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$(z_1, \dots, z_n) \in (\mathrm{L}^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in \mathrm{I}$ and $(\log(|z_1|), \dots, \log(|z_n|)) = (x_1, \dots, x_n)$.

3.3. Nonarchimedean amoebas of hypersurfaces

3.3.1. — Let K be a field endowed with a nonarchimedean absolute value. Let R be the valuation ring of K, let k its residue field and $\mathrm{red} \colon \mathrm{R} \to k$ the reduction morphism; it maps the maximal ideal to 0 and induces a morphism of groups from $\mathrm{R}^\times$ to $k^\times$.

The map from $v \colon \mathrm{K}^\times$ to $\mathbf{R}$ given by $a \mapsto -\log(|a|)$ is a morphism of groups. Let Γ be its image. One says that the given valued field K is split if we are given a *section* of the surjective map v. Such a section does not exist in general, but it does exist in the following two important cases:

– Assume that K is discretely valued. Then R is a discrete valuation ring. If t is a given generator of its maximal ideal, one has $\Gamma = \mathbf{Z} \log(|t|)$ and the map $n \log(|t|) \mapsto t^n$ is a section as required.

– If K is algebraically closed, then such a section also exists, by an abstract homological algebra argument. Indeed, in this case, $\mathrm{R}^\times$ is a divisible abelian group, hence an injective $\mathbf{Z}$-module. In a more elementary way, one can also use the fact that Γ is a uniquely divisible abelian group, hence a $\mathbf{Q}$-vector space. It then suffices to choose, in a compatible manner, nth roots of a given element of $\mathrm{K}^\times$.

Let $\gamma \in \Gamma$ and let $a \in \mathrm{K}^\times$ be such that $\log(|a|) = \gamma$. Let us choose inductively elements $a_n \in \mathrm{K}^\times$ such that $a_1 = a$ and $(a_n)^n = a_{n-1}$ for all integers $n \geqslant 2$. In particular $(a_n)^{n!} = a$ for all $n \geqslant 1$. Moreover, if $m \geqslant n$, then n divides $m!$ and we see by induction that $(a_m)^{m!/n} = (a_n)^{n!/n} = (a_n)^{(n-1)!}$. Then there is a unique morphism of groups from $\mathbf{Q}\gamma$ to $\mathrm{K}^\times$ that maps $\frac{m}{n}\gamma$ to $(a_n)^{(n-1)!m}$ for all $m, n \in \mathbf{Z}$ such that $n \geqslant 1$.

If K is a split valued field, then we can extend the morphism of groups $\mathrm{red}\colon \mathrm{R}^\times \to k^\times$ to a morphism of monoids $\rho\colon \mathrm{K} \to k$, by setting $\rho(a) = \mathrm{red}(a/|a|)$. Note that ρ restricts to a morphism of groups from $\mathrm{K}^\times$ to $k^\times$. Moreover, the map $a \mapsto (-\log(|a|), \rho(a))$ is a group isomorphism from $\mathrm{K}^\times$ to $\Gamma \times k^\times$.

Definition **(3.3.2)**. — *Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ be a nonzero Laurent polynomial; write $f = \sum c_m \mathrm{T}^m$.*

a) *The* tropical polynomial *associated with f is the map*

$$\tau_f : \mathrm{R}^n \to \mathbf{R}, \quad x \mapsto \sup_m \left(\log(|c_m|) + \langle m, x\rangle\right).$$

b) *The* tropical hypersurface *defined by f is the subset $\mathscr{T}_f$ of all $x \in \mathbf{R}^n$ such that there exist two distinct elements $m \in \mathbf{Z}^n$ such that $\tau_f(x) = $* $\log(|c_m|) + \langle m, x\rangle$.

Notations K corps valué
$|\cdot| : K \to \mathbb{R}_+$
$R = \{a \in K \mid |a| \leq 1\}$
anneau de valuation

k = corps résiduel $= R/\underbrace{m_R}_{\text{idéal maximal}}$

(rare) $\mathrm{red} : R \to k$
$a \mapsto \bar{a}$

$\Gamma = \log(|K^\times|)$ groupe de valeurs

K scindé : si on s'est donné une section de $K^\times \to \Gamma$

cette section est notée $\gamma \mapsto t^\gamma$
$\log(|t^\gamma|) = \gamma$
$t^{\gamma+\gamma'} = t^\gamma t^{\gamma'}$

c) *(Assuming that the valued field* K *is split.) For* $x \in \mathbf{R}^n$*, the* initial form *of* f *at* x *is the Laurent polynomial*

$$\mathrm{in}_x(f) = \sum_{\tau_f(x)=\log(|c_m|)+\langle m,x\rangle} \rho(c_m)\mathrm{T}^m. \qquad \in k[T_1^{\pm 1},\dots,T_n^{\pm 1}]$$

Recall that the *support* of a Laurent polynomial $f = \sum c_m \mathrm{T}^m$ is the set $\mathrm{S}(f)$ of all $m \in \mathbf{Z}^n$ such that $c_m \neq 0$, and that the Newton polytope of f is the convex hull NP_f of $\mathrm{S}(f)$ in $\mathbf{R}^n$.

$S(f)$ support de f $\quad \{m \mid c_m \neq 0\}$

$S_x(f) = \{m \mid \tau_f(x) = \log|c_m| + \langle m, x\rangle\}$

NP_f polytope de Newton $= \mathrm{conv}(S(f))$

$NP_{f,x} = \mathrm{conv}(S_x(f))$

For $x \in \mathbf{R}^n$, we will often denote by $\mathrm{S}_x(f)$ the subset of $\mathrm{S}(f)$ consisting of those m such that $\tau_f(x) = \log(|c_m|) + \langle m, x\rangle$; this is the support of the initial form $\mathrm{in}_x(f)$; its convex hull is then a sub-polytope $\mathrm{NP}_{f,x}$ of NP_f.

With this notation, the tropical hypersurface $\mathscr{T}_f$ is the set of all $x \in \mathbf{R}^n$ such that $\mathrm{S}_x(f)$ has at least two elements, equivalently, $\mathrm{NP}_{f,x}$ is not a point. When K is a split valued field, this is also equivalent to the property that $\mathrm{in}_x(f)$ is not a monomial.

The preceding concepts make sense when $f = 0$: one has $\mathrm{S}(f) = \varnothing$ (no nonzero monomials), $\tau_f(x) = -\infty$ (supremum of an empty family), and $\mathrm{in}_x(f) = 0$, but the tropical variety $\mathscr{T}_f$ should be defined as $\mathbf{R}^n$.

Remark **(3.3.3)**. — Let $\varphi : \mathbf{G}_{\mathrm{m}_\mathrm{K}}^n \to \mathbf{G}_{\mathrm{m}_\mathrm{K}}^p$ be a *monomial* morphism, given at the level of Laurent polynomials by a morphism of K-algebras $\varphi^* : \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_p^{\pm1}] \to \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ of the form $\mathrm{T}_j \mapsto$

On the other hand, for two distinct elements m, q of $\mathrm{S}(f)$, the polyhedron $\mathrm{P}_m \cap \mathrm{P}_q$ is contained in the hyperplane defined by the nontrivial affine equation $\log(|c_m|) + \langle m, x\rangle = \log(|c_q|) + \langle q, x\rangle$, so that $\mathrm{P}_m \cap \mathrm{P}_q$ is disjoint from the interior of P_m. In particular, if $\mathrm{Card}(\mathrm{S}_x(f)) \geqslant 2$, then x does not belong to the interior of P_m.

This proves that $\mathbf{R}^n$ is the union of those polyhedra P_m which have dimension n, and that the union of their interiors is the set of all $x \in \mathbf{R}^n$ such that $\mathrm{S}_x(f)$ is reduced to one element.

Consequently, the tropical hypersurface $\mathscr{T}_f$, which is its complementary subset, is the union of the $(n-1)$-dimensional faces of these polyhedra P_m, and they are Γ-strict convex polyhedra. □

Theorem **(3.3.6)** (Kapranov). — *Let* $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a Laurent polynomial. The following four subsets of* $\mathbf{R}^n$ *coincide:*

$\mathbb{R}$ — (i) *The tropical hypersurface* $\mathscr{T}_f$;

k — (ii) *Assuming that the valued field* K *is split, the set of all* $x \in \mathbf{R}^n$ *such that* $\mathrm{in}_x(f)$ *is not a monomial;*

K (iii) *The set of all* $x \in \mathbf{R}^n$ *such that there exists a valued extension* L *of* K *and a point* $z \in (\mathrm{L}^\times)^n$ *such that* $f(z) = 0$ *and* $x = \cancel{\lambda}(z)$; log(|z|)

(iv) *The image of* $\mathscr{V}(f)$ *in* $\mathscr{M}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$ *by the tropicalization map* $\lambda \colon \mathscr{M}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$, $p \mapsto (\log(p(\mathrm{T}_1)), \dots, \log(p(\mathrm{T}_n)))$.

If L *is an algebraically closed extension of* K, *endowed with a nontrivial absolute value extending that of* K, *they also coincide with the set:*

(v) *The closure of the set of all $x \in \mathbf{R}^n$ such that there exists a point $z \in (\mathrm{L}^\times)^n$ such that $f(z) = 0$ and $x = \lambda(z)$.*

(vi) si $\Gamma = \mathbb{R}$ et L est algébriquement clos, $\{x \mid \exists z \in (\mathbb{C}^\times)^n\ x = \lambda(z),\ f(x) = 0\}$.

Proof. — Let $\mathrm{S}_1 = \mathscr{T}_f, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5$ be these subsets. Write $f = \sum c_m \mathrm{T}^m$.

Let $x \in \mathbf{R}^n$. Let $m \in \mathbf{Z}^n$; the monomial T^m appears in $\mathrm{in}_x(f)$ if and only if $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$. Consequently, $\mathrm{in}_x(f)$ is a monomial if and only if the supremum defining $\tau_f(x)$ is reached only once. This proves that $\mathrm{S}_1 = \mathrm{S}_2$.

The equality $\mathrm{S}_3 = \mathrm{S}_4$ follows from the discussion in §3.2.9.

Let L be a valued extension of K, let $z \in (\mathrm{L}^\times)^n$ be a point such that $f(z) = 0$ and let $x = \lambda(z)$. One has $\sum c_m z^m = 0$. Since the absolute value is nonarchimedean, the supremum of all $|c_m z^m|$ must be attained twice. Since $\lambda(c_m z^m) = \log(|c_m|) + \langle m, x\rangle$, this shows that there exist two distinct elements $m, q \in \mathbf{Z}^n$ such that $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$. In other words, x belongs to the hypersurface $\mathscr{T}_f$. This proves that one has $\mathrm{S}_3 \subset \mathrm{S}_1$.

By definition, the set S_5 is the closure of a subset of S_3; since S_1 is closed, one also has $\mathrm{S}_5 \subset \mathrm{S}_1$.

By the corollary to the lifting proposition below, one has $\mathscr{T}_f \cap \Gamma^n \in \mathrm{S}_5$. Since K is algebraically closed and its valuation is nontrivial, the group Γ is a non zero $\mathbf{Q}$-subspace of $\mathbf{R}$; in particular, it is dense in $\mathbf{R}$. On the other hand, since $\mathscr{T}_f$ is a Γ-strict polyhedral subspace of $\mathbf{R}^n$, its subset $\mathscr{T}_f \cap \Gamma^n$ is dense in $\mathscr{T}_f$. Since $\mathscr{T}_f$ is closed in $\mathbf{R}^n$, this implies that $S_1 = \mathscr{T}_f \subset S_5$.

Using Gauss absolute values (proposition 3.1.10) and example 3.1.12, there exists an algebraically closed valued extension L of K whose value group Γ_L contains the coordinates of x. By the corollary of the lifting proposition, there exists $z \in (L^\times)^n$ such that $f(z) = 0$ and $\lambda(z) = x$; in other words, one has $x \in S_3$. Consequently, $S_1 \subset S_3$. This concludes the proof of the theorem. □

Proposition **(3.3.7)** (Lifting). — *Assume that* K *is an algebraically closed valued field with residue field* k. *Let* $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *be a Laurent polynomial. We assume that the coefficients of* f *belong to the valuation ring of* K *and that its reduction* $\varphi \in k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *is nonzero.*

For every $\alpha \in (k^\times)^n$ *such that* $\varphi(\alpha) = 0$, *there exists* $a \in (R^\times)^n$ *such that* $\rho(a) = \alpha$ *and* $f(a) = 0$. *Moreover, if* f *is irreducible, then the set of such* a *is Zariski dense in the closed subscheme* $V(f)$ *of* $\mathbf{G}_{m_K}^n$.

$f \in R[T_1^{\pm}, \dots, T_n^{\pm 1}]$
$0 \neq \varphi \in k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$
$\varphi(\alpha) = 0 \quad \alpha \in (k^\times)^n$
$\rightsquigarrow \exists\, a \in (R^\times)^n \quad \mathrm{red}(a) = \alpha$
$f(a) = 0$

Proof. — We do the proof by induction on n.

Let us first assume that $n = 1$. Since K is algebraically closed, we may write $f = cT^m \prod_{j=1}^q (T - a_j)$, for some $c \in K^\times$, $m \in \mathbf{Z}$, $q \in \mathbf{N}$ and $a_1, \dots, a_q \in K^\times$. If $|a_j| > 1$, we write $T - a_j = -a_j(1 - a_j^{-1}T)$, so that

$$f = \underbrace{c \prod_{|a_j|>1} (-a_j)}_{c'} T^m \prod_{|a_j|>1} (1 - a_j^{-1}T) \prod_{|a_j|\leqslant 1} (T - a_j).$$

Densité de ces solutions a pour la topologie de Zariski :

$g \in K(T_1^{\pm}, \dots, T_n^{\pm 1})$
tel que $g(a) = 0$ pour tous les a comme dans la prop.,
alors $f \mid g$ dans $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

(f irréductible)

- $n = 1$: $f = c\,\overbrace{T_1^{m}\,(T_1}\, - a_1)$
  $a = a_1$ est le seul relèvement possible.
  $(m_1 = m'_1 \Rightarrow m = m')$

- Cas général : g non multiple de $f = \sum c_{d,m'}\, T_1^{d}\, T_2^{m_2'} \cdots$

Gauss A anneau factoriel de corps des fractions K.
$f \in A[T]$ irréductible
$\Leftrightarrow \begin{cases} f \in A \text{ irréductible} \\ \text{ou } ct(f) = 1 \ \& \ f \text{ est irréductible dans } K[T] \end{cases}$

Let $c' = c\prod_{|a_j|>1}(-a_j)$. If $|c'| < 1$, then this formula shows that f reduces to 0 in $k[\mathrm{T}^{\pm 1}]$, contradicting the stated hypothesis that $\varphi \neq 0$. If $|c'| > 1$, the coefficients of $(c')^{-1}f$ belong to the maximal ideal of the valuation ring of K, so that the reduction of $(c')^{-1}f$ is zero; on the other hand, we see that this reduction is equal to $\mathrm{T}^m\prod_{|a_j|\leqslant 1}(\mathrm{T}-\overline{a_j})$. then the relation $(c')^{-1}f$ and the hypothesis Consequently, $|c'| = 1$ and $\varphi = \rho(c')\mathrm{T}^m(\prod_{|a_j|<1}\mathrm{T})\prod_{|a_j|=1}(\mathrm{T}-\rho(a_j))$. Since φ vanishes at α, there must exist $j \in \{1,\dots,q\}$ such that $|a_j| = 1$ and $\rho(a_j) = \alpha$. This proves the proposition in this case.

To do the induction step, we first perform a multiplicative Noether normalization theorem to reduce to the case where the map $m \mapsto m_1$ from the support $\mathrm{S}(f)$ of f to $\mathbf{Z}$ is injective. To see that it is possible, we make an invertible monomial change of variables $\mathrm{T}_1 \to \mathrm{T}_1$, $\mathrm{T}_2 \to \mathrm{T}_2\mathrm{T}_1^q, \dots, \mathrm{T}_{n-1} \to \mathrm{T}_{n-1}\mathrm{T}_1^{q^{n-2}}, \mathrm{T}_n \to \mathrm{T}_n\mathrm{T}_1^{q^{n-1}}$ for some integer q, chosen to be large enough so that $q > |m_j - m'_j|$ for all $m, m' \in \mathrm{S}(f)$ and all $j \in \{1,\dots,n\}$. This change of variables transforms the Laurent polynomial f into the polynomial

$$f_q = \sum_{m\in\mathrm{S}(f)} c_m \mathrm{T}_1^{\varphi(m)}\mathrm{T}_2^{m_2}\dots\mathrm{T}_n^{m_n},$$

where

$$\varphi(m) = m_1 + qm_2 + q^2m_3 + \dots + q^{n-1}m_n.$$

$|a'_j| < 1$ $\quad |a_j| \leq 1$

$$f = c'\,T^m \underbrace{\prod_j (1 - T a'_j) \prod_j (T - a_j)}_{\in R[T],\ \text{de réduction} \neq 0}$$

$\underline{|c'| = 1}$

car : $|c'| < 1$: la réduction de f est nulle absurde

: $|c'| > 1$ $\quad \underbrace{(c')^{-1}}_{\text{de réduction nulle, absurde.}} f = T^m \prod (1 - T a'_j) \prod (T - a_j)$

$$\varphi = \gamma' \cdot T^m \prod_{j : |a_j|<1} T \cdot \prod_{j : |a_j|=1} (T - \alpha_j)$$

$\mathrm{red}(a_j) = 0 \qquad \alpha_j = \mathrm{red}(a_j)$

$\varphi(\alpha) = 0 \Rightarrow \exists j \quad |a_j| = 1, \ \alpha = \alpha_j$

relèvement : $\underline{a_j}$

To do the induction step, we first perform a multiplicative Noether normalization theorem to reduce to the case where the map $m \mapsto m_1$ from the support $\mathrm{S}(f)$ of f to $\mathbf{Z}$ is injective. To see that it is possible, we make an invertible monomial change of variables $T_1 \to T_1$, $T_2 \to T_2T_1^q, \dots, T_{n-1} \to T_{n-1}T_1^{q^{n-2}}, T_n \to T_nT_1^{q^{n-1}}$ for some integer q, chosen to be large enough so that $q > |m_j - m'_j|$ for all $m, m' \in \mathrm{S}(f)$ and all $j \in \{1, \dots, n\}$. This change of variables transforms the Laurent polynomial f into the polynomial

$$f_q = \sum_{m \in \mathrm{S}(f)} c_m T_1^{\varphi(m)} T_2^{m_2} \dots T_n^{m_n},$$

where

$$\varphi(m) = m_1 + qm_2 + q^2m_3 + \dots + q^{n-1}m_n.$$

Let $m, m' \in \mathrm{S}(f)$ be such that $m \neq m'$; let $k \in \{1, \dots, n\}$ be such that $m_j = m'_j$ for $j > k$ and $m_k \neq m'_k$; then one has

$$\varphi(m') - \varphi(m) = \sum_{j=1}^{n} q^{j-1}(m'_j - m_j) = \sum_{j=1}^{k-1} q^{j-1}(m'_j - m_j) + q^{k-1}(m'_k - m_k).$$

In absolute value, the last term is at least q^{k-1}, because $m'_k \neq m_k$; on the other hand, the first one is bounded from the above by

$$\sum_{j=1}^{k-1} q^{j-1}(q-1) = (q-1)\frac{q^{k-1}-1}{q-1} = q^{k-1} - 1,$$

hence $|\varphi(m') - \varphi(m)| \geqslant 1$.

$$\begin{array}{ccc} \mathrm{S}(f) \subset \mathbb{Z}^n & & m \\ \downarrow & & \downarrow \\ \text{injectif?}\ \mathbb{Z} & & m_1 \end{array}$$

$$f = \sum c_m T^m = \sum c_m T_1^{m_1} T_2^{m_2} \dots T_n^{m_n}$$

$$f(T_1, T_2T_1^q, \dots, T_nT_1^{q^{n-1}}) = \sum c_m \underbrace{T_1^{m_1+qm_2 \dots}}_{1} T_2^{m_2} \dots T_n^{m_n}$$

coeff. $\varphi(m) = m_1 + qm_2 + q^2m_3 + \dots + q^{n-1}m_n$

si q est assez grand, unicité du développement en base q donne $m \neq m' \Rightarrow \varphi(m) \neq \varphi(m')$

Let $m, m' \in \mathrm{S}(f)$ be such that $m \neq m'$; let $k \in \{1, \dots, n\}$ be such that $m_j = m'_j$ for $j > k$ and $m_k \neq m'_k$; then one has

$$\varphi(m') - \varphi(m) = \sum_{j=1}^{n} q^{j-1}(m'_j - m_j) = \sum_{j=1}^{k-1} q^{j-1}(m'_j - m_j) + q^{k-1}(m'_k - m_k).$$

In absolute value, the last term is at least q^{k-1}, because $m'_k \neq m_k$; on the other hand, the first one is bounded from the above by

$$\sum_{j=1}^{k-1} q^{j-1}(q-1) = (q-1)\frac{q^{k-1}-1}{q-1} = q^{k-1} - 1,$$

hence $|\varphi(m') - \varphi(m)| \geqslant 1$.

Assume that this property holds. In other words, if f is written as a Laurent polynomial in T_1, with coefficients Laurent polynomials in $\mathrm{T}_2, \dots, \mathrm{T}_n$, then all of these coefficients are monomials.

Then fix any lifting $a' = (a_2, \dots, a_n) \in (\mathrm{R}^\times)^{n-1}$ of $\alpha' = (\alpha_2, \dots, \alpha_n)$. The polynomial f is not a monomial; otherwise φ would be a monomial and would not vanish at α. Thanks to the property imposed on the exponents of f, the one-variable Laurent polynomial $f(\mathrm{T}, a')$ is not a monomial either; its reduction is $\varphi(\mathrm{T}, \alpha')$ and vanishes at α_1. By the $n = 1$ case, there exists $a_1 \in \mathrm{R}^\times$ such that $\rho(a_1) = \alpha_1$ and $f(a_1, a') = 0$.

To prove the density, we let Z be the Zariski closure in $\mathbf{G}_{\mathrm{m}_\mathrm{K}}^n$ of the set of these elements $a \in (\mathrm{R}^\times)^n$ such that $f(a) = 0$ and $\rho(a) = \alpha$. By

$f = \sum_m c_m T^m$

$= \sum_d \Big(\sum_{m \text{ tq } m_1 = d} c_m T^m \Big)$

$m = (d, m')$ $\quad d \in \mathbb{Z}$, $m' \in \mathbb{Z}^{n-1}$

$= \sum_d T_1^d \underbrace{\sum_{m'} c_{(d,m')} T_2^{m'_2} \cdots T_n^{m'_n}}_{\text{monôme !}}$

on choisit $a_2, \dots, a_n \in R^\times$

tels que $\bar{a}_2 = \alpha_2, \dots, \bar{a}_n = \alpha_n$

monôme $(a_2, \dots, a_n) \neq 0$

$|c_{(d,m')} a_2^{m'_2} \cdots a_n^{m'_n}| = |c_{(d,m')}| \leq$

$a' = (a_2, \dots, a_n)$

$f(T, a')$ une variable

vérifie les hypothèses de la prop.

$\rightsquigarrow a_1$ tq $\bar{a}_1 = \alpha_1$

$f(a_1, a') = 0$.

Élimination / th. des zéros de Hilbert / lemme de normalisation de Noether

Normalisation de Noether

A algèbre de type fini sur un corps k. ($\exists a_1, \dots, a_n \in A$ $A = k[a_1, \dots, a_n]$)

Il existe des éléments $a_1, \dots, a_m \in A$ algébriquement indépendants ($f \neq 0 \Rightarrow f(a_1, \dots, a_m) \neq 0$)

($k[T_1, \dots, T_m] \hookrightarrow A$, $T_j \longmapsto a_j$ injectif)

tels que A soit entière sur $k[a_1, \dots, a_m]$

(de manière équivalente, A est un $k[a_1, \dots, a_m]$ module de type fini)

si k est infini on peut prendre pour a_i des comb. linéaires de générateurs de A)

(En général $a_i \to a_i - a_n^{q^i}$)

$A = k[T_1, \dots, T_n] / I$ $\quad I$ idéal.

$k[T_1, \dots, T_m] \simeq k[f_1, \dots, f_m] \hookrightarrow A$ entière

(si k infini, f linéaire)

Géométriquement (k alg. clos) $\quad k^n \longrightarrow k^m$, $a \longmapsto (f(a))$ $\quad V(I) \xrightarrow[\text{surjectif}]{\text{fini}} \mathbb{A}^m_k$

Exercice : $I = (T_1 T_2 - 1)$

definition, the ideal $\mathscr{I}(Z)$ of Z is the set of all Laurent polynomials $h \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ such that $h(a) = 0$ for all these a. One has $f \in \mathscr{I}(Z)$ by construction, hence $(f) \subset \mathscr{I}(Z)$. To prove that $Z = \mathscr{V}(f)$, it suffices to prove that $\mathscr{I}(Z) = (f)$.

Let $g \in \mathscr{I}(Z) - (f)$. Since $K[T_2^{\pm 1}, \dots, T_n^{\pm 1}]$ is a unique factorization domain and f is irreducible in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, Gauss's theorem shows that f is either a unit, or irreducible in the one-variable polynomial ring $K(T_2, \dots, T_n)[T_1, T_1^{\pm 1}]$. Since g does not belong to (f), the polynomials f and g are coprime in this principal ideal domain and there exist polynomials $u, v \in K(T_2, \dots, T_n)[T_1, T_1^{\pm 1}]$ such that $uf + vg$ is a nonzero element of $K(T_2, \dots, T_n)$. Multiplying by a common denominator, this furnishes a nonzero element h of $(f, g) \cap K[T_2^{\pm 1}, \dots, T_n^{\pm 1}]$. Let $a' \in (R^\times)^{n-1}$ be such that $\rho(a') = \alpha'$. By what precedes, there exists $a \in R^n$ of the form $a = (t, a')$ such that $f(t, a') = 0$ and $\rho(a) = \alpha$; by assumption, $g(a) = 0$, hence $h(a') = 0$. This contradicts the fact that these elements a' are Zariski-dense in $\mathbf{G}_{m_K}^{n-1}$ (lemma 3.3.8 below). □

($f \neq 0$)

Lemma **(3.3.8)**. — *Let* K *be a field, let* $A_1, \dots, A_n$ *be subsets of* K *and let* $A = A_1 \times \dots \times A_n$. *Let* $f \in K[T_1, \dots, T_n]$. *If* $\operatorname{Card}(A_j) > \deg_{T_j}(f)$ *for all* j, *then there exists* $a \in A$ *such that* $f(a) \neq 0$.

In particular, if $A_1, \dots, A_n$ *are infinite, then* A *is Zariski dense in* $\mathbf{A}^n$.

$A = K[T_2^{\pm 1}, \dots, T_n^{\pm 1}]$

corps de fraction $K(T_2, \dots, T_n) = L$

g non multiple de f

⟹ ______ dans $L[T_1^{\pm 1}]$

relation de Bézout

$uf + vg = 1$ dans $L[T_1^{\pm 1}]$

puis relation

$uf + vg = h$ $(\neq 0)$

$h \in K[T_2, \dots, T_n]$

$u, v \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$f(a) = g(a) \Rightarrow h(a) = 0$.

or a est de la forme

$(a_1, a_2, \dots, a_n)$

où $a' = (a_2, \dots, a_n)$ est

arbitraire tel que $|a_j| = 1$

$\overline{a_j} = \alpha_j'$

infinité de choix

de la forme $A_2 \times \dots \times A_n$

Proof. — If $n = 1$, this amounts to the fact that a polynomial in one variable has no more roots than its degree. We then prove the result by induction on n, writing $f = f_0 + f_1\mathrm{T}_1 + \cdots + f_m\mathrm{T}_1^m$, for $f_0, \dots, f_m \in \mathrm{K}[\mathrm{T}_2, \dots, \mathrm{T}_n]$, where $m = \deg_{\mathrm{T}_1}(f)$, hence $f_m \neq 0$. By induction, there exists $a_2 \in \mathrm{A}_2, \dots, a_n \in \mathrm{A}_n$ such that $f_m(a_2, \dots, a_n) \neq 0$. This implies that the polynomial $f(\mathrm{T}, a_2, \dots, a_n)$ has degree m. Since $\mathrm{Card}(\mathrm{A}_1) > \deg_{\mathrm{T}_1}(f) = m$, there exists $a_1 \in \mathrm{A}_1$ such that $f(a_1, a_2, \dots, a_n) \neq 0$, as was to be shown. □

affine (vi)

$$\begin{cases} f(a) = 0 \\ \lambda(a) = x \\ \rho(a) = \alpha \end{cases}$$

Corollary **(3.3.9)**. — *Assume that* K *is an algebraically closed split valued field and let $x \in \Gamma^n$. Then, for every $\alpha \in (k^\times)^n$ such that $\mathrm{in}_x(f)(\alpha) = 0$, there exists $a \in (\mathrm{K}^\times)^n$ such that $\lambda(a) = x$ and $f(a) = 0$. Moreover, if f is irreducible, then the set of such a is Zariski dense in $\mathscr{V}(f)$.*

Proof. — By assumption, there exists $b \in (\mathrm{K}^\times)^n$ such that $\lambda(b) = x$. Let $c \in (\mathrm{R}^\times)^n$ be such that $\rho(c) = \rho(b)$. Since $\lambda(c) = 1$, we then have $\lambda(bc^{-1}) = \lambda(b) = x$ and $\rho(bc^{-1}) = \rho(b)\rho(c)^{-1} = 1$. Replacing b by bc^{-1}, we now assume that $\lambda(b) = x$ and $\rho(b) = 1$. Let $g(\mathrm{T}) = f(b_1\mathrm{T}_1, \dots, b_n\mathrm{T}_n)$; writing $f = \sum_{m\in \mathrm{S}(f)} c_m\mathrm{T}^m$, we have $g = \sum_{m\in\mathrm{S}(f)} c_m b^m \mathrm{T}^m$. Consequently, for every $z \in \mathbf{R}^n$, one has

$$\tau_g(z) = \sup(\log(|c_m|) + \langle m, x + z\rangle) = \tau_f(x + z).$$

This also shows that $S_z(g) = S_{x+z}(f)$ and that

$$\mathrm{in}_z(g) = \sum_{m\in S_z(g)} \rho(c_m b^m) T^m = \sum_{m\in S_{x+z}(f)} \rho(c_m) T^m = \mathrm{in}_{x+z}(f).$$

In particular, $x \in \mathscr{T}_f$ if and only if $0 \in \mathscr{T}_g$, $\mathrm{in}_x(f)(\alpha) = 0$ if and only if $\mathrm{in}_0(g)(\alpha) = 0$, and $g(a) = 0$ if and only if $f(ab) = 0$, where $ab = (a_1 b_1, \dots, a_n b_n)$.

By the lifting proposition, there exists $a \in (R^\times)^n$ such that $\rho(a) = \alpha$ and $g(a) = 0$; then $ab \in (K^\times)^n$ satisfies $\rho(ab) = \alpha$ and $f(ab) = 0$.

Moreover, if f is irreducible, then g is irreducible as well, the set of such elements a is Zariski dense in $\mathscr{V}(g)$, hence the set of such elements ab is Zariski dense in $\mathscr{V}(f)$. □

3.4. Monomial ideals

Definition **(3.4.1)**. — *An ideal of* $K[T_1, \dots, T_n]$ *is said to be* monomial *if it is generated by a set of monomials.*

Observe that if an ideal I is generated by a family (f_i) of monomials, then a monomial f belongs to I if and only if it is divisible by some f_i.

Lemma **(3.4.2)**. — *Let* I *be an ideal of* $K[T_1, \dots, T_n]$. *The following properties are equivalent:*

(i) *The ideal* I *is monomial;*

f_i monoms $\quad f_i = c_i T^{m_i}$

$c_m T^m = f = \sum f_i g_i$

T^m apparaît dans un $f_i g_i$

$\Rightarrow$ $m \geqslant m_i$ $\quad [(m - m_i)_j \geqslant 0.]$

$m - m_i \in \mathbb{N}^n$

$\Leftrightarrow$ T^{m_i} divise T^m

La théorie des idéaux monomiaux est très proche de celle des semi-groupes

(ii) *For every polynomial $f \in \mathrm{I}$, every monomial that appears in f belongs to* I.

If I is an ideal of $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, we shall sometimes consider the ideal J generated by all monomials which belong to I; it is the largest monomial ideal contained in I.

Proof. — (i)⇒(ii). Assume that I is monomial. Let $f \in \mathrm{I}$; we may write $f = \sum_{i=1}^m f_i g_i$, where f_i is a monomial in a given generating family of I and $g_i \in \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. Let $c\mathrm{T}^m$ be a (nonzero) monomial that appears in f. There exists $i \in \{1, \dots, m\}$ such that m belongs to the support of $f_i g_i$; since every monomial of $f_i g_i$ is divisible by the monomial f_i, this implies that f_i divides T^m, hence $c\mathrm{T}^m \in (\mathrm{I})$.

(ii)⇒(i). Let (f_i) be a generating family of I. By assumption, all the monomials of the f_i belong to I. The family consisting of all of these monomials generates an ideal which is contained in I by assumption, and which contains I since it contains all of the f_i. □

Example **(3.4.3)**. — The ideal generated by a subfamily $(\mathrm{T}_i)_{i\in\mathrm{S}}$ of the indeterminates is a monomial ideal. It is also prime, since the quotient ring, isomorphic to the polynomial ring $\mathrm{K}[(\mathrm{T}_i)_{i\notin\mathrm{S}}]$ in the other indeterminates, is an integral domain.

Conversely, all prime monomial ideals are of this form. Let indeed I be a prime monomial ideal of $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ and let S be the set of all $i \in \{1, \dots, n\}$ such that $\mathrm{T}_i \in \mathrm{I}$; let us prove that $\mathrm{I} = ((\mathrm{T}_i)_{i\in\mathrm{S}})$. The

$J \subset I$

J monomial, le plus grand idéal monomial contenu dans J.

(i) ⇒ (ii) $c T^m \in S(f)$

$\exists i \quad c T^m \in S(f_i g_i) = m_i + S(g_i)$

$\exists i \quad m \geq m_i$

alors $T^m = T^{m_i} \cdot T^{m-m_i} \in I$

(ii) ⇒ (i) $\quad f_i = \sum_{m \in S(f_i)} c_m^{(i)} T^m \quad I = (f_i)$

par hypothèse $T^m \in I$ si $m \in S(f_i)$

$\Rightarrow J \supset (T^m, m \in \cup S(f_i))$

$\Rightarrow J \supset I$

Ex. $(T_1, \dots, T_m)$ idéal monomial premier

Décomposition primaire des idéaux
(Lasker, Noether)

inclusion $((\mathrm{T}_i)_{i\in\mathrm{S}}) \subset \mathrm{I}$ is obvious. Conversely, let $f \in \mathrm{I}$ and let us prove that $f \in ((\mathrm{T}_i)_{i\in\mathrm{S}})$. Since all monomials of f belong to I, we may assume that f is a monomial; write $f = c\mathrm{T}^m = c\mathrm{T}_1^{m_1}\dots\mathrm{T}_n^{m_n}$. If none of the indeterminates that appear in f belong to I, then neither does their product, by definition of a prime ideal. Consequently, there exists $i \in \mathrm{S}$ such that $m_i \geqslant 1$, and $f \in (\mathrm{T}_i) \subset ((\mathrm{T}_i)_{i\in\mathrm{S}})$.

Proposition **(3.4.4)**. — a) *The sum and the intersection of a family of monomial ideals is a monomial ideal.*

b) *The radical of a monomial ideal is a monomial ideal.*

c) *Every monomial ideal has a primary decomposition which consists of monomial ideals. In particular, the prime ideals associated with a monomial ideal are monomial ideals.*

Proof. — a) The case of a sum follows directly from the definition. Let (I_j) be a family of monomial ideals and let $\mathrm{I} = \bigcap_j \mathrm{I}_j$. Let $f \in \mathrm{I}$ and let $c\mathrm{T}^m$ be a monomial that appears in f. Fix an index j; since $f \in \mathrm{I}_j$ and I_j is a monomial ideal, we have $c\mathrm{T}^m \in \mathrm{I}_j$. Consequently, $c\mathrm{T}^m \in \mathrm{I}$. This proves that I is a monomial ideal.

b) Let I be a monomial ideal and let $\mathrm{J} = \sqrt{\mathrm{I}}$; let us prove that J is a monomial ideal. Let $f \in \mathrm{J}$ and let us prove that every monomial of f belongs to J. Subtracting from f its monomials that belong to J, we may assume that no monomial of f belongs to J; assume, arguing by contradiction, that $f \neq 0$ and write $f = \sum c_m\mathrm{T}^m$. Let $m \in \mathbf{N}^m$ be a

Décomposition primaire (Lasker-Noether)

- A anneau (noethérien)

  $Q \subset A$ idéal est primaire si $1 \notin A$

  et $ab \in Q$, $a \notin Q \Rightarrow \exists n$ $b^n \in Q$.

  Cela entraîne $\sqrt{Q} = P$ est un idéal premier.

- Si P est maximal et $\sqrt{Q} = P$, alors Q est primaire.

  Mais il y a des idéaux de radical premier qui ne sont pas primaires.

- Tout idéal est intersection d'une famille finie d'idéaux primaires.

  $I = \bigcap Q_\alpha$ $\qquad P_\alpha = \sqrt{Q_\alpha}$

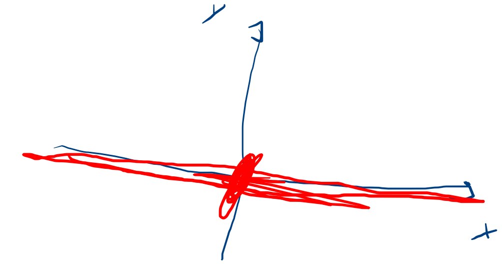
$\mathrm{Spec}(A) \supset V(I) = \bigcup V(P_\alpha)$

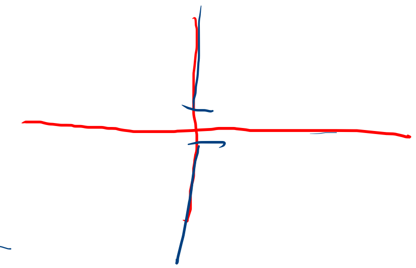
les composants irréductibles de $V(I)$ sont les éléments minimaux des $V(P_\alpha)$

Exemple $\quad y = 0$ ou $(x = 0$ et $y^2 = 0)$

$I = (y) \cap (x, y^2)$

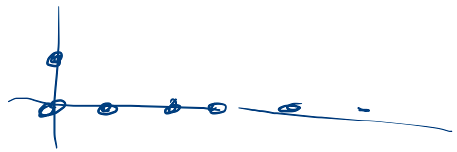
$= (xy, y^2)$



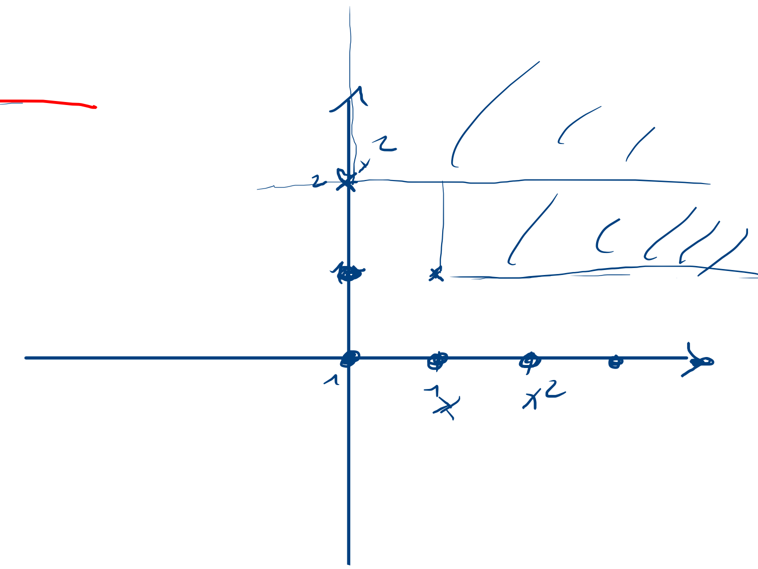


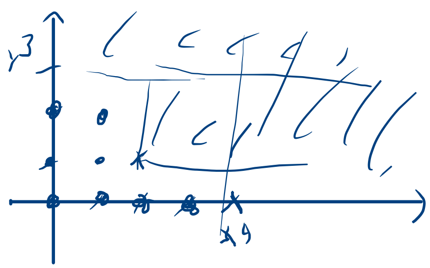
$(1,1)$
$(0,2)$

$M(I) =$



$(x^2 y, x^4, y^3)$





dessous d'escalier

vertex of the Newton polytope of f, so that $c_m \neq 0$ and $\mathrm{T}^m \notin \mathrm{J}$. Then for every integer $s \geqslant 1$, the exponent sm is a vertex of the Newton polytope of f^s, because $\mathrm{NP}_{f^s} = s\mathrm{NP}_f$, and the coefficient of T^{sm} in f^s is equal to c_m^s. Since I is a monomial ideal, one has $c_m^s \mathrm{T}^{sm} \in \mathrm{I}$; by the definition of the radical, one has $\mathrm{T}^m \in \mathrm{J}$, a contradiction.

c) Let I be a monomial ideal and let us consider a primary decomposition $\mathrm{I} = \bigcap_\alpha \mathrm{I}_\alpha$ of I. For every α, let P_α be the radical of I_α, let J_α be the largest monomial ideal in I_α.

Let Q_α be the radical of J_α. It is the largest monomial ideal contained in P_α. Indeed, if a monomial T^m belongs to P_α, then there exists $s \geqslant 1$ such that $\mathrm{T}^{sm} \in \mathrm{I}_\alpha$, hence $\mathrm{T}^{sm} \in \mathrm{J}_\alpha$, hence $\mathrm{T}^m \in \mathrm{Q}_\alpha$.

Let us prove that Q_α is a prime ideal. It is contained in P_α, hence is not equal to (1). Let $f, g \in \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ be such that $fg \in \mathrm{Q}_\alpha$; subtracting from f and g all of their monomials that belong to Q_α, we may assume that they have no monomial in Q_α; assuming that $f \neq 0$, we need to prove that g belongs to Q_α. We may assume that $g \neq 0$. The Newton polytope of fg is equal to the Minkowski sum of the Newton polytopes and f and g. Considering a vertex of the Newton polytope of fg, we get two monomials $c_m \mathrm{T}^m$ of f, and $d_q \mathrm{T}^q$ of g, such that their product $c_m d_q \mathrm{T}^{m+q}$ is a monomial of fg, and their power $(c_m d_q)^s \mathrm{T}^{s(m+q)}$ is a monomial of $(fg)^s$, for every integer $s \geqslant 1$. Since Q_α is the radical of J_α, there exists s such that $(fg)^s \in \mathrm{J}_\alpha$; since J_α is a monomial ideal, one then has $\mathrm{T}^{s(m+q)} \in \mathrm{J}_\alpha \subset \mathrm{I}_\alpha$, hence

$T^{m+q} \in P_\alpha$. The monomial T^m does not belong to P_α, hence $T^q \in P_\alpha$, hence $T^q \in Q_\alpha$.

We now prove that J_α is a Q_α-primary ideal. Similarly, we consider $f, g \in K[T_1, \dots, T_n]$ such that $fg \in J_\alpha$ and $f \notin Q_\alpha$, and prove that $g \in J_\alpha$. Subtracting from f and g all monomials that belong to Q_α and J_α respectively, we reduce ourselves to the case where no monomial of f belongs to Q_α, and no monomial of g belongs to J_α. Assume that $f, g \neq 0$; as above, there are monomials $c_m T^m$ of f and $d_q T^q$ of g such that $c_m d_q T^{m+q}$ is a monomial of fg. Since J_α is a monomial ideal, one has $T^{m+q} \in J_\alpha \subset I_\alpha$. Since $T^m \notin Q_\alpha$ and T^m is a monomial, one has $T^m \notin P_\alpha$. Since I_α is P_α-primary, one then has $T^q \in I_\alpha$, hence $T^q \in J_\alpha$, a contradiction.

Let us now prove that $I = \bigcap_\alpha J_\alpha$. One has $J_\alpha \subset I_\alpha$ for all α, hence $\bigcap_\alpha J_\alpha \subset \bigcap_\alpha I_\alpha = I$. To prove the other inclusion, let $f \in I$ and let us prove that $f \in J_\alpha$ for all α. Since I is a monomial ideal, it suffices to treat the case where f is a monomial. Then for every α, one has $f \in I_\alpha$, hence $f \in J_\alpha$ since f is a monomial. Consequently, $f \in \bigcap_\alpha J_\alpha$. □

Theorem **(3.4.5)** (Maclagan, 2001). — *Let* K *be a field and let* $\mathscr{F}$ *be an infinite set of monomial ideals in* $K[T_1, \dots, T_n]$. *There exists a strictly decreasing sequence of elements of* $\mathscr{F}$.

Proof. — The set of monomial prime ideals is finite. Considering minimal primary decompositions consisting of monomial ideals and successively extracting infinite subsets, we reduce to the case where all ideals in $\mathscr{F}$ are primary with respect to the same prime ideal, $(T_1, \dots, T_m)$. Replacing K by the field $K(T_{m+1}, \dots, T_n)$, we are reduced to the case where all ideals in $\mathscr{F}$ are primary with respect to the maximal ideal $(T_1, \dots, T_n)$.

For every monomial ideal I, let M(I) be the set of $m \in \mathbf{N}^n$ such that $T^m \notin I$.

If $I \in \mathscr{F}$, there exists an integer $N \geqslant 1$ such that $(T_1^N, \dots, T_n^N) \subset I$, so that the set M(I) is contained in $[0; N]^n$; in particular, M(I) is finite.

Observe that the inclusion $I \subset J$ is equivalent to the inclusion $M(J) \subset M(I)$. We will first prove by contradiction that there are ideals $I, J \in \mathscr{F}$ such that $I \subsetneq J$. Assume otherwise.

Let J_0 be the intersection of all ideals in $\mathscr{F}$ and choose $I_1 \in \mathscr{F}$. For every $I \in \mathscr{F}$ such that $I \neq I_1$, one has $I_1 \not\subset I$, so that there exists $m \in M(I_1)$ such that $T^m \in I$. Since $\mathscr{F}$ is infinite and $M(I_1)$ is finite, there exists an infinite subset $\mathscr{F}_1$ of $\mathscr{F}$ and a nonempty subset M_1 of $M(I_1)$ such that for all $I \in \mathscr{F}_1$ and all $m \in \mathbf{N}^n$, $m \in M_1$ if and only if $m \in M(I_1)$ and $T^m \in I$; let then J_1 be the intersection of all ideals I, for $I \in \mathscr{F}_1$. One has $J_0 \subset J_1$, by construction. On the other hand, if $m \in M_1$, then $T^m \in I$ for every $I \in \mathscr{F}_1$, but $T^m \notin I_1$, so that $T^m \in J_1$ and $T^m \notin J_0$, so that $J_0 \subsetneq J_1$.

réduction au cas où $I \in \mathcal{F} \Rightarrow \sqrt{I} = (T_1, \dots, T_n)$

$M(I) = \{m \mid T^m \notin I\}$

dessous d'escalier

$I \subset J \Leftrightarrow M(J) \subset M(I)$

$J_0 = \bigcap_{I \in \mathcal{F}} I$

$\big(M(I_1) \not\subset M(I)\big)$

$I \not\subset I_1$

$I_1 \in \mathcal{F}$

pour tout $I \neq I_1$, il existe $m \in M(I_1)$ tq $m \notin M(I)$

fini

$\mathcal{F}_1$ tq $I \in \mathcal{F}_1 \Rightarrow m_1 \notin M(I)$

$T^{m_1} \in I$

$m_1 \in M(I_1)$ fixé

$J_0 \subsetneq J_1 = \bigcap_{I \in \mathcal{F}_1} I$

$T^m \in J_1 - J_0$

$T^{m_1} \in J_1$

$T^m \in I \quad \forall I \in \mathcal{F}_1$

$m \notin M(I)$

$T^m \notin J_0$

$\exists I \; T^m \notin I$

$\exists I \quad m \in M(I)$

par récurrence

suite str. croissante d'idéaux

(absurde)

Iterating this construction, we construct a strictly increasing sequence (J_k) of ideals in $K[T_1, \dots, T_n]$. This contradicts the fact that this ring is noetherian.

Consequently, in any infinite set of monomial ideals which are primary with respect to the maximal ideal, we can find two ideals which are contained one in another.

Let us now construct a strictly decreasing sequence of ideals in such a set $\mathscr{F}$. Since the ring $K[T_1, \dots, T_n]$ is noetherian, the set $\mathscr{F}$ has finitely many maximal elements; for one of them, say I_1, the set $\mathscr{F}_1$ of ideals $I \in \mathscr{F}$ such that $I \subsetneq I_1$ is infinite. Applying this construction with $\mathscr{F}_1$ instead of $\mathscr{F}$, we obtain an ideal $I_2 \in \mathscr{F}_1$ such that $I_1 \subsetneq I_2$ and an infinite subset of $\mathscr{F}_2$ consisting of ideals contained in $\mathscr{F}$. Iterating this construction, we obtain the desired decreasing sequence. □

3.5. Initial ideals and Gröbner bases

mêmes notations standard

Let K be a valued field, let R be its valuation ring and let k be its residue field. It will be important below to admit the case where the valuation of K is trivial; in fact, we will apply the theory to polynomials with coefficients in k, when viewed as a trivially valued field.

Let $\Gamma = \log(|K^\times|)$ be the value group of K; it is a subgroup of $\mathbf{R}$.

We assume implicitly that *the valued field* K *is split*, denoting by $\gamma \mapsto t^\gamma$ a morphism of groups from Γ to $K^\times$; one has $\log(|t^\gamma|) = \gamma$ for all $\gamma \in \Gamma$. We also write $\rho : K^\times \to k^\times$ for the group morphism given by $a \mapsto \mathrm{red}(at^{-\log(|a|)})$.

(non nul)

3.5.1. — To a polynomial $f \in K[T_0, \dots, T_n]$, we have attached a tropical polynomial $\tau_f : \mathbf{R}^{n+1} \to \mathbf{R}$ as well as, for every $x \in \mathbf{R}^{n+1}$, an initial form $\mathrm{in}_x(f) \in k[T_0, \dots, T_n]$. The exponents of the monomials of $\mathrm{in}_x(f)$ are exponents of monomials of f; in particular, if f is homogeneous of degree d, then so is $\mathrm{in}_x(f)$.

Definition **(3.5.2)**. — *Let* I *be an ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *The* initial ideal *of* I *at* x *is the ideal of* $k[T_0, \dots, T_n]$ *generated by all initial forms* $\mathrm{in}_x(f)$, *for* $f \in$ I. *It is denoted by* $\mathrm{in}_x(\mathrm{I})$.

Lemma **(3.5.3)**. — *Let* I *be an ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *If* I *is a homogeneous ideal, then* $\mathrm{in}_x(\mathrm{I})$ *is a homogeneous ideal.*

Proof. — Let J be the ideal of $k[T_0, \dots, T_n]$ generated by the initial forms $\mathrm{in}_x(f)$, for all *homogeneous* polynomials $f \in$ I; one has $\mathrm{J} \subset \mathrm{in}_x(\mathrm{I})$, and J is a homogeneous ideal. Let $f \in$ I and let $f = \sum_{d \in \mathbf{N}} f_d$ be its decomposition as a sum of homogeneous polynomials, f_d being of degree d. Since I is a homogeneous ideal, one has $f_d \in$ I. By

Inspiré par la théorie des bases de Gröbner (où il n'y a pas de valuation)

Ordre sur les monômes $<$

$f \in K[T] \rightsquigarrow lt(f) = c_m T^m$

$f = \sum c_m T^m$

$m = \sup(S(f))$

$I \subset K[T]$

$\mathrm{in}(I) = \langle lt(f), f \in I \rangle$

idéal monomial

Mais il y a une déformation de $V(I)$ sur $V(\mathrm{in}(I))$ si bien que l'on peut comparer partiellement la géométrie algébrique de $V(I)$ et $V(\mathrm{in}_x I)$)

ex d'ordre $m < m'$ si $\langle m, x \rangle < \langle m', x \rangle$

$x \in \mathbb{R}^n$ coord. très différentes (bien $\mathbb{Q}$-indépendantes) definition of the tropical polynomial, one has

$$\tau_f(x) = \sup_{d\in\mathbf{N}}(\tau_{f_d}(x)).$$

Let D be the set of all $d \in \mathbf{N}$ such that $f_d \neq 0$ and $\tau_f(x) = \tau_{f_d}(x)$. By definition of $\mathrm{in}_x(f)$, one then has

$$\mathrm{in}_x(f) = \sum_{d\in\mathrm{D}} \mathrm{in}_x(f_d),$$

because of the exponents of the monomials appearing in the polynomials f_d are pairwise distinct. In particular, $\mathrm{in}_x(f) \in \mathrm{J}$. This proves that $\boxed{\mathrm{in}_x(\mathrm{I}) = \mathrm{J}}$ is a homogeneous ideal. □

3.5.4. — The initial ideal at 0, $\mathrm{in}_0(\mathrm{I})$, is the image in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ of the ideal $\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ by the reduction morphism. Let indeed J be this ideal. For every $f \in \mathrm{I}$, written as $f = \sum c_m \mathrm{T}^m$, one has $\tau_f(0) = \sup_m \log(|c_m|)$ and $\mathrm{in}_0(f)$ is the image of the element $f t^{-\tau_f(0)} \in \mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, so that $\mathrm{in}_0(f) \in \mathrm{J}$. On the other hand, if $f \in \mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, then either $\tau_f(0) < 0$, in which case the image of f in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ is zero, or $\tau_f(0) = 0$, in which case $\mathrm{in}_0(f)$ is the image of f. This proves that $\mathrm{J} = \mathrm{in}_0(\mathrm{I})$.

Moreover, $\mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]/(\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n])$ is a torsion free R-module, hence is *flat*, because R is a valuation ring. In the case where I is a homogeneous ideal, this says that the family $\mathrm{Proj}(\mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]/(\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n])) \to \mathrm{Spec}(\mathrm{R})$ is a flat morphism

$f = \sum c_m T^m = \sum_d f_d$ $\qquad f_d = \sum_{|m|=d} c_m T^m$

$\tau_f(x) = \sup_{m\in S(f)} (\log|c_m| + \langle m, x\rangle)$

$= \sup_d \underbrace{\sup_{|m|=d} (\ \_\_\_\ )}_{\tau_{f_d}(x)}$

$\mathrm{in}_x(f) \in$,

$f \in I$

$\mathrm{in}_x(I) \supset J' = \langle \mathrm{in}_x(f),\ f \text{ homogène} \rangle$

$= \langle \mathrm{in}_x(f) \rangle$

$f \in I \Rightarrow \mathrm{in}_x(f) \in J$.

$\Rightarrow \langle \mathrm{in}_x(f) \rangle \subset J$

Idéal initial en $x=0$

$f = \sum c_m T^m$

$\tau_f(0) = \sup_m \log(|c_m|) = 0$ (sinon $f \rightsquigarrow \lambda f$)

- $|c_m| \leq 1$ $\quad \sup |c_m| = 1$

$f \in R[T]$ $\quad \bar{f} = red(f) \neq 0$ (même contexte que la prop. de relèvement)

$$in_0(f) = \sum_{\substack{m \\ |c_m|=1}} \bar{c_m} T^m = \bar{f}$$

- si $\tau_f(0) = \log_{p \in S(f)} |c_p| \Rightarrow c_p^{-1} f$

$$in_0(f) = \sum_{\substack{m \\ |c_m|=|c_p|}} \rho(c_m) T^m = \sum_m \rho(c_m/c_p) T^m \cdot \rho(c_p)$$

$$\overline{c_m/c_p}\; T^m \cdot \underbrace{\quad}_{\neq 0}$$

$$= red(c_p^{-1} f)$$

$$in_0(I) = red(I \cap R[T_0, \dots, T_n])$$

of projective schemes; its generic fiber is $\operatorname{Proj}(\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I}) = \mathrm{V}(\mathrm{I})$, and its closed fiber is $\operatorname{Proj}(k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_0(\mathrm{I})) = \mathrm{V}(\mathrm{in}_0(\mathrm{I}))$. This flatness has the following important consequences:[2]

– The Hilbert functions of I and $\mathrm{in}_0(\mathrm{I})$ coincide. Explicitly, for every integer d, one has

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_0(\mathrm{I}))_d);$$

– If $\mathrm{V}(\mathrm{I})$ is integral, then $\mathrm{V}(\mathrm{in}_0(\mathrm{I}))$ is equidimensional, of the same dimension.

3.5.5. — Let $x \in \mathbf{R}^{n+1}$; let us assume that the coordinates of x belong to the value group Γ. For every $j \in \{0,\dots,n\}$, fix $a_j \in \mathrm{K}^\times$ such that $\log(|a_j|) = x_j$; let also $\alpha_j = \rho(a_j)$ for every j.

For every $f = \sum c_m \mathrm{T}^m \in \mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$, one has $f(a\mathrm{T}) = \sum c_m a^m \mathrm{T}^m$, so that

$$\tau_{f(a\mathrm{T})}(0) = \sup_m (\log(|c_m|) + \langle m, x\rangle) = \tau_f(x),$$

as well as

$$\mathrm{in}_0(f(a\mathrm{T})) = \sum_{m\in \mathrm{S}_f(x)} \rho(c_m a^m)\mathrm{T}^m = \sum_{m\in \mathrm{S}_f(x)} \rho(c_m)\alpha^m \mathrm{T}^m = \mathrm{in}_x(f)(\alpha\mathrm{T}).$$

Let φ_a be the K-algebra automorphism of $\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$ given by $\varphi_a(f) = f(a_0\mathrm{T}_0,\dots,a_n\mathrm{T}_n)$ and let ψ_α be the k-algebra automorphism

[2] Maybe write an appendix with material from commutative algebra and algebraic geometry that is used in the notes.

of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ given by $\psi_\alpha(f) = f(\alpha_0\mathrm{T}, \dots, \alpha_n\mathrm{T})$. By the preceding computation, we have $\psi_\alpha(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_0(\varphi_a(\mathrm{I}))$ is the image of the ideal $\varphi_a(\mathrm{I}) \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$.

This change of variables will allow to reduce properties of the initial ideal $\mathrm{in}_x(\mathrm{I})$ to the case of $x = 0$. In particular, it immediately implies the following lemma.

$x = 0$

la fonction de Hilbert est polynomiale à partir d'un certain rang

$\dim V(I) =$ degré de ce polynôme

Lemma **(3.5.6)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$ *be such that its coordinates belong to the value group of* K.

faux si $x \notin \Gamma^n$

a) *The initial ideal* $\mathrm{in}_x(\mathrm{I})$ *is the set of all* $\mathrm{in}_x(f)$, *for* $f \in \mathrm{I}$;

b) *If* V(I) *is integral, then* $\mathrm{V}(\mathrm{in}_x(\mathrm{I}))$ *is equidimensional, of the same dimension;*

c) *The Hilbert functions of* I *and* $\mathrm{in}_x(\mathrm{I})$ *coincide. Explicitly, for every integer* d, *one has*

non trivial

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d).$$

x quelconque

changement de variable $T \to aT$

$\lambda(a) = x$

One of the goals of the theory that we develop now is to extend these properties to an arbitrary $x \in \mathbf{R}^{n+1}$.

si $x \in \Gamma^n$

exercice

Remark **(3.5.7)**. — Let $x_1, x_2 \in \mathbf{R}$ be nonzero real numbers, $\mathbf{Q}$-linearly independent, such that $(\mathbf{Q}x_1 + \mathbf{Q}x_2) \cap v(\mathrm{K}^\times) = 0$. Let $\mathrm{I} = (\mathrm{T}_1, \mathrm{T}_2)$. One has $\mathrm{in}_x(\mathrm{I}) \subset (\mathrm{T}_1, \mathrm{T}_2)$, and the relations $\mathrm{in}_x(\mathrm{T}_1) = \mathrm{T}_1$ and $\mathrm{in}_x(\mathrm{T}_2) = \mathrm{T}_2$ imply that $\mathrm{in}_x(\mathrm{I}) = (\mathrm{T}_1, \mathrm{T}_2)$.

On the other hand, let $f \in \mathrm{K}[\mathrm{T}_1, \mathrm{T}_2]$, written $\sum c_m \mathrm{T}^m$, and let $m, n \in \mathbf{N}^2$ be elements such that $\log(|c_m|) + \langle x, m\rangle = \log(|c_n|) + \langle x, n\rangle = \tau_f(x)$. Then $\log(|c_m/c_n|) + x_1(m_1 - n_1) + x_2(m_2 - n_2) = 0$, so that $c_m/c_n \in \mathrm{R}^\times$, $m_1 = n_1$ and $m_2 = n_2$; this proves that $\mathrm{in}_x(f)$ is a monomial. In that case, the set of polynomials of the form $\mathrm{in}_x(f)$, for $f \in \mathrm{I}$, is not an ideal of I. In particular, the statement of Lemma 2.4.2 in Maclagan & Sturmfels (2015) is incorrect (this is signaled in the *errata* of that reference).

The next lemma is a weakening of the expected property.

Beware:

The definition of the nonarchimedean amoebas has been modified so as to be more consistent with the definition in the archimedean case. I made the necessary corrections up to here, but there are certainly inconsistencies below.

Lemma **(3.5.8)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$.

a) *Every element of* $\mathrm{in}_x(\mathrm{I})$ *is the sum of polynomials of the form* $\mathrm{in}_x(f)$*, for* $f \in \mathrm{I}$*;*

b) *Let* $f, g \in \mathrm{I}$*. If the supports of* $\mathrm{in}_x(f)$ *and* $\mathrm{in}_x(g)$ *are not disjoint, then there exists* $h \in \mathrm{I}$ *such that* $\mathrm{in}_x(h) = \mathrm{in}_x(f) + \mathrm{in}_x(g)$*. If* $\tau_f(x) = \tau_g(x)$*, one may even take* $h = f + g$*;*