TOPICS IN TROPICAL GEOMETRY

Antoine Chambert-Loir

Cours annoté - 15 février 2021

Antoine Chambert-Loir
Université Paris-Diderot.
E-mail : Antoine.Chambert-Loir@math.univ-paris-diderot.fr

Version of February 15, 2021, 13h33
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only. Since $f_j \in \mathrm{I}$, we have proved that $x \notin \mathscr{T}_{\mathrm{X}}$. This proves assertion c).

In fact, the same argument also allows to deduce assertion a) in full. We may indeed apply it to every element f of a tropical basis of I_{L}, and every point $x \in \mathbf{R}^n$. For a given f, there are only finitely many possible families $(m_1, \dots, m_r)$ as above, so that when x varies, the procedure furnishes finitely Laurent polynomials in I. The collection (f_i) of these Laurent polynomials is a tropical basis of I, as sought for.

For every i, $\mathscr{T}_{f_i}$ is a Γ-strict polyhedral subspace of $\mathbf{R}^n$, hence so is their intersection $\mathscr{T}_{\mathrm{X}}$. This proves b) and concludes the proof of the proposition. □

Remark **(3.7.7)**. — Let V be a closed subvariety of $(\mathbf{C}^*)^n$ and let I be its ideal in $\mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. Let us endow the field $\mathbf{C}$ with the trivial valuation. Then $\mathscr{T}_{\mathrm{V(I)}}$ coincides with the tropical variety $\mathscr{T}_{\mathrm{V}}$ of definition 2.6.3. This proves that $\mathscr{T}_{\mathrm{V}}$ is a $\mathbf{Q}$-rational polyhedral set. In particular, theorem 2.6.6 applies to V, and this concludes the proof of the Bieri–Groves theorem (theorem 2.6.5).

Theorem **(3.7.8)** (Einsiedler, Kapranov & Lind, 2006)

Let I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* X *be the closed subscheme* V(I) *of* ${\mathbf{G}_{\mathrm{m}}}^n$. *The following three subsets of* $\mathbf{R}^n$ *coincide:*

(i) *The tropical variety* $\mathscr{T}_{\mathrm{X}}$*;*

S_1 (i) La variété tropicale $\mathscr{T}_X$

S_2 (ii) *The set of all $x \in \mathbf{R}^n$ such that there exists a valued extension L of K and a point $z \in X(L) \subset (L^\times)^n$ such that $x = \lambda(z)$;*

S_3 (iii) *The image of $X^{an} = \mathscr{V}(I) \subset (\mathbf{G}_m{}^n)^{an}$ by the tropicalization map $p \mapsto (\log(p(T_1)), \dots, \log(p(T_n)))$.*

If the valuation of K admits a splitting, they also coincide with:

S_4 (iv) *The set of all $x \in \mathbf{R}^n$ such that $\mathrm{in}_x(I) \neq K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$;*

For any algebraically closed extension L of K, endowed with an absolute value extending that of K which is nontrivial, they also coincide with:

S_5^L (v) *The closure of the set of all $x \in \mathbf{R}^n$ such that there exists a point $z \in X(L) \subset (L^\times)^n$ such that $x = \lambda(z)$.*

Proof. — Let us denote these subsets of $\mathbf{R}^n$ by $S_1 = \mathscr{T}_X, S_2, S_3, S_4, S_5^L$. As for the proof of theorem 3.3.6, some inclusions are essentially formal. The equality $S_2 = S_3$ has been proved in §3.2.9. If the valuation has a splitting, the equality $\mathrm{in}_x(I) = K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ is equivalent to the existence of $f \in I$ such that $\mathrm{in}_x(I)$ is invertible, that is, a monomial. This proves that $S_1 = S_4$. By definition, S_5^L is the closure of a subset of S_2; since S_3 is closed, one has $S_5^L \subset S_3$. Finally, for every $f \in I$, one has $S_2 \subset \mathscr{T}_f$, hence $S_2 \subset \mathscr{T}_X = S_1$.

The rest of the proof follows from the results proved below. We first establish (lemma 3.7.9) that the dimension of $\mathscr{T}_X$ is at most that of V(I). Under the assumption that K is algebraically closed and its valuation is nontrivial, this is then used to prove that for every point $x \in \mathscr{T}_X \cap \Gamma^n$, there exists $z \in X(K)$ such that $\lambda(z) = x$

$\mathscr{T}_X = \bigcap_{f \in I(X)} \mathscr{T}_f$ Variété tropicale

$f = \sum c_m T^m$ $\tau_f(x) = \sup(\log|c_m| + \langle m, x\rangle)$

$\mathscr{T}_f = \{x \in \mathbb{R}^n \mid \exists\, m \neq m'$, $\log|c_m| + \langle m, x\rangle = \log|c_{m'}| + \langle m', x\rangle = \tau_f(x)\}$

$\lambda : X^{an} \xrightarrow{\cancel{\text{propre}}} \mathbb{R}^n$

$p \mapsto (\log p(T_1), \dots, \log p(T_n))$ tropicalisation

$\mathscr{T}_f = \{x \mid \mathrm{in}_x(f)$ n'est pas un monôme$\}$

$\mathscr{T}_X = \{x \mid \mathrm{in}_x(I)$ ne contient pas de monôme$\}$

$S_2 = S_3$

$S_1 = S_4$

$S_5^L \subset \overline{S_2} = \overline{S_3} = S_3 \subset S_1 = S_4$

$f = \sum c_m T^m$ $f(z) = 0$ $p(f) = 0$

car la borne supérieure des $p(c_m)\, p(T_1)^{m_1} \cdots p(T_n)^{m_n}$ ne peut être atteinte en un seul m

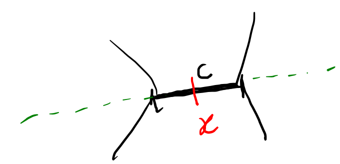
Se ramener au cas des hypersurfaces par des projections bien choisies.

(proposition 3.7.10). In this case, this implies the inclusion $\mathscr{T}_X \subset S_5^K$, hence the equality of all five sets.

In the general case, let us consider an algebraically closed valued extension K′ of K whose value group is nontrivial; in particular, the valuation admits a splitting. By the case already proved, the subsets $S'_1, \dots, S'_4, S'_5 = S_5^{K'}$ of $\mathbf{R}^n$ corresponding to the ideal $I_{K'}$ of $K'[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ deduced from I satisfy the equalities $S'_1 = S'_2 = S'_3 = S'_4 = S'_5$. The inclusions $S'_1 \subset S_1$ and $S'_5 \subset S'_2 \subset S_2 = S_3$ follow from the definitions, and the equality $S'_1 = S_1$ has been proved in proposition 3.7.6, *c*). One then obtains the missing inclusion $S_1 = S'_1 \subset S'_5 \subset S_3$, and that will conclude the proof of the theorem. □

$\mathscr{T}_X = \mathscr{T}_{X_L}$ $\left\}\begin{array}{l} L \text{ alg. clos, donc scindé} \\ \Gamma = \log|L^\times| \text{ dense.} \end{array}\right.$ $(\Leftrightarrow \Gamma \neq \{0\})$

$\mathscr{T}_X$ est polyédral car il existe des bases tropicales $(f_1, \dots, f_r)$ $\mathscr{T}_X = \mathscr{T}_{f_1} \cap \dots \cap \mathscr{T}_{f_r}$



C cellule maximale de la déc. de gröbner de $\mathscr{T}_X$

Lemma **(3.7.9).** — *The polyhedral set $\mathscr{T}_X$ has dimension at most* dim(X).

$\tilde{C} \subset \mathscr{T}_{X^h} \subset \mathbb{R}^{n+1}$ \ $\mathrm{in}_{\tilde{x}}(I^h)$ est constant sur relint $\tilde{C}$)

Using theorem 3.7.8, we shall prove later (theorem 3.8.4) that the dimension of $\mathscr{T}_X$ is equal to dim(X).

$C = \{x \mid (0,x) \in \tilde{C}\}$

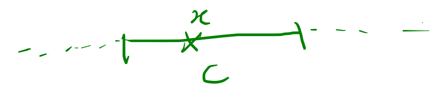
Proof. — Thanks to proposition 3.7.6, we may assume that the valuation of K has a splitting and its image Γ is dense in **R**. Then a point $x \in \mathbf{R}^n$ belongs to $\mathscr{T}_X$ if and only if $\mathrm{in}_x(I) = (1)$, if and only if there exists $f \in I$ such that $\mathrm{in}_x(f) = 1$.

Let then C be a maximal cell of the Gröbner polyhedral decomposition of $\mathscr{T}_X$ and let $m = \dim(C)$. Since C is a Γ-strict polyhedron, and since Γ is dense in **R**, there exists a point x in the relative interior of C whose coordinates belong to Γ. Up to a monomial

$\mathrm{relint}(C) \cap \Gamma^n \neq \emptyset$

aff span(C) est dirigé par un sous espace vectoriel Γ-rationnel de $\mathbb{R}^n$.

$\dim(C) = d$



$L = \mathbb{R}^m \times 0$.

change of variables, we may assume that the affine span of C is $x + (\mathbf{Z}^m \times \{(0, \dots, 0)\})$. Fix a finite generating family $(f_1, \dots, f_r)$ of $\mathrm{in}_x(\mathrm{I})$ such that no nontrivial subpolynomial of the f_j belongs to $\mathrm{in}_x(\mathrm{I})$.

Let $y \in \mathbf{R}^n$ such that $y_{m+1} = \dots = y_n = 0$. It follows from proposition 3.6.7 and a homogeneization-dehomogeneization argument that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_x(\mathrm{I})$. Since $\mathrm{in}_y(f_j)$ is a subpolynomial of f_j, nonzero if $f_j \neq 0$, this implies that $\mathrm{in}_y(f_j) = f_j$ for all j. Apply this remark when y is one of the first m vectors $e_1, \dots, e_m$ of the canonical basis of $\mathbf{R}^n$. Writing $f_j = \sum c_m \mathrm{T}^m$, one has $\tau_{f_j}(e_i) = \sup_{m \in \mathrm{S}(f)} m_j = \deg_{\mathrm{T}_1}(f_j)$ (recall that the residue field k is endowed with the trivial absolute value). The relation $\mathrm{in}_{e_i}(f_j) = f_j$ implies that f_j is a power of T_i multiplied by a polynomial in the other variables. In other words, there exists a Laurent polynomial $g_j \in \mathrm{K}[\mathrm{T}_{m+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ and $p \in \mathbf{Z}^m$ such that $f_j = \mathrm{T}_1^{p_1} \dots \mathrm{T}_m^{p_m} g_j$. Letting J be the ideal of $\mathrm{K}[\mathrm{T}_{m+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ generated by $g_1, \dots, g_r$, one has $\mathrm{V}(\mathrm{in}_x(\mathrm{I})) = \mathbf{G}_{\mathrm{m}_k}^m \times \mathrm{V}(\mathrm{J})$. Since $\mathrm{in}_x(\mathrm{I}) \neq (1)$, one has $\mathrm{J} \neq (1)$ and $\dim(\mathrm{V}(\mathrm{in}_x(\mathrm{I}))) = m + \dim(\mathrm{V}(\mathrm{J})) \geqslant m$. On the other hand, $\dim(\mathrm{V}(\mathrm{in}_x(\mathrm{I}))) = \dim(\mathrm{V}(\mathrm{I}))$. This concludes the proof. □

Proposition **(3.7.10)**. — *Assume that the field* K *is algebraically closed and that its value group* Γ *is nontrivial. Let* $x \in \mathscr{T}_{\mathrm{X}} \cap \Gamma^n$. *Then there exists* $z \in \mathrm{X}(\mathrm{K})$ *such that* $\lambda(z) = x$. *If, moreover,* X *is irreducible, then the set of such* z *is Zariski-dense in* X.

$L = \{ y \mid \mathrm{in}_y(\mathrm{in}_x(I)) = \mathrm{in}_x(I) \}$.

contient $e_1 = (1, 0, \dots, 0)$, $e_d = (0, \dots, 0, 1, 0, \dots 0)$ $\underbrace{}_{d}$

$\mathrm{in}_x(I) = \langle f_j \rangle$

$\mathrm{in}_{e_i}(f_j) = f_j$

$f_j = \sum c_m T^m$ $\qquad c_m \in k^\times$

$\tau_{e_i}(f_j) = \sup_m \langle e_i, m \rangle = m_i$

une seule valeur m_i

$f_j = T_i^{m_i} \, f_j'(T')$ autres variables

$f_j = \underbrace{T_1^{m_1} \dots T_d^{m_d}}\, g_j(T')$

$J = \langle g_j \rangle \subset k[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]$ $T_{d+1}, \dots, T_n$

$\mathrm{in}_x(I) = J \cdot k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$in_x(I) = J \cdot k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

$J \subset k[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]$

$V(in_x(I)) = \mathbb{G}_{m,k}^d \times_k V(J)$

est <u>horizontal</u> de dim $\geq d$.

($V(J) \neq \emptyset$, sinon $\cancel{\forall}$ $1 \in J$, donc $1 \in in_x(I)$, donc $x \notin \mathcal{C}_X$)

Or $\dim V(in_x(I)) \leq \dim V(I) = \dim(X)$

(même polynôme de Hilbert)

donc $\underbrace{d \leq \dim(X)}$

Proposition **(3.7.10)**. — *Assume that the field* $\bar{K}$ *is algebraically closed and that its value group* Γ *is nontrivial. Let* $x \in \mathscr{T}_X \cap \Gamma^n$. *Then there exists* $z \in X(K)$ *such that* $\lambda(z) = x$. *If, moreover,* X *is irreducible, then the set of such* z *is Zariski-dense in* X.

présisé $S_5^L \supset S_1$.

$x \in \left[S_1 \cap \Gamma^n = S_5^L \right.$ $\overset{?}{\Rightarrow} \exists z \in X(L)$
$\lambda(z) = x$.

dense dans S_1

• $X = \cup X_j \Rightarrow \mathcal{C}_X = \cup \mathcal{C}_{X_j}$
composants irréductibles
, irréductible.

• récurrence sur n. $I = (f)$ $I = (0)$
cas de base $n = \dim(X) + 1$, ou $\dim(X)$
hypersurfaces. trivial

Proof. — It suffices to treat the case where X is irreducible. Replacing the ideal I of X by its radical $\sqrt{I}$ does not change $\mathscr{T}_X$, nor the set X(K). We may thus assume that I is a prime ideal of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

The proof of this proposition is by induction on n; we will make use of the case of hypersurfaces, already proved in theorem 3.3.6. The proposition is obvious if $I = (0)$.

Assume that $\dim(X) = n - 1$. We first recall that there exists $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ such that $I = (f)$. Indeed, let f be a nonzero element of I; it is a product of irreducible elements, and one of them belongs to I, since I is prime. We can thus assume that f is irreducible; since $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ is a unique factorization domain, the ideal (f) is then prime. The inclusion $(f) \subset I$ implies an inclusion $X \subset V(f) \subsetneq \mathbf{G}_{\mathrm{m}K}^n$ of irreducible sets. Since $\dim(X) = n - 1$, this implies $X = V(f)$, hence $I = (f)$. Consequently, the proposition follows from corollary 3.3.9 in this case.

We now assume that $\dim(X) < n - 1$. Let $x \in \mathscr{T}_X \cap \Gamma^n$. To prove the existence of $z \in X(K)$ such that $\lambda(z) = x$, we shall project X to $\mathbf{G}_{\mathrm{m}K}^{n-1}$. Take a nonzero element $f \in I$. Up to a permutation of the variables, we may assume that f is not a monomial in T_n. We then make a monomial change of variables given by $T_1 \to T_1$, $T_2 \to T_2 T_1^q, \dots, T_n \to T_n T_1^{q^{n-1}}$, as in the proof of proposition 3.3.7, so as assuming that, when written as a polynomial in T_1, every coefficient of f is a monomial in the other variables. This implies

hypersurfaces irréductibles X de Spec(A) si A est un anneau factoriel

$X = V(I)$
I idéal premier de hauteur 1
$0 \subsetneq J \subset I$ premier
$\Rightarrow J = I$.

$\Rightarrow \exists f \in A$ irréductible
$I = (f)$.

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premier $I \ni f \neq 0$,
$f = \prod f_i$ $\quad f_i$ irréductibles
$\exists i \quad f_i \in I$ $\quad$ remplace f par f_i

$I \ni f$ irréductible
$0 \neq (f) \subset I$
A factoriel $\Rightarrow (f)$ est premier
$ht(I) = 1 \Rightarrow I = (f)$.

$\dim(X) = n - 1 \Rightarrow I = (f)$
$X = V(f)$
$\mathscr{T}_X = \mathscr{T}_f$
$\mathscr{T}_X \cap \Gamma^n = \{\lambda(z),\ z \in X(K)\}$
si K est alg clos
val. non triviale.

that the projection morphism p from $\mathbf{G}_{\mathrm{m}_\mathrm{K}}^n$ to $\mathbf{G}_{\mathrm{m}_\mathrm{K}}^{n-1}$ (forgetting the first coordinate) induces an integral, hence finite, morphism from X to its image. This image X′ is then a closed integral subscheme of $\mathbf{G}_{\mathrm{m}_\mathrm{K}}^{n-1}$. One has $p(x) \in \mathscr{T}_{\mathrm{X}'}$, so that there exists $z' \in \mathrm{X}'(\mathrm{K})$ such that $\lambda(z') = p(x)$. By finiteness, the point z' lifts to a point $z \in \mathrm{X}(\mathrm{K})$, but not all lifts will satisfy $\lambda(z) = x$. We force this property by making use of the diversity of possible projections, using multiple change of variables as above. Let $\pi \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ be a linear map of the form $(x_1, \dots, x_n) \mapsto (x_2 + qx_1, \dots, x_n + q^{n-1}x_1)$; one has $\mathrm{Ker}(\pi) = \mathbf{R}(-1, q, \dots, q^{n-1})$. We shall choose the integer q so that $\pi^{-1}(\pi(x)) \cap \mathscr{T}_\mathrm{X} = \{x\}$. Let $x' \in \pi^{-1}(\pi(x)) \cap \mathscr{T}_\mathrm{X}$ be such that $x' \neq x$; let C be polyhedron of a given polyhedral decomposition of $\mathscr{T}_\mathrm{X}$ such that $x' \in \mathscr{T}_\mathrm{X}$; then $x' - x \in \mathrm{Ker}(\pi)$, so that there exists $t \in \mathbf{R}$ such that $x' - x = t(-1, q, \dots, q^{n-1})$, and the line $\mathbf{R}(-1, q, q^2, \dots, q^{n-1}, 1)$ meets $\mathrm{C} - x$ in a nonzero point. On the other hand, since $\dim(\mathbf{R}\mathrm{C} + \mathbf{R}x) \leqslant n-1$, it is contained in a nontrivial affine hyperplane with equation, say $a_1x_1 + \dots + a_nx_n = b$, and for all but finitely many $q \in \mathbf{Z}$, one has $-a_1 + a_2q + \dots + a_nq^{n-1} \neq 0$. We may thus impose that $\pi^{-1}(\pi(x)) \cap \mathscr{T}_\mathrm{X} = \{x\}$.

By induction, the set V′ of elements $z' \in \mathrm{X}'(\mathrm{K})$ such that $\lambda(z') = \pi(x)$ is Zariski-dense in X′. Since $p \colon \mathrm{X} \to \mathrm{X}'$ is a surjective morphism of irreducible schemes and K is algebraically closed, the inverse image $\mathrm{V} = p^{-1}(\mathrm{V}')$ of V′ is Zariski-dense in X(K). For every $z \in \mathrm{V}$,

CHOIX D'UNE PROJECTION (= DE q) TELLE QUE

$\pi^{-1}(\pi(x)) = \{x\}$ $\forall x \in \mathscr{C}_X$

We now assume that $\dim(\mathrm{X}) < n-1$. Let $x \in \mathscr{T}_\mathrm{X} \cap \Gamma^n$. To prove the existence of $z \in \mathrm{X}(\mathrm{K})$ such that $\lambda(z) = x$, we shall project X to $\mathbf{G}_{\mathrm{m}_\mathrm{K}}^{n-1}$. Take a nonzero element $f \in \mathrm{I}$. Up to a permutation of the variables, we may assume that f is not a monomial in T_n. We then make a monomial change of variables given by $\mathrm{T}_1 \to \mathrm{T}_1$, $\mathrm{T}_2 \to \mathrm{T}_2\mathrm{T}_1^q, \dots, \mathrm{T}_n \to \mathrm{T}_n\mathrm{T}_1^{q^{n-1}}$, as in the proof of proposition 3.3.7, so as assuming that, when written as a polynomial in T_1, every coefficient of f is a monomial in the other variables. This implies

Choix d'une projection $\mathbb{G}_m^n \to \mathbb{G}_m^{n-1}$ un peu comme dans la preuve du théorème pour les hypersurfaces

$(z_1, \dots, z_n) \mapsto (z_2 z_1^q, z_3 z_1^{q^2}, \dots, z_n z_1^{q^{n-1}})$

$z \mapsto z'$

$x' = p(x)$

$\lambda \downarrow \qquad \downarrow \lambda$

$\mathbb{R}^n \xrightarrow{\pi_q} \mathbb{R}^{n-1}$

$\mathrm{Ker}(\pi) = (-1, +q, q^2, \dots, q^{n-1})$

$(x_1, \dots, x_n) \longmapsto (x_2 + qx_1, x_3 + q^2 x_1, \dots, x_n + q^{n-1}x_1)$

$x \in \mathscr{C}$ $\qquad$ $\mathscr{C}$ $\quad x'$

$\lambda(z) = x$??

$x = \tilde{x}$

$x' = \pi(x)$

par le dessus, il existe $z' \in X'(K)$

$\lambda(z') = x'$

$$\begin{array}{ccccc} V \quad z & X \subset \mathbb{G}_m^n & \xrightarrow{p_q} & \mathbb{G}_m^{n-1} & z' \quad V' \\ & \downarrow \quad \downarrow \lambda & & \downarrow \lambda & \\ & \mathscr{C}_X \subset \mathbb{R}^n & \xrightarrow{\pi_q} & \mathbb{R}^{n-1} & \\ \tilde{x} & x & \rightsquigarrow & x' & \end{array}$$

$\tilde{x} = \lambda(z)$

$\pi(\tilde{x}) = x'$

$\tilde{x} \in \pi^{-1}(\quad \pi(x)) = \{x\}$

$\tilde{x} = x.$

$X' = \overline{p(X)}$

$= p(X).$

$z' \in X'(K)$

$p : X(K) \twoheadrightarrow X'(K)$

$z \in X(K)$

Pour q bien choisi

$f = \sum c_m T^m$

$= \sum c_m T_1^{\varphi(m)} (T_2 T_1^{-q})^{m_2} \cdots (T_n T_1^{q^{n-1}})^{m_n}$

$S(f) \to \mathbb{Z}$

$m \mapsto \varphi(m)$ injective.

$d(T_1^{-1})$ dans $k[T^{\pm 1}]/(f)$

est entier sur $k[T_2^{\pm 1}, \dots, T_n^{\pm 1}]$

$\Rightarrow p|_X$ est entier donc fini

$\Rightarrow p|_X$ est fermé

et $p(X) = X'$

$V = p^{-1}(V')$ convient

si V' est dense pour la topologie de Zariski,

alors V aussi

one has $\lambda(z) \in \mathscr{T}_X$ and $\pi(\lambda(z)) = \lambda(p(z)) = \pi(x)$ since $p(z) \in V'$, so that $\lambda(z) = x$. This concludes the proof. □

3.8. Dimension of tropical varieties

Proposition **(3.8.1)**. — *Let* K *be a valued field and let* X *be a closed subscheme of* $\mathbf{G}_{\mathrm{m}K}^n$. *Let* $p\colon \mathbf{G}_{\mathrm{m}K}^n \to \mathbf{G}_{\mathrm{m}K}^m$ *be a monomial morphism of tori, let* $\pi\colon \mathbf{R}^n \to \mathbf{R}^m$ *be the corresponding linear map and let* $Y = \overline{p(X)}$ *be the schematic image of* X *under* p. *One has* $\mathscr{T}_Y = \pi(\mathscr{T}_X)$.

$(z_1,\dots,z_n) \mapsto (z^{a_1},\dots,z^{a_m})$, $a_i \in \mathbb{Z}^n$

Proof. — Write

$$\mathbf{G}_{\mathrm{m}K}^n = \operatorname{Spec}(K[T_1^{\pm1},\dots,T_n^{\pm1}]) \quad\text{and}\quad \mathbf{G}_{\mathrm{m}K}^m = \operatorname{Spec}(K[S_1^{\pm1},\dots,S_m^{\pm1}]);$$

the morphism p corresponds to a morphism of K-algebras

$p^*(S_i) = T^{a_i}$

$$p^*\colon K[S_1^{\pm1},\dots,S_m^{\pm1}] \to K[T_1^{\pm1},\dots,T_n^{\pm1}].$$

By assumption, $p^*(S_j)$ is a monomial, for every j. Let I be the ideal of X and let $J = (p^*)^{-1}(I)$, so that the morphism p^* induces an injective morphism of K-algebras, still denoted by p^*:

$$K[S_1^{\pm1},\dots,S_m^{\pm1}]/J \to K[T_1^{\pm1},\dots,T_n^{\pm1}]/I.$$

Then p maps $X = V(I)$ into a Zariski-dense subset of $Y = V(J)$; moreover, by Chevalley's theorem (Mumford (1994), corollary 2, p. 51)[3],

[3]Add a reference in the yet-to-be-written appendix

$$\begin{array}{ccccc} \mathbb{G}_m^n & & \xrightarrow{p} & & \mathbb{G}_m^m \\ \cup & & & & \cup \\ V(I) = X & & \xrightarrow{p|_X} & & Y = \overline{p(X)} = V(J) \\ \| & & & & \| \\ \operatorname{Spec}(K[T^{\pm1}]/I) & & & & \operatorname{Spec}(K[S^{\pm1}]/J) \end{array}$$

$$K[S^{\pm1}] \xrightarrow{p^*} K[T^{\pm1}] \to K[T^{\pm1}]/I$$

$$\boxed{J = (p^*)^{-1}(I)}$$

J est le plus grand idéal par lequel ce morphisme se factorise

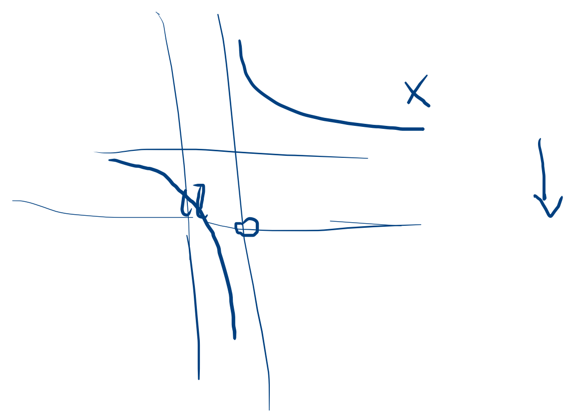
Ker(ce morphisme) !!

p(X) est une partie Zariski dense de Y. par construction de Y

Th: p(X) contient un ouvert dense de Y

Exemple typique

$$\mathbb{G}_m^2 \xrightarrow{p} \mathbb{G}_m \qquad (T_1, T_2) \longmapsto T_2$$



$$X = V\big((T_1-1)(T_2-1)-1\big) \qquad Y = \overline{p(X)}$$

$$p(X) = Y - \{1\}.$$

$$\mathbb{A}^2 \xrightarrow{p} \mathbb{A}^1$$

$$(x,y) \longmapsto x$$

$$(xy-1) \longrightarrow \text{image} = \mathbb{A}^1 - \{0\}.$$

$p^*(s_j) = T^{a_j}$

$\pi(x_1, \dots, x_n) = (\langle a_1, x\rangle, \dots, \langle a_m, x\rangle)$

$p(z_1, \dots, z_n) = (z^{a_1}, \dots, z^{a_m})$

the pointwise image $p(X)$ of X contains a dense open subscheme Y′ of Y.

For every valued extension L of K and every $z \in X(L)$, one has $\lambda(p(z)) = \pi(\lambda(z))$; this implies that $\pi(\mathscr{T}_X) \subset \mathscr{T}_Y$. Conversely, let $y \in \mathscr{T}_Y$. Fix an algebraically closed valued extension L of K which is non trivially valued. By proposition 3.7.10, the set of points $t \in Y(L)$ such that $\lambda(t) = y$ is Zariski-dense in Y. Consequently, it meets the dense open subscheme Y′ of Y; let thus choose $t \in Y'(L)$ such that $\lambda(t) = y$. Since L is algebraically closed, there exists $z \in X(L)$ such that $p(z) = t$. Then $\lambda(z) \in \mathscr{T}_X$ and $\pi(\lambda(z)) = \lambda(p(z)) = \lambda(t) = y$, which proves that $\mathscr{T}_Y \subset \pi(\mathscr{T}_X)$. ☐

(X, irréductible et Y)

Proposition **(3.8.2)**. — *Let* X *be a closed subscheme of* $\mathbf{G}_{m\,K}^n$ *such that* $\mathscr{T}_X$ *is finite. Then* X *is finite.*

Proof. — We argue by induction on n. The result is obvious if $n = 0$. One has $X \neq \mathbf{G}_{m\,K}^n$ for, otherwise, one would have $\mathscr{T}_X = \mathbf{R}^n$; consequently, $I(X) \neq 0$. Choosing a nonzero Laurent polynomial $f \in I(X)$, we may find an adequate monomial projection $p : \mathbf{G}_{m\,K}^n \to \mathbf{G}_{m\,K}^{n-1}$ that induces a finite morphism from X to $\mathbf{G}_{m\,K}^{n-1}$, and let Y be its image. By proposition 3.8.1, the tropical variety $\mathscr{T}_Y$ is finite. By induction this implies that Y is finite. Since $p : X \to Y$ is finite, this implies that X is finite as well. ☐

$$\begin{tikzcd}
X = V(I) \arrow[d] & \subset & \mathbb{G}_m^n \arrow[r, "\lambda"] \arrow[d, "p"] & \mathbb{R}^n \arrow[d, "\pi"] & \supset \mathscr{T}_X \\
Y = V(J) & \subset & \mathbb{G}_m^m \arrow[r, "\lambda"] & \mathbb{R}^m & \supset \mathscr{T}_Y
\end{tikzcd}$$

<u>Th.</u> $\left(\pi(\mathscr{T}_X) = \mathscr{T}_Y. \right.$

L alg clos $|\cdot|_L \neq 0$.

$y \in \mathscr{T}_Y$

$\{ w \in Y(L) \,,\, \lambda(w) = y \}$ est dense dans Y pour la top. de Zariski.

donc rencontre l'ensemble $p(X)$ (qui contient un ouvert dense)

$z \in X(L)$

tel que $p(z) = w$

$\lambda(w) = y$

alors $x = \lambda(z)$

$\pi(x) = y$.

Proposition **(3.8.2)**. — *Let* X *be a closed subscheme of* $\mathbf{G}_{\mathrm{m}_K}^n$ *such that* $\mathscr{T}_X$ *is finite. Then* X *is finite.*

Proof. — We argue by induction on n. The result is obvious if $n = 0$. One has $X \neq \mathbf{G}_{\mathrm{m}_K}^n$ for, otherwise, one would have $\mathscr{T}_X = \mathbf{R}^n$; consequently, $I(X) \neq 0$. Choosing a nonzero Laurent polynomial $f \in I(X)$, we may find an adequate monomial projection $p : \mathbf{G}_{\mathrm{m}_K}^n \to \mathbf{G}_{\mathrm{m}_K}^{n-1}$ that induces a finite morphism from X to $\mathbf{G}_{\mathrm{m}_K}^{n-1}$, and let Y be its image. By proposition 3.8.1, the tropical variety $\mathscr{T}_Y$ is finite. By induction this implies that Y is finite. Since $p : X \to Y$ is finite, this implies that X is finite as well. □

• $I(X) = 0 \Rightarrow X = \mathbb{G}_m^n$
$\Rightarrow \mathscr{T}_X = \mathbb{R}^n \Rightarrow n = 0.$
$\Rightarrow X$ est fini.

• $I(X) \neq (0)$ $(n \geq 1)$
où $f \in I(X)$
projection bien choisie

$$\begin{tikzcd} \mathbb{G}_m^n \arrow[r, "p"] \arrow[d] & \mathbb{G}_m^{n-1} \arrow[d] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^{n-1} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p_{|X}", "\text{fini}"'] \arrow[d] & X' \arrow[d] \\ \mathscr{T}_X \arrow[r] & \mathscr{T}_{X'} \end{tikzcd}$$

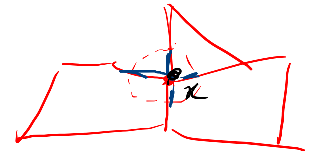
$\mathscr{T}_{X'} = \pi(\mathscr{T}_X) \overset{(\text{prop.})}{\Rightarrow} \mathscr{T}_{X'}$ est fini

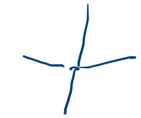
$\underset{(\text{récurrence})}{\Rightarrow}$ X' est fini

$\underset{(\text{composition de morphismes finis})}{\Rightarrow}$ X est fini

En particulier $\dim(X) = 0$ du th.

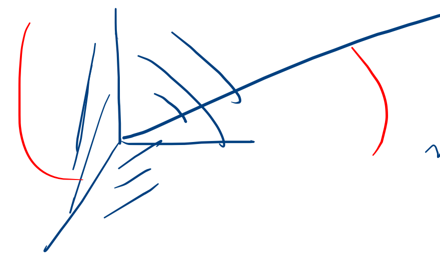
$\dim(\mathscr{T}_X) = \dim(X)$





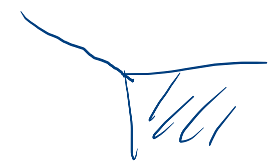
Lemma **(3.8.3)**. — *Let* K *be a split valued field. Let* I *be an ideal of* $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ *and let* $X = V(I)$. *Let* $x \in \mathscr{T}_X$. *Then* $\mathrm{Star}_x(\mathscr{T}_X) = \mathscr{T}_{V(\mathrm{in}_x(I))}$.

Proof. — Fix a tropical basis $(f_1, \dots, f_r)$ of I. Recall that the polyhedral set $\mathrm{Star}_x(\mathscr{T}_X)$ is the set of $y \in \mathbf{R}^n$ such that $x + \varepsilon y \in \mathscr{T}_X$ for $\varepsilon > 0$ small enough.



polyédral de dim 2
mais avec une cellule
maximale de dim 1

Let $y \in \mathbf{R}^n$ be such that $y \notin \mathrm{Star}_x(\mathscr{T}_X)$. Then, for every $\varepsilon > 0$ small enough, one has $x + \varepsilon y \notin \mathscr{T}_X$, hence there exists i such that $\mathrm{in}_{x+\varepsilon y}(f_i)$ is a monomial. On the other hand, for all $\varepsilon > 0$ small enough, one has $\mathrm{in}_{x+\varepsilon y}(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$. Consequently, $\mathrm{in}_y(\mathrm{in}_x(f_i))$ is a monomial and $y \notin \mathscr{T}_{V(\mathrm{in}_x(I))}$.



Conversely, let $y \in \mathbf{R}^n$ be such that $y \notin \mathscr{T}_{V(\mathrm{in}_x(I))}$. By definition, there exists $g \in \mathrm{in}_x(I)$ such that $\mathrm{in}_y(g)$ is a monomial. There is a finite family $(f_1, \dots, f_r)$ in I such that the initial forms $\mathrm{in}_x(f_1), \dots, \mathrm{in}_x(f_r)$ have disjoint supports and $g = \sum \mathrm{in}_x(f_j)$. Since $\mathrm{in}_y(g)$ is a monomial, there exists $j \in \{1, \dots, r\}$ such that $\mathrm{in}_y(\mathrm{in}_x(f_j))$ contains this monomial, and by the disjointness property of the supports, $\mathrm{in}_y(\mathrm{in}_x(f_j))$ is a monomial. For $\varepsilon > 0$ small enough, one has $\mathrm{in}_y(\mathrm{in}_x(f_j)) = \mathrm{in}_{x+\varepsilon y}(f_j)$, hence $x + \varepsilon y \notin \mathscr{T}_X$ for $\varepsilon > 0$ small enough and $y \notin \mathrm{Star}_x(\mathscr{T}_X)$. This proves the other inclusion $\mathrm{Star}_x(\mathscr{T}_X) \subset \mathscr{T}_{V(\mathrm{in}_x(I))}$. $\square$

si toutes les comp. irréductibles de
X ont dimension d,
alors, les cellules maximales
de $\mathscr{T}_X$ ont dim. d.

Theorem **(3.8.4)**. — *Let* K *be a valued field, let* I *be an ideal of* $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ *and let* $X = V(I)$. *One has* $\dim(\mathscr{T}_X) = \dim(X)$. *More precisely, if* X *is nonempty and every irreducible component of* X *has dimension p, then the tropical variety $\mathcal{T}_X$ is a purely p-dimensional polyhedral set.*

Proof. — We start by copying the proof of lemma 3.7.9. We may assume that the valuation of K has a splitting and that its image Γ is dense in $\mathbf{R}$. We consider a maximal cell C in the Gröbner polyhedral decomposition of $\mathcal{T}_X$ and a point x which belongs to the relative interior of C. By a monomial change of coordinates, we may assume that the affine span of C is $x + (\mathbf{R}^m \times \{(0,\dots,0)\})$. If I is the ideal of X, there exists an ideal J of $k[T_{m+1}^{\pm1},\dots,T_n^{\pm1}]$ such that $\mathrm{in}_x(I) = \langle J\rangle$ so that $V(\mathrm{in}_x(I)) = \mathbf{G}_{\mathrm{m}_k}^m \times V(J)$. Moreover, if $p\colon \mathbf{G}_{\mathrm{m}_K}^n \to \mathbf{G}_{\mathrm{m}_K}^{n-m}$ is the projection $(z_1,\dots,z_n) \mapsto (z_{m+1},\dots,z_n)$, and $\pi\colon \mathbf{R}^n \to \mathbf{R}^{n-m}$, $(x_1,\dots,x_n) \mapsto (x_{m+1},\dots,x_n)$ is the corresponding linear projection, one has $V(J) = p(V(\mathrm{in}_x(I)))$, hence $\mathcal{T}_{V(J)} = \pi(\mathcal{T}_{V(\mathrm{in}_x(I))})$. Since x belongs to the relative interior of C, one has

$$\mathcal{T}_{V(\mathrm{in}_x(I))} = \mathrm{Star}_x(\mathcal{T}_X) = \mathrm{affsp}(C) - x.$$

Its image under π is equal to 0, hence $\mathcal{T}_{V(J)} = \{0\}$. By proposition 3.8.2, this implies that V(J) is finite, so that $V(\mathrm{in}_x(I)) = \mathbf{G}_{\mathrm{m}_k}^m \times V(J)$ has dimension m. Since V(I) is irreducible, one then has $\dim(V(I)) = \dim(V(\mathrm{in}_x(I))) = m$. □

Remark **(3.8.5)**. — Let I be an ideal of $\mathbf{Q}[T_1^{\pm1},\dots,T_n^{\pm1}]$ and $X = V(I)$. Every absolute value v of $\mathbf{Q}$ gives rise to a corresponding tropical variety $\mathcal{T}_{X,v}$ in $\mathbf{R}^n$. Let us prove that for all but finitely many prime

$\dim(\mathcal{C}_X) \subset \dim(X)$.

On reprend la preuve de l'inégalité. (de dim d) X $\mathcal{C}_X$

C cellule maximale de projection convenable $\mathbb{G}_m^n \to \mathbb{G}_m^{n-d}$

~~I(x)~~ $\mathrm{in}_x(I) = \cancel{I} = J.\,k[T_1^{\pm1},\dots,T_n^{\pm1}]$

$J \subset k[T_{d+1}^{\pm1},\dots,T_n^{\pm1}]$

$J = \cancel{I}\ \mathrm{in}_x(I) \cap k[T_{d+1}^{\pm1},\dots,T_n^{\pm1}]$.

$$\begin{array}{ccccc} \mathbb{G}_m^n & & V(\mathrm{in}_x(I)) & & \mathcal{C}_{V(\mathrm{in}_x(I))} \\ p\downarrow & & p\downarrow & & \Downarrow \\ \mathbb{G}_m^{n-d} & & Y = V(J) & & \mathcal{C}_Y \end{array}$$

$V(\mathrm{in}_x(I)) = \mathbb{G}_m^d \times V(J)$.

$\dim(\mathcal{C}_Y) = 0$ sinon $\boxed{\dim \mathcal{C}_{V(\mathrm{in}_x(I))} > d}$

donc $\dim(Y) = 0$

donc $\dim V(\mathrm{in}_x(I)) = d$.

Remark **(3.8.5)**. — Let I be an ideal of $\mathbf{Q}[T_1^{\pm1},\dots,T_n^{\pm1}]$ and $X = V(I)$. Every absolute value v of $\mathbf{Q}$ gives rise to a corresponding tropical variety $\mathscr{T}_{X,v}$ in $\mathbf{R}^n$. Let us prove that for all but finitely many prime numbers p, the tropical variety $\mathscr{T}_{X,p}$ associated with the p-adic absolute value coincides with the tropical variety $\mathscr{T}_{X,0}$ associated with the trivial absolute value. Also recall from example 3.1.7 that $\mathscr{T}_{X,0}$, the non-archimedean amoeba of X associated with the trivial valuation on X, is the logarithmic limit set of the complex (archimedean) amoeaba of X.

The case where $X = V(f)$ is a hypersurface, where $f \in \mathbf{Q}[T_1^{\pm1},\dots,T_n^{\pm1}]$ is a nonzero Laurent polynomial, follows from the description of $\mathscr{T}_X$ as the non-smooth locus of the tropical polynomial τ_f. Indeed, if $f = \sum_{m\in S(f)} c_m T^m$, one has

$$\tau_{f,p}(x) = \sup_{m\in S(f)} \log(|c_m|_p) + \langle m, x\rangle,$$

for every prime number p. For al but finitely many primes p, one has $|c_m|_p = 1 = |c_m|_0$ for every $m \in S(f)$. Consequently, $\tau_{f,p} = \tau_{f,0}$ for all but finitely many prime numbers f, whence the equality $\mathscr{T}_{X,p} = \mathscr{T}_{X,0}$.

Let us now prove the general case.

Let $x \in \mathbf{R}^n$ such that $x \notin \mathscr{T}_{X,0}$. By definition, there exists $f \in I$ such that $\mathrm{in}_{x,|\cdot|_0}(f) = 1$. Write $f = \sum c_m T^m$. For $m \in S(f)$, the set of prime numbers p such that $|c_m|_p \neq 1$ is finite. For any prime number p outside of the union of these finite sets, one has $\mathrm{in}_{x,|\cdot|_p}(f) = \mathrm{in}_{x,|\cdot|_0}(f) = 1$, hence $x \notin \mathscr{T}_{X,p}$. This proves the existence of a finite set S of prime numbers such that for every prime number p such that $p \notin S$, one has $\mathscr{T}_{X,p} \subset \mathscr{T}_{X,0}$.

$\mathbb{Q}$. valeurs absolues p-adiques

archimédienne

triviale.

amibe non archimédienne $\mathcal{A}_p$

amibe "complexe" $\mathcal{A}_\infty$

amibe $\mathcal{A}_0$.

- pour $p \gg 0$, $\mathcal{A}_p = \mathcal{A}_0$
- $\lambda(\mathcal{A}_\infty) = \mathcal{A}_0$.

$X = V(f)$ $\quad f = \sum c_m T^m \in \mathbb{Q}[T^{\pm1}]$

$\mathcal{A}_p \leftrightarrow \tau_p(x) = \sup_{m\in S(f)} (\log|c_m|_p + \langle m,x\rangle)$

$p \gg 0$ $\quad \tau_p(x) = \sup_{m\in S(f)} \langle m,x\rangle = \tau_0(x)$

$x \notin \mathcal{A}_0 \Rightarrow x \notin \mathcal{A}_p$, $p \gg 0$.

$\Updownarrow$

$\exists f \in I$, $x \notin \mathcal{C}_{f,0} \Rightarrow x \notin \mathcal{C}_{f,p}$ ($p \gg 0$) $\Rightarrow x \notin \mathcal{C}_{X,p}$ ($p \gg 0$).

To prove the other inclusion, we argue by induction on n. The result is obvious if $\dim(X) = n$, and it corresponds to the case of hypersurfaces if $\dim(X) = n-1$; let us now assume that $\dim(X) < n-1$. Since the polyhedral sets $\mathscr{T}_{X,p}$ and $\mathscr{T}_{X,0}$ have the same dimension, namely $\dim(X)$, As in the proof of theorem 3.7.8, there exists a monomial morphism $q\colon {\mathbf{G}_{\mathrm{m}}}^n \to {\mathbf{G}_{\mathrm{m}}}^{n-1}$, whose associated linear map $\chi\colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ is surjective and is such that $\chi^{-1}(y) \cap \mathscr{T}_{X,0}$ has at most one point, for every $y \in \mathbf{R}^{n-1}$. Let $Y = \overline{q(X)}$. One has $\chi(\mathscr{T}_{X,p}) = \mathscr{T}_{Y,p}$ for every prime number p, and $\chi(\mathscr{T}_{X,0}) = \mathscr{T}_{Y,0}$. By induction, up to enlarging the finite set S, we may assume that $\mathscr{T}_{Y,p} = \mathscr{T}_{Y,0}$ for all prime numbers p such that $p \notin S$. This implies that $\mathscr{T}_{X,p} = \mathscr{T}_{X,0}$ for all such prime numbers p. Let indeed $x \in \mathscr{T}_{X,0}$ and let $y = q(x) \in \mathscr{T}_{Y,0}$. By what precedes, one has $y \in \mathscr{T}_{Y,p} = \chi(\mathscr{T}_{X,p})$, so that there exists $x' \in \mathscr{T}_{X,p}$ such that $y = q(x')$. Since $\mathscr{T}_{X,p} \subset \mathscr{T}_{X,0}$, one has $x' \in \mathscr{T}_{X,0}$. By the choice of the linear map q, this implies that $x' = x$, hence $x \in \mathscr{T}_{X,p}$, as was to be shown.

Beware:

The definition of the nonarchimedean amoebas has been modified so as to be more consistent with the definition in the archimedean case. I made the necessary corrections up to here, but there are certainly inconsistencies below.
