TOPICS
IN TROPICAL GEOMETRY

Antoine Chambert-Loir

Antoine Chambert-Loir
Université Paris-Diderot.
E-mail : `Antoine.Chambert-Loir@math.univ-paris-diderot.fr`

Version of February 11, 2021, 13h10
The most up-do-date version of this text should be accessible online at address http://webusers.imj-prg.fr/~antoine.chambert-loir/enseignement/2019-20/gt/toptrop.pdf

3.6. The Gröbner polyhedral decomposition associated with an ideal

3.6.1. — Let I be a homogeneous ideal in $K[T_0, \dots, T_n]$. For $x \in \mathbf{R}^{n+1}$, let $C'_x(I)$ be the set of $y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_y(I) = \mathrm{in}_x(I)$ and let $C_x(I)$ be its closure in $\mathbf{R}^{n+1}$. Let $e = (1, \dots, 1) \in \mathbf{R}^{n+1}$.

Here is the main theorem

Theorem **(3.6.2)**. — *Let* I *be a homogeneous ideal in* $K[T_0, \dots, T_n]$. *The sets* $C_x(I)$ *form a* Γ*-strict and* $\mathbf{R}e$*-invariant polyhedral decomposition of* $\mathbf{R}^{n+1}$.

Proposition **(3.6.3)**. — *Let* $x \in \mathbf{R}^{n+1}$.

a) *The set* $C_x(I)$ *is a closed* Γ*-strict and* $\mathbf{R}e$*-invariant polyhedron in* $\mathbf{R}^{n+1}$;

b) *If* $\mathrm{in}_x(I)$ *is a monomial ideal, then* $C'_x(I)$ *is the interior of* $C_x(I)$;

c) *If* $\mathrm{in}_x(I)$ *is not a monomial ideal, then there exists* $y \in \mathbf{R}^{n+1}$ *such that* $\mathrm{in}_y(\mathrm{in}_x(I))$ *is a monomial ideal; for every such* y, *the polyhedron* $C_x(I)$ *is a face of* $C_y(I)$.

Proof. — Fix $y \in \mathbf{R}^{n+1}$ satisfying the conditions of proposition 3.5.11, *c*), small enough so that $\mathrm{in}_{x+y}(I) = \mathrm{in}_y(\mathrm{in}_x(I))$ is a monomial ideal. As a consequence of *b*), it will be enough to assume that $\mathrm{in}_y(\mathrm{in}_x(I))$ is a monomial ideal and y is small enough.

K corps valué, R, k, Γ, scindé $K^\times \to \Gamma$

$I \subset K[T_0, \dots, T_n]$ idéal homogène

$x \in R^{n+1} \rightsquigarrow \mathrm{in}_x(I) = \langle \mathrm{in}_x(f), f \in I \rangle$
$\subset k[T_0, \dots, T_n]$ idéal homogène

$C'_x(I) = \{ y \in R^{n+1}, \mathrm{in}_y(I) = \mathrm{in}_x(I) \}$

$C_x(I) = \overline{C'_x(I)}$ cellule

· $C_x(I)$ est un polyèdre Γ-strict Re invariant

$C'_x(I) = \mathrm{relint}(C_x(I))$

- décomposition polyédrale
  - l'intersection de deux cellules est vide ou une face de chacune d'elles.
  - finie

cellules de dim $n+1$: ($\ominus$ $C'_x(I)$ ouvert)

$\Leftrightarrow$ idéaux initiaux monomiaux

Let $z = x + y$. Fix a finite family $(f_1, \dots, f_r)$ in I such that the polynomials $\mathrm{in}_z(f_i)$ are monomials and generate $\mathrm{in}_z(\mathrm{I})$; we may also assume that $\mathrm{in}_z(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$ for all i.

For each i, let $m_i \in \mathbf{N}^{n+1}$ be such that $\mathrm{in}_z(f_i) = \mathrm{T}^{m_i}$. By the argument explained in the proof of lemma 3.5.13, there exists a unique polynomial $g_i \in \mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, homogeneous of degree $|m_i|$, such that $\mathrm{T}^{m_i} - g_i \in \mathrm{I}$, and such that no monomial appearing in g_i belongs to $\mathrm{in}_z(\mathrm{I})$; write $g_i = \sum c_{i,m}\mathrm{T}^m$ and set $f_i = \mathrm{T}^{m_i} - g_i$. Since T^{m_i} is the only monomial appearing in f_i that belongs to the monomial ideal $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$, one has $\mathrm{T}^{m_i} = \mathrm{in}_y(\mathrm{in}_x(f_i))$. The family (f_i) is thus a Gröbner basis for I at z.

Lemma **(3.6.4)**. — *With the preceding notation, the set* $\mathrm{C}'_z(\mathrm{I})$ *is defined by the strict inequalities*

$$\langle m - m_i, \cdot\rangle + \log(|c_{i,m}|) < 0,$$

for all $i \in \{1, \dots, r\}$ *and all* $m \in \mathbf{N}^{m+1}$ *in the support of* f_i. *The set* $\mathrm{C}_z(\mathrm{I})$ *is the* Γ*-strict polyhedron defined by the inequalities*

$$\langle m - m_i, \cdot\rangle + \log(|c_{i,m}|) \leqslant 0,$$

for $i \in \{1, \dots, r\}$ *and* $m \in \mathbf{N}^{m+1}$ *in the support of* f_i.

Proof. — Let $w \in \mathrm{C}'_z(\mathrm{I})$. By definition of $\mathrm{C}'_z(\mathrm{I})$, one has $\mathrm{in}_w(f_i) \in \mathrm{in}_w(\mathrm{I}) = \mathrm{in}_z(\mathrm{I})$, so that the only monomial that can appear in $\mathrm{in}_w(f_i)$ is T^{m_i}, hence $\log(|c_{i,m}|) + \langle m, w\rangle < \langle m_i, w\rangle$ for all i and all m such that $m \neq m_i$. In the other direction, if w statisfies these inequalities, then $\mathrm{in}_w(f_i) = \mathrm{T}^{m_i}$ for all i, hence $\mathrm{in}_w(\mathrm{I})$ contains $\mathrm{in}_z(\mathrm{I})$. Since both of these ideals have the same Hilbert function, they have to be equal and $w \in \mathrm{C}'_z(\mathrm{I})$.

Let P be the closed convex polyhedron in $\mathbf{R}^{n+1}$ defined by the inequalities $\langle m - m_i, \cdot\rangle + \log(|c_{i,m}|) \leqslant 0$, for all i and $m \neq m_i$. By what precedes, one has $\mathring{\mathrm{P}} = \mathrm{C}'_z(\mathrm{I})$. Since $\mathring{\mathrm{P}}$ is nonempty (it contains z), one has $\mathrm{P} = \mathrm{C}_z(\mathrm{I})$. □

Lemma **(3.6.5)**. — *The set* $\mathrm{C}_x(\mathrm{I})$ *is the smallest face of the polyhedron* $\mathrm{C}_z(\mathrm{I})$ *that contains* x.

Proof. — By the choice of y, one has $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$ for all ε such that $0 < \varepsilon < 1$. In particular, $x + \varepsilon y \in \mathrm{C}'_z(\mathrm{I})$. If we let ε go to 0, we obtain $x \in \mathrm{C}_z(\mathrm{I})$.

Let $x' \in \mathrm{C}'_x(\mathrm{I})$; since $\mathrm{in}_{x'}(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$, the preceding analysis still applies when one replaces the point x with x', so that

$$\mathrm{in}_{x'+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_{x'}(\mathrm{I})) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$$

for $\varepsilon > 0$ small enough. Then $x' + \varepsilon y \in \mathrm{C}'_z(\mathrm{I})$ and $x' \in \mathrm{C}_z(\mathrm{I})$. Taking the closure, we obtain $\mathrm{C}_x(\mathrm{I}) \subset \mathrm{C}_z(\mathrm{I})$.

Moreover, T^{m_i} is the only monomial in the support of f_i that belongs to $\mathrm{in}_y(\mathrm{in}_{x'}(f_i))$; this implies that $\mathrm{in}_y(\mathrm{in}_{x'}(f_i)) = \mathrm{T}^{m_i}$. On the other hand, the polynomial $\mathrm{in}_{x'}(f_i) - \mathrm{in}_x(f_i)$ belongs to $\mathrm{in}_x(\mathrm{I})$, and none of its monomials belongs to $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$, by the definition of f_i.

Its initial form at y must vanish, which implies that $\mathrm{in}_{x'}(f_i) = \mathrm{in}_x(f_i)$. Since T^{m_i} appears in $\mathrm{in}_{x'}(f_i)$, this shows that $\tau_{f_i}(x') = \langle m_i, x'\rangle$, so that $\log(|c_{i,m}|) + \langle m, x'\rangle = \langle m_i, x'\rangle$ for every m such that T^m is in the support of $\mathrm{in}_x(f_i)$; on the other hand, if T^m is not in that support, then $\log(|c_{i,m}|) + \langle m, x'\rangle < \langle m_i, x'\rangle$. Conversely, these inequalities imply that $\mathrm{in}_{x'}(f_i) = \mathrm{in}_x(f_i)$ for all i, so that $\mathrm{in}_{x'}(\mathrm{I}) \supset \langle(\mathrm{in}_x(f_i))_i\rangle = \mathrm{in}_x(\mathrm{I})$. Since both ideals $\mathrm{in}_x(\mathrm{I})$ and $\mathrm{in}_{x'}(\mathrm{I})$ have the same Hilbert function, we obtain the equality $\mathrm{in}_{x'}(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$.

This proves that $\mathrm{C}_x(\mathrm{I})$ is contained in the face of $\mathrm{C}_z(\mathrm{I})$ defined by the equalities $\log(|c_{i,m}|) + \langle m - m_i, x'\rangle = 0$ for all i and all m such that $m \neq m_i$ and $\log(|c_{i,m}|) + \langle m - m_i, x\rangle = 0$. Conversely, if w is a point of this face, then every point of the open segment $]x; w[$ belongs to $\mathrm{C}'_x(\mathrm{I})$, hence w belongs to $\mathrm{C}_x(\mathrm{I})$.

Finally, a face of $\mathrm{C}_z(\mathrm{I})$ containing a point x is obtained by replacing, in the system of affine inequalities defining this polyhedron, by the corresponding equalities some of those inequalities which are equalities at x. The smallest such face is obtained in replacing all possible such inequalities. By the previous description, this is exactly $\mathrm{C}_x(\mathrm{I})$. □

Lemma 3.6.5 proves part c) of proposition 3.6.3. The formulas of lemma 3.6.4 prove that $\mathrm{C}_z(\mathrm{I})$ is a closed Γ-strict polyhedron in $\mathbf{R}^{n+1}$. Moreover, since f_i is homogeneous, one has $\langle m_i - m, w + te\rangle = \langle m_i - m, w\rangle$ for every $w \in \mathbf{R}^{n+1}$, every $t \in \mathbf{R}$ and every $m \in \mathbf{N}^{m+1}$

such that $c_{i,m} \neq 0$, so that e belongs to the lineality space of $\mathrm{C}_z(\mathrm{I})$. Since $\mathrm{C}_x(\mathrm{I})$ is a face of $\mathrm{C}_z(\mathrm{I})$, the same properties hold for $\mathrm{C}_x(\mathrm{I})$.

Let us finally assume that $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal. For every monomial f belonging to $\mathrm{in}_x(\mathrm{I})$, one has $\mathrm{in}_y(f) = f$, so that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ contains $\mathrm{in}_x(\mathrm{I})$. This implies that $\mathrm{in}_z(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ contains $\mathrm{in}_x(\mathrm{I})$, hence $\mathrm{in}_z(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$ since both ideals have the same Hilbert function. As a consequence, $\mathrm{C}'_x(\mathrm{I}) = \mathrm{C}'_z(\mathrm{I})$, hence $\mathrm{C}_x(\mathrm{I}) = \mathrm{C}_z(\mathrm{I})$. By the formulas of lemma 3.6.4, $\mathrm{C}'_z(\mathrm{I})$ is a the closure of a nonempty convex open subset of $\mathbf{R}^{n+1}$, and every point of $\mathrm{C}_z(\mathrm{I}) - \mathrm{C}'_z(\mathrm{I})$ belongs to a face of $\mathrm{C}_z(\mathrm{I})$. This proves that $\mathrm{C}'_z(\mathrm{I})$ is the interior of $\mathrm{C}_z(\mathrm{I})$. □

Lemma **(3.6.6)** (Maclagan). — *The set of monomial ideals in* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *which are of the form* $\mathrm{in}_x(\mathrm{I})$*, for some* $x \in \mathbf{R}^{n+1}$*, is finite.*

Proof. — Let $\mathscr{F}$ be this set of ideals. If $\mathscr{F}$ were infinite, there would exist, by theorem 3.4.5, two elements $x, y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_x(\mathrm{I})$ and $\mathrm{in}_y(\mathrm{I})$ are monomial ideals and $\mathrm{in}_x(\mathrm{I}) \subsetneq \mathrm{in}_y(\mathrm{I})$. This contradicts the fact that these two ideals have the same Hilbert function. □

Proof of theorem 3.6.2. — Let $\mathscr{C}$ be the set of all subsets of $\mathbf{R}^{n+1}$ the form $\mathrm{C}_x(\mathrm{I})$, for some $x \in \mathbf{R}^{n+1}$. The sets $\mathrm{C}_x(\mathrm{I})$ are Γ-strict convex polyhedra in $\mathbf{R}^{n+1}$. Since $x \in \mathrm{C}_x(\mathrm{I})$ for all x, their union is equal to $\mathbf{R}^{n+1}$. If $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal, then $\mathrm{C}_x(\mathrm{I})$ has dimension $n+1$; otherwise, $\mathrm{C}_x(\mathrm{I})$ is a face of a polyhedron of the form $\mathrm{C}_w(\mathrm{I})$. By lemma 3.6.6, the set $\mathscr{C}$ is finite. Consequently, the set of initial ideals $\mathrm{in}_x(\mathrm{I})$ is finite, when x varies in $\mathbf{R}^{n+1}$.

Let $x, y \in \mathbf{R}^{n+1}$. The preceding description shows that $x \in \mathrm{C}_y(\mathrm{I})$ if and only if $\mathrm{C}_x(\mathrm{I}) \subset \mathrm{C}_y(\mathrm{I})$. In this case, $\mathrm{C}_x(\mathrm{I})$ and $\mathrm{C}_y(\mathrm{I})$ are faces of a common $(n+1)$-dimensional polyhedron of the form $\mathrm{C}_z(\mathrm{I})$; in particular, $\mathrm{C}_x(\mathrm{I})$ is a face of $\mathrm{C}_y(\mathrm{I})$.

If F is a face of $\mathrm{C}_y(\mathrm{I})$, choose x in the relative interior of F; then F and $\mathrm{C}_x(\mathrm{I})$ are faces of $\mathrm{C}_y(\mathrm{I})$ which both have the point x in their relative interiors; necessarily, $\mathrm{F} = \mathrm{C}_x(\mathrm{I})$.

Let $x, y \in \mathbf{R}^{n+1}$. For every point $z \in \mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$, one has $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{C}_x(\mathrm{I})$, since $z \in \mathrm{C}_x(\mathrm{I})$, and $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{C}_y(\mathrm{I})$, since $z \in \mathrm{C}_y(\mathrm{I})$, so that $\mathrm{C}_z(\mathrm{I}) \subset \mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$. This proves that $\mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$ is a union of faces of $\mathrm{C}_x(\mathrm{I})$. However, a union of faces of a polyhedron is convex if and only it has a unique maximal element — so that one of these faces contains all of them. As a consequence, $\mathrm{C}_x(\mathrm{I}) \cap \mathrm{C}_y(\mathrm{I})$ is a face of $\mathrm{C}_x(\mathrm{I})$, and it belongs to $\mathscr{C}$. □

Proposition **(3.6.7)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *Let* $\mathrm{L} = \mathrm{affsp}(\mathrm{C}_x(\mathrm{I})) - x$ *be the minimal vector subspace of* $\mathbf{R}^{n+1}$ *such that* $\mathrm{C}_x(\mathrm{I}) \subset x + \mathrm{L}$. *One has* $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_x(\mathrm{I})$ *for every* $y \in \mathrm{L}$.

Proof. — Let $g \in \mathrm{in}_x(\mathrm{I})$; let $f_1, \dots, f_r \in \mathrm{I}$ be such that g decomposes as a sum $\sum_{i=1}^r \mathrm{in}_x(f_i)$ of initial forms with pairwise disjoint supports.

Prop. $C_x(I)$ $x + L =$ aff sp$(C_x(I))$

x

$\mathrm{in}_y(\mathrm{in}_x(I)) = \mathrm{in}_x(I)$
pour tout $y \in L$.

$g \in \mathrm{in}_x(I)$

$g = \mathrm{in}_x(f_1) + \dots + \mathrm{in}_x(f_r)$
supports des $\mathrm{in}_x(f_i)$ disjoints

For all i and all $\varepsilon > 0$ small enough, one has ~~$\tau_{f_i}(x+\varepsilon y) = \varepsilon\tau_{\mathrm{in}_x(f_i)}(y)$~~ $\tau_{f_i}(x) +$ and $\mathrm{in}_{x+\varepsilon y}(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$. Let J be the subset of $\{1,\dots,r\}$ consisting of all i where $\tau_{\mathrm{in}_x(f_i)}(y)$ is maximal; by the disjointness of the supports of the polynomials $\mathrm{in}_x(f_i)$, one has

$$\mathrm{in}_y(g) = \sum_{i\in \mathrm{J}} \mathrm{in}_y(\mathrm{in}_x(f_i)) = \sum_{i\in\mathrm{J}} \mathrm{in}_{x+\varepsilon y}(f_i).$$

If ε is small enough, one has $x+\varepsilon y \in \mathrm{C}'_x(\mathrm{I})$, hence $\mathrm{in}_{x+\varepsilon y}(f_i) \in \mathrm{in}_x(\mathrm{I})$ for all i, so that $\mathrm{in}_y(g) \in \mathrm{in}_x(\mathrm{I})$. This implies the inclusion $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) \subset \mathrm{in}_x(\mathrm{I})$. Since these two homogeneous ideals have the same Hilbert functions, one has equality. □

$g \in \mathrm{in}_x(I)$

$\Downarrow$

$\mathrm{in}_y(g) \in \mathrm{in}_x(I)$

$\boxed{\mathrm{in}_y(\mathrm{in}_x(I)) \subset \mathrm{in}_x(I)}$

inclusion d'idéaux homogènes ayant même fonction de Hilbert

donc égalité

3.7. Tropicalization of algebraic varieties

The goal of this section is to generalize theorem 3.3.6 to all ideals of $\mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}]$. We first recall how to pass from ideals of this ring to homogeneous ideals of $\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$, and back.

3.7.1. — Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}]$. The support in $\mathbf{Z}^n$ of the homogeneous Laurent polynomial $f(\mathrm{T}_1/\mathrm{T}_0,\dots,\mathrm{T}_n/\mathrm{T}_0)$ have an infimum, say $p = (p_0,\dots,p_n)$. Explicitly, if $\mathrm{S}(f)$ is the support of f and $f = \sum_{m\in\mathrm{S}(f)} c_m\mathrm{T}^m$, then

$$f(\mathrm{T}_1/\mathrm{T}_0,\dots,\mathrm{T}_n/\mathrm{T}_0) = \sum_{m\in\mathrm{S}(f)} c_m \mathrm{T}_0^{-m_1-\dots-m_n}\mathrm{T}_1^{m_1}\dots\mathrm{T}_n^{m_n},$$

Prop. $C_x(I)$ $x+L = \mathrm{aff\,sp}(C_x(I))$

$\mathrm{in}_y(\mathrm{in}_x(I)) = \mathrm{in}_x(I)$ pour tout $y \in L$.

$g \in \mathrm{in}_x(I)$

$g = \mathrm{in}_x(f_1) + \dots + \mathrm{in}_x(f_r) = \sum c_m T^m$

supports des $\mathrm{in}_x(f_i)$ disjoints.

? $\mathrm{in}_y(g)$

$\tau_g(y) = \sup_m \log|c_m| + \langle m, y\rangle$

$= \sup_{1\le i\le r}\left(\sup_{m\in \mathrm{Supp}(\mathrm{in}_x(f_i))} \log|c_m| + \langle m,y\rangle\right)$

$= \sup_i \tau_{\mathrm{in}_x(f_i)}(y)$

$J = \{ j \mid \tau_g(y) = \tau_{\mathrm{in}_x(f_j)}(y)\}$

$\mathrm{in}_{x+\varepsilon y}(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$ $\varepsilon > 0$ assez petit

$\mathrm{in}_y(g) = \sum_{i\in J} \mathrm{in}_y\, \mathrm{in}_x(f_i) = \sum \underline{\mathrm{in}_{x+\varepsilon y}}(f_i)$

$C'_x(I) = \mathrm{relint}\ C_x(I)$ $\Rightarrow$

$x + \varepsilon y \in C'_x(I)$ si $\varepsilon > 0$ est assez petit

$\Rightarrow \mathrm{in}_{x+\varepsilon y}(f_i) \in \mathrm{in}_x(I)$

$\Rightarrow \mathrm{in}_y(g) \in \mathrm{in}_x(I)$

3.7. Tropicalization of algebraic varieties

The goal of this section is to generalize theorem 3.3.6 to all ideals of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. We first recall how to pass from ideals of this ring to homogeneous ideals of $K[T_0, \dots, T_n]$, and back.

3.7.1. — Let $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. The support in $\mathbf{Z}^n$ of the homogeneous Laurent polynomial $f(T_1/T_0, \dots, T_n/T_0)$ have an infimum, say $p = (p_0, \dots, p_n)$. Explicitly, if $S(f)$ is the support of f and $f = \sum_{m \in S(f)} c_m T^m$, then

$$f(T_1/T_0, \dots, T_n/T_0) = \sum_{m \in S(f)} c_m T_0^{-m_1 - \dots - m_n} T_1^{m_1} \dots T_n^{m_n},$$

homogène de degré 0

Procédé d'homogénéisation

$T_1 + T_2^3 + T_1 T_2^{-1}$

$\rightsquigarrow T_1 T_0^2 + T_2^3 + T_0 T_1 T_2$

$\leftarrow$ $(T_0 = 1)$

$T_1 T_2^{-1} + T_2^3 T_1^2$

$= \underbrace{T_1 T_2^{-1}}_{\text{monôme}} (1 + T_1 T_2^4)$

monômes ≥ 0 premiers entre eux

$\rightsquigarrow T_0^5 + T_1 T_2^4$

$\leftarrow$ $T_0 = 1$ multiplication par un monôme

Géométriquement :

algèbre $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$\mathrm{Spec}(\quad) = \mathbb{G}_{m,K}^n$ groupe multiplicatif de dim n

analogue schématique de $(\mathbb{C}^\times)^n$

algèbre graduée $K[T_0, \dots, T_n]$

$\mathrm{Proj}(K[T_0, \dots, T_n]) = \mathbb{P}_{n,K}$

espace projectif de dim n sur K,

analogue schématique de $\mathbb{C}\mathbb{P}_n = \mathbb{P}_n(\mathbb{C}) = (\mathbb{C}^{n+1} - \{0\}) / \mathbb{C}^\times$

$I \subset K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$X = V(I) \subset \mathbb{G}_m^n \rightsquigarrow Y = \overline{X}$ adhérence schématique

$X = Y \cap \mathbb{G}_m^n \longleftarrow Y = V(J)$

$I^h = J =$ idéal homogène engendré par ~~des~~ f^h, $f \in I$.

$I = J \cdot K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$I^h \in J$ $\qquad I^h = \{ f \mid \exists \text{ monôme } T^m,\ T^m f \in J \}$

so that $p_0 = -\deg(f)$ and $p_j = \operatorname{ord}_{T_j}(f)$ for $j \in \{1, \dots, n\}$. Let then f^h be the polynomial $T^{-p} f(T_1/T_0, \dots, T_n/T_0)$; it is the unique homogeneous polynomial in $K[T_0, \dots, T_n]$ such that $\operatorname{ord}_{T_j}(f^h) = 0$ for every $j \in \{0, \dots, n\}$ and $f = T_1^{p_1} \dots T_n^{p_n} f^h(1, T_1, \dots, T_n)$.

3.7.2. — Let I be an ideal in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. The ideal I^h generated by all polynomials f^h, for $f \in I$, is a homogeneous ideal of $K[T_0, \dots, T_n]$. The ring morphism $K[T_0, \dots, T_n] \to K[T_1, \dots, T_n]$ with kernel $(T_0 - 1)$ corresponds to setting to 1 the homogeneous coordinate T_0, it identifies the invertibility locus of T_0 in $\mathbf{P}^n_K$ with the affine space $\mathbf{A}^n_K$. The locus of invertibility of $T_0 \dots T_n$ is defined by requiring further that the other homogeneous coordinates are invertible too: this is an open subscheme of $\mathbf{P}^n_K$ which is is naturally isomorphic to $\mathbf{G}_{m_K}^n$ and corresponds to the ring morphism $f \mapsto f(1, T_1, \dots, T_n)$ from $K[T_0, \dots, T_n]$ to $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

Ideals of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ correspond to closed subschemes of $\mathbf{G}_{m_K}^n = \operatorname{Spec}(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}])$. Homogeneous ideals of $K[T_0, \dots, T_n]$ correspond to closed subschemes of $\mathbf{P}^n_K = \operatorname{Proj}(K[T_0, \dots, T_n])$. Then $V(I^h)$ is the Zariski closure of $V(I)$.

As a consequence, several geometric properties of $V(I)$ are preserved when passing to $V(I^h)$:

– If $V(I)$ is irreducible, then so is $V(I^h)$;
– If $V(I)$ is integral, then so is $V(I^h)$;

$\mathbb{G}_m^n$ $\quad$ $\mathbb{P}_n$

$V(I) \rightsquigarrow V(I^h)$

préserve
- dimension
- intégrité
- irréductibilité
- équidimensionalité

Adhérence schématique

. X schéma, S partie de X (ou sous-schéma de X)

. $S \subset X$ $\quad \overline{S}$ = plus petit sous schéma fermé de X contenant S.

$$\mathcal{I}_{\overline{S}} = \langle f \mid f = 0 \text{ sur } S \rangle$$

si X est affine $X = \mathrm{Spec}(A)$

si S est un sous-schéma fermé de l'un ouvert U de X

comment calculer $\overline{S}$?

exemple $\quad U = D(f) = \mathrm{Spec}(A_f)$

$S = V(J) \quad J \subset A_f$.

$$\overline{S} = V(I) \qquad I = \{ a \in A \mid \tfrac{a}{1} \in J \}$$

le plus grand idéal de A tel que $I \cdot A_f \subset J$

– One has $\dim(V(I^h)) = \dim(V(I))$;
– If $V(I)$ is equidimensional, then so is $V(I^h)$.

3.7.3. — Let $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$; let $p \in \mathbf{Z}^n$ be such that

$$f = T_1^{p_1} \dots T_n^{p_n} f^h(1, T_1, \dots, T_n).$$

Let $x' \in \mathbf{R}^n$ and let $x = (0, x') \in \mathbf{R}^{n+1}$; then the definitions of the tropical polynomials and of the initial forms imply that $\tau_f(x) = \langle p, x\rangle + \tau_{f^h}(x')$ and $\mathrm{in}_x(f) = T_1^{p_1} \dots T_n^{p_n} \mathrm{in}_{x'}(f^h)$. In particular, $\mathrm{in}_{x'}(f^h) \in \mathrm{in}_x(I)^h$. Every homogeneous element of I^h is of the form $T^m f^h$ for some elements $f \in I$ and $m \in \mathbf{Z}^n$, one then has $\mathrm{in}_{x'}(T^m f^h) = T^m \mathrm{in}_{x'}(f^h) \in \mathrm{in}_x(I)^h$, hence $\mathrm{in}_{x'}(I^h) \subset \mathrm{in}_x(I)^h$. Conversely, if $f \in I$, then there exists $m \in \mathbf{Z}^{n+1}$ such that $T^m \mathrm{in}_x(f)^h = \mathrm{in}_{x'}(f^h)$, hence $T^m \mathrm{in}_x(f)^h \in \mathrm{in}_{x'}(I^h)$. This proves the relation

$$\mathrm{in}_x(I)^h = (\mathrm{in}_{x'}(I^h) : (T_0 \dots T_n)^\infty) = K[T_0, \dots, T_n] \cap \mathrm{in}_{x'}(I^h)_{T_0 \dots T_n}.$$

In any case, identifying $\mathbf{R}^n$ with $\{0\} \times \mathbf{R}^n$, the Gröbner decomposition Σ_{I^h} of $\mathbf{R}^{n+1}$ associated with the ideal I^h furnishes a similar decomposition Σ_I of $\mathbf{R}^n$. When x varies in an open cell of this decomposition, and $x' = (0, x)$, the initial ideal $\mathrm{in}_{x'}(I^h)$ is constant, hence the initial ideal $\mathrm{in}_x(I)$ is constant. The reader shall be cautious not to state an indue converse assertion: for example, $\mathrm{in}_x(I) = (1)$ only means that $\mathrm{in}_{x'}(I^h)$ contains a monomial, but the different initial ideals $\mathrm{in}_{x'}(I^h)$ can be very different.

$f \rightsquigarrow f^h$

$f = T^p f^h(1, T_1, \dots, T_n)$

$\tau_f(x) = \langle p, x \rangle + \tau_{f^h}(x')$

$x \in \mathbb{R}^n$ $\quad x' = (0, x)$.

$\mathrm{in}_x(f) = T^p \, \mathrm{in}_{x'}(f^h)(1, T_1, \dots, T_n)$

[$\mathrm{in}_{x'}(I^h) \subset \mathrm{in}_x(I)^h$.

$\mathrm{in}_x(I)^h = \{ f \mid \exists T^m.\ f \cdot T^m \in \mathrm{in}_{x'}(I^h) \}$

$I^h \rightsquigarrow$ décomposition de Gröbne de $\mathbb{R}^{n+1}$

$C_{x'}(I^h)$

$\rightsquigarrow$ déc. de Gröbner de $\mathbb{R}^n = \{0\} \times \mathbb{R}^n$

$C_{x'}(I^h) \cap (x'_0 = 0)$

cellules de $\mathbb{R}^n$ tq idéal initial $\mathrm{in}_{x'}(I^h)$ est constant.

($\Rightarrow$ $\mathrm{in}_x(I)$ est constant mais pas réciproquement)

Definition **(3.7.4)**. — *Let* K *be a valued field, let* I *be an ideal of* $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ *and let* X *be the closed subscheme* V(I) *of* $\mathbf{G}_m{}^n$.

a) *The* tropical variety $\mathscr{T}_X$ *of* X *is the intersection, for all* $f \in I$, *of the tropical hypersurfaces* $\mathscr{T}_f$.

b) *A* tropical basis *of* I *is a finite family* $(f_1, \dots, f_m)$ *in* I *such that* $\mathscr{T}_X = \bigcap_{i=1}^m \mathscr{T}_{f_i}$.

By definition, for $x \in\in \mathbf{R}^n$, one has $x \notin \mathscr{T}_X$ if and only if there exists $f \in I$ such that the supremum defining $\tau_f(x)$ is achieved at a single monomial $c_m T^m$ of f. This also means that $NP_{f,x}$ is reduced to a point, or, if K is split, that the initial form $\mathrm{in}_x(f)$ is a monomial.

Replacing f by $c_m^{-1} T^{-m} f$, we may assume that $\tau_f(x) = 0$, which is achieved uniquely at the monomial 1, that is $NP_{f,x} = \{0\}$. From the point of view of initial forms, this means that $\mathrm{in}_x(f) = 1$.

Proposition **(3.7.5)**. — *Let* I *be an ideal of* $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ *and let* X *be the closed subscheme* V(I). *Let* (X_j) *be the family of its irreducible components; for every* j, *let* $I_j = I(X_j)$ *be the prime ideal defining* X_j. *One has* $\mathscr{T}_X = \bigcup_j \mathscr{T}_{X_j}$.

Proof. — The ideals I_j are the minimal prime ideals of $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ containing I; as a consequence, their intersection $J = \bigcap_j I_j$ is the radical of I, the set of all elements $f \in K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ such that there exists $m \geqslant 1$ such that $f^m \in I$.

généralise la définition donnée sur $\mathbb{C}$

· $X = V(I)$

$\mathscr{T}_X = \bigcap_{f \in I} \mathscr{T}_f \subset \mathbb{R}^n$.

· base tropicale $(f_1, \dots, f_m)$ $\quad \mathscr{T}_X = \mathscr{T}_{f_1} \cap \dots \cap \mathscr{T}_{f_m}$.

$f = \sum c_m T^m \quad \tau_f(x) = \sup_m \log|c_m| + \langle m, x\rangle$

$\mathscr{T}_f = \{x \mid$ τ_f est affine au voisinage de x

($\Leftrightarrow$ $S_x(f)$ est réduit à un élément

$\Leftrightarrow$ $NP_{f,x}$ est un point)

$\Leftrightarrow$ $\mathrm{in}_x(f)$ est un monôme.

$x \notin \mathscr{T}_X \Leftrightarrow \exists f \in I$ $\quad \mathrm{in}_x(f)$ est un monôme

$\Leftrightarrow$ ——— $\quad \mathrm{in}_x(f) = 1$

$X = \bigcup_j X_j$ composantes irréductibles.

$I_{X_j} = I_j$ idéaux premiers minimaux contenant $I = I_X$

$\sqrt{I_X} = \bigcap_{j=1} I_j$

$\mathscr{T}_X = \bigcup \mathscr{T}_{X_j}$

For every j, one has $\mathrm{I} \subset \mathrm{I}_j$, hence $\mathscr{T}_{\mathrm{X}_j} \subset \mathscr{T}_{\mathrm{X}}$. Consequently, $\bigcup \mathscr{T}_{\mathrm{X}_j} \subset \mathscr{T}_{\mathrm{X}}$. Conversely, let $x \in \mathbf{R}^n - \bigcup \mathscr{T}_{\mathrm{X}_j}$. For every j, there exists $f_j \in \mathrm{I}_j$ such that $\mathrm{in}_x(f_j) = 1$. Let $f = \prod f_j$; one has $f \in \bigcap \mathrm{I}_j = \mathrm{J}$, hence there exists $m \in \mathbf{N}$ such that $f^m \in \mathrm{I}$. Then $\mathrm{in}_x(f^m) = \prod_j \mathrm{in}_x(f_j)^m = 1$, hence $\mathrm{in}_x(\mathrm{I}) = 1$ and $x \notin \mathscr{T}_{\mathrm{X}}$. □

Proposition **(3.7.6)**. — *Let* K *be a valued field, let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* X *be the closed subscheme* V(I) *of* $\mathbf{G}_{\mathrm{m}}{}^n$.

a) *The ideal* I *admits a tropical basis.* ← ok

$\mathscr{T}_X = \bigcap \mathscr{T}_{f_j}$

b) *The tropical variety* $\mathscr{T}_{\mathrm{X}}$ *is a* Γ*-strict polyhedral subspace of* $\mathbf{R}^n$. ←

c) *For every valued extension* L *of* K, *one has* $\mathscr{T}_{\mathrm{X}_{\mathrm{L}}} = \mathscr{T}_{\mathrm{X}}$. ←

Proof. — We first prove assertion *a*) under the assumption that there is a splitting of the valuation $\mathrm{K}^\times \to \Gamma$.

Let I^{h} be the homogeneous ideal of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ associated with I. Let $\mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$ be the set of all $x \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ does not contain any monomial.

If $x \notin \mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$, then $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ contains a monomial, say T^m, hence there exists $f \in \mathrm{I}^{\mathrm{h}}$ such that $\mathrm{in}_x(f) = \mathrm{T}^m$. Then $\mathrm{in}_y(f) = \mathrm{T}^m$ for every $y \in \mathbf{R}^{n+1}$ close enough to x, so that $\mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$ is closed in $\mathbf{R}^{n+1}$.

If $x \in \mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$, then the open cell $\mathrm{C}'_x(\mathrm{I}^{\mathrm{h}})$ is contained in $\mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$ as well, and its closure $\mathrm{C}_x(\mathrm{I}^{\mathrm{h}})$ too. Consequently, $\mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$ is a union of some cells of the Gröbner polyhedral decomposition $\Sigma_{\mathrm{I}^{\mathrm{h}}}$. In particular, it is a Γ-strict polyhedral subset of $\mathbf{R}^{n+1}$.

$x \notin \bigcup \mathscr{T}_{X_j}$

$\Rightarrow \exists f_j \in I_j$

$\mathrm{in}_x(f_j) = 1$.

$f = \prod f_j \in \bigcap I_j = \sqrt{I_X}$

$\exists p \quad f^p \in I_X$

$\mathrm{in}_x(f^p) = \prod \mathrm{in}_x(f_j)^p = 1$.

donc $x \notin \mathscr{T}_X$

(on peut récrire l'argument en termes de polytopes de Newton

→ pas besoin qu'il soit scindé)

$\mathscr{T}^h_X = \{x \in \mathbb{R}^{n+1} \mid \mathrm{in}_x(I^h)$ ne contient pas de monôme $\}$

$\mathscr{T}_X = \{x \in \mathbb{R}^n \mid \mathrm{in}_x(I)$ ne contient pas de monôme $\}$.

$= \{x \in \mathbb{R}^n \mid x' = (0,x) \in \mathscr{T}^h_X\}$.

réunion de cellules de la décomposition de Gröbner de I^h.

→ polyédral.

Γ-strict.

donc $\mathscr{T}_X$ est polyédral, Γ-strict.

Let $x \in \mathbf{R}^n$ and let $x' = (0, x)$. Then $\operatorname{in}_x(\mathrm{I}) = (1)$ if and only if there exists $f \in \mathrm{I}$ such that $\operatorname{in}_x(f) = 1$; then $\operatorname{in}_{x'}(f^{\mathrm{h}})$ is a monomial in $\mathrm{T}_1, \dots, \mathrm{T}_n$ multiplied by a polynomial in T_0. By homogeneity, $\operatorname{in}_{x'}(f^{\mathrm{h}})$ is a monomial. Conversely, if $\operatorname{in}_{x'}(f^{\mathrm{h}})$ is a monomial, then $\operatorname{in}_x(f)$ is a monomial as well. This proves that $\mathscr{T}_{\mathrm{X}}$ is the set of $x \in \mathbf{R}^n$ such that $(0, x) \in \mathscr{T}_{\mathrm{X}}^{\mathrm{h}}$.

b

Moreover, for every cell $\mathrm{C}'_x(\mathrm{I}^{\mathrm{h}})$ such that the corresponding initial ideal $\operatorname{in}_x(\mathrm{I}^{\mathrm{h}})$ contains a monomial, we may choose $f \in \mathrm{I}$ such that $\operatorname{in}_x(f^{\mathrm{h}})$ is a monomial. The family (f_i) of these polynomials satisfies the required condition.

a)

$x \in \mathbb{R}^{n+1}$

si $\operatorname{in}_x(I^h)$ contient un monôme, choisissons f_x tq $\operatorname{in}_x(f)$ soit un monôme.

$\underline{y \in C'_x(I^h)}$

équation de $C'_x(I^h)$

on voit que $\operatorname{in}_y(f_x)$ est encore ce monôme.

nombre fini de ces mauvaises cellules

⇝ $f_1, \dots, f_m \in I^h$ tq

$x \in$ mauvaise cellule

$\Leftrightarrow \exists j$ $\operatorname{in}_x(f_j)$ est un monôme

Autrement dit

$\mathcal{C}_X = \bigcap_{j=1}^{m} \mathcal{C}_{f_j}$

K' (alg clos) valué (scindé)

L

K

To prove c), we may assume that the valuation of the field L has a splitting, so that assertion a) holds for $\mathscr{T}_{\mathrm{X}_{\mathrm{L}}}$.

il y a une base tropicale sur K'

⇓

il y a une base tropicale sur L

⇓

sur K

The inclusion $\mathscr{T}_{\mathrm{X}_{\mathrm{L}}} \subset \mathscr{T}_{\mathrm{X}}$ follows from the definition. Indeed, if $x \in \mathbf{R}^n - \mathscr{T}_{\mathrm{X}}$, there exists $f \in \mathrm{I}$ such that the supremum defining $\tau_f(x)$ is reached for one monomial only, and the same property holds for f viewed as an element of I_{L}, so that $x \notin \mathscr{T}_{\mathrm{X}_{\mathrm{L}}}$.

Conversely, let $x \in \mathbf{R}^n - \mathscr{T}_{\mathrm{X}_{\mathrm{L}}}$ and let $f = \sum c_m \mathrm{T}^m \in \mathrm{I}_{\mathrm{L}}$ be such that the supremum defining $\tau_f(x)$ is reached at only one monomial. Let us consider an expression $f = \sum_{j=1}^r a_j f_j$, where $a_j \in \mathrm{L}$ and $f_j \in \mathrm{I}$, and the integer r is minimal. Let $\mathrm{S} \subset \mathbf{Z}^n$ be the union of the supports of the f_j and let us consider the $r \times \mathrm{S}$ matrix A given by the coefficients of these Laurent polynomials. Among all finite families

$I \subset I_L$

$\Rightarrow \mathcal{C}_{X_L} \subset \mathcal{C}_X$

Supposons que $X_L = V(I_L)$ a une base tropicale.

Démontrons que $X = V(I)$

et $\mathcal{C}_{X_L} = \mathcal{C}_X$

Conversely, let $x \in \mathbf{R}^n - \mathscr{T}_{X_L}$ and let $f = \sum c_m T^m \in I_L$ be such that the supremum defining $\tau_f(x)$ is reached at only one monomial. Let us consider an expression $f = \sum_{j=1}^r a_j f_j$, where $a_j \in L$ and $f_j \in I$, and the integer r is minimal. Let $S \subset \mathbf{Z}^n$ be the union of the supports of the f_j and let us consider the $r \times S$ matrix A given by the coefficients of these Laurent polynomials. Among all finite families

On a prouvé $\mathscr{C}_{X_L} \subset \mathscr{C}_X$.

$x \notin \mathscr{C}_{X_L}$; démontrons $x \in \mathscr{C}_X$

il existe $f \in I_L$ $\quad S_x(f) = \{m\}$

$in_X(f) =$ monôme.

$f = a_1 f_1 + \cdots + a_r f_r \qquad a_j \in L \quad f_j \in I.$

(minimale $\Rightarrow$ $f_1, \dots, f_r$ sont lin. indép. sur L $\Leftarrow$ K) $\qquad f_j = \sum_{m \in S} c_{j,m} T^m$

$S(f) \subset S = \bigcup_j S(f_j) \qquad A \in \mathrm{mat}_{r \times S}(K) \qquad A = \underset{r}{\updownarrow}\overbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_r \end{bmatrix}}^{\mathrm{Card}(S)} = \begin{bmatrix} c_{1,m} \cdots \\ \vdots \\ c_{r,m} \cdots \end{bmatrix}$ rang r.

$\exists \quad R \subset S \quad \mathrm{Card}(R) = r \rightsquigarrow A^R = (c_{j,m})_{\substack{1 \le j \le r \\ m \in R}} \qquad A^R$ inversible $\rightsquigarrow U = (A^R)^{-1}$

$\det(A^R) \neq 0$

prendre R qui maximise $\quad \log|\det(A^R)| + \sum_{m \in R} \langle m, x \rangle$

$UA \Leftrightarrow$ combinaison linéaire des f_j (opération sur les lignes)

on se ramène au cas : $\boxed{f_j = T^{m_j} + \sum_{m \neq m_1, \dots, m_r} c_{jm} T^m}$

$R = (m_1, \dots, m_r)$

$f = \sum a_j f_j = \sum a_j T^{m_j} + \sum_{m \notin R} \left(\sum_j a_j c_{jm}\right) T^m$

$\mathrm{R} = (m_1, \dots, m_r)$ in S, let us choose one such that the quantity

$$\log(|\det(A^{\mathrm{R}})|) + \sum_{j=1}^{r} \langle m_j, x \rangle$$

is maximal, where A^{R} is the $r \times r$ submatrix of A with columns $m_1, \dots, m_r$. Since A has rank r, the matrix A^{R} is invertible and there exists a matrix $U \in \mathrm{GL}(r, \mathrm{K})$ such that $(UA)^{\mathrm{R}} UA^{\mathrm{R}} = I_r$. Then

$$\log(|\det((UA)^{\mathrm{R}})|) + \sum_{j=1}^{r} \langle m_j, x \rangle = \log(|\det(U)|) + \log(|\det(A^{\mathrm{R}})|) + \sum_{j=1}^{r} \langle m_j, x \rangle$$

is maximal. For $i \in \{1, \dots, r\}$ and $m \in \mathrm{S} - \mathrm{R}$, exchanging the columns m and m_i replaces the above quantity by

$$\log(|(UA)_{i,m}|) + \sum_{j=1}^{r} \langle m_j, x \rangle + \langle m, x \rangle - \langle m_i, x \rangle,$$

so that

$$\log(|(UA)_{i,m}|) + \langle m, x \rangle \geqslant \langle m_i, x \rangle.$$

Replacing the polynomials $f_1, \dots, f_r$ by the polynomials whose coefficients are given by the matrix UA, we may assume that there are Laurent polynomials $g_j = \sum_{m \in \mathrm{S}-\mathrm{R}} c_{j,m} \mathrm{T}^m$ with support contained in $\mathrm{S} - \mathrm{R}$ (for $j \in \{1, \dots, r\}$) such that $f_j = \mathrm{T}^{m_j} + g_j$ and such that

Explicitons le choix de R.

si on remplace R
par $R - \{m_j\} \cup \{m\} = R'$
pour $m \in S - R$
on diminue la quantité maximisée sur R.

$$\det(UA^R) = \det(U) \cdot \det(A^R)$$

$$R' : \log|\det(UA^{R'})| + \sum_{m \in R'} \langle m, x \rangle$$

$$\begin{array}{cccc} & R \; m_j & S - R & \\ UA = & \begin{bmatrix} 1 & & & \\ & \ddots & & c_{j,m} \\ & & 1 & \end{bmatrix} \end{array}$$

$$\det(UA^{R'}) = c_{j,m}$$

$$\log|c_{jm}| + \sum_{m \in R - \{m_j\} \cup \{m\}} \langle m, x \rangle \leq \cancel{\log|1|} + \sum_{m \in R} \langle m, x \rangle$$

$$\boxed{\log|c_{jm}| - \langle m_j, x \rangle + \langle m, x \rangle \leq 0}$$

$\log(|c_{j,m}|) + \langle m, x\rangle \leqslant \langle m_j, x\rangle$. Then

$$f = \sum_{j=1}^{r} a_j \mathrm{T}^{m_j} + \sum_{m\in \mathrm{S}-\mathrm{R}} \overbrace{\Big(\sum_{j=1}^{r} a_j c_{j,m}\Big)}^{c_m}\mathrm{T}^m,$$

so that

$$\tau_f(x) \geqslant \sup_j(\log(|a_j|) + \langle m_j, x\rangle).$$

Then, for every $m \in \mathrm{S}-\mathrm{R}$ and every $j \in \{1,\dots,r\}$, one has the inequality

$$\log(|a_j c_{j,m}|) + \langle m, x\rangle \leqslant \log(|a_j|) + \langle m_j, x\rangle,$$

so that $\log(|c_{j,m}|) + \langle m, x\rangle \leqslant \langle m_j, x\rangle$ and $\tau_f(x) = \sup_j(\log(|a_j|) + \langle m_j, x\rangle)$.

By the assumption $x \notin \mathscr{T}_f$, there exists a unique $j \in \{1,\dots,r\}$ such that $\tau_f(x) = \log(|a_j|) + \langle m_j, x\rangle$, and $\log(|c_m|) + \langle m, x\rangle < \tau_f(x)$ for every $m \in \mathrm{S}-\mathrm{R}$. For $i \in \{1,\dots,r\}$ such that $i \neq j$, one thus has $\log(|a_i|) + \langle m_i, x\rangle < \log(|a_j|) + \langle m_j, x\rangle$. Then for every $m \in \mathrm{S}-\mathrm{R}$, one has

$$\log(|a_i c_{i,m}|) + \langle m, x\rangle \leqslant \log(|a_i|) + \langle m_i, x\rangle < \log(|a_j|) + \langle m_j, x\rangle,$$

so that $\log(|c_m|) = \log(|a_j c_{j,m}|)$. Since $\log(|c_m|) + \langle m, x\rangle < \tau_f(x) = \log(|a_j|) + \langle m_j, x\rangle$, one has $\log(|c_{j,m}|) + \langle m, x\rangle < \langle m_j, x\rangle$. This proves that the supremum defining $\tau_{f_j}(x)$ is reached for the monomial m_j

$|c_m| = |\sum a_i c_{i,m}| = |a_j c_{jm}| \rightsquigarrow \boxed{\log|c_{jm}| + \langle m, x\rangle < \langle m_j, x\rangle}$

$$f = \sum_{j=1}^{r} a_j f_j = \underbrace{\sum_{j=1}^{r} a_j T^{m_j}} + \sum_{m\in S-R} \Big(\sum a_j c_{jm}\Big) T^m$$

? $\tau_f(x)$ est atteint en un seul monôme.

déjà . $\tau_f(x) \geq \sup_j (\log|a_j| + \langle m_j, x\rangle)$

. $|\sum a_j c_{jm}| \leq \sup_j |a_j|\,|c_{jm}|$

$\log|\sum a_j c_{jm}| \leq \sup_j \log|a_j| + \log|c_{jm}|$

$\leq (\sup_j \log|a_j| + \langle m_j, x\rangle) - \langle m, x\rangle$

$\log|\sum a_j c_{jm}| + \langle m, x\rangle \leq \sup_j \log|a_j| + \langle m_j, x\rangle \leq \tau_f(x)$

$\Rightarrow \tau_f(x) = \sup_j (\log|a_j| + \langle m_j, x\rangle)$

$x \notin \mathscr{T}_f$ $\exists!\, j$ qui maximise $\log|a_j| + \langle m_j, x\rangle$ $\rightsquigarrow$ $x \in \mathscr{T}_{f_j}$

only. Since $f_j \in \mathrm{I}$, we have proved that $x \notin \mathscr{T}_{\mathrm{X}}$. This proves assertion c).

In fact, the same argument also allows to deduce assertion a) in full. We may indeed apply it to every element f of a tropical basis of I_{L}, and every point $x \in \mathbf{R}^n$. For a given f, there are only finitely many possible families $(m_1, \dots, m_r)$ as above, so that when x varies, the procedure furnishes finitely Laurent polynomials in I. The collection (f_i) of these Laurent polynomials is a tropical basis of I, as sought for.

For every i, $\mathscr{T}_{f_i}$ is a Γ-strict polyhedral subspace of $\mathbf{R}^n$, hence so is their intersection $\mathscr{T}_{\mathrm{X}}$. This proves b) and concludes the proof of the proposition. □

Remark **(3.7.7)**. — Let V be a closed subvariety of $(\mathbf{C}^*)^n$ and let I be its ideal in $\mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. Let us endow the field $\mathbf{C}$ with the trivial valuation. Then $\mathscr{T}_{\mathrm{V(I)}}$ coincides with the tropical variety $\mathscr{T}_{\mathrm{V}}$ of definition 2.6.3. This proves that $\mathscr{T}_{\mathrm{V}}$ is a $\mathbf{Q}$-rational polyhedral set. In particular, theorem 2.6.6 applies to V, and this concludes the proof of the Bieri–Groves theorem (theorem 2.6.5).

Theorem **(3.7.8)** (Einsiedler, Kapranov & Lind, 2006)

Let I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* X *be the closed subscheme* V(I) *of* $\mathbf{G}_{\mathrm{m}}{}^n$. *The following three subsets of* $\mathbf{R}^n$ *coincide:*

(i) *The tropical variety* $\mathscr{T}_{\mathrm{X}}$;

$\bigcap \mathcal{T}_{X_L} \subset \bigcap \mathcal{T}_X \to \mathcal{T}_X \subset \mathcal{T}_{X_L} \to$ égalité $\mathcal{T}_X = \mathcal{T}_{X_L}$

$x \rightsquigarrow f \in I_L \rightsquigarrow f_1, \dots, f_r \in I$

$\rightsquigarrow R \rightsquigarrow g_j$

$\mathcal{T}_{X_L}$ a une base tropicale $\sum a_j f_i$

→ il suffit de prendre f parmi un ens. fini

→ R n'a qu'un nb fini de possibilités

on le prend tous.

pour chacun : une autre décomposition de $f_r = \sum b_j^R g_j^R$

$\exists R, j$ tq $x \in \mathcal{T}_{g_j^R}$

base tropicale : prendre tous les g_j^R.

$f = \sum c_m T^m$
$f \in I$
$f(z) = 0$
$\lambda(z) \notin \mathcal{T}_f$

$\lambda : (L^\times)^n \to \mathbf{R}^n$
$z \mapsto \log(|z_1|, \dots, |z_n|)$

$X^{an} \xrightarrow{\lambda} \mathbf{R}^n$ $\quad \mathcal{T}_X = \lambda(X^{an})$

variété tropicale — amibe non archimédienne

(ii) *The set of all $x \in \mathbf{R}^n$ such that there exists a valued extension* L *of* K *and a point $z \in X(L) \subset (L^\times)^n$ such that $x = \lambda(z)$;*

(iii) *The image of $X^{an} = \mathscr{V}(I) \subset (\mathbf{G}_m{}^n)^{an}$ by the tropicalization map $p \mapsto (+\log(p(T_1)), \dots, +\log(p(T_n)))$.*

If the valuation of K *admits a splitting, they also coincide with:*

(iv) *The set of all $x \in \mathbf{R}^n$ such that $\operatorname{in}_x(I) \neq K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$;*

If $\cancel{K}$ is algebraically closed and its valuation nontrivial, they also coincide with:

L

(v) *The closure of the set of all $x \in \mathbf{R}^n$ such that there exists a point $z \in X(\cancel{K}) \subset (\cancel{K}^\times)^n$ such that $x = \lambda(z)$.*

L L

Proof. — Let us denote these subsets of $\mathbf{R}^n$ by $S_1 = \mathscr{T}_X, S_2, S_3, S_4, S_5$. As for the proof of theorem 3.3.6, some inclusions are essentially formal. The equality $S_2 = S_3$ has been proved in §3.2.9. If the valuation has a splitting, the equality $\operatorname{in}_x(I) = K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ is equivalent to the existence of $f \in I$ such that $\operatorname{in}_x(I)$ is invertible, that is, a monomial. This proves that $S_1 = S_4$. By definition, S_5 is the closure of a subset of S_2; since S_3 is closed, one has $S_5 \subset S_3$. Finally, for every $f \in I$, one has $S_2 \subset \mathscr{T}_f$, hence $S_2 \subset \mathscr{T}_X = S_1$.

étend le th. 3.3.6 (Kapranov) du cas $X = V(f)$

$S_2 = S_3$: description de $X^{an} = \mathcal{M}(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I)$

$S_1 = S_4$ définition de $\mathcal{T}_X$

$S_5 \subset \overline{S_2} = S_3 \subset S_1$

Il reste à prouver $S_1 \subset S_5$.

The rest of the proof follows from the results proved below. We first establish (lemma 3.7.9) that the dimension of $\mathscr{T}_X$ is at most that of V(I). Under the assumption that K is algebraically closed and its valuation is nontrivial, this is then used to prove that for every point $x \in \mathscr{T}_X \cap \Gamma^n$, there exists $z \in X(K)$ such that $\lambda(z) = x$