TOPICS IN TROPICAL GEOMETRY

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As a consequence, one has

$$p(a+b) \leqslant (n+1)^{1/n} \sup(p(a), p(b)).$$

When $n \to +\infty$, we obtain the upper bound $p(a+b) \leqslant \sup(p(a), p(b))$; this proves that p is nonarchimedean. □

Example **(3.1.6)**. — A theorem of Ostrowski describes the multiplicative seminorms on the field $\mathbf{Q}$ of rational numbers.

a) The usual absolute value $|\cdot|$, and its powers $|\cdot|^r$ for $r \in]0;1]$;

b) For every prime number p, the p-adic absolute value $|\cdot|_p$, and its powers $|\cdot|_p^r$, for all $r \in]0;+\infty[$;

c) The trivial absolute value $|\cdot|_0$ defined by $|0|_0 = 0$ and $|a|_0 = 1$ for all $a \in \mathbf{Q}^\times$.

Example **(3.1.7)**. — Let $\mathscr{U}$ be an ultrafilter on $\mathbf{N}$ that contains the Fréchet filter: $\mathscr{U}$ is a set of $\mathfrak{P}(\mathbf{N})$ satisfying the following properties, for $A, B \subset \mathbf{N}$:

(i) If $\complement A$ is finite, then $A \in \mathscr{U}$;

(ii) If $A \subset B$ and $A \in \mathscr{U}$, then $B \in \mathscr{U}$;

(iii) If $A, B \in \mathscr{U}$, then $A \cap B \in \mathscr{U}$;

(iv) $\varnothing \notin \mathscr{U}$.

(v) If $A \notin \mathscr{U}$, then $\complement A \in \mathscr{U}$;

Lien entre amibes archimédiennes
et amibes non archimédiennes

Bieri Groves : on a admis temporairement que $\mathcal{T}_V$ (tropicalisation) était un ens. polyédral $\mathbb{Q}$-rationnel.

Ex. 3.1.7 corps des complexes « non standard » avec une valeur absolue non archimédienne qui reflète la géométrie « à grande distance ».

$A \in \mathcal{U} \Rightarrow A \subset \mathbb{N}$ « grande »

$A \notin \mathcal{U}$ « petite ».

In more elementary terms, elements of $\mathscr{U}$ are the subsets of $\mathbf{N}$ which are almost sure with respect to some $0/1$-valued finitely additive probability, and for which finite sets have probability 0.

$A \in \mathcal{U} \Rightarrow \mu(A) = 1$
$A \notin \mathcal{U} \Rightarrow \mu(A) = 0$
μ est une mesure finiment additive.

The existence of ultrafilters follows from Zorn's theorem, the set of subsets of $\mathfrak{P}(\mathbf{N})$ satisfying (i)–(iv) being inductive with respect to inclusion.

Members of a chosen (ultra)filter are sorts of neighborhoods of infinity. In particular, one can define the notion of *convergence along* $\mathscr{U}$ for a sequence (a_n): $\lim_{n,\mathscr{U}}(a_n) = a$ if for every neighborhood V of a, the set of $n \in \mathbf{N}$ such that $a_n \in \mathrm{V}$ belongs to $\mathscr{U}$. Every sequence with valued in a *compact* (Hausdorff) topological space has a unique limit along $\mathscr{U}$.

($\mathcal{U}$ ultrafiltre)

$(\mathbb{C}^*)^n \subset K_t^{*n} \xrightarrow{\log|\cdot|} \mathbb{R}^n$
$V(f)$ $\quad \mathcal{U}_{K_t}(f)$

Fix a sequence $t = (t_n)$ of strictly positive real numbers converging to $+\infty$. ($t_n = n$)

Let B_t, resp. Z_t, be the set of all sequences $(a_n) \in \mathbf{C}^{\mathbf{N}}$ such that $\lim_{n,\mathscr{U}} |a_n|/t_n < \infty$, resp. $\lim_{n,\mathscr{U}} |a_n|/t_n = 0$. The set B_t is a subring of the product ring $\mathbf{C}^{\mathbf{N}}$, and Z_t is a maximal ideal of B_t. The quotient $\mathbf{C}$-algebra $\mathrm{K}_t = \mathrm{B}_t/\mathrm{Z}_t$ is an algebraically closed field. The map $(a_n) \mapsto \lim_{n,\mathscr{U}} |a_n|/t_n$ gives rise to an absolute value on K_t which restricts to the trivial absolute value on $\mathbf{C}$. In particular, it is nonarchimedean.

$\frac{\log|a_n|}{t_n} \to \alpha \quad \frac{\log|b_n|}{t_n} \to \beta \quad \dots$

$P = T^d + \alpha_1 T^{d-1} + \dots + \alpha_0 \in K_t[T] \Rightarrow \alpha = 0$ ou $\beta = 0$

$\alpha_j = cl(a^{(j)}) \quad a^{(j)} \in B_t$

$P_n = T^d + a_n^{(1)} T^{d-1} + \dots + a_n^{(d)} \in \mathbb{C}[T]$

z_n racines $\quad (z_n) \in Z_t$

$a^{(j)} \in B \Rightarrow |a_n^{(j)}| \ll t_n^{c}$

$\Rightarrow |z_n| \ll t_n^{c'}$

sa classe est une racine de P.

The study of the logarithmic limit set of a complex variety amounts more or less to the study of the nonarchimedean amoeba of the associated K_t-variety.

that the requested map is continuous on $\mathscr{V}(J)$, as the restriction of a continuous map. $\square$

Theorem **(3.2.7)**. — *Let* R *be a finitely generated* K*-algebra and let* $f = (f_1, \dots, f_n)$ *be a generating family. The continuous map* $\mathscr{M}(R)$ *to* $\mathbf{R}^n$ *given by* $p \mapsto (p(f_1), \dots, p(f_n))$ *is proper. In particular,* $\mathscr{M}(R)$ *is a locally compact topological space.*

Proof. — Let $\varphi \colon K[T_1, \dots, T_n] \to R$ be the unique morphism of K-algebras such that $\varphi(T_j) = f_j$ for all $j \in \{1, \dots, n\}$. Since it induces a closed embedding of $\mathscr{M}(R)$ into $\mathscr{M}(K[T_1, \dots, T_n])$, it suffices to treat the case where $R = K[T_1, \dots, T_n]$ and $f_j = T_j$ for all j.

For $r \in \mathbf{R}$, the set V_r of all $p \in \mathscr{M}(R)$ such that $p(T_j) < r$ for all j is open in $\mathscr{M}(R)$ and the union of all V_r is equal to $\mathscr{M}(R)$. Moreover, the closure of V_r is contained in the set W_r of all $p \in \mathscr{M}(R)$ such that $p(T_j) \leqslant r$ for all j. Consequently, to prove that $\mathscr{M}(R)$ is locally compact, it suffices to prove that W_r is compact.

The map $j \colon \mathscr{M}(R) \to \mathbf{R}_+^R$ given by $p \mapsto (p(f))$ is continuous, by definition of the topology of $\mathscr{M}(R)$ and of the product topology. It is injective, by the definition of a seminorm. Moreover, its image is the subset of $\mathbf{R}_+^R$ defined by the relations in the definition of a multiplicative seminorm, each of them defining a closed subset of $\mathbf{R}_+^R$ since it involves only finitely many elements of R. Finally, j is a

$\varphi : k[T_1, \dots, T_n] \twoheadrightarrow R \qquad J = \mathrm{Ker}(\varphi)$

$T_j \mapsto f_j$

$\mathcal{M}(R) \xrightarrow{\varphi^*} \mathcal{M}(k[T_1, \dots, T_n])$

$p \longmapsto (f \mapsto p(\varphi(f))) = p \circ \varphi$

φ^* identifie $\mathcal{M}(R)$ au fermé $V(J)$ de $\mathcal{M}(k[T])$

= semi norms p tq $p(f) = 0 \ \forall f \in J$

$p \mapsto (p(f_1), \dots, p(f_n))$

$\mathbb{R}^n$

$(p(T_1), \dots, p(T_n))$

$r \in \mathbb{R}_+^*$

$V_r \subset W_r \subset \mathcal{M}(R)$

p tq $p(T_j) \leq r \ \forall j$

p tq $p(T_j) < r \ \forall j$

V_r ouvert $\qquad W_r$ fermé

$\bigcup V_r = \mathcal{M}(k[T])$

homeomorphism onto its image. Indeed, the inverse bijection associates to a family $c = (c_f)$ the multiplicative seminorm $f \mapsto c_f$. To prove that j^{-1} is continuous, it suffices to prove that for every $f \in R$, the composition $c \mapsto j^{-1}(c)(f)$ is continuous; since this map is the restriction of the projection $c \mapsto c_f$, this is indeed the case, by the definition of the product topology.

For $f \in R$, set $\|f\|_r = \sup_m |c_m| r^{|m|}$, where $f = \sum c_m T^m \in R$ and $|m| = m_1 + \cdots + m_n$. For every $p \in W_r$, one has $p(f) \leqslant \|f\|_r$, so that $j(W_r) \subset \prod_{f\in R}[0; \|f\|_r]$. According to Tikhonov's theorem, the latter set is compact, as a product of compact sets; consequently, W_r is homeomorp

compact.

By what precede

by the map $p \mapsto$ (

are locally compact, this implies that this map is proper (Bourbaki (1971), chap 1, §10, nº 3, prop. 7). □

Corollary **(3.2.8)**. — *Let* $X = \operatorname{Spec}(K[T_1^{\pm1}, \dots, T_n^{\pm1}])$. *The map* $\lambda \colon X^{an} \to \mathbf{R}^n$ *given by* $p \mapsto (\log(p(T_1)), \dots, \log(p(T_n)))$ *is surjective and proper. In particular, for every ideal* I *of* $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$, $\lambda(\mathscr{V}(I))$ *is a closed subset of* $\mathbf{R}^n$.

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$W_r = \{p \mid p(T_j) \le r \ \forall j\}$ compact

analyse fonctionnelle

(compacité de la boule unité pour la topologie faible-* (Banach–Alaoglu).

$M(R) \supset W_r$ · injective · continue $\longrightarrow \prod [0, \|f\|_r]$ compact (Tychonoff) $\subset \prod_{f\in k(T)} \mathbb{R}_+$

$f \in k[T]$

$p \longmapsto (p(f))$

$\|f\|_r = \sup_m |c_m| r^{|m|}$ si $f = \sum_m c_m T^m$

continuité : $p \longmapsto p(f)$ continue.

homéo. de W_r sur son image.

famille (c_f) tq

$c_a = |a|$ si $f = a \in K$

$c_{f+g} \le \sup(c_f, c_g)$

$c_{fg} = c_f c_g$

fermé pour la topologie produit

Corollary **(3.2.8).** — *Let* $\mathrm{X} = \mathrm{Spec}(\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}])$. *The map* $\lambda\colon \mathrm{X}^{\mathrm{an}} \to \mathbf{R}^n$ *given by* $p \mapsto (\log(p(\mathrm{T}_1)),\dots,\log(p(\mathrm{T}_n)))$ *is surjective and proper. In particular, for every ideal* I *of* $\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}]$, $\lambda(\mathscr{V}(\mathrm{I}))$ *is a closed subset of* $\mathbf{R}^n$.

Proof. — Let $x \in \mathbf{R}^n$ and let v_x be the Gauss absolute value of $\mathrm{K}(\mathrm{T}_1,\dots,\mathrm{T}_n)$ such that $v_x(\mathrm{T}_j) = e^{x_j}$ for all j. One has $\lambda(v_x) = x$, so that λ is surjective.

By theorem 3.2.7, the map

$$p \mapsto (\log(p(\mathrm{T}_1)),\dots,\log(p(\mathrm{T}_n)),\log(p(\mathrm{T}_1^{-1})),\log(p(\mathrm{T}_n^{-1})))$$

from $\mathrm{X}^{\mathrm{an}} \to \mathbf{R}^{2n}$ is continuous and proper. Its image is contained in the closed subspace L of $\mathbf{R}^{2n}$ defined by the equations $x_1 = x_{n+1}, x_2 = x_{n+2}, x_n = x_{2n}$, so that λ induces a continuous and proper map from X^{an} to L. The corollary follows from the fact that the linear projection $(x_1,\dots,x_{2n}) \mapsto (x_1,\dots,x_n)$ from $\mathbf{R}^{2n}$ to $\mathbf{R}^n$ induces an homeomorphism from L to $\mathbf{R}^n$. ☐

3.2.9. — The scheme $\mathrm{X} = \mathrm{Spec}(\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}])$ is the n-dimensional torus over K, the algebraic-geometry analogue of the complex manifold $(\mathbf{C}^*)^n$. The map λ is then the analogue of the tropicalization map $(\mathbf{C}^*)^n \to \mathbf{R}^n$, $(z_1,\dots,z_n) \mapsto (\log(|z_1|),\dots,\log(|z_n|))$ studied in chapter 2.

If I is an ideal of $\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}]$, then the closed subscheme V(I) of X has a Berkovich analytification $\mathscr{V}(\mathrm{I})$, naturally a closed subspace of $\mathrm{X}^{\mathrm{an}} = \mathscr{M}(\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}])$, and its image $\lambda(\mathscr{V}(\mathrm{I}))$ is the *tropicalization* of V(I).

In the algebraic geometry of schemes, one makes a careful distinction between the scheme X (or its closed subscheme V(I)) and its set

$X = \mathrm{Spec}(K[\underbrace{T_1^{\pm1},\dots,T_n^{\pm1}}_{R}]) =$

$X^{an} = \mathcal{M}(R)$

$\lambda : X^{an} \to \mathbb{R}^n$ continue propre

$p \mapsto (\log p(T_j))_j$

Conséquence

$I \subset R$ $\quad V(I) \subset X^{an}$

$= \mathcal{M}(R/I)$

$\lambda(V(I)) \subset \mathbb{R}^n$

fermé

tropicalisation de $V(I)$ / $V(I)$ Spec(R/I)

R est engendré par $T_1,\dots,T_n, T_1^{-1},\dots,T_n^{-1}$

$X^{an} \to \mathbb{R}^n \times \mathbb{R}^n$, $p \mapsto (p(T_1),\dots,p(T_n), p(T_1^{-1}),\dots,p(T_n^{-1}))$

continue et propre

image $= \{(p_1,\dots,p_n, \frac{1}{p_1},\dots,\frac{1}{p_n})$, $p_j \in \mathbb{R}_+^*\}$

(valeurs absolues de Gauss)

$$\begin{tikzcd}
\mathcal{M}(K[T_1^{\pm 1},\dots,T_n^{\pm 1}]) \arrow[r] \arrow[d, "p"'] \arrow[dr, "\lambda"] & \text{hyperbole}(p_1 p_{n+1} = \dots = p_n p_{2n} = 1) \subset \mathbb{R}_+^{2n} \arrow[r, "\log"] & \mathbb{R}^n \times \mathbb{R}^n \supset L \arrow[d] \\
(\mathrm{Pol}\ p(T_1), \dots) & \mathbb{R}^n & \mathbb{R}^n
\end{tikzcd}$$

L : $x_1 + x_{n+1} = \dots = x_n + x_{2n} = 0$

$I \underset{\text{idéal}}{\subset} K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

tropicalisation $\lambda(V(I))$ fermé dans $\mathbb{R}^n$

plus intéressante et plus naturelle que, par exemple,

$$\{(\log|z_1|, \dots, \log|z_n|) \mid z \in (K^\times)^n \cap V(I)\}.$$

même si K est algébriquement clos.

of points $X(K)$ with values in a given field. One has a natural identification of $X(K)$ with $(K^\times)^n$, an n-tuple $(z_1, \dots, z_n) \in (K^\times)^n$ being identified with the images of $T_1, \dots, T_n$ by a morphism of K-algebras from $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ to K; more generally, for any K-algebra L, the set $X(L)$ identifies with $(L^\times)^n$. Then, the set $V(I)(L)$ identifies with the set of elements $(z_1, \dots, z_n) \in (L^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in I$.

Similarly, a point in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ is a multiplicative seminorm p on this K-algebra. Its kernel $J_p = \{f\,;\, f(p) = 0\}$ is a prime ideal of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ and p induces a multiplicative norm on the quotient K-algebra $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/J_p$, and then on its field of fractions L_p which is a an extension of K endowed with an absolute value that extends the absolute value on K. The field L_p is generated by the images $z_1, \dots, z_n$ of $T_1, \dots, T_n$ by the morphism of K-algebras $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \to L_p$, and the condition $p \in \mathscr{V}(I)$ is equivalent to the condition $I \subset J_p$, or to the condition $f(z_1, \dots, z_n) = 0$ for all $f \in I$. Conversely, any valued extension L of K and any family $(z_1, \dots, z_n) \in (L^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in I$ gives rise to a point in $\mathscr{V}(I)$, given by the multiplicative seminorm $f \mapsto |f(z_1, \dots, z_n)|$ on $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

Consequently, the tropicalization of $V(I)$ is the set of all $x \in \mathbf{R}^n$ for which there exists a valued extension L of K and a family

$(z_1, \dots, z_n) \in (\mathrm{L}^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in \mathrm{I}$ and $(\log(|z_1|), \dots, \log(|z_n|)) = (x_1, \dots, x_n)$.

3.3. Nonarchimedean amoebas of hypersurfaces

3.3.1. — Let K be a field endowed with a nonarchimedean absolute value. Let R be the valuation ring of K, let k its residue field and red : R $\to k$ the reduction morphism; it maps the maximal ideal to 0 and induces a morphism of groups from $\mathrm{R}^\times$ to $k^\times$.

The map from $v : \mathrm{K}^\times$ to $\mathbf{R}$ given by $a \mapsto -\log(|a|)$ is a morphism of groups. Let Γ be its image. One says that the given valued field K is split if we are given a *section* of the surjective map v. Such a section does not exist in general, but it does exist in the following two important cases:

$\Gamma \simeq \mathbb{Z}$

– Assume that K is discretely valued. Then R is a discrete valuation ring. If t is a given generator of its maximal ideal, one has $\Gamma = \mathbf{Z}\log(|t|)$ and the map $n\log(|t|) \mapsto t^n$ is a section as required.

$\Gamma \simeq \mathbb{Q}\cdot\omega$.

– If K is algebraically closed, then such a section also exists, by an abstract homological algebra argument. Indeed, in this case, $\mathrm{R}^\times$ is a divisible abelian group, hence an injective $\mathbf{Z}$-module. In a more elementary way, one can also use the fact that Γ is a uniquely divisible abelian group, hence a $\mathbf{Q}$-vector space. It then suffices to choose, in a compatible manner, nth roots of a given element of $\mathrm{K}^\times$.

. K corps + valeur absolue non archimédienne

. $R = \{a \in K,\ |a| \leq 1\}$

anneau de valuation

. M = idéal maximal $= \{a \mid |a| < 1\}$

. $k = R/M$ corps résiduel.

v valuation : $a \mapsto -\log(|a|)$

$v(a+b) \geq \inf(v(a), v(b))$

Γ groupe de valeurs de K

$= \log(|K^\times|) \subset \mathbb{R}$

image de v

: $a \mapsto \log(|a|)$

K scindé : si $K^\times \to \Gamma$, $a \mapsto \log(|a|)$

a une section.

$\gamma \mapsto t^\gamma$ où $t^\gamma \in K^\times$

$\log|t^\gamma| = \gamma$

morphisme de groupes

$t^{\gamma+\gamma'} = t^\gamma t^{\gamma'}$

Let $\gamma \in \Gamma$ and let $a \in \mathrm{K}^\times$ be such that $\log(|a|) = \gamma$. Let us choose inductively elements $a_n \in \mathrm{K}^\times$ such that $a_1 = a$ and $(a_n)^n = a_{n-1}$ for all integers $n \geqslant 2$. In particular $(a_n)^{n!} = a$ for all $n \geqslant 1$. Moreover, if $m \geqslant n$, then n divides $m!$ and we see by induction that $(a_m)^{m!/n} = (a_n)^{n!/n} = (a_n)^{(n-1)!}$. Then there is a unique morphism of groups from $\mathbf{Q}\gamma$ to $\mathrm{K}^\times$ that maps $\frac{m}{n}\gamma$ to $(a_n)^{(n-1)!m}$ for all $m, n \in \mathbf{Z}$ such that $n \geqslant 1$.

If K is a split valued field, then we can extend the morphism of groups $\mathrm{red}\colon \mathrm{R}^\times \to k^\times$ to a morphism of monoids $\rho\colon \mathrm{K} \to k$, by setting $\rho(a) = \mathrm{red}(at^{-v(a)})$. Note that ρ restricts to a morphism of groups from $\mathrm{K}^\times$ to $k^\times$.

$R \xrightarrow{\text{réd}} k = R/M$

$R^\times \to k^\times$ morph. de groupes

$\cap$ $\quad = K^\times \quad \rho \quad k$ morph. de monoïdes $\quad K \ni 0$

$K^\times \to R^\times$

$a \mapsto a \cdot t^{+v(a)}$

$\log|t^\gamma| = \gamma$

$\log|t^{+v(a)}| = +v(a)$

$= -\log|a|$

Definition **(3.3.2)**. — *Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ be a Laurent polynomial; write $f = \sum c_m \mathrm{T}^m$.*

changements de signes

a) *The* tropical polynomial *associated with f is the map*

$$\tau_f : \mathbf{R}^n \to \mathbf{R}, \quad x \mapsto \sup_m \left(\log(|c_m|) + \langle m, x\rangle\right).$$

b) *The* tropical hypersurface *defined by f is the subset $\mathscr{T}_f$ of all $x \in \mathbf{R}^n$ such that there exist two distinct elements $m \in \mathbf{Z}^n$ such that $\tau_f(x) = \log(|c_m|) + \langle m, x\rangle$.*

$f = \sum c_m T^m$

• $\tau_f(x) = \sup_m \big(\underbrace{\log|c_m| + \langle m, x\rangle}_{\log|c_m a^m|}\big)$

si $a \in (L^\times)^n$ $\quad L \supset K$ corps valué

$|f(a)| \le \tau_f(x)$ si $x = \lambda(a)$

c) *(Assuming that the valued field* K *is split.) For* $x \in \mathbf{R}^n$, *the* initial form *of* f *at* x *is the Laurent polynomial*

$$\operatorname{in}_x(f) = \sum_{\substack{\tau_f(x)=\log(|c_m|)+\langle m,x\rangle \\ m \in S_x(f)}} \overset{\neq 0}{\rho(c_m)} \mathrm{T}^m.$$

Recall that the *support* of a Laurent polynomial $f = \sum c_m \mathrm{T}^m$ is the set $\mathrm{S}(f)$ of all $m \in \mathbf{Z}^n$ such that $c_m \neq 0$, and that the Newton polytope of f is the convex hull NP_f of $\mathrm{S}(f)$ in $\mathbf{R}^n$.

We will occasionally define $\mathrm{S}_x(f)$ to be the subset of $\mathrm{S}(f)$ consisting of those m such that $\tau_f(x) = \log(|c_m|) + \langle m, x\rangle$; this is the support of the initial form $\operatorname{in}_x(f)$; its convex hull is then a sub-polytope $\mathrm{NP}_{f,x}$ of NP_f.

tout le temps.

With this notation, the tropical hypersurface $\mathscr{T}_f$ is the set of all $x \in \mathbf{R}^n$ such that $\mathrm{S}_x(f)$ has at least two elements, equivalently, $\mathrm{NP}_{f,x}$ is not a point. When K is a split valued field, this is also equivalent to the property that $\operatorname{in}_x(f)$ is not a monomial (or zero).

Remark **(3.3.3)**. — Let $\varphi : \mathbf{G}_{\mathrm{m}_\mathrm{K}}^n \to \mathbf{G}_{\mathrm{m}_\mathrm{K}}^p$ be a *monomial* morphism, given at the level of Laurent polynomials by a morphism of K-algebras $\varphi^* : \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_p^{\pm1}] \to \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ of the form $\mathrm{T}_j \mapsto a_j \mathrm{T}^{e_j}$, where $a_1, \dots, a_p \in \mathrm{K}^\times$ and $e_1, \dots, e_p \in \mathbf{Z}^n$. If L is an extension of K, this morphism φ maps a point $z = (z_1, \dots, z_n) \in (\mathrm{L}^\times)^n$ to the point $\varphi(z) = (a_1 z^{e_1}, \dots, a_p z^{e_p})$.

$f = \sum_m c_m T^m$

$S(f)$

$S_x(f) = \{m \text{ tq } \log|c_m| + \langle m, x\rangle = \tau_f(x)\}$

$\operatorname{in}_x(f) = \sum_{m \in S_x(f)} \rho(c_m) T^m \in k[T^{\pm 1}]$

$NP_f = \operatorname{conv}(S(f))$

$NP_{f,x} = \operatorname{conv}(S_x(f))$

$\operatorname{in}_x(f)$ est un monôme

$\iff$ $S_x(f)$ est un singleton

$\iff$ $x \notin \mathscr{T}_f$

This morphism gives rise to an affine map $\varphi_\tau : \mathbf{R}^m \to \mathbf{R}^p$, given by $x = (x_1, \dots, x_n) \mapsto (\log(|a_1|) + \langle e_1, x\rangle, \dots)$ and to monomial morphism $\varphi_\rho : \mathbf{G}_{\mathrm{m}_k}^n \to \mathbf{G}_{\mathrm{m}_k}^p$ given by $z \mapsto (\alpha_1 z^{e_1}, \dots, z_p^{e_p})$, where $\alpha_1 = \rho(a_1), \dots, \alpha_p = \rho(a_p)$.

Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_p^{\pm1}]$; write $f = \sum_{m\in\mathbf{Z}^p} c_m \mathrm{T}^m$ so that

$$\varphi^*(f) = \sum_{m\in\mathbf{Z}^p} c_m a_1^{m_1} \dots a_p^{m_p} \mathrm{T}^{m_1e_1+\dots+m_pe_p}.$$

If the rank of $(e_1, \dots, e_p) \in \mathrm{M}_{n,p}(\mathbf{Z})$ is equal to p, then all exponents $m_1e_1 + \dots + m_pe_p$ are pairwise distinct. This implies that

$$\begin{aligned}\tau_{\varphi^*(f)}(x) &= \sup_m \Big(\log(|c_m|) + m_1 \log(|a_1|) + \dots + m_p \log(|a_p|) \\ &\qquad\qquad + \langle m_1e_1 + \dots + m_pe_p, x\rangle \Big) \\ &= \sup_m \Big(\log(|c_m|) + m_1(\log(|a_1|) + \langle e_1, x\rangle) + \dots \\ &\qquad\qquad + m_p(\log(|a_p|) + \langle e_p, x\rangle) \Big) \\ &= \sup_m \Big(\log(|c_m|) + m_1y_1 + \dots + m_py_p \Big),\end{aligned}$$

where $y_j = \log(|a_j|) + \langle e_j, x\rangle)$ for $j \in \{1, \dots, p\}$. This shows that

$$\boxed{\tau_{\varphi^*(f)} = \tau_f \circ \varphi_\tau.}$$

Remark **(3.3.3)**. — Let $\varphi : \mathbf{G}_{\mathrm{m}_\mathrm{K}}^n \to \mathbf{G}_{\mathrm{m}_\mathrm{K}}^p$ be a *monomial* morphism, given at the level of Laurent polynomials by a morphism of K-algebras $\varphi^* : \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_p^{\pm1}] \to \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ of the form $\mathrm{T}_j \mapsto a_j\mathrm{T}^{e_j}$, where $a_1, \dots, a_p \in \mathrm{K}^\times$ and $e_1, \dots, e_p \in \mathbf{Z}^n$. If L is an extension of K, this morphism φ maps a point $z = (z_1, \dots, z_n) \in (\mathrm{L}^\times)^n$ to the point $\underline{\varphi(z) = (a_1z^{e_1}, \dots, a_pz^{e_p})}$.

$\varphi_\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ $\quad x = (x_1, \dots, x_n) \mapsto \left(\log|a_j| + \langle e_j, x\rangle\right)_j$

(affine)

$\varphi_\rho : \mathbb{G}_{m,k}^n \to \mathbb{G}_{m,k}^m$

$z \longmapsto (\rho(a_1)z^{e_1}, \dots, \rho(a_p)z^{e_p})$

$f \in K[T_1^{\pm1}, \dots, T_p^{\pm1}]$

$\varphi^*(f) \in K[T_1^{\pm1}, \dots, T_n^{\pm1}]$

$x \in \mathbb{R}^n$

$\downarrow$

$\varphi_\tau(x) = y \in \mathbb{R}^p$

$$\boxed{\begin{array}{c}\mathrm{in}_x(\varphi^*(f)) \\ \| \\ \mathrm{in}_y(f)\end{array}}$$

If K is a split valued field, we obtain similarly that

$$\operatorname{in}_x(\varphi^*(f)) = \sum_m \rho(c_m)\alpha_1^{m_1}\dots\alpha_p^{m_p}\mathrm{T}^{m_1e_1+\dots+m_pe_p} = \varphi_\rho^*(\operatorname{in}_{\varphi_\tau(x)}(f)).$$

généralise
$\deg(fg) = \deg(f)+\deg(g)$ dans $K(T_1 \dots T_n)$

Lemma **(3.3.4)**. — *Let $f, g \in \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ be nonzero Laurent polynomials and let $h = fg$. For every $x \in \mathbf{R}^n$, one has the following relations:*

(i) *$\tau_h(x) = \tau_f(x) + \tau_g(x)$;*
(ii) *$\mathrm{NP}_{h,x} = \mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$;* $\left(NP_h = NP_f + NP_g \right)$
(iii) *If K is a split valued field, then $\operatorname{in}_x(h) = \operatorname{in}_x(f)\operatorname{in}_x(g)$.*

Proof. — Write $f = \sum a_p \mathrm{T}^p$, $g = \sum b_q \mathrm{T}^q$ and $h = \sum c_m \mathrm{T}^m$. Let μ be a vertex of $\mathrm{NP}_{h,x}$ and let $\xi \in \mathbf{R}^n$ be such that $\langle m, \xi\rangle < \langle \mu, \xi\rangle$ for every $m \in \mathrm{NP}_{h,x}$ such that $m \neq \mu$. For $m \in \mathbf{Z}^n$, one has $c_m = \sum_{p+q=m} a_p b_q$, hence

$$\begin{aligned}\log(|c_m|) + \langle m, x\rangle &\leqslant \sup_{p+q=m} (\log(|a_p|) + \langle p, x\rangle) + (\log(|b_q|) + \langle q, x\rangle) \\ &\leqslant \sup_p(\log(|a_p|) + \langle p, x\rangle) + \sup_q(\log(|c_q|) + \langle q, x\rangle) \\ &= \tau_f(x) + \tau_g(x).\end{aligned}$$

This proves that $\tau_h(x) \leqslant \tau_f(x) + \tau_g(x)$.

On the other hand, if $p \in \mathrm{NP}_{f,x}$ and $q \in \mathrm{NP}_{g,x}$ are such that $p + q = \mu$, we have $\langle p, \xi\rangle + \langle q, \xi\rangle = \langle \mu, \xi\rangle$, so that the face of $\mathrm{NP}_{h,x}$ defined by ξ contains the Minkowski sum of the faces of $\mathrm{NP}_{f,x}$ and $\mathrm{NP}_{g,x}$ defined by ξ. This implies that these faces are vertices:

m sommet de $NP_{h,x} = \operatorname{conv}(S_x(h))$.

$$c_m = \sum_{p+q=m} a_p b_q.$$

$$|c_m| \le \sup_{p+q=m} |a_p|\,|b_q|.$$

$$\log|c_m| + \langle m, x\rangle \le \sup_{p+q=m} \Big(\underbrace{\log|a_p| + \langle p, x\rangle}_{\le \tau_f(x)} + \underbrace{\log|b_q| + \langle q, x\rangle}_{\le \tau_g(x)} \Big)$$

$$\tau_h(x) \le \tau_f(x) + \tau_g(x).$$

m sommet de $NP_{h,x}$

$\Rightarrow$ au plus une décomposit. $m = p+q$, $p \in N_{f,x}$, $q \in N_{g,x}$.

$$\Rightarrow \log|c_m| + \langle m, x\rangle = \log|a_p| + \langle p, x\rangle + \log|b_q| + \langle q, x\rangle = \tau_f(x) + \tau_g(x)$$

$f = f' + f''$ $\qquad f' = \sum_{m \in S_x(f)} a_m T^m$ $\qquad f'' = \sum_{m \notin S_x(f)} a_m T^m$

$g = g' + g''$

$h = f'g' +$ (trois autres termes)

coefficient plus petits.
qui ne comptent pas pour τ_h, dans le premier terme.
s'il n'y a pas de compens.

permet de supposer $S(f) = S_x(f)$ et $S(g) = S_x(g)$

$NP(h) \subset NP(f) + NP(g)$ $\qquad S(h) \subset S(f) + S(g)$

Prouvons l'égalité. $\qquad \xi \in \mathbb{R}^n$ qui définit un sommet m de $S(h)$

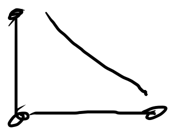
- $\langle m', \xi \rangle < \langle m, \xi \rangle \quad \forall m' \in S(h),\ m' \neq m$
- face de $S(f)$ définie par ξ $\quad F$
  face de $S(g)$ définie par ξ $\quad G$

$NP(h) \supset F + G$ $\qquad$ sur $F+G$, ξ est maximal

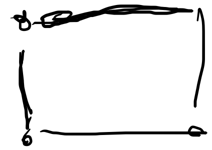
$F + G \subset$ face $\{m\}$.

$\Rightarrow F = \{p\} \quad G = \{q\}$.

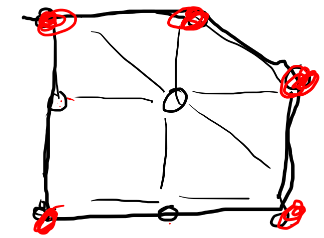
$m = p + q$.
unique décomposition
$c_m = a_p b_q$



$f \quad 1 + T_1 + T_2$



$g \quad 1 - T_1 + T_2 - T_1 T_2$



h

there exists a unique decomposition $\mu = p + q$, where p and q are vertices of $\mathrm{NP}_{f,x}$ and $\mathrm{NP}_{g,x}$ respectively. In the formula for c_μ, the term $a_p b_q$ has absolute value given by $\log(|a_p|) + \log(|b_q|) = \tau_f(x) + \tau_g(x) - \langle \mu, x\rangle$, while the absolute value of all other terms is strictly smaller. This shows that $\log(|c_\mu|) = \tau_f(x) + \tau_g(x) - \langle \mu, x\rangle$, hence $\tau_h(x) = \tau_f(x) + \tau_g(x)$.

This also shows that $\mathrm{NP}_{h,x}$ is equal to the Minkowski sum of $\mathrm{NP}_{f,x}$ and $\mathrm{NP}_{g,x}$. The arguments of the first part of the proof prove that $\mathrm{NP}_{h,x} \subset \mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$, while the second part of the proof shows that every vertex of $\mathrm{NP}_{h,x}$ belongs to the latter sum.

Let $m \in \mathbf{Z}^n$. If $\log(|c_m|) + \langle m, x\rangle < \tau_h(x)$, then the monomial T^m does not appear in $\mathrm{in}_x(h)$.

Otherwise, since $\tau_h(x) = \tau_f(x) + \tau_g(x)$, one has $\log(|a_p|) + \log(|b_q|) \leqslant \log(|c_m|)$ for every pair (p, q) such that $p + q = m$, and equality is achieved for at least one pair. In other words,

$v(a_p) + v(b_q) \geqslant v(c_m)$, with some pairs achieving equality. Consequently,

$$\begin{aligned} t^{-v(c_m)} c_m &= t^{-v(c_m)} \sum_{p+q=m} a_p b_q \\ &= \sum_{\substack{p+q=m \\ p\in \mathrm{S}_x(f) \\ q \in \mathrm{S}_x(g)}} t^{-v(a_p)} a_p t^{-v(b_q)} b_q \\ &\quad + \sum_{\text{other terms}} t^{v(a_p)+v(b_q)-v(c_m)} t^{-v(a_p)} a_p t^{-v(a_q)} b_q, \end{aligned}$$

a relation between elements of R. The reduction of the left hand side modulo the maximal ideal is the coefficient $\rho(c_m)$ of T^m in $\mathrm{in}_x(h)$. Similarly, if $p \in \mathrm{S}_x(f)$, then the reduction of $t^{-v(a_p)} a_p$ is $\rho(a_p)$; if $q \in \mathrm{S}_x(g)$, then the reduction of $t^{-v(b_q)} b_q$ is $\rho(b_q)$. On the other hand, if $v(a_p) + v(b_q) > v(c_m)$, then the reduction of corresponding term on the right hand side is zero. Consequently,

$$\rho(c_m) = \sum_{\substack{p+q=m \\ p\in \mathrm{S}_x(f) \\ q \in \mathrm{S}_x(g)}} \rho(a_p)\rho(b_q).$$

Since $\mathrm{in}_x(f) = \sum_{p\in \mathrm{S}_x(f)} \rho(a_p)\mathrm{T}^p$ and $\mathrm{in}_x(g) = \sum_{q\in \mathrm{S}_x(g)} \rho(b_q)\mathrm{T}^q$, this proves the coefficient of T^m in $\mathrm{in}_x(h)$ is equal to the coefficient

$c_m = \sum a_p b_q$

$t^{-v(c_m)} c_m = \sum_{p+q=m} t^{-v(a_p)} a_p \, t^{-v(b_q)} b_q$

$t^{v(a_p)+v(b_q)-v(c_m)}$

$= \sum_{\substack{p+q=m \\ p\in S_x(f) \\ q\in S_x(g)}} t^{-v(a_p)} a_p \, t^{-v(b_q)} b_q + \sum \underbrace{\text{autres termes}}_{|\ | < 1}$

$\rho(c_m) = \overline{t^{-v(c_m)} c_m} = \sum_{\substack{p\in S_x(f) \\ q\in S_x(g) \\ p+q=m}} \rho(a_p)\rho(b_q)$

$m \in S_x(h)$

$\mathrm{in}_x(h) = \mathrm{in}_x(f)\,\mathrm{in}_x(g)$

$m \in S_x(f) \iff x \in P_m$

of T^m in the product of $\mathrm{in}_x(f)$ and $\mathrm{in}_x(g)$. Consequently, $\mathrm{in}_x(h) = \mathrm{in}_x(f)\mathrm{in}_x(g)$, as claimed. □

$S_x(f) = \{m\}$.

$m \neq q \in S(f) \Rightarrow \log|c_q| + \langle q, x\rangle < \log|c_m| + \langle m, x\rangle$

l'inégalité stricte persiste au voisinage.

$S_y(f) = \{m\}$ au voisinage de x.

$\Rightarrow x \in \mathrm{int}(P_m)$.

cf description de l'épine

Proposition **(3.3.5)**. — *Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ be a Laurent polynomial. The associated tropical hypersurface $\mathscr{T}_f$ is a closed Γ-strict polyhedral subset of $\mathbf{R}^n$, purely of dimension $n-1$. More precisely, there exists a Γ-strict polyhedral decomposition of $\mathbf{R}^n$ the $(n-1)$-dimensional polyhedra of which $\mathscr{T}_f$ is the union.*

même preuve.

Proof. — Write $f = \sum c_m \mathrm{T}^m$; let $\mathrm{S}(f)$ be the support of f; for $x \in \mathbf{R}^n$, let $\mathrm{S}_x(f)$ be the set of all $m \in \mathrm{S}(f)$ such that $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$.

For every $m \in \mathrm{S}(f)$, let P_m be the the set of $x \in \mathbf{R}^n$ such that $m \in \mathrm{S}_x(f)$. Since P_m is defined in $\mathbf{R}^n$ by the affine inequalities $\log(|c_q|) + \langle q, x\rangle \leqslant \log(|c_m|) + \langle m, x\rangle$ for all $q \in \mathrm{S}(f)$, it is a convex polyhedron. The slopes of these affine forms are integers, and their constant terms are elements of the value group $\Gamma = \log(|\mathrm{K}^\times|)$ of K; consequently, P_m is a Γ-strict convex polyhedron. By construction, these polyhedra cover $\mathbf{R}^n$.[1]

If $\mathrm{S}_x(f)$ is reduced to an element m, then then there exists an open neighborhood V of x such that $\mathrm{S}_y(f) = \{m\}$ for all $y \in \mathrm{V}$; in particular, V is is disjoint from the other polyhedra P_q, and it is contained in the interior of P_m.

[1] Vérifier la terminologie sur les polyèdres stricts

$\lambda : \mathcal{M}(K[T_1^{\pm1},\dots,T_n^{\pm1}]) \longrightarrow \mathbb{R}^n$
$p \longmapsto (\log p(T_j))_j$

$\lambda_L : (L^\times)^n \longrightarrow \mathbb{R}^n$
$z \longmapsto (\log|z_j|)$

On the other hand, for two distinct elements m, q of $\mathrm{S}(f)$, the polyhedron $\mathrm{P}_m \cap \mathrm{P}_q$ is contained in the hyperplane defined by the nontrivial affine equation $\log(|c_m|) + \langle m, x\rangle = \log(|c_q|) + \langle q, x\rangle$, so that $\mathrm{P}_m \cap \mathrm{P}_q$ is disjoint from the interior of P_m. In particular, if $\mathrm{Card}(\mathrm{S}_x(f)) \geqslant 2$, then x does not belong to the interior of P_m.

This proves that $\mathbf{R}^n$ is the union of those polyhedra P_m which have dimension n, and that the union of their interiors is the set of all $x \in \mathbf{R}^n$ such that $\mathrm{S}_x(f)$ is reduced to one element.

Consequently, the tropical hypersurface $\mathcal{T}_f$, which is its complementary subset, is the union of the $(n-1)$-dimensional faces of these polyhedra P_m, and they are Γ-strict convex polyhedra. □

• $S_1 = S_2$ car $x \in \mathcal{T}_f \iff \mathrm{Card}(S_x(f)) \geqslant 2$
monômes de $\mathrm{in}_x(f)$.

• $S_3 = S_4$: $p \in \mathcal{M}(R)$ $\quad$ $\boxed{S_4 \subset S_3}$
$\mathrm{Fr}(K[T_1^{\pm1},\dots,T_n^{\pm1}]/\ker(p)) = L$
$a_i \in L^\times$ classe de T_i
si $p \in \mathcal{V}(f)$
alors $f(a) = 0$.

Theorem **(3.3.6)** (Kapranov). — *Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}]$ be a Laurent polynomial. The following four subsets of $\mathbf{R}^n$ coincide:*

S_1 (i) *The tropical hypersurface $\mathcal{T}_f$;* géométrie tropicale

approche de MacLagan

S_2 (ii) *Assuming that the valued field* K *is split, the set of all $x \in \mathbf{R}^n$ such that $\mathrm{in}_x(f)$ is not a monomial;*

S_3 (iii) *The set of all $x \in \mathbf{R}^n$ such that there exists a valued extension* L *of* K *and a point $z \in (\mathrm{L}^\times)^n$ such that $f(z) = 0$ and $x \in v(z)$;* $\lambda(z)$

S_4 (iv) *The image of $\mathcal{V}(f)$ in $\mathcal{M}(\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}])$ by the tropicalization map $\lambda : \mathcal{M}(\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}])$, $p \mapsto (\log(p(\mathrm{T}_1)),\dots,\log(p(\mathrm{T}_n)))$.*

approche espaces de Berkovich

$S_3 \subset S_4$ $\quad$ $f \mapsto |f(z)|$
semi norme mult. sur $K[T^{\pm1}]$

If L *is an algebraically closed extension of* K*, endowed with a nontrivial absolute value extending that of* K*, they also coincide with the set:*

• $S_5^L \subset \overline{S_4} = S_4$ (tropicalisation)

S_5^L (v) *The closure of the set of all $x \in \mathbf{R}^n$ such that there exists a point $z \in (\mathrm{L}^\times)^n$ such that $f(z) = 0$ and $x = \lambda(z)$.* Kapranov.

• $S_4 \subset S_1$ $\quad$ $f = \sum c_m T^m$ $\quad$ $f(z) = 0$
deux monômes maximaux au moins
si $S_x(f) = \{m\}$ $\quad$ $\lambda(z) = x$
$\|\log|f(z)|\| = \tau_f(x)$

(v) *The closure of the set of all* $x \in \mathbf{R}^n$ *such that there exists a point* $z \in (\mathrm{L}^\times)^n$ *such that* $f(z) = 0$ *and* $x = \lambda(z)$.

Proof. — Let $\mathrm{S}_1 = \mathscr{T}_f, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5$ be these subsets. Write $f = \sum c_m \mathrm{T}^m$.

Let $x \in \mathbf{R}^n$. Let $m \in \mathbf{Z}^n$; the monomial T^m appears in $\mathrm{in}_x(f)$ if and only if $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$. Consequently, $\mathrm{in}_x(f)$ is a monomial if and only if the supremum defining $\tau_f(x)$ is reached only once. This proves that $\mathrm{S}_1 = \mathrm{S}_2$.

The equality $\mathrm{S}_3 = \mathrm{S}_4$ follows from the discussion in §3.2.9.

Let L be a valued extension of K, let $z \in (\mathrm{L}^\times)^n$ be a point such that $f(z) = 0$ and let $x = \lambda(z)$. One has $\sum c_m z^m = 0$. Since the absolute value is nonarchimedean, the supremum of all $|c_m z^m|$ must be attained twice. Since $\lambda(c_m z^m) = \log(|c_m|) + \langle m, x\rangle$, this shows that there exist two distinct elements $m, q \in \mathbf{Z}^n$ such that $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$. In other words, x belongs to the hypersurface $\mathscr{T}_f$. This proves that one has $\mathrm{S}_3 \subset \mathrm{S}_1$.

By definition, the set S_5 is the closure of a subset of S_3; since S_1 is closed, one also has $\mathrm{S}_5 \subset \mathrm{S}_1$.

By the corollary to the lifting proposition below, one has $\mathscr{T}_f \cap \Gamma^n \in \mathrm{S}_5$. Since K is algebraically closed and its valuation is nontrivial, the group Γ is a non zero $\mathbf{Q}$-subspace of $\mathbf{R}$; in particular, it is dense in $\mathbf{R}$. On the other hand, since $\mathscr{T}_f$ is a Γ-strict polyhedral subspace of $\mathbf{R}^n$,

$S_5^L \subset S_3 = S_4 \subset S_1 = S_2$

Relèvement + corollaire (facile)

$S_1 \cap \Gamma_L^n \subset S_5^L \subset S_1$.

S_1 est une réunion de polyèdres Γ-stricts.

$\Rightarrow S_1 \cap \Gamma_L^n$ est dense dans S_1

$\Rightarrow S_5^L = S_1$ car S_5^L est fermé its subset $\mathscr{T}_f \cap \Gamma^n$ is dense in $\mathscr{T}_f$. Since $\mathscr{T}_f$ is closed in $\mathbf{R}^n$, this implies that $S_1 = \mathscr{T}_f \subset S_5$.

Using Gauss absolute values (proposition 3.1.10) and example 3.1.12, there exists an algebraically closed valued extension L of K whose value group Γ_L contains the coordinates of x. By the corollary of the lifting proposition, there exists $z \in (L^\times)^n$ such that $f(z) = 0$ and $\lambda(z) = x$; in other words, one has $x \in S_3$. Consequently, $S_1 \subset S_3$. This concludes the proof of the theorem. □

Proposition **(3.3.7)** (Lifting). — *Assume that* K *is an algebraically closed valued field with residue field k. Let $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be a Laurent polynomial. We assume that the coefficients of f belong to the valuation ring of* K *and that its reduction $\varphi \in k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ is nonzero.*

For every $\alpha \in (k^\times)^n$ such that $\varphi(f) = 0$, there exists $a \in (R^\times)^n$ such that $\rho(a) = \alpha$ and $f(a) = 0$. Moreover, if f is irreducible, then the set of such a is Zariski dense in the closed subscheme $V(f)$ *of* $\mathbf{G}_{m_K}^n$.

Proof. — We do the proof by induction on n.

Let us first assume that $n = 1$. Since K is algebraically closed, we may write $f = cT^m \prod_{j=1}^q (T - a_j)$, for some $c \in K^\times$, $m \in \mathbf{Z}$, $q \in \mathbf{N}$ and $a_1, \dots, a_q \in K^\times$. If $|a_j| > 1$, we write $T - a_j = -a_j(1 - a_j^{-1}T)$, so that

$$f = c \prod_{|a_j|>1} (-a_j) T^m \prod_{|a_j|>1} (1 - a_j^{-1}T) \prod_{|a_j|\leqslant 1} (T - a_j).$$

$f = \sum c_m T^m \quad \sup_m |c_m| = 1.$
$\varphi = \sum \bar{c}_m T^m$
$\alpha \in (k^\times)^n \quad \varphi(\alpha) = 0.$
$\Rightarrow \exists a \in K^{\times n} \quad |a_1| = \dots = |a_n| = 1.$
$\bar{a}_j = \alpha_j \quad f(a) = 0.$

Let $c' = c\prod_{|a_j|>1}(-a_j)$. If $|c'| < 1$, then this formula shows that f reduces to 0 in $k[\mathrm{T}^{\pm 1}]$, contradicting the stated hypothesis that $\varphi \neq 0$. If $|c'| > 1$, the coefficients of $(c')^{-1}f$ belong to the maximal ideal of the valuation ring of K, so that the reduction of $(c')^{-1}f$ is zero; on the other hand, we see that this reduction is equal to $\mathrm{T}^m\prod_{|a_j|\leqslant 1}(\mathrm{T}-\overline{a_j})$. then the relation $(c')^{-1}f$ and the hypothesis Consequently, $|c'| = 1$ and $\varphi = \rho(c')\mathrm{T}^m(\prod_{|a_j|<1}\mathrm{T})\prod_{|a_j|=1}(\mathrm{T}-\rho(a_j))$. Since φ vanishes at α, there must exist $j \in \{1,\dots,q\}$ such that $|a_j| = 1$ and $\rho(a_j) = \alpha$. This proves the proposition in this case.

To do the induction step, we first perform a multiplicative Noether normalization theorem to reduce to the case where the map $m \mapsto m_1$ from the support $\mathrm{S}(f)$ of f to $\mathbf{Z}$ is injective. To see that it is possible, we make an invertible monomial change of variables $\mathrm{T}_1 \to \mathrm{T}_1$, $\mathrm{T}_2 \to \mathrm{T}_2\mathrm{T}_1^q, \dots, \mathrm{T}_{n-1} \to \mathrm{T}_{n-1}\mathrm{T}_1^{q^{n-2}}, \mathrm{T}_n \to \mathrm{T}_n\mathrm{T}_1^{q^{n-1}}$ for some integer q, chosen to be large enough so that $q > |m_j - m'_j|$ for all $m, m' \in \mathrm{S}(f)$ and all $j \in \{1,\dots,n\}$. This change of variables transforms the Laurent polynomial f into the polynomial

$$f_q = \sum_{m\in\mathrm{S}(f)} c_m \mathrm{T}_1^{\varphi(m)}\mathrm{T}_2^{m_2}\dots\mathrm{T}_n^{m_n},$$

where

$$\varphi(m) = m_1 + qm_2 + q^2m_3 + \dots + q^{n-1}m_n.$$

Let $m, m' \in \mathrm{S}(f)$ be such that $m \neq m'$; let $k \in \{1, \dots, n\}$ be such that $m_j = m'_j$ for $j > k$ and $m_k \neq m'_k$; then one has

$$\varphi(m') - \varphi(m) = \sum_{j=1}^{n} q^{j-1}(m'_j - m_j) = \sum_{j=1}^{k-1} q^{j-1}(m'_j - m_j) + q^{k-1}(m'_k - m_k).$$

In absolute value, the last term is at least q^{k-1}, because $m'_k \neq m_k$; on the other hand, the first one is bounded from the above by

$$\sum_{j=1}^{k-1} q^{j-1}(q-1) = (q-1)\frac{q^{k-1}-1}{q-1} = q^{k-1} - 1,$$

hence $|\varphi(m') - \varphi(m)| \geqslant 1$.

Assume that this property holds. In other words, if f is written as a Laurent polynomial in T_1, with coefficients Laurent polynomials in $\mathrm{T}_2, \dots, \mathrm{T}_n$, then all of these coefficients are monomials.

Then fix any lifting $a' = (a_2, \dots, a_n) \in (\mathrm{R}^\times)^{n-1}$ of $\alpha' = (\alpha_2, \dots, \alpha_n)$. The polynomial f is not a monomial; otherwise φ would be a monomial and would not vanish at α. Thanks to the property imposed on the exponents of f, the one-variable Laurent polynomial $f(\mathrm{T}, a')$ is not a monomial either; its reduction is $\varphi(\mathrm{T}, \alpha')$ and vanishes at α_1. By the $n = 1$ case, there exists $a_1 \in \mathrm{R}^\times$ such that $\rho(a_1) = \alpha_1$ and $f(a_1, a') = 0$.

To prove the density, we let Z be the Zariski closure in $\mathbf{G}_{\mathrm{m}\,\mathrm{K}}^{\,n}$ of the set of these elements $a \in (\mathrm{R}^\times)^n$ such that $f(a) = 0$ and $\rho(a) = \alpha$. By definition, the ideal $\mathscr{I}(\mathrm{Z})$ of Z is the set of all Laurent polynomials $h \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ such that $h(a) = 0$ for all these a. One has $f \in \mathscr{I}(\mathrm{Z})$ by construction, hence $(f) \subset \mathscr{I}(\mathrm{Z})$. To prove that $\mathrm{Z} = \mathscr{V}(f)$, it suffices to prove that $\mathscr{I}(\mathrm{Z}) = (f)$.

Let $g \in \mathscr{I}(\mathrm{Z}) - (f)$. Since $\mathrm{K}[\mathrm{T}_2^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ is a unique factorization domain and f is irreducible in $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$, Gauss's theorem shows that f is either a unit, or irreducible in the one-variable polynomial ring $\mathrm{K}(\mathrm{T}_2, \dots, \mathrm{T}_n)[\mathrm{T}_1, \mathrm{T}_1^{\pm 1}]$. Since g does not belong to (f), the polynomials f and g are coprime in this principal ideal domain and there exist polynomials $u, v \in \mathrm{K}(\mathrm{T}_2, \dots, \mathrm{T}_n)[\mathrm{T}_1, \mathrm{T}_1^{\pm 1}]$ such that $uf + vg$ is a nonzero element of $\mathrm{K}(\mathrm{T}_2, \dots, \mathrm{T}_n)$. Multiplying by a common denominator, this furnishes a nonzero element h of $(f, g) \cap \mathrm{K}[\mathrm{T}_2^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. Let $a' \in (\mathrm{R}^\times)^{n-1}$ be such that $\rho(a') = \alpha'$. By what precedes, there exists $a \in \mathrm{R}^n$ of the form $a = (t, a')$ such that $f(t, a') = 0$ and $\rho(a) = \alpha$; by assumption, $g(a) = 0$, hence $h(a') = 0$. This contradicts the fact that these elements a' are Zariski-dense in $\mathbf{G}_{\mathrm{m}_\mathrm{K}}^{n-1}$ (lemma 3.3.8 below). □

Lemma **(3.3.8)**. — *Let* K *be a field, let* $\mathrm{A}_1, \dots, \mathrm{A}_n$ *be subsets of* K *and let* $\mathrm{A} = \mathrm{A}_1 \times \dots \times \mathrm{A}_n$. *Let* $f \in \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. *If* $\mathrm{Card}(\mathrm{A}_j) > \deg_{\mathrm{T}_j}(f)$ *for all* j, *then there exists* $a \in \mathrm{A}$ *such that* $f(a) \neq 0$.

In particular, if $\mathrm{A}_1, \dots, \mathrm{A}_n$ *are infinite, then* A *is Zariski dense in* $\mathbf{A}^n$.

Proof. — If $n = 1$, this amounts to the fact that a polynomial in one variable has no more roots than its degree. We then prove the result by induction on n, writing $f = f_0 + f_1 T_1 + \cdots + f_m T_1^m$, for $f_0, \dots, f_m \in K[T_2, \dots, T_n]$, where $m = \deg_{T_1}(f)$, hence $f_m \neq 0$. By induction, there exists $a_2 \in A_2, \dots, a_n \in A_n$ such that $f_m(a_2, \dots, a_n) \neq 0$. This implies that the polynomial $f(T, a_2, \dots, a_n)$ has degree m. Since $\mathrm{Card}(A_1) > \deg_{T_1}(f) = m$, there exists $a_1 \in A_1$ such that $f(a_1, a_2, \dots, a_n) \neq 0$, as was to be shown. □

$S_1 \cap T_L^n \subseteq S_5^L$

Corollary **(3.3.9)**. — *Assume that K is an algebraically closed split valued field and let $x \in \Gamma^n$. Then, for every $\alpha \in (k^\times)^n$ such that $\mathrm{in}_x(f)(\alpha) = 0$, there exists $a \in (K^\times)^n$ such that $\lambda(a) = x$ and $f(a) = 0$. Moreover, if f is irreducible, then the set of such a is Zariski dense in $\mathscr{V}(f)$.*

changement de variables.

$x \in \Gamma^n \Rightarrow \exists b \in (K^\times)^n \begin{cases} \lambda(b) = x \\ \rho(b) = 1 \end{cases}$

$f(a) = 0 \iff g(b^{-1}a) = 0$

$g(T) = f(bT)$

décale $\tau_0(g) = \tau_x(f)$

$\mathrm{in}_0(g) = \mathrm{in}_x(f)$

on applique à g le résultat de relèvent.

Proof. — By assumption, there exists $b \in (K^\times)^n$ such that $\lambda(b) = x$. Let $c \in (R^\times)^n$ be such that $\rho(c) = \rho(b)$. Since $\lambda(c) = 1$, we then have $\lambda(bc^{-1}) = \lambda(b) = x$ and $\rho(bc^{-1}) = \rho(b)\rho(c)^{-1} = 1$. Replacing b by bc^{-1}, we now assume that $\lambda(b) = x$ and $\rho(b) = 1$. Let $g(T) = f(b_1 T_1, \dots, b_n T_n)$; writing $f = \sum_{m \in S(f)} c_m T^m$, we have $g = \sum_{m \in S(f)} c_m b^m T^m$. Consequently, for every $z \in \mathbf{R}^n$, one has

$$\tau_g(z) = \sup(\log(|c_m|) + \langle m, x + z \rangle) = \tau_f(x + z).$$

This also shows that $S_z(g) = S_{x+z}(f)$ and that

$$\mathrm{in}_z(g) = \sum_{m \in S_z(g)} \rho(c_m b^m) T^m = \sum_{m \in S_{x+z}(f)} \rho(c_m) T^m = \mathrm{in}_{x+z}(f).$$

In particular, $x \in \mathscr{T}_f$ if and only if $0 \in \mathscr{T}_g$, $\mathrm{in}_x(f)(\alpha) = 0$ if and only if $\mathrm{in}_0(g)(\alpha) = 0$, and $g(a) = 0$ if and only if $f(ab) = 0$, where $ab = (a_1 b_1, \dots, a_n b_n)$.

By the lifting proposition, there exists $a \in (R^\times)^n$ such that $\rho(a) = \alpha$ and $g(a) = 0$; then $ab \in (K^\times)^n$ satisfies $\rho(ab) = \alpha$ and $f(ab) = 0$.

Moreover, if f is irreducible, then g is irreducible as well, the set of such elements a is Zariski dense in $\mathscr{V}(g)$, hence the set of such elements ab is Zariski dense in $\mathscr{V}(f)$. □

Beware:

The definition of the nonarchimedean amoebas has been modified so as to be more consistent with the definition in the archimedean case. I made the necessary corrections up to here, but there are certainly inconsistencies below.

3.4. Monomial ideals

Definition **(3.4.1)**. — *An ideal of* $K[T_1, \dots, T_n]$ *is said to be* monomial *if it is generated by a set of monomials.*