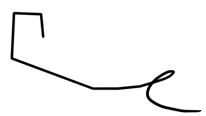
TOPICS
IN TROPICAL GEOMETRY



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CHAPTER 3

NONARCHIMEDEAN AMOEBAS

3.1. Seminorms

Definition **(3.1.1)**. — *Let* R *be a ring. A* seminorm *on* R *is a map* $p : \mathrm{R} \to \mathbf{R}_+$ *satisfying the following properties:*

(i) *One has* $p(0) = 0$ *and* $p(1) \leqslant 1$*;*
(ii) *For every* $a, b \in \mathrm{A}$*, one has* $p(a - b) \leqslant p(a) + p(b)$*;*
(iii) *For every* $a, b \in \mathrm{A}$*, one has* $p(ab) \leqslant p(a)p(b)$*.*

One says that the seminorm p *is* radical *or* power-multiplicative *if, moreover, it satisfies*

(iv) *For every* $a \in \mathrm{A}$ *and* $n \in \mathbf{N}$*, one has* $p(a^n) = p(a)^n$*.*

One says that the seminorm p *is* multiplicative *if:*

(v) *For every* $a, b \in \mathrm{A}$*, one has* $p(ab) = p(a)p(b)$*.*

One says that the seminorm p *is a* norm*, or an* absolute value*, if* $p(a) = 0$ *implies* $a = 0$*.*

semi norme

$|0| = 0$
$|a \pm b| \leq |a| + |b|$
$|ab| \leq |a| \cdot |b|$
$|1| \leq 1$

valeur absolue.

$|a| \cdot |b| = |ab|$
$|1| = 1$

corps valué = corps + valeur absolue

One has $p(a) \leqslant p(a)p(1)$ for all $a \in \mathrm{R}$; if $p \neq 0$, this implies $1 \leqslant p(1)$ hence $p(1) = 1$.

Taking $a = 0$ in (ii), one has $p(-b) \leqslant p(b)$, hence $p(-b) = p(b)$ for all b. Consequently, $p(a+b) \leqslant p(a) + p(b)$ for all $a, b \in \mathrm{R}$.

Example **(3.1.2)**. — Let R be a ring and let p be a seminorm on R. Let $\mathrm{P} = \{a \in \mathrm{R}\,;\, p(a) = 0\}$. Let $a, b \in \mathrm{P}$; then $p(a+b) \leqslant p(a) + p(b) = 0$, hence $p(a+b) = 0$ and $a + b \in \mathrm{P}$. Let $a \in \mathrm{R}$ and $b \in \mathrm{P}$; then $p(ab) \leqslant p(a)p(b) = 0$, hence $ab \in \mathrm{P}$. This proves that P is an ideal of R.

For every $a \in \mathrm{R}$ and every $b \in \mathrm{P}$, one has $p(a+b) \leqslant p(a)$, and $p(a) = p((a+b)-b) \leqslant p(a+b)$, so that $p(a+b) = p(a)$. Consequently, p passes to the quotient and defines a seminorm on R/P.

If p is radical, then P is a radical ideal. Let indeed $a \in \mathrm{R}$ and $n \in \mathbf{N}$ be such that $a^n \in \mathrm{P}$; then $p(a)^n = p(a^n) = 0$, hence $p(a) = 0$ and $a \in \mathrm{P}$.

Assume that p is multiplicative and $p \neq 0$, and let us show that P is a prime ideal. Since $p \neq 0$, one has $\mathrm{P} \neq \mathrm{R}$. Let also $a, b \in \mathrm{R}$ be such that $ab \in \mathrm{P}$; then $p(ab) = p(a)p(b) = 0$, hence either $p(a) = 0$ and $a \in \mathrm{P}$, or $p(b) = 0$ and $b \in \mathrm{P}$.

Example **(3.1.3)**. — Let R be a ring, let S be a multiplicative subset of R, let R_{S} be the associated fraction ring. Let p be a multiplicative seminorm on R such that $p(s) \neq 0$ for every $s \in \mathrm{S}$. There exists a unique map $p' : R_S \to \mathbf{R}_+$ such that $p'(a/s) = p(a)/p(s)$ for every $a \in A$ and every $s \in S$. (Indeed, if $a/s = b/t$, for $a, b \in R$ and $s, t \in S$, there exists $u \in S$ such that $atu = bsu$; then $p(a)p(t)p(u) = p(b)p(s)p(u)$, hence $p(a)/p(s) = p(b)/p(t)$.) It is clear that p' is multiplicative: $p'((a/s)(b/t)) = p'(ab/st) = p(ab)/p(st) = (p(a)/p(s)) \cdot (p(b)/p(t))$. Moreover, let $a, b \in R$ and $s, t \in S$; then $(a/s) + (b/t) = (at + bs)/st$, so that

$$\begin{aligned} p'(\frac{a}{s} + \frac{b}{t}) &= p'(\frac{at+bs}{st}) = \frac{p(at+bs)}{p(st)} \\ &\leqslant \frac{p(at)+p(bs)}{p(st)} = \frac{p(a)}{p(s)} + \frac{p(b)}{p(t)} \\ &= p'(\frac{a}{s}) + p'(\frac{b}{t}). \end{aligned}$$

In particular, any absolute value on an integral domain extends uniquely to an absolute value on its field of fractions.

Definition **(3.1.4)**. — *Let* R *be a ring and let* p *be a seminorm on* R. *One says that the seminorm* p *is* nonarchimedean, *or* ultrametric, *if one has* $p(a+b) \leqslant \sup(p(a), p(b))$ *for every* $a, b \in R$.

The terminology *ultrametric* refers to the property that p satisfies an inequality stronger than the triangular inequality. The terminology *nonarchimedean* alludes to the fact that it implies that $p(na) \leqslant p(a)$ for every $n \in \mathbf{N}$: no matter how many times one adds an element,

(La théorie des seminormes (multiplicatives) se ramène à celle des normes.
· $|\ |$ semi norme sur R
$P = \{a \in R \mid |a| = 0\}$ idéal de R
⇝ norme quotient sur R/P.
· (multiplicatif) : P est un idéal premier
R/P anneau intègre
⇝ valeur absolue sur $\operatorname{Frac}(R/P)$)

it never gets higher than the initial size. The following example explains the relations between these two properties.

Lemma **(3.1.5)**. — *Let* R *be a ring and let p be a seminorm on* R.

a) *If p is nonarchimedean, then $p(na) \leqslant p(a)$ for every $n \in \mathbf{Z}$ and every $a \in \mathrm{R}$.*

b) *Conversely, let us assume that p is radical and that $p(n) \leqslant 1$ for every $n \in \mathbf{N}$. Then p is nonarchimedean.*

Proof. — The first assertion is proved by an obvious inductive argument. Let us prove the second one. Let $a, b \in \mathrm{R}$. For every $n \in \mathbf{N}$, one has

$$\begin{aligned} p(a+b)^n &= p((a+b)^n) \leqslant p(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}) \\ &\leqslant \sum_{k=0}^{n} p(\binom{n}{k}) p(a)^k p(b)^{n-k} \leqslant \sum_{k=0}^{n} p(a)^k p(b)^{n-k} \\ &\leqslant (n+1) \sup(p(a), p(b))^n. \end{aligned}$$

As a consequence, one has

$$p(a+b) \leqslant (n+1)^{1/n} \sup(p(a), p(b)).$$

When $n \to +\infty$, we obtain the upper bound $p(a+b) \leqslant \sup(p(a), p(b))$; this proves that p is nonarchimedean. ☐

Example **(3.1.6)**. — A theorem of Ostrowski describes the multiplicative seminorms on the field $\mathbf{Q}$ of rational numbers.

a) The usual absolute value $|\cdot|$, and its powers $|\cdot|^r$ for $r \in]0;1]$;

b) For every prime number p, the p-adic absolute value $|\cdot|_p$, and its powers $|\cdot|_p^r$, for all $r \in]0;+\infty[$;

c) The trivial absolute value $|\cdot|_0$ defined by $|0|_0 = 0$ and $|a|_0 = 1$ for all $a \in \mathbf{Q}^\times$.

3.1.7. — Let K be a nonarchimedean *valued field*, that is, a field endowed with a nonarchimedean absolute value.

Let R be the set of $a \in \mathrm{K}$ such that $|a| \leqslant 1$. Then R is a subring of K, and K is its fraction field. More precisely, for every $a \in \mathrm{K}^\times$, then either $a \in \mathrm{R}$ (if $|a| \leqslant 1$), or $1/a \in \mathrm{R}$ (when $|a| \geqslant 1$), which means that R is a valuation ring. It is called the *valuation ring* of K.

An element $a \in \mathrm{R}$ is invertible in R if and only if $|a| = 1$. As a consequence, the ring R is a local ring and the set M of all $a \in \mathrm{R}$ such that $|a| < 1$ is its unique maximal ideal. The field $k = \mathrm{R}/\mathrm{M}$ is called the *residue field* of K.

If the absolute value of K is trivial, then $\mathrm{R} = \mathrm{K}$, $\mathrm{M} = 0$ and $k = \mathrm{K}$.

In this context, the map from $\mathrm{K}^\times$ to the ordered abelian group $\mathbf{R}$ given by $v\colon a \mapsto -\log(|a|)$ is a group morphism which satisfies the property $v(a+b) \geqslant \inf(v(a), v(b))$ for all $a, b \in \mathrm{K}$ such that $a, b, a+b \neq 0$; in other words, v is a *valuation* on K. In this context, one also defines $v(0) = +\infty$.

Weil, basic number theory

référence → Dwork, Gerotto, Sullivan. An introduction to G-functions

non-références: Engler-Prestel Valuations
Ribenboim
Bourbaki, Alg commutative.
l'aspect "valuation générale" est privilégié

K corps + valeur absolue non archimédienne

$R = \{a \in K, |a| \le 1\}$ sous-anneau de K

$a, b \in R$: $|a+b| \le \sup(|a|,|b|) \le 1$, $|ab| = |a||b| \le 1$, $|1| = 1$, $|0| = 0 \le 1$

anneau de valuation de K

pour tout $a \in K^\times$, ou bien $a \in R$, ou bien $1/a \in R$

$|a| \le 1$ ou $|a| \geq 1$

3.1.7. — Let K be a nonarchimedean *valued field*, that is, a field endowed with a nonarchimedean absolute value.

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K

$R = \{a \in K \text{ tq } |a| \leq 1\}$ — anneau de valuation de K

$M = \{a \in K \text{ tq } |a| < 1\}$ — idéal de R

unique idéal maximal : $M = R - R^\times$ — anneau local

$a \in R^\times \iff |a| = 1.$

$k = R/M$ est un corps — corps résiduel de K.

(Cas trivial : $|a| = 1$ pour tout $a \in K^\times$
$R = K$, $M = 0$; $k = K$.)

Théorie additive

$v(a) = -\log(|a|)$

$v : K \longrightarrow \mathbb{R} \cup \{+\infty\}$ — morphisme de monoïdes $(K, \times) \longrightarrow (\mathbb{R} \cup \{+\infty\}, +)$

- $v(ab) = v(a) + v(b)$
  $v(1) = 0$
  $v(0) = +\infty$
- $v(a+b) \geqslant \inf(v(a), v(b))$

Les éléments de grande valuation sont petits

Exercice

- si $|a| \neq |b|$, alors
  $|a+b| = \sup(|a|, |b|)$
- si $v(a) \neq v(b)$, alors
  $v(a+b) = \inf(v(a), v(b))$

Valuation générale :
Γ groupe abélien ordonné, additivement $(,+)$

$v : K \longrightarrow \Gamma \cup \{+\infty\} = \Gamma_*$ monoïde

$(+\infty > \gamma \quad \forall \gamma \in \Gamma) \quad (+\infty + \gamma = +\infty)$

morphisme de monoïdes $v(ab) = v(a) + v(b)$
$v(1) = 1$

inégalité triangulaire :

$$v(a+b) \geq \inf(v(a), v(b))$$

Normalisation peut-être plus naturelle (R Huber)

$v : K \to \Gamma \cup \{0\}$ $\quad \Gamma$ noté multiplicativement voire additivement

$0 \leq \gamma \quad \forall \gamma \in \Gamma$

morphisme de monoïde

$$v(a+b) \leq \sup(v(a), v(b))$$

The minus sign in the definition of v is sometimes annoying, at least it creates confusion, so that some authors define an abstract valuation on a field K as a morphism λ from $K^\times$ to an ordered abelian group Γ such that $\lambda(a+b) \leqslant \sup(\lambda(a), \lambda(b))$ for all $a, b \in K$ such that $a, b, a+b \neq 0$. In this context, one also sets $\lambda(0)$ to be an additional element smaller than any element of Γ.

Conversely, let K be a field and let R be a valuation ring of K. For $a, b \in K^\times$, write $a \preceq b$ if there exists $u \in R$ such that $a = bu$; this is a preordering relation on $K^\times$ and it induces an ordering relation on the quotient abelian group $K^\times/R^\times$ and the canonical morphism $\lambda \colon K^\times \to K^\times/R^\times$ is a valuation. Indeed, let $a, b \in K^\times$ be such that $a+b \neq 0$ and set $u = b/a$; if $u \in R$, then $b = au$ and $a+b = a(1+u)$, so that $(a+b) \preceq a$; otherwise, $v = 1/u \in R$, $a = bv$ and $a+b = b(1+v)$ so that $(a+b) \preceq b$; in both cases, we have shown that $\lambda(a+b) \leqslant \sup(\lambda(a), \lambda(b))$.

Example **(3.1.8)**. — Let K be a nonarchimedean valued field and let R be its valuation ring. It follows from the property of a valuation ring that for every $a, b \in R$, either $a \in bR$ or $b \in aR$, according to whether $|a| \leqslant |b|$ or $|b| \leqslant |a|$. In particular, every finitely generated ideal of R is principal.

If M is finitely generated, then M is a principal ideal. Let $\pi \in M$ be such that $M = \pi R$; one has $|\pi| < 1$; moreover, for $a \in R$, either $|a| \leqslant |\pi|$, or $|a| = 1$. Let $a \in R - \{0\}$; there exists a largest integer $n \in$

Inégalité triangulaire

$a, b \in K^\times$ $\qquad v(a+b) \leq \sup(v(a), v(b))$

$r = a/b \qquad a = br \qquad$ si $\underline{r \in R}$ $\qquad a \precsim b$

$v(a) \leq v(b)$

$a + b = br + b = b(\underbrace{1 + r}_{\in R})$

$(a+b) \precsim b$

$v(a+b) \leq \sup(v(a), v(b)) = v(b)$

sinon $1/r \in R$

$b = a/r$

$v(a+b) \leq v(a)$

$= \sup(v(a), v(b))$.

Example **(3.1.8)**. — Let K be a nonarchimedean valued field and let R be its valuation ring. It follows from the property of a valuation ring that for every $a, b \in R$, either $a \in bR$ or $b \in aR$, according to whether $|a| \leqslant |b|$ or $|b| \leqslant |a|$. In particular, every finitely generated ideal of R is principal. (vrai pour tout anneau de valuation)

If M is finitely generated, then M is a principal ideal. Let $\pi \in M$ be such that $M = \pi R$; one has $|\pi| < 1$; moreover, for $a \in R$, either $|a| \leqslant |\pi|$, or $|a| = 1$. Let $a \in R - \{0\}$; there exists a largest integer $n \in$ **N** such that $|a| \leqslant |\pi|^n$. One thus has $|\pi| < |a/\pi^n| \leqslant 1$, so that $|a/\pi^n| = 1$ and there exists $u \in R^\times$ such that $a = u\pi^n$.

uniformisante (uniformizer)

As a consequence, all ideals of R are of the form $\pi^n R$, for some unique $n \in$ **N**. In particular, R is a principal ideal domain. The map $v : K^\times \to$ **Z** given by $v(a) = n$ if and only if $aR = \pi^n R$ is a (normalized) discrete valuation on K.

• $a, b \in R$ $v(a) \leq v(b)$ ou $v(b) \leq v(a)$

$aR + bR = aR$ si $v(b) \leq v(a)$

$= bR$ si $v(a) \leq v(b)$

• Il y a des anneaux de valuation non noethériens.

$\pi \in R$ tq $M = \pi R$. $|\pi| < 1$

si $a \in R$ vérifie $|a| > |\pi|$ alors $a \in R^\times$ et $|a| = 1$

$a \in R$, n maximal tq $|a| \geq |\pi|^n$

$\Rightarrow |a| = |\pi|^n$

$\Rightarrow \exists u \in R^\times$ tq $a = \pi^n u$.

la valuation est essentiellement

$R \longrightarrow \mathbb{N}\, v(\pi)$

$u\pi^n \longmapsto n\, v(\pi)$

• R est un anneau principal

anneau de valuation discrète

si $I = (a_1, a_2, \dots)$. $a_i = \pi^{n_i} u_i$, $u_i \in R^\times$, $n_i \in \mathbb{N}$

$= (\pi^n)$ $n = \inf(n_1, n_2, \dots)$

groupe de valeurs $\simeq \mathbb{Z}$

$\mathbf{N}$ such that $|a| \leqslant |\pi|^n$. One thus has $|\pi| < |a/\pi^n| \leqslant 1$, so that $|a/\pi^n| = 1$ and there exists $u \in \mathrm{R}^\times$ such that $a = u\pi^n$.

As a consequence, all ideals of R are of the form $\pi^n\mathrm{R}$, for some unique $n \in \mathbf{N}$. In particular, R is a principal ideal domain. The map $v \colon \mathrm{K}^\times \to \mathbf{Z}$ given by $v(a) = n$ if and only if $a\mathrm{R} = \pi^n\mathrm{R}$ is a (normalized) discrete valuation on K.

Proposition **(3.1.9)**. — *Let* K *be a field endowed with a nonarchimedean absolute value* $|\cdot|$ *and let* $r = (r_1, \dots, r_n)$ *be a family of strictly positive real numbers. There is a unique absolute value* p_r *on* $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$ *such that for every polynomial* $f = \sum c_m \mathrm{T}^m$, *one has*

$$p_r(f) = \sup_{m \in \mathbf{N}^n} |c_m| r_1^{m_1} \dots r_n^{m_n}.$$

Its restriction to $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ *is the largest absolute value such that* $p_r(\mathrm{T}_j) = r_j$ *for* $j \in \{1, \dots, n\}$ *and which restricts to the given absolute value on* K.

Proof. — To prove the first assertion, it suffices to prove that the given formula defines an absolute value on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, because it then extends uniquely to its fraction field $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$. One has $p_r(0) = 0$; conversely, if $f = \sum c_m \mathrm{T}^m$ is such that $p_r(f) = 0$, then $|c_m| = 0$ for all m, hence $f = 0$. One also has $p_r(1) = 1$.

Let $f = \sum c_m \mathrm{T}^m$ and $g = \sum d_m \mathrm{T}^m$ be two polynomials.

Then $f+g=\sum(c_m+d_m)\mathrm{T}^m$; for every m,

$$|c_m+d_m|r_1^{m_1}\dots r_n^{m_n} \leqslant \left(\sup(|c_m|,|d_m|)r_1^{m_1}\dots r_n^{m_n}\right) \leqslant \sup(p_r(f),p_r(g)),$$

so that $p_r(f+g) \leqslant \sup(p_r(f),p_r(g))$.

Moreover, $fg=\sum_m(\sum_{p+q=m} c_p d_q)\mathrm{T}^m$. For every m, one has

$$\left|\sum_{p+q=m} c_p d_q\right| r^m \leqslant \sup_{p+q=m} |c_p||d_q|r^p r^q \leqslant p_r(f)p_r(g),$$

so that $p_r(fg) \leqslant p_r(f)p_r(g)$. This shows that p_r is a norm on $\mathrm{K}[\mathrm{T}_1,\dots,\mathrm{T}_n]$, and it remains to prove that p_r is multiplicative.

Let P be the convex hull of the set of all $p\in\mathbf{N}^n$ such that $p_r(f)=|c_p|r^p$, and let Q be the convex hull of the set of all $q\in\mathbf{N}^n$ such that $p_r(g)=|d_q|r^q$. Let m be a vertex of $\mathrm{P}+\mathrm{Q}$; then there is a vertex a of P, and a vertex b of Q such that $m=a+b$. In particular, if $m=p+q$, for $p\in\mathrm{P}$ and $q\in\mathrm{Q}$, then $p=a$ and $q=b$, so that the coefficient of T^m in fg is the sum of $c_a d_b$ and of other elements $c_p d_q$, where either $|c_p|r^q<|c_a|r^a$, or $|d_q|r^q<|d_b|r^b$ (or both) This implies that

$$\left|\sum_{p+q=m} c_p d_q\right| r^m = |c_a d_b|r^m = |c_a|r^a\,|d_b|r^b = p_r(f)p_r(g).$$

Consequently, $p_r(fg)=p_r(f)p_r(g)$ and p_r is a multiplicative seminorm on $\mathrm{K}[\mathrm{T}_1,\dots,\mathrm{T}_n]$. □

3.1.10. — Let K be a field endowed with an absolute value. The map $(a, b) \mapsto |a - b|$ is a distance on K.

Let $\widehat{\mathrm{K}}$ be the completion of K for this distance. Let us recall its definition. One starts from the ring S of all Cauchy sequences in K and the subset M of all Cauchy sequences which converge to 0. It is obvious that M is an additive subgroup of M; since a Cauchy sequence is bounded, it is an ideal of S, and $\widehat{\mathrm{K}}$ is the quotient ring S/M. Let $j \colon \mathrm{K} \to \widehat{\mathrm{K}}$ be the map such that $j(a)$ is the class of the constant sequence with value a; it is a morphism of rings.

For $a, b \in \mathrm{K}$, one has

$$\big||a| - |b|\big| \leqslant |a - b|.$$

This implies that for every Cauchy sequence (a_n) in K, the sequence $(|a_n|)$ is a Cauchy sequence in $\mathbf{R}$; in particular, it converges. It induces a map $|\cdot| \colon \widehat{\mathrm{K}} \to \mathbf{R}$ which is a multiplicative seminorm on $\widehat{\mathrm{K}}$ such that $|j(a)| = |a|$ for every $a \in \mathrm{K}$.

Let $a = (a_n)$ be a Cauchy sequence in K which does not converge to 0; by definition, there exists $\varepsilon > 0$ and arbitrarily large integers n such that $|a_n| \geqslant \varepsilon$. Since (a_n) is a Cauchy sequence, there exists an integer p such that $|a_n - a_m| \leqslant \varepsilon/2$ for all integers $m, n \geqslant p$. Taking $m \geqslant p$ such that $|a_m| \geqslant \varepsilon$, it follows that $|a_n| \geqslant \varepsilon/2$ for all integers $n \geqslant p$. In particular, one has $|a| \geqslant \varepsilon/2$. Consequently, the seminorm on $\widehat{\mathrm{K}}$ is an absolute value.

Deux autres constructions
* complétion
* clôture algébrique

K corps valué
⇝ distance sur K $d(a,b) = |a-b|$
$\widehat{K}$ complétion de K
c'est un corps valué

$\widehat{K} = S/M$ automatiquement bornée

anneau $S = \{$suites de Cauchy dans $K\}$

n $M = \{$suites qui convergent vers $0\}$

$a = (a_n)$ $b = (b_n)$ $ab = (a_n b_n)$

$K \to S$ $a \mapsto (a, a, a, \dots)$

M idéal maximal

$a \in S$ $b \in M$ $\Rightarrow ab \in M$

$a_n b_n \overset{?}{\to} 0$ car a est bornée.

Idéal **maximal** : $a = (a_n)$ qui ne tend pas vers 0

- $\exists\, \varepsilon > 0$ tq $|a_n| \geq \varepsilon$ pour une infinité d'entiers n.
- critère de Cauchy : $(\exists p)$ pour $m, n \geq p$ $|a_m - a_n| \leq \varepsilon/2$

$\Rightarrow$ $|a_m| \geq |a_n| - |a_m - a_n| \geq \varepsilon/2$ si $|a_n| \geq \varepsilon/2$
$m, n \geq p$

$b_n = \begin{cases} 1/a_n & \text{si } n \geq p \\ 0 & \text{sinon} \end{cases}$ $a_n b_n$ est stationnaire vers 1.

(b_n) est de Cauchy car $|b_n - b_m| = \dfrac{|a_n - a_m|}{|a_n|\,|a_m|} \leq \dfrac{4}{\varepsilon^2} |a_m - a_n|$

pour $m, n \geq p$

$b = (b_n)$ $b \in R$ $ab - 1 \in M$

$\Rightarrow$ M est un idéal maximal
R/M est un corps.

$\big|\, |a| - |b| \,\big| \leq |a - b|$ $\Rightarrow$ si $a \in S$, la suite $(|a_n|)$ est une suite de Cauchy dans $\mathbb{R}$, donc converge

$\rightsquigarrow$ semi-norme $|\cdot|$ sur S $\rightsquigarrow$ valeur absolue sur $\hat{K} = S/M$
$M = \mathrm{Ker}$ (seminorme)

Si K est non archimédien :

* $\hat{K}$ aussi

* $a \in S$, $\begin{cases} m, n \geq p \Rightarrow |a_m - a_n| \leq \varepsilon/2 \\ |a_m| \geq \varepsilon \quad \text{pour une infinité de } m \end{cases}$

$$\left.\begin{aligned} |a_m| &\geq \varepsilon \\ |a_m - a_n| &\leq \varepsilon/2 \end{aligned}\right] \Rightarrow |a_n| = |a_m|$$

donc la suite $(|a_n|)$ est stationnaire

les valeurs de la valeur absolue sur $\hat{K}$ sont les mêmes que celle sur K.

Set $b_n = 0$ for $n < p$ and $b_n = 1/a_n$ for $n \geqslant p$. The inequalities

$$\left|b_m - b_n\right| = \frac{|a_n - a_m|}{|a_m||a_n|} \leqslant \frac{4}{\varepsilon^2}|a_n - a_m|,$$

for $m, n \geqslant p$, imply that $b = (b_n)$ is a Cauchy sequence. Moreover, ab converges to 1, hence the equality $[a][b] = j(1)$ in $\widehat{\mathrm{K}}$. This proves that $\widehat{\mathrm{K}}$ is a field.

Assume that the initial absolute value of K is nonarchimedean. The obtained absolute value on $\widehat{\mathrm{K}}$ is then nonarchimedean as well. Moreover, with the previous notation, we have $|a_n| = |a_m|$ for all integers $m, n \geqslant p$: if the Cauchy sequence (a_n) does not converge to 0, then the sequence $(|a_n|)$ is eventually constant. In particular, the value group of $\widehat{\mathrm{K}}$ is the same as that of K.

Example **(3.1.11)**. — Let K be a nonarchimedean valued field. It is known that the absolute value of K extends to an absolute value on any algebraic extension of K.

More precisely, if K is complete, then for every algebraic extension L of K, there exists a *unique* extension absolute value on L that extends the absolute value of K. I refer to **?**, theorem 5.1, for a detailed proof. Let us just mention that when the extension $\mathrm{K} \to \mathrm{L}$ is finite, the absolute value of L is given by the formula

$$|b|_{\mathrm{L}} = |\mathrm{N}_{\mathrm{L}/\mathrm{K}}(b)|^{1/[\mathrm{L}:\mathrm{K}]},$$

for every $b \in \mathrm{L}$.

(Dwork, Gerotto, Sullivan, 1994)

passage à une extension algébrique.

si K est complet, sa valeur absolue s'étend de manière unique à toute extension algébrique de K.

1) il suffit de traiter le cas des extensions finies.

2) — quasigaloisiennes (= normales)
Aut(L/K) agit transitivement sur les conjugués d'un élément
→ transf. K linéaires d'un esp. vect. de dim finie
→ automatiquement continus,
les conjugués ont même valeur absolue qu'un élément

$b \in L$ de pol. minimal $f = (T - b_1)\cdots(T - b_n)$

$|b_i| = |b| \Rightarrow |b| = |b_1 \cdots b_n|^{1/n} = |N_{L/K}(b)|^{1/[L:K]}$

Il reste à prouver que cette formule $|b|_L = |N_{L/K}(b)|^{1/[L:K]}$ convient

[utilise que K est complet]

- <u>multiplicatif</u> $N(bb') = N(b)\, N(b')$
- <u>sous-additivité</u> ? $|1+b|_L \leq 1$ si $|b|_L \leq 1$.

pol. ~~caractéristique~~ minimal de b $\quad f = T^n + a_1 T^{n-1} + \dots + a_n = \prod_{j=1}^{n} (T - b_j)$

$n = [L:K]$

~~dét~~

$P_b = f^m$

f irréductible

$$N_{L/K}(b) = (-1)^n a_n$$

On prouve que si $|a_n| \leq 1$, alors $|a_1|, \dots, |a_{n-1}| \leq 1$.

<u>par l'absurde</u> sinon il existe i tq $|a_i| \geq |a_j|$ $\forall j$

et $|a_i| > 1$. $\quad$ i minimal

$$\frac{f}{a_i} = \frac{1}{a_i} T^n + \frac{a_1}{a_i} T^{n-1} + \dots + 1 \cdot T^{n-i} + \dots + \frac{a_n}{a_i} \in R[T]$$

modulo M $\quad \frac{f}{a_i} \equiv T^{n-i} + \dots$

$$f \equiv \left(T^{n-i} + \frac{a_{i-1}}{a_i} T^{n+1-i} + \dots + \frac{a_n}{a_i} \right) \times \left(1 + a_i T^i \right)$$

factorisation approchée de f $\quad f \equiv g h \quad (\bar g, \bar h) = 1$ car $\bar h = 1$

$\rightsquigarrow$ méthode de Newton / lemme de Hensel

fournit une factorisation exacte $f = g' h'$

$$\begin{cases} g \equiv g' \mod M \\ h \equiv h' \mod M \end{cases}$$

cette factorisation contredit l'irréductibilité de f.

$b \in L^\times$, $|N_{L/K}(b)| \leq 1 \Rightarrow$ le pol. minimal de b appartient à $R[T]$

$\Rightarrow$ le pol. car. f aussi

$$N_{L/K}(1+b) = \pm f(-1) \in R$$

$$\Rightarrow |N_{L/K}(1+b)| \leq 1.$$

3.2. The analytic spectrum of a ring

Definition **(3.2.1)** **(?)** — *Let* K *be a field endowed with a nonarchimedean absolute value and let* R *be a* K*-algebra. The* analytic spectrum *of* R *is the set of all multiplicative seminorms on* R *which restrict to the given absolute value on* K*, endowed with the coarsest topology for which the maps from* R *to* **R**, $f \mapsto p(f)$*, are continuous, for every* $f \in$ R*. It is denoted by* $\mathscr{M}(\mathrm{R})$.

If R is the ring of an affine K-scheme X, hence X = Spec(R), then the analytic spectrum of R is also called the (Berkovich) *analytification* of X, and is denoted by X^{an}.

Let J be a ideal of R and let $\mathscr{V}(\mathrm{J})$ be the subset of $\mathscr{M}(\mathrm{R})$ consisting of all seminorms p such that $p(f) = 0$ for every $f \in \mathrm{J}$. It is a closed subset of $\mathscr{M}(\mathrm{R})$, For each $f \in \mathrm{R}$, the set of all seminorms p on R such that $p(f) = 0$ is closed, as the preimage of the closed set $\{0\}$ by the continuous map $f \mapsto p(f)$ on $\mathscr{M}(\mathrm{R})$. Therefore, $\mathscr{V}(\mathrm{J})$ is the intersection of a family of closed subsets of $\mathscr{M}(\mathrm{R})$, hence is closed.

If X = Spec(R), the following proposition shows that $\mathscr{V}(\mathrm{J})$ identifies with the analytification of V(J) = Spec(R/J).

Proposition **(3.2.2)**. — *Let* K *be a field endowed with nonarchimedean absolute value.*

a) *If* $\varphi : \mathrm{R} \to \mathrm{S}$ *is a morphism of* K*-algebras, then the map* $\varphi^* : p \mapsto p \circ \varphi$ *is a continuous map from* $\mathscr{M}(\mathrm{S})$ *to* $\mathscr{M}(\mathrm{R})$.

Analogue non archimédien de la tropicalisation

$(\mathbb{C}^\times)^n \longrightarrow \mathbb{R}^n$ $(z_1, \dots, z_n) \mapsto (\log|z_j|)$

si K est un corps valué non archimédien.

Point crucial :

$(K^\times)^n \longrightarrow \mathbb{R}^n$ $(a_1, \dots, a_n) \mapsto (\log|a_j|)$

ne suffit pas

par exemple * parce que K n'est pas forcément alg clos

* $\log|a_j|$ peut être dans un sous groupe strict de $\mathbb{R}$ ($K = \mathbb{Q}_p$, $\overline{\mathbb{Q}_p}$, $|K^\times| = p^{\mathbb{Z}}$, ou $p^{\mathbb{Q}}$)

et on ne peut pas espérer voir apparaître des ens. polyédraux ...

Kapranov (- Einsiedler - Lind)

| prennent une adhérence dans $\mathbb{R}^n$ au moment opportun.
| on perd le caractère intuitif

Spectre analytique traduit le fait que l'on peut vouloir regarder tous les corps valués qui contiennent K.

schéma anneau R ⇝ Spec (R) = {idéaux premiers de R}

$R = K[T_1, \dots, T_n] / (f_1, \dots, f_m)$

Spec (R) = {idéaux premiers de $K[T_1, \dots, T_n]$ qui contiennent les f_j}

$(a_1, \dots, a_n)$ est solution du système d'équations polynomiales ($f_j = 0$) dans le corps $\kappa(P)$

P → $a_1, \dots, a_n \in R/P$ anneau quotient intègre

$\subset \kappa(P)$ corps des fractions $= \mathrm{Frac}(R/P)$

↓ classes des T_i

$f_j \in P$

$f_j(a_1, \dots, a_n) = 0$ dans $\kappa(P)$

Definition **(3.2.1)** **(?)** — *Let* K *be a field endowed with a nonarchimedean absolute value and let* R *be a* K*-algebra. The* analytic spectrum *of* R *is the set of all multiplicative seminorms on* R *which restrict to the given absolute value on* K*, endowed with the coarsest topology for which the maps from* R *to* **R**, $f \mapsto p(f)$*, are continuous, for every* $f \in$ R*. It is denoted by* $\mathscr{M}$(R).

If R is the ring of an affine K-scheme X, hence X = Spec(R), then the analytic spectrum of R is also called the (Berkovich) *analytification* of X, and is denoted by X^{an}.

Let J be a ideal of R and let $\mathscr{V}$(J) be the subset of $\mathscr{M}$(R) consisting of all seminorms p such that $p(f) = 0$ for every $f \in$ J. It is a closed subset of $\mathscr{M}$(R), For each $f \in$ R, the set of all seminorms p on R such that $p(f) = 0$ is closed, as the preimage of the closed set $\{0\}$ by the continuous map $f \mapsto p(f)$ on $\mathscr{M}$(R). Therefore, $\mathscr{V}$(J) is the intersection of a family of closed subsets of $\mathscr{M}$(R), hence is closed.

If X = Spec(R), the following proposition shows that $\mathscr{V}$(J) identifies with the analytification of V(J) = Spec(R/J).

base d'ouverts :

$U_{\Omega,f} = \{ p \mid p(f) \in \Omega \}$

pour $\begin{cases} f \in R \\ \Omega \subset \mathbb{R} \text{ ouvert} \end{cases}$

$X = \operatorname{Spec}(R) \qquad \mathcal{M}(R) = X^{an}$

$\mathcal{M}(R) = \{ p : R \to \mathbb{R}_+ \mid p(0) = 0,\ p(1) = 1,\ p(ab) = p(a)\,p(b)\ \forall a,b,\ p(a+b) \leq \sup(p(a), p(b)) \}$

un point de $\mathcal{M}(R)$ est une seminorme.

évaluation $f \in R$ $\quad ev_f : \mathcal{M}(R) \to \mathbb{R}_+$, $\quad p \longmapsto p(f)$

Topologie sur $\mathcal{M}(R)$ = la moins fine telle que toutes ces applications d'évaluation sont continues.

Pour qu'une application d'un espace topologique X $\varphi : X \to \mathcal{M}(R)$ dans $\mathcal{M}(R)$ soit continue, il faut et il suffit que pour tout $f \in R$, l'application $x \mapsto \varphi(x)(f)$ soit continue.

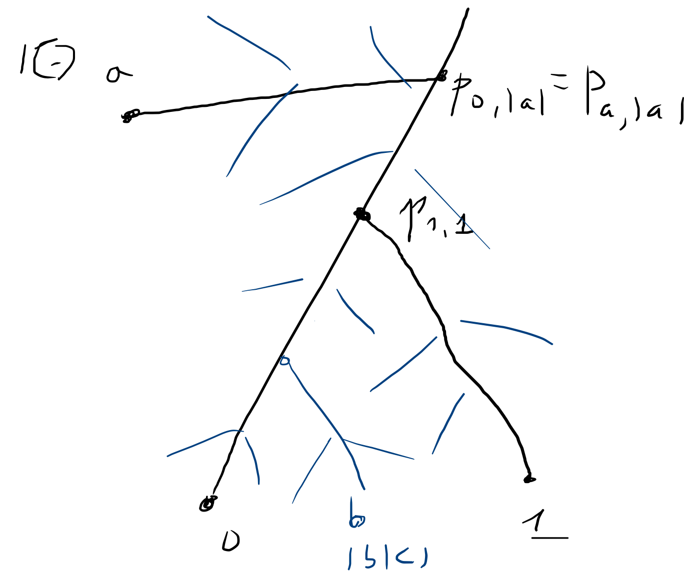
$ev_f \circ \varphi$

K (complet, alg. clos)

$R = K[T]$

semi normes : . $a \in K$ $\quad p_a : f \longmapsto |f(a)|$

(J. Poineau la géométrie du brocoli)



Gauss . $r \in \mathbb{R}_+$, $a \in K$
$$\left\{ \begin{array}{l} f = \sum c_m (T-a)^m \\ p_{a,r}(f) = \sup_m |c_m| r^m \end{array} \right.$$

(il y en a parfois d'autres).

$\mathbb{R}_+ \longrightarrow \mathcal{M}(R) \qquad r \longmapsto p_{0,r}$ est continue

car $r \longmapsto p_{0,r}(f) = \sup |c_m| r^m$ est continue.

$f = \sum c_m T^m$

$p_{0,0}(f) = |f(0)|$ $\qquad p_{0,0} = p_0$

$r \mapsto p_{1,r}$

$p_{1,0} = p_1$

$p_{0,1} = p_{1,0}$

$$\boxed{p_{0,r} = p_{1,r} \iff r \geq 1}$$

Let J be a ideal of R and let $\mathscr{V}(J)$ be the subset of $\mathscr{M}(R)$ consisting of all seminorms p such that $p(f) = 0$ for every $f \in J$. It is a closed subset of $\mathscr{M}(R)$, For each $f \in R$, the set of all seminorms p on R such that $p(f) = 0$ is closed, as the preimage of the closed set $\{0\}$ by the continuous map $f \mapsto p(f)$ on $\mathscr{M}(R)$. Therefore, $\mathscr{V}(J)$ is the intersection of a family of closed subsets of $\mathscr{M}(R)$, hence is closed.

If X = Spec(R), the following proposition shows that $\mathscr{V}(J)$ identifies with the analytification of V(J) = Spec(R/J).

Proposition **(3.2.2)**. — *Let* K *be a field endowed with nonarchimedean absolute value.*

a) *If* $\varphi : R \to S$ *is a morphism of* K*-algebras, then the map* $\varphi^* : p \mapsto p \circ \varphi$ *is a continuous map from* $\mathscr{M}(S)$ *to* $\mathscr{M}(R)$.

$f \in R$ $\quad \{ p \mid p(f) = 0 \}$ est une partie fermée de $\mathcal{M}(R)$

$\mathcal{V}(J) = \{ p \mid p(f) = 0 \ \forall f \in J \}$ est fermée

$J \subset R$ idéal

seminorms multiplicatives sur R tq $p(f) = 0 \quad \forall f \in J$

$\Leftrightarrow$ semi norms multiplicatives sur R/J.

$\mathcal{M}(R) \supset \mathcal{V}(J) \simeq \mathcal{M}(R/J)$ homéomorphe

preuve. exercice de topologie produit.

b) *If φ is surjective, then φ^* induces a homeomorphism from $\mathscr{M}(\mathrm{S})$ to its image, which is a closed subset of $\mathscr{M}(\mathrm{R})$.*

Proof. — a) To prove that φ^* is continuous, it suffices, by the definition of the topology of $\mathscr{M}(\mathrm{R})$, to prove that for every $f \in \mathrm{R}$, the map $p \mapsto \varphi^*(p) = p \circ \varphi(f)$ from $\mathscr{M}(\mathrm{S})$ to $\mathbf{R}$ is continuous. But this follows from the fact the definition of the topology of $\mathscr{M}(\mathrm{S})$.

b) Assume that φ is surjective and let $\mathrm{J} = \mathrm{Ker}(\varphi)$. Multiplicative seminorms on S then correspond, via φ, to multiplicative seminorms on R which vanish on J. Consequently, φ^* is injective and its image is the closed subset $\mathscr{V}(\mathrm{J})$ of $\mathscr{M}(\mathrm{R})$ consisting of all seminorms p such that $p(f) = 0$ for every $f \in \mathrm{J}$. Let us prove that the inverse bijection, $(\varphi^*)^{-1} \colon \mathscr{V}(\mathrm{J}) \to \mathscr{M}(\mathrm{S})$, is continuous. By the definition of the topology of $\mathscr{M}(\mathrm{S})$, it suffices to prove that for every $f \in \mathrm{S}$, the map from $\mathscr{V}(\mathrm{J})$ to $\mathbf{R}$ given by $p \mapsto (\varphi^*)^{-1}(p)(f)$ is continuous. Let $g \in \mathrm{R}$ be such that $f = \varphi(g)$. For every $q \in \mathscr{M}(\mathrm{S})$, one has $\varphi^*(q) = q \circ \varphi$, hence $\varphi^*(q)(g) = q \circ \varphi(g) = q(f)$; if $p = \varphi^*(q) \in \mathscr{V}(\mathrm{J})$, one thus has $q = (\varphi^*)^{-1}(p)$ and $(\varphi^*)^{-1}(p)(f) = p(g)$. By definition of the topology of $\mathscr{M}(\mathrm{R})$, the map $p \mapsto p(g)$ is continuous on $\mathscr{M}(\mathrm{R})$, so that the requested map is continuous on $\mathscr{V}(\mathrm{J})$, as the restriction of a continuous map. □

Theorem **(3.2.3)**. — *Let* R *be a finitely generated* K*-algebra and let $f = (f_1, \dots, f_n)$ be a generating family. The continuous map $\mathscr{M}(\mathrm{R})$ to $\mathbf{R}^n$*

Theorem **(3.2.3)**. — *Let* R *be a finitely generated* K*-algebra and let* $f = (f_1, \dots, f_n)$ *be a generating family. The continuous map* $\mathscr{M}(\mathrm{R})$ *to* $\mathbf{R}^n$ *given by* $p \mapsto (p(f_1), \dots, p(f_n))$ *is proper. In particular,* $\mathscr{M}(\mathrm{R})$ *is a locally compact topological space.*

Proof. — Let $\varphi : \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n] \to \mathrm{R}$ be the unique morphism of K-algebras such that $\varphi(\mathrm{T}_j) = f_j$ for all $j \in \{1, \dots, n\}$. Since it induces a closed embedding of $\mathscr{M}(\mathrm{R})$ into $\mathscr{M}(\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n])$, it suffices to treat the case where $\mathrm{R} = \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ and $f_j = \mathrm{T}_j$ for all j.

For $r \in \mathbf{R}$, the set V_r of all $p \in \mathscr{M}(\mathrm{R})$ such that $p(\mathrm{T}_j) < r$ for all j is open in $\mathscr{M}(\mathrm{R})$ and the union of all V_r is equal to $\mathscr{M}(\mathrm{R})$. Moreover, the closure of V_r is contained in the set W_r of all $p \in \mathscr{M}(\mathrm{R})$ such that $p(\mathrm{T}_j) \leqslant r$ for all j. Consequently, to prove that $\mathscr{M}(\mathrm{R})$ is locally compact, it suffices to prove that W_r is compact.

The map $j : \mathscr{M}(\mathrm{R}) \to \mathbf{R}_+^{\mathrm{R}}$ given by $p \mapsto (p(f))$ is continuous, by definition of the topology of $\mathscr{M}(\mathrm{R})$ and of the product topology. It is injective, by the definition of a seminorm. Moreover, its image is the subset of $\mathbf{R}_+^{\mathrm{R}}$ defined by the relations in the definition of a multiplicative seminorm, each of them defining a closed subset of $\mathbf{R}_+^{\mathrm{R}}$ since it involves only finitely many elements of R. Finally, j is a homeomorphism onto its image. Indeed, the inverse bijection associates to a family $c = (c_f)$ the multiplicative seminorm $f \mapsto c_f$. To prove that j^{-1} is continuous, it suffices to prove that for every $f \in \mathrm{R}$, the composition $c \mapsto j^{-1}(c)(f) = c_f$ is continuous, which is true by the definition of the product topology.

$\mathcal{M}(R) \longrightarrow \mathbb{R}^n$

$p \longmapsto (p(f_1), \dots, p(f_n))$

. $R = K[T]$

$\mathcal{M}(R) \to \mathbb{R}$, $p \mapsto p(T)$

. cette application est ~~propre~~ continue

- universellement fermée
- image réciproque d'un compact est compacte

. $\mathcal{M}(R)$ est localement compact.

$R = K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ anneau des pol de Laurent

$\lambda : \mathcal{M}(R) \longrightarrow \mathbb{R}^n$

$p \longmapsto (\log p(T_1), \dots, \log p(T_n))$

continue et propre

$f \in K[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] = R$

$V(f) \subset \mathcal{M}(R) = \mathbb{G}_m^n$

fermé

l'amibe non archimédienne de f

sera $\mathcal{A}_f = \lambda(V(f)) \subset \mathbb{R}^n$

(fermé).

On va démontrer que c'est un ensemble polyédral.

(Kapranov) (généralisation à tous les idéaux de R.)

et le munir de structure supplémentaires. (poids) vérifiant des relations remarquables (condition d'équilibre)

lien avec Bieri-Groves via l'analyse non standard.