TOPICS
IN TROPICAL GEOMETRY

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$b_j \in \Gamma$. Each face of P is defined by adding inequalities of the form $-f_j \leqslant 0$, for j in some subset I of J; this shows that they are rational. Similarly, the recession cone of P is defined by the linear inequalities $\varphi_j \leqslant 0$; it is thus Q-rational. The lineality space is defined by the equalities $\varphi_j = 0$, for $j \in \mathrm{J}$, hence is Q-rational. Finally, since the affine span of P is defined by all the implicit equalities $f_j = 0$ in the given system, it is (Q, Γ)-rational as well. □

Example **(1.7.5)**. — Let C be a Q-rational cone.

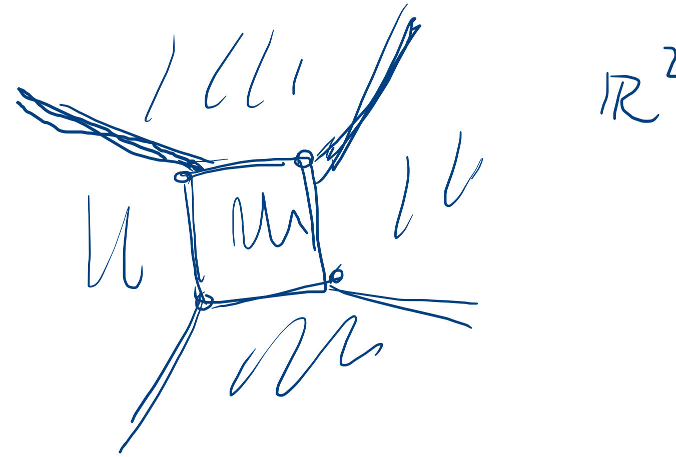
Assume that $\dim(\mathrm{C}) = 1$. In this case, $\operatorname{affsp}(\mathrm{C}) = \mathrm{C} - \mathrm{C}$. Since $\operatorname{affsp}(\mathrm{C})$ is a Q-rational line, there exists $v \in \mathrm{Q}^n$ such that $\operatorname{affsp}(\mathrm{C}) = \mathrm{R}v$. Up to replacing v by $-v$, one then has $\mathrm{C} = \mathrm{R}_+ v$.

In the general case, the extremal rays of C are themselves Q-rational cones, hence of the form $\mathrm{R}_+ v$ for some $v \in \mathrm{Q}^n$. Given proposition 1.6.4, this implies that C is the polyhedral convex cone generated by a finite family of vectors in Q^n.

1.8. Polyhedral subspaces, fans

Definition **(1.8.1)**. — *Let* V *be a finite dimensional* R*-vector space and let* S *be a subspace of* V.

a) *One says that* S *is a* polyhedral subspace *of* V *if it is a finite union of polyhedra in* V*;*



b) *A* polyhedral decomposition *of* S *is a finite set* $\mathscr{C}$ *of polyhedra satisfying the following properties:*

(i) *The union of all polyhedra in* $\mathscr{C}$ *is equal to* S*;*

(ii) *Every face of a polyhedron in* $\mathscr{C}$ *belongs to* $\mathscr{C}$*;*

(iii) *The intersection of every two polyhedra* P, Q *in* $\mathscr{C}$ *is either empty, or a face of both of them.*

The set S *is also called the support of the polyhedral decomposition* $\mathscr{C}$*, and is denoted by* $|\mathscr{C}|$*.*

c) *A* fan *is a polyhedral decomposition all of which polyhedra are cones.*

Remark **(1.8.2)**. — a) If a finite union of polyhedral cones is a convex cone, then it is a polyhedral cone. In other words, a convex cone is a polyhedral subset if and only if it is a polyhedral cone, so that the terminology is not ambiguous.

b) A polyhedral decomposition is determined by its maximal polyhedra, all other are faces of them. Since a face of a cone is a cone, a polyhedral decomposition is a fan if and only if its maximal polyhedra are cones.

c) Let $\mathscr{C}$ be a polyhedral decomposition of a polyhedral subspace S. For every $x \in$ S and every polyhedron P $\in \mathscr{C}$ such that $x \in$ P, either x belongs to a facet of P, or x belongs to the relative interior of P, bot not simultaneously. Consequently, the relative interiors of the polyhedra in $\mathscr{C}$ are pairwise disjoint, and their union is S.

$T^{m+q} \in P_\alpha$. The monomial T^m does not belong to P_α, hence $T^q \in P_\alpha$, hence $T^q \in Q_\alpha$.

We now prove that J_α is a Q_α-primary ideal. Similarly, we consider $f, g \in K[T_1, \dots, T_n]$ such that $fg \in J_\alpha$ and $f \notin Q_\alpha$, and prove that $g \in J_\alpha$. Subtracting from f and g all monomials that belong to Q_α and J_α respectively, we reduce ourselves to the case where no monomial of f belongs to Q_α, and no monomial of g belongs to J_α. Assume that $f, g \neq 0$; as above, there are monomials $c_m T^m$ of f and $d_q T^q$ of g such that $c_m d_q T^{m+q}$ is a monomial of fg. Since J_α is a monomial ideal, one has $T^{m+q} \in J_\alpha \subset I_\alpha$. Since $T^m \notin Q_\alpha$ and T^m is a monomial, one has $T^m \notin P_\alpha$. Since I_α is P_α-primary, one then has $T^q \in I_\alpha$, hence $T^q \in J_\alpha$, a contradiction.

Let us now prove that $I = \bigcap_\alpha J_\alpha$. One has $J_\alpha \subset I_\alpha$ for all α, hence $\bigcap_\alpha J_\alpha \subset \bigcap_\alpha I_\alpha = I$. To prove the other inclusion, let $f \in I$ and let us prove that $f \in J_\alpha$ for all α. Since I is a monomial ideal, it suffices to treat the case where f is a monomial. Then for every α, one has $f \in I_\alpha$, hence $f \in J_\alpha$ since f is a monomial. Consequently, $f \in \bigcap_\alpha J_\alpha$. □

Theorem **(3.4.5)** (Maclagan, 2001). — *Let* K *be a field and let* $\mathscr{F}$ *be an infinite set of monomial ideals in* $K[T_1, \dots, T_n]$. *There exists a strictly decreasing sequence of elements of* $\mathscr{F}$.

Proof. — The set of monomial prime ideals is finite. Considering minimal primary decompositions consisting of monomial ideals and successively extracting infinite subsets, we reduce to the case where all ideals in $\mathscr{F}$ are primary with respect to the same prime ideal, $(T_1, \dots, T_m)$. Replacing K by the field $K(T_{m+1}, \dots, T_n)$, we are reduced to the case where all ideals in $\mathscr{F}$ are primary with respect to the maximal ideal $(T_1, \dots, T_n)$.

For every monomial ideal I, let M(I) be the set of $m \in \mathbf{N}^n$ such that $T^m \notin I$.

If $I \in \mathscr{F}$, there exists an integer $N \geqslant 1$ such that $(T_1^N, \dots, T_n^N) \subset I$, so that the set M(I) is contained in $[0; N]^n$; in particular, M(I) is finite.

Observe that the inclusion $I \subset J$ is equivalent to the inclusion $M(J) \subset M(I)$. We will first prove by contradiction that there are ideals $I, J \in \mathscr{F}$ such that $I \subsetneq J$. Assume otherwise.

Let J_0 be the intersection of all ideals in $\mathscr{F}$ and choose $I_1 \in \mathscr{F}$. For every $I \in \mathscr{F}$ such that $I \neq I_1$, one has $I_1 \not\subset I$, so that there exists $m \in M(I_1)$ such that $T^m \in I$. Since $\mathscr{F}$ is infinite and $M(I_1)$ is finite, there exists an infinite subset $\mathscr{F}_1$ of $\mathscr{F}$ and a nonempty subset M_1 of $M(I_1)$ such that for all $I \in \mathscr{F}_1$ and all $m \in \mathbf{N}^n$, $m \in M_1$ if and only if $m \in M(I_1)$ and $T^m \in I$; let then J_1 be the intersection of all ideals I, for $I \in \mathscr{F}_1$. One has $J_0 \subset J_1$, by construction. On the other hand, if $m \in M_1$, then $T^m \in I$ for every $I \in \mathscr{F}_1$, but $T^m \notin I_1$, so that $T^m \in J_1$ and $T^m \notin J_0$, so that $J_0 \subsetneq J_1$.

$\mathcal{F}_1 \ni I = Q_1 \cap \dots \cap Q_m$ — Q_j primaires, monomiaux

$\sqrt{I} = P_1 \cap \dots \cap P_m$ — $P_j = \sqrt{Q_j}$ premiers monomiaux.

nombre fini de P_j possibles

, parmi les él. de $\mathcal{F}$,
il y en a un ens. infini
qui ont une décomposition d'un même type,
avec les mêmes P_j — on se ramène au cas où $\mathcal{F}$...

• On fixe P idéal premier monomial.

$\mathcal{F}_P = \{Q,$ Q est une composante P-primaire d'un él de $\mathcal{F}\}$

$(\sqrt{Q} = P)$

si $\mathcal{F}_P$ est fini, l'un des Q apparaît une infinité de fois, on se ramène au cas où $\mathcal{F}_P = \{Q\}$.

sinon $\mathcal{F}_P$ est infini.
on construit une suite décroissante
on remplace $\mathcal{F}_1$ par les termes de cette suite

→

Iterating this construction, we construct a strictly increasing sequence (J_k) of ideals in $K[T_1, \dots, T_n]$. This contradicts the fact that this ring is noetherian.

Consequently, in any infinite set of monomial ideals which are primary with respect to the maximal ideal, we can find two ideals which are contained one in another.

Let us now construct a strictly decreasing sequence of ideals in such a set $\mathscr{F}$. Since the ring $K[T_1, \dots, T_n]$ is noetherian, the set $\mathscr{F}$ has finitely many maximal elements; for one of them, say I_1, the set $\mathscr{F}_1$ of ideals $I \in \mathscr{F}$ such that $I \subsetneq I_1$ is infinite. Applying this construction with $\mathscr{F}_1$ instead of $\mathscr{F}$, we obtain an ideal $I_2 \in \mathscr{F}_1$ such that $I_1 \subsetneq I_2$ and an infinite subset of $\mathscr{F}_2$ consisting of ideals contained in $\mathscr{F}$. Iterating this construction, we obtain the desired decreasing sequence. □

3.5. Initial ideals and Gröbner bases

Let K be a valued field, let R be its valuation ring and let k be its residue field. It will be important below to admit the case where the valuation of K is trivial; in fact, we will apply the theory to polynomials with coefficients in k, when viewed as a trivially valued field.

Let $\Gamma = \log(|K^\times|)$ be the value group of K; it is a subgroup of $\mathbf{R}$.

We assume implicitly that *the valued field* K *is split*, denoting by $\gamma \mapsto t^\gamma$ a morphism of groups from Γ to $K^\times$; one has $\log(|t^\gamma|) = \gamma$ for all $\gamma \in \Gamma$. We also write $\rho \colon K^\times \to k^\times$ for the group morphism given by $a \mapsto \operatorname{red}(at^{-\log(|a|)})$.

3.5.1. — With a polynomial $f \in K[T_0, \dots, T_n]$, we have associated a tropical polynomial $\tau_f : \mathbf{R}^{n+1} \to \mathbf{R}$ as well as, for every $x \in \mathbf{R}^{n+1}$, an initial form $\operatorname{in}_x(f) \in k[T_0, \dots, T_n]$. The exponents of the monomials of $\operatorname{in}_x(f)$ are exponents of monomials of f; in particular, if f is homogeneous of degree d, then so is $\operatorname{in}_x(f)$.

Definition **(3.5.2)**. — *Let* I *be an ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *The* initial ideal *of* I *at* x *is the ideal of* $k[T_0, \dots, T_n]$ *generated by all initial forms* $\operatorname{in}_x(f)$, *for* $f \in I$. *It is denoted by* $\operatorname{in}_x(I)$.

Lemma **(3.5.3)**. — *Let* I *be an ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *If* I *is a homogeneous ideal, then* $\operatorname{in}_x(I)$ *is a homogeneous ideal.*

Proof. — Let J be the ideal of $k[T_0, \dots, T_n]$ generated by the initial forms $\operatorname{in}_x(f)$, for all *homogeneous* polynomials $f \in I$; one has $J \subset \operatorname{in}_x(I)$, and J is a homogeneous ideal. Let $f \in I$ and let $f = \sum_{d \in \mathbf{N}} f_d$ be its decomposition as a sum of homogeneous polynomials, f_d being of degree d. Since I is a homogeneous ideal, one has $f_d \in I$. By definition of the tropical polynomial, one has

$$\tau_f(x) = \sup_{d\in\mathbf{N}}(\tau_{f_d}(x)).$$

Let D be the set of all $d \in \mathbf{N}$ such that $f_d \neq 0$ and $\tau_f(x) = \tau_{f_d}(x)$. By definition of $\mathrm{in}_x(f)$, one then has

$$\mathrm{in}_x(f) = \sum_{d\in\mathrm{D}} \mathrm{in}_x(f_d),$$

because of the exponents of the monomials appearing in the polynomials f_d are pairwise distinct. In particular, $\mathrm{in}_x(f) \in \mathrm{J}$. This proves that $\mathrm{in}_x(\mathrm{I}) = \mathrm{J}$ is a homogeneous ideal. □

3.5.4. — The initial ideal at 0, $\mathrm{in}_0(\mathrm{I})$, is the image in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ of the ideal $\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ by the reduction morphism. Let indeed J be this ideal. For every $f \in \mathrm{I}$, written as $f = \sum c_m \mathrm{T}^m$, one has $\tau_f(0) = \sup_m \log(|c_m|)$ and $\mathrm{in}_0(f)$ is the image of the element $f t^{-\tau_f(0)} \in \mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, so that $\mathrm{in}_0(f) \in \mathrm{J}$. On the other hand, if $f \in \mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, then either $\tau_f(0) < 0$, in which case the image of f in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ is zero, or $\tau_f(0) = 0$, in which case $\mathrm{in}_0(f)$ is the image of f. This proves that $\mathrm{J} = \mathrm{in}_0(\mathrm{I})$.

Moreover, $\mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]/(\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n])$ is a torsion free R-module, hence is *flat*, because R is a valuation ring. In the case where I is a homogeneous ideal, this says that the family $\mathrm{Proj}(\mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]/(\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n])) \to \mathrm{Spec}(\mathrm{R})$ is a flat morphism of projective schemes; its generic fiber is $\operatorname{Proj}(\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I}) = \mathrm{V}(\mathrm{I})$, and its closed fiber is $\operatorname{Proj}(k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_0(\mathrm{I})) = \mathrm{V}(\mathrm{in}_0(\mathrm{I}))$. This flatness has the following important consequences:[2]

– The Hilbert functions of I and $\mathrm{in}_0(\mathrm{I})$ coincide. Explicitly, for every integer d, one has

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_0(\mathrm{I}))_d);$$

– If $\mathrm{V}(\mathrm{I})$ is integral, then $\mathrm{V}(\mathrm{in}_0(\mathrm{I}))$ is equidimensional, of the same dimension.

3.5.5. — Let $x \in \mathbf{R}^{n+1}$; let us assume that the coordinates of x belong to the value group Γ. For every $j \in \{0,\dots,n\}$, fix $a_j \in \mathrm{K}^\times$ such that $\log(|a_j|) = x_j$; let also $\alpha_j = \rho(a_j)$ for every j.

For every $f = \sum c_m \mathrm{T}^m \in \mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$, one has $f(a\mathrm{T}) = \sum c_m a^m \mathrm{T}^m$, so that

$$\tau_{f(a\mathrm{T})}(0) = \sup_m(\log(|c_m|) + \langle m, x\rangle) = \tau_f(x),$$

as well as

$$\mathrm{in}_0(f(a\mathrm{T})) = \sum_{m\in \mathrm{S}_f(x)} \rho(c_m a^m)\mathrm{T}^m = \sum_{m\in \mathrm{S}_f(x)} \rho(c_m)\alpha^m \mathrm{T}^m = \mathrm{in}_x(f)(\alpha\mathrm{T}).$$

Let φ_a be the K-algebra automorphism of $\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$ given by $\varphi_a(f) = f(a_0\mathrm{T}_0,\dots,a_n\mathrm{T}_n)$ and let ψ_α be the k-algebra automorphism

[2] Maybe write an appendix with material from commutative algebra and algebraic geometry that is used in the notes.

K corps valué
R anneau de valuation
k corps résiduel
$\Gamma = \log |K^\times| \subset \mathbb{R}$ groupe de valeurs.

$I \subset K[T_0, \dots, T_n]$ idéal (homogène)
$x \in \mathbb{R}^{n+1} \rightsquigarrow \mathrm{in}_x(I) \subset k[T_0, \dots, T_n] = \langle \mathrm{in}_x(f), f \in I \rangle$

$x \in \Gamma^{n+1}$

of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ given by $\psi_\alpha(f) = f(\alpha_0\mathrm{T}, \dots, \alpha_n\mathrm{T})$. By the preceding computation, we have $\psi_\alpha(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_0(\varphi_a(\mathrm{I}))$ is the image of the ideal $\varphi_a(\mathrm{I}) \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$.

This change of variables will allow to reduce properties of the initial ideal $\mathrm{in}_x(\mathrm{I})$ to the case of $x = 0$. In particular, it immediately implies the following lemma.

Lemma **(3.5.6)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$ *be such that its coordinates belong to the value group of* K.

a) *The initial ideal* $\mathrm{in}_x(\mathrm{I})$ *is the set of all* $\mathrm{in}_x(f)$, *for* $f \in \mathrm{I}$;

b) *If* $\mathrm{V}(\mathrm{I})$ *is integral, then* $\mathrm{V}(\mathrm{in}_x(\mathrm{I}))$ *is equidimensional, of the same dimension;*

c) *The Hilbert functions of* I *and* $\mathrm{in}_x(\mathrm{I})$ *coincide. Explicitly, for every integer* d, *one has*

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d).$$

$\varphi_I(d) = \dim_K \left(K[T_0, \dots, T_n]/I \right)_d = \dim_K \left(K[T_0 \dots T_n]_d / I_d \right)$

One of the goals of the theory that we develop now is to extend these properties to an arbitrary $x \in \mathbf{R}^{n+1}$.

Remark **(3.5.7)**. — Let $x_1, x_2 \in \mathbf{R}$ be nonzero real numbers, $\mathbf{Q}$-linearly independent, such that $(\mathbf{Q}x_1 + \mathbf{Q}x_2) \cap v(\mathrm{K}^\times) = 0$. Let $\mathrm{I} = (\mathrm{T}_1, \mathrm{T}_2)$. One has $\mathrm{in}_x(\mathrm{I}) \subset (\mathrm{T}_1, \mathrm{T}_2)$, and the relations $\mathrm{in}_x(\mathrm{T}_1) = \mathrm{T}_1$ and $\mathrm{in}_x(\mathrm{T}_2) = \mathrm{T}_2$ imply that $\mathrm{in}_x(\mathrm{I}) = (\mathrm{T}_1, \mathrm{T}_2)$.

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On the other hand, let $f \in \mathrm{K}[\mathrm{T}_1, \mathrm{T}_2]$, written $\sum c_m \mathrm{T}^m$, and let $m, n \in \mathbf{N}^2$ be elements such that $\log(|c_m|) + \langle m, x\rangle = \log(|c_n|) + \langle x, n\rangle = \tau_f(x)$. Then $\log(|c_m/c_n|) + x_1(m_1 - n_1) + x_2(m_2 - n_2) = 0$, so that $c_m/c_n \in \mathrm{R}^\times$, $m_1 = n_1$ and $m_2 = n_2$; this proves that $\mathrm{in}_x(f)$ is a monomial. In that case, the set of polynomials of the form $\mathrm{in}_x(f)$, for $f \in \mathrm{I}$, is not an ideal of I. In particular, the statement of Lemma 2.4.2 in Maclagan & Sturmfels (2015) is incorrect (this is signaled in the *errata* of that reference).

The next lemma is a weakening of the expected property.

Lemma **(3.5.8)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$.

a) *Every element of* $\mathrm{in}_x(\mathrm{I})$ *is a sum of polynomials of the form* $\mathrm{in}_x(f)$, *for* $f \in \mathrm{I}$.

b) *Let* $f, g \in \mathrm{I}$. *If the supports of* $\mathrm{in}_x(f)$ *and* $\mathrm{in}_x(g)$ *are not disjoint, then there exists* $h \in \mathrm{I}$ *such that* $\mathrm{in}_x(h) = \mathrm{in}_x(f) + \mathrm{in}_x(g)$. *If* $\tau_f(x) = \tau_g(x)$ *and* $\mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0$, *then one may even take* $h = f + g$.

c) *Let* $m \in \mathbf{N}^{n+1}$; *if* $\mathrm{T}^m \in \mathrm{in}_x(\mathrm{I})$, *then there exists* $f \in \mathrm{I}$ *such that* $\mathrm{T}^m = \mathrm{in}_x(f)$.

Proof. — a) Let $f \in \mathrm{I}$, let $\alpha \in k^\times$ and let $m \in \mathbf{N}^{n+1}$. Let $a \in \mathrm{R}^\times$ be such that $\rho(a) = \alpha$. One has $\tau_{a\mathrm{T}^m f}(x) = \tau_f(x) + \langle m, x\rangle$ and $\mathrm{in}_x(a\mathrm{T}^m f) = \alpha \mathrm{T}^m \mathrm{in}_x(f)$ (this is an elementary instance of lemma 3.3.4). This proves that the set of initial forms is stable under

$\varphi \in \mathrm{in}_x(\mathrm{I})$

$\Rightarrow \exists f_1, \dots, f_p \in \mathrm{I}$

tq 1) $\varphi = \sum_{j=1}^{p} \mathrm{in}_x(f_j)$

2) $S(\mathrm{in}_x(f_j)) \cap S(\mathrm{in}_x(f_i)) = \emptyset$ $\forall i \neq j$

φ monôme $\Rightarrow \varphi = \mathrm{in}_x(f)$

a) $\varphi = \sum g_j \, \mathrm{in}_x(f_j)$

$g_j = \sum \underbrace{c_{m,j} \mathrm{T}^m}$

$\mathrm{in}_x(c\mathrm{T}^m f) = \gamma \mathrm{T}^m \mathrm{in}_x(f)$

$\varphi = \sum \mathrm{in}_x(c_{m_j} \mathrm{T}^m f_j)$

b)

$\Gamma \to K^\times$
$\gamma \mapsto t^\gamma$

$\rho : K^\times \to k^\times$

$\mu \in S_x(f) \cap S_x(g)$

$f = \sum c_m T^m$
$g = \sum d_m T^m$

cas où $|c_\mu| = |d_\mu| = 1$.

multiplication by a monomial. In particular, the additive monoid it generates in $k[T_0, \dots, T_n]$ is an ideal of $k[T_0, \dots, T_n]$.

b) Write $f = \sum c_m T^m$, $g = \sum d_m T^m$ and let $\mu \in \mathbf{N}^{n+1}$ be a common point of the supports of $\mathrm{in}_x(f)$ and of $\mathrm{in}_x(g)$. This means that $\log(|c_\mu|) + \langle \mu, x\rangle = \tau_f(x) = \sup_m(\log(|c_m|) + \langle m, x\rangle)$ and $\log(|d_\mu|) + \langle \mu, x\rangle = \tau_g(x) = \sup_m(\log(|d_m|) + \langle m, x\rangle)$. In particular, $\tau_f(x) - \tau_g(x) = \log(|c_\mu/d_\mu|)$. Replacing f by $ft^{-\log(|c_\mu|)}$ and g by $gt^{-\log(|d_\mu|)}$ does not change $\mathrm{in}_x(f)$ and $\mathrm{in}_x(g)$ and allows us to assume that $|c_\mu| = |d_\mu| = 1$ and $\boxed{\tau_f(x) = \tau_g(x)} = \langle \mu, x\rangle$.

If $\mathrm{in}_x(f) + \mathrm{in}_x(g) = 0$, then we take $h = 0$.

Let us now assume that $\boxed{\mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0}$ and let $h = f + g = \sum(c_m + d_m)T^m$. For all m, one has $\log(|c_m|) + \langle m, x\rangle \leqslant \tau_f(x)$ and $\log(|d_m|) + \langle m, x\rangle \leqslant \tau_g(x) = \tau_f(x)$, so that $\log(|c_m + d_m|) + \langle m, x\rangle \leqslant \tau_f(x)$ and $\boxed{\tau_h(x) \leqslant \tau_f(x).}$

Let $m \in \mathbf{N}^{m+1}$.

Let us assume that $m \in S_x(f) - S_x(g)$. Then $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$ but $\log(|d_m|) + \langle m, x\rangle < \tau_g(x) = \tau_f(x)$; we then have $|d_m| < |c_m|$, hence $|c_m + d_m| = |c_m|$ and $\log(|c_m + d_m|) + \langle m, x\rangle = \tau_f(x)$. This implies that $\tau_h(x) = \tau_f(x)$. Moreover, $\rho(c_m + d_m) = \rho(c_m)$ is the coefficient of T^m in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ and in $\mathrm{in}_x(h)$.

Similarly, if $m \in S_x(g) - S_x(f)$, then $|c_m + d_m| = |d_m| > |c_m|$, $\tau_h(x) = \tau_f(x)$, and $\rho(c_m + d_m) = \rho(d_m)$ is the coefficient of T^m in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ and in $\mathrm{in}_x(h)$.

$\rho(c) = \mathrm{red}(c\, t^{-\log|c|})$

$\rho(ct^\gamma) = \rho(c)$

$\log|c_m| + \langle m, x\rangle \leq \tau_f(x)$
$\log|d_m| + \langle m, x\rangle \leq \tau_g(x)$
$\log|c_m + d_m| + \langle m, x\rangle \leq \tau_f(x)$.

$\tau_h(x) \leq \tau_f(x)$

$m \in \mathbb{N}^{n+1}$

$$\begin{cases} \tau_f(x) = \tau_g(x) \\ h = f + g \\ \mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0 \end{cases}$$

Cinq cas.

① $m \in S_x(f) - S_x(g)$

$\log|c_m| + \langle m, x\rangle = \tau_f(x)$

$\log|d_m| + \langle m, x\rangle < \tau_g(x)$

$|d_m| < |c_m|$, donc $|c_m + d_m| = |c_m|$

$\Rightarrow \tau_h(x) = \tau_f(x)$

le coeff de T^m dans $\mathrm{in}_x(h)$ est égal à celui de $\mathrm{in}_x(f)$.
(celui de $\mathrm{in}_x(g)$ est nul)

② $m \in S_x(g) - S_x(f)$

cas symétrique, $\tau_h(x) = \tau_g(x) = \tau_f(x)$

le coeff. de T^m dans $\mathrm{in}_x(h)$ est celui de $\mathrm{in}_x(g)$
(celui de $\mathrm{in}_x(f)$ est nul).

③ $m \in S_x(f) \cap S_x(g)$ et le coeff de T^m dans $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ n'est pas nul
$\rho(c_m) + \rho(d_m)$

on a $\begin{cases} \log|c_m| = \log|d_m| = \tau_f(x) - \langle m, x\rangle \\ \underbrace{\rho(c_m) + \rho(d_m)}_{\neq 0} = \mathrm{red}(c_m t^{-\log|c_m|}) + \mathrm{red}(d_m t^{-\log|d_m|}) \end{cases}$

$= \mathrm{red}((c_m + d_m)\, t^{-\log|c_m|}) \Rightarrow |c_m + d_m| = |c_m|$

$\Rightarrow \tau_h(x) = \tau_f(x)$ et le coeff. de T^m dans $\mathrm{in}_x(h)$ est $\rho(c_m) + \rho(d_m)$

<u>Pause</u> : . l'un de ces trois cas se produit
prendre m dans le support de $in_X(f) + in_X(g)$.

$\tau_h(x) = \tau_f(x)$ est démontré.

Deux derniers cas

4 $m \in S_X(f) \cap S_X(g)$ mais le coeff de T^m dans $in_X(f) + in_X(g)$ est nul.

même raisonnement qu'en 3

$$\text{red}\left((c_m + d_m)\, t^{-\log|c_m|}\right) = \rho(c_m) + \rho(d_m) = 0$$

donc $|c_m + d_m| < |c_m|$

donc $\log|c_m + d_m| + \langle m, x\rangle < \log|c_m| + \langle m, x\rangle$
$= \tau_h(x)$

donc $m \notin \bar{S}_X(h)$

T^m n'apparaît pas dans $in_X(h)$.

5 $m \notin S_X(f) \cup S_X(g)$; alors $\log|c_m|, \log|d_m| < \tau_f(x) - \langle m, x\rangle$

alors $\log|c_m + d_m| + \langle m, x\rangle < \tau_f(x) = \tau_h(x)$ donc $m \in S_X(h)$.

If $m \in \mathrm{S}_x(f) \cap \mathrm{S}_x(g)$ and $\rho(c_m) + \rho(d_m) \neq 0$, then $\log(|c_m|) + \langle m, x \rangle = \tau_f(x) = \tau_g(x) = \log(|d_m|) + \langle m, x \rangle$, so that $|c_m| = |d_m|$ and

$$\begin{aligned} 0 &\neq \rho(c_m) + \rho(d_m) \\ &= \mathrm{red}(c_m t^{-\log(|c_m|)}) + \mathrm{red}(d_m t^{-\log(|c_m|)}) \\ &= \mathrm{red}((c_m + d_m) t^{-\log(|c_m|)}), \end{aligned}$$

so that $|c_m + d_m| = |c_m|$. Then $\tau_h(x) = \tau_f(x)$ and $\rho(c_m + d_m) = \rho(c_m) + \rho(d_m)$ is the coefficient of T^m in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ and in $\mathrm{in}_x(h)$.

Since $\mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0$, by assumption, at least one of these three cases appears. This already proves that $\tau_h(x) = \tau_f(x)$.

Two possibilities remain for $m \in \mathbf{N}^{m+1}$.

If $m \notin \mathrm{S}_x(f) \cup \mathrm{S}_x(g)$, then $\log(|c_m|) + \langle m, x \rangle < \tau_f(x)$ and $\log(|d_m|) + \langle m, x \rangle < \tau_g(x)$, so that $\log(|c_m + d_m|) + \langle m, x \rangle < \tau_f(x) = \tau_h(x)$. Then m does not appear in $\mathrm{in}_x(f)$, $\mathrm{in}_x(g)$ or $\mathrm{in}_x(h)$.

Let us finally assume that $m \in \mathrm{S}_x(f) \cap \mathrm{S}_x(g)$ and $\rho(c_m) + \rho(d_m) = 0$. As above, one has $|c_m| = |d_m|$ and $\rho(c_m) + \rho(d_m)$ is the reduction of $(c_m + d_m) t^{-\log(|c_m|)}$. This implies that $|c_m + d_m| < |c_m|$, hence T^m does not appear in $\mathrm{in}_x(h)$, and neither does it appear in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$.

c) Let $\varphi \in \mathrm{in}_x(\mathrm{I})$ and let $(f_i)_{1 \leqslant i \leqslant p}$ be a finite family of minimal cardinality of elements of I such that $\varphi = \sum_{i=1}^{p} \mathrm{in}_x(f_i)$. By minimality of p, one has $\mathrm{in}_x(f_i) \neq 0$ for all i. Applying b), we deduce from the minimality of p that for all $i \neq j$, the supports of $\mathrm{in}_x(f_i)$ and $\mathrm{in}_x(f_j)$

f homogène

$f = \sum_{d=0}^{\infty} f_d \quad f_d$ homogène de degré d

$\mathrm{in}_x(f) = \sum_{d \in D} \mathrm{in}_x(f_d)$

$D = \{ d \mid \tau_{f_d}(x) = \tau_f(x) \}$

are disjoint. The support of their sum, $\sum_i \mathrm{in}_x(f_i) = \varphi$, is then the union of their supports, hence it has at least p elements.

If φ is a monomial, this implies that $p = 1$, so that there exists $f \in \mathrm{I}$ such that $\varphi = \mathrm{in}_x(f)$. □

Definition **(3.5.9)**. — *Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *A finite family* $(f_1, \dots, f_m)$ *of elements of* I *is called a* Gröbner basis *for* I *at* x *if the initial forms* $\mathrm{in}_x(f_j)$ *at* x *generate the initial ideal* $\mathrm{in}_x(\mathrm{I})$ *of* I *at* x.

Since the initial ideal $\mathrm{in}_x(\mathrm{I})$ is generated by the polynomials of the form $\mathrm{in}_x(f)$, for $f \in \mathrm{I}$, the existence of a Gröbner basis follows from the noetherian property of the ring $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ (aka, Hilbert's finite basis theorem).

Assume, moreover, that I is homogeneous and let $(f_1, \dots, f_m)$ be a Gröbner basis for I at a point x. For every j, $\mathrm{in}_x(f_j)$ is ~~the~~ a sum of the initial forms of the homogeneous components of f_j, as we saw in the proof of lemma 3.5.3. This implies that the homogeneous components of the f_j constitute a Gröbner basis for I at x.

$$\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n] \xrightarrow{\mathrm{in}_x} k[\mathrm{T}_0, \dots, \mathrm{T}_n] \xrightarrow{\mathrm{in}_y} k[\mathrm{T}_0, \dots, \mathrm{T}_n]$$

In the next lemma, we consider initial forms of polynomials of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$; this means that the field k is considered as a valued field, for the trivial absolute value.

Lemma **(3.5.10)**. — *Let* $x \in \mathbf{R}^{n+1}$ *and let* $f \in \mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. *There exists a strictly positive real number* δ *such that for every* $y \in \mathbf{R}^{n+1}$ *such that* $\|y\| < \delta$, *one has* $\mathrm{in}_{x+y}(f) = \mathrm{in}_y(\mathrm{in}_x(f))$ *and* $\tau_f(x+y) = \tau_{\mathrm{in}_x(f)}(y)$.

$f = \sum c_m T^m$ $S(f) = \{m \,/\, c_m \neq 0\}$

• $\tau_f(x) = \sup_m \log|c_m| + \langle m, x\rangle$

$S_x(f) = \{m \mid \log|c_m| + \langle m, x\rangle = \tau_f(x)\}$
$= S(\mathrm{in}_x(f))$

Proof. — Write $f = \sum c_m \mathrm{T}^m$; let $\mathrm{S}(f)$ be its support and let $\mathrm{S}_x(f)$ be the set of all $m \in \mathrm{S}(f)$ such that

$$\log(|c_m|) + \langle m, x\rangle = \tau_f(x) = \sup_m(\log(|c_m|) + \langle m, x\rangle),$$

so that one has $\mathrm{in}_x(f) = \sum_{m\in \mathrm{S}_x(f)} \rho(c_m)\mathrm{T}^m$. Let then $\mathrm{S}_{x,y}(f)$ be the set of all $m \in \mathrm{S}_x(f)$ such that

$$\langle m, y\rangle = \sup_{m\in \mathrm{S}_x(f)} \langle m, y\rangle = \tau_{\mathrm{in}_x(f)}(y),$$

$\mathrm{in}_x(f) = \sum_{m \in S_x(f)} \rho(c_m)\, T^m$.

• $\tau_{\mathrm{in}_x(f)}(y) = \sup_{m \in S_x(f)} \langle m, y\rangle$.

so that $\mathrm{in}_y(\mathrm{in}_x(f)) = \sum_{m\in \mathrm{S}_{x,y}(f)} \rho(c_m)\mathrm{T}^m$.

Let ε be a strictly positive real number such that $\log(|c_m|) + \langle m, x\rangle < \tau_f(x) - \varepsilon$ for $m \in \mathrm{S}(f) - \mathrm{S}_x(f)$. Let also $\delta > 0$ be such $|\langle m, y\rangle| < \varepsilon/2$ for every $m \in \mathrm{S}(f)$ and every $y \in \mathbf{R}^{n+1}$ such that $\|y\| < \delta$. For every such y and every $m \in \mathrm{S}_{x,y}(f)$, one then has

$\|y\| < \delta$

• $\tau_f(x+y)$

$$\log(|c_m|) + \langle m, x+y\rangle = (\log(|c_m|) + \langle m, x\rangle) + \langle m, y\rangle = \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y).$$

$\log|c_m| + \langle m, x+y\rangle$
$= (\log|c_m| + \langle m, x\rangle) + \langle m, y\rangle$.

In particular, one has $\tau_f(x+y) \geqslant \tau_{\mathrm{in}_x(f)}(y)$. If $m \in \mathrm{S}(f) - \mathrm{S}_x(f)$, one has

On va prouver : $\boxed{\text{maximal}}$ si $m \in S_{x,y}(f)$ (si y est assez petit)

$m \in S(f)$

$$\begin{aligned}\log(|c_m|) + \langle x+m, y\rangle &= (\log(|c_m|) + \langle m, x\rangle) + \overbrace{\langle m, y\rangle}^{< \varepsilon/2} \\ &< \tau_f(x) - \varepsilon + \langle m, y\rangle \\ &< \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y).\end{aligned}$$

$\tau_{\mathrm{in}_x(f)}(y) > -\varepsilon/2$

$m \in S(f)$

① $m \in S_{x,y}(f)$ $\qquad \left.\begin{array}{l}\log|c_m| + \langle m, x\rangle = \tau_f(x) \\ \langle m, y\rangle = \tau_{m_x(f)}(y)\end{array}\right\}$ $\log|c_m| + \langle m, x+y\rangle = \tau_f(x) + \tau_{m_x(f)}(y)$

② $m \in S_x(f) - S_{x,y}(f)$ $\qquad \left.\begin{array}{l}\log|c_m| + \langle m, x\rangle = \tau_f(x) \\ \langle m, y\rangle < \tau_{m_x(f)}(y)\end{array}\right\}$ $<$

③ $m \in S(f) - S_x(f)$

$$\left.\begin{array}{ll}\cdot \ \log|c_m| + \langle m, x\rangle < \tau_f(x) - \varepsilon & (\text{déf. de } \varepsilon) \\ \cdot \ \langle m, y\rangle < \frac{\varepsilon}{2} & (\text{déf. de } y) \\ \tau_{m_x(f)}(y) > -\frac{\varepsilon}{2} & (\text{——})\end{array}\right\} <$$

Donc $\tau_f(x+y) = \tau_f(x) + \tau_{in_x(f)}(y)$ et $in_{x+y}(f) = in_y(in_x(f))$

M_y = idéal engendré par tous les monômes appartenant à $in_y(I)$.

(Indeed, $\tau_{\mathrm{in}_x(f)}(y) > -\varepsilon/2$ and $\langle m, y\rangle < \varepsilon/2$.) On the other hand, if $m \in \mathrm{S}_x(f) - \mathrm{S}_{x,y}(f)$, then

$$\begin{aligned}\log(|c_m|) + \langle x+m, y\rangle &= (\log(|c_m|) + \langle m, x\rangle) + \langle m, y\rangle \\ &= \tau_f(x) + \langle m, y\rangle \\ &< \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y).\end{aligned}$$

$(M_y \subset M_z \Rightarrow M_y = M_z)$

$\Rightarrow in_y(I) = M_y$

This proves that $\tau_f(x+y) = \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y)$ and that

$$\mathrm{in}_{x+y}(f) = \sum_{m \in \mathrm{S}_{x,y}(f)} \rho(c_m)\mathrm{T}^m = \mathrm{in}_y(\mathrm{in}_x(f)).$$

□

Beware:
The definition of the nonarchimedean amoebas has been modified so as to be more consistent with the definition in the archimedean case. I made the necessary corrections up to here, but there are certainly inconsistencies below.

Proposition **(3.5.11)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. *For any* $y \in \mathbf{R}^{n+1}$, *let* M_y *be the largest monomial ideal contained in* $\mathrm{in}_y(\mathrm{I})$.

a) *Let* $y \in \mathbf{R}^{n+1}$ *be such that* M_y *is maximal among the ideals of this form. Then* $\mathrm{in}_y(\mathrm{I}) = \mathrm{M}_y$ *— in particular,* $\mathrm{in}_y(\mathrm{I})$ *is a monomial ideal.*

b) *Assume, moreover that the valuation of* K *is trivial. Then there exists* $\delta > 0$ *such that for every* $z \in \mathbf{R}^{n+1}$ *such that* $\|z\| < \delta$*, one has one has* $\mathrm{in}_{y+z}(\mathrm{I}) = \mathrm{in}_y(\mathrm{I}) = \mathrm{M}_y$*;*

c) *Let* $x \in \mathbf{R}^{n+1}$*. Let* $y \in \mathbf{R}^{n+1}$ *be such that* $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ *is maximal among the ideals of this form. Then there exists a finite family* (f_i) *in* I *such that the polynomials* $\mathrm{in}_y(\mathrm{in}_x(f_i))$ *generate the ideal* $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$*. Moreover, there exists* $\delta > 0$ *such that for every* $\varepsilon \in \mathbf{R}$ *such that* $0 < \varepsilon < \delta$*, one has* $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$*, and this ideal is monomial.*

Proof. — a) By construction, the ideal M_y is generated by a family (T^{m_i}) of monomials belonging to $\mathrm{in}_y(\mathrm{I})$; by Hilbert's basis theorem, this family can be assumed to be finite. By lemma 3.5.8, there exists, for every i a polynomial $f_i \in \mathrm{I}$ such that $\mathrm{T}^{m_i} = \mathrm{in}_y(f_i)$.

We now argue by contradiction and consider $f \in \mathrm{I}$ such that $\mathrm{in}_y(f) \notin \mathrm{M}_y$. If a monomial appearing in f belongs to M_y, we choose $g \in \mathrm{I}$ such that $\mathrm{in}_y(g)$ is that monomial; by lemma 3.5.8, there exists $h \in \mathrm{I}$ such that $\mathrm{in}_y(f) - \mathrm{in}_y(g) = \mathrm{in}_y(h)$, and that monomial does not appear in $\mathrm{in}_y(h)$; moroever, $\mathrm{in}_y(h) \notin \mathrm{M}_y$. Repeating this argument, we assume that no monomial of $\mathrm{in}_y(f)$ belongs to M_y.

Let now μ be a vertex of the Newton polytope of $\mathrm{in}_y(f)$ and let $z \in \mathbf{R}^{n+1}$ be the coefficients of a linear form defining μ. In other words, μ belongs to the support of $\mathrm{in}_y(f)$, and for every other m in this support, one has $\langle m, z\rangle < \langle \mu, z\rangle$. Then $\mathrm{in}_z(\mathrm{in}_y(f))$ is the monomial of exponent μ in $\mathrm{in}_y(f)$. By lemma 3.5.10, for $z \in \mathbf{R}^{n+1}$

On a $M_y \subset in_y(I)$
Prouver $in_y(I) = M_y$.
Il suffit de prouver $in_y(f) \in M_y$ $\forall f \in I$

$\rightsquigarrow$ $in_y(f)$ n'a aucun monôme dans M_y.

$NP(in_y(f)) = conv(S_y(f))$

μ sommet $\quad$ z définissant ce sommet μ

z

$in_y(f) = \sum c_m T^m$

$in_z(in_y(f)) = c_\mu T^\mu$

$c_\mu T^\mu = in_{\varepsilon z}(in_y(f)) = in_{y+\varepsilon z}(f)$

$S_y(f)$

such that $\|z\|$ is small enough, one has $\mathrm{in}_{y+z}(f) = \mathrm{in}_z(\mathrm{in}_y(f))$. Similarly, if $\|z\|$ is small enough, then for every i, one has $\mathrm{in}_{y+z}(f_i) = \mathrm{in}_z(\mathrm{in}_y(f_i)) = \mathrm{in}_y(f_i)$ since $\mathrm{in}_y(f_i)$ is a monomial. This implies that M_{y+z} contains M_y. On the other hand, the monomial T^μ belongs to M_{y+z} but not to M_y. This contradicts the hypothesis that M_y is maximal among the ideals of this form.

b) The ideal $\mathrm{in}_y(\mathrm{I})$ is generated by the monomials $\mathrm{in}_y(f_i)$. For $z \in \mathbf{R}^{n+1}$ such that $\|z\|$ is small enough, one has $\mathrm{in}_{y+z}(f_i) = \mathrm{in}_z(\mathrm{in}_y(f_i)) = \mathrm{in}_y(f_i)$ since $\mathrm{in}_y(f_i)$ is a monomial and the valuation of K is trivial. Consequently, $\mathrm{in}_{y+z}(\mathrm{I})$ contains the monomial ideal $\mathrm{in}_y(\mathrm{I})$. By maximality, the equality follows.

c) Let us apply the first part of the proposition to the ideal $\mathrm{in}_x(\mathrm{I})$ of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ and choose $y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ is maximal for this property — it is then a monomial ideal, by *a*). We shall prove that $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ for $\varepsilon > 0$ small enough.

Let $(g_1, \dots, g_m)$ be a finite family of elements of $\mathrm{in}_x(\mathrm{I})$ such that $\mathrm{in}_y(g_i)$ is a monomial, for every i, and such these monomials generate $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$. Fix i. As in the proof of lemma 3.5.8, *c*), there exists a finite family $(f_{i,j})_j$ of elements of I, with pairwise disjoint supports, such that $g_i = \sum_j \mathrm{in}_x(f_{i,j})$. Then the polynomials $\mathrm{in}_y(\mathrm{in}_x(f_{i,j}))$ have pairwise disjoint supports, and there exists a unique j such that the monomial $\mathrm{in}_y(g_i)$ appears in $\mathrm{in}_y(\mathrm{in}_x(f_{i,j}))$, in which case

$g_1, \dots, g_m \in \mathrm{in}_x(I)$
$\mathrm{in}_y(g_i)$ monôme
$\langle \mathrm{in}_y(g_1), \dots, \mathrm{in}_y(g_m)\rangle = \mathrm{in}_y(\mathrm{in}_x(I))$

$g_i = \sum \mathrm{in}_x(f_{ij})$ supports disjoints
$\mathrm{in}_y(g_i) = \mathrm{in}_y(\mathrm{in}_x(f_{i,j_i}))$

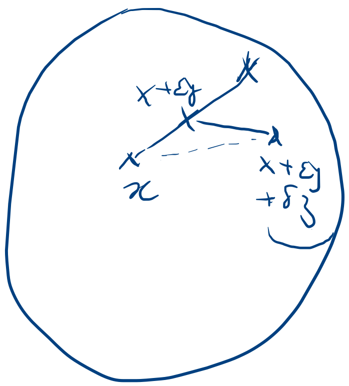
$\mathrm{in}_y(g_i) = \mathrm{in}_y(\mathrm{in}_x(f_{i,j}))$. This shows that there exists a finite family (f_i) in I such that $\mathrm{in}_y(\mathrm{in}_x(f_i))$ is a monomial for each i, and such that these monomials generate the ideal $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$.

Let then $\delta > 0$ be such that $\mathrm{in}_{x+\varepsilon y}(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$ for every i and every $\varepsilon \in \mathbf{R}$ such that $0 < \varepsilon < \delta$; in particular, $\mathrm{in}_y(\mathrm{in}_x(f_i)) \in \mathrm{in}_{x+\varepsilon y}(\mathrm{I})$, hence $\boxed{\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) \subset \mathrm{in}_{x+\varepsilon y}(\mathrm{I}).}$ Let us assume that the inclusion is strict. Then, there exists $f \in \mathrm{I}$ such that $\mathrm{in}_{x+\varepsilon y}(f)$ does not belong to the monomial ideal $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$. Subtracting from f an adequate linear combination of the the f_i, we may moreover assume that no monomial of $\mathrm{in}_{x+\varepsilon y}(f)$ belongs to $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$.

Let $z \in \mathbf{R}^{n+1}$ be such that $\mathrm{in}_z(\mathrm{in}_{x+\varepsilon y}(f))$ is a monomial. (In other words, z does not belong to the tropical hypersurface associated with $\mathrm{in}_{x+\varepsilon y}(f)$.) For $\delta > 0$ small enough (depending on y, ε, f), one then has $\mathrm{in}_{x+\varepsilon y+\delta z}(f) = \mathrm{in}_z(\mathrm{in}_{x+\varepsilon y}(f))$, hence is a nonzero monomial. However, applying b), we observe that if $\varepsilon y + \delta z$ is small enough (depending on x and I uniquely), then that monomial belongs to $\mathrm{in}_{\varepsilon y+\delta z}(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$, a contradiction which concludes the proof that $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$. $\square$

Theorem **(3.5.12)**. — *Let* I *be a* homogeneous *ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. *For every* $x \in \mathbf{R}^{n+1}$, *the Hilbert functions of* I *and* $\mathrm{in}_x(\mathrm{I})$ *are equal: for every integer* d, *one has*

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d).$$



$$\mathrm{in}_{\varepsilon y+\delta z}(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_{y+\frac{\delta}{\varepsilon}z}(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$$

petit

Lemma **(3.5.13)**. — *The conclusion of theorem 3.5.12 holds if* $\mathrm{in}_x(\mathrm{I})$ *is a monomial ideal.*

Proof. — Fix $d \in \mathbf{N}$.

Let M be the set of $m \in \mathbf{N}^{n+1}$ such that $|m| = d$ and $\mathrm{T}^m \notin \mathrm{in}_x(\mathrm{I})$. Let us prove that the family $(\mathrm{T}^m)_{m\in\mathrm{M}}$ is free in $(\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d$. Let $(c_m)_{m\in\mathrm{M}}$ be a family in K such that $\sum_{m\in\mathrm{M}} c_m\mathrm{T}^m \in \mathrm{I}$. Then there exist a family $(\tilde c_m)_{m\in\mathrm{M}}$ in k such that $\mathrm{in}_x(f) = \sum \tilde c_m \mathrm{T}^m$. By definition, one has $\mathrm{in}_x(f) \in \mathrm{in}_x(\mathrm{I})$, and since $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal, one has $\tilde c_m\mathrm{T}^m \in \mathrm{in}_x(\mathrm{I})$ for every $m \in \mathrm{M}$. Since $\mathrm{T}^m \notin \mathrm{in}_x(\mathrm{I})$ for $m \in \mathrm{M}$, this implies $\tilde c_m = 0$, hence $\mathrm{in}_x(f) = 0$ and $f = 0$. As a consequence, one has

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) \geqslant \mathrm{Card}(\mathrm{M}).$$

On the other hand, since the homogeneous ideal $\mathrm{in}_x(\mathrm{I})$ is generated by monomials, one has

$$\mathrm{Card}(\mathrm{M}) = \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d),$$

so that

$$\boxed{\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) \geqslant \mathrm{Card}(\mathrm{M}) \geqslant \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d).}$$

In the other direction, let now M′ be the set of $m \in \mathbf{N}^{n+1}$ such that $|m| = d$ and $\mathrm{T}^m \in \mathrm{in}_x(\mathrm{I})$. For every $m \in \mathrm{M}'$, there exists $f_m \in \mathrm{I}$ such that $\mathrm{in}_x(f_m) = \mathrm{T}^m$. Since I is a homogeneous ideal, we may also assume that f_m is homogeneous of degree d. Multiplying f_m by an

$d \in \mathbb{N}$ fixé

· $M = \{m,\ |m| = d,\ T^m \notin \mathrm{in}_x(I)\}$.

$(T^m, m \in M)$ est une base de $k[T_0, .., T_n]_d / I_d$.

· $(T^m, m \in M)$ libre dans $K[T_0, .., T_n]_d / I_d$.

supp $f = \sum_{m\in M} c_m T^m \in I - \{0\}$.

$\mathrm{in}_x(f) = \sum_{m \in M} \tilde c_m T^m \in \mathrm{in}_x(I)$

$(\tilde c_m = \rho(c_m)$ ou $0)$

$\Rightarrow T^m \in \mathrm{in}_x(I)$ si $\tilde c_m \neq 0$

impossible, donc $\tilde c_m = 0$

donc $\mathrm{in}_x(f) = 0$, donc $f = 0$

$a_{mm} = 1$

element of $R^\times$, we may assume that $f_m = T^m + \sum_{p\neq m} a_{m,p} T^p$. Let us prove that the family $(f_m)_{m\in M'}$ is free. Let $(c_m)_{m\in M'}$ be a family in K such that $\sum c_m f_m = 0$.

Let $\mu \in M'$ such that $\log(|c_\mu|) + \langle \mu, x\rangle$ is maximal. Considering the coefficient of T^μ in $\sum c_m f_m$, one has $(c_\mu \neq 0)$

$$c_\mu + \sum_{m\neq\mu} c_m a_{m,\mu} = 0.$$

By ultrametricity, there exists $m \neq \mu$ such that $|c_\mu| \leqslant |c_m a_{m,\mu}|$, and then

$$\log(|c_m|) + \langle m, x\rangle \leqslant \log(|c_\mu| + \langle \mu, x\rangle \leqslant \log(|c_m|) + \log(|a_{m,\mu}|) + \langle \mu, x\rangle,$$

so that

$$\boxed{\langle m, x\rangle \leqslant \log(|a_{m,\mu}|) + \langle \mu, x\rangle,}$$

contradicting the hypothesis that $\mathrm{in}_x(f_m)$ is the monomial T^m.

Consequently,

$$\dim_K(K[T_0, \dots, T_n]_d \cap I) \geqslant \mathrm{Card}(M') = \dim_K(k[T_0, \dots, T_n]_d \cap \mathrm{in}_x(I)).$$

Since I is a homogeneous ideal, one has

$$\begin{aligned}
&\dim_K((K[T_0, \dots, T_n]/I)_d) \\
&\qquad = \dim_K(K[T_0, \dots, T_n]_d) - \dim(K[T_0, \dots, T_n]_d \cap I) \\
&\qquad \leqslant \dim_k(k[T_0, \dots, T_n]_d) - \dim(k[T_0, \dots, T_n]_d \cap I) \\
&\qquad = \dim_k((k[T_0, \dots, T_n]/\mathrm{in}_x(I))_d).
\end{aligned}$$

$M' = \{m,\ |m| = d,\ T^m \in \mathrm{in}_x(f)\}$

$T^m = \mathrm{in}_x(f_m) \qquad f_m \in I$

$(f_m)_{m\in M'}$ est libre

$\sum c_m f_m = 0 = \Big(c_\mu + \sum_{m\neq\mu} c_m a_{m,\mu}\Big) T^\mu + \sum_{n\neq\mu} (\dots) T^n$

$\tau_x(f_m) = \sup(\log|a_{m,p}| + \langle p, x\rangle) \geqslant \langle m, x\rangle$

$\mathrm{in}_x(f_m) = T^m$

$S_x(f_m) = \{m\}$

$\log|a_{m,p}| + \langle p, x\rangle < \langle m, x\rangle$

$p \neq m$

This concludes the proof of the lemma. □

Proof of theorem 3.5.12. — We fix $y \in \mathbf{R}^{n+1}$ and $\varepsilon > 0$ such that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_{x+\varepsilon y}(\mathrm{I})$ is a monomial ideal.

Applying lemma 3.5.13 to the ideal I of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ and the point $x + \varepsilon y$, we have

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_{x+\varepsilon y}(\mathrm{I}))_d).$$

Applying that lemma to the ideal $\mathrm{in}_x(\mathrm{I})$ of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ and the point y, we have

$$\begin{aligned} \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d) &= \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})))_d) \\ &= \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_{x+\varepsilon y}(\mathrm{I}))_d). \end{aligned}$$

This shows that

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d),$$

as claimed. □

Corollary **(3.5.14).** — *Let* I *be a* homogeneous *ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, *let* $x \in \mathbf{R}^{n+1}$ *and let* $(f_1, \dots, f_m)$ *be a Gröbner basis of* I *at* x. *Then* $\mathrm{I} = (f_1, \dots, f_m)$.

Proof. — Let J be the homogeneous ideal of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ generated by the homogeneous components of $f_1, \dots, f_m$. One has $\mathrm{J} \subset \mathrm{I}$, because these homogeneous components belong to I. Moreover, for every j, the initial form $\mathrm{in}_x(f_j)$ is the sum of the initial forms of

comparer les f. de Hilbert en x ?

$x \longrightarrow x + \varepsilon y$

monomial

in

$\mathrm{In} \xrightarrow{=} \mathrm{in}_x(\mathrm{I})$

cas particulier

cas particulier

$\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_{\varepsilon y}(\mathrm{in}_x(\mathrm{I}))$

$\mathrm{in}_x(\mathrm{I}) = \mathrm{in}_x(\mathrm{J})$

$\boxed{\mathrm{I} \supset \mathrm{J}} = \langle f_1, \dots, f_m \rangle$ (homogènes)

$\mathrm{in}_x(\mathrm{I}) = \langle \mathrm{in}_x(f_j) \rangle$

$\mathrm{in}_x(\mathrm{I}) = \langle \mathrm{in}_x(f_j) \rangle \subset \mathrm{in}_x(\mathrm{J}) \subset \mathrm{in}_x(\mathrm{I})$

$C'_x(I) = \{y \mid in_y(I) = in_x(I)\}$

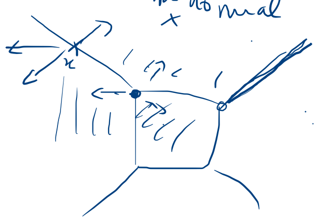
$C_x(I)$ = adhérence

· $\mathbf{R}e + C_x(I) = C_x(I)$

$\mathbf{R}e \subset$ linsp $(C_x(I))$

· équations de $C_x(I)$ dans $\mathbf{R}^{n+1}$ affines : partie linéaire à coeff. entiers constante dans Γ

$in_x(I)$ monomial



the homogeneous components of f_j, so that $\mathrm{in}_x(f_j) \subset \mathrm{in}_x(\mathrm{J})$. As a consequence, $\mathrm{in}_x(\mathrm{I}) \subset \mathrm{in}_x(\mathrm{J})$, hence the equality $\mathrm{in}_x(\mathrm{J}) = \mathrm{in}_x(\mathrm{I})$. By theorem 3.5.12, the homogeneous ideals I and J have the same Hilbert functions. Since $\mathrm{J} \subset \mathrm{I}$, this implies $\mathrm{J} = \mathrm{I}$. □

3.6. The Gröbner polyhedral decomposition associated with an ideal

3.6.1. — Let I be a homogeneous ideal in $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. For $x \in \mathbf{R}^{n+1}$, let $\mathrm{C}'_x(\mathrm{I})$ be the set of $y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_y(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$ and let $\mathrm{C}_x(\mathrm{I})$ be its closure in $\mathbf{R}^{n+1}$. Let $e = (1, \dots, 1) \in \mathbf{R}^{n+1}$.

Here is the main theorem

Theorem **(3.6.2)**. — *Let* I *be a homogeneous ideal in* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. *The sets* $\mathrm{C}_x(\mathrm{I})$ *form a* Γ*-strict and* $\mathbf{R}e$*-invariant polyhedral decomposition of* $\mathbf{R}^{n+1}$.

Proposition **(3.6.3)**. — *Let* $x \in \mathbf{R}^{n+1}$.

a) *The set* $\mathrm{C}_x(\mathrm{I})$ *is a closed* Γ*-strict and* $\mathbf{R}e$*-invariant polyhedron in* $\mathbf{R}^{n+1}$;

b) *If* $\mathrm{in}_x(\mathrm{I})$ *is a monomial ideal, then* $\mathrm{C}'_x(\mathrm{I})$ *is the interior of* $\mathrm{C}_x(\mathrm{I})$;

$in_{x+y}(I)$

c) *If* $\mathrm{in}_x(\mathrm{I})$ *is not a monomial ideal, then there exists* $y \in \mathbf{R}^{n+1}$ *such that* $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ *is a monomial ideal; for every such* y, *the polyhedron* $\mathrm{C}_x(\mathrm{I})$ *is a face of* $\mathrm{C}_y(\mathrm{I})$.

$C_{x+y}(I)$

Proof. — Fix $y \in \mathbf{R}^{n+1}$ satisfying the conditions of proposition 3.5.11, c), small enough so that $\mathrm{in}_{x+y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ is a monomial ideal. As a consequence of b), it will be enough to assume that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ is a monomial ideal and y is small enough.

Let $z = x + y$. Fix a finite family $(f_1, \dots, f_r)$ in I such that the polynomials $\mathrm{in}_z(f_i)$ are monomials and generate $\mathrm{in}_z(\mathrm{I})$; we may also assume that $\mathrm{in}_z(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$ for all i.

For each i, let $m_i \in \mathbf{N}^{n+1}$ be such that $\mathrm{in}_z(f_i) = \mathrm{T}^{m_i}$. By the argument explained in the proof of lemma 3.5.13, there exists a unique polynomial $g_i \in \mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, homogeneous of degree $|m_i|$, such that $\mathrm{T}^{m_i} - g_i \in \mathrm{I}$, and such that no monomial appearing in g_i belongs to $\mathrm{in}_z(\mathrm{I})$; write $g_i = \sum c_{i,m}\mathrm{T}^m$ and set $f_i = \mathrm{T}^{m_i} - g_i$. Since T^{m_i} is the only monomial appearing in f_i that belongs to the monomial ideal $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$, one has $\mathrm{T}^{m_i} = \mathrm{in}_y(\mathrm{in}_x(f_i))$. The family (f_i) is thus a Gröbner basis for I at z.

Lemma **(3.6.4)**. — *With the preceding notation, the set $\mathrm{C}'_z(\mathrm{I})$ is defined by the strict inequalities*

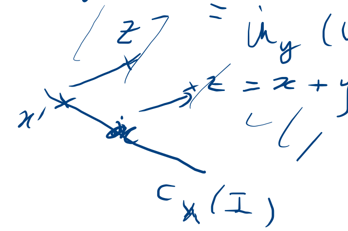
$$\langle m - m_i, \cdot\rangle < \log(|c_{i,m}|),$$

for all $i \in \{1, \dots, r\}$ and all $m \in \mathbf{N}^{m+1}$ in the support of f_i. The set $\mathrm{C}_z(\mathrm{I})$ is the Γ-strict polyhedron defined by the inequalities

$$\langle m - m_i, \cdot\rangle \leqslant \log(|c_{i,m}|),$$

for $i \in \{1, \dots, r\}$ and $m \in \mathbf{N}^{m+1}$ in the support of f_i.

$\mathrm{in}_{x'+y}(I) = \mathrm{in}_y(\mathrm{in}_{x'}(I))$
$= \mathrm{in}_y(\mathrm{in}_x(I)) = \mathrm{in}_z(I)$



$\mathrm{in}_z(I)$ monomial
$= \langle \mathrm{in}_z(f_i)\rangle$ $\mathrm{in}_z(f_i) = T^{m_i}$
$I \ni f_i = T^{m_i} - \sum_{m \neq m_i} c_{i,m} T^m$

$\mathrm{in}_z(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$
donne des inégalités sur les coefficients c_{im}.

Proof. — Let $w \in \mathrm{C}'_z(\mathrm{I})$. By definition of $\mathrm{C}'_z(\mathrm{I})$, one has $\mathrm{in}_w(f_i) \in \mathrm{in}_w(\mathrm{I}) = \mathrm{in}_z(\mathrm{I})$, so that the only monomial that can appear in $\mathrm{in}_w(f_i)$ is T^{m_i}, hence $\log(|c_{i,m}|) + \langle m, w\rangle < \langle m_i, w\rangle$ for all i. In the other direction, if w statisfies these inequalities, then $\mathrm{in}_w(f_i) = \mathrm{T}^{m_i}$ for all i, hence $\mathrm{in}_w(\mathrm{I})$ contains $\mathrm{in}_z(\mathrm{I})$. Since both of these ideals have the same Hilbert function, they have to be equal and $w \in \mathrm{C}'_z(\mathrm{I})$.

Let P be the closed convex polyhedron in $\mathbf{R}^{n+1}$ defined by the inequalities $\langle m - m_i, \cdot\rangle \leqslant \log(|c_{i,m}|)$, for all i and m. By what precedes, one has $\mathrm{C}'_z(\mathrm{I}) \subset \mathrm{P}$, hence $\mathrm{C}'_z(\mathrm{I}) \subset \mathring{\mathrm{P}}$ and $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{P}$. Replacing an inequality in the definition set of P by the corresponding equality amounts to intersecting P with a hyperplane; since P has nonempty interior, this defines a strict face of P. This implies that $\mathring{\mathrm{P}} = \mathrm{C}'_z(\mathrm{I})$, and then $\mathrm{P} = \mathrm{C}_z(\mathrm{I})$, since it is the closure of its interior. □

Lemma **(3.6.5)**. — *The set* $\mathrm{C}_x(\mathrm{I})$ *is a face of the polyhedron* $\mathrm{C}_z(\mathrm{I})$.

Proof. — By the choice of y, one has $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$ for all ε such that $0 < \varepsilon < 1$. In particular, $x + \varepsilon y \in \mathrm{C}'_z(\mathrm{I})$. If we let ε go to 0, we obtain $x \in \mathrm{C}_z(\mathrm{I})$.

Let $x' \in \mathrm{C}'_x(\mathrm{I})$; since $\mathrm{in}_{x'}(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$, the preceding analysis still applies when one replaces the point x with x', so that

$$\mathrm{in}_{x'+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_{x'}(\mathrm{I})) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$$

for $\varepsilon > 0$ small enough. Then $x' + \varepsilon y \in \mathrm{C}'_z(\mathrm{I})$ and $x' \in \mathrm{C}_z(\mathrm{I})$. Taking the closure, we obtain $\mathrm{C}_x(\mathrm{I}) \subset \mathrm{C}_z(\mathrm{I})$.

Moreover, T^{m_i} is the only monomial in the support of f_i that belongs to $\operatorname{in}_y(\operatorname{in}_{x'}(f_i))$; this implies that $\operatorname{in}_y(\operatorname{in}_{x'}(f_i)) = T^{m_i}$. On the other hand, the polynomial $\operatorname{in}_{x'}(f_i) - \operatorname{in}_x(f_i)$ belongs to $\operatorname{in}_x(I)$, and none of its monomials belongs to $\operatorname{in}_y(\operatorname{in}_x(I))$, by the definition of f_i. Its initial form at y must vanish, which implies that $\operatorname{in}_{x'}(f_i) = \operatorname{in}_x(f_i)$. Since T^{m_i} appears in $\operatorname{in}_{x'}(f_i)$, this shows that $\tau_{f_i}(x') = \langle m_i, x'\rangle$, so that $\log(|c_{i,m}|) + \langle m, x'\rangle = \langle m_i, x'\rangle$ for every m such that T^m is in the support of $\operatorname{in}_x(f_i)$; on the other hand, if T^m is not in that support, then $\log(|c_{i,m}|) + \langle m, x'\rangle < \langle m_i, x'\rangle$. Conversely, these inequalities imply that $\operatorname{in}_{x'}(f_i) = \operatorname{in}_x(f_i)$ for all i, so that $\operatorname{in}_{x'}(I) \supset \langle(\operatorname{in}_x(f_i))_i\rangle = \operatorname{in}_x(I)$. Since both ideals $\operatorname{in}_x(I)$ and $\operatorname{in}_{x'}(I)$ have the same Hilbert function, we obtain the equality $\operatorname{in}_{x'}(I) = \operatorname{in}_x(I)$.

This proves that $C_x(I)$ is contained in the face of $C_z(I)$ defined by the equalities $\log(|c_{i,m}|) + \langle m, x'\rangle = \langle m_i, x'\rangle$. Conversely, if w is a point of this face, then every point of the open segment $]x; w[$ belongs to $C'_x(I)$, hence w belongs to $C_x(I)$. $\square$

[3]

Lemma 3.6.5 proves part c) of proposition 3.6.3. The formulas of lemma 3.6.4 prove that $C_z(I)$ is a closed Γ-strict polyhedron in $\mathbf{R}^{n+1}$. Moreover, since f_i is homogeneous, one has $\langle m_i - m, w + te\rangle = \langle m_i - m, w\rangle$ for every $w \in \mathbf{R}^{n+1}$, every $t \in \mathbf{R}$ and every $m \in \mathbf{N}^{m+1}$

[3]Est-ce que $C'_x(I)$ est toujours l'intérieur relatif de $C_x(I)$?

such that $c_{i,m} \neq 0$, so that e belongs to the lineality space of $\mathrm{C}_z(\mathrm{I})$. Since $\mathrm{C}_x(\mathrm{I})$ is a face of $\mathrm{C}_z(\mathrm{I})$, the same properties hold for $\mathrm{C}_x(\mathrm{I})$.

Let us finally assume that $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal. For every monomial f belonging to $\mathrm{in}_x(\mathrm{I})$, one has $\mathrm{in}_y(f) = f$, so that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ contains $\mathrm{in}_x(\mathrm{I})$. This implies that $\mathrm{in}_z(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ contains $\mathrm{in}_x(\mathrm{I})$, hence $\mathrm{in}_z(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$ since both ideals have the same Hilbert function. As a consequence, $\mathrm{C}'_x(\mathrm{I}) = \mathrm{C}'_z(\mathrm{I})$, hence $\mathrm{C}_x(\mathrm{I}) = \mathrm{C}_z(\mathrm{I})$. By the formulas of lemma 3.6.4, $\mathrm{C}'_z(\mathrm{I})$ is a the closure of a nonempty convex open subset of $\mathbf{R}^{n+1}$, and every point of $\mathrm{C}_z(\mathrm{I}) - \mathrm{C}'_z(\mathrm{I})$ belongs to a face of $\mathrm{C}_z(\mathrm{I})$. This proves that $\mathrm{C}'_z(\mathrm{I})$ is the interior of $\mathrm{C}_z(\mathrm{I})$. □

égalité des fonctions de Hilbert

Th de Maclagan

Lemma **(3.6.6)**. — *The set of monomial ideals in* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *which are of the form* $\mathrm{in}_x(\mathrm{I})$, *for some* $x \in \mathbf{R}^{n+1}$, *is finite.*

$\mathrm{in}_x(\mathrm{I}) \subset \mathrm{in}_y(\mathrm{I})$

$\Rightarrow \mathrm{in}_x(\mathrm{I}) = \mathrm{in}_y(\mathrm{I})$.

Proof. — Let $\mathscr{F}$ be this set of ideals. If $\mathscr{F}$ were infinite, there would exist, by theorem 3.4.5, two elements $x, y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_x(\mathrm{I})$ and $\mathrm{in}_y(\mathrm{I})$ are monomial ideals and $\mathrm{in}_x(\mathrm{I}) \subsetneq \mathrm{in}_y(\mathrm{I})$. This contradicts the fact that these two ideals have the same Hilbert function. □

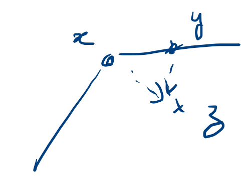
Proof of theorem 3.6.2. — Let $\mathscr{C}$ be the set of all subsets of $\mathbf{R}^{n+1}$ the form $\mathrm{C}_x(\mathrm{I})$, for some $x \in \mathbf{R}^{n+1}$. The sets $\mathrm{C}_x(\mathrm{I})$ are Γ-strict convex polyhedra in $\mathbf{R}^{n+1}$. Since $x \in \mathrm{C}_x(\mathrm{I})$ for all x, their union is equal to $\mathbf{R}^{n+1}$. If $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal, then $\mathrm{C}_x(\mathrm{I})$ has dimension $n+1$; otherwise, $\mathrm{C}_x(\mathrm{I})$ is a face of a polyhedron of the form $\mathrm{C}_w(\mathrm{I})$. By lemma 3.6.6, the set $\mathscr{C}$ is finite. Consequently, the set of initial ideals $\mathrm{in}_x(\mathrm{I})$ is finite, when x varies in $\mathbf{R}^{n+1}$.

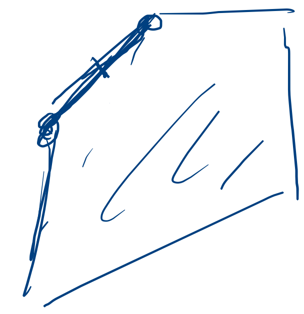
Let $x, y \in \mathbf{R}^{n+1}$. The preceding description shows that $x \in \mathrm{C}_y(\mathrm{I})$ if and only if $\mathrm{C}_x(\mathrm{I}) \subset \mathrm{C}_y(\mathrm{I})$. In this case, $\mathrm{C}_x(\mathrm{I})$ and $\mathrm{C}_y(\mathrm{I})$ are faces of a common $(n+1)$-dimensional polyhedron of the form $\mathrm{C}_z(\mathrm{I})$; in particular, $\mathrm{C}_x(\mathrm{I})$ is a face of $\mathrm{C}_y(\mathrm{I})$. [4]

If F is a face of $\mathrm{C}_y(\mathrm{I})$, choose x in the relative interior of F; then F and $\mathrm{C}_x(\mathrm{I})$ are faces of $\mathrm{C}_y(\mathrm{I})$ which both have the point x in their relative interiors; necessarily, $\mathrm{F} = \mathrm{C}_x(\mathrm{I})$.

Let $x, y \in \mathbf{R}^{n+1}$. For every point $z \in \mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$, one has $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{C}_x(\mathrm{I})$, since $z \in \mathrm{C}_x(\mathrm{I})$, and $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{C}_y(\mathrm{I})$, since $z \in \mathrm{C}_y(\mathrm{I})$, so that $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$. This proves that $\mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$ is a union of faces of $\mathrm{C}_x(\mathrm{I})$. However, a union of faces of a polyhedron is convex if and only it has a unique maximal element — so that one of these faces contains all of them. As a consequence, $\mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$ is a face of $\mathrm{C}_x(\mathrm{I})$, and it belongs to $\mathscr{C}$. □

$x \in C_y(I) \iff C_x(I) \subset C_y(I)$





3.7. Tropicalization of algebraic varieties

The goal of this section is to generalize theorem 3.3.6 to all ideals of $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. We first recall how to pass from ideals of this ring to homogeneous ideals of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, and back.

[4]It needs to be explained more clearly