = -1/\log(u_k)$, the sequence (h_k) converges to 0 and consists of strictly positive real numbers. Then, $h_k \lambda(z'_k) = (h_k \lambda(z_k), h_k \lambda(u_k))$ converges to $(0, -1) = x$. This proves that $x \in \lambda_\infty(\mathrm{V})$. □

Lemma **(2.6.8)**. — *Let $t \in \mathbf{R}_+$ and let $x = (0, \dots, 0, -t)$; if $x \in \mathscr{T}_\mathrm{V}$, then $x \in \lambda_\infty(\mathrm{V})$.*

Proof. — The result is obvious if $x = 0$. Since both $\mathscr{T}_\mathrm{V}$ and $\lambda_\infty(\mathrm{V})$ are invariant by multiplication by a positive real number, we may assume that $x = (0, \dots, 0, -1)$.

$x \in \mathcal{T}_V \overset{??}{\Rightarrow} x \in \lambda_\infty(V)$

? $z_k \in V$

$h_k \to 0$

$h_k \lambda(z_k) \to (0, \dots, 0, -1)$

$h_k \to 0$

$\lambda(z_k) \approx (\underbrace{\cancel{x_1}, \dots, \cancel{x_{n-1}}}_{??}, \underbrace{-\frac{1}{h_k}}_{\to -\infty})$

$z_{k,n} \to 0$

$z \rightsquigarrow z_0 \in V_0$

$R \supset R', \xrightarrow{\varphi} R_0 = R'/(T_n)$

La construction $V \rightsquigarrow V' \rightsquigarrow V_0$ construit une limite en géométrie algébrique des tranches $V \cap (T_n = \varepsilon)$ quand $\varepsilon \to 0$.

$\boxed{V_0 \neq \emptyset}$ $\Leftrightarrow I_0 \neq R_0$, $\boxed{1 \notin I_0}$?

Par l'absurde soit $f \in R'$ (I') tel que $\varphi(f) = f(T_1, \dots, T_{n-1}, 0) = 1$

Let us prove that $I_0 \neq (1)$. Otherwise, there exists $f \in I' = I \cap R'$ such that $\varphi(f) = 1$; let S be the support of f and write $f = \sum_{m \in S} c_m T^m$, so that

$$\varphi(f) = \sum_{\substack{m \in S \\ m_n = 0}} c_m T_1^{m_1} \dots T_{n-1}^{m_{n-1}}.$$

$f = \sum_{m \in S} c_m T^m$ $\qquad \varphi(f) = \sum_{\substack{m \in S \\ m_n = 0}} c_m T_1^{m_1} \dots T_{n-1}^{m_{n-1}}$

$S = \operatorname{supp}(f)$

Since $f \in I'$, one has $S \subset \mathbf{Z}^{n-1} \times \mathbf{N}$, so that $\langle x, m \rangle = -m_n \leqslant 0$ for all $m \in S$. From the equality $\varphi(f) = 1$, we see that there exists $m \in S$ such that $m_n = 0$ and $(m_1, \dots, m_{n-1}) = 0$, that is, $0 \in S$. In particular, $\sup_{m \in S} \langle x, m \rangle = 0$.

Since $x \in \mathscr{T}_f$, there are at least two distinct elements $m, m' \in S$ such that $0 = \langle x, m \rangle = \langle x, m' \rangle$, that is, $m_n = m'_n = 0$. Then $(m_1, \dots, m_{n-1}) \neq (m'_1, \dots, m'_{n-1})$, hence $\varphi(f)$ is not a monomial, contrary to the hypothesis $\varphi(f) = 1$. Consequently, $V_0 \neq \emptyset$. Let $z \in (\mathbf{C}^*)^{n-1}$ be a point such that $(z, 0) \in V_0$.

$x \in (0, \dots, 0, -1) \in \mathscr{T}_V = \bigcap_{h \in I} \mathscr{T}_h$ $\qquad I \cap R' = I' \subset I$

$f \in I'$ donc $x = (0, \dots, 0, -1) \in \mathscr{T}_f$

le sup de $-m_n = \langle x, m \rangle$ est atteint en deux points $m, m' \in S$

$f \in I'$ $\quad m_n \geq 0$ $\qquad -m_n \leq 0$ $\qquad$ c'est 0 :

$1 = \varphi(f) \neq 0$ $\qquad \exists\, m \quad -m_n = 0$ $\qquad$ $m \neq m' \in S$, $m_n = 0$ $\to$ deux monômes dans $\varphi(f)$

$\varphi(f) = 1$. Consequently, $V_0 \neq \varnothing$. Let $z \in (\mathbf{C}^*)^{n-1}$ be a point such that $(z, 0) \in V_0$.

By definition, V is a dense open subset of V′ for the Zariski topology. It is therefore an open subset of V′ for the classical topology. Moreover, a basic but nontrivial result of algebraic geometry asserts it is also dense; see, for example, (Mumford, 1994), p. 58, theorem 1. Consequently, there is a sequence (z'_k) of points of V such that $z_k \to (z, 0)$. If one writes $z'_k = (z_k, u_k)$, with $z_k \in (\mathbf{C}^*)^{n-1}$ and $u_k \in \mathbf{C}^*$, this means that $z_k \to z$ and $u_k \to 0$. In particular, $\lambda(z_k) \to \lambda(z)$ and $\lambda(u_k) \to -\infty$; For k large enough, one thus has $\log(u_k) < 0$; removing a few terms, we assume that $\log(u_k) < 0$ for all k; setting $h_k = -1/\log(u_k)$, the sequence (h_k) converges to 0 and consists of strictly positive real numbers. Then, $h_k\lambda(z'_k) = (h_k\lambda(z_k), h_k\lambda(u_k))$ converges to $(0, -1) = x$. This proves that $x \in \lambda_\infty(V)$. ☐

$(z,0) \in V_0 = (\mathbb{C}^*)^{n-1} \times \{0\} \cap V'$

V' = adhérence de V.

Fait $V' - V_0$ est un ouvert dense de V', non seulement pour la topologie de Zariski, mais aussi pour la topologie complexe.

$V' \supset V_0 \ni (z,0)$ est limite de points de $V' - V_0 = V$

$(V' \cap (\mathbb{C}^*)^{n-1} = V)$

$(z,0) = \lim z'_k$

$(z_k, u_k) = z_k \in V$

$(u_k \neq 0)$

$z_k \to z$

$\lambda(z_k) \to \lambda(z)$

$\lambda(u_k) \to -\infty$

$h_k > 0$ pour k assez grand.

$\boxed{\frac{1}{-\lambda(u_k)}} \lambda(z'_k) \to (0, \cdot, 0, -1)$

Si $V = \mathcal{V}(f)$
on écrit $(z,0) \in \mathbb{C}^{*n-1} \times \mathbb{C}$
tq $f(z,0) = 0$
comme limite de solutions
$f(z_k) = 0$
$z_{k,n} \neq 0$.

passage de $x = (0, \dots, 0, -1)$
à $x \in \mathcal{T}_V$
arbitraire

Proposition **(2.6.9)**. — *Assume that $\mathscr{T}_V$ is a* **Q**-*rational polyhedral subset of* $\mathbf{R}^n$. *Then* $\mathscr{T}_V \subset \lambda_\infty(V)$.

Proof. — Since $\mathscr{T}_V$ is a **Q**-rational conic polyhedral subset of $\mathbf{R}^n$, its rational points $\mathbf{Q}^n \cap \mathscr{T}_V$ are dense in $\mathscr{T}_V$. Since $\lambda_\infty(V)$ is closed in $\mathbf{R}^n$, it thus suffices to prove that every point of $\mathbf{Q}^n \cap \mathscr{T}_V$ belongs to $\lambda_\infty(V)$. Let $x \in \mathbf{Q}^n \cap \mathscr{T}_V$. If $x = 0$, then $x \in \lambda_\infty(V)$; let us then assume that $x \neq 0$. By the classification of matrices over **Z**, there exists $A \in \mathrm{GL}_n(\mathbf{Z})$ such that $A^{-1}x = (0, \dots, 0, -t)$, where $t \in \mathbf{Q}$. Performing the monomial change of variables given by A, we are reduced to the case of $x = (0, \dots, 0, -1)$. The proposition follows from the preceding lemma. □

$\mathcal{T}_V = \bigcup_{i=1}^{m} P_i$

P_i polyèdres $\mathbb{Q}$-rationnels

$\Rightarrow \overline{P_i \cap \mathbb{Q}^n} = P_i$

il suffit de prouver q4

$\mathcal{T}_V \cap \mathbb{Q}^n \subset \lambda_\infty(V)$

$x \in \mathcal{T}_V \cap \mathbb{Q}^n$

caractère conique

$x \in \mathcal{T}_V \cap \mathbb{Z}^n$
$(x_1, \dots, x_n)$ premiers entre eux.

2.7. Missing

Following Forsberg, Passare & Tsikh (2000); Passare & Rullgård (2004); Passare & Tsikh (2005):

– The connected components of the complement of the amoeba are maximal open sets on which the Ronkin function is affine.

– (Limit of the amoebas is the tropical hypersurface, it is purely $(n-1)$-dimensional;) maybe explain the balancing condition, at least the local concavity, maybe not.

changement de base
$A \in GL_n(\mathbb{Z})$

$x' = Ax = (0, \dots, 0, -1)$

$\leftrightarrow$ changement de variable monomial dans $\mathbb{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$T_i = S^{\alpha_i}$ $\quad \alpha_i \in \mathbb{Z}^n$ $\quad \leftrightarrow \quad$ $S_j = T^{\beta_j}$ $\quad \beta_j \in \mathbb{Z}^n$

$\leftrightarrow$ variété $W \subset (\mathbb{C}^*)^n$
après changement
de variables.

$(0, \dots, 0, -1) \in \mathcal{T}_W$
donc — $\in \lambda_\infty(W)$

Norm/Seminorms sur un anneau

$a \mapsto |a| = p(a)$

$$\begin{cases} |0| = 0 \\ |1| \leq 1 \\ |a \pm b| \leq |a| + |b| \\ |ab| \leq |a| \cdot |b| \end{cases}$$

(radical)

$|a^n| = |a|^n$ $n \geq 0$.

(multiplicative)

$|ab| = |a|\,|b|$

$|a| \leq |a|\,|1|$

si $|1| < 1$

on a $|a| = 0$ pour tout a

$p = |\cdot|$

noyau d'une seminorme sur R

$\{a \mid |a| = 0\} = \mathrm{Ker}(p)$

idéal de R

p radicale

$\Rightarrow \mathrm{Ker}(p)$

est un idéal radical

CHAPTER 3

NONARCHIMEDEAN AMOEBAS

3.1. Seminorms

Definition **(3.1.1)**. — *Let* R *be a ring. A* seminorm *on* R *is a map* $p \colon \mathrm{R} \to \mathbf{R}_+$ *satisfying the following properties:*

(i) *One has* $p(0) = 0$ *and* $p(1) \leqslant 1$;
(ii) *For every* $a, b \in \mathrm{A}$, *one has* $p(a - b) \leqslant p(a) + p(b)$;
(iii) *For every* $a, b \in \mathrm{A}$, *one has* $p(ab) \leqslant p(a)p(b)$.
One says that the seminorm p *is* radical *or* power-multiplicative *if, moreover, it satisfies*
(iv) *For every* $a \in \mathrm{A}$ *and* $n \in \mathbf{N}$, *one has* $p(a^n) = p(a)^n$.
One says that the seminorm p *is* multiplicative *if:* $p \neq 0$ et
(v) *For every* $a, b \in \mathrm{A}$, *one has* $p(ab) = p(a)p(b)$.
One says that the seminorm p *is a* norm *if* $p(a) = 0$ *implies* $a = 0$.

One has $p(a) \leqslant p(a)p(1)$ for all $a \in \mathrm{R}$; if $p \neq 0$, this implies $1 \leqslant p(1)$ hence $p(1) = 1$.

Taking $a = 0$ in (ii), one has $p(-b) \leqslant p(b)$, hence $p(-b) = p(b)$ for all b. Consequently, $p(a + b) \leqslant p(a) + p(b)$ for all $a, b \in \mathrm{R}$.

Example **(3.1.2)**. — Let R be a ring and let p be a seminorm on R. Let $\mathrm{P} = \{a \in \mathrm{R}\,;\, p(a) = 0\}$. Let $a, b \in \mathrm{P}$; then $p(a+b) \leqslant p(a)+p(b) = 0$, hence $p(a+b) = 0$ and $a+b \in \mathrm{P}$. Let $a \in \mathrm{R}$ and $b \in \mathrm{P}$; then $p(ab) \leqslant p(a)p(b) = 0$, hence $ab \in \mathrm{P}$. This proves that P is an ideal of R.

For every $a \in \mathrm{R}$ and every $b \in \mathrm{P}$, one has $p(a + b) \leqslant p(a)$, and $p(a) = p((a + b) - b) \leqslant p(a + b)$, so that $p(a + b) = p(a)$. Consequently, p passes to the quotient and defines a seminorm on R/P.

If p is radical, then P is a radical ideal. Let indeed $a \in \mathrm{R}$ and $n \in \mathbf{N}$ be such that $a^n \in \mathrm{P}$; then $p(a)^n = p(a^n) = 0$, hence $p(a) = 0$ and $a \in \mathrm{P}$.

Assume that p is multiplicative and $p \neq 0$, and let us show that P is a prime ideal. Since $p \neq 0$, one has $\mathrm{P} \neq \mathrm{R}$. Let also $a, b \in \mathrm{R}$ be such that $ab \in \mathrm{P}$; then $p(ab) = p(a)p(b) = 0$, hence either $p(a) = 0$ and $a \in \mathrm{P}$, or $p(b) = 0$ and $b \in \mathrm{P}$.

p : $|\cdot|$ est multiplicative (non nulle)
$\Rightarrow$ Ker(p) est un idéal premier.
$ab \in$ Ker(p)
$|ab| = 0$
$= |a| \cdot |b|$
donc $|a| = 0$ ou $|b| = 0$.

Example **(3.1.3)**. — Let R be a ring, let S be a multiplicative subset of R, let $\mathrm{R_S}$ be the associated fraction ring. Let p be a multiplicative seminorm on R such that $p(s) \neq 0$ for every $s \in \mathrm{S}$. There exists a unique map $p' : \mathrm{R_S} \to \mathbf{R}_+$ such that $p'(a/s) = p(a)/p(s)$ for every $a \in \mathrm{A}$ and every $s \in \mathrm{S}$. (Indeed, if $a/s = b/t$, for $a, b \in \mathrm{R}$ and $s, t \in \mathrm{S}$, there exists $u \in \mathrm{S}$ such that $atu = bsu$; then $p(a)p(t)p(u) = p(b)p(s)p(u)$, hence $p(a)/p(s) = p(b)/p(t)$.) It is clear that p' is multiplicative: $p'((a/s)(b/t)) = p'(ab/st) = p(ab)/p(st) = (p(a)/p(s)) \cdot (p(b)/p(t))$. Moreover, let $a, b \in \mathrm{R}$ and $s, t \in \mathrm{S}$; then $(a/s) + (b/t) = (at + bs)/st$, so that

$$\begin{aligned} p'(\frac{a}{s} + \frac{b}{t}) &= p'(\frac{at+bs}{st}) = \frac{p(at+bs)}{p(st)} \\ &\leqslant \frac{p(at)+p(bs)}{p(st)} = \frac{p(a)}{p(s)} + \frac{p(b)}{p(t)} \\ &= p'(\frac{a}{s}) + p'(\frac{b}{t}). \end{aligned}$$

$|\cdot|$ semi norme multiplicative sur $\mathbb{R}$
$\rightsquigarrow$ semi norme sur $R_S = S^{-1}R$ multiplicative.
$\left|\frac{a}{s}\right| = \frac{|a|}{|s|}$
condition : $|s| \neq 0$ si $s \in S$
$S \cap$ Ker$(|\cdot|) = \emptyset$

Definition **(3.1.4)**. — *Let* R *be a ring and let p be a seminorm on* R. *One says that the seminorm p is* nonarchimedean, *or* ultrametric, *if one has $p(a+b) \leqslant \sup(p(a), p(b))$ for every $a, b \in \mathrm{R}$.*

The terminology *ultrametric* refers to the property that p satisfies an inequality stronger than the triangular inequality. The terminology *nonarchimedean* alludes to the fact that it implies that $p(na) \leqslant p(a)$ for every $n \in \mathbf{N}$: no matter how many times one adds an element, it never gets higher than the initial size. The following example explains the relations between these two properties.

Lemma **(3.1.5)**. — *Let* R *be a ring and let p be a seminorm on* R.

Lemma **(3.1.5)**. — *Let* R *be a ring and let* p *be a seminorm on* R.

a) *If* p *is nonarchimedean, then* $p(na) \leqslant p(a)$ *for every* $n \in \mathbf{Z}$ *and every* $a \in \mathrm{R}$.

b) *Conversely, let us assume that* p *is radical and that* $p(n) \leqslant 1$ *for every* $n \in \mathbf{N}$. *Then* p *is nonarchimedean.*

Proof. — The first assertion is proved by an obvious inductive argument. Let us prove the second one. Let $a, b \in \mathrm{R}$. For every $n \in \mathbf{N}$, one has

$$\begin{aligned} p(a+b)^n = p((a+b)^n) &\leqslant p(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}) \\ &\leqslant \sum_{k=0}^{n} p(\binom{n}{k}) p(a)^k p(b)^{n-k} \leqslant \sum_{k=0}^{n} p(a)^k p(b)^{n-k} \\ &\leqslant (n+1)\sup(p(a), p(b))^n. \end{aligned}$$

As a consequence, one has

$$p(a+b) \leqslant (n+1)^{1/n} \sup(p(a), p(b)).$$

When $n \to +\infty$, we obtain the upper bound $p(a+b) \leqslant \sup(p(a), p(b))$; this proves that p is nonarchimedean. □

Proposition **(3.1.6)**. — *Let* K *be a field endowed with a nonarchimedean absolute value* $|\cdot|$ *and let* $r = (r_1, \dots, r_n)$ *be a family of strictly positive real numbers. There is a unique absolute value* p_r *on* $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$ *such that for every polynomial* $f = \sum c_m \mathrm{T}^m$, *one has*

$$p_r(f) = \sup_{m \in \mathbf{N}^n} |c_m| r_1^{m_1} \dots r_n^{m_n}.$$

Its restriction to $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ *is the largest absolute value such that* $p_r(\mathrm{T}_j) = r_j$ *for* $j \in \{1, \dots, n\}$ *and which restricts to the given absolute value on* K.

Proof. — To prove the first assertion, it suffices to prove that the given formula defines an absolute value on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, because it then extends uniquely to its fraction field $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$. One has $p_r(0) = 0$; conversely, if $f = \sum c_m \mathrm{T}^m$ is such that $p_r(f) = 0$, then $|c_m| = 0$ for all m, hence $f = 0$. One also has $p_r(1) = 1$.

Let $f = \sum c_m \mathrm{T}^m$ and $g = \sum d_m \mathrm{T}^m$ be two polynomials.

$$(a+b)^n = \sum \binom{n}{k} a^k b^{n-k}$$

$$|(a+b)^n| \le \sum \underbrace{\left|\binom{n}{k}\right|}_{\le 1} |a^k| \, |b^{n-k}|$$

$$\le \sum_{k=0}^{n} |a|^k |b|^{n-k}$$

$$\le (n+1) \sup(|a|, |b|)^n$$

$|\cdot|$ radicale

$$\Rightarrow |a+b|^n = |(a+b)^n| \le (n+1) \sup(|a|, |b|)^n$$

$$\Rightarrow |a+b| \le \underbrace{(n+1)^{1/n}}_{\xrightarrow[n\to\infty]{} 1} \sup(|a|, |b|)$$

Then $f + g = \sum (c_m + d_m)\mathrm{T}^m$; for every m,

$$|c_m + d_m| r_1^{m_1} \dots r_n^{m_n} \leqslant \left(\sup(|c_m|, |d_m|) r_1^{m_1} \dots r_n^{m_n}\right) \leqslant \sup(p_r(f), p_r(g)),$$

so that $p_r(f+g) \leqslant \sup(p_r(f), p_r(g))$.

Moreover, $fg = \sum_m (\sum_{p+q=m} c_p d_q)\mathrm{T}^m$. For every m, one has

$$\left| \sum_{p+q=m} c_p d_q \right| r^m \leqslant \sup_{p+q=m} |c_p||d_q| r^p r^q \leqslant p_r(f) p_r(g),$$

so that $p_r(fg) \leqslant p_r(f)p_r(g)$. This shows that p_r is a norm on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, and it remains to prove that p_r is multiplicative.

Let P be the convex hull of the set of all $p \in \mathbf{N}^n$ such that $p_r(f) = |c_p|r^p$, and let Q be the convex hull of the set of all $q \in \mathbf{N}^n$ such that $p_r(g) = |d_q|r^q$. Let a and b be vertices of P and Q respectively, defined by linear forms φ and ψ on $\mathbf{R}^n$; let $m = a + b$. Then m is a vertex of the polytope P + Q, defined by the linear form $\varphi + \psi$, so that the coefficient of T^m in fg is the sum of $c_a d_b$ and of other elements $c_p d_q$, where $|c_p|r^q < |c_a|r^a$ and $|d_q|r^q < |d_b|r^b$. This implies that

$$\left| \sum_{p+q=m} c_p d_q \right| r^m = |c_a d_b| r^m = |c_a| r^a \, |d_b| r^b = p_r(f) p_r(g).$$

Consequently, $p_r(fg) = p_r(f)p_r(g)$ and p_r is a multiplicative seminorm on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. ☐

Example **(3.1.7)**. — A theorem of Ostrowski describes the multiplicative seminorms on the field $\mathbf{Q}$ of rational numbers.

a) The usual absolute value $|\cdot|$, and its powers $|\cdot|^r$ for $r \in]0; 1]$;

b) For every prime number p, the p-adic absolute value $|\cdot|_p$, and its powers $|\cdot|_p^r$, for all $r \in]0; +\infty[$;

c) The trivial absolute value $|\cdot|_0$ defined by $|0|_0 = 0$ and $|a|_0 = 1$ for all

D. $|p|_p = 1/p$

. $|n|_p = 1$

si $n \in \mathbb{Z} - p\mathbb{Z}$