TOPICS
IN TROPICAL GEOMETRY

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INTRODUCTION

Tropical geometry has been invented at the end of the xx$^{\text{th}}$ century by different group of scientists, with totally different motivations. It first appeared in computer science (Gaubert, Quadrat,. . .) for questions of network optimization, under the names of $(\max, +)$-algebra, exotic algebra, and, finally, tropical algebra, in reference to their Brazilian colleague Imre Simon. In functional analysis, it is also known as Maslov dequantization, the phenomenon that happens to the ring laws of **R**, written on log-log paper, are renormalized by letting the basis of logarithm ("Plancks's constant") go to 0:

$$a +_h b = h \log(e^{a/h} + e^{b/h}) \to \sup(a, b)$$

and

$$a \cdot_h b = h \log(e^{a/h} \cdot e^{b/h}) \to a + b.$$

Slightly later, it also appeared in algebraic geometry, first for questions of real geometry (Viro) or enumerative geometry (counting curves satisfying some incidence conditions), for example in Mikhalkin's correspondence theorem, and then in other contexts as well, such as the fine study of linear systems on algebraic curves.

To be true, tropical geometry has its origin in much older works: in some sense, it was founded by Newton in his analysis of Puiseux series expansion of singularities of curves, a theory that, suitably generalized, is at the basis of the study of valued fields.

In all of these examples, complicated phenomena in analysis or geometry are studied by reducing them to piecewise linear phenomena. For example, the Bergman and Bieri-Groves theory of amoebas of complex algebraic varieties shows that, drawn on log-log paper and at a large scale, complex algebraic varieties look like polyhedra.

This course is grounded in algebraic geometry and will try to expose various examples where this piecewise linear point of view illuminates algebraic geometry.

A first part of the course will be focused on these amoebas; we will also need to introduce polyhedral geometry and other themes of algebraic geometry, such as Gröbner bases, toric varieties, valued fields, Berkovich spaces… I expect that this part of the course will be of interest for students of various origins, hence will try to minimize the requested background.

In a second part, we will prove more specialized theorem in algebraic geometry, which precisely is too early to say.

These notes are in a moving state. Some parts may be incomplete, some parts may be false; some may even may be both. Do not hesitate to complain about any inaccuracy, imprecision, mistake, or misunderstanding you might be aware of.

CHAPTER 1

POLYHEDRAL GEOMETRY

The fundamental idea underlying tropical geometry is to understand phenomena in algebraic geometry that are governed by piecewise linear structures. This first chapter describes the basic notions in polyhedral geometry.

My basic reference was the book of Schrijver (1998).

1.1. Algebraic setting

Classical polyhedral geometry is the study of subspaces of $\mathbf{R}^n$ defined by affine inequalities, in the same way that affine algebra is the study of subspaces of $\mathbf{R}^n$ defined by affine equalities. However, it will be important later to restrict the subspaces we consider by assuming that they are defined by affine inequalities whose multiplicative coefficients are rational numbers.

Therefore, we consider the following general setting.

1.1.1. — We fix an *ordered field* R. In other words, R is a field, endowed with total order relation $<$ such that the following properties hold:

a) For all $a, b, c \in \mathrm{R}$ such that $a \leqslant b$, one has $a + c \leqslant b + c$;
b) For all $a, b \in \mathrm{R}$ such that $0 \leqslant a, b$, one has $0 \leqslant ab$.

If $a \in \mathrm{R}$ satisfies $a > 0$, then $2a = a + a > a + 0 = a$ and, by induction, $na > 0$ for all integers n such that $n \geqslant 1$. Similarly, if $a < 0$, then $na < 0$ for all $n \geqslant 1$. This proves that R has characteristic zero, hence its prime subfield is $\mathbf{Q}$.

We also fix a subfield Q of R.

1.1.2. Examples. — For tropical geometry, the main example will be given by R = $\mathbf{R}$ and Q = $\mathbf{Q}$.

For classical polyhedral geometry, one simply takes R = Q = $\mathbf{R}$.

In the theory of valued fields, Q = $\mathbf{Q}$ while the group R may be arbitrary.

For the relation with analytic geometry, it may be more natural to take R = $\mathbf{R}_+^*$ (as a multiplicative group) and Q = $\mathbf{Q}$, the structure of R as a Q-module being given by $a \cdot x = x^a$. In this setting, it is even useful to consider the additional datum of a Q-submodule Γ of R.

1.1.3. — We will sometimes assume that R is archimedean, that is, for every $a \in \mathrm{R}$ such that $a > 0$, there exists an integer n such that $na > 1$. In fact, according to the classification of complete archimedean ordered fields, this implies that R is a subfield of $\mathbf{R}$.

A variant of this assumption would be that Q is unbounded in R.

A stronger property is that Q is dense in R, but it is not equivalent in general.

Exercise **(1.1.4)**. — Let E be an ordered field and let F = E(t) be the field of rational functions in one indeterminate t.

a) Show that there exists an ordering on F for which a rational function $f \in \mathrm{E}(t)$ is strictly positive if $f(n)$ is strictly positive for all large enough integers n.

b) With respect to this ordering, the field F is nonarchimedean, and $t > n$ for every integer n. (One says that t is *infinite*.)

c) Assume that E is nonarchimedean. Show that for every infinite element a of E, one has $t < a$. Prove that E is unbounded in F but not dense.

1.1.5. — Let $n \in \mathbf{N}$ and let us consider the vector space R^n.

We also define a (partial) ordering relation $\leqslant$ on R^n as follows: for $x = (x_1, \dots, x_n) \in \mathrm{R}^n$, we write $0 \leqslant x$ if $0 \leqslant x_k$ for all $k \in \{1, \dots, n\}$.

Let us observe that if $x, y \in \mathrm{R}^n$ satisfy $0 \leqslant x$ and $0 \leqslant y$, then $0 \leqslant x + y$. Moreover, if $x \in \mathrm{R}^n$ and $a \in \mathrm{Q}$ satisfy $0 \leqslant x$ and $0 \leqslant a$, then $0 \leqslant ax$.

We say that a linear form f on R^n is positive, and write $f \geqslant 0$, if its coefficients are positive, in other words, if it is positive on the vectors of the canonical basis.

An R-linear form f on R^n is Q-*rational* if it is of the form $(x_1, \dots, x_n) \mapsto a_1x_1 + \dots + a_nx_n$, with $a_1, \dots, a_n \in Q$.

An affine form f on R^n is Q-*rational* if it is of the form $(x_1, \dots, x_n) \mapsto a_1x_1 + \dots + a_nx_n + b$, with $a_1, \dots, a_n \in Q$ and $b \in R$. If, moreover, $b \in Q$, then we say that it is *strictly* Q-*rational*.

Its ordering induces on the field R a natural topology, for which a basis of open subsets is given by the open intervals $]a; b[$, for $a, b \in R$ such that $a < b$. We then endow R^n with the product topology. Since affine maps are continuous, this allows to endow every finite dimensional affine space over R with a canonical topology.

Definition **(1.1.6)**. — *Let* V *be an* R*-vector space and let* C *be a subset of* V.

a) *One says that* C *is* convex *if for all* $x, y \in C$ *and all* $a \in R$ *such that* $0 \leqslant a \leqslant 1$, *one has* $(1-a)x + ay \in C$;

b) *One says that* C *is a* cone *if* $0 \in C$ *and if for all* $x \in C$ *and all* $a \in R$ *such that* $0 \leqslant a$, *one has* $ax \in C$.

The intersection of a family (C_i) of convex subsets of R^n (resp. of cones in R^n) is itself convex (resp. a cone).

Consequently, for every subset A of V, there exists a smallest convex subset (resp. a smallest convex cone) in R^n that contains A; it is called the *convex hull* of A (resp. the convex cone generated by A) and is denoted by conv(A) (resp. cone(A)).

Alternatively, conv(A) is the set of all points in V of the form $a_1x_1 + \dots + a_mx_m$, for $x_1, \dots, x_m \in A$ and $a_1, \dots, a_m \in R$ such that $0 \leqslant a_i$ for all i and $a_1 + \dots + a_m = 1$. Similarly, cone(A) is the set of all points in V of the form $a_1x_1 + \dots + a_mx_m$, for $x_1, \dots, x_m \in A$ and $a_1, \dots, a_m \in R$ such that $0 \leqslant a_i$ for all i.

One has $\operatorname{conv}(\varnothing) = \varnothing$ and $\operatorname{cone}(\varnothing) = \{0\}$.

The convex hull of a finite subset is called a *polytope*. The convex cone generated by a finite subset is called a *polyhedral cone*.

Example **(1.1.7)**. — The set of all positive elements in R^n is a polyhedral cone, it is generated by the vectors of the canonical basis of R^n. Similarly the set of all positive linear forms on R^n is a polyhedral cone of the dual space; the coordinate forms identify this dual space with R^n, and this identifies the positive cone of $(\mathrm{R}^n)^*$ with the positive cone of R^n.

Definition **(1.1.8)**. — *Let* V *be a finite dimensional* R*-vector space and let* P *be a subset of* V.

One says that P *is a* polyhedron *if there exists a finite family* $(f_1, \dots, f_m)$ *of affine forms on* V *such that* P *is the set of all* $x \in \mathrm{V}$ *such that* $f_k(x) \leqslant 0$ *for* $k \in \{1, \dots, m\}$.

In this setting, we say that P is *defined* by the affine forms f_k, or, to avoid ambiguities, by the affine inequalities $f_k \leqslant 0$. Note that a polyhedron is convex and closed.

Remark **(1.1.9)**. — A polyhedron which can be defined by linear forms is a convex cone.

Conversely, let C be a cone and let f be a linear form on V which is bounded from above on C, and let us prove that $f(x) \leqslant 0$ for every $x \in \mathrm{C}$.

By assumption, there exists $a \in \mathrm{R}$ such that $f(x) \leqslant a$ for every $x \in \mathrm{C}$. Since $0 \in \mathrm{C}$, one has $0 = f(0) \leqslant a$. Let us argue by contradiction and let $x \in \mathrm{C}$ be such that $f(x) > 0$; taking any element $t \in \mathrm{R}$ such that $tf(x) > a$, we obtain $f(tx) > a$; since $tx \in \mathrm{C}$, this contradicts the definition of a.

In particular, if a polyhedron is a cone, then it can be defined by linear forms.

1.2. The Farkas lemmas

In linear algebra, a vector v belongs to the subspace generated by some set A if and only if every linear form f that vanishes on A vanishes on v as well. The Farkas lemma is the counterpart of this result in polyhedral geometry. Actually, this lemma is rather a constellation of similarly looking results. We will derive them from the following general theorem, borrowed from Schrijver (1998), whose proof is inspired by the *simplex method* in linear programming.

Theorem **(1.2.1)**. — *Let* V *be an* R*-vector space, let* A *be a finite subset of* V *and let* $v \in \mathrm{V}$. *The following assertions are equivalent:*

(i) *There exist an integer* m*, an independent family* $(u_1, \dots, u_m)$ *in* A *and positive elements* $a_1, \dots, a_m \in \mathrm{R}$ *such that* $v = \sum_{i=1}^{m} a_i u_i$*;*

(ii) *One has* $v \in \mathrm{cone}(\mathrm{A})$*;*

(iii) *For all linear forms* f *on* V *such that* $f(u) \geqslant 0$ *for all* $u \in \mathrm{A}$*, one has* $f(v) \geqslant 0$*;*

(iv) *Let* t *be the dimension of the vector subspace generated by* $\mathrm{A} \cup \{v\}$*; for all linear forms* f *on* V *such that* $f(u) \geqslant 0$ *for all* $u \in \mathrm{A}$*, either* $f(v) \geqslant 0$*, or the vector subspace generated by* $\mathrm{A} \cap \mathrm{Ker}(f)$ *has dimension* $< t - 1$.

Proof. — The implications (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(iv) are obvious. Let us prove the remaining implication (iv)⇒(i).

First of all, one has $v \in \mathrm{vect}(\mathrm{A})$. Otherwise, there would exist a linear form f on V such that $\mathrm{A} \subset \mathrm{Ker}(f)$ and $f(v) = 1$; then the linear form $-f$ would contradict (iv), since $\dim(\mathrm{vect}(\mathrm{A} \cap \mathrm{Ker}(f))) = \dim(\mathrm{vect}(\mathrm{A})) = t - 1$. In particular, one has $t = \dim(\mathrm{vect}(\mathrm{A}))$.

Let $\mathrm{B}_0 \subset \mathrm{A}$ be a basis of vect(A). We will construct by induction a (possibly finite) sequence of bases (B_k) of vect(A) consisting of elements of A. We endow the set A with a total ordering $\preceq$. Assume that B_k is given and write $v \in \mathrm{vect}(\mathrm{A})$ as an R-linear combination, $v = \sum_{u \in \mathrm{B}_k} a_u u$, of the elements of B_k. If $a_u \geqslant 0$ for all $u \in \mathrm{B}_k$, the assertion (i) is proven. Otherwise, let $u \in \mathrm{B}_k$ be an element such that $a_u < 0$, chosen to be minimal for the given ordering of A. Let f be a linear form on V such that $f(u) = 1$ and $\mathrm{B}_k - \{u\} \subset \mathrm{Ker}(f)$; in particular, $f(v) = a_u < 0$. The subspace generated by $\mathrm{A} \cap \mathrm{Ker}(f)$ contains the $(t-1)$-dimensional subspace generated by $\mathrm{B}_k - \{u\}$. By the contrapositive of assumption (iv), there exists $w \in \mathrm{A}$ such that $f(w) < 0$; let w be the smallest such element for the given ordering of A. One has $w \notin \mathrm{vect}(\mathrm{B}_k - \{u\})$, so that $\mathrm{B}_{k+1} = \mathrm{B}_k \cup \{w\} - \{u\}$ is a basis of vect(A).

If the sequence (B_k) is finite, then assertion (i) holds. Otherwise, since the set of subsets of A is finite, the same basis appears twice; without loss of generality, we assume that $\mathrm{B}_0 = \mathrm{B}_s$, for some integer $s > 0$, and then the sequence is periodic : $\mathrm{B}_{k+s} = \mathrm{B}_k$ for all $k \in \mathbf{N}$. Let w be the largest element of A which is removed from one of the bases $\mathrm{B}_0, \dots, \mathrm{B}_{s-1}$ to construct the next one, and assume that it is removed at step p: $w \in B_p$ but $w \notin B_{p+1}$. Since $B_s = B_0$, this element is restored at some later step, say q, such that $p < q < p + s$, that is, $w \notin B_q$ but $w \in B_{q+1}$.

Let f be the linear form considered at step q; we have shown in the construction that $f(v) < 0$. Now write $v = \sum_{u \in B_p} a_u u$ (for some $a_u \in R$) as a linear combination of elements of B_p; then $f(v) = \sum_{u \in B_p} a_u f(u)$, and we will derive a contradiction by showing that all terms are positive.

Let $u \in B_p$. If $u \succ w$, then u is untouched by the construction process, hence $u \in B_q$ and $f(u) = 0$; then $a_u f(u) = 0$.

Assume that $u = w$. The addition of w at step q asserts that $f(w) < 0$; moreover, the removal of w at step p asserts that $a_w < 0$; in particular, $a_w f(w) > 0$.

Assume finally that $u \prec w$. Since w is the minimal element of B_p such that $a_w < 0$, we have $a_u \geqslant 0$; similarly, w is the minimal element of A such that $f(w) < 0$, so that $f(u) \geqslant 0$, hence $a_u f(u) \geqslant 0$.

Then $f(v) = \sum_{u \in B_p} a_u f(u) \geqslant a_w f(w) > 0$, while we have seen that $f(v) < 0$. This contradiction shows that the sequence (B_k) is finite, and this concludes the proof. □

A first corollary of the theorem is the Carathéodory theorem:

Corollary **(1.2.2)** (Carathéodory theorem). — *Let* V *be an* R*-vector space and let* A *be a subset of* V*. For every vector* $v \in \operatorname{cone}(A)$*, there exists a linearly independent subset* A′ *of* A *such that* $v \in \operatorname{cone}(A')$.

Proof. — Since $v \in \operatorname{cone}(A)$, there exists a finite subset $\{v_1, \dots, v_m\}$ of A such that $v \in \operatorname{cone}(v_1, \dots, v_m)$. This allows to assume that A is finite. By assumption, there exist positive elements of R, $a_1, \dots, a_m$, such that $v = a_1 v_1 + \dots + a_m v_m$. The corollary thus follows from implication (ii)⇒(i) in theorem 1.2.1. □

Corollary **(1.2.3)** (Farkas lemma, version 1). — *Let* $\varphi \colon R^m \to R^n$ *be a linear map and let* $v \in R^n$*. There exists* $x \in R^m$ *such that* $\varphi(x) = v$ *and* $x \geqslant 0$ *if and only if* $f(v) \geqslant 0$ *for all linear forms* f *on* R^n *such that* $f \circ \varphi \geqslant 0$.

Proof. — Let $x \in R^m$ be such that $x \geqslant 0$ and $\varphi(x) = v$; let then f be a linear form on R^n such that $f \circ \varphi \geqslant 0$. Then $f(v) = f(\varphi(x)) = f \circ \varphi(x) \geqslant 0$ since $x \geqslant 0$.

Conversely, assume that $f(v) \geqslant 0$ for all linear forms f on R^n such that $f \circ \varphi \geqslant 0$. Let $(e_1, \dots, e_m)$ be the canonical basis of R^m; for every $k \in \{1, \dots, m\}$, let $v_k = \varphi(e_k)$, so that $\varphi(x_1, \dots, x_m) = x_1 v_1 + \cdots + x_m v_m$ for all $x \in \mathrm{R}^m$. The assumption says that every linear form f on R^n which is positive on $v_1, \dots, v_m$ is positive on v. By the implication (iii)⇒(ii) of theorem 1.2.1, there exist positive elements $x_1, \dots, x_m \in \mathrm{R}$ such that $v = x_1 v_1 + \cdots + x_m v_m$. Let $x = (x_1, \dots, x_m)$; one has $x \geqslant 0$ and $\varphi(x) = v$. □

Corollary **(1.2.4)** (Farkas lemma, version 2). — *Let $\varphi \colon \mathrm{R}^m \to \mathrm{R}^n$ be a linear map and let $v \in \mathrm{R}^n$. There exists $x \in \mathrm{R}^m$ such that $\varphi(x) \leqslant v$ if and only if $f(v) \geqslant 0$ for all linear forms f on R^n such that $f \geqslant 0$ and $f \circ \varphi = 0$.*

Proof. — This corollary is deduced from the previous one by rewriting the given problem.

Let $\psi \colon \mathrm{R}^{2m+n} \to \mathrm{R}^n$ be the linear map defined by $\psi(x, x', y) = \varphi(x) - \varphi(x') + y$. For $x \in \mathrm{R}^m$, we can write $x = x_+ - x_-$, with $x_+, x_- \geqslant 0$, so that there exists $x \in \mathrm{R}^m$ such that $\varphi(x) \leqslant v$ if and only if there exists $z \in \mathrm{R}^{2m+n}$ such that $z \geqslant 0$ and $\psi(z) = v$. By the previous corollary, this is equivalent to the inequality $f(v) \geqslant 0$ for every linear form f on R^n such that $f \circ \psi \geqslant 0$. But $f \circ \psi \geqslant 0$ means that f, $f \circ \varphi$ and $f \circ (-\varphi)$ are positive, that is, $f \geqslant 0$ and $f \circ \varphi = 0$. This concludes the proof. □

Corollary **(1.2.5)** (Farkas lemma, version 3). — *Let $\varphi \colon \mathrm{R}^m \to \mathrm{R}^n$ be a linear map and let $v \in \mathrm{R}^n$. There exists $x \in \mathrm{R}^m$ such that $x \geqslant 0$ and $\varphi(x) \leqslant v$ if and only if $f(v) \geqslant 0$ for all linear forms f on R^n such that $f \geqslant 0$ and $f \circ \varphi \geqslant 0$.*

Proof. — Let $\psi \colon \mathrm{R}^{m+n} \to \mathrm{R}^n$ be the linear map given by $\psi(x, y) = \varphi(x) + y$. Since $\psi(x, v - \varphi(x)) = v$, there exists $x \in \mathrm{R}^m$ such that $x \geqslant 0$ and $\varphi(x) \leqslant v$ if and only if there exists $z \in \mathrm{R}^{m+n}$ such that $z \geqslant 0$ and $\psi(z) = v$. By the first corollary, this is equivalent to the condition that $f(v) \geqslant 0$ for every linear form on R^n such that $f \circ \psi \geqslant 0$, which means precisely that $f \circ \varphi$ and f are positive. □

Corollary **(1.2.6)**. — *Let n be an integer and let C be a convex cone in R^n. The following propositions are equivalent:*

(i) *The cone* C *is polyhedral (that is, there exist a finite family of vectors* $(v_1, \dots, v_m)$ *in* R^n *such that* $\mathrm{C} = \mathrm{cone}(v_1, \dots, v_m)$*);*

(ii) *There exists a finite family of linear forms* $(f_1, \dots, f_m)$ *on* R^n *such that* C *is defined by the inequalities* $f_j(x) \leqslant 0$*;*

(iii) *The cone* C *is a polyhedron.*

Proof. — (i)⇒(ii). Assume that C is polyhedral, that is, there are vectors $v_1, \dots, v_m \in \mathrm{C}$ such that $\mathrm{C} = \mathrm{cone}(v_1, \dots, v_m)$ and let us prove that it is a polyhedron.

By the equivalence (ii)⇔(iii) in theorem 1.2.1, a vector v belongs to C if and only if $f(v_i) \geqslant 0$ for all i, so that C is defined by the (possibly infinite) family of all linear forms f on R^n such that $f(v_i) \geqslant 0$ for all i. We need to show that a finite subfamily still defines C. To that aim, we will rather make use of the equivalence with assertion (iv) of theorem 1.2.1.

We first assume that the v_i generate R^n as a vector subspace. Let Φ be the set of all nonzero linear forms f on R^n such that $f(v_i) \geqslant 0$ for all i, and such that $\mathrm{Ker}(f)$ has a basis among the v_i. Up to a normalization factor (which can be set by imposing that f takes the value 1 on some of the v_i), this set Φ is finite. Equivalence (iv)⇔(ii) in theorem 1.2.1 asserts that a vector $v \in \mathrm{R}^n$ belongs to C if and only if $f(v) \geqslant 0$ for every $f \in \Phi$. Consequently, Φ is a finite set of linear forms that defines the convex cone C, which is therefore polyhedral.

The general case is similar. Let $\mathrm{V} = \mathrm{vect}(v_1, \dots, v_m)$ and let Φ_1 be the set of all nonzero linear forms f on R^n such that $f(v_i) \geqslant 0$ for all i, and such that $\mathrm{V} \cap \mathrm{Ker}(f)$ has a basis among the v_i. Modulo addition of a linear form on R^n which vanishes on V and multiplication by a strictly positive element of R, this set is finite Let Φ be the set obtained by adjoining to Φ_1 a basis of the space of linear forms on R^n which vanish on V as well as their additive inverses. Let us show that $\mathrm{cone}(v_1, \dots, v_m)$ is the set of all vectors $v \in \mathrm{R}^n$ such that $f(v) \geqslant 0$ for every $f \in \Phi$.

If $v \in \mathrm{cone}(v_1, \dots, v_m)$, then $f(v) \geqslant 0$ for every $f \in \Phi$, obviously. Conversely, let $v \in \mathrm{R}^n$ be such that $f(v) \geqslant 0$ for every $f \in \Phi$. To prove that $v \in \mathrm{cone}(v_1, \dots, v_m)$, we use the implication (iv)⇒(i) in theorem 1.2.1. Using the forms of $\Phi - \Phi_1$, we already see that $v \in \mathrm{vect}(v_1, \dots, v_m)$. In particular, the integer t of assertion (iv) is given by $t = \dim(\mathrm{vect}(\mathrm{V} + \mathrm{R}v)) = \dim(\mathrm{V})$. Let then f be a linear form on R^n such that $f(v_i) \geqslant 0$ for all i and $\dim(\mathrm{vect}(\{v_1, \dots, v_m\} \cap \mathrm{Ker}(f))) \geqslant t-1$. If $\dim(\mathrm{vect}(\{v_1, \dots, v_m\} \cap \mathrm{Ker}(f))) = t$, then f vanishes on V and $f(v) = 0$. Otherwise, $\mathrm{Ker}(f)$ draws a hyperplane on V which, by assumption, has a basis consisting of vectors of $\{v_1, \dots, v_m\}$. Then, modulo addition of a linear form vanishing on V and multiplication by a strictly positive scalar, we may assume that $f \in \Phi_1$ (this does not change the sign of $f(v)$), and then the hypothesis on v implies that $f(v) \geqslant 0$. Using the implication (iv)⇒(i), We conclude that $v \in \mathrm{cone}(v_1, \dots, v_m)$.

(ii)⇒(i). Let us assume that C is the set of all $x \in \mathrm{R}^n$ such that $f_j(x) \geqslant 0$ for all $j \in \{1, \dots, p\}$ and let D be the polyhedral cone $\mathrm{cone}(f_1, \dots, f_p)$ they generate in $(\mathrm{R}^n)^*$. By the implication (i)⇒(ii) applied to the cone D, there exist elements $v_1, \dots, v_m \in \mathrm{R}^n$ such that a linear form f on R^n belongs to D if and only if $f(v_i) \geqslant 0$ for all i. Let $\mathrm{C}' = \mathrm{cone}(v_1, \dots, v_m)$, and let us prove that $\mathrm{C} = \mathrm{C}'$. For every i, one has $f_j(v_i) \geqslant 0$ for all i, because $f_j \in \mathrm{D}$; this implies that $v_i \in \mathrm{C}$. Consequently, $\mathrm{C}' \subset \mathrm{C}$. Let $v \in \mathrm{C}$ and let f be a linear form on R^n such that $f(v_i) \geqslant 0$. We then have $f \in \mathrm{D}$, so that there exist positive elements $a_1, \dots, a_p \in \mathrm{R}$ such that $f = a_1 f_1 + \dots + a_p f_p$, hence $f(v) = \sum a_i f_i(v) \geqslant 0$. By theorem 1.2.1, this implies that $v \in \mathrm{C}'$, hence $\mathrm{C} \subset \mathrm{C}'$.

The equivalence (ii)⇔(iii) is remark 1.1.9. □

1.3. Polyhedra and polytopes

In corollary 1.2.6, we have seen that convex polyhedral cones can be defined in two ways, either from the inside, as the cone generated by a finite family of vectors, or from the outside, as defined by a finite set of linear inequalities. Here, we extend this description to polyhedra, by reducing their study to the case of cones.

1.3.1. — Let V be an R-vector space. Let us embed V into $\mathrm{V}' = \mathrm{V} \times \mathrm{R}$ by the affine map $\iota \colon x \mapsto (x, 1)$. An affine map $\mathrm{V} \to \mathrm{R}$ is of the form $f(x) = \varphi(x) + b$ for $\varphi \in \mathrm{V}^*$ and $b \in \mathrm{R}$, hence it extends to the linear form $f' \colon \mathrm{V}' \to \mathrm{R}$ given by $f'(x, t) = \varphi(x) + tb$. Then the inequality $f(x) \leqslant 0$ in V is equivalent to the inequality $f'(y) \leqslant 0$ in V′ together with the equality $t = 1$. In this way, every polyhedron in V is viewed as the intersection with the affine hyperplane $\{t = 1\}$ of a convex polyhedral cone of V'.

We first apply this method to extend the Farkas lemma to polytopes.

Proposition **(1.3.2)**. — *Let* V *be an* $\mathbf{R}$*-vector space, let* A *be a finite subset of* V *and let* $v \in V$*. The following assertions are equivalent:*

(i) *There exists an integer* m*, an affinely independent family* $(u_0, \dots, u_m)$ *in* A *and positive elements* $a_0, \dots, a_m \in \mathbf{R}$ *such that* $v = \sum_{i=1}^m a_i u_i$ *and* $1 = \sum_{i=0}^m a_i$*;*

(ii) *One has* $v \in \operatorname{conv}(A)$*;*

(iii) *For all linear forms* f *on* V *and all* $b \in \mathbf{R}$ *such that* $f(u) \geqslant b$ *for all* $u \in A$*, one has* $f(v) \geqslant b$*.*

Proof. — Let $V' = V \times \mathbf{R}$; for $u \in V$, set $u' = (u, 1)$, and let A' be the set of all u', for $u \in A$. Condition (i) is equivalent to the existence of an integer m, an independent family $(u'_0, \dots, u'_m)$ in A' and positive elements $a_0, \dots, a_m \in \mathbf{R}$ such that $v' = \sum_{i=0}^m a_i u_i$. Condition (ii) is equivalent to $v' \in \operatorname{cone}(A')$. All linear forms f' on V' are of the form $(x, t) \mapsto \varphi(x) - bt$, for $\varphi \in V^*$ and $b \in \mathbf{R}$, that is are associated to an affine form $f \colon x \mapsto \varphi(x) - b$ on V; condition (iii) is then equivalent to the inequality $f'(v') \geqslant 0$ for all linear forms f' on V' such that $f'(u') \geqslant 0$ for all $u' \in A'$. The proposition is then a consequence of theorem 1.2.1. ☐

Corollary **(1.3.3)** (Carathéodory theorem). — *Let* V *be an* $\mathbf{R}$*-vector space and let* A *be a subset of* V*. For every* $v \in \operatorname{conv}(A)$*, there exists an affinely independent subset* A' *of* A *such that* $v \in \operatorname{conv}(A')$*.*

Theorem **(1.3.4)**. — *Let* V *be a finite dimensional* $\mathbf{R}$*-vector space and let* P *be a subset of* V*. Then* P *is a polyhedron if and only if there exist a polytope* Q *and a polyhedral convex cone* C *such that* $P = Q + C$*.*

Proof. — Let P be defined in V by a finite family of affine inequalities of the form $f_j(x) \leqslant b_j$, where f_j is a linear form on V and $b_j \in \mathbf{R}$. Let then P' be the polyhedral convex cone in $V \times \mathbf{R}$ (with coordinates (x, t)) defined by the linear inequalities $f_j(x) - b_j t \leqslant 0$ and $-t \leqslant 0$. By corollary 1.2.6, the cone P' is generated by a finite family of vectors of the form $(x_i, t_i)_{i \in I}$, where $x_i \in V$ and $t_i \in \mathbf{R}$. Necessarily, $t_i \geqslant 0$; since P' is a cone, we may assume that $t_i \in \{0, 1\}$. Let I' be the set of all $i \in I$ such that $t_i = 0$, and let $I'' = I - I'$ be the complementary subset. Let Q be the polytope in V, convex hull of the x_i for $i \in I'$, and let C be the convex cone in V generated by the x_i for $i \in I''$. Let us now show that $P = Q + C$.

Let $x \in V$. One has $x \in P$ if and only if $(x, 1) \in P$, if and only if there is a family (a_i) of positive elements of R such that $(x, 1) = \sum_i a_i(x_i, t_i)$, which is equivalent to the relations

$$x = \sum_{i \in I'} a_i x_i + \sum_{i \in I''} a_i x_i \quad \text{and} \quad \sum_{i \in I'} a_i = 1.$$

This writes x as the sum of the element $x' = \sum_{i \in I'} a_i x_i$ of P′ and of the element $x'' = \sum_{i \in I''} a_i x_i$ of C, and conversely.

Let us now assume that $P = Q + C$, where Q is a polytope in V and C is a polyhedral cone in V. Let $(x_i)_{i \in I}$ be a finite family of vectors of which Q is the convex hull, and let $(y_j)_{j \in J}$ be a finite family of vectors such that $C = \operatorname{cone}(y_j)$. Let P′ be the convex polyhedral cone in $V \times R$ generated by the vectors $(x_i, 1)$, for $i \in I$, and $(y_j, 0)$, for $j \in J$. By corollary 1.2.6, this cone is defined in $V \times R$ by a finite family of linear inequalities, say $f'_k(x, t) \leqslant 0$. For every k, the function f_k on V given by $f_k(x) = f'_k(x, 1)$ is an affine form.

Now, a vector $x \in V$ belongs to P if and only if the vector $(x, 1)$ of $V \times R$ belongs to P′, that is, if and only if $f_k(x) \leqslant 0$ for all k. This proves that P is a polyhedron. □

Corollary **(1.3.5)**. — *Let $\varphi : V \to W$ be an affine map between finite dimensional* R*-vector spaces.*

a) *The image $\varphi(P)$ of a polyhedron $P \subset V$ is a polyhedron in* W.
b) *The inverse image $\varphi^{-1}(P)$ of a polyhedron $P \subset W$ is a polyhedron in* V.

Proof. — a) By the theorem, there exist a polytope Q and a polyhedral convex cone C in V such that $P = Q + C$. Then $\varphi(P) = \varphi(Q) + \varphi(C)$. Obviously, $\varphi(Q)$ is a polytope and $\varphi(P)$ is a polyhedral convex cone in W. Consequently, $\varphi(P)$ is a polyhedron in W.

b) This follows from the definition of a polyhedron: if P is defined in W by affine inequalities $f_j(y) \leqslant 0$, then $\varphi^{-1}(P)$ is defined in V by the affine inequalities $f_j \circ \varphi(x) \leqslant 0$. □

Corollary **(1.3.6)**. — *The Minkowski sum* $P + Q$ *of two polyhedra is a polyhedron.*

Proof. — The subset $P \times Q$ of $V \times V$ is a polyhedron: if P is defined by affine forms f_j and Q is defined by affine forms g_k, then $P \times Q$ is defined by the forms $f_j \circ p_1$ and $g_k \circ p_2$, where $p_1, p_2 \colon V \times V \to V$ are the two projections. Then $P + Q$ is the image of this polyhedron $P \times Q$ under the addition map, $V \times V \to V$, which is affine. □

Corollary **(1.3.7)**. — *A polyhedron is a polytope if and only if it is bounded.*

Proof. — It is obvious that a polytope is bounded. Conversely, if P is a polyhedron, let us write $P = Q + C$, where Q is a polytope and C is a polyhedral convex cone. If $P = \varnothing$, then it is the convex hull of the empty family; let us thus assume that P is nonempty — then Q is nonempty as well. If $C \neq 0$, then C is unbounded, because it is a cone, which implies, since Q is nonempty, that P is unbounded as well. This concludes the proof. □

Definition **(1.3.8)**. — *Let* V *be a finite dimensional* R*-vector space and let* P *be a nonempty polyhedron in* V. *The* recession cone *of* P *is the set of all* $y \in V$ *such that* $x + ty \in P$ *for all* $x \in P$ *and all positive* $t \in R$*; we denote it by* recc(P).

Proposition **(1.3.9)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V.

a) *If* P *is defined by affine inequalities* $f_j(x) \leqslant b_j$ *(where* f_j *is a linear form on* V *and* $b_j \in R$*), then* recc(P) *is defined by the linear inequalities* $f_j(x) \leqslant 0$ *in* V. *In particular,* recc(P) *is a polyhedral convex cone in* V.

b) *For every* $x \in P$, *the recession cone of* P *is also characterized by:*

$$\mathrm{recc}(P) = \{y \in V\,;\, \exists x \in P,\ \forall t \geqslant 0,\ x + ty \in P\}.$$

If R *is archimedean, then one also has*

$$\mathrm{recc}(P) = \{y \in V\,;\, \forall x \in P,\ x + y \in P\}.$$

c) *The recession cone* recc(P) *of* P *is the unique cone* C *for which there exists a polytope* Q *such that* $P = Q + C$.

Proof. — a) Let $y \in V$ be such that $f_j(y) \leqslant 0$ for all j. Then, for every $x \in P$ and every $t \in R$ such that $t \geqslant 0$, one has $f_j(x+ty) = f_j(x)+tf_j(y) \leqslant b_j$ for all j, hence $x+ty \in P$. Conversely, let $y \in \operatorname{recc}(P)$ and fix some element $x \in P$. Then for all j and all $t \geqslant 0$, one has $f_j(x)+tf_j(y) \leqslant b_j$; when $t \to \infty$, this implies $f_j(y) \leqslant 0$.

b) Let C' and C'' be these two other sets.

The inclusion $\operatorname{recc}(P) \subset C'$ is obvious, because P is nonempty. Conversely, let $y \in C'$ and let $x \in P$ be such that $x+ty \in P$ for all $t \in R$ such that $t \geqslant 0$. With the notation of *a*), for all j, one has $f_j(x)+tf_j(y) \leqslant b_j$ for all $t \geqslant 0$ hence $f_j(y) \leqslant 0$; consequently, $y \in \operatorname{recc}(P)$.

One has $\operatorname{recc}(P) \subset C''$: if $x \in P$ and $y \in \operatorname{recc}(P)$, then $x+y \in P$, hence $y \in C''$. Conversely, let us assume that R is archimedean and let us prove that $C'' \subset \operatorname{recc}(P)$. If $y \in C''$, then we see by induction that $x+ny \in P$ for all $x \in P$ and all $n \geqslant 0$. Let $t \in R$ such that $t \geqslant 0$. Since R is archimedean, there exists $n \in \mathbf{N}$ such that $t \leqslant n$; then x and $x+ny$ belong to P. The expression

$$x+ty = (1-\frac{t}{n})x + \frac{t}{n}(x+ny)$$

shows that $x+ty$ is a convex combination of x and $x+ny$; since P is convex, this shows that $x+ty \in P$.

c) By theorem 1.3.4, there exist a polytope Q and a polyhedral convex cone C such that $P = Q + C$. From these relations, we see that $P + C = P$ hence, C being a cone, the inclusion $C \subset \operatorname{recc}(P)$.

Let now $y \in \operatorname{recc}(P)$. Let f be any linear form such that $f(x) \leqslant 0$ for all $x \in C$; and let us show that $f(y) \leqslant 0$; by theorem 1.2.1, this will imply that $y \in C$, hence the inclusion $\operatorname{recc}(P) \subset C$.

For all $x \in Q$ and $t \in R$ such that $t > 0$, we can write $x+ty = x_t + ty_t$, with $x_t \in Q$ and $y_t \in C$. Then $y = (x_t - x)/t + y_t$, hence $f(y) = f(x_t-x)/t + f(y_t)$; since Q is bounded, $f(x_t - x)$ is bounded as well; moreover, $f(y_t) \leqslant 0$. When $t \to \infty$, we obtain $f(y) \leqslant 0$, as claimed. □

Definition **(1.3.10)**. — *Let* P *be a nonempty polyhedron. The* lineality space *of* P *is the intersection* $\operatorname{recc}(P) \cap (-\operatorname{recc}(P))$*; we denote it by* $\operatorname{linsp}(P)$.

Proposition **(1.3.11)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V.

a) *The lineality space of* P *is the largest vector subspace* W *of* V *such that* $P + W = P$.

b) *Let* $\varphi : V \to R^n$ *be a linear map and let* $v \in R^n$ *be such that* P *is the polyhedron defined by* $\varphi(x) \leqslant v$. *Then* $\operatorname{linsp}(P) = \operatorname{Ker}(\varphi)$.

Proof. — As the intersection of two convex cones, linsp(P) is itself a convex cone. By construction, it is stable under $x \mapsto -x$; this implies that it is a vector subspace of V. If W is a vector subspace of V such that $P + W = P$, then one has $x + R_+ y \subset P$ for all $y \in W$ and all $x \in P$, hence $y \in \operatorname{recc}(P)$, so that $W \subset \operatorname{recc}(P)$; since, moreover, $W = -W$, this implies $W \subset \operatorname{linsp}(P)$, as claimed.

If $P = \{x \in V\,;\, \varphi(x) \leqslant v\}$, we have seen that $C = \{x \in V\,;\, \varphi(x) \leqslant 0\}$. Then $\operatorname{linsp}(P) = C \cap (-C)$ is the set of $x \in V$ such that $\varphi(x) \leqslant 0$ and $\varphi(-x) \leqslant 0$; the latter condition means $\varphi(x) \geqslant 0$; combined, they are thus equivalent to $\varphi(x) = 0$. □

1.3.12. — Let P be a nonempty polyhedron in a finite dimensional R-vector space V. We define the dimension of P as the dimension of the affine subspace affsp(P) it spans. We also define the *relative interior* of P to be its interior inside affsp(P); we denote it by relint(P).

Let P be defined by a finite family $(f_j)_{j\in J}$ of affine inequalities, that is, P is the set of $x \in V$ such that $f_j(x) \leqslant 0$ for all j. Let $i \in J$.

One says that the inequality $f_i(x) \leqslant 0$ is *redundant* if the family $(f_j)_{j\in J-\{i\}}$ defines the same polyhedron. It the system is minimal, then it has no redundant inequality.

One says that $f_i = 0$ is an *implicit equality* for this system if one has $f_i(x) = 0$ for all $x \in P$. If $f_i = \varphi_i - b_i$, where φ_i is a linear form on V and $b_i \in R$, one also says that $\varphi_i = b_i$ is an implicit equality.

Let I be the set of all $j \in J$ such that $f_j = 0$ is an implicit equality for this system. Assume that $I \neq J$.

Proposition **(1.3.13)**. — *Let* P *be a nonempty polyhedron defined by a system of affine inequalities* $(f_j)_{j\in J}$ *in a finite dimensional vector space* V, *and let* $(f_j)_{j\in I}$ *be the subfamily of those which are implicit equalities.*

a) *The affine subspace spanned by* P, affsp(P), *is defined in* V *by the equalities* $(f_j = 0)_{j\in I}$.

b) *The relative interior of* P *is defined in* affsp(P) *by the strict inequalities* $(f_j < 0)_{j\in J-I}$.

c) *The polyhedron* P *is the closure of its relative interior.*

Proof. — We first treat the case where I = J. Then P coincides with the affine space defined by the vanishing of the affine forms f_j, so that P = affsp(P). In this case, relint(P) = P and all three assertions hold.

For the rest of the proof, we assume that I ≠ J. For every $j \in J - I$, let us choose a point $x_j \in P$ such that $f_j(x_j) < 0$; let then $\xi = (\sum_{j\in J-I} x_j)/\mathrm{Card}(J-I)$ be their center of mass (here use that I ≠ J!). Since P is convex, one has $\xi \in P$. Moreover, for all $j \in J - I$, one has

$$f_j(\xi) = (\sum_{k\in J-I} f_j(x_k)/\mathrm{Card}(J-I) \leqslant f_j(x_j)/\mathrm{Card}(J-I) < 0.$$

Explicitly, we have constructed a point $\xi \in P$ at which all inequalities which are not implicit equalities are strict.

Let L be the affine subspace of V defined by the implicit equalities $f_j = 0$, for all $j \in I$.

Let U be the subset of P defined by the strict inequalities $f_j < 0$, for $j \in J - I$. By construction, one has $\xi \in U$.

We need to prove that U = relint(P) and L = affsp(P).

By classical linear algebra, the affine subspace spanned by P is the subspace of V defined by the affine forms which equal to zero on P. Let f be an affine form on V such that $f \equiv 0$ on P. Since $f \leqslant 0$ on P, corollary 1.4.2 implies that there exist positive elements $a_j \in \mathbf{R}$ and $c \in \mathbf{R}$ such that $f = -c + \sum a_j f_j$. One then has

$$0 = f(\xi) = -c + \sum a_j f_j(\xi) = -c + \sum_{j\in J-I} a_j f_j(\xi),$$

so that $c = 0$ and $a_j = 0$ for all $j \in J - I$. In other words, f is a linear combination of the affine forms $(f_j)_{j\in I}$ and $f \equiv 0$ on L. This proves that L = affsp(P).

Since U is open in L, one has U ⊂ relint(P). Conversely, let $x \in$ relint(P). The line (ξx) is contained in L, and the trace of relint(P) on this line is an open interval containin x. Consequently, there exists $t > 1$

such that $(1-t)\xi + tx \in P$. Then, for every $j \in J - I$, one has

$$f_j((1-t)\xi + tx) = (1-t)f_j(\xi) + tf_j(x) \leqslant 0,$$

which implies, because $1-t < 0$ and $f_j(\xi) < 0$, that $f_j(x) < 0$. This proves that $x \in U$, hence $U = \operatorname{relint}(P)$.

To conclude the proof of the proposition, it remains to prove that P is the closure of relint(P). Since P is defined by large inequalities, it is closed. Conversely, let $x \in P$; for $t \in \mathbf{R}$ such that $0 < t < 1$, one has $(1-t)\xi+tx \in \operatorname{relint}(P)$, because $f_j((1-t)\xi+tx) = (1-t)f_j(\xi)+tf_j(x) < 0$. When $t \to 1$, one has $(1-t)\xi + tx \to x$, so that x belongs to the closure of relint(P). ☐

1.3.14. — For every subset C of V, one defines a subset C° of V^* as the set of all linear forms f on V such that $f(x) \leqslant 1$ for all $x \in C$. This subset is called the *polar* of C.

The mapping $C \mapsto C^\circ$ reverses inclusions: if C, D are subsets of V such that $C \subset D$, then $D^\circ \subset C^\circ$. Moreover, one has $C^\circ = \operatorname{conv}(C)^\circ$.

If C is a cone, then applying the relation $f(x) \leqslant 1$ to all multiples of x shows that C° is the set of all $f \in V$ such that $f(x) \leqslant 0$ for all $x \in C$. In that case, C° is a convex cone. Assume that C is a polyhedral convex cone and let $u_1, \dots, u_k \in V$ be such that $C = \operatorname{cone}(u_1, \dots, u_k)$. Then C° is defined by the linear inequalities $f(u_j) \leqslant 0$, for $j \in \{1, \dots, k\}$. Consequently, C° is polyhedral convex cone in V^*.

If C is a polytope, say $C = \operatorname{conv}(v_1, \dots, v_m)$, then C° is defined by the affine inequalities $f(v_j) \leqslant 1$, for $j \in \{1, \dots, m\}$, hence it is a polyhedron.

If C is a polyhedron, written as $C = Q + \operatorname{recc}(C)$, then we see that $C^\circ = Q^\circ \cap \operatorname{recc}(C)^\circ$ is a polyhedron.

Assume that $V = \mathbf{R}^n$ (or that it is endowed with a Q-structure, for some subfield Q of R). If C is a Q-rational polyhedron, defined by affine forms with rational linear part *and* constant term, then its polar subset is again a rational polyehdron. Indeed, the vertices $v_1, \dots, v_m$ of C belong to $\mathbf{Q}^n$, and its recession cone is generated by vectors $u_1, \dots, u_k$ in $\mathbf{Q}^n$. The claim then follows from the fact that C° is defined in $\mathbf{R}^n$ by the affine inequalities $f(v_i) \leqslant 1$ (for $i \in \{1, \dots, m\}$) and $f(u_j) \leqslant 0$ (for $j \in \{1, \dots, k\}$).

1.3.15. — If L is a vector subspace of V, then L° is a vector subspace of V^*, it identifies with the dual of V/L. If one identifies V with its bidual V^{**} via the canonical morphism, one then has $L = L^{\circ\circ}$.

More generally, the mapping $C \mapsto C^\circ$ gives rise to a duality of polyhedra when they contain the origin, and of polyhedral convex cones.

Proposition **(1.3.16)** (Convex duality). — *If* C *is a polyhedron of* V *that contains* 0*, then* $C^{\circ\circ} = C$.

Proof. — The inclusion $C \subset C^{\circ\circ}$ follows from the definitions; let us prove the converse inclusion. Identify V with R^m and write C as the set of all $x \in R^m$ such that $\varphi(x) \leqslant v$, for some linear map $\varphi : R^m \to R^n$ and some $v \in R^n$. Since $0 \in C$, one has $v_j \geqslant 0$ for all j. If $v_j > 0$, then $\varphi_j/v_j \in C^\circ$ and the condition $x \in C^{\circ\circ}$ implies that $\varphi_j(x) \leqslant v_j$. Otherwise, if $v_j = 0$, then $t\varphi_j \in C^\circ$ for all $t \in R$ such that $t \geqslant 0$; if $x \in C^{\circ\circ}$, then $t\varphi_j(x) \leqslant 1$ for all $t \geqslant 0$, hence $\varphi_j(x) \leqslant 0$. This proves the claimed equality. □

The inclusions $\mathrm{linsp}(C) \subset C \subset \mathrm{affsp}(C)$ furnish inclusions $\mathrm{affsp}(C)^\circ \subset C^\circ \subset \mathrm{linsp}(C)^\circ$. In fact, with the previous notation, one has $\mathrm{linsp}(C^\circ) = C^\circ \cap (-C^\circ)$ is the set of all $f \in V^*$ such that $f(v_j) = 0$ for all j, that is $\mathrm{linsp}(C^\circ) = \mathrm{affsp}(C)^\circ$. By duality, we see that $\mathrm{linsp}(C)^\circ = \mathrm{affsp}(C^\circ)$.

1.4. Linear programming

Proposition **(1.4.1)** (Linear programming). — *Let* $\varphi : R^m \to R^n$ *be a linear map, let* $v \in R^n$ *and let* C *be the set of* $x \in R^m$ *such that* $\varphi(x) \leqslant v$*. Let* f *be a linear form on* R^m *and let* D *be the set of all positive linear forms* g *on* R^n *such that* $f = g \circ \varphi$.

a) *Assume that both* C *and* D *are nonempty. Then there exist* $\xi \in C$ *and* $\gamma \in D$ *such that*

$$f(\xi) = \sup_{x\in C} f(x) = \inf_{g\in D} g(v) = \gamma(v).$$

b) *Let* $\xi \in C$ *and* $\gamma \in D$*. The following are equivalent:*

(i) *One has* $f(\xi) = \sup_{x\in C} f(x)$ *and* $\gamma(v) = \inf_{g\in D} g(v)$*;*
(ii) *One has* $f(\xi) = \gamma(v)$*;*
(iii) *One has* $\gamma(v - \varphi(\xi)) = 0$*;*

(iv) *If the* jth *component of* γ *is strictly positive, then the* jth *component of the inequality* $\varphi(\xi) \leqslant v$ *is an equality.*

c) *If* C *is nonempty but* D *is empty, then* $\sup_{x\in C} f(x) = +\infty$. *If* D *is nonempty but* C *is empty, then* $\inf_{\gamma\in D} \gamma(v) = -\infty$.

Proof. — a) For $x \in C$ and $g \in D$, one has $\varphi(x) \leqslant v$, hence $f(x) = g \circ \varphi(x) \leqslant g(v)$ since g is positive. Conversely, we prove that there exist $\xi \in C$ and $\gamma \in D$ such that $f(\xi) \geqslant \gamma(v)$. Writing ξ under the form $x - x'$, for positive $x, x' \in R^m$, this problem is equivalent to finding positive $x, x' \in R^m$ and a positive linear form g on R^n satisfying the system of inequalities $\psi(x, x', g) \leqslant (v, 0, f, -f)$, where $\psi : R^{2m} \times (R^n)^* \to R^n \times R \times (R^m)^{*2}$ is the linear map given by

$$\psi(x, x', g) = (\varphi(x) - \varphi(x'), -f(x) + f(x') + g(v), g \circ \varphi, -g \circ \varphi).$$

By corollary 1.2.5, this is then equivalent to proving that $g(v) + f(u) - f(u') \geqslant 0$ for all positive linear forms g on R^n, all positive $t \in R$, all positive vectors $u, u' \in R^m$ such that $g \circ \varphi - tf = 0$ and $tv + \varphi(u) - \varphi(u') \geqslant 0$.

Let us prove tht this last assertion holds. First assume that $t > 0$; then $f = t^{-1} g \circ \varphi$, so that

$$g(v) + f(u) - f(u') = t^{-1} g(tv + \varphi(u - u')) \geqslant 0$$

since $g \geqslant 0$ and $tv + \varphi(u) - \varphi(u') \geqslant 0$. Let us now assume that $t = 0$, so that $g \circ \varphi = 0$ and $\varphi(u) - \varphi(u') \geqslant 0$. Fix $\xi \in C$ and $\gamma \in D$. Since $\varphi(\xi) \leqslant v$ and $g \geqslant 0$, we then have

$$g(v) + f(u) - f(u') \geqslant g(\varphi(\xi)) + \gamma \circ \varphi(u - u') \geqslant 0,$$

since $g \circ \varphi = 0$, $\gamma \geqslant 0$ and $\varphi(u - u') \geqslant 0$.

b) The implication (i)⇒(ii) follows from part *a*). Since

$$\gamma(v - \varphi(\xi)) = \gamma(v) - g \circ \varphi(\xi) = \gamma(v) - f(\xi),$$

assertions (ii) and (iii) are equivalent. Assume then that $\gamma(v) = f(\xi)$. For every $x \in C$ and every $g \in D$, one has

$$f(x) = g \circ \varphi(x) \leqslant g(v),$$

because g is positive and $\varphi(x) \leqslant v$. In particular, $f(x) \leqslant \gamma(v) = f(\xi)$, so that $f(\xi) = \sup_{x\in C} f(x)$. Similarly, $g(v) \geqslant f(\xi) = \gamma(v)$, so that $\gamma(v) = \inf_{g\in D} g(v)$. This proves (i).

By assumption, $\varphi(\xi) \leqslant v$ and γ is positive, so that $\gamma(v - \varphi(\xi)) \geqslant 0$. Consequently, the equality $\gamma(v - \varphi(\xi)) = 0$ is equivalent to the fact that for every integer $j \in \{1, \dots, n\}$ such that $\varphi_j(\xi) < v_j$, one has $\gamma_j = 0$. This proves the equivalence of (iii) and (iv).

c) Assume that $D = \varnothing$. By corollary 1.2.3 applied to $\varphi^{t} : (\mathbf{R}^n)^* \to (\mathbf{R}^m)^*$ and to the vector $f \in (\mathbf{R}^m)^*$ there exists $\xi \in \mathbf{R}^m$ such that $\varphi(\xi) \geqslant 0$ and $f(x) < 0$. Let $x \in C$. For every $t \in \mathbf{R}$ such that $t \geqslant 0$, the vector $x - t\xi \in \mathbf{R}^m$ satisfies $\varphi(x - t\xi) = \varphi(x) - t\varphi(\xi) \leqslant v$, hence $x - t\xi \in C$, so that

$$\sup_{y\in C} f(y) \geqslant \sup_{t\leqslant 0} f(x - t\xi) = f(x) + \sup_{t\leqslant 0}(-tf(\xi)) = +\infty.$$

Similarly, assume that $C = \varnothing$. By corollary 1.2.4, there exists a positive linear form g on $\mathbf{R}^n$ such that $g \circ \varphi = 0$ and $g(v) < 0$. Let also $\gamma \in D$. For every $t \in \mathbf{R}$ such that $t \geqslant 0$, one has $\gamma + tg \geqslant 0$ and $(\gamma + tg) \circ \varphi = \gamma \circ \varphi = f$, so that $\gamma + tg \in D$; moreover, $(\gamma + tg)(v) = \gamma(v) + tg(v)$, so that

$$\inf_{h\in D} h(v) \leqslant \inf_{t\geqslant 0}(\gamma + tg)(v) = \gamma(v) + \inf_{t\geqslant 0} tg(v) = -\infty.$$

This concludes the proof. ☐

Corollary **(1.4.2)**. — *Let $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ be a linear map, let $v \in \mathbf{R}^n$; let f be a linear form on $\mathbf{R}^m$ and let $c \in \mathbf{R}$. Assume that there exists $x \in \mathbf{R}^m$ such that $\varphi(x) \leqslant v$, and that one has $f(x) \leqslant c$ for every such x. Then there exists a positive linear form g on $\mathbf{R}^n$ such that $f = g \circ \varphi$ and $g(v) \leqslant c$.*

More explicitly, every inequality $f(x) \leqslant c$ which is implied by a *consistent* system of inequalities $\varphi_j(x) \leqslant v_j$ is *trivially* implied by them, in the sense that a positive linear combination of them is of the form $f(x) \leqslant b$, with $b \leqslant c$.

Proof. — With the notation of proposition 1.4.1, the set C is nonempty and $f(x)$ is bounded from above on C. By part *c*), the set D is nonempty as well, and part *a*) implies that there exist $\xi \in \mathbf{R}^m$ such that $\varphi(\xi) \leqslant v$ and a positive linear form γ on $\mathbf{R}^n$ such that $\gamma \circ \varphi = f$ and $f(\xi) = \gamma(v)$. Then $\gamma(v) = f(\xi) \leqslant c$. ☐

Corollary **(1.4.3).** — *Let us retain the notation of proposition 1.4.1, assuming that* C *and* D *are both nonempty. For every $j \in \{1, \dots, n\}$, exactly one of the following assertions holds:*

(i) *There exists $\xi \in C$ such that $f(\xi) = \sup_{x \in C} f(x)$ and $\varphi_j(\xi) < v_j$;*
(ii) *There exists $\gamma \in D$ such that $\gamma(v) = \inf_{g \in D} g(v)$ and $\gamma_j > 0$.*

Proof. — That (i) and (ii) cannot hold simultaneously follows from property (iv) of proposition 1.4.1.

Let $c = \sup_{x \in C} f(x) = \inf_{g \in D} g(v)$; let $\xi \in C$ and $\gamma \in D$ be such that $f(\xi) = \gamma(v) = c$. Assume that (i) does not hold, that is, that for every $x \in C$ such that $f(\xi) = \sup_{x \in C} f(x)$, one has $\varphi_j(\xi) = v_j$. In other words, the inequality $\varphi_j(\xi) \geqslant v_j$ is implied by the inequalities $\varphi_i(\xi) \leqslant v_i$ and $f(x) \geqslant c$.

By proposition 1.4.1 applied to the linear map $\varphi' \colon R^m \times R \to R^n \times R$ defined by $\varphi'(x, t) = (\varphi(x), -f(x))$, the vector $v' = (v, -c)$, and the linear form $-\varphi_j$, there exist positive elements $a_1, \dots, a_m, t \in R$ such that $\sum_{i=1}^m a_i \varphi_i - tf = -\varphi_j$ and $\sum_{i=1}^m a_i v_i - tc \leqslant -v_j$. Let $a' \in R^m$ be given by $a'_i = a_i$ for $i \neq j$ and $a'_j = a_j + 1$; then $\sum_{i=1}^m a'_i \varphi_i = tf$ and $\sum_{i=1}^m a'_i v_i \leqslant tc$; moreover, $a'_j \geqslant 1$. Let $g, g' \colon R^m \to R$ be the linear forms given by $g(y_1, \dots, y_m) = \sum_{i=1}^m a_i y_i$ and $g'(y_1, \dots, y_m) = \sum_{i=1}^m a'_i y_i$, so that $g \circ \varphi = tf - \varphi_j$, $g' \circ \varphi = tf$, $g(v) = tc - v_j$ and $g'(v) = tc$; moreover, g and g' are positive.

First assume that $t = 0$. Then the linear form $\gamma' = \gamma + g'$ is positive and satisfies $\gamma' \circ \varphi = \gamma \circ \varphi = f$; moreover, $\gamma'(v) = \gamma(v) = c$ and the jth component of γ' is strictly positive, so that (ii) holds.

Now assume that $t > 0$. Then $\gamma' = t^{-1} g'$ is a positive linear form such that $\gamma' \circ \varphi = f$ and $\gamma'(v) = c$; moreover, the jth component of γ' is strictly positive, so that (ii) holds. ☐

Corollary **(1.4.4).** — *With the notation of proposition 1.4.1, assume that* C *and* D *are nonempty. There exists a linear form γ on R^n such that, if* J *is the set of $j \in \{1, \dots, n\}$ such that the jth component of γ is strictly positive, the family $(\varphi_j)_{j \in J}$ is linearly independent.*

Proof. — Fix $\xi \in C$ and $\gamma \in D$ such that $f(\xi) = g(v)$. Let $a_1, \dots, a_n$ be the coordinates of γ; they are positive and one has $f = \gamma \circ \varphi =$

$\sum_{j=1}^{n} a_j\varphi_j$ and $f(\xi) = g(v) = \sum_{j=1}^{n} a_j v_j$. In other words, in the space $(\mathrm{R}^n)^* \times \mathrm{R}$, the vector $(f, f(\xi))$ belongs to the cone generated by the vectors (φ_j, v_j). By Carathéodory's theorem (corollary 1.2.2), there exist a subset J of $\{1, \dots, m\}$ and positive elements $b_j \in \mathrm{R}$ (for $j \in \mathrm{J}$) such that $(f, g(v)) = \sum_{j\in\mathrm{J}} b_j(\varphi_j, v_j)$ and such that the family $(\varphi_j, v_j)_{j\in\mathrm{J}}$ is linearly independent in $(\mathrm{R}^n)^* \times \mathrm{R}$. We may even assume that $b_j > 0$ for all $j \in \mathrm{J}$.

Let $\varphi' : \mathrm{R}^n \to \mathrm{R}^\mathrm{J}$ be the linear map $x \mapsto (\varphi_j(x))_{j\in\mathrm{J}}$, let $v' = (v_j)_{j\in\mathrm{J}} \in \mathrm{R}^\mathrm{J}$ and let γ' be the positive linear form on R^J given by $\gamma'(y) = \sum b_j y_j$. One has $\varphi'(\xi) \leqslant v'$ and $\gamma' \circ \varphi = f$, so that ξ and γ' belong to the sets C′ and D′ associated by proposition 1.4.1 with the linear map φ', the vector v' and the form f. For all $j \in \mathrm{J}$, one has $b_j > 0$, hence part *b*) of that proposition implies that $\varphi_j(\xi) = v_j$. As a consequence, the families $(\varphi_j, \varphi_j(\xi))_{j\in\mathrm{J}}$ and $(\varphi_j)_{j\in\mathrm{J}}$ have the same rank, which proves that the latter family is linearly independent. Then, the linear form γ_1 on R^n defined by $\gamma_1(y) = \sum_{j\in\mathrm{J}} b_j y_j$ satisfies the requirements of the corollary. □

1.5. Faces, facets, vertices

1.5.1. — Let P be a nonempty polyhedron in a finite dimensional R-vector space V. Let $f \in \mathrm{V}^*$ be a linear form on V (possibly, $f = 0$). If f is bounded from above on P, then linear programming (proposition 1.4.1) implies that there exists $x \in \mathrm{P}$ such that $f(x) = \sup_\mathrm{P}(f)$. The subspace of P defined by the equality $f = \sup_\mathrm{P}(f)$ is then a polyhedron (it suffices to add the inequality $-f \leqslant -\sup_\mathrm{P}(f)$ to a system defining P), which we call the *face* of P defined by f.

According to this definition, a face is never empty. However, some authors (actually, *most* authors) let the empty set be a face of a polyhedron.

If $f = 0$, then the face of P defined by f is equal to P.

Let us assume that $f \neq 0$. Then the affine hyperplane $\{f = \sup_\mathrm{P}(f)\}$ is called the the *supporting hyperplane* defined by f; it separates the two closed halfspaces $\{f \geqslant \sup_\mathrm{P}(f)\}$ and $\{f \leqslant \sup_\mathrm{P}(f)\}$ in V, and P is contained in the latter.

The faces of P are ordered by inclusion. Faces of P which are maximal among those distinct from P are called *facets*. Faces of dimension 0 (these are the faces which are reduced to a point) are called *vertices*.

Proposition **(1.5.2)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V.

a) *Let* $(f_j)_{j\in J}$ *be a finite family of affine forms on* V *that defines* P *(that is,* $x \in P$ *if and only if* $f_j(x) \leqslant 0$ *for all* $j \in J$*). Then a nonempty subset* F *of* P *is a face of* P *if and only if there exists a subset* I *of* J *such that* $F = \{x \in P\,;\ \forall j \in I,\ f_j(x) = 0\}$.

b) *The set of faces of* P *is finite and nonempty.*

c) *Let* F *be a face of* P*. Then a subset* F′ *of* F *is a face of* P *if and only if it is a face of* F.

Proof. — a) We may assume that $V = R^n$. For every $j \in J$, write $f_j = \varphi_j - b_j$, where φ_j is a linear form on R^n and $b_j \in R$. Let $\varphi\colon R^n \to R^J$ be the corresponding linear map and let $b = (b_j) \in R^J$.

Let F be a face of P. Let f be a linear form on V which is bounded from above on P and let $F = \{x \in P\,;\ f(x) = \sup_P(f)\}$ be the corresponding face. By linear programming (proposition 1.4.1), there exist a positive linear form g on R^n such that $g \circ \varphi = f$ and $\xi \in F$ such that $f(\xi) = \sup_P(f) = \varphi(b)$. Let (a_j) be the coordinates of g so that $f = g \circ \varphi = \sum a_j\varphi_j$; let I be the subset of J consisting of those j such that $a_j > 0$. By proposition 1.4.1, *b*), one has $F = \{x \in P\,;\ \forall j \in I,\ \varphi_j(x) = b_j\}$.

Conversely, let I be a subset of J and let F be the subset of P defined the equations $\varphi_j(x) = b_j$ for $j \in I$, and $\varphi_j(x) \leqslant b_j$ for $j \in J - I$. Assume that $F \neq \varnothing$. Let $f = \sum_{i\in I} \varphi_j$ and $c = \sum_{i\in I} b_j$. For $x \in P$, one has $f(x) \leqslant c$, and $f(x) = c$ if and only if $x \in F$. Since $F \neq \varnothing$, this proves that F is the face of P defined by f.

b) This follows from the preceding assertion since the set of all subsets of J is finite and nonempty. (In any case, P is a face of itself.)

c) Retain the notation from *a*) and let I be a subset of J such that F is defined in P by the equalities $f_j(x) = 0$ for $j \in I$. We may view F as a polyhedron by adding to the system defining P the inequalities $-f_j(x) \leqslant 0$ (for $j \in I$).

If a subset F′ of F is a face of P, it is defined in P by additional equalities $f_j(x) = 0$, for j in a subset I′ of J. If we add to these the inequalities $-f_j(x) \leqslant 0$, for $j \in$ I, this still defines F′, because F′ ⊂ F. Consequently, F′ is a face of F.

Conversely, if F′ is a face of F, let it be defined in F by additional equalities $f_j(x) = 0$, for j in a subset I'_1 of J, as well as by additional equalities $-f_j(x) = 0$, for j in a subset I'_2 of I. Let $\mathrm{I}' = \mathrm{I}'_1 \cup \mathrm{I}'_2$; then F′ is defined in P by the additional equalities $f_j(x) = 0$, for $j \in$ I′. This proves that F′ is a face of P. □

Proposition **(1.5.3)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V*. Let* $(f_j)_{j\in \mathrm{J}}$ *be a finite family of affine forms that defines* P *(that is,* P *is the set of* $x \in$ V *such that* $f_j(x) \leqslant 0$ *for all* $j \in$ J*); assume that it is minimal, that is, has no redundant inequality. For every* $j \in$ J*, let* $\mathrm{F}_j = \{x \in \mathrm{P}\,;\, f_j(x) = 0\}$.

a) *Let* $j \in$ J*. If* $\mathrm{F}_j = \mathrm{P}$*, then* $f_j(x) = 0$ *is an implicit equality in this system; otherwise,* F_j *is a facet of* P.

b) *If* F *is a facet of* P*, then there exists a unique* $j \in$ J *such that* $\mathrm{F}_j = \mathrm{F}$ *(and* $\mathrm{F}_j \neq \mathrm{P}$*).*

Proof. — a) Let I be the subset of all $i \in$ J such that $f_i(x) = 0$ is an implicit equality in the given system defining P; in other words, $i \in$ I if and only if $\mathrm{F}_i = \mathrm{P}$.

Let $j \in$ J. Assume that F_j is empty. Then there exists a family $(a_i)_{i\in\mathrm{J}}$ of positive elements in R, as well as a strictly positive element $c \in$ R, such that $f_j = -c + \sum_{i\in\mathrm{I}} a_i f_i$. If we had $a_j \geqslant 1$, we would write $\sum_{i\neq j} a_i f_i(x) + (a_j - 1) f_j(x) = c$ on P, hence $c \leqslant 0$, a contradiction, so that $a_j < 1$. Then $(1 - a_j) f_j = -c + \sum_{i\neq j} a_i f_i$, so that $(1 - a_j) f_j(x) \leqslant -c < 0$ for all $x \in$ P; this implies that the inequality $f_j(x) \leqslant 0$ is redundant in the given system defining P. As a consequence, F_j is not empty.

By the previous proposition, this proves that F_j is a face of P, and that the only faces of P containing F_j are F_j and P. If $\mathrm{F}_j \neq \mathrm{P}$, then F_j is a facet of P; otherwise, $\mathrm{F}_j = \mathrm{P}$, and $j \in$ I.

b) Let F be a facet of P and let K be a subset of J such that F is defined by additional equalities $f_k(x) = 0$ for $k \in$ K. Since F ≠ P, the set K is not contained in I. Let then $k \in K - I$; one has $F_k \neq \varnothing$ and $F \subset F_k \subsetneq P$. Since F is a facet, this implies that $F = F_k$.

Let now $i \in J - I$ and let $x \in \operatorname{relint}(P)$, so that $f_j(x) < 0$ for all $j \in J - I$. Since no inequality in the given system is redundant, there exists $y \in V$ such that $f_i(y) > 0$ and $f_j(y) \leqslant 0$ for all $j \in J$ such that $j \neq i$. Since f_i is affine and $f_i(y) > 0 > f_i(x)$, there exists a unique point z on the segment $[x; y]$ such that $f_i(z) = 0$; moreover, $z \neq x, y$. For $j \in J - \{i\}$, one has $f_j(x) < 0$ and $f_j(y) \leqslant 0$, so that $f_j(z) < 0$. This shows that $z \in F_i$ and that $z \notin F_j$ for any $j \in J - \{i\}$. In particular, $F_j \neq F_i$ for distinct i, j in $J - I$. □

Corollary **(1.5.4)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V.

a) *Every face of* P *is the intersection of some family of facets of* P*;*
b) *The facets of* P *have dimension* $\dim(P) - 1$*;*
c) *The polyhedron* P *has no facets if and only if it is an affine subspace;*
d) *The relative interior is the complement in* P *of the union of all facets.*

Proof. — We fix a finite family $(f_j)_{j \in J}$ of affine forms on V that defines P; we also assume that it is minimal, so that no inequality is redundant. Let I be the subset of J consisting of all $i \in J$ such that $f_i(x) = 0$ is an implicit equality on P.

a) Let F be a face of P. By proposition 1.5.2, we know that there exists a subset K of J such that $F = \{x \in P\,;\ \forall k \in K,\ f_k(x) = 0\}$. Then F is the intersection of the facets F_k, for $k \in K - I$.

b) Let $j \in J - I$. The facet $F_j = P \cap \{f_j(x) = 0\}$ is defined by the family of inequalities $f_i(x) \leqslant 0$, for $i \in J$, to which we add the inequality $-f_j(x) \leqslant 0$. On this new system, there are two new implicit equalities, namely $f_j(x) = 0$ and $-f_j(x) = 0$ — which are equivalent — but there are none other, because $F_j \not\subset F_i$. for $i \in J - I$ and $j \neq i$.

The dimension of P, being the dimension of the affine subspace defined by the implicit equalities in the given system, is equal to $\dim(V) - \operatorname{rank}((f_i)_{i \in I})$. Similarly, the dimension of F_j is equal to $\dim(V) - \operatorname{rank}((f_i)_{i \in I \cup \{j\}})$. In particular, $\dim(P) - 1 \leqslant \dim(F_j) \leqslant \dim(P)$. Since $j \notin I$, the form f_j is not a linear combination of the forms f_i, for $i \in I$, so that $\dim(F_j) = \dim(P) - 1$, as was to be shown.

c) Assume that the polyhedron P is an affine subspace. Then every linear form which is bounded from above on P has to be constant, hence defines the face P itself. This proves that P is the only face of P, and P has no facets.

Conversely, if P has no facets, then I = J, all inequalities in the given system that defines P give rise to implicit equalities and P is an affine subspace.

d) This is a consequence of proposition 1.3.13, *b*). □

Corollary **(1.5.5)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V*. A face* F *of* P *is minimal if and only if it is an affine subspace; it is then a principal homogeneous space under the lineality space of* P*.*

In particular, if linsp(P) = 0, this proves that the minimal faces of the polyhedron P are its vertices. Conversely, if P has vertices, then they are minimal faces, hence linsp(P) = 0.

Proof. — Let F be a face of P. Since a face of F is a face of P, F is a minimal face of P if and only if it has no facets, that is, if and only if it is an affine subspace of V. Assume that it is the case.

Let $(f_j)_{j\in J}$ be a finite family of affine forms that defines P; we may assume that it is minimal so that no inequality is redundant in this system. For every $j \in J$, write $f_j = \varphi_j - b_j$, where φ_j is a linear form on V and $b_j \in R$. By proposition 1.3.11, linsp(P) is the set of all $x \in V$ such that $\varphi_j(x) = 0$ for all j.

Let I be a minimal subset of J such that $F = \{x \in P\,;\, \forall i \in I,\ f_i(x) = 0\}$. Then F is defined by the system of inequalities given by the union of the families $(-f_i)_{i\in I}$ and $(f_j)_{j\in J}$. These forms are obviously bounded from above on F. Since, by assumption, F has no facets, each of them is constant on F, hence F is a translate of linsp(P). □

Corollary **(1.5.6)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V*. Let* $\mathscr{F}_*(P)$ *be the set of faces of* P *to which we add the empty set; ordered by inclusion, this is a catenary lattice.*

Recall that a *lattice* is an ordered set of which every finite subset has a least upper bound (*join*) and a greatest lower bound (*meet*); a lattice is *catenary* if and only if all maximal totally ordered subsets have the same cardinality.

Proof. — The ordered set $\mathscr{F}_*(P)$ as a unique minimal element, $\varnothing$, and a unique maximal element, P. If nonempty, then the intersection of two faces is again a face, and this face is the largest face of P which is contained in both of them. By induction, the intersection of any subset of $\mathscr{F}_*(P)$ belongs to $\mathscr{F}_*(P)$, and it is their greatest lower bound. For $F, F' \in \mathscr{F}_*(P)$, the intersection of all $G \in \mathscr{F}_*(P)$ such that $F \subset G$ and $F' \subset G$, belongs to $\mathscr{F}_*(P)$, and is their least upper bound. This proves that $\mathscr{F}_*(P)$ is a lattice.

Let us consider a maximal totally ordered subset in $\mathscr{F}_*(P)$. It can be written on the form $\{F_0, \dots, F_m\}$, where $F_0, \dots, F_m$ are either faces of P or the empty set such that $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m$. Since it is maximal, $F_0 = \varnothing$ and F_1 is a minimal face of P, so that $\dim(F_1) = \dim(\operatorname{linsp}(P))$. Moreover, by maximality, for every $j \in \{1, \dots, m\}$, F_{j-1} is a facet of F_j, so that $\dim(F_j) = \dim(F_{j-1}) + 1$. Still by maximality, one has $F_m = P$. Finally, $\dim(P) = \dim(F_1) + (m-1) = \dim(\operatorname{linsp}(P)) + (m-1)$, so that $m = 1 + \dim(P) - \dim(\operatorname{linsp}(P))$. This concludes the proof. ☐

Remark **(1.5.7)**. — Let C be a polyhedral convex cone in a finite dimensional R-vector space V. Observe that every face of C is a polyhedral convex cone. Indeed, since C is a *cone*, and not only a polyhedron, there exists a finite family $(\varphi_j)_{j\in J}$ of *linear* forms defining C, that is, such that C is the set of $x \in V$ such that $\varphi_j(x) \leqslant 0$ for all j. Let F be a face of C; there exists a subset I of J such that an element $x \in P$ belongs to F if and only if $\varphi_i(x) = 0$ for all $i \in I$. Thus F is defined by the inequalities $\varphi_j(x) \leqslant 0$ (for $j \in J$) and $-\varphi_i(x) \leqslant 0$ (for $i \in I$); this proves that F is a polyhedral convex cone.

In particular, the lineality space linsp(C) of C is the unique minimal face of C.

1.6. Vertices, extremal rays

In this section, we show how a polyhedron of trivial lineality space can be reconstructed from its vertices and its faces of dimension 1 (extremal rays). We start with the case of polytopes.

Theorem **(1.6.1)**. — *Let* V *be a finite dimensional* R*-vector space and let* P *be a polytope in* V.

a) *The set* S *of vertices of* P *is the smallest subset of* P *such that* P = conv(S)*;*

b) *For every face* F *of* P*, its vertices are vertices of* P*; in particular, there exists a subset* T *of* S *such that* F = conv(T).

Proof. — We may assume that $P \neq \varnothing$. We then note that the vertices of a face of P are faces of P of dimension 0, hence are are vertices of P.

Since $P \neq \varnothing$, the set of faces of P is not empty, so that P has minimal faces, which are points because P is a polytope, hence linsp(P) = 0.

Let us prove that P = conv(S). The inclusion $\operatorname{conv}(S) \subset P$ follows from the fact that P is convex; let us prove the converse assertion by induction on dim(P). If dim(P) = 0, then P is reduced to a point $\{x\}$, and $S = \{x\}$, so that the result holds. Since the vertices of a face F of P are themselves vertices of F, we may assume by induction that all faces of P distinct from P are of the form conv(T), for some subset T of S. Let then $x \in P$ and let us prove that $x \in \operatorname{conv}(S)$. If $x \in S$, then we are done. Otherwise, choose $v \in S$ and consider the set L of all $t \in \mathbf{R}$ such that $t \geqslant 0$ and $x + t(x - v) \in P$. Let $(f_j)_{j\in J}$ be a minimal family of affine forms that defines P and let I be the set of all $j \in J$ such that $f_j(v) < f_j(x)$. For $t \in \mathbf{R}$, one has $f_j(x + t(x - v)) = f_j(x) + t(f_j(x) - f_j(v))$, so that $t \in L$ if and only

$$0 \leqslant t \leqslant \inf_{j\in I} \frac{-f_j(x)}{f_j(x) - f_j(v))}.$$

Since P is a polytope, the set L is bounded polyhedron of $\mathbf{R}_+$, hence one has $L \neq [0; +\infty[$; in particular, $I \neq \varnothing$. Let $a = \inf_{j\in I}(-f_j(x)/(f_j(x) - f_j(v)))$ and let $i \in I$ be such that $a = -f_i(x)/(f_i(x) - f_i(v))$. Since $f_i(x) \neq f_i(v)$, the affine form $f_i(x)$ is not constant on P, hence it does not give rise to an implicit equality in the given system that defines P, and the point $x + a(x - v)$ belongs to the facet of P defined by f_i. By induction, this point belongs to conv(S). Then the relation

$$x = \frac{1}{1+a}(x + a(x - v)) + \frac{a}{1+a} v$$

proves that $x \in \operatorname{conv}(S)$ as well.

Let T be a subset of P such that $P \subset \operatorname{conv}(T)$. Let $v \in S$ and let us show that $v \in T$; we argue by contradiction. Since $\{v\}$ is a face of P, there exists a linear form φ on V such that $\varphi(x) < \varphi(v)$ for every $x \in P - \{v\}$. Write $v = \sum_{x\in T} a_x x$ for some family $(a_x)_{x\in T}$ with finite support such that $a_x \geqslant 0$ for all x, and $\sum_{x\in T} a_x = 1$. If $v \notin T$, then $\varphi(x) < \varphi(v)$ for all $x \in T$, and there exists $x \in T$ such that $a_x > 0$ since $\sum_{x\in T} a_x = 1$; we then have $\varphi(v) = \sum_{x\in T} a_x\varphi(x) < \varphi(v)$, a contradiction which proves that $v \in T$. □

Example **(1.6.2)** (Birkhoff, von Neumann). — Doubly stochastic matrices $A \in M_n(\mathbf{R})$ are matrices with positive coefficients and such that the sums of each row and of each column are equal to 1. The subset P of $\mathbf{R}^{n^2}$ consisting of doubly stochastic matrices is defined by the system

$$\begin{cases} \sum_{j=1}^n a_{ij} = 1 & (1 \leqslant j \leqslant n) \\ \sum_{i=1}^n a_{ij} = 1 & (1 \leqslant i \leqslant n) \\ a_{ij} \geqslant 0 & (1 \leqslant i, j \leqslant n), \end{cases}$$

hence is a polyhedron. Since it is contained in $[0;1]^{n^2}$, it is bounded, hence is a polytope.

The $2n$ equalities of the given system define the affine span of P, because none of the remaining inequalities ($a_{ij} \geqslant 0$) give rise to implicit equalities: indeed, the matrix all of which coefficients are $1/n$ belongs to P. These $2n$ equalities are not linearly independent, the sum of the first n is equal to the sum of the remaining ones, but their rank is $2n-1$: the elements $a_{i,j}$ for $1 \leqslant i, j \leqslant n-1$ can be chosen at will, the given equations then furnish $a_{i,n}$ (for $1 \leqslant i \leqslant n-1$) and $a_{n,j}$ (for $1 \leqslant j \leqslant n$). In particular, this polytope P has dimension $n^2 - (2n-1) = (n-1)^2$.

Let us prove that the vertices of P are the permutation matrices, that is, the matrices of the form $A_\sigma = (a_{i,j})$ where $a_{i,j} = 1$ if $j = \sigma(i)$, and $a_{i,j} = 0$ otherwise, for some permutation $\sigma \in \mathfrak{S}_n$.

Let $\sigma \in \mathfrak{S}_n$ and let φ be the linear form on $\mathbf{R}^{n^2}$ defined by $\varphi(a_{ij}) = \sum_{i=1}^n a_{i,\sigma(i)}$. For every $A \in P$, one has $\varphi(A) \leqslant n$, and $\varphi(A) = n$ if and only if $A = A_\sigma$ is the permutation matrix defined above. This proves that the permutation matrices are vertices of P.

We prove the converse inclusion by induction on n. Let v be a vertex of P. As a zero-dimensional face of P, it must be defined by n^2 linearly independent linear forms from the systems. Since the first $2n$ equalities are not independent have rank $2n-1$, at least n^2-2n+1 of the remaining inequalities $a_{i,j} \geqslant 0$ are equalities at v: there exist n^2-2n+1 pairs (i,j) such that $a_{ij} = 0$ on v or, equivalently, at most $2n-1$ coefficients of v are nonzero. Since the sum of the coefficients of every row is 1, every row has a nonzero coefficients; if all rows had at least two nonzero coefficients, this would make $2n$ nonzero coefficients, so that at least one row has only one nonzero coefficient, which is equal to 1. Then all other coefficients of the column of that coefficient, being positive and of sum zero, are equal to 0. By renumbering the system, we assume that $a_{n,n} = 1$ and $a_{i,n} = a_{n,i} = 0$ for $i \in \{1,\dots,n-1\}$. Moreover, $\mathrm{P}' = \mathrm{P} \cap \{a_{n,n} = 1\}$ is a face of P (defined by the linear form $a_{n,n}$) which is identified with the set of doubly stochastic matrices of size $n-1$. By induction, the vertex v (viewed in $\mathrm{R}^{(n-1)^2}$) is associated with the permutation matrix corresponding to a permutation $\sigma' \in \mathfrak{S}_{n-1}$; then v is the permutation matrix associated with the permutation $\sigma \in \mathfrak{S}_n$ that extends σ' (and fixes n).

1.6.3. — Let P be a nonempty polyhedron in a finite dimensional R-vector space V. Assume for the moment that linsp(P) = $\{0\}$, the general case will be explained later.

In this case, the minimal faces of P are its vertices, and their dimension is 0. Let F be a face of P such that dim(P) = 1. Then the affine span $\langle \mathrm{F} \rangle$ is an affine line; let $\varphi \colon \mathrm{R} \to \langle \mathrm{F} \rangle$ be an affine bijection. Then $\varphi^{-1}(\mathrm{F})$ is a nonempty polyhedron in R, distinct from R because then F would be an affine subspace, hence have no facet, thus contradicting the hypothesis that minimal faces of P have dimension 0.

Consequently, two possibilities remain:

– Either there exists $a \in \mathrm{R}$ such that $\varphi^{-1}(\mathrm{F}) = [a;+\infty[$ or $\varphi^{-1}(\mathrm{F}) =]-\infty;a]$: the face F is a half-line, and we say that it is an *extremal ray* of P;
– Or there are $a, b \in \mathrm{R}$ such that $a < b$ and $\varphi^{-1}(\mathrm{F}) = [a;b]$: the face F is a segment; we say that it is an *edge* of P.

The first case corresponds to the case where F is unbounded, and the second one to the case where F is bounded.

Proposition **(1.6.4)**. — *Let* C *be a cone in* V *such that* linsp(C) = 0 *and let* S *be the set of all faces* F *of* C *such that* dim(F) = 1.

a) *For every face* $F \in S$ *and every vector* $v_F \in F - \{0\}$, *one has* $F = R_+ v_F$;
b) *Choose vectors as above; one has* $C = \operatorname{cone}((v_F)_{F\in S})$.
c) *Let* T *be a set of vectors such that* $C = \operatorname{cone}(T)$; *then, for every face* $F \in S$, *there exists* $w \in T$ *and* $t > 0$ *such that* $v_F = tw$.

Proof. — *a*) Let $F \in S$ be a face of C such that $\dim(F) = 1$. Let f be a linear form defining F: it is bounded from above on P and $F = \{x \in P\,;\, f(x) = \sup_P(f)\}$. Since C is a cone, f is negative on C, so that $\sup_P(f) = 0$, hence F is itself a cone. Let F_0 be a facet of F; it is again a cone and $0 \in F_0$. One has $\dim(F_0) = 0$, hence $F_0 = \{0\}$.

By the description of extremal rays of polyhedra, one then has $F = R_+ v$ for every $v \in F - \{0\}$.

b) Let us choose such a family $(v_F)_{F\in S}$ and let $C' = \operatorname{cone}((v_F))$ be the polyhedral convex cone it generates; let us prove that $C = C'$. Since C is a convex cone, one has $C' \subset C$; we prove the other inclusion by induction on dim(C).

If $\dim(C) = 0$, then $C = \{0\}$ and the assertion is trivial (actually, S is empty and $C' = \{0\}$).

Assume that $\dim(C) > 0$. In this case, C has faces of dimension 1, so that $S \neq \varnothing$; fix $F \in S$.

Let G be a face of C which is distinct from C. Its faces of dimension 1 are faces of C; by induction, this proves that G is contained in C′. Let $x \in C$. The set of all $t \in R$ such that $t \geqslant 0$ and $x - tv_F \in C$ is a polyhedron in R of the form $[0; a]$, for some $a \in R$, because $v_F \neq 0$ and linsp(C) = 0. Necessarily, $x - av_F \in C$ belongs to a face of C; by induction, it belongs to C′. Then $x = av_F + (x - av_F) \in C'$, as was to be shown.

c) Let f be a linear form on V which defines F in C: since C is a cone, one has $f = 0$ on F and $f < 0$ on $C - F$. Write $v_F = \sum_{w\in T} \lambda_w w$, for a family (λ_w) of positive elements of T with finite support. One has $\sum \lambda_w f(w) = f(v_F) = 0$. All terms of the sum are negative, so that they all vanish. Since $v_F \neq 0$, there exists $w \in T$ such that $\lambda_w \neq 0$; then $f(w) = 0$, which implies that w is a positive multiple of v_F. □

Proposition **(1.6.5)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V*; assume that* $\operatorname{linsp}(P) = 0$*. Let* S *be the set of vertices of* P *and let* T *be the set of its extremal rays.*

a) *The set* S *is not empty.*

b) *Every extremal ray* F *of* P *contains a unique vertex* x_F *of* P*; choose a vector* $v_F \in V$ *such that* $F = x_F + R_+ v_F$.

c) *Every edge* F *of* P *is of the form* $[x_F; y_F]$*, for two distinct vertices* x_F, y_F *of* P.

d) *One has* $P = \operatorname{conv}(S) + \operatorname{cone}((v_F)_{F \in T})$.

e) *The recession cone of* P *is the polyhedral convex generated by the vectors* v_F*, for* $F \in T$.

Proof. — Since P is nonempty and $\operatorname{linsp}(P) = \{0\}$, its minimal faces are vertices; consequently, S is not empty.

Let F be face of P such that $\dim(F) = 1$. Since $\dim(F) = 1$ and $\operatorname{linsp}(P) = 0$, minimal faces of P are vertices so that F has a facet, which is necessarily reduced to a vertex $\{x_F\}$ of P. Since $\dim(\operatorname{affsp}(F)) = 1$, it follows that F is either of the form $x_F + \mathbf{R}_+ v_F$, for some nonzero vector $v_F \in V$ (and one can take $v_F = y_F - x_F$ for every $y_F \in F - \{x_F\}$), or of the form $[x_F; y_F]$. In the latter case, y_F is vertex of F, hence it is a vertex of P. This proves the description of *a*) and *c*).

Let $F \in T$ be an extremal ray of P. For every $t \in R_+$, one has $x_F + t v_F \in F$, hence $x_F + t v_F \in P$. Consequently, $v_F \in \operatorname{recc}(P)$ and $\operatorname{cone}((v_F)_{F \in T}) \subset \operatorname{recc}(P)$.

Let us prove that $P = \operatorname{conv}(S) + \operatorname{cone}((v_F)_{F \in T})$ by induction on $\dim(P)$. The inclusion $\operatorname{conv}(S) + \operatorname{cone}((v_F)_{F \in T}) \subset P$ follows from the convexity of P and the fact that its recession cone contains the vectors v_F. Let $x \in P$. If x belongs to a face of P, then it belongs to the desired set, by the induction hypothesis. Otherwise, x belongs to the relative interior of P.

If $\operatorname{relint}(P) = \{x\}$, then $P = \{x\}$, $S = \{x\}$ and $T = \varnothing$, in which case the proposition is clear. Otherwise, let y be another point of the relative interior of P and let $v = y - x$. Since $\mathrm{linsp}(P) = \{0\}$, the line (xy) meets P in a polyhedron which is either a half-line, or a segment.

Assume $T \neq \varnothing$ and let $F \in T$. Since $v_F \in \mathrm{recc}(P)$ and $\mathrm{linsp}(P) = \{0\}$, one has $v_F \notin \mathrm{recc}(P)$ and the trace of P on the line $x + \mathbf{R}v_F$ is a polyhedron of the form $x - av_F + \mathbf{R}_+ v_F$, for some $a \in \mathbf{R}_+$ with $a > 0$. Necessarily, $x - av_F$ belongs to some face of P hence, by induction, belongs to $\mathrm{conv}(S) + \mathrm{cone}((v_F)_{F\in T})$. Then $x = (x - av_F) + av_F$ belongs to $\mathrm{conv}(S) + \mathrm{cone}((v_F)_{F\in T})$ as well.

If T is empty, all 1-dimensional faces of P are bounded and the following lemma asserts that P is a polytope. Then $\mathrm{recc}(P) = 0$, and $P = \mathrm{conv}(S)$ by theorem 1.6.1. This proves *d*), and proposition 1.3.9 then implies *e*), which concludes the proof of the proposition. □

Lemma **(1.6.6)**. — *Let* P *be a nonempty polyhedron of a finite dimensional vector space such that* $\mathrm{linsp}(P) = \{0\}$. *Then* P *is a polytope if and only if all 1-dimensional faces of* P *are bounded;*

Proof. — One implication is obvious, so let us assume that all 1-dimensional faces of P are bounded. To prove that P is a polytope, it suffices, by corollary 1.3.7, to prove that P is bounded, and we argue by induction on the dimension of P. We may assume that $\mathrm{affsp}(P) = V$ and that $\dim(P) \geqslant 2$. Let $C = \mathrm{recc}(P)$ be the recession cone of P; the origin is a face of C and there exists a non-zero linear form f on V which is strictly positive on $C \setminus \{0\}$. Let $x \in P$ and let v be nonzero vector in $\mathrm{Ker}(f)$; in particular, $v \notin C$ and $-v \notin C$. The intersection of the line $x + \mathbf{R}v$ with P is thus a bounded polyhedron, of the form $[x'; x'']$. The points x' and x'' belong to lower dimensional faces F' and F'' of P. By induction, F' and F'' are bounded, and the point x belongs to their convex hull. Since the set of faces of P is finite, this proves that P is bounded, as claimed. □

1.7. Rational polyhedra

1.7.1. — Let Q be a subfield of R and let Γ be a Q-vector subspace of R.

A linear form on $\mathbf{R}^n$ is said to be Q-rational if it is of the form $x \mapsto a_1 x_1 + \cdots + a_n x_n$ for some $a_1, \ldots, a_n \in Q$.

An affine form on R^n is said to be (Q,Γ)-rational if its linear part is Q-rational and its constant term belongs to Γ.

If Q and Γ are clear from the context, we also use the simpler expression "rational".

Definition **(1.7.2)**. — *A polyhedron* P *in* R^n *is said to be* (Q,Γ)-rational *(or, simply rational) if it can be defined by affine forms which are* (Q,Γ)*-rational.*

Example **(1.7.3)**. — a) Let V be a vector subspace of R^n.

Assume that V is Q-rational and let us choose a minimal family $(f_1,\dots,f_t)$ of rational linear forms that define V in R^n. They are Q-linearly independent, hence the Q-subspace of Q^n defined by these linear forms is a vector subspace of dimension $n-t$; let us choose a basis $(v_1,\dots,v_{n-t})$ in Q^n. Then $(v_1,\dots,v_{n-t})$ is a family of R-linearly independent vectors of V, so that $\dim(\mathrm{V}) \geqslant n-t$. Since V is defined by t linear forms, one has $\dim(\mathrm{V}) \leqslant n-t$, hence equality. Consequently, V has a basis consisting of vectors in Q^n.

Conversely, let $(v_1,\dots,v_{n-t})$ be a basis of V consisting of vectors in Q^n. Let $(f_1,\dots,f_t)$ be independent linear forms on Q^n which vanish on them. Since $(f_1,\dots,f_t)$ are still R-linearly independent, the vector subspace of R^n that they define has dimension $(n-t)$. Since it contains $v_1,\dots,v_{n-t}$, it is equal to V. This proves that V is Q-rational.

b) Let V be an affine subspace of R^n.

Assume that V is (Q,Γ)-rational. Moreover, it follows from gaussian elimination that the a system of (Q,Γ)-rational affine equations has a solution v in Γ^n, if it has any solution at all. Then $\mathrm{V}-v$ is a rational vector subspace of R^n.

Conversely, assume that there exists $v \in \Gamma^n$ such that $\mathrm{V}-v$ is a rational vector subspace of R^n. Let $(\varphi_1,\dots,\varphi_t)$ be a family of rational linear forms on R^n which define $\mathrm{V}-v$. Since the affine forms $\varphi_j(v)+\varphi_j$, for $j \in \{1,\dots,t\}$, are (Q,Γ)-rational and define V, we conclude that V is (Q,Γ)-rational.

Proposition **(1.7.4)**. — *Let* P *be a nonempty rational polyhedron.*

a) *All faces of* P *are rational polyhedra.*

b) *The recession cone of* P, recc(P)*, its lineality space* linsp(P)*, the affine subspace it spans,* affsp(P)*, are rational.*

Proof. — Let $(f_j)_{j\in J}$ be a finite family of rational affine forms defining P; write $f_j = \varphi_j + b_j$, where φ_j is a Q-rational linear form and $b_j \in \Gamma$. Each face of P is defined by adding inequalities of the form $-f_j \leqslant 0$, for j in some subset I of J; this shows that they are rational. Similarly, the recession cone of P is defined by the linear inequalities $\varphi_j \leqslant 0$; it is thus Q-rational. The lineality space is defined by the equalities $\varphi_j = 0$, for $j \in J$, hence is Q-rational. Finally, since the affine span of P is defined by all the implicit equalities $f_j = 0$ in the given system, it is (Q, Γ)-rational as well. ☐

Example **(1.7.5)**. — Let C be a Q-rational cone.

Assume that $\dim(C) = 1$. In this case, $\operatorname{affsp}(C) = C - C$. Since affsp(C) is a Q-rational line, there exists $v \in Q^n$ such that $\operatorname{affsp}(C) = Rv$. Up to replacing v by $-v$, one then has $C = R_+ v$.

In the general case, the extremal rays of C are themselves Q-rational cones, hence of the form $R_+ v$ for some $v \in Q^n$. Given proposition 1.6.4, this implies that C is the polyhedral convex cone generated by a finite family of vectors in Q^n.

1.8. Polyhedral subspaces, fans

Definition **(1.8.1)**. — *Let* V *be a finite dimensional* R*-vector space and let* S *be a subspace of* V.

a) *One says that* S *is a* polyhedral subspace *of* V *if it is a finite union of polyhedra in* V;

b) *One says that a map f from* S *to a finite dimensional* R*-vector space* V′ *is* piecewise affine *if there is a finite family $\mathscr{C}$ of polyhedra of* V *of which* S *is the union and such that for every* $C \in \mathscr{C}$*, there exists an affine map* $f_C \colon V \to V'$ *which coincides with f on* C.

1.8.2. — Let S be a polyhedral subspace. Let $\mathscr{C}$ be a finite set of polyhedra of which S is the union. The supremum $\sup_{C\in\mathscr{C}} \dim(C)$ does not depend of the choice of the set $\mathscr{C}$ such that $\mathscr{C} = \bigcup_{C\in\mathscr{C}}$;

it is called the *dimension* of S, and is denoted by dim(S). Let indeed $\mathscr{C}'$ be another finite set of polyhedra such that $S = \bigcup_{C\in\mathscr{C}'} C$. If $\mathscr{C} = \varnothing$, then $S = \varnothing$ and $\mathscr{C}' = \varnothing$ as well; in this case, the supremum is $-\infty$. Let $C \in \mathscr{C}$. Then $C = \bigcup_{D\in\mathscr{C}'}(C \cap D)$ writes C as a union of polyhedra of dimension $\leqslant \dim(D)$ contained in affsp(C). Necessarily, one of them has nonempty interior in affsp(C), which implies that there exists $D \in \mathscr{C}'$ such that $\dim(C \cap D) = \dim(C)$. Then $\dim(C) \leqslant \dim(D)$ and $\dim(C) \leqslant \sup_{D\in\mathscr{C}'} \dim(D)$. Consequently, $\sup_{C\in\mathscr{C}} \dim(C) \leqslant \sup_{D\in\mathscr{C}'} \dim(D)$, and the other inequality follows by symmetry.

Lemma **(1.8.3)**. — *Let* V, V′ *be finite dimensional* R*-vector spaces. Let* S *be a polyhedral subspace of* V*, let* S′ *be a polyhedral subspace of* V′ *and let* $f\colon S \to V$ *be a piecewise affine map.*

a) *The intersection, the union of two polyhedral subspaces of* V *is a polyhedral subspace. One has* $\dim(S \cup S') = \sup(\dim(S), \dim(S'))$ *and* $\dim(S \cap S') \leqslant \inf(\dim(S), \dim(S'))$.

b) *The image* $f(S)$ *is a polyhedral subspace of* V′ *such that* $\dim(f(S)) \leqslant \dim(S)$.

c) *The preimage* $f^{-1}(S')$ *is a polyhedral subspace of* V.

Proof. — These assertions follow immediately from the analogous assertions for polyhedra and affine maps. □

1.8.4. — Let $f, g\colon S \to R$ be piecewise affine maps; then $f + g$, cf, $\inf(f, g)$ and $\sup(f, g)$ are piecewise affine.

Let $f\colon S \to R$ be a map. Then f is piecewise affine if and only if its graph Γ_f is a polyhedral subspace of $S \times R$, if and only if its epigraph Γ_f^+ (the set of all (x, t) such that $t \geqslant f(x)$) is a polyhedral subspace of $S \times R$.

If f is piecewise affine and bounded from the above, then its maximal level set $\{x \in S\,;\, f(x) = \sup(f)\}$ is a polyhedral subset of S.

Definition **(1.8.5)**. — *Let* V *be a finite dimensional* R*-vector space and let* S *be a subspace of* V*. A* polyhedral decomposition *of* S *is a finite set* $\mathscr{C}$ *of polyhedra satisfying the following properties:*

a) *The union of all polyhedra in* $\mathscr{C}$ *is equal to* S*;*

b) *Every face of a polyhedron in* $\mathscr{C}$ *belongs to* $\mathscr{C}$*;*

c) *The intersection of every two polyhedra* P, Q *in* $\mathscr{C}$ *is either empty, or a face of both of them.*

The set S *is also called the support of the polyhedral decomposition* $\mathscr{C}$*, and is denoted by* $|\mathscr{C}|$*.*

A fan *is a polyhedral decomposition all of which polyhedra are cones.*

1.8.6. — One defines analogously the notions of a rational polyhedral subspace, of a rational piecewise affine function on such a polyhedral subspace, or of a rational polyhedral decomposition of a rational polyhedral subspace.

Remark **(1.8.7)**. — a) If a finite union of polyhedral cones is a convex cone, then it is a polyhedral cone. In other words, a convex cone is a polyhedral subset if and only if it is a polyhedral cone, so that the terminology is not ambiguous.

b) A polyhedral decomposition is determined by its maximal polyhedra, all other are faces of them. Since a face of a cone is a cone, a polyhedral decomposition is a fan if and only if its maximal polyhedra are cones.

c) Let $\mathscr{C}$ be a polyhedral decomposition of a polyhedral subspace S. For every $x \in S$ and every polyhedron $P \in \mathscr{C}$ such that $x \in P$, either x belongs to a facet of P, or x belongs to the relative interior of P, bot not simultaneously. Consequently, the relative interiors of the polyhedra in $\mathscr{C}$ are pairwise disjoint, and their union is S.

1.8.8. — Let S be a polyhedral subspace of V and let $\mathscr{C}, \mathscr{C}'$ be polyhedral decompositions of S. One says that $\mathscr{C}$ is finer than $\mathscr{C}'$ if every polyhedron in $\mathscr{C}'$ is the union of some polyhedra in $\mathscr{C}$. Equivalently, for every point $x \in S$ and every polyhedron $P' \in \mathscr{C}'$ such that $x \in P'$, there exists a polyhedron $P \in \mathscr{C}$ such that $x \in P$ and $P \subset P'$.

Note that classical presentations of polyhedral subspaces start from polyhedral decompositions and defines *polyhedral complexes* as polyhedral decompositions up to refinement.

Proposition **(1.8.9)**. — a) *For every finite set* $\mathscr{S}$ *of polyhedra in* V*, there exists a polyhedral decomposition* $\mathscr{C}$ *of* V *such that every polyhedron* $P \in \mathscr{S}$ *is a union of polyhedra in* $\mathscr{C}$*.*

b) *Every polyhedral subspace has a polyhedral decomposition.*

c) *Moreover, two polyhedral decompositions of a polyhedral subspace admit a common refinement.*

Proof. — a) Write S as a finite union of polyhedra $(\mathrm{P}_i)_{i\in\mathrm{I}}$ and, for each of them, let $(f_{i,j})_{j\in\mathrm{J}_i}$ be a finite family of affine forms that defines it. Let J be the disjoint union of the sets J_i, that is, the set of pairs (i,j) with $i \in \mathrm{I}$ and $j \in \mathrm{J}_i$. For every family $\varepsilon = (\varepsilon_{i,j}) \in \{=, \leqslant, \geqslant\}^{\mathrm{J}}$, let P_ε be the set of all points $x \in \mathrm{V}$ such that $f_{i,j}(x)\ \varepsilon_{i,j}\ 0$ for all $(i,j) \in \mathrm{J}$; it is a polyhedron in V. Moreover, the family (P_ε) is stable under taking faces and intersections, its union is equal to V. For every i, the polyhedron P_i is the union of the polyhedra P_ε for all ε such that $\varepsilon_{i,j} = \leqslant$ for all $j \in \mathrm{J}_i$. Consequently, S is the union of a subfamily of the family (P_ε). This proves that the set of polyhedra of the form P_ε is a polyhedral decomposition of S.

b) Let S be a polyhedral subspace of V. Let $(\mathrm{P}_i)_{i\in\mathrm{I}}$ be finite family of polyhedra such that $\mathrm{S} = \bigcup_{i\in\mathrm{I}} \mathrm{P}_i$. Apply *a*) to the family (P_i); we obtain a polyhedral decomposition $\mathscr{C}$ of V such that every polyhedron P_i is a union of polyhedra in $\mathscr{C}$. This implies that S is a union of polyhedra in $\mathscr{C}$, as claimed.

c) Let $\mathscr{C}$ and $\mathscr{C}'$ be polyhedral decompositions of S. The polyhedra $\mathrm{P} \cap \mathrm{P}'$, for $\mathrm{P} \in \mathscr{C}$ and $\mathrm{P}' \in \mathscr{C}'$ cover S. By *b*), there exists a polyhedral decomposition $\mathscr{C}''$ of S such that every polyhedron $\mathrm{P} \cap \mathrm{P}'$ is a union of some polyhedra in $\mathscr{C}''$. This polyhedral decomposition is finer than $\mathscr{C}$ and $\mathscr{C}'$.[1] □

Corollary **(1.8.10)**. — *Let $f : \mathrm{V} \to \mathrm{V}'$ be a linear map between finite dimensional* **R***-vector spaces. Let $\mathscr{S}$ be a finite set of polyhedra in* V *and let $\mathscr{S}'_1$ be a finite set of polyhedra in* V′. *There exists a polyhedral decomposition $\mathscr{C}$ of* V *and a polyhedral decomposition $\mathscr{C}'$ of* V′ *satisfying the following properties:*

(i) *Every polyhedron* $\mathrm{S} \in \mathscr{S}$ *is a union of polyhedra that belong to* $\mathscr{C}$;

(ii) *Every polyhedron* $\mathrm{S}' \in \mathscr{S}'$ *is a union of polyhedra that belong to* $\mathscr{C}'$;

(iii) *For every* $\mathrm{C} \in \mathscr{C}$, $f(\mathrm{C})$ *belongs to* $\mathscr{C}'$;

(iv) *For every* $\mathrm{C}' \in \mathscr{C}'$, $f^{-1}(\mathrm{C}')$ *is a union of polyhedra belonging to* $\mathscr{C}$.

[1] It is plausible that the $\mathrm{P} \cap \mathrm{P}'$ already form a polyhedral decomposition of S.

Proof. — We start from a polyhedral decomposition $\mathscr{C}_1$ of V such that every polyhedron $S \in \mathscr{S}$. is a union of polyhedra in $\mathscr{C}_1$. Let then $\mathscr{C}'$ be a polyhedral decomposition of V′ such that every polyhedron of the form $f(C)$, for $C \in \mathscr{C}_1$, the subspace $f(V)$, and every polyhedron $S' \in \mathscr{S}'$ are union of polyhedra of $\mathscr{C}'$.

Let $C \in \mathscr{C}_1$ and $C' \in \mathscr{C}'$, let $S = C \cap f^{-1}(C')$; then $f(S) = f(C) \cap C'$ is a union of polyhedra in $\mathscr{C}'$ which are faces of C′. Since it is convex, it is a face of C′; replacing C′ by $f(S)$, we may assume that $f(S) = C'$.

Let us show that the set $\mathscr{C}$ of all non-empty polyhedra of V of the form $C \cap f^{-1}(C')$, for $C \in \mathscr{C}_1$ and $C' \in \mathscr{C}'$ such that $f(C) \supset C'$, is a polyhedral decomposition of V. It covers V; moreover, the intersection of two of its members is either empty, or a an element of $\mathscr{C}$. Let us consider two members of $\mathscr{C}$, say $S = C \cap f^{-1}(C')$ and $T = D \cap f^{-1}(D')$ such that $T \subset S$; let us prove that T is a face of S. As above, we may assume $f(S) = C'$ and $f(T) = D'$. One thus has $D' \subset C'$, hence D′ is a face of C′, and $T = D \cap f^{-1}(D')$ is a face of $D \cap f^{-1}(C')$. We are reduced to proving the assertion when $D' = C'$. Since $\mathscr{C}_1$ is a polyhedral decomposition, $C \cap D$ is a face of C and of D, and it belongs to $\mathscr{C}_1$. Then $T = S \cap T = (C \cap D) \cap f^{-1}(C')$ is a face of $S = C \cap f^{-1}(C')$.

The polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ satisfy properties (i) and (ii) by construction. Let $S \in \mathscr{C}$; as we have seen, $f(S) \in \mathscr{C}'$; this proves (iii). Finally, let $C' \in \mathscr{C}'$; writing $V = \bigcup_{C \in \mathscr{C}_1} C$, we have $f^{-1}(C') = \bigcup_{C \in \mathscr{C}_1} (C \cap f^{-1}(C'))$, which shows that property (iv) holds as well. □

Corollary **(1.8.11)**. — *Let* S *be a polyhedral subspace of* V *and let* $f \colon S \to V'$ *be a piecewise affine map. Then* $f(S)$ *is a polyhedral subspace of* V′, *and there exist polyhedral decompositions* $\mathscr{C}$ *of* S *and* $\mathscr{C}'$ *of* S′ *such that the following properties hold:*

a) *For every* $C' \in \mathscr{C}'$, *the set of all* $C \in \mathscr{C}$ *such that* $f(C) \subset C'$ *is a polyhedral decomposition of* $f^{-1}(C')$;

b) *For every* $C \in \mathscr{C}$, *the map* $f|_C$ *is the restriction of an affine map from* V *to* V′.

1.9. Regular polyhedral decompositions and the Legendre transform

Proposition **(1.9.1)**. — *Let* P *be a polyhedron of dimension* d *in a finite dimensional* R*-vector space* V *and let* $f : \mathrm{P} \to \mathrm{R}$ *be a piecewise affine* convex *function. There exists a unique polyhedral decomposition* $\mathscr{C}$ *of* P *satisfying the following properties:*

(i) *The restriction of* f *to any polyhedron* $\mathrm{C} \in \mathscr{C}$ *coincides with an affine function* f_{C} *on* V*;*

(ii) *If* C, D *are distinct polyhedra of dimension* d *of* $\mathscr{C}$*, then* $f_{\mathrm{D}} < f_{\mathrm{C}}$ *on* $\mathring{\mathrm{C}}$*.*

Moreover, one has the following properties:

(iii) *One has* $f = \sup_{\mathrm{C}\in\mathscr{C}} f_{\mathrm{C}}$*;*

(iv) *All maximal polyhedra of* $\mathscr{C}$ *have dimension* d*;*

(v) *The union of the interiors of the polyhedra of* $\mathscr{C}$ *is the differentiability locus of* f*.*

This polyhedral decomposition is called the *regular* polyhedral decomposition of P associated with f.

Proof. — Replacing the ambient vector space by the affine span of P, we assume that P has nonempty interior in R^n. Let U be the set of $x \in \mathrm{P}$ such that f is affine in a neighborhood of x. Let $\mathscr{D}$ be a finite polyhedral decomposition of P such that $f|_{\mathrm{D}}$ is affine, for every polyhedron $\mathrm{D} \in \mathscr{D}$. This open subset of P contains the union of the interiors of the maximal cells of $\mathscr{D}$; it follows that U is dense in P and that its set of connected components is finite.

On U, the function f is everywhere differentiable and its differential is locally constant; consequently, for every connected component E of U, there exists an affine function f_{E} on R^n such that $f|_{\mathrm{E}} = f_{\mathrm{E}}$. Since the graph of a convex function is always above its tangents, one has $f(x) \geqslant \sup_{\mathrm{E}} f_{\mathrm{E}}(x)$ for every $x \in \mathrm{P}$. This implies that for every connected component E of U and every $x \in \mathrm{E}$, one has $f_{\mathrm{E}}(x) = f(x) \geqslant \sup_{\mathrm{F}} f_{\mathrm{F}}(x)$. This proves the relation $f(x) = \sup_{\mathrm{E}} f_{\mathrm{E}}(x)$ on U, hence on P because U is dense in P.

Let E be a connected component of U. The function $f - f_{\mathrm{E}}$ is convex, positive, and vanishes on E. The set of points $x \in \mathrm{P}$ such that $f(x) = f_{\mathrm{E}}(x)$ is thus convex and contains $\overline{\mathrm{E}}$.

Let us prove that if E, F are two distinct connected components, then the linear parts of f_E and f_F are not equal; otherwise, $f_E - f_F$ would be constant. If it were strictly positive, then one would have $f_E(x) > f_F(x)$ for all $x \in F$, contradiction; if it were strictly negative, one would have a similar contradiction. So $f_E = f_F$. But then the convexity of f implies that f coincides with f_E on the convex hull of $E \cup F$, which is an open subset of R^n containing E and F. By definition of a connected component, this implies E = F.

Let E and F be two distinct connected components of U; let us prove that $f_F(x) < f_E(x)$ for every $x \in E$. Otherwise, one would have $f_F(x) = f_E(x)$ at some point $x \in E$. Then the affine half-space defined by $f_F > f_E$ contains x on its boundary; for x' close to x there, one has $f(x') \geqslant f_F(x') > f_E(x') = f(x')$, a contradiction.

It follows that $\overline{E}$ is defined in P by the inequalities $f_E(x) \geqslant f_F(x)$ for all connected components F of U. In particular, $\overline{E}$ is a polyhedron.

If E and F are distinct connected components, then $\overline{E} \cap \overline{F}$ is defined in P by the equality $f_E = f_F$ and the inequalities $f_E \geqslant f_G$ for all other components. If non-empty, this is the face of $\overline{E}$ associated with the affine form $f_F - f_E$ which is maximal there, and the face of $\overline{F}$ associated with the affine form $f_E - f_F$.

The desired polyhedral decomposition of P is thus the family of polyhedra $\overline{E}$ and their faces. □

1.9.2. — Let P be a nonempty polytope in a finite dimensional vector space V and let $f : P \to R$ be a convex piecewise affine function. The regular decomposition $\mathscr{C}$ of P associated with f admits an alternative, possibly more concrete, description.

Write $\langle \cdot, \cdot \rangle$ for the duality between V and its dual space V^* and let P^* be the subset of V^* consisting of all linear forms y such that $x \mapsto \langle x, y \rangle - f(x)$ is bounded from above on P. The *Legendre transform* is the function on P^* given by

$$f^*(y) = \sup_{x \in P} \langle x, y \rangle - f(x).$$

If one adds to f an affine form $x \mapsto \langle x, \eta\rangle + \beta$, this changes P^* to $\mathrm{P}^* + \eta$ and f^* to $y \mapsto f^*(y - \eta) - \beta$, as one sees on the relation

$$\langle x, y\rangle - \big(f(x) + \langle x, \eta\rangle + \beta\big) = \sup_{x\in\mathrm{P}} \langle x, y - \eta\rangle - f(x) - \beta.$$

Consider similarly the function $g : x \mapsto f(x-\xi)$ on $\mathrm{P}+\xi$. The relation

$$\langle x, y\rangle - g(x) = \langle x, y\rangle - f(x - \xi) = \big(\langle x - \xi, y\rangle - f(x - \xi)\big) + \langle \xi, y\rangle$$

shows that this does not change P^* but changes f^* to $y \mapsto f^*(y) + \langle \xi, y\rangle$.

Proposition **(1.9.3)** (Legendre duality). — *The function f^* on P^* is convex and piecewise affine. Moreover, one has $f^{**} = f$ on P.*

Proof. — For $\mathrm{C} \in \mathscr{C}$, let $y_{\mathrm{C}} \in \mathrm{V}^*$ and $a_{\mathrm{C}} \in \mathbf{R}$ be such that $f(x) = f_{\mathrm{C}}(x) = \langle x, y_{\mathrm{C}}\rangle - a_{\mathrm{C}}$ for $x \in \mathrm{C}$, so that

$$f(x) = \sup_{\mathrm{C}\in\mathscr{C}} \big(\langle x, y_{\mathrm{C}}\rangle - a_{\mathrm{C}}\big).$$

For $y \in \mathrm{V}^*$, the function $x \mapsto \langle x, y\rangle - f(x)$ is bounded from above on C if and only if $x \mapsto \langle x, y - y_{\mathrm{C}}\rangle$ is bounded from above on C, which means that $y - y_{\mathrm{C}} \in \operatorname{recc}(\mathrm{C})^\circ$. In this case, one has

$$\sup_{x\in\mathrm{C}} \langle x, y - y_{\mathrm{C}}\rangle + a_{\mathrm{C}} = \sup_{x\in\mathrm{V}(\mathrm{C})} (\langle x, y - y_{\mathrm{C}}\rangle + a_{\mathrm{C}}),$$

where $\mathrm{V}(\mathrm{C})$ is the set of vertices of C. Consequently, $\mathrm{P}^* = \bigcap_{\mathrm{C}\in\mathscr{C}}(y_{\mathrm{C}} + \operatorname{recc}(\mathrm{C})^\circ)$ and

$$f^*(y) = \sup_{\mathrm{C}\in\mathscr{C}} \sup_{x\in\mathrm{V}(\mathrm{C})} (\langle x, y - y_{\mathrm{C}}\rangle + a_{\mathrm{C}})$$

for all $y \in \mathrm{P}^*$. In particular, f^* is piecewise affine and convex.

For $x \in \mathrm{P}$ and $y \in \mathrm{P}^*$, one has $\langle x, y\rangle \leqslant f(x) + f^*(y)$. This implies that f^{**} is defined on P and that $f^{**}(x) \leqslant f(x)$ for all $x \in \mathrm{P}$.

To prove the equality, we may subtract from f any affine form that it takes on a maximal dimensional polyhedron $\mathrm{C} \in \mathscr{C}$. Indeed, both sides of the equality to prove are changed in the same way. This implies that $f \geqslant 0$ on P.

Let $\xi \in \mathrm{P}$. Then $\xi \in \mathrm{P}^{**}$ and $f^{**}(\xi) \leqslant f(\xi)$. Let $\beta \in \mathbf{R}$ be such that $\beta < f(\xi)$. Since the epigraph of f is a polyhedron which does not contain (ξ, β), there exists an affine form on $\mathrm{V} \times \mathbf{R}$, say $(x, t) \mapsto \langle x, \eta\rangle + at + c$ which is strictly negative at (ξ, β) but positive at all points

(x,t) such that $x \in \mathrm{P}$ and $t \geqslant f(x)$. Necessarily, $a > 0$. Dividing the given affine form by a, we assume that $a = 1$. The inequalities $\langle x, \eta\rangle + f(x) + c \geqslant 0$ for $x \in \mathrm{P}$ imply that $-\eta \in \mathrm{P}^*$ and that $f^*(-\eta) \leqslant c$. Then,

$$\sup_{y\in\mathrm{P}^*} \langle \xi, y\rangle - f^*(y) \geqslant \langle \xi, -\eta\rangle - f^*(-\eta) \geqslant -\langle \xi, \eta\rangle - c > \beta.$$

Consequently, $f^{**}(\xi) > \beta$. Since $\beta < f(\xi)$ is arbitrary, this implies that $f^{**}(\xi) = f(\xi)$.

Let now $\xi \in \mathrm{V} - \mathrm{P}$ and let us prove that $\xi \notin \mathrm{P}^{**}$. Similarly as above, there exists an affine form on V, say $(x,t) \mapsto \langle x, \eta\rangle + c$ which is strictly negative at ξ but positive on P. Let $t > 0$. Since $f \geqslant 0$, one has $\langle x, -\eta\rangle - tf(x) \leqslant c$ for all $x \in \mathrm{P}$, hence $-\eta/t \in \mathrm{P}^*$ and $f^*(-\eta/t) \leqslant c/t$. Then,

$$\sup_{y\in\mathrm{P}^*} \langle \xi, y\rangle - f^*(y) \geqslant \langle \xi, -\eta/t\rangle - f^*(-\eta/t) \geqslant -\frac{1}{t}(\langle \xi, \eta\rangle + c).$$

When t tends to 0, this implies that $\xi \notin \mathrm{P}^{**}$, as was to be shown. □

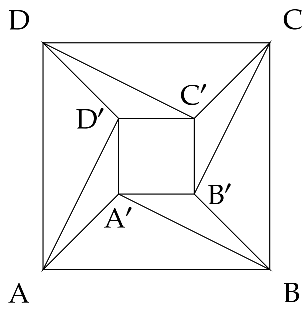
1.9.4. — Let Q be the epigraph of f, the set of all $(x,t) \in \mathrm{P} \times \mathrm{R}$ such that $t \geqslant f(x)$. Since f is convex and piecewise affine, Q is a convex polyhedral subset of $\mathrm{V} \times \mathrm{R}$, hence it is a polyhedron. Let G be a face of Q and F be its projection to V. The linear form defining G takes the form $(x,t) \mapsto \langle x, y\rangle + at$, for some $y \in \mathrm{V}^*$ and $a \in \mathrm{R}$; let b be its value on G. Since that linear form is bounded from above on P, one has $a \leqslant 0$. There are two cases.

Assume that $a < 0$. Dividing y and b by $-a$, we may assume that $a = -1$. Then we see that F is the set of all $x \in \mathrm{P}$ such that $\langle x, y\rangle - f(x) = f^*(y)$, and G is the set of all (x,t) for $x \in \mathrm{F}$ and $t = f(x)$. In particular, the restriction of f to F is affine, given by $x \mapsto \langle x, y\rangle - f^*(y)$. More precisely, F is the set of all $x \in \mathrm{P}$ such that $f(x) + f^*(y) = \langle x, y\rangle$. We say that G is a “horizontal face”, or a “lower face” of Q.

When E ranges over the regular polyhedral decomposition of P associated with f, the intersections $\mathrm{E} \cap \mathrm{F}$ are polyhedra which cover F. At least one of them, say E, has dimension $\dim(\mathrm{F})$, and then $f(x) = f_{\mathrm{E}}(x)$ on F, hence $\mathrm{F} \subset \mathrm{E}$. In fact, for all $x \in \mathrm{E} - \mathrm{F}$, one has $\langle x, y\rangle - f_{\mathrm{E}}(x) =$

$\langle x, y\rangle - f(x) < f^*(y)$, so that F is the face of E defined by the linear form $x \mapsto \langle x, y\rangle - f_{\mathrm{E}}(x)$.

Assume now that $a = 0$. Then the linear form y on V defines a face F of P, and G is the set of all $(x, t) \in \mathrm{F} \times \mathbf{R}$ such that $t \geqslant f(x)$. Since G is a polyhedron, the restriction of f to F is affine, and G is a "vertical face" of Q.



Remark **(1.9.5)**. — Here is a classical example of a polyhedral decomposition of a square which is *not* regular. To fix ideas assume that the vertices of the outer square are $\mathrm{A} = (0,0)$, $\mathrm{B} = (3,0)$, $\mathrm{C} = (3,3)$ and $\mathrm{D} = (0,3)$, while the vertices of the inner square are $\mathrm{A}' = (1,1)$, $\mathrm{B}' = (2,1)$, $\mathrm{C}' = (2,2)$ and $\mathrm{D}' = (2,2)$. Let f be convex piecewise affine function on the square whose loci of affinity are as depicted on the figure. Subtracting an affine function, we may assume that f is identically 0 on the inner square A′B′C′D′. Let a, b, c, d be the values of f at the corresponding vertices of the outer square, so that $a = f(0,0)$, $b = f(3,0)$,... On the triangle AA′D′, one has $f(x,y) = a(1-x)$, because $f(\mathrm{A}) = a$ while $f(\mathrm{A}') = f(\mathrm{D}') = 0$; similarly, on the triangle BA′B′, one has $f(x,y) = b(1-y)$. Since A does not belong to the triangle BA′B′, one has $a = f(\mathrm{A}) \geqslant b$. By symmetry, one has $b \geqslant c$, $c \geqslant d$ and $d \geqslant a$, hence the equality $a = b = c = d$. It follows that f is given by $f(x,y) = a(1-x)$ on the triangle ADD′, which implies that it is affine on the trapezoid AA′D′D, a contradiction.

1.9.6. — We will now show that the regular polyhedral decomposition $\mathscr{C}$ of P associated with a convex piecewise function f induces a "dual" (regular) polyhedral decomposition $\mathscr{C}^*$ of P^*. We first need a definition.

Definition **(1.9.7)**. — *Let* P *be a polyedron of* V *and let* P^* *be a polyhedron of* V^*. *Let* $\mathscr{C}$ *be a polyhedral decomposition of* P *and let* $\mathscr{C}^*$ *be a polyhedral decomposition of* P^*. *A* duality *is a bijection from* $\mathscr{C}$ *to* $\mathscr{C}'$, *denoted by* $\mathrm{C} \mapsto \mathrm{C}^*$, *satisfying the two properties:*

(i) *If* $\mathrm{C}, \mathrm{D} \in \mathscr{C}$, *then* $\mathrm{C} \subset \mathrm{D}$ *if and only if* $\mathrm{D}^* \subset \mathrm{C}^*$;

(ii) *For* $C, D \in \mathscr{C}$ *such that* $C \subset D$, *one has* $\mathrm{cone}(D - C)^{\circ} = \mathrm{cone}(C^* - D^*)$.

Taking $D = C$ in (ii), observe that $\mathrm{cone}(C - C)$ is the vector subspace of V directing $\mathrm{affsp}(C)$; its polar in V^* is the vector subspace directing $\mathrm{affsp}(C^*)$. In particular, these conditions imply that $\dim(C) + \dim(C^*) = \dim(V)$ for all $C \in \mathscr{C}$.

1.9.8. — Let P be a nonempty polyhedron of V and let $f : P \to \mathbf{R}$ be a convex piecewise affine function. Let $f^* : P^* \to \mathbf{R}$ be the Legendre transform of f.

For $x \in P$, define C^*_x as the subset of V^* consisting of all $y \in P^*$ such that $\langle x, y \rangle = f(x) + f^*(y)$. Similarly, for $y \in P^*$, let C_y be the set of all $x \in P$ such that $\langle x, y \rangle = f(x) + f^*(y)$. For fixed x (resp. for fixed y), the function $f(x) + f^*(y) - \langle x, y \rangle$ is convex, piecewise affine, positive, and vanishes at some point y (resp. x); consequently, its vanishing locus is a polyhedron, so that the sets C^*_x and C_y are polyhedra.

In fact, it follows from the description of the regular polyhedral decomposition of P associated with f done in §1.9.4 that it is the family of all C_y, for $y \in P^*$. Similarly, the family of all C^*_x, for $x \in P$, is the regular polyhedral decomposition $\mathscr{C}^*$ of P^* associated with f^*.

Proposition **(1.9.9).** — *For* $C \in \mathscr{C}$, *the intersection* C^* *of all* C^*_x, *for* $x \in C$, *is an element of* $\mathscr{C}^*$. *Moreover, the map* $C \mapsto C^*$ *from* $\mathscr{C}$ *to* $\mathscr{C}^*$ *is a duality of polyhedral decompositions.*

Proof. — For $x \in P$ and $y \in P^*$, set

$$P(x, y) = f(x) + f^*(y) - \langle x, y \rangle.$$

Let $x', x'' \in P$ be such that $C^*_{x'} \cap C^*_{x''}$ is nonempty; let x be any point on the open segment $]x'; x''[$ and let us prove that $C^*_{x'} \cap C^*_{x''} = C^*_x$. Let $y \in C^*_{x'} \cap C^*_{x''}$; by convexity and positivity of $P(\cdot, y)$, one has $y \in C^*_x$; this proves the inclusion $C^*_{x'} \cap C^*_{x''} \subset C^*_x$. Let then $z \in C^*_x$ and write

$x = (1-t)x' + tx''$. One has

$$\begin{aligned} 0 = \mathrm{P}(x,z) &= f(x) + f^*(z) - \langle x, z\rangle \\ &= \big(f(x) - (1-t)f(x') - tf(x'')\big) + (1-t)\left(f(x') + f^*(z) - \langle x', z\rangle\right) \\ &\qquad + t\left(f(x'') + f^*(z) - \langle x'', z\rangle\right) \\ &= \big(f(x) - (1-t)f(x') - tf(x'')\big) + (1-t)\mathrm{P}(x', z) + t\mathrm{P}(x'', z). \end{aligned}$$

Since these three terms are positive and $0 < t < 1$, this implies that $\mathrm{P}(x', z) = \mathrm{P}(x'', z) = 0$, hence $z \in \mathrm{C}^*_{x'} \cap \mathrm{C}^*_{x''}$. As a consequence, one has $\mathrm{C}^* = \mathrm{C}^*_x$ for any point x in the relative interior of C. This proves that C^* belongs to $\mathscr{C}^*$.

Let C, D be elements of $\mathscr{C}$ such that $\mathrm{C} \subset \mathrm{D}$. It follows from the construction that one has $\mathrm{D}^* \subset \mathrm{C}^*$.

Let us prove that $\operatorname{cone}(\mathrm{C}^* - \mathrm{D}^*) \subset \operatorname{cone}(\mathrm{D} - \mathrm{C})^\circ$. Let $y \in \mathrm{D}^*$, $y' \in \mathrm{C}^*$, $x \in \mathrm{C}$ and $x' \in \mathrm{D}$; one has

$$\langle x' - x, y' - y\rangle = -\mathrm{P}(x', y') + \mathrm{P}(x, y') + \mathrm{P}(x', y) - \mathrm{P}(x, y) = -\mathrm{P}(x', y') \leqslant 0.$$

This implies that $y' - y \in \operatorname{cone}(\mathrm{D} - \mathrm{C})^\circ$; consequently, $\operatorname{cone}(\mathrm{C}^* - \mathrm{D}^*) \subset \operatorname{cone}(\mathrm{D} - \mathrm{C})^\circ$.

Conversely, let $z \in \operatorname{cone}(\mathrm{D} - \mathrm{C})^\circ$ and let us prove that $z \in \operatorname{cone}(\mathrm{C}^* - \mathrm{D}^*)$. Choose $y \in \mathrm{V}^*$ such that $\mathrm{D} = \mathrm{C}_y$. There exists $t > 0$ such that f^* is affine on the segment $[y; y + tz]$; let us prove that $[y; y + tz] \subset \mathrm{D}$.

Let y' be any point of the open segment $]y; y + tz[$ and let $x \in \mathrm{C}_{y'}$, so that $\mathrm{P}(x, y') = 0$. Since the function $s \mapsto \mathrm{P}(x, y + sz) = f(x) + f^*(y + sz) - \langle x, y + sz\rangle$ is affine on $[0; t]$, positive, and vanishes at some interior point, it vanishes identically. This proves that $\mathrm{P}(x, y) = \mathrm{P}(y + tz) = 0$; in particular, $x \in \mathrm{D}$. Let $x' \in \mathrm{C}$; one has $\langle x - x', z\rangle \leqslant 0$, by definition of z. Then,

$$\mathrm{P}(x', y') = -\mathrm{P}(x, y') + \mathrm{P}(x, y) + \mathrm{P}(x', y') - \mathrm{P}(x', y) = \langle x - x', y' - y\rangle \leqslant 0,$$

so that $\mathrm{P}(x', y') = 0$ and $y' \in \mathrm{C}^*$. Since $y \in \mathrm{D}^*$, this implies that $z \in \operatorname{cone}(\mathrm{C}^* - \mathrm{D}^*)$. □

1.9.10. — Let V be a finite dimensional R-vector space, let S be a finite subset of V^* and let $\varphi : \mathrm{S} \to \mathbf{R}$ be any function. Let $f : \mathrm{V} \to \mathrm{R}$ be the convex piecewise affine function defined by

$$f(x) = \sup_{m\in S} \left(\langle x, m\rangle - \varphi(m)\right).$$

The "tropical polynomials" are particular examples of such functions.

Let P be the polytope conv(S). Let us prove that P is the support of the Legendre transform f^*.

Let $y \in V^*$ such that $y \notin P$. By the Farkas lemma for polytopes (prop. 1.3.2), there exists $x \in V$ such that $\langle x, y\rangle > a$, where $a = \sup_{m\in S}\langle x, m\rangle$. For $t \geqslant 0$, one has

$$f(tx) = \sup_{m\in S} \left(\langle tx, m\rangle - \varphi(m)\right) \leqslant ta - \inf_{m\in S} \varphi(m),$$

hence

$$\langle tx, y\rangle - f(tx) \geqslant t\left(\langle x, y\rangle - a\right) - \inf_{m\in S} \varphi(m).$$

When $t \to +\infty$, this proves that f^* is not defined at y.

Conversely, let $m \in S$. Since $f(x) \geqslant \langle x, m\rangle - \varphi(m)$, one has $\langle x, m\rangle - f(x) \leqslant \varphi(m)$, hence m belongs to the support of $f^*(m)$. By convexity, P is contained in the support of f^* and $f^*(m) \leqslant \varphi(m)$.

If $f^*(m) < \varphi(m)$, the definition of the Legendre transform implies that for all $x \in V$, one has $\langle x, m\rangle - f(x) \leqslant f^*(m) \leqslant \varphi(m)$, hence $f(x) > \langle x, m\rangle - \varphi(m)$. Consequently, denoting by T the set of $m \in S$ such that $f^*(m) = \varphi(m)$, one has

$$f(x) = \sup_{m\in T} \left(\langle x, m\rangle - \varphi(m)\right).$$

Let $\mathscr{C}$ and $\mathscr{C}^*$ be the regular polyhedral decompositions of V and P associated with f and f^*. The polyhedra of $\mathscr{C}^*$ are of the form C^*_x, for some $x \in V$, where C^*_x is the set of all $y \in P$ such that $\langle x, y\rangle = f(x) + f^*(y)$. By the duality theorem, the vertices of the polyhedra of $\mathscr{C}^*$ correspond to d-dimensional cells of $\mathscr{C}$, and those are of the form $P_m = \{x\,;\, f(x) = \langle x, m\rangle - \varphi(m)\rangle\}$. In particular, they belong to T.

This also implies that the polyhedral decomposition $\mathscr{C}^*$ can be constructed from the function φ, as the projections of the horizontal faces of the polyhedron $Q = \operatorname{conv}((m, -\varphi(m))_{m\in S} + \mathbf{R}_+ e$ of $V^* \times R$.

1.10. The normal fan of a polyhedron

1.10.1. — Let P be a nonempty polyhedron in a finite dimensional R-vector space V and let $v \in P$. Let $N_v(P)$ be the set of all linear forms f on V such that $f(x) \leqslant f(v)$ for all $x \in P$; equivalently, f is negative on the translated polytope $P - v$. In other words, $N_v(P)$ is the polar set $\operatorname{cone}(P - v)^\circ$ of $\operatorname{cone}(P - v)$.

We recall that it is a polyhedral convex cone — we call it the *normal cone of* P *at* v. Indeed, fix a decomposition $P = \operatorname{conv}((x_i)_{i\in I}) + \operatorname{cone}((y_j)_{j\in J})$, where $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ are finite families in V. Then a linear form f belongs to $N_v(P)$ if and only if $f(x_i) \leqslant f(v)$ for all $i \in I$ and $f(y_j) = 0$ for all $j \in J$, a finite set of linear inequalities.

By convex duality (proposition 1.3.16), one has $\operatorname{cone}(P - v) = N_v(P)^\circ$.

1.10.2. — Let $\Phi_P \subset V^*$ be the set of all linear form on V which are bounded from above on P. A linear form f on V belongs to Φ_P if and only if $f(y_j) \leqslant 0$ for all $j \in J$; then $\sup_P(f) = \sup_{i\in I} f(x_i)$. In particular, Φ_P is a polyhedral cone in V^*; if P is a polytope, then $\Phi_P = V^*$. In general, Φ_P is the polar set to the recession cone of P.

1.10.3. — For $f \in \Phi_P$, let then $P_f = \{x \in P\,;\, f(x) = \sup_P(f)\}$ be the corresponding face of P. If I_f is the set of $i \in I$ such that $f(x_i) = \sup_P(f)$, and J_f is the set of $j \in J$ such that $f(y_j) = 0$, one has

$$P_f = \operatorname{conv}((x_i)_{i\in I_f}) + \operatorname{cone}((y_j)_{j\in J_f}).$$

Let $g \in \Phi_P$ be a second linear form. One has $P_f \subset P_g$ if and only if the points x_i, for $i \in I_f$, and the points y_j, for $j \in J_f$, belong to P_g, which means that $g(x_i) \geqslant g(x_k)$ and $g(y_j) = 0$ for all $i \in I_f$ and all $k \in I$, and $g(x_j) = 0$ for all $j \in J_f$. This description shows that the set of all linear forms $g \in \Phi_P$ such that $P_f \subset P_g$ is a polyhedral cone $N_f(P)$ in V^*.

Definition **(1.10.4)**. — *Let* F *be a face of* P*. The* normal cone *of* P *along* F *is the set of all linear forms* f *on* V *which are bounded from above on* P *and such that* P_f *contains* F.

If $F = \{v\}$ is a vertex v of P, then we recover the normal cone of P at v.

Proposition **(1.10.5)**. — *Let* F *and* G *be faces of* P. *The inclusions* $N_F(P) \subset N_G(P)$ *and* $G \subset F$ *are equivalent.*

Proof. — Let $f, g \in \Phi_P$ be linear forms such that $F = P_f$ and $G = P_g$. By definition, a linear form $h \in \Phi_P$ belongs to $N_F(P)$ if and only $F = P_f \subset P_h$. Taking $h = f$, we see that $f \in N_F(P)$. If $N_F(P) \subset N_G(P)$, then $f \in N_G(P)$, hence $P_f = F \supset G$. Conversely, if $G \subset F$ and $h \in N_F(P)$, then $P_h \supset F$, hence $P_h \subset G$ and $h \in N_G(P)$. □

Theorem **(1.10.6)**. — *Let* P *be a polyhedron in a finite dimensional* $\mathbf{R}$*-vector space* V *and let* Φ_P *be the set of all linear forms on* V *which are bounded from above on* P. *The set of all cones* $N_F(P)$, *for* F *in the set of faces of* P, *is a polyhedral fan with support* Φ_P.

Proof. — By construction, one has $N_F(P) \subset \Phi_P$ for every face F of P. Conversely, if $f \in \Phi_P$ and F is the face of P defined by f, then $f \in N_F(P)$. This proves that the union of the cones $N_F(P)$ is equal to Φ_P.

Let F', F'' be two faces of P. By definition, a linear form $f \in \Phi_P$ belongs to $N_{F'}(P)$ if and only if P_f contains F'. Consequently, $N_{F'}(P) \cap N_{F''}(P)$ is the set of all linear forms $f \in \Phi_P$ such that P_f contains the smallest face F of P that contains both of F' and F'' (in other words, $F = \sup(F', F'')$). This shows that $N_{F'}(P) \cap N_{F''}(P) = N_F(P)$.

To conclude the proof, it remains to show that if F and G are faces of P such that $N_F(P) \subset N_G(P)$, then $N_F(P)$ is a face of $N_G(P)$. By proposition 1.10.5, one has $G \subset F$. Let $f, g \in \Phi_P$ be linear forms such that $F = P_f$ and $G = P_g$. For $h \in \Phi_P$, the condition $h \in N_G(P)$ means that $h(x) \leqslant h(y)$ for every $x \in P$ and every $y \in G$; this implies that h is constant on G. Requiring moreover that $h \in N_F(P)$ imposes the additional inequalities $h(x) \leqslant h(y)$ for $x \in P$ and $y \in F$. In particular, h has to be constant on F. Conversely, if $h \in N_G(P)$ is constant on F, then it takes on F the same value that it takes on G, since $G \subset F$, and then $h(x) \leqslant h(y)$ for all $x \in P$ and $y \in F$. It remains to see that these conditions actually define a face of $N_G(P)$. Let us write $F = \operatorname{conv}((x_i)) + \operatorname{cone}((y_j))$, for two finite families $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$; let v be the mean of the x_i and let w be the sum of the y_j; let also $z \in G$. Then for all $h \in N_G(P)$, one has $h(x_i) \leqslant h(z)$ for all i, and $h(y_j) \leqslant 0$ for all j. Consequently, $h(v + w) \leqslant h(z)$, and the equality $h(v + w) = h(z)$ implies that $h(x_i) = h(z)$ for all i and $h(y_j) = 0$ for all j, hence h is constant on F. This proves that $N_F(P)$ is the face of $N_G(P)$ defined by the linear form $h \mapsto h(v+w)$. □

Definition **(1.10.7)**. — *Let* P *be a nonempty polyhedron in a finite dimensional* R*-vector space* V *and let* Φ_P *be the set of all linear forms on* V *which are bounded from above on* P*. The fan consisting of all cones* $N_F(P)$*, for all faces* F *of* P*, is called the* normal fan *of* P*.*

Remark **(1.10.8)**. — In fact, this section can be seen as a particular case of the preceding one applied to the function $f \equiv 0$ on the polyhedron P. Then, Φ_P is the support of the Legendre transform f^*, and the normal fan is the regular polyhedral decomposition of Φ_P associated with f^*.

CHAPTER 2

ARCHIMEDEAN AMOEBAS

2.1. The tropicalization map

2.1.1. — Let $\lambda\colon (\mathbf{C}^*)^n \to \mathbf{R}^n$ be the map given by

$$(z_1,\dots,z_n)\mapsto (\log(|z_1|),\dots,\log(|z_n|)).$$

We say that λ is the *tropicalization map*.

Lemma **(2.1.2)**. — *The tropicalization map $\lambda\colon (\mathbf{C}^*)^n \to \mathbf{R}^n$ is continuous, open and proper.*

Proof. — The continuity of λ follows from the continuity of the logarithm map.

The image of the open disk $D(z,r)$ (for $r\leqslant |z|$) by the map $z\mapsto |z|$ from $\mathbf{C}^*$ to $\mathbf{R}_+^*$ is the open interval $]|z|-r,|z|+r[$, so that this map is open. Since the logarithm is an homeomorphism from $\mathbf{R}_+^*$ to $\mathbf{R}$, it follows that the map $z\mapsto \log(|z|)$ from $\mathbf{C}^*$ to $\mathbf{R}$ is open. Then a product topololgy argument implies that the map λ is open.

By Bourbaki (1971), chap. 1, §, n° 2, th. 1, to prove that the map λ is proper, it suffices to show that for every sequence (z_m) in $(\mathbf{C}^*)^n$ such that $\lambda(z_m)$ converges to some point $u\in\mathbf{R}^n$, there exists $z\in(\mathbf{C}^*)^n$ which is a limit point of (z_m) such that $\lambda(z)=u$. Since $|z_{m,j}|$ converges to e^{u_j} for all $j\in\{1,\dots,n\}$, the sequence (z_m) is bounded in $\mathbf{C}^n$. Up to considering a subsequence, we may thus assume that it converges in $\mathbf{C}^n$. Then its limit z satisfies $|z_j|=e^{u_j}$ for every j, hence $z\in(\mathbf{C}^*)^n$ and $(z_m)_m$ converges to z. Moreover, one has $\lambda(z)=u$. □

2.1.3. — One aspect of tropical geometry consists in studying subsets of $(\mathbf{C}^*)^n$ via their images by λ. When V is an algebraic subvariety, this image $\lambda(V)$ is the *tropicalization* of V, often called its *amoeba* by reference to its "tentacular" appeareance.

Remark **(2.1.4)**. — It would be more natural to define the tropicalization map as the map $\tau : \mathbf{C}^n \to \mathbf{R}_+^n$ defined by $\tau(z_1, \dots, z_n) = (|z_1|, \dots, |z_n|)$. Indeed, this map is *semi-algebraic*, ie, can be defined only polynomials and quantifiers in the real and imaginary parts x_j and y_j of z_j: $(t_1, \dots, t_n) \in \mathbf{R}^n$ belongs to $\tau(V)$ if and only if there exist $x_1, \dots, x_n, y_1, \dots, t_n \in \mathbf{R}$ such that $x_1^2 + y_1^2 = t_1^2, \dots, x_n^2 + y_n^2 = t_n^2$ and $(x_1 + iy_1, \dots, x_n + iy_n) \in V$. Moreover, if V is an algebraic subset of $\mathbf{C}^n$, then the latter condition can be expressed as polynomial equations in $x_1, \dots, x_n, y_1, \dots, y_n$.

Then, by Tarski's theorem, a fundamental result of real algebraic geometry, the set $\tau(V)$ is semi-algebraic, that is, can be defined using only polynomial equations (and inequations). An crucial fact of real algebraic geometry is that such sets have good algebraic properties (stability under intersection, union, or complement, as well as under taking images by semi-algebraic maps) and good topological properties, both *local* (such as local contractibility) and *global* (for example, finiteness of the set of connected components, finite dimensionality of its homology and cohomology).

For $V \in (\mathbf{C}^*)^n$, one has $\tau(V) \subset (\mathbf{R}_+^*)^n$, and $\lambda(V)$ is the image of $\tau(V)$ under the (real) logarithm map, $(t_1, \dots, t_n) \mapsto (\log(t_1), \dots, \log(t_n))$. Since this map is a homeomorphism from $(\mathbf{R}_+^*)^n$ to $\mathbf{R}^n$, the tropicalization $\lambda(V)$ of V has the same topological properties.

2.1.5. — One says that a subset D of $(\mathbf{C}^*)^n$ is a *Reinhardt domain* if for every $z \in D$ and every $u \in (\mathbf{C}^*)^n$ such that $|u_1| = \dots = |u_n| = 1$, one has $uz = (u_1 z_1, \dots, u_n z_n) \in D$. Thus D is a Reinhardt domain if and only if $D = \lambda^{-1}(\lambda(D))$.

If D is a Reinhardt domain, and U is its interior, then U is a Reinhardt domain and $\lambda(U)$ is the interior of $\lambda(D)$. Indeed, $\lambda(U)$ is open since λ is an open map. Moreover, let $x \in D$ such that $\lambda(x)$ belongs to the interior of $\lambda(D)$; let us prove that $x \in U$. Let O be an open neighborhood of $\lambda(x)$ contained in $\lambda(D)$; then $\lambda^{-1}(O)$ is an open subset of $(\mathbf{C}^*)^n$ containing x and contained in $\lambda^{-1}(\lambda(D)) = D$. This proves that $x \in U$.

2.2. Laurent series and their convergence domains

2.2.1. — Every monoid M gives rise to a convolution algebra $\mathbf{C}^{(\mathrm{M})}$ whose underlying vector space is the group of all functions $f : \mathrm{M} \to \mathbf{C}$ with finite support, and with multiplication given by convolution: $f * g(m) = \sum_{m=p+q} f(p)g(q)$ for $f, g \in \mathbf{C}^{(\mathrm{M})}$ and $m \in \mathrm{M}$. Denoting by T^m the function that takes the value 1 at m, and 0 elsewhere, we may write $f = \sum f(m)\mathrm{T}^m$ for all $f \in \mathbf{C}^{(\mathrm{M})}$; moreover one has $\mathrm{T}^m * \mathrm{T}^n = \mathrm{T}^{m+n}$, and we thus recover the classical point of view on the algebra associated with a monoid; the classical notation is $\mathbf{C}[\mathrm{M}]$.

The most classical example of this situation are the algebra of polynomials in n variables, $\mathbf{C}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, corresponding to the monoid $\mathrm{M} = \mathbf{N}^n$, where T_i is the element $\mathrm{T}^{\varepsilon_i}$. Another classical example the algebra of *Laurent polynomials* in n variables, corresponding to the monoid $\mathrm{M} = \mathbf{Z}^n$; it can also be viewed as a localization of the algebra of polynomials, and is often denoted by $\mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$.

2.2.2. — If we remove the support condition in the definition of $\mathbf{C}^{(\mathrm{M})}$, we only get a vector space in general, because the sum defining the convolution product may be infinite. If the fibers of the addition law of M are finite, then one obtains an *enlarged monoid algebra*, classically denoted by $\mathbf{C}[\![\mathrm{M}]\!]$.

This property holds when $\mathrm{M} = \mathbf{N}^n$ — we then obtain the algebra of formal power series $\mathbf{C}[\![\mathrm{T}_1, \dots, \mathrm{T}_n]\!]$ — but does not hold when $\mathrm{M} = \mathbf{Z}^n$ (if $n \geqslant 1$).

Nevertheless, we call an element of $\mathbf{C}^{\mathbf{Z}^n}$ a *formal Laurent series* in the n variables $\mathrm{T}_1, \dots, \mathrm{T}_n$. We conform to the tradition and write it as $f = \sum_{m \in \mathbf{Z}^n} a_m \mathrm{T}^m$, where $\mathrm{T}^m = \mathrm{T}_1^{m_1} \dots \mathrm{T}_n^{m_n}$ for $m \in \mathbf{Z}^n$.

Definition **(2.2.3)**. — *Let $f = \sum_{m \in \mathbf{Z}^n} a_m \mathrm{T}^m$ be a formal power series in n variables.*

a) *The domain of absolute convergence of f is the set of all $z \in (\mathbf{C}^*)^n$ such that the family $(a_m z^m)_{m \in \mathbf{Z}^n}$ is summable; it is denoted by $\mathscr{C}_f$.*

b) *Its interior is denoted by $\mathscr{U}_f$ and is called the open domain of absolute convergence of f.*

c) *The domain of boundedness of f is the set of all $z \in (\mathbf{C}^*)^n$ such that the family $(a_m z^m)_{m\in\mathbf{Z}^n}$ is bounded; we denote it by $\mathscr{B}_f$.*

Let $\lambda \colon (\mathbf{C}^*)^n \to \mathbf{R}^n$ be the map given by

$$(z_1,\dots,z_n) \mapsto (\log(|z_1|),\dots,\log(|z_n|)).$$

One says that a subset D of $(\mathbf{C}^*)^n$ is a *Reinhardt domain* if for every $z \in \mathrm{D}$ and every $u \in (\mathbf{C}^*)^n$ such that $|u_1| = \dots = |u_n| = 1$, one has $uz = (u_1z_1,\dots,u_nz_n) \in \mathrm{D}$. Then D is a Reinhardt domain if and only if $\mathrm{D} = \lambda^{-1}(\lambda(\mathrm{D}))$.

Proposition **(2.2.4)**. — *Let $f = \sum_{m\in\mathbf{Z}^n} a_m \mathrm{T}^m$ be a formal power series in n variables.*

a) *The domain of absolute convergence $\mathscr{C}_f$ of f, its interior $\mathscr{U}_f$, and the domain of boundednes $\mathscr{B}_f$ of f are Reinhardt domains in $(\mathbf{C}^*)^n$;*

b) *The domains $\mathscr{C}_f$ and $\mathscr{B}_f$ have the same interior;*

c) *The function $z \mapsto \sum_{m\in\mathbf{Z}^n} a_m z^m$ is holomorphic on $\mathscr{U}_f$;*

d) *The tropicalizations $\lambda(\mathscr{B}_f)$, $\lambda(\mathscr{C}_f)$ and $\lambda(\mathscr{U}_f)$ of $\mathscr{B}_f$, $\mathscr{C}_f$ and $\mathscr{U}_f$ are convex subsets of $\mathbf{R}^n$; moreover, $\lambda(\mathscr{U}_f)$ is the interior of $\lambda(\mathscr{B}_f)$ and of $\lambda(\mathscr{C}_f)$.*

Proof. — a) That $\mathscr{C}_f$ and $\mathscr{B}_f$ be Reinhardt domains is obvious, since the property for a family to be summable (resp. bounded) only depends on the absolute values of its terms. Since the interior of a Reinhardt domain is a Reinhardt domain again, $\mathscr{U}_f$ is a Reinhardt domain.

b) Since $\mathscr{C}_f \subset \mathscr{B}_f$, the interior of $\mathscr{C}_f$ is contained in that of $\mathscr{B}_f$. Let w be a point in the interior of $\mathscr{B}_f$; let r be a strictly positive real number such that $z \in \mathscr{B}_f$ for every z such that $|z_j - w_j| \leqslant r$ for every j. (In particular, $r \leqslant |w_j|$ for all j.) Choosing z_j with the same argument than w_j, we see that the sequences $(a_m(|w_1| \pm r)^{m_1} \dots (|w_n| \pm r)^{m_n})$ are bounded; let c be an upper bound. Let $z \in (\mathbf{C}^*)^n$ be such that $|w_j| - r < |z_j| < |w_j| + r$ for every j; let $\theta = \sup_j(\sup(|z_j|/(|w_j|+r), (|w_j|-r)/|z_j|))$; one has $0 < \theta < 1$. If $m_j \geqslant 0$, then $|z_j|^{m_j} \leqslant \theta^{m_j}(|w_j|+r)^{m_j}$; if $m_j \leqslant 0$, then $|z_j|^{m_j} \leqslant \theta^{-m_j}(|w_j| - r)^{m_j}$. Consequently, $|a_m z^m| \leqslant c\theta^{|m|}$ for all $m \in \mathbf{Z}^n$, hence $z \in \mathscr{C}_f$. This implies that $w \in \mathscr{U}_f$. These estimates also show that $\mathscr{U}_f$ is a Reinhardt domain.

c) The previous estimate shows the normal convergence of the series $\sum a_m z^m$ on every closed ball contained in $\mathscr{U}_f$. Its sum is thus a holomorphic function on that open set.

d) By §2.1.5, the set $\lambda(\mathscr{U}_f)$ is the interior of both $\lambda(\mathscr{B}_f)$ and $\lambda(\mathscr{C}_f)$. Since $\mathscr{U}_f$ is a Reinhardt domain, one has $\mathscr{B}_f = \lambda^{-1}(\lambda(\mathscr{B}_f))$.

Let us prove that $\lambda(\mathscr{B}_f)$ is convex in $\mathbf{R}^n$. Let $u, w \in \mathscr{B}_f$ and let $t \in [0;1]$; any point $z \in (\mathbf{C}^*)^n$ such that $|z_j| = |u_j|^{1-t}|w_j|^t$ satisfies $\lambda(z) = (1-t)\lambda(u)+t\lambda(w)$. For every $m \in \mathbf{Z}^n$, one has $|a_m z^m| = (|a_m u^m|)^{1-t}(|a_m w^m|)^t$, the sequences $(a_m u^m)_m$ and $(a_m w^m)_m$ are bounded, hence the squence $(a_m z^m)_m$ is bounded and the point z belongs to $\mathscr{B}_f$. This proves that $\lambda(\mathscr{B}_f)$ is convex.

For $x, y \in \mathbf{R}_+$ and $t \in [0;1]$, one has $x^{1-t}y^t \leqslant (1-t)x + ty$ (Young inequality), so that $|a_m z^m| \leqslant (1-t)|a_m u^m| + t|a_m w^m|$ for all $m \in \mathbf{Z}^n$. The same argument then shows that $\lambda(\mathscr{C}_f)$ is convex.

Finally, the interior of a convex set is convex, hence $\lambda(\mathscr{U}_f)$ is convex.

□

Theorem **(2.2.5)**. — *Let* U *be a nonempty connected open Reinhardt domain in* $(\mathbf{C}^*)^n$ *and let* $\varphi \colon \mathrm{U} \to \mathbf{C}$ *be a holomorphic function. There exists a unique Laurent series* $f \in \mathbf{C}[\![\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]\!]$ *such that* $\mathrm{U} \subset \mathscr{U}_f$ *and such that* $f(z) = \varphi(z)$ *for every* $z \in \mathrm{U}$.

Proof. — Let us first assume that $n = 1$. Then there are real numbers a, b such that $0 \leqslant a < b$ and such that $\mathrm{U} = \{z \in \mathbf{C}\,;\, a < |z| < b\}$. Let $r, s \in \mathbf{R}$ be such that $a < r < s < b$; for any point $z \in \mathbf{C}$ such that $r < |z| < s$, the residue theorem implies that

$$\varphi(z) = \frac{1}{2\pi i}\int_{|u|=s} \frac{\varphi(u)}{u-z}\,du - \frac{1}{2\pi i}\int_{|u|=r} \frac{\varphi(u)}{u-z}\,du.$$

If $|u| = s$ and $|z| < s$, we write

$$\frac{\varphi(u)}{u-z} = \frac{\varphi(u)}{u}\,\frac{1}{1-(z/u)} = \sum_{k=0}^{\infty} \varphi(u) z^k u^{k-1}.$$

The first integral has then the expansion as a power series in z

$$\frac{1}{2\pi i}\int_{|u|=s} \frac{\varphi(u)}{u-z}\,du = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i}\int_{|u|=s} \varphi(u) u^{k-1}\,du$$

which is normally convergent on every compact subset of the domain defined by $|z| < s$. Similarly, if $|u| = r$ and $|z| > r$, we write

$$\frac{\varphi(u)}{u-z} = -\frac{\varphi(u)}{z}\,\frac{1}{1-(u/z)} = -\sum_{k=0}^{\infty} \varphi(u)u^k z^{-k-1},$$

which gives the expansion

$$\frac{1}{2\pi i}\int_{|u|=r} \frac{\varphi(u)}{u-z}\,du = -\sum_{k=0}^{\infty} z^{-k-1}\frac{1}{2\pi i}\int_{|u|=r} \varphi(u)u^k\,du$$

of the second integral as a power series in z^{-1} (with no constant term) that converges normally on every compact subset of the domain defined by $|z| > r$. Let us define a Laurent series $f = \sum_{k\in\mathbf{Z}} a_k \mathrm{T}^k$ by

$$a_k = \begin{cases} \frac{1}{2\pi i}\int_{|u|=s} \varphi(u)u^{k-1}\,du & \text{if } k \geqslant 0, \\ \frac{1}{2\pi i}\int_{|u|=r} \varphi(u)u^{-k-1}\,du & \text{if } k \leqslant -1. \end{cases}$$

Its open domain of convergence contains the annulus $r < |z| < s$ and the holomorphic function it defines on that domain coincides with φ.

Uniqueness follows from the fact that, conversely, the coefficients a_k of a formal Laurent series f with $\mathscr{U}_f \neq \varnothing$ can be recovered by this formula, just replacing φ by f. If two Laurent series converging on a common open Reinhardt domain defined the same holomorphic function, they would thus be equal.

Consequently, all formal power series defined as above but in a different annulus contained in U coincide with f. In particular, the open domain of convergence of f contains U.

In higher dimension, the argument is analogous, but makes use of the multiple integral

$$\varphi(z) = \sum_{\varepsilon\in\{+,-\}^n} \frac{1}{(2\pi i)^n}\iint_{|u_j|=r_j^{\varepsilon}} \frac{\varphi(u)}{\prod(u_j - z_j)}\,du,$$

where $r_j^{\pm}$ are real numbers such that $0 < r_j^- < |z| < r_j^+$ such that U contains the product of the closed annuli consisting of all $w \in \mathbf{C}^n$ such that $r_j^- \leqslant |w_j| \leqslant r_j^+$ for all j.

We then develop $1/(u_j - z_j)$ as a power series of z_j/u_j if $\varepsilon_j = +$ and u_j/z_j if $\varepsilon_j = -$. In this way, we obtain a Laurent series f whose open domain of convergence contains the polyannulus with inner radii r_j^- and outer radii r_j^+ and such that $f(z) = \varphi(z)$ for all z in that domain.

The uniqueness part is analogous: if f is a formal Laurent power series with nonempty open domain of convergence $\mathcal{U}_f$, then its coefficients can be recovered by those formulas (replacing φ by f) with well chosen inner and closed radii. This implies that $f^{-1}(0)$ has empty interior in that domain if $f \neq 0$; otherwise one could choose these radii so that f evaluates as 0 in such a polyannulus, hence $f = 0$. By connectedness of U, all formal Laurent series defined as above, associated with small enough polyannuli contained in U, coincide with a single one, whose open domain of convergence thus contains U. □

Exercise **(2.2.6)**. — Let U be a nonempty convex open subset of $\mathbf{R}^n$. Prove that there exists a formal Laurent series f such that $\mathcal{U}_f = \lambda^{-1}(\mathrm{U})$.

2.3. The amoeba of a hypersurface

The following definition is due to Gelfand, Kapranov & Zelevinsky (1994).

Definition **(2.3.1)**. — *Let $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ be a Laurent polynomial and let $\mathrm{V}_f = f^{-1}(0)$ be the hypersurface of $(\mathbf{C}^*)^n$ it defines. The* amoeba *of f is the image $\lambda(\mathrm{V}_f)$ of V_f by the tropicalization map; we denote it by $\mathscr{A}_f$.*

Example **(2.3.2)**. — If $n = 1$ and $f \neq 0$, then V_f is a finite subset of $\mathbf{C}^*$, so that $\mathscr{A}_f$ is a finite set of points.

[Expliquer ce qui se passe dans ce cas là]

Example **(2.3.3)** (Amoeba of a line). — Assume that $n = 2$ and $f = \mathrm{T}_1 + \mathrm{T}_2 - 1$. Then a pair $(u_1, u_2) \in \mathbf{R}^2$ belongs to $\mathscr{A}_f$ if and only if there exist $z_1, z_2 \in \mathbf{C}^*$ such that $|z_1| = e^{u_1}$, $|z_2| = e^{u_2}$ and $z_1 + z_2 = 1$. Writing $1 = z_1 + z_2$, the triangular inequality implies implies $1 \leqslant e^{u_1} + e^{u_2}$; writing $z_1 = 1 - z_2$, it implies $e^{u_1} \leqslant 1 + e^{u_2}$; writing $z_2 = 1 - z_1$, it implies $e^{u_2} \leqslant 1 + e^{u_1}$. Conversely, the conjunction of these three inequalities implies that there exists a (possibly flat) triangle with lengths $1, e^{u_1}, e^{u_2}$. Let $a, b, c \in \mathbf{C}$ be the vertices of this triangle; up to reordering, one has $|c - b| = 1$, $|b - a| = e^{u_1}$ and $|c - a| = e^{u_2}$, then $z_1 = (a - b)/(c - b)$ and

$z_2 = (c-a)/(c-b)$ satisfy $|z_1| = e^{u_1}$, $|z_2| = e^{u_2}$ and $z_1 + z_2 = 1$, hence $(u_1, u_2) \in \mathscr{A}_f$.

We have represented the amoeba in figure 1, together with three half-lines with equations $u_1 = u_2 \geqslant 0$, $u_2 \leqslant 0 = u_1$ and $u_1 \leqslant 0 = u_2$. These half-lines correspond to the asymptotic directions of the line V_f in $(\mathbf{C}^*)^2$. Namely, a sequence of points (z_m) in V_f converges to infinity either if it is unbounded in $\mathbf{C}^2$, or if $(z_{m,1})$ has 0 as a limit point (hence (z_m) has $(0,1)$ as a limit point), or if $z_{m,2}$ has 0 as a limit point, so that (z_m) has $(1,0)$ has a limit point. Up to extracting subsequences, we can assume either that (z_m) tends to infinity in $\mathbf{C}^2$, or that it tends to $(1,0)$, or that it tends to $(0,1)$. In the first case, the relation $z_{m,1} + z_{m,2} = 1$ implies that $z_{m,1}/z_{m,2}$ converges to 1, and $\log(|z_{m,1}|) - \log(|z_{m,2}|)$ converges to 0; we obtain the first half-line. In the second case, $\log(|z_{m,2}|)$ tends to $-\infty$, while $\log(|z_{m,1}|)$ converges to 0; we obtain the second half-line. In the last case, a similar argument furnishes the third half-line.

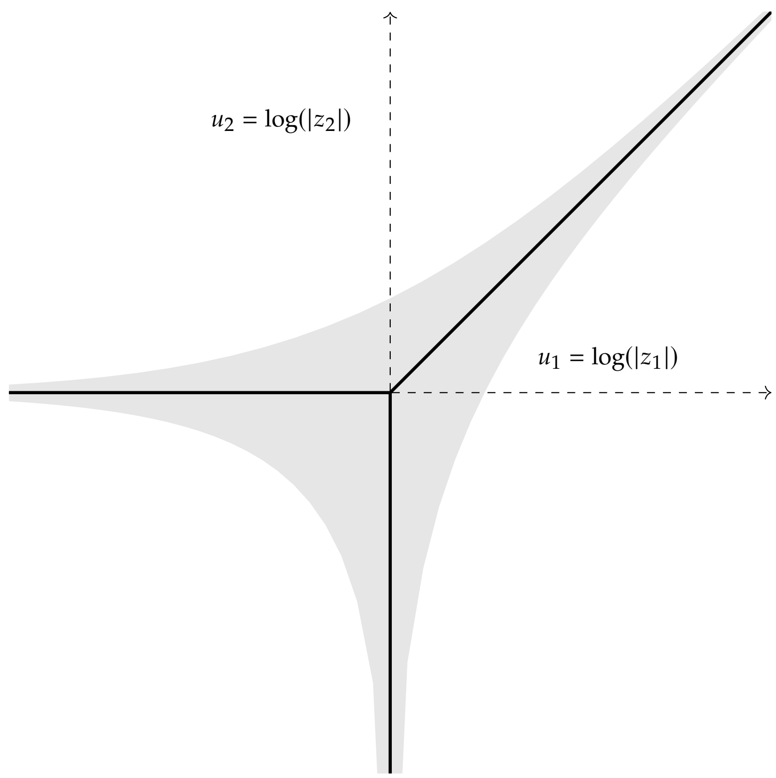


FIGURE 1. Amoeba of the polynomial $\mathrm{T}_1 + \mathrm{T}_2 - 1$

Theorem **(2.3.4)** (Gelfand, Kapranov, Zelevinski). — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ *be a Laurent polynomial.*

a) *The amoeba* $\mathscr{A}_f$ *of* f *is a closed subset of* $\mathbf{R}^n$.

b) *The connected components of its complement* $\complement\mathscr{A}_f$ *are open and convex.*

c) *For each connected component* E *of* $\complement\mathscr{A}_f$, *there exists a unique formal Laurent series* g_{E} *whose open domain of convergence is equal to* $\lambda^{-1}(\mathrm{E})$, *and such* $g_{\mathrm{E}}(z)f(z) = 1$ *for all* $z \in \lambda^{-1}(\mathrm{E})$. *Moreover, if* E *and* E′ *are distinct connected components of* $\complement\mathscr{A}_f$, *then* $g_{\mathrm{E}} \neq g_{\mathrm{E}'}$.

Proof. — Since V_f is closed in $(\mathbf{C}^*)^n$ and the tropicalization map λ is proper, hence closed, the amoeba of f is a closed subset of $\mathbf{R}^n$. Let U be its complementary subset; it is open, hence its connected components are open too ($\mathbf{R}^n$ is locally connected). By construction, $\lambda^{-1}(\mathrm{U})$ is the largest Reinhardt domain in $(\mathbf{C}^*)^n$ which is disjoint from V_f. Since the fibers of λ are connected (they are polycircles) and λ is proper, the connected components of $\lambda^{-1}(\mathrm{U})$ are the open sets $\lambda^{-1}(\mathrm{E})$, where E ranges over the set of connected components of U.

Let $\varphi\colon \lambda^{-1}(\mathrm{U}) \to \mathbf{C}$ be the holomorphic function $z \mapsto 1/f(z)$ on $\lambda^{-1}(\mathrm{U})$. Let E be a connected component of U; by proposition 2.2.4, there exists a unique formal Laurent series g_{E} whose open domain of convergence $\mathscr{U}_{\mathrm{E}}$ contains $\lambda^{-1}(\mathrm{E})$ and such that $g_{\mathrm{E}}(z)f(z) = 1$ for every $z \in \lambda^{-1}(\mathrm{E})$.

Let E, E′ be connected components of U such that $g_{\mathrm{E}} = g_{\mathrm{E}'}$. Then $\mathscr{U}_{\mathrm{E}} = \mathscr{U}_{\mathrm{E}'}$, so that $\lambda(\mathscr{U}_{\mathrm{E}}) = \lambda(\mathscr{U}_{\mathrm{E}'})$ is a convex open subset of $\mathbf{R}^n$ that contains E and E′. Since a convex set is connected, this implies that $\mathrm{E} = \mathrm{E}'$, and the map $\mathrm{E} \mapsto g_{\mathrm{E}}$ is injective.

More generally, if $\mathscr{U}_{\mathrm{E}}$ and $\mathscr{U}_{\mathrm{E}'}$ have a nonempty intersection, proposition 2.2.4 implies that g_{E} and $g_{\mathrm{E}'}$ both are the Laurent series expansion of the function φ on this intersection, hence $g_{\mathrm{E}} = g_{\mathrm{E}'}$

When E ranges over the set of connected components of U, the sets $\lambda(\mathscr{U}_{\mathrm{E}})$ are pairwise disjoint convex open subsets and their union is equal to U. Since $\lambda(\mathscr{U}_{\mathrm{E}})$ contains E, one necessarily has, $\lambda(\mathscr{U}_{\mathrm{E}}) = \mathrm{E}$ for all E; in partiular, E is convex. □

2.4. The order of a connected component of the complement

Definition **(2.4.1)**. — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a non zero Laurent polynomial; write* $f = \sum_{m \in \mathbf{Z}^n} c_m \mathrm{T}^m$. *The* Newton polytope *of* f *is the convex hull in* $\mathbf{R}^n$ *of the set of* $m \in \mathbf{Z}^n$ *such that* $c_m \neq 0$. *We denote it by* NP_f.

Since NP_f is the convex hull of points in $\mathbf{Z}^n$, it is a $\mathbf{Q}$-rational polytope.

Proposition **(2.4.2)**. — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a non zero Laurent polynomial, let* $\mathscr{A}_f$ *be its amoeba and let* E *be a connected component of its complement* $\complement \mathscr{A}_f$.

a) *There exists an element* $\nu_f^{\mathrm{E}} \in \mathbf{Z}^n$ *such that*

$$\nu_{f,j}^{\mathrm{E}} = \frac{1}{(2\pi i)^n} \int_{\lambda^{-1}(x)} z_j \frac{\partial_j f(z_1, \dots, z_n)}{f(z_1, \dots, z_n)} \frac{dz_1 \dots dz_n}{z_1 \dots z_n}$$

for every $x \in \mathrm{E}$ *and every* $j \in \{1, \dots, n\}$.

b) *Moreover, for every* $z \in \lambda^{-1}(\mathrm{E})$ *and every* $s \in \mathbf{Z}^n - \{0\}$, *the expression*

$$\langle s, \nu_f^{\mathrm{E}} \rangle = \sum_{j=1}^{n} s_j \nu_{f,j}^{\mathrm{E}}$$

is the number of zeroes minus the order of the pole at the origin of the one-variable Laurent polynomial

$$t \mapsto f(z_1 t^{s_1}, \dots, z_n t^{s_n})$$

within the unit disk $\{|t| \leqslant 1\}$.

Let us recall that for $x \in \complement \mathscr{A}_f$, $\lambda^{-1}(x)$ is the product of the circles with center 0 and radius e^{x_j} in $\mathbf{C}$. If we parameterize these circles as $z_j = e^{x_j + i t_j}$, for $t_j \in [0; 2\pi]$, we see that the measure $dz_1 \dots dz_n / (2\pi i)^n z_1 \dots z_n$ is the Haar measure of $\lambda^{-1}(x)$ normalized so that it has total mass 1.

Proof. — We will need some classic facts of complex analysis that we recall first. First of all, if $\omega = \sum \omega_j(z)\, dz_j$ is a closed holomorphic 1-form on an open subset Ω of $\mathbf{C}^n$, then for each path γ in Ω, that is, each $\mathscr{C}^1$-function $[0; 1] \to \Omega$, the path integral

$$\int_\gamma \omega = \sum_{j=1}^{n} \int_0^1 \omega_j(\gamma(t)) \gamma_j'(t)\, dt$$

is an element of **C** which does only depend on the strict homotopy class of γ (that is, up to homotopies fixing the endpoints of γ). In the case γ is a closed path, the integral even only depends on the homology class of γ.

One important case happens for $\omega = \varphi^{-1}(z)d\varphi(z)$, where φ is a holomorphic function on Ω which does not vanish. In this case, the path integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\varphi(z)}\,d\varphi(z) = \frac{1}{2\pi i}\int_0^1 \sum_{j=1}^n \frac{\partial_j\varphi}{\varphi}(\gamma(t))\gamma_j'(t)\,dt$$

is an integer. Let us indeed pose, for $u \in [0;1]$,

$$\psi(u) = \int_0^u \sum_{j=1}^n \frac{\partial_j\varphi}{\varphi}(\gamma(t))\gamma_j'(t)\,dt.$$

Formally, $\psi(u) = \int_0^u d\log(\varphi(t)) = \log(\varphi(\gamma(u))) - \log(\varphi(\gamma(0)))$, so that $e^{\psi(u)} = \varphi(\gamma(u))\varphi(\gamma(0))^{-1}$. To prove the latter formula, we show that $u \mapsto \varphi(\gamma(u))e^{-\psi(u)}$ is constant; this follows readily from the fact that it is $\mathscr{C}^1$ and that its derivative is zero:

$$\sum_{j=1}^n \partial_j\varphi(\gamma(u))\gamma_j'(u)e^{-\psi(u)} - \varphi(\gamma(u))\psi'(u)e^{-\psi(u)} = 0.$$

Then $e^{\psi(1)} = \varphi(\gamma(1))\varphi(\gamma(0))^{-1} = 1$. Consequently, $\psi(1) \in 2\pi i\mathbf{Z}$. This is the *argument principle*.

In the case where Ω is the complement of a finite set in a simply connected domain of **C** and φ is meromorphic, then this integer is equal to the number of zeroes and poles (poles being counted negatively) each of them multiplied by the index of the closed path γ (Rouché's theorem).

We now prove the proposition. Let $x \in \mathrm{E}$. By the Fubini theorem, the integral on the right hand side can be rewritten as

$$\frac{1}{(2\pi)^{n-1}}\iint \left(\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} e^{x_j+it_j}\frac{\partial_j f}{f}(e^{x_1+it_1},\dots,e^{x_n+it_n})\,dt_j\right)dt_1\dots$$

For $z \in \lambda^{-1}(\mathrm{E})$, the map $t \mapsto (z_1, \dots, z_{j-1}, z_j e^{it}, z_{j+1}, \dots, z_n)$ is a closed path in $\lambda^{-1}(\mathrm{E})$, so that the inner integral

$$\frac{1}{2\pi i} \int_0^{2\pi} i z_j e^{it} \frac{\partial_j f}{f}(z_1, \dots, z_{j-1}, z_j e^{it}, z_{j+1}, \dots, z_n)\, dt$$

is an integer. Since it varies continuously when z varies in $\complement \mathscr{A}_f$, it is constant on the connected set $\lambda^{-1}(\mathrm{E})$. If we integrate furthermore with respect to the remaining variables, we obtain that the given integral is an integer $\nu^{\mathrm{E}}_{f,j}$ that does not depend on the choice of x. This concludes the proof of the first part of the proposition.

Let $z \in (\mathbf{C}^\times)^n$ be such that $\lambda(z) \in \mathrm{E}$ and let $s \in \mathbf{Z}^n$. As explained above, Rouché's theorem implies that the number $\nu(z, s)$ of zeroes (minus the order of the pole at the origin) of the meromorphic function $\varphi \colon u \mapsto f(z_1 u^{s_1}, \dots, z_n u^{s_n})$ within the unit disk is given by

$$\nu(z, s) = \frac{1}{2\pi i} \int_0^{2\pi} i e^{it} \frac{\varphi'}{\varphi}(e^{it})\, dt.$$

It is thus equal to

$$\begin{aligned}
\nu(z, s) &= \frac{1}{2\pi} \int_0^{2\pi} e^{it} \sum_{j=1}^n s_j z_j e^{i(s_j - 1)t} \frac{\partial_j f}{f}(z_1 e^{i s_1 t}, \dots, z_n e^{i s_n t})\, dt \\
&= \sum_{j=1}^n s_j \frac{1}{2\pi} \int_0^{2\pi} z_j e^{i s_j t} \frac{\partial_j f}{f}(z_1 e^{i s_1 t}, \dots, z_j e^{i s_j t}, \dots, z_n e^{i s_n t})\, dt \\
&= \frac{1}{2\pi} \int_\gamma \frac{1}{f(z)}\, df(z),
\end{aligned}$$

where γ is the closed path in $\lambda^{-1}(\mathrm{E})$ given by $t \mapsto (z_1 e^{i s_1 t}, \dots, z_n e^{i s_n t})$. For every $j \in \{1, \dots, n\}$, let γ_j be the closed path in $\lambda^{-1}(\mathrm{E})$ given by $t \mapsto (z_1, \dots, z_{j-1}, z_j e^{it}, z_{j+1}, \dots, z_n)$. Since the closed path γ is homologous to the sum $\sum_{j=1}^n s_j \gamma_j$, one has

$$\nu(z, s) = \frac{1}{2\pi i} \int_\gamma \frac{1}{f(z)}\, df(z) = \sum_{j=1}^n s_j \frac{1}{2\pi i} \int_{\gamma_j} \frac{1}{f(z)}\, df(z).$$

Now, it follows from the definition of $\nu^{\mathrm{E}}_{f,j}$ that one has $\nu(z, s) = \sum_{j=1}^n s_j \nu^{\mathrm{E}}_{f,j}$. The concludes the proof. □

Definition **(2.4.3)**. — *The vector* $\nu_f^{\mathrm{E}} \in \mathbf{Z}^n$ *characterized by proposition 2.4.2 is called the* order of the component E *of the complement of the amoeba* $\mathscr{A}_f$.

Theorem **(2.4.4)**. — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a non zero Laurent polynomial and let* $\mathscr{A}_f$ *be its amoeba.*

a) *For every connected component* E *of the complement* $\complement \mathscr{A}_f$ *of the amoeba, the vector* ν_f^{E} *belongs to the Newton polytope of* f. *Moreover, the normal cone of* NP_f *at* ν_f^{E} *is the largest* **Q***-rational convex cone* C *of* $\mathbf{R}^n$ *such that* $\mathrm{E} + \mathrm{C} \subset \mathrm{E}$.

b) *If* E *and* E′ *are distinct components of* $\complement \mathscr{A}_f$, *then* $\nu_f^{\mathrm{E}} \neq \nu_f^{\mathrm{E}'}$.

c) *In particular, the set of connected components of* $\complement \mathscr{A}_f$ *is finite.*

d) *For every vertex* μ *of the Newton polytope of* f, *there exists a connected component of* $\complement \mathscr{A}_f$ *such that* $\nu_f^{\mathrm{E}} = \mu$.

Assertion *a*) implies that the normal cone $\mathrm{N}_{\nu_f^{\mathrm{E}}}(\mathrm{NP}_f)$ of the Newton polytope NP_f at the vertex ν_f^{E} is contained in the recession cone of the component E. In fact, we will prove in corollary 2.5.8 below that there is equality: $\mathrm{N}_{\nu_f^{\mathrm{E}}}(\mathrm{NP}_f) = \mathrm{recc}(\mathrm{E})$. This is not obvious because the recession cone of an arbitrary convex open subset of $\mathbf{R}^n$ might not be rational.

Proof. — a) Write $f = \sum_{m \in \mathbf{Z}^n} c_m \mathrm{T}^m$ and let V be the support of f, that is, the set of $m \in \mathbf{Z}^n$ such that $c_m \neq 0$. By Farkas lemma (proposition 1.3.2), to prove that ν_f^{E} belongs to $\mathrm{NP}_f = \mathrm{conv}(\mathrm{V})$, it suffices to prove that for every linear form h on $\mathbf{R}^n$ and every vector $m \in \mathrm{V}$, one has $h(\nu_f^{\mathrm{E}}) \leqslant h(m)$. Writing $h(x) = \sum s_j x_j$, it suffices, by density, to prove this inequality when $(s_1, \dots, s_n) \in \mathbf{Q}^n$, and, by homogeneity, to prove it when $s = (s_1, \dots, s_n) \in \mathbf{Z}^n$. The assertion is obvious if $s = 0$; we therefore assume that $s \neq 0$.

Let $z \in \lambda^{-1}(\mathrm{E})$ and let φ be the meromorphic function on **C** given by $\varphi(t) = f(z_1 t^{s_1}, \dots, z_n t^{s_n})$. One has $h(\nu_f^{\mathrm{E}}) = \langle s_j \nu_{f,j}^{\mathrm{E}} \rangle$; by proposition 2.4.2, it is equal to the number of zeroes, minus the order of the pole at the origin, of the function φ on the unit disk. On the other hand,

$$\varphi(t) = \sum_{m \in \mathbf{Z}^n} c_m z^m t^{s_1 m_1 + \dots + s_n m_n}$$

is a Laurent polynomial of degree at most $\sup_{m\in V}\langle s, m\rangle$. This is in particular an upper bound for the total number of zeroes of this polynomial minus the order of the pole at origin, hence the conclusion that $\nu_f^E \in NP_f$.

Let C be the cone of NP_f at ν_f^E; it is the cone generated by the translated polyhedron $NP_f - \nu_f^E$ (in other words, we choose ν_f^E for the origin of the affine space V). This is a **Q**-rational polyhedral convex cone; moreover, a linear form h on $\mathbf{R}^n$ belongs to $N_{\nu_f^E}(NP_f)$ if and only if it is negative on $NP_f - \nu_f^E$, that is, if and only if $h(m - \nu_f^E) \leqslant 0$ for every $m \in V$; since $\nu_f^E \in \operatorname{conv}(V)$, this is equivalent to $h(\nu_f^E) = \sup_{m\in V} h(m)$.

Let h be a **Q**-rational linear form on $\mathbf{R}^n$; let $s \in \mathbf{Q}^n$ be such that $h(x) = \sum s_j x_j$ for $x \in \mathbf{R}^n$. Let us show that $h \in N_{\nu_f^E}(NP_f)$ if and only if the half-line $E + \mathbf{R}_+ s$ is contained in E. By homogeneity, we may assume that $s \in \mathbf{Z}^n$. By the definition of a connected component, saying that $E + \mathbf{R}_+ s$ is contained in E means that $E + \mathbf{R}_+ s$ does not meet $\mathscr{A}_f$. By proposition 2.4.2, this is equivalent to the fact that the Laurent polynomial $\varphi(zT^s) = \sum c_m z^m T^{\langle s,m\rangle}$ has no zero t such that $|t| \geqslant 1$. Its degree is $\leqslant \sup_{m\in V}\langle s, m\rangle$, with equality if the arguments of the components of z are well chosen. Since its number of zeroes within the unit disk, minus the order of the pole at the origin, is equal to $\langle s, \nu_f^E\rangle$, this shows that $E + \mathbf{R}_+ s$ is disjoint from $\mathscr{A}_f$ if and only if $\langle s, \nu_f^E\rangle = \sup_{m\in V}\langle s, m\rangle$, that is, if and only if $h \in N_{\nu_f^E}(NP_f)$.

b) Since E and E′ are nonempty open subsets of $\mathbf{R}^n$, they contain rational points; let us thus choose $z, z' \in (\mathbf{C}^*)^n$ such that $\lambda(z) \in E \cap \mathbf{Q}^n$ and $\lambda(z') \in E' \cap \mathbf{Q}^n$. Let also $s \in \mathbf{Z}^n$ such that $\lambda(z') - \lambda(z)$ is a positive multiple of s, say $\lambda(z') - \lambda(z) = rs$, for $r \in \mathbf{R}_+^*$; we can assume that $z' = ze^{rs}$. Since $E \neq E'$, one has $\lambda(z') \neq \lambda(z)$, hence $s \neq 0$.

Let $u \in (\mathbf{S}_1)^n$. We have seen that $\langle s, \nu_f^E\rangle$ and $\langle s, \nu_f^{E'}\rangle$ are the number of zeroes within the unit disk (minus the order of the pole at origin) of the Laurent polynomials $t \mapsto f(zt^s)$ and $t \mapsto f(z'ut^s)$. Since $z' = ze^{rs}$, one has $z'ut^s = zu(e^r t)^s$, hence $\langle s, \nu_f^{E'}\rangle$ is the number of zeroes (minus the order of the pole at the origin) of the function $t \mapsto f(zut^s)$ in the disk of radius e^r. If one had $\langle s, \nu_f^E\rangle = \langle s, \nu_f^{E'}\rangle$, the function $t \mapsto f(zut^s)$

would not vanish in the open annulus $1 < |t| \leqslant e^r$. Since this holds for all u, the function f would not vanish on the inverse image by λ of the segment $[\lambda(z), \lambda(z')]$, and this segment would not meet the amoeba of f, a contradiction.

c) Since the Newton polytope of f is compact, it contains only finitely many points of $\mathbf{Z}^n$, hence the assertion by *a*) and *b*).

d) Write $f = \sum c_m \mathrm{T}^m$. By definition, NP_f is the convex hull of the set of $m \in \mathbf{Z}^n$ such that $c_m \neq 0$. Since μ is a vertex of NP_f, one has $\mu \in \mathbf{Z}^n$ and $c_\mu \neq 0$. Since $\{\mu\}$ is a face of NP_f, there exists a linear form on $\mathbf{R}^n$ such that $\varphi(m) < \varphi(\mu)$ for every $m \in \mathrm{NP}_f$ such that $m \neq \mu$. In particular, for every $m \in \mathbf{Z}^n$ such that $c_m \neq 0$ and $m \neq \mu$, one has $\varphi(m) < \varphi(\mu)$. Write $\varphi(y) = \sum x_j y_j$ for $y \in \mathbf{R}^n$ and let $z \in (\mathbf{C}^*)^n$ such that $\log(|z_j|) = x_j$ for every j. By construction, one has $|z^m| < |z^\mu|$ for every $m \in \mathbf{Z}^n$ as above. Up to replacing z by a large enough power z^t, we may assume that the inequality

$$\sum_{m \neq \mu} |c_m z^m| < |c_\mu z^\mu|$$

holds. By continuity, the set W of all $w \in (\mathbf{C}^*)^n$ such that

$$\sum_{m \neq \mu} |c_m w^m| < |c_\mu w^\mu|$$

is an open neighborhood of z; it is also a Reinhardt domain. Let g be the Laurent polynomial given by

$$g(\mathrm{T}) = \sum_{m \neq \mu} c_m c_\mu^{-1} \mathrm{T}^{m-\mu};$$

one has $f = c_\mu \mathrm{T}^\mu (1+g)$, and $|g(w)| < 1$ for all $w \in \mathrm{W}$. Consequently, there exists a holomorphic function u on W such that $e^{u(w)} = 1 + g(w)$ for all $w \in \mathrm{W}$. Moreover, W is disjoint from $\mathscr{A}_f$; let E be the connected component of z in $\complement \mathscr{A}_f$. Writing $z_j \frac{\partial_j f}{f}(z) = \mu_j + z_j \partial_j u$ in the definition of the order of E, the integral gives $\nu_j^{\mathrm{E}} = \mu_j$. Consequently, $\mu = \nu^{\mathrm{E}}$ is the order of E. □

2.5. The Ronkin function of a Laurent polynomial

Definition **(2.5.1)**. — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a nonzero Laurent polynomial. The* Ronkin function *of* f *is the function on* $\mathbf{R}^n$ *defined by*

$$\mathrm{R}_f(x) = \frac{1}{(2\pi i)^n} \int_{\lambda^{-1}(x)} \log(|f(z_1, \dots, z_n)|) \frac{dz_1 \dots dz_n}{z_1 \dots z_n}$$

for $x \in \mathbf{R}^n$.

Let us recall that $\lambda^{-1}(x)$ is product of the circles with center 0 and radius e^{x_j}, and that the measure $dz_1 \dots dz_n/(2\pi i)^n z_1 \dots z_n$ is its normalized Haar measure.

Example **(2.5.2)**. — Assume that $n = 1$. Let then $a_1, \dots, a_p$ be the roots of f in $\mathbf{C}^*$, repeated according to their multiplicites, and let m be the order of f at 0, so that

$$f(\mathrm{T}) = c\mathrm{T}^m \prod_{j=1}^{p} (\mathrm{T} - a_j),$$

for some $c \in \mathbf{C}^*$. Using the classic integral, for $a \in \mathbf{C}$ and $r \in \mathbf{R}_+$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log(|re^{i\theta} - a|)\, d\theta &= \begin{cases} \log(r) & \text{if } r \geqslant |a|, \\ \log(a) & \text{otherwise} \end{cases} \\ &= \log(\sup(r, |a|)), \end{aligned}$$

one has

$$\mathrm{R}_f(x) = \log(|c|) + mx + \sum_{j=1}^{p} \sup(x, \log(|a_j|)).$$

(This is Jensen's formula.) Consequently, the Ronkin function is piecewise affine, increasing, convex, with a slope change at each point of the amoeba of f.

Theorem **(2.5.3)** (Ronkin). — *The Ronkin function of* f *is convex (hence continuous) on* $\mathbf{R}^n$. *For every connected component* E *of* $\complement \mathscr{A}_f$, R_f *is affine on* E, *and its differential is given by* $\langle \nu_f^{\mathrm{E}}, \cdot \rangle$.

Proof. — We use a bit of complex analysis. First of all, recall that the function $u : z \mapsto \log(|f(e^{z_1}, \dots, e^{z_n})|)$ on $\mathbf{C}^n$ is plurisubharmonic (psh), as the logarithm of the absolute value of a holomorphic function. It is also invariant by translation under the lattice $2\pi\mathbf{Z}^n$. As a mean of translates of this function, the function on $\mathbf{C}^n$ defined by

$$\begin{aligned} u^*(z) &= (2\pi)^{-n} \iiint_0^{2\pi} u(z_1 + i\theta_1, \dots, z_n + i\theta_n)\, d\theta_1 \dots d\theta_n \\ &= (2\pi)^{-n} \iiint_0^{2\pi} \log(|f(e^{z_1+i\theta_1}, \dots, e^{z_n+i\theta_n})|)\, d\theta_1 \dots d\theta_n \end{aligned}$$

is psh. By construction, this function is independent of the imaginary parts of its arguments and coincides with R_f on $\mathbf{R}^n$.

If u^* were smooth, its psh nature would be detected by the positivity of the differential form of type $(1,1)$,

$$i\partial\overline{\partial}u^* = \sum_{j,k=1}^{n} \frac{\partial^2 u^*}{\partial z_j \partial \overline{z}_k} i dz_j \wedge d\overline{z}_k,$$

equivalently, by the positivity of the hermitian matrix $(\partial^2 u^*/\partial z_j \partial \overline{z}_k)$. Since u^* is invariant under imaginary translations, one has $\frac{\partial u^*}{\partial z_j} = \frac{1}{2}\frac{\partial u^*}{\partial x_j}$ and $\frac{\partial u^*}{\partial \overline{z}_j} = \frac{1}{2}\frac{\partial u^*}{\partial x_j}$. Consequently, the Hessian matrix $(\partial^2 \mathrm{R}_f/\partial x_j \partial x_j)$ is symmetric positive at each point, which would prove that R_f is convex.

The general case follows from the case of a smooth function by an approximation argument. Indeed, as any psh function, $\log(|f|)$ is a decreasing limit of smooth psh functions; then u is a decreasing limit of smooth psh functions which are invariant by translations under the lattice $2\pi\mathbf{Z}^n$, and u^* is a decreasing limit of smooth psh functions which are invariant under imaginary translations. By what precedes, R_f is a decreasing limit of smooth convex functions, hence it is convex.

Let E be a connected component of $\complement \mathscr{A}_f$. Let U_E be the open subset of $\mathbf{C}^n$ consisting of points $(z_1, \dots, z_n)$ with real part in E. On U_E, the function $z \mapsto f^{-1}(e^{z_1}, \dots, e^{z_n})$ is holomorphic, hence the function $-u$ is psh, and the same argument as above implies that $-\mathrm{R}_f$ is convex on E. Since both R_f and $-\mathrm{R}_f$ are convex on E, it follows that R_f is affine on E.

To compute the differential of R_f on E, we may differentiate under the integral sign in the formula

$$\mathrm{R}_f(x) = \frac{1}{(2\pi)^n} \iiint_0^{2\pi} \log(|f(e^{x_1+it_1}, \dots, e^{x_n+it_n})|)\, dt.$$

Since the function $\log(|f|)$ is $\mathscr{C}^\infty$ on $\lambda^{-1}(\mathrm{E})$ and the integral ranges over a compact set, this is valid. With any local determination of the logarithm, one has $\log(|f|) = \mathfrak{R}(\log(f))$, so that $\partial_j(\log(|f|))$ is the real part of $\frac{1}{f}\partial_j f$. Consequently, $\partial_j \mathrm{R}_f(x)$ is the real part of

$$\frac{1}{(2\pi)^n} \iiint_0^{2\pi} ie^{x_j+it_j} \frac{\partial_j f}{f}(e^{x_1+it_1}, \dots, e^{x_n+it_n}))\, dt = \nu_{f,j}^{\mathrm{E}}.$$

Since $\nu_{f,j}^{\mathrm{E}} \in \mathbf{Z}$, this implies the relation

$$\frac{\partial \mathrm{R}_f}{\partial x_j}(x) = \nu_{f,j}^{\mathrm{E}}$$

for every $x \in \mathrm{E}$, and this concludes the proof. □

Remark **(2.5.4)**. — One can prove that the connected components of $\complement \mathscr{A}_f$ are the maximal open subsets of $\mathbf{R}^n$ on which the Ronkin function R_f is affine. We refer to Passare & Rullgård (2004) for the proof.

2.5.5. — Let E be a connected component of $\complement \mathscr{A}_f$ and let ν_f^{E} be its order. There exists a unique complex number c_f^{E} such that

$$\mathrm{R}_f(x) = c_f^{\mathrm{E}} + \langle \nu_f^{\mathrm{E}}, x\rangle$$

for every $x \in \mathrm{E}$. Let then

$$\mathrm{S}_f(x) = \sup_{\mathrm{E}\in\pi_0(\complement\mathscr{A}_f)} \left(c_f^{\mathrm{E}} + \langle \nu_f^{\mathrm{E}}, x\rangle\right),$$

where E ranges over the set $\pi_0(\complement \mathscr{A}_f)$ of connected components of $\complement \mathscr{A}_f$. This is a piecewise affine function; as the supremum of a family of affine, hence convex, functions, it is convex. It is viewed as an approximation of the Ronkin function of f. We call it the *Passare–Rullgård function* of f.

Definition **(2.5.6)** (PASSARE & RULLGÅRD, 2004). — *The* spine $\mathscr{S}_f$ *of the Laurent polynomial* f *is the set of points of* $\mathbf{R}^n$ *at which the Passare–Rullgård function of* f, S_f, *is not smooth.*

Theorem **(2.5.7)**. — a) *The Ronkin and the Passare–Rullgård functions of* f *coincide outside of* $\mathscr{A}_f$.

b) *One has* $\mathscr{S}_f \subset \mathscr{A}_f$*: the spine of* f *is contained in its amoeaba.*

c) *More precisely, the spine* $\mathscr{S}_f$ *is a purely* $(n-1)$*-dimensional polyhedral subspace of* $\mathbf{R}^n$ *and is a deformation retract of* $\mathscr{A}_f$.

Proof. — a) Let E, F be connected components of $\complement\mathscr{A}_f$; let $x \in \mathrm{E}$ and $y \in \mathrm{F}$. Let us consider the restriction of R_f to the real line (xy): for $t \in \mathbf{R}$, we set $u(t) = \mathrm{R}_f((1-t)x + ty)$. The function u is convex; it is given by $u(t) = c_f^{\mathrm{E}} + \langle v_f^{\mathrm{E}}, (1-t)x + ty\rangle$ in a neighborhood of 0, and by $u(t) = c_f^{\mathrm{F}} + \langle v_f^{\mathrm{F}}, (1-t)x + ty\rangle$ in a neighborhood of 1. By the classical theory of convex functions of one variable, the graph of u at 0 is above the tangent line at 1: $u(0) \geqslant u(1) - u'(1)$. In other words, one has the inequality

$$c_f^{\mathrm{E}} + \langle v_f^{\mathrm{E}}, x\rangle \geqslant c_f^{\mathrm{F}} + \langle v_f^{\mathrm{F}}, y\rangle - \langle v_f^{\mathrm{F}}, y - x\rangle = c_f^{\mathrm{F}} + \langle v_f^{\mathrm{F}}, x\rangle.$$

When F ranges over $\pi_0(\complement\mathscr{A}_f)$, this furnishes the desired equality

$$\mathrm{S}_f(x) = c_f^{\mathrm{E}} + \langle v_f^{\mathrm{E}}, x\rangle = \mathrm{R}_f(x) \quad \text{on E}.$$

b) In particular, S_f is smooth on $\complement\mathscr{A}_f$. By definition of the spine, this proves the inclusion $\mathscr{S}_f \subset \mathscr{A}_f$.

c) For every connected component E of $\complement\mathscr{A}_f$, let us denote by E′ the set of points $x \in \mathbf{R}^n$ such that $\mathrm{S}_f(x) = c_f^{\mathrm{E}} + \langle v_f^{\mathrm{E}}, x\rangle$. It is a polyhedron in $\mathbf{R}^n$ that contains E, and the function S_f is affine, hence smooth, on the interior of E′.

When E ranges over $\pi_0(\complement\mathscr{A}_f)$, these polyhedra cover $\mathbf{R}^n$. Let E and F be distinct connected components of $\complement\mathscr{A}_f$; then the gradient of S_f is v_f^{E} on the interior of E′, and v_f^{F} on the interior of F′. Since $v_f^{\mathrm{E}} \neq v_f^{\mathrm{F}}$ by, this shows that the interiors of E′ and F′ are disjoint. At a point x of $\mathrm{E}' \cap \mathrm{F}'$, the Passare–Rullgård function cannot be smooth: considering points of E′ that approach x, we see that its gradient should be equal to v_f^{E}, considering points of F', it should be equal to ν_f^F. As a consequence, the spine $\mathscr{S}_f$ is the union of faces of the polyhedra E'. This also shows that it is a purely $(n-1)$-dimensional polyhedral subspace of $\mathbf{R}^n$.

For every $E \in \pi_0(\complement \mathscr{A}_f)$, let us choose a point $x_E \in E$. Let U_E be the set of points $x \in E' - \{x_E\}$ such that the half-line $[x_E, x)$ meets the boundary $\partial(E')$ of E'; this is an open subset of $E' - \{x_E\}$ containing $\partial(E')$. For any $x \in U_E$, there is a largest real number $\tau(x)$ such that $x + \tau(x)(x - x_E) \in E'$; the function $\tau \colon U_E \to \mathbf{R}_+^*$ is continuous. Setting $\sigma_E(x,t) = x_E + t\tau(x)(x - x_E)$, for $t \in [0;1]$ and $x \in U_E$ is a retraction by deformation of U_E onto $\partial(E')$.

Let U be the union of these sets U_E, when E ranges over $\pi_0(\complement \mathscr{A}_F)$; the maps σ_E define a continuous map $\sigma \colon U \times [0;1] \to \mathbf{R}^n$ which is a retraction by deformation of U onto $\mathscr{S}_f$.

To prove that $\mathscr{S}_f$ is a retraction by deformation of $\mathscr{A}_f$, it suffices to prove that U contains $\mathscr{A}_f$. Let $x \in \mathscr{A}_f - \mathscr{S}_f$, let E be the (unique) connected component of $\complement \mathscr{A}_f$ such that $x \in E'$ and let $\xi = x - x_E$; if $x \notin U$, then the half-line $x_E + \mathbf{R}_+\xi$ does not meet $\partial(E')$, which means that it is contained in E', hence $\xi \in \operatorname{recc}(E')$. This implies that $\langle \nu_f^E, \xi\rangle \geqslant \langle \nu_f^F, \xi\rangle$ for every other component F of $\complement \mathscr{A}_f$. Since every vertex of NP_f is of the form ν_f^F, for some component F, this is equivalent to the inequalities $\langle \nu_f^E, \xi\rangle \geqslant \langle \mu, \xi\rangle$ for every $\mu \in NP_f$. As a consequence, ξ belongs to the normal cone $N_{\nu_f^E}(NP_f)$ of NP_f at the point ν_f^E. By theorem 2.4.4, one has $E + N_{\nu_f^E}(NP_f) \subset E$; in particular $x_E + \mathbf{R}_+\xi \subset E$, contradicting the hypothesis that $x_E + \xi = x \in \mathscr{A}_f$. □

As a corollary of an argument in the proof, we can strengthen the first assertion of theorem 2.4.4.

Corollary **(2.5.8)**. — *For every connected component* E *of* $\complement \mathscr{A}_f$, *the normal cone* $N_{\nu_f^E}(NP_f)$ *is the largest cone* C *such that* $E + C \subset E$.

Proof. — Since NP_f is a polytope with vertices in $\mathbf{Z}^n$, it is $\mathbf{Q}$-rational and its normal cone $N_{\nu_f^E}(NP_f)$ at ν_f^E is generated by vectors in $\mathbf{Q}^n$. We have seen in the proof of theorem 2.4.4 that these vectors belong to the recession cone of E, hence the inclusion $E + N_{\nu_f^E}(NP_f) \subset E$.

Conversely, let $x \in \mathrm{E}$ and let $\xi \in \mathbf{R}^n$ be such that $x + \mathbf{R}_+\xi \subset \mathrm{E}$. Then $x + \mathbf{R}_+\xi \subset \mathrm{E}'$, with the notation of the proof, and we have seen how this implies that $\xi \in \mathrm{N}_{\nu_f^{\mathrm{E}}}(\mathrm{NP}_f)$. This concludes the proof. □

2.6. The logarithmic limit set of a variety

Definition **(2.6.1)**. — *Let* V *be an algebraic subvariety of* $(\mathbf{C}^*)^n$. *The* logarithmic limit set *of* V *is the set of points* $x \in \mathbf{R}^n$ *such that there exists sequences* $(x_k) \in \lambda(\mathrm{V})$ *and* $(h_k) \in \mathbf{R}_+^*$ *such that* $h_k \to 0$ *and* $h_k x_k \to x$. *We denote it by* $\lambda_\infty(\mathrm{V})$.

This set has been introduced by Bergman (1971) who gave a description of the set when V is a hypersurface. His work has been completed by Bieri & Groves (1984).

It is also called the *asymptotic cone* of $\lambda(\mathrm{C})$, and can be defined as the limit of the closed subsets $h\lambda(\mathrm{V})$, when $h \to 0$ (restricted to $h > 0$) for the topology defined by the Hausdorff distance on compact sets.

In this section, we describe $\lambda_\infty(\mathrm{V})$ when $\mathrm{V} = \mathscr{V}(f)$ is defined by a nonzero Laurent polynomial in $\mathbf{C}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$.

Lemma **(2.6.2)**. — *Let* V *be a nonempty closed algebraic subvariety of* $(\mathbf{C}^*)^n$. *Then its logarithmic limit set* $\lambda_\infty(\mathrm{V})$ *is a closed conic subset of* $\mathbf{R}^n$.

Proof. — Since V is nonempty, one has $\lambda(\mathrm{V}) \neq \varnothing$; one then may choose x_k to be equal to a given element of $\lambda(\mathrm{V})$ and $h_k = 1/k$; this shows that $0 \in \lambda_\infty(\mathrm{V})$.

Let $x \in \lambda_\infty(\mathrm{V})$; write $x = \lim h_k x_k$, with $x_k \in \lambda(\mathrm{V})$ and $(h_k) \to 0$. For every $t > 0$, one has $tx = \lim(th_k)x_k$, and $th_k \to 0$, so that $tx \in \lambda_\infty(\mathrm{V})$.

This proves that $\lambda_\infty(\mathrm{V})$ is a cone. Let us prove that it is closed.

Let $(x^{(m)})$ be a sequence of points of $\lambda_\infty(\mathrm{V})$ that converges to a point $x \in \mathbf{R}^n$ and let us prove that $x \in \lambda_\infty(\mathrm{V})$. For every m, choose a point $x_m \in \lambda(\mathrm{V})$ and a real number h_m such that $0 < h_m < 1/m$ and $\left\|x^{(m)} - h_m x_m\right\| < 1/m$. Then $\|x - h_m x_m\| < \left\|x - x^{(m)}\right\| + 1/m$, so that $x = \lim h_m x_m$, hence $x \in \lambda_\infty(\mathrm{V})$. This proves that $\lambda_\infty(\mathrm{V})$ is closed. □

Definition **(2.6.3)**. — *Let* $f \in \mathbf{C}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ *be a nonzero Laurent polynomial and let* $\mathrm{S} \subset \mathbf{Z}^n$ *be its support. The* tropical variety *defined by* f *is the* *set of all points* $x \in \mathbf{R}^n$ *such that* $\sup_{m\in S}\langle x, m\rangle$ *is attained for at least two values of* $m \in V$. *We denote it by* $\mathscr{T}_f$.

It follows from the definition of $\mathscr{T}_f$ that it is a closed $\mathbf{Q}$-rational cone (non convex, in general).

In general, if V is a closed subvariety of $(\mathbf{C}^*)^n$, one defines its *tropical variety* $\mathscr{T}_V$ as the intersection of all $\mathscr{T}_f$, for $f \in \mathscr{I}(V) - \{0\}$, wher $\mathscr{I}(V)$ is the *ideal* of V, namely the ideal of all Laurent polynomials $f \in \mathbf{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ such that $f|_V \equiv 0$.

If $V \subset W$, one has $\mathscr{I}(W) \subset \mathscr{I}(V)$, hence $\mathscr{T}_V \subset \mathscr{T}_W$.

The tropical variety $\mathscr{T}_V$ is a closed conic subset of $\mathbf{R}^n$, as an intersection of a family of such subsets.

Lemma **(2.6.4)**. — *Assume that* $V = \mathscr{V}(f)$ *is a hypersurface defined by a nonzero Laurent polynomial* $f \in \mathbf{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. *Then* $\mathscr{T}_V = \mathscr{T}_f$. *In particular,* $\mathscr{T}_V$ *is a* $\mathbf{Q}$*-rational polyhedral set.*

Proof. — It suffices to prove that $\mathscr{T}_f \subset \mathscr{T}_{fg}$ for every nonzero Laurent polynomial g. One has $NP_{fg} = NP_f + NP_g$; indeed, if $m \in \mathbf{Z}^n$ is a vertex of NP_{fg}, it must be a vertex of both NP_f and NP_g. In other words, if a linear form defines a nonpunctual face of NP_f, then it defines a nonpunctual face of NP_{fg}; this means exactly that $\mathscr{T}_f \subset \mathscr{T}_{fg}$. □

Using Gröbner bases and the notion of nonarchimedean amoebas, we shall prove in the next chapter (remark 3.7.7) a conjecture put forward by Bergman (1971) and proved by Bieri & Groves (1984) that there is a finite family (f_i) of Laurent polynomials such that $\mathscr{T}_V = \bigcap_i \mathscr{T}_{f_i}$. In particular, $\mathscr{T}_V$ is a $\mathbf{Q}$-rational polyhedral set. The motivation for the work of Bieri & Groves (1984) came from the following consequence regarding the logarithmic limit set of an algebraic variety.

Theorem **(2.6.5)** (Bieri & Groves, 1984). — *For every closed subvariety* V *of* $(\mathbf{C}^*)^n$, *the tropical variety of* V *coincides with its logarithmic limit set:* $\mathscr{T}_V = \lambda_\infty(V)$.

For the moment, we need to content ourselves with the weakest result.

Theorem **(2.6.6)** (Bergman, 1971). — *Let* V *be a closed subvariety such that* $\mathscr{T}_{\mathrm{V}}$ *is a* **Q**-*rational polyhedral set. Then* $\mathscr{T}_{\mathrm{V}} = \lambda_\infty(\mathrm{V})$. *In particular, for every non zero Laurent polynomial* $f \in \mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$, *one has* $\mathscr{T}_f = \lambda_\infty(\mathscr{V}(f))$.

We split the proof of this equality as two inclusions. The proof of the first one is relatively elementary, the second will require a bit of algebraic geometry.

Proposition **(2.6.7)**. — *One has* $\lambda_\infty(\mathrm{V}) \subset \mathscr{T}_{\mathrm{V}}$.

Proof. — It suffices to prove that $\lambda_\infty(\mathscr{V}(f)) \subset \mathscr{T}_f$ for every non zero Laurent polynomial f. Fix $x \in \mathbf{R}^n$. Let S be the support of f and write $f = \sum_{m\in \mathrm{S}} c_m \mathrm{T}^m$; let S_x be the set of $m \in \mathrm{S}$ such that $\langle x, m\rangle = \sup_{m\in\mathrm{S}}\langle x, m\rangle$. By definition, $x \in \mathscr{T}_f$ if and only if $\mathrm{Card}(\mathrm{S}_x) \geqslant 2$. Let us assume that $x \notin \mathscr{T}_f$, that is, $\mathrm{Card}(\mathrm{S}_x) = 1$, and let us prove that $x \notin \lambda_\infty(\mathscr{V}(f))$. We argue by contradiction, assuming that there is a sequence (z_k) in $\mathscr{V}(f)$ and a sequence (h_k) of strictly positive real numbers such that $h_k \to 0$ and $h_k\lambda(z_k) \to x$. Let $\mu \in \mathrm{S}$ be the unique element such that $\mathrm{S}_x = \{\mu\}$. By assumption, one has $\langle x, m\rangle < \langle x, \mu\rangle$ for every $m \in \mathrm{S} - \{\mu\}$. Let $\varepsilon > 0$ be such that $\langle x, m\rangle < \langle x, \mu\rangle - \varepsilon$ for every $m \in \mathrm{S} - \{\mu\}$; by continuity, this inequality holds in a neighborhood U of x. For k large enough such that $h_k\lambda(z_k) \in \mathrm{U}$, one then has

$$\log(z_k^{m-\mu}) = \langle \lambda(z_k), m - \mu\rangle = h_k^{-1}\langle h_k\lambda(z_k), m-\mu\rangle \leqslant -h_k^{-1}\varepsilon$$

for all $m \in \mathrm{S} - \{\mu\}$. Since h_k tends to 0, this shows that $\log(|z_k^{m-\mu}|$ converges to $-\infty$, hence $|z_k^{m-\mu}|$ converges to 0. From the equality $f(z_k) = 0$, we deduce that

$$1 = -\sum_{m\in\mathrm{S}-\{\mu\}} \frac{c_m}{c_\mu} z_k^{m-\mu}.$$

By the preceding estimate, the right hand side of the previous equality converges to 0, whence the desired contradiction. □

Lemma **(2.6.8)**. — *Let* $t \in \mathbf{R}_+$ *and let* $x = (0, \dots, 0, -t)$; *if* $x \in \mathscr{T}_{\mathrm{V}}$, *then* $x \in \lambda_\infty(\mathrm{V})$.

Proof. — The result is obvious if $x = 0$. Since both $\mathscr{T}_{\mathrm{V}}$ and $\lambda_\infty(\mathrm{V})$ are invariant by multiplication by a positive real number, we may assume that $x = (0, \dots, 0, -1)$.

Let $R_0 = \mathbf{C}[T_1^{\pm 1}, \dots, T_{n-1}^{\pm 1}]$, let $R = R_0[T_n^{\pm 1}]$ and $R' = R_0[T_n]$; let $\varphi \colon R' \to R_0$ be the unique morphism of R_0-algebras such that $\varphi(T_n) = 0$. These rings R, R′ and R_0 are respectively viewed as the rings of functions on the algebraic varieties $(\mathbf{C}^*)^n$, $(\mathbf{C}^*)^{n-1} \times \mathbf{C}$ and $(\mathbf{C}^*)^{n-1} \times \{0\}$. Let $I = \mathscr{I}(V)$ be the ideal of V in R; let $I' = I \cap R'$ and let $I_0 = \varphi(I')$. Geometrically, I′ is the ideal of the Zariski closure V′ of V in $(\mathbf{C}^*)^{n-1} \times \mathbf{C}$, and I_0 is the ideal of $V_0 = V' \cap (\mathbf{C}^*)^{n-1} \times \{0\}$.

Let us prove that $I_0 \neq (1)$. Otherwise, there exists $f \in I' = I \cap R'$ such that $\varphi(f) = 1$; let S be the support of f and write $f = \sum_{m \in S} c_m T^m$, so that

$$\varphi(f) = \sum_{\substack{m \in S \\ m_n = 0}} c_m T_1^{m_1} \dots T_{n-1}^{m_{n-1}}.$$

Since $f \in I'$, one has $S \subset \mathbf{Z}^{n-1} \times \mathbf{N}$, so that $\langle x, m \rangle = -m_n \leqslant 0$ for all $m \in S$. From the equality $\varphi(f) = 1$, we see that there exists $m \in S$ such that $m_n = 0$ and $(m_1, \dots, m_{n-1}) = 0$, that is, $0 \in S$. In particular, $\sup_{m \in S} \langle x, m \rangle = 0$.

Since $x \in \mathscr{T}_f$, there are at least two distinct elements $m, m' \in S$ such that $0 = \langle x, m \rangle = \langle x, m' \rangle$, that is, $m_n = m'_n = 0$. Then $(m_1, \dots, m_{n-1}) \neq (m'_1, \dots, m'_{n-1})$, hence $\varphi(f)$ is not a monomial, contrary to the hypothesis $\varphi(f) = 1$. Consequently, $V_0 \neq \varnothing$. Let $z \in (\mathbf{C}^*)^{n-1}$ be a point such that $(z, 0) \in V_0$.

By definition, V is a dense open subset of V′ for the Zariski topology. It is therefore an open subset of V′ for the classical topology. Moreover, a basic but nontrivial result of algebraic geometry asserts it is also dense; see, for example, (Mumford, 1994), p. 58, theorem 1. Consequently, there is a sequence (z'_k) of points of V such that $z_k \to (z, 0)$. If one writes $z'_k = (z_k, u_k)$, with $z_k \in (\mathbf{C}^*)^{n-1}$ and $u_k \in \mathbf{C}^*$, this means that $z_k \to z$ and $u_k \to 0$. In particular, $\lambda(z_k) \to \lambda(z)$ and $\lambda(u_k) \to -\infty$; For k large enough, one thus has $\log(u_k) < 0$; removing a few terms, we assume that $\log(u_k) < 0$ for all k; setting $h_k = -1/\log(u_k)$, the sequence (h_k) converges to 0 and consists of strictly positive real numbers. Then, $h_k \lambda(z'_k) = (h_k \lambda(z_k), h_k \lambda(u_k))$ converges to $(0, -1) = x$. This proves that $x \in \lambda_\infty(V)$. □

Proposition **(2.6.9)**. — *Assume that* $\mathscr{T}_V$ *is a* $\mathbf{Q}$*-rational polyhedral subset of* $\mathbf{R}^n$*. Then* $\mathscr{T}_V \subset \lambda_\infty(V)$*.*

Proof. — Since $\mathscr{T}_V$ is a $\mathbf{Q}$-rational conic polyhedral subset of $\mathbf{R}^n$, its rational points $\mathbf{Q}^n \cap \mathscr{T}_V$ are dense in $\mathscr{T}_V$. Since $\lambda_\infty(V)$ is closed in $\mathbf{R}^n$, it thus suffices to prove that every point of $\mathbf{Q}^n \cap \mathscr{T}_V$ belongs to $\lambda_\infty(V)$. Let $x \in \mathbf{Q}^n \cap \mathscr{T}_V$. If $x = 0$, then $x \in \lambda_\infty(V)$; let us then assume that $x \neq 0$. By the classification of matrices over $\mathbf{Z}$, there exists $A \in GL_n(\mathbf{Z})$ such that $A^{-1}x = (0, \dots, 0, -t)$, where $t \in \mathbf{Q}$. Performing the monomial change of variables given by A, we are reduced to the case of $x = (0, \dots, 0, -1)$. The proposition follows from the preceding lemma. □

2.7. Missing

Following Forsberg, Passare & Tsikh (2000); Passare & Rullgård (2004); Passare & Tsikh (2005):

– The connected components of the complement of the amoeba are maximal open sets on which the Ronkin function is affine.

– (Limit of the amoebas is the tropical hypersurface, it is purely $(n-1)$-dimensional;) maybe explain the balancing condition, at least the local concavity, maybe not.

CHAPTER 3

NONARCHIMEDEAN AMOEBAS

3.1. Seminorms

Definition **(3.1.1)**. — *Let* R *be a ring. A* seminorm *on* R *is a map* $p \colon \mathrm{R} \to \mathbf{R}_+$ *satisfying the following properties:*

(i) *One has* $p(0) = 0$ *and* $p(1) \leqslant 1$;
(ii) *For every* $a, b \in \mathrm{A}$, *one has* $p(a - b) \leqslant p(a) + p(b)$;
(iii) *For every* $a, b \in \mathrm{A}$, *one has* $p(ab) \leqslant p(a)p(b)$.

One says that the seminorm p *is* radical *or* power-multiplicative *if, moreover, it satisfies*

(iv) *For every* $a \in \mathrm{A}$ *and* $n \in \mathbf{N}$, *one has* $p(a^n) = p(a)^n$.

One says that the seminorm p *is* multiplicative *if:*

(v) *For every* $a, b \in \mathrm{A}$, *one has* $p(ab) = p(a)p(b)$.

One says that the seminorm p *is a* norm, *or an* absolute value, *if* $p(a) = 0$ *implies* $a = 0$.

One has $p(a) \leqslant p(a)p(1)$ for all $a \in \mathrm{R}$; if $p \neq 0$, this implies $1 \leqslant p(1)$ hence $p(1) = 1$.

Taking $a = 0$ in (ii), one has $p(-b) \leqslant p(b)$, hence $p(-b) = p(b)$ for all b. Consequently, $p(a + b) \leqslant p(a) + p(b)$ for all $a, b \in \mathrm{R}$.

Example **(3.1.2)**. — Let R be a ring and let p be a seminorm on R. Let $\mathrm{P} = \{a \in \mathrm{R}\,;\, p(a) = 0\}$. Let $a, b \in \mathrm{P}$; then $p(a+b) \leqslant p(a)+p(b) = 0$, hence $p(a+b) = 0$ and $a+b \in \mathrm{P}$. Let $a \in \mathrm{R}$ and $b \in \mathrm{P}$; then $p(ab) \leqslant p(a)p(b) = 0$, hence $ab \in \mathrm{P}$. This proves that P is an ideal of R.

For every $a \in \mathrm{R}$ and every $b \in \mathrm{P}$, one has $p(a + b) \leqslant p(a)$, and $p(a) = p((a + b) - b) \leqslant p(a + b)$, so that $p(a + b) = p(a)$. Consequently, p passes to the quotient and defines a seminorm on R/P.

If p is radical, then P is a radical ideal. Let indeed $a \in \mathrm{R}$ and $n \in \mathbf{N}$ be such that $a^n \in \mathrm{P}$; then $p(a)^n = p(a^n) = 0$, hence $p(a) = 0$ and $a \in \mathrm{P}$.

Assume that p is multiplicative and $p \neq 0$, and let us show that P is a prime ideal. Since $p \neq 0$, one has $\mathrm{P} \neq \mathrm{R}$. Let also $a, b \in \mathrm{R}$ be such that $ab \in \mathrm{P}$; then $p(ab) = p(a)p(b) = 0$, hence either $p(a) = 0$ and $a \in \mathrm{P}$, or $p(b) = 0$ and $b \in \mathrm{P}$.

Example **(3.1.3)**. — Let R be a ring, let S be a multiplicative subset of R, let R_S be the associated fraction ring. Let p be a multiplicative seminorm on R such that $p(s) \neq 0$ for every $s \in \mathrm{S}$. There exists a unique map $p' \colon \mathrm{R}_\mathrm{S} \to \mathbf{R}_+$ such that $p'(a/s) = p(a)/p(s)$ for every $a \in \mathrm{A}$ and every $s \in \mathrm{S}$. (Indeed, if $a/s = b/t$, for $a, b \in \mathrm{R}$ and $s, t \in \mathrm{S}$, there exists $u \in \mathrm{S}$ such that $atu = bsu$; then $p(a)p(t)p(u) = p(b)p(s)p(u)$, hence $p(a)/p(s) = p(b)/p(t)$.) It is clear that p' is multiplicative: $p'((a/s)(b/t)) = p'(ab/st) = p(ab)/p(st) = (p(a)/p(s)) \cdot (p(b)/p(t))$. Moreover, let $a, b \in \mathrm{R}$ and $s, t \in \mathrm{S}$; then $(a/s) + (b/t) = (at + bs)/st$, so that

$$\begin{aligned} p'(\frac{a}{s} + \frac{b}{t}) = p'(\frac{at+bs}{st}) &= \frac{p(at+bs)}{p(st)} \\ &\leqslant \frac{p(at) + p(bs)}{p(st)} = \frac{p(a)}{p(s)} + \frac{p(b)}{p(t)} \\ &= p'(\frac{a}{s}) + p'(\frac{b}{t}). \end{aligned}$$

In particular, any absolute value on an integral domain extends uniquely to an absolute value on its field of fractions.

Definition **(3.1.4)**. — *Let* R *be a ring and let* p *be a seminorm on* R*. One says that the seminorm* p *is* nonarchimedean*, or* ultrametric*, if one has* $p(a+b) \leqslant \sup(p(a), p(b))$ *for every* $a, b \in \mathrm{R}$.

Assume that p is ultrametric and let $a, b \in \mathrm{R}$ be such that $p(a) \neq p(b)$. If, say, $p(a) > p(b)$, we have $p(a+b) \leqslant p(a)$; moreover, from the relation $a = (a+b)+(-b)$, we deduce that $p(a) \leqslant \sup(p(a+b), p(-b)) = \sup(p(a+b), p(b)) \leqslant p(a)$. Since $p(a) > p(b)$, this implies that $p(a+b) = p(a) = \sup(p(a), p(b))$. By symmetry, the same relation holds if $p(a) < p(b)$.

The terminology *ultrametric* refers to the property that p satisfies an inequality stronger than the triangular inequality. The terminology *nonarchimedean* alludes to the fact that it implies that $p(na) \leqslant p(a)$ for every $n \in \mathbf{N}$: no matter how many times one adds an element, it never gets higher than the initial size. The following lemma explains the relations between these two properties.

Lemma **(3.1.5)**. — *Let* R *be a ring and let* p *be a seminorm on* R.

a) *If* p *is nonarchimedean, then* $p(na) \leqslant p(a)$ *for every* $n \in \mathbf{Z}$ *and every* $a \in \mathrm{R}$.

b) *Conversely, let us assume that* p *is radical and that* $p(n) \leqslant 1$ *for every* $n \in \mathbf{N}$. *Then* p *is nonarchimedean.*

Proof. — The first assertion is proved by an obvious inductive argument. Let us prove the second one. Let $a, b \in \mathrm{R}$. For every $n \in \mathbf{N}$, one has

$$\begin{aligned}
p(a+b)^n = p((a+b)^n) &\leqslant p(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}) \\
&\leqslant \sum_{k=0}^{n} p(\binom{n}{k}) p(a)^k p(b)^{n-k} \leqslant \sum_{k=0}^{n} p(a)^k p(b)^{n-k} \\
&\leqslant (n+1) \sup(p(a), p(b))^n.
\end{aligned}$$

As a consequence, one has

$$p(a+b) \leqslant (n+1)^{1/n} \sup(p(a), p(b)).$$

When $n \to +\infty$, we obtain the upper bound $p(a+b) \leqslant \sup(p(a), p(b))$; this proves that p is nonarchimedean. ☐

Example **(3.1.6)**. — A theorem of Ostrowski describes the multiplicative seminorms on the field $\mathbf{Q}$ of rational numbers.

a) The usual absolute value $|\cdot|$, and its powers $|\cdot|^r$ for $r \in]0;1]$;

b) For every prime number p, the p-adic absolute value $|\cdot|_p$, and its powers $|\cdot|_p^r$, for all $r \in]0;+\infty[$;

c) The trivial absolute value $|\cdot|_0$ defined by $|0|_0 = 0$ and $|a|_0 = 1$ for all $a \in \mathbf{Q}^\times$.

Example **(3.1.7)**. — Let $\mathscr{U}$ be an ultrafilter on $\mathbf{N}$ that contains the Fréchet filter: $\mathscr{U}$ is a set of $\mathfrak{P}(\mathbf{N})$ satisfying the following properties, for $\mathrm{A}, \mathrm{B} \subset \mathbf{N}$:

(i) If $\complement \mathrm{A}$ is finite, then $\mathrm{A} \in \mathscr{U}$;
(ii) If $\mathrm{A} \subset \mathrm{B}$ and $\mathrm{A} \in \mathscr{U}$, then $\mathrm{B} \in \mathscr{U}$;
(iii) If $\mathrm{A}, \mathrm{B} \in \mathscr{U}$, then $\mathrm{A} \cap \mathrm{B} \in \mathscr{U}$;
(iv) $\varnothing \notin \mathscr{U}$.
(v) If $\mathrm{A} \notin \mathscr{U}$, then $\complement \mathrm{A} \in \mathscr{U}$;

In more elementary terms, elements of $\mathscr{U}$ are the subsets of $\mathbf{N}$ which are almost sure with respect to some $0/1$-valued finitely additive probability, and for which finite sets have probability 0.

The existence of ultrafilters follows from Zorn's theorem, the set of subsets of $\mathfrak{P}(\mathbf{N})$ satisfying (i)–(iv) being inductive with respect to inclusion.

Members of a chosen (ultra)filter are sorts of neighborhoods of infinity. In particular, one can define the notion of *convergence along* $\mathscr{U}$ for a sequence (a_n): $\lim_{n,\mathscr{U}}(a_n) = a$ if for every neighborhood V of a, the set of $n \in \mathbf{N}$ such that $a_n \in \mathrm{V}$ belongs to $\mathscr{U}$. Every sequence with values in a *compact* (Hausdorff) topological space has a unique limit along $\mathscr{U}$; in particular, every real valued sequence converges along $\mathscr{U}$ to an element in $\mathbf{R} \cup \{\pm\infty\}$.

Fix a sequence $t = (t_n)$ of strictly positive real numbers converging to $+\infty$.

Let B_t, resp. Z_t, be the set of all sequences $(a_n) \in \mathbf{C}^{\mathbf{N}}$ such that $\lim_{n,\mathscr{U}} \log(|a_n|)/t_n < \infty$, resp. $\lim_{n,\mathscr{U}} \log(|a_n|)/t_n = 0$. The set B_t is a subring of the product ring $\mathbf{C}^{\mathbf{N}}$, and Z_t is a maximal ideal of B_t. The quotient $\mathbf{C}$-algebra $\mathrm{K}_t = \mathrm{B}_t/\mathrm{Z}_t$ is an algebraically closed field. The map $(a_n) \mapsto \lim_{n,\mathscr{U}} |a_n|^{1/t_n}$ gives rise to an absolute value on K_t which restricts to the trivial absolute value on $\mathbf{C}$. In particular, it is nonarchimedean.

The study of the logarithmic limit set of a complex variety amounts more or less to the study of the nonarchimedean amoeba of the associated K_t-variety.

3.1.8. — Let K be a nonarchimedean *valued field*, that is, a field endowed with a nonarchimedean absolute value.

Let R be the set of $a \in K$ such that $|a| \leqslant 1$. Then R is a subring of K, and K is its fraction field. More precisely, for every $a \in K^\times$, then either $a \in R$ (if $|a| \leqslant 1$), or $1/a \in R$ (when $|a| \geqslant 1$), which means that R is a valuation ring. It is called the *valuation ring* of K.

An element $a \in R$ is invertible in R if and only if $|a| = 1$. As a consequence, the ring R is a local ring and the set M of all $a \in R$ such that $|a| < 1$ is its unique maximal ideal. The field $k = R/M$ is called the *residue field* of K.

If the absolute value of K is trivial, then $R = K$, $M = 0$ and $k = K$.

In this context, the map from $K^\times$ to the ordered abelian group $\mathbf{R}$ given by $v\colon a \mapsto -\log(|a|)$ is a group morphism which satisfies the property $v(a+b) \geqslant \inf(v(a), v(b))$ for all $a, b \in K$ such that $a, b, a+b \neq 0$; in other words, v is a *valuation* on K. In this context, one also defines $v(0) = +\infty$.

The minus sign in the definition of v is sometimes annoying, at least it creates confusion by reversing the inequalities. As we shall see below, a valuation gives rise to a topology, but an element of large valuation is small. For this reason, some authors such as Huber (1993) define an abstract valuation on a field K as a morphism λ from $K^\times$ to an ordered multiplicative abelian group Γ such that $\lambda(a+b) \leqslant \sup(\lambda(a), \lambda(b))$ for all $a, b \in K$ such that $a, b, a+b \neq 0$. In this context, one also sets $\lambda(0)$ to be an additional element 0 which is strictly smaller than any element of Γ.

Conversely, let K be a field and let R be a valuation ring of K. For $a, b \in K^\times$, write $a \preceq b$ if there exists $u \in R$ such that $a = bu$; this is a preordering relation on $K^\times$ and it induces an ordering relation on the quotient abelian group $K^\times/R^\times$ and the canonical morphism $\lambda\colon K^\times \to K^\times/R^\times$ is a valuation. Indeed, let $a, b \in K^\times$ be such that $a+b \neq 0$ and set $u = b/a$; if $u \in R$, then $b = au$ and $a + b = a(1+u)$, so that $(a+b) \preceq a$; otherwise, $v = 1/u \in R$, $a = bv$ and $a + b = b(1+v)$ so that $(a+b) \preceq b$; in both cases, we have shown that $\lambda(a+b) \leqslant \sup(\lambda(a), \lambda(b))$.

Example **(3.1.9)**. — Let K be a nonarchimedean valued field and let R be its valuation ring. It follows from the property of a valuation ring that for every $a, b \in R$, either $a \in bR$ or $b \in aR$, according to whether $|a| \leqslant |b|$ or $|b| \leqslant |a|$. In particular, every finitely generated ideal of R is principal.

If M is finitely generated, then M is a principal ideal. Let $\pi \in \mathrm{M}$ be such that $\mathrm{M} = \pi\mathrm{R}$; one has $|\pi| < 1$; moreover, for $a \in \mathrm{R}$, either $|a| \leqslant |\pi|$, or $|a| = 1$. Let $a \in \mathrm{R} - \{0\}$; there exists a largest integer $n \in \mathbf{N}$ such that $|a| \leqslant |\pi|^n$. One thus has $|\pi| < |a/\pi^n| \leqslant 1$, so that $|a/\pi^n| = 1$ and there exists $u \in \mathrm{R}^\times$ such that $a = u\pi^n$.

As a consequence, all ideals of R are of the form $\pi^n\mathrm{R}$, for some unique $n \in \mathbf{N}$. In particular, R is a principal ideal domain. The map $v : \mathrm{K}^\times \to \mathbf{Z}$ given by $v(a) = n$ if and only if $a\mathrm{R} = \pi^n\mathrm{R}$ is a (normalized) discrete valuation on K.

Proposition **(3.1.10)**. — *Let* K *be a field endowed with a nonarchimedean absolute value* $|\cdot|$ *and let* $r = (r_1, \dots, r_n)$ *be a family of strictly positive real numbers. There is a unique absolute value* p_r *on* $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$ *such that for every polynomial* $f = \sum c_m \mathrm{T}^m$*, one has*

$$p_r(f) = \sup_{m \in \mathbf{N}^n} |c_m| r_1^{m_1} \dots r_n^{m_n}.$$

Its restriction to $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ *is the largest absolute value such that* $p_r(\mathrm{T}_j) = r_j$ *for* $j \in \{1, \dots, n\}$ *and which restricts to the given absolute value on* K.

This absolute value is called the *Gauss absolute value* (with parameters r). Indeed, its multiplicativity is essentially equivalent to the multiplicativity of the content of two polynomials with coefficients in a unique factorization domain.

Proof. — To prove the first assertion, it suffices to prove that the given formula defines an absolute value on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, because it then extends uniquely to its fraction field $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$. One has $p_r(0) = 0$; conversely, if $f = \sum c_m \mathrm{T}^m$ is such that $p_r(f) = 0$, then $|c_m| = 0$ for all m, hence $f = 0$. One also has $p_r(1) = 1$.

Let $f = \sum c_m \mathrm{T}^m$ and $g = \sum d_m \mathrm{T}^m$ be two polynomials.

Then $f + g = \sum (c_m + d_m)\mathrm{T}^m$; for every m,

$$|c_m + d_m| r_1^{m_1} \dots r_n^{m_n} \leqslant \left(\sup(|c_m|, |d_m|) r_1^{m_1} \dots r_n^{m_n}\right) \leqslant \sup(p_r(f), p_r(g)),$$

so that $p_r(f + g) \leqslant \sup(p_r(f), p_r(g))$.

Moreover, $fg = \sum_m (\sum_{p+q=m} c_p d_q) \mathrm{T}^m$. For every m, one has

$$\left|\sum_{p+q=m} c_p d_q\right| r^m \leqslant \sup_{p+q=m} |c_p||d_q| r^p r^q \leqslant p_r(f) p_r(g),$$

so that $p_r(fg) \leqslant p_r(f)p_r(g)$. This shows that p_r is a norm on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$, and it remains to prove that p_r is multiplicative.

Let P be the convex hull of the set of all $p \in \mathbf{N}^n$ such that $p_r(f) = |c_p| r^p$, and let Q be the convex hull of the set of all $q \in \mathbf{N}^n$ such that $p_r(g) = |d_q| r^q$. Let m be a vertex of $\mathrm{P} + \mathrm{Q}$; then there is a vertex a of P, and a vertex b of Q such that $m = a + b$. In particular, if $m = p + q$, for $p \in \mathrm{P}$ and $q \in \mathrm{Q}$, then $p = a$ and $q = b$, so that the coefficient of T^m in fg is the sum of $c_a d_b$ and of other elements $c_p d_q$, where either $|c_p| r^q < |c_a| r^a$, or $|d_q| r^q < |d_b| r^b$ (or both) This implies that

$$\left|\sum_{p+q=m} c_p d_q\right| r^m = |c_a d_b| r^m = |c_a| r^a \, |d_b| r^b = p_r(f) p_r(g).$$

Consequently, $p_r(fg) = p_r(f)p_r(g)$ and p_r is a multiplicative seminorm on $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. ☐

3.1.11. — Let K be a field endowed with an absolute value. The map $(a, b) \mapsto |a - b|$ is a distance on K.

Let $\widehat{\mathrm{K}}$ be the completion of K for this distance. Let us recall its definition. One starts from the ring S of all Cauchy sequences in K and the subset M of all Cauchy sequences which converge to 0. It is obvious that M is an additive subgroup of M; since a Cauchy sequence is bounded, it is an ideal of S, and $\widehat{\mathrm{K}}$ is the quotient ring S/M. Let $j \colon \mathrm{K} \to \widehat{\mathrm{K}}$ be the map such that $j(a)$ is the class of the constant sequence with value a; it is a morphism of rings.

For $a, b \in \mathrm{K}$, one has

$$\big||a| - |b|\big| \leqslant |a - b|.$$

This implies that for every Cauchy sequence (a_n) in K, the sequence $(|a_n|)$ is a Cauchy sequence in $\mathbf{R}$; in particular, it converges. It induces a map $|\cdot| \colon \widehat{\mathrm{K}} \to \mathbf{R}$ which is a multiplicative seminorm on $\widehat{\mathrm{K}}$ such that $|j(a)| = |a|$ for every $a \in \mathrm{K}$.

Let $a = (a_n)$ be a Cauchy sequence in K which does not converge to 0; by definition, there exists $\varepsilon > 0$ and arbitrarily large integers n such that $|a_n| \geqslant \varepsilon$. Since (a_n) is a Cauchy sequence, there exists an integer p such that $|a_n - a_m| \leqslant \varepsilon/2$ for all integers $m, n \geqslant p$. Taking $m \geqslant p$ such that $|a_m| \geqslant \varepsilon$, it follows that $|a_n| \geqslant \varepsilon/2$ for all integers $n \geqslant p$. In particular, one has $|a| \geqslant \varepsilon/2$. Consequently, the seminorm on $\widehat{\mathrm{K}}$ is an absolute value.

Set $b_n = 0$ for $n < p$ and $b_n = 1/a_n$ for $n \geqslant p$. The inequalities

$$\left|b_m - b_n\right| = \frac{|a_n - a_m|}{|a_m||a_n|} \leqslant \frac{4}{\varepsilon^2}|a_n - a_m|,$$

for $m, n \geqslant p$, imply that $b = (b_n)$ is a Cauchy sequence. Moreover, ab converges to 1, hence the equality $[a][b] = j(1)$ in $\widehat{\mathrm{K}}$. This proves that $\widehat{\mathrm{K}}$ is a field.

Assume that the initial absolute value of K is nonarchimedean. The obtained absolute value on $\widehat{\mathrm{K}}$ is then nonarchimedean as well. Moreover, with the previous notation, we have $|a_n| = |a_m|$ for all integers $m, n \geqslant p$: if the Cauchy sequence (a_n) does not converge to 0, then the sequence $(|a_n|)$ is eventually constant. In particular, the value group of $\widehat{\mathrm{K}}$ is the same as that of K.

Example **(3.1.12)**. — Let K be a nonarchimedean valued field. It is known that the absolute value of K extends to an absolute value on any algebraic extension of K.

More precisely, if K is complete, then for every algebraic extension L of K, there exists a *unique* extension absolute value on L that extends the absolute value of K. I refer to Dwork *et al* (1994), theorem 5.1, for a detailed proof. Let us just mention that when the extension $\mathrm{K} \to \mathrm{L}$ is finite, the absolute value of L is given by the formula

$$|b|_{\mathrm{L}} = |\mathrm{N}_{\mathrm{L}/\mathrm{K}}(b)|^{1/[\mathrm{L}:\mathrm{K}]},$$

for every $b \in \mathrm{L}$.

3.2. The analytic spectrum of a ring

Definition **(3.2.1)** (Berkovich, 1990). — *Let* K *be a field endowed with a nonarchimedean absolute value and let* R *be a* K*-algebra. The* analytic spectrum *of* R *is the set* $\mathscr{M}(\mathrm{R})$ *of all multiplicative seminorms on* R *which restrict to the given absolute value on* K*, endowed with the coarsest topology for which the maps from* $\mathscr{M}(\mathrm{R})$ *to* $\mathbf{R}$*,* $p \mapsto p(f)$*, are continuous, for every* $f \in \mathrm{R}$*.*

If R is the ring of an affine K-scheme X, hence X = Spec(R), then the analytic spectrum of R is also called the (Berkovich) *analytification* of X, and is denoted by X^{an}.

3.2.2. — Since the evaluation maps $p \mapsto p(f)$ are continuous, for every $a, b \in \mathbf{R}$ such that $a < b$, the set of all multiplicative seminorms p such that $p(f) < b$ (resp. that $p(f) > a$) is an open subset of $\mathscr{M}(\mathrm{R})$. On the other hand, the definition of the topology of $\mathscr{M}(\mathrm{R})$ implies that for every $p \in \mathscr{M}(\mathrm{R})$, a subset V of $\mathscr{M}(\mathrm{R})$ is a neighborhood of p if and only if there exists a finite family $(f_1, \dots, f_m)$ in R, real numbers $(a_1, \dots, a_m)$ and $(b_1, \dots, b_m)$ such that $a_j < p(f_j) < b_j$ for all j, and such that for every $q \in \mathscr{M}(\mathrm{R})$ such that $a_j < q(f_j) < b_j$ for all j, one has $q \in \mathrm{V}$.

Another way to understand the topology of $\mathscr{M}(\mathrm{R})$ is its universal property: if X is a topological space, then a map $\varphi \colon \mathrm{X} \to \mathscr{M}(\mathrm{R})$ is continuous if and only if the map from X to $\mathbf{R}$ given by $x \mapsto \varphi(x)(f)$ is continuous, for every $f \in \mathrm{R}$.

Example **(3.2.3)**. — Let K be a nonarchimedean valued field, let n be a positive integer and let $\mathrm{R} = \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ be the K-algebra of Laurent polynomials in n indeterminates.

a) Every point $z \in (\mathrm{K}^\times)^n$ induces a multiplicative seminorm p_z on R, given by $p_z(f) = |f(z)|$. For every $f \in \mathrm{R}$, the map from $(\mathrm{K}^\times)^n$ to $\mathbf{R}$ given by $z \mapsto |f(z)|$ is continuous. Consequently, the map $z \mapsto p_z$ from $(\mathrm{K}^\times)^n$ to $\mathscr{M}(\mathrm{R})$ is continuous. It is also injective because for $z \in (\mathrm{K}^\times)^n$ and $a \in \mathrm{K}$, one has $p_z(\mathrm{T}_j - a) = |z_j - a| = 0$ if and only if $a = z_j$. One can even show that it induces a homeomorphism from $(\mathrm{K}^\times)^n$ onto its image.

b) For every $r \in (\mathbf{R}_+^*)^n$, the Gauss absolute value p_r on $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$ induces an element of $\mathscr{M}(\mathrm{R})$.

Let $f \in \mathrm{R}$. The formula defining p_r in proposition 3.1.10 shows that the map $r \mapsto p_r(f)$ from $(\mathbf{R}_+^*)^n$ to $\mathbf{R}$ is continuous, as the supremum of a finite family of continuous functions. Consequently, the map $r \mapsto p_r$, from $(\mathbf{R}_+^*)^n$ to $\mathscr{M}(\mathrm{R})$, is continuous. Using the relation $p_r(\mathrm{T}_j) = r_j$, one can also show that this map induces a homeomorphism onto its image which is a closed subset of $\mathscr{M}(\mathrm{R})$.

3.2.4. — Let K be a nonarchimedean valued field and let R be a K-algebra. Let $p \in \mathscr{M}(\mathrm{R})$ be a multiplicative seminorm on R which restricts to the given absolute value on K. The kernel $\mathrm{Ker}(p)$ of p is a prime ideal of R, hence a point of Spec(R), the classical spectrum of R. This furnishes a map $\pi \colon \mathscr{M}(\mathrm{R}) \to \mathrm{Spec}(\mathrm{R})$.

This map is surjective. Indeed, let P be a prime ideal of R; the residue ring R/P is an integral domain; its fraction field $\kappa(\mathrm{P})$ is an extension of K. Using absolute values on completions, algebraic extensions and Gauss norms, we see that there exists an absolute value on this field which extends the given absolute value on K. This absolute value restricts to a multiplicative seminorm on R with kernel P.

3.2.5. — Let J be a ideal of R and let $\mathscr{V}(\mathrm{J})$ be the subset of $\mathscr{M}(\mathrm{R})$ consisting of all seminorms p such that $p(f) = 0$ for every $f \in \mathrm{J}$. It is a closed subset of $\mathscr{M}(\mathrm{R})$, For each $f \in \mathrm{R}$, the set of all seminorms p on R such that $p(f) = 0$ is closed, as the preimage of the closed set $\{0\}$ by the continuous map $f \mapsto p(f)$ on $\mathscr{M}(\mathrm{R})$. Therefore, $\mathscr{V}(\mathrm{J})$ is the intersection of a family of closed subsets of $\mathscr{M}(\mathrm{R})$, hence is closed.

If $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, the following proposition shows that $\mathscr{V}(\mathrm{J})$ identifies with the analytification of $\mathrm{V}(\mathrm{J}) = \mathrm{Spec}(\mathrm{R}/\mathrm{J})$.

Proposition **(3.2.6)**. — *Let* K *be a field endowed with nonarchimedean absolute value.*

a) *If* $\varphi \colon \mathrm{R} \to \mathrm{S}$ *is a morphism of* K*-algebras, then the map* $\varphi^* \colon p \mapsto p \circ \varphi$ *is a continuous map from* $\mathscr{M}(\mathrm{S})$ *to* $\mathscr{M}(\mathrm{R})$.

b) *If* φ *is surjective, then* φ^* *induces a homeomorphism from* $\mathscr{M}(\mathrm{S})$ *to its image, which is a closed subset of* $\mathscr{M}(\mathrm{R})$.

Proof. — a) To prove that φ^* is continuous, it suffices, by the definition of the topology of $\mathscr{M}(\mathrm{R})$, to prove that for every $f \in \mathrm{R}$, the map

$p \mapsto \varphi^*(p)(f) = p(\varphi(f))$ from $\mathscr{M}(\mathrm{S})$ to $\mathbf{R}$ is continuous. But this follows from the fact the definition of the topology of $\mathscr{M}(\mathrm{S})$.

b) Assume that φ is surjective and let $\mathrm{J} = \mathrm{Ker}(\varphi)$. Multiplicative seminorms on S then correspond, via φ, to multiplicative seminorms on R which vanish on J. Consequently, φ^* is injective and its image is the closed subset $\mathscr{V}(\mathrm{J})$ of $\mathscr{M}(\mathrm{R})$ consisting of all seminorms p such that $p(f) = 0$ for every $f \in \mathrm{J}$. Let us prove that the inverse bijection, $(\varphi^*)^{-1} \colon \mathscr{V}(\mathrm{J}) \to \mathscr{M}(\mathrm{S})$, is continuous. By the definition of the topology of $\mathscr{M}(\mathrm{S})$, it suffices to prove that for every $f \in \mathrm{S}$, the map from $\mathscr{V}(\mathrm{J})$ to $\mathbf{R}$ given by $p \mapsto (\varphi^*)^{-1}(p)(f)$ is continuous. Let $g \in \mathrm{R}$ be such that $f = \varphi(g)$. For every $q \in \mathscr{M}(\mathrm{S})$, one has $\varphi^*(q) = q \circ \varphi$, hence $\varphi^*(q)(g) = q \circ \varphi(g) = q(f)$; if $p = \varphi^*(q) \in \mathscr{V}(\mathrm{J})$, one thus has $q = (\varphi^*)^{-1}(p)$ and $(\varphi^*)^{-1}(p)(f) = p(g)$. By definition of the topology of $\mathscr{M}(\mathrm{R})$, the map $p \mapsto p(g)$ is continuous on $\mathscr{M}(\mathrm{R})$, so that the requested map is continuous on $\mathscr{V}(\mathrm{J})$, as the restriction of a continuous map. □

Theorem **(3.2.7)**. — *Let* R *be a finitely generated* K*-algebra and let* $f = (f_1, \dots, f_n)$ *be a generating family. The continuous map* $\mathscr{M}(\mathrm{R})$ *to* $\mathbf{R}^n$ *given by* $p \mapsto (p(f_1), \dots, p(f_n))$ *is proper. In particular,* $\mathscr{M}(\mathrm{R})$ *is a locally compact topological space.*

Proof. — Let $\varphi \colon \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n] \to \mathrm{R}$ be the unique morphism of K-algebras such that $\varphi(\mathrm{T}_j) = f_j$ for all $j \in \{1, \dots, n\}$. Since it induces a closed embedding of $\mathscr{M}(\mathrm{R})$ into $\mathscr{M}(\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n])$, it suffices to treat the case where $\mathrm{R} = \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ and $f_j = \mathrm{T}_j$ for all j.

For $r \in \mathbf{R}$, the set V_r of all $p \in \mathscr{M}(\mathrm{R})$ such that $p(\mathrm{T}_j) < r$ for all j is open in $\mathscr{M}(\mathrm{R})$ and the union of all V_r is equal to $\mathscr{M}(\mathrm{R})$. Moreover, the closure of V_r is contained in the set W_r of all $p \in \mathscr{M}(\mathrm{R})$ such that $p(\mathrm{T}_j) \leqslant r$ for all j. Consequently, to prove that $\mathscr{M}(\mathrm{R})$ is locally compact, it suffices to prove that W_r is compact.

The map $j \colon \mathscr{M}(\mathrm{R}) \to \mathbf{R}_+^{\mathrm{R}}$ given by $p \mapsto (p(f))$ is continuous, by definition of the topology of $\mathscr{M}(\mathrm{R})$ and of the product topology. It is injective, by the definition of a seminorm. Moreover, its image is the subset of $\mathbf{R}_+^{\mathrm{R}}$ defined by the relations in the definition of a multiplicative seminorm, each of them defining a closed subset of $\mathbf{R}_+^{\mathrm{R}}$ since it involves only finitely many elements of R. Finally, j is a homeomorphism onto its image. Indeed, the inverse bijection associates to a family $c = (c_f)$ the multiplicative seminorm $f \mapsto c_f$. To prove that j^{-1} is continuous, it suffices to prove that for every $f \in \mathrm{R}$, the composition $c \mapsto j^{-1}(c)(f)$ is continuous; since this map is the restriction of the projection $c \mapsto c_f$, this is indeed the case, by the definition of the product topology.

For $f \in \mathrm{R}$, set $\|f\|_r = \sup_m |c_m| r^{|m|}$, where $f = \sum c_m \mathrm{T}^m \in \mathrm{R}$ and $|m| = m_1 + \cdots + m_n$. For every $p \in \mathrm{W}_r$, one has $p(f) \leqslant \|f\|_r$, so that $j(\mathrm{W}_r) \subset \prod_{f \in \mathrm{R}} [0; \|f\|_r]$. According to Tikhonov's theorem, the latter set is compact, as a product of compact sets; consequently, W_r is homeomorphic to a closed subset of a compact set, hence is compact.

By what precedes, the inverse image of a compact subset of $\mathbf{R}^n$ by the map $p \mapsto (p(\mathrm{T}_1), \dots, p(\mathrm{T}_n))$ is compact. Since $\mathscr{M}(\mathrm{R})$ and $\mathbf{R}^n$ are locally compact, this implies that this map is proper (Bourbaki (1971), chap 1, §10, n° 3, prop. 7). □

Corollary **(3.2.8)**. — *Let* $\mathrm{X} = \mathrm{Spec}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$. *The map* $\lambda \colon \mathrm{X}^{\mathrm{an}} \to \mathbf{R}^n$ *given by* $p \mapsto (\log(p(\mathrm{T}_1)), \dots, \log(p(\mathrm{T}_n)))$ *is surjective and proper. In particular, for every ideal* I *of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$, $\lambda(\mathscr{V}(\mathrm{I}))$ *is a closed subset of* $\mathbf{R}^n$.

Proof. — Let $x \in \mathbf{R}^n$ and let v_x be the Gauss absolute value of $\mathrm{K}(\mathrm{T}_1, \dots, \mathrm{T}_n)$ such that $v_x(\mathrm{T}_j) = e^{x_j}$ for all j. One has $\lambda(v_x) = x$, so that λ is surjective.

By theorem 3.2.7, the map

$$p \mapsto (\log(p(\mathrm{T}_1)), \dots, \log(p(\mathrm{T}_n)), \log(p(\mathrm{T}_1^{-1})), \log(p(\mathrm{T}_n^{-1})))$$

from $\mathrm{X}^{\mathrm{an}} \to \mathbf{R}^{2n}$ is continuous and proper. Its image is contained in the closed subspace L of $\mathbf{R}^{2n}$ defined by the equations $x_1 = x_{n+1}, x_2 = x_{n+2}, x_n = x_{2n}$, so that λ induces a continuous and proper map from X^{an} to L. The corollary follows from the fact that the linear projection $(x_1, \dots, x_{2n}) \mapsto (x_1, \dots, x_n)$ from $\mathbf{R}^{2n}$ to $\mathbf{R}^n$ induces an homeomorphism from L to $\mathbf{R}^n$. □

3.2.9. — The scheme $\mathrm{X} = \mathrm{Spec}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$ is the n-dimensional torus over K, the algebraic-geometry analogue of the complex manifold $(\mathbf{C}^*)^n$. The map λ is then the analogue of the tropicalization map $(\mathbf{C}^*)^n \to \mathbf{R}^n, (z_1, \dots, z_n) \mapsto (\log(|z_1|), \dots, \log(|z_n|))$ studied in chapter 2.

If I is an ideal of $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$, then the closed subscheme V(I) of X has a Berkovich analytification $\mathscr{V}(I)$, naturally a closed subspace of $X^{an} = \mathscr{M}(K[T_1^{\pm1}, \dots, T_n^{\pm1}])$, and its image $\lambda(\mathscr{V}(I))$ is the *tropicalization* of V(I).

In the algebraic geometry of schemes, one makes a careful distinction between the scheme X (or its closed subscheme V(I)) and its set of points X(K) with values in a given field. One has a natural identification of X(K) with $(K^\times)^n$, an n-tuple $(z_1, \dots, z_n) \in (K^\times)^n$ being identified with the images of $T_1, \dots, T_n$ by a morphism of K-algebras from $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ to K; more generally, for any K-algebra L, the set X(L) identifies with $(L^\times)^n$. Then, the set V(I)(L) identifies with the set of elements $(z_1, \dots, z_n) \in (L^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in I$.

Similarly, a point in $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ is a multiplicative seminorm p on this K-algebra. Its kernel $J_p = \{f\,;\, f(p) = 0\}$ is a prime ideal of $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$ and p induces a multiplicative norm on the quotient K-algebra $K[T_1^{\pm1}, \dots, T_n^{\pm1}]/J_p$, and then on its field of fractions L_p which is a an extension of K endowed with an absolute value that extends the absolute value on K. The field L_p is generated by the images $z_1, \dots, z_n$ of $T_1, \dots, T_n$ by the morphism of K-algebras $K[T_1^{\pm1}, \dots, T_n^{\pm1}] \to L_p$, and the condition $p \in \mathscr{V}(I)$ is equivalent to the condition $I \subset J_p$, or to the condition $f(z_1, \dots, z_n) = 0$ for all $f \in I$. Conversely, any valued extension L of K and any family $(z_1, \dots, z_n) \in (L^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in I$ gives rise to a point in $\mathscr{V}(I)$, given by the multiplicative seminorm $f \mapsto |f(z_1, \dots, z_n)|$ on $K[T_1^{\pm1}, \dots, T_n^{\pm1}]$.

Consequently, the tropicalization of V(I) is the set of all $x \in \mathbf{R}^n$ for which there exists a valued extension L of K and a family $(z_1, \dots, z_n) \in (L^\times)^n$ such that $f(z_1, \dots, z_n) = 0$ for all $f \in I$ and $(\log(|z_1|), \dots, \log(|z_n|)) = (x_1, \dots, x_n)$.

3.3. Nonarchimedean amoebas of hypersurfaces

3.3.1. — Let K be a field endowed with a nonarchimedean absolute value. Let R be the valuation ring of K, let k its residue field and $\mathrm{red}\colon R \to k$ the reduction morphism; it maps the maximal ideal to 0 and induces a morphism of groups from $R^\times$ to $k^\times$.

The map from $v\colon \mathrm{K}^\times$ to $\mathbf{R}$ given by $a \mapsto -\log(|a|)$ is a morphism of groups. Let Γ be its image. One says that the given valued field K is split if we are given a *section* of the surjective map v. Such a section does not exist in general, but it does exist in the following two important cases:

– Assume that K is discretely valued. Then R is a discrete valuation ring. If t is a given generator of its maximal ideal, one has $\Gamma = \mathbf{Z}\log(|t|)$ and the map $n\log(|t|) \mapsto t^n$ is a section as required.

– If K is algebraically closed, then such a section also exists, by an abstract homological algebra argument. Indeed, in this case, $\mathrm{R}^\times$ is a divisible abelian group, hence an injective $\mathbf{Z}$-module. In a more elementary way, one can also use the fact that Γ is a uniquely divisible abelian group, hence a $\mathbf{Q}$-vector space. It then suffices to choose, in a compatible manner, nth roots of a given element of $\mathrm{K}^\times$. Let $\gamma \in \Gamma$ and let $a \in \mathrm{K}^\times$ be such that $\log(|a|) = \gamma$. Let us choose inductively elements $a_n \in \mathrm{K}^\times$ such that $a_1 = a$ and $(a_n)^n = a_{n-1}$ for all integers $n \geqslant 2$. In particular $(a_n)^{n!} = a$ for all $n \geqslant 1$. Moreover, if $m \geqslant n$, then n divides $m!$ and we see by induction that $(a_m)^{m!/n} = (a_n)^{n!/n} = (a_n)^{(n-1)!}$. Then there is a unique morphism of groups from $\mathbf{Q}\gamma$ to $\mathrm{K}^\times$ that maps $\frac{m}{n}\gamma$ to $(a_n)^{(n-1)!m}$ for all $m, n \in \mathbf{Z}$ such that $n \geqslant 1$.

If K is a split valued field, then we can extend the morphism of groups $\mathrm{red}\colon \mathrm{R}^\times \to k^\times$ to a morphism of monoids $\rho\colon \mathrm{K} \to k$, by setting $\rho(a) = \mathrm{red}(a/|a|)$. Note that ρ restricts to a morphism of groups from $\mathrm{K}^\times$ to $k^\times$. Moreover, the map $a \mapsto (-\log(|a|), \rho(a))$ is a group isomorphism from $\mathrm{K}^\times$ to $\Gamma \times k^\times$.

Definition **(3.3.2)**. — *Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ be a nonzero Laurent polynomial; write $f = \sum c_m \mathrm{T}^m$.*

a) *The* tropical polynomial *associated with f is the map*

$$\tau_f\colon \mathrm{R}^n \to \mathbf{R}, \quad x \mapsto \sup_m \left(\log(|c_m|) + \langle m, x\rangle\right).$$

b) *The* tropical hypersurface *defined by f is the subset $\mathscr{T}_f$ of all $x \in \mathbf{R}^n$ such that there exist two distinct elements $m \in \mathbf{Z}^n$ such that $\tau_f(x) = \log(|c_m|) + \langle m, x\rangle$.*

c) *(Assuming that the valued field* K *is split.) For* $x \in \mathbf{R}^n$*, the* initial form *of* f *at* x *is the Laurent polynomial*

$$\mathrm{in}_x(f) = \sum_{\tau_f(x)=\log(|c_m|)+\langle m,x\rangle} \rho(c_m)\mathrm{T}^m.$$

Recall that the *support* of a Laurent polynomial $f = \sum c_m \mathrm{T}^m$ is the set $\mathrm{S}(f)$ of all $m \in \mathbf{Z}^n$ such that $c_m \neq 0$, and that the Newton polytope of f is the convex hull NP_f of $\mathrm{S}(f)$ in $\mathbf{R}^n$.

For $x \in \mathbf{R}^n$, we will often denote by $\mathrm{S}_x(f)$ the subset of $\mathrm{S}(f)$ consisting of those m such that $\tau_f(x) = \log(|c_m|) + \langle m, x\rangle$; this is the support of the initial form $\mathrm{in}_x(f)$; its convex hull is then a sub-polytope $\mathrm{NP}_{f,x}$ of NP_f.

With this notation, the tropical hypersurface $\mathscr{T}_f$ is the set of all $x \in \mathbf{R}^n$ such that $\mathrm{S}_x(f)$ has at least two elements, equivalently, $\mathrm{NP}_{f,x}$ is not a point. When K is a split valued field, this is also equivalent to the property that $\mathrm{in}_x(f)$ is not a monomial.

The preceding concepts make sense when $f = 0$: one has $\mathrm{S}(f) = \varnothing$ (no nonzero monomials), $\tau_f(x) = -\infty$ (supremum of an empty family), and $\mathrm{in}_x(f) = 0$, but the tropical variety $\mathscr{T}_f$ should be defined as $\mathbf{R}^n$.

Remark **(3.3.3)**. — Let $\varphi : \mathbf{G}_{\mathrm{m}_\mathrm{K}}^n \to \mathbf{G}_{\mathrm{m}_\mathrm{K}}^p$ be a *monomial* morphism, given at the level of Laurent polynomials by a morphism of K-algebras $\varphi^* : \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_p^{\pm1}] \to \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ of the form $\mathrm{T}_j \mapsto a_j \mathrm{T}^{e_j}$, where $a_1, \dots, a_p \in \mathrm{K}^\times$ and $e_1, \dots, e_p \in \mathbf{Z}^n$. If L is an extension of K, this morphism φ maps a point $z = (z_1, \dots, z_n) \in (\mathrm{L}^\times)^n$ to the point $\varphi(z) = (a_1 z^{e_1}, \dots, a_p z^{e_p})$.

This morphism gives rise to an affine map $\varphi_\tau : \mathbf{R}^m \to \mathbf{R}^p$, given by $x = (x_1, \dots, x_n) \mapsto (\log(|a_1|) + \langle e_1, x\rangle, \dots)$ and to monomial morphism $\varphi_\rho : \mathbf{G}_{\mathrm{m}_k}^n \to \mathbf{G}_{\mathrm{m}_k}^p$ given by $z \mapsto (\alpha_1 z^{e_1}, \dots, z_p^{e_p})$, where $\alpha_1 = \rho(a_1), \dots, \alpha_p = \rho(a_p)$.

Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_p^{\pm1}]$; write $f = \sum_{m \in \mathbf{Z}^p} c_m \mathrm{T}^m$ so that

$$\varphi^*(f) = \sum_{m\in\mathbf{Z}^p} c_m a_1^{m_1} \dots a_p^{m_p} \mathrm{T}^{m_1 e_1 + \dots + m_p e_p}.$$

If the rank of $(e_1, \dots, e_p) \in \mathrm{M}_{n,p}(\mathbf{Z})$ is equal to p, then all exponents $m_1e_1 + \dots + m_pe_p$ are pairwise distinct. This implies that

$$\begin{aligned}\tau_{\varphi^*(f)}(x) &= \sup_m \Big(\log(|c_m|) + m_1 \log(|a_1|) + \dots + m_p \log(|a_p|) \\ &\qquad\qquad + \langle m_1e_1 + \dots + m_pe_p, x\rangle \Big) \\ &= \sup_m \Big(\log(|c_m|) + m_1(\log(|a_1|) + \langle e_1, x\rangle) + \dots \\ &\qquad\qquad + m_p(\log(|a_p|) + \langle e_p, x\rangle)\Big) \\ &= \sup_m \Big(\log(|c_m|) + m_1y_1 + \dots + m_py_p \Big),\end{aligned}$$

where $y_j = \log(|a_j|) + \langle e_j, x\rangle)$ for $j \in \{1, \dots, p\}$. This shows that

$$\tau_{\varphi^*(f)} = \tau_f \circ \varphi_\tau.$$

If K is a split valued field, we obtain similarly that

$$\mathrm{in}_x(\varphi^*(f)) = \sum_m \rho(c_m)\alpha_1^{m_1} \dots \alpha_p^{m_p} \mathrm{T}^{m_1e_1+\dots+m_pe_p} = \varphi_\rho^*(\mathrm{in}_{\varphi_\tau(x)}(f)).$$

Lemma **(3.3.4)**. — *Let $f, g \in \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ be nonzero Laurent polynomials and let $h = fg$. For every $x \in \mathbf{R}^n$, one has the following relations:*

(i) $\tau_h(x) = \tau_f(x) + \tau_g(x)$;
(ii) $\mathrm{NP}_{h,x} = \mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$;
(iii) *If* K *is a split valued field, then* $\mathrm{in}_x(h) = \mathrm{in}_x(f)\mathrm{in}_x(g)$.

Proof. — Write $f = \sum a_p \mathrm{T}^p$, $g = \sum b_q \mathrm{T}^q$ and $h = \sum c_m \mathrm{T}^m$. For $m \in \mathbf{Z}^n$, one has $c_m = \sum_{p+q=m} a_pb_q$, hence

$$\begin{aligned}\log(|c_m|) + \langle m, x\rangle &\leqslant \sup_{p+q=m} (\log(|a_p|) + \langle p, x\rangle) + (\log(|b_q|) + \langle q, x\rangle) \\ &\leqslant \sup_p(\log(|a_p|) + \langle p, x\rangle) + \sup_q(\log(|c_q|) + \langle q, x\rangle) \\ &= \tau_f(x) + \tau_g(x).\end{aligned}$$

This proves that $\tau_h(x) \leqslant \tau_f(x) + \tau_g(x)$.

In order to prove that equality holds, we observe that the set of all $m \in \mathbf{Z}^m$ such that $\log(|c_m|) + \langle m, x \rangle = \tau_f(x) + \tau_g(x)$ is contained in the sum $\mathrm{S}_x(f) + \mathrm{S}_x(g)$. Let then μ be a vertex of the Minkowski sum $\mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$ and let $\xi \in \mathbf{R}^n$ be such that $\langle m, \xi \rangle < \langle \mu, \xi \rangle$ for every $m \in \mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$ such that $m \neq \mu$. The face of $\mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$ defined by ξ contains the Minkowski sum of the faces of $\mathrm{NP}_{f,x}$ and $\mathrm{NP}_{g,x}$ defined by ξ; consequently, these faces are vertices and there exists a unique pair (p, q), with $p \in \mathrm{S}_x(f)$ and $q \in \mathrm{S}_x(g)$, such that $\mu = p + q$. The initial computation then implies that

$$\log(|c_\mu|) + \langle \mu, x \rangle = (\log(|a_p|) + \langle p, x \rangle) + (\log(|b_q|) + \langle q, x \rangle),$$

so that $\tau_h(x) = \tau_f(x) + \tau_g(x)$. It also shows that $\mathrm{NP}_{f,x} + \mathrm{NP}_{g,x} \subset \mathrm{NP}_{h,x}$. However, we have seen that every vertex of $\mathrm{NP}_{h,x}$ belongs to $\mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$, which implies the equality $\mathrm{NP}_{h,x} = \mathrm{NP}_{f,x} + \mathrm{NP}_{g,x}$.

The relation regarding initial forms is a refinement of these properties.

Let $m \in \mathbf{Z}^n$. If $\log(|c_m|) + \langle m, x \rangle < \tau_h(x)$, then the monomial T^m does not appear in $\mathrm{in}_x(h)$.

Otherwise, since $\tau_h(x) = \tau_f(x) + \tau_g(x)$, one has $\log(|a_p|) + \log(|b_q|) \leqslant \log(|c_m|)$ for every pair (p, q) such that $p + q = m$, and equality is achieved for at least one pair. Consequently,

$$\begin{aligned} t^{-\log(|c_m|)} c_m &= t^{-\log(|c_m|)} \sum_{p+q=m} a_p b_q \\ &= \sum_{\substack{p+q=m \\ p \in \mathrm{S}_x(f) \\ q \in \mathrm{S}_x(g)}} t^{-\log(|a_p|)} a_p t^{-\log(|b_q|)} b_q \\ &\qquad + \sum_{\text{other terms}} t^{-\log(|c_m|) + \log(|a_p|) + \log(|b_q|)} t^{-\log(|a_p|)} a_p t^{-\log(|b_q|)} b_q, \end{aligned}$$

a relation between elements of R. The reduction of the left hand side modulo the maximal ideal is the coefficient $\rho(c_m)$ of T^m in $\mathrm{in}_x(h)$. Similarly, if $p \in \mathrm{S}_x(f)$, then the reduction of $t^{-\log(|a_p|)} a_p$ is $\rho(a_p)$; if $q \in \mathrm{S}_x(g)$, then the reduction of $t^{-\log(|b_q|)} b_q$ is $\rho(b_q)$. On the other hand, if $\log(|a_p|) + \log(|b_q|) < \log(|c_m|)$, then the reduction of corresponding term on the right hand side is zero. Consequently,

$$\rho(c_m) = \sum_{\substack{p+q=m \\ p\in S_x(f) \\ q\in S_x(g)}} \rho(a_p)\rho(b_q).$$

Since $\mathrm{in}_x(f) = \sum_{p\in S_x(f)} \rho(a_p)T^p$ and $\mathrm{in}_x(g) = \sum_{q\in S_x(g)} \rho(b_q)T^q$, this proves the coefficient of T^m in $\mathrm{in}_x(h)$ is equal to the coefficient of T^m in the product of $\mathrm{in}_x(f)$ and $\mathrm{in}_x(g)$. Consequently, $\mathrm{in}_x(h) = \mathrm{in}_x(f)\mathrm{in}_x(g)$, as claimed. □

Proposition **(3.3.5)**. — *Let $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be a Laurent polynomial. The associated tropical hypersurface $\mathscr{T}_f$ is a closed Γ-strict polyhedral subset of $\mathbf{R}^n$, purely of dimension $n-1$. More precisely, there exists a Γ-strict polyhedral decomposition of $\mathbf{R}^n$ the $(n-1)$-dimensional polyhedra of which $\mathscr{T}_f$ is the union.*

Proof. — Write $f = \sum c_m T^m$; let $S(f)$ be the support of f; for $x \in \mathbf{R}^n$, let $S_x(f)$ be the set of all $m \in S(f)$ such that $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$.

For every $m \in S(f)$, let P_m be the the set of $x \in \mathbf{R}^n$ such that $m \in S_x(f)$. Since P_m is defined in $\mathbf{R}^n$ by the affine inequalities $\log(|c_q|) + \langle q, x\rangle \leqslant \log(|c_m|) + \langle m, x\rangle$ for all $q \in S(f)$, it is a convex polyhedron. The slopes of these affine forms are integers, and their constant terms are elements of the value group $\Gamma = \log(|K^\times|)$ of K; consequently, P_m is a Γ-strict convex polyhedron. By construction, these polyhedra cover $\mathbf{R}^n$.[1]

If $S_x(f)$ is reduced to an element m, then then there exists an open neighborhood V of x such that $S_y(f) = \{m\}$ for all $y \in V$; in particular, V is is disjoint from the other polyhedra P_q, and it is contained in the interior of P_m.

On the other hand, for two distinct elements m, q of $S(f)$, the polyhedron $P_m \cap P_q$ is contained in the hyperplane defined by the nontrivial affine equation $\log(|c_m|) + \langle m, x\rangle = \log(|c_q|) + \langle q, x\rangle$, so that $P_m \cap P_q$ is disjoint from the interior of P_m. In particular, if $\mathrm{Card}(S_x(f)) \geqslant 2$, then x does not belong to the interior of P_m.

[1]Vérifier la terminologie sur les polyèdres stricts

This proves that $\mathbf{R}^n$ is the union of those polyhedra P_m which have dimension n, and that the union of their interiors is the set of all $x \in \mathbf{R}^n$ such that $\mathrm{S}_x(f)$ is reduced to one element.

Consequently, the tropical hypersurface $\mathscr{T}_f$, which is its complementary subset, is the union of the $(n-1)$-dimensional faces of these polyhedra P_m, and they are Γ-strict convex polyhedra. ☐

Theorem **(3.3.6)** (Kapranov). — *Let* $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *be a Laurent polynomial. The following three subsets of* $\mathbf{R}^n$ *coincide:*

(i) *The tropical hypersurface* $\mathscr{T}_f$*;*

(ii) *The set of all* $x \in \mathbf{R}^n$ *such that there exists a valued extension* L *of* K *and a point* $z \in (\mathrm{L}^\times)^n$ *such that* $f(z) = 0$ *and* $x = \lambda(z)$*;*

(iii) *The image of* $\mathscr{V}(f)$ *in* $\mathscr{M}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$ *by the tropicalization map* $\lambda \colon \mathscr{M}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$*,* $p \mapsto (\log(p(\mathrm{T}_1)), \dots, \log(p(\mathrm{T}_n)))$*.*

Assuming that the valued field K *is split, they concide with:*

(iv) *The set of all* $x \in \mathbf{R}^n$ *such that* $\mathrm{in}_x(f)$ *is not a monomial.*

If L *is an algebraically closed extension of* K*, endowed with a nontrivial absolute value extending that of* K*, they also coincide with the set:*

(v) *The closure of the set of all* $x \in \mathbf{R}^n$ *such that there exists a point* $z \in (\mathrm{L}^\times)^n$ *such that* $f(z) = 0$ *and* $x = \lambda(z)$*.*

Proof. — Let $\mathrm{S}_1 = \mathscr{T}_f, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5$ be these subsets. Write $f = \sum c_m \mathrm{T}^m$.

Let $x \in \mathbf{R}^n$. Let $m \in \mathbf{Z}^n$; the monomial T^m appears in $\mathrm{in}_x(f)$ if and only if $\log(|c_m|) + \langle m, x \rangle = \tau_f(x)$. Consequently, $\mathrm{in}_x(f)$ is a monomial if and only if the supremum defining $\tau_f(x)$ is reached only once. This proves that $\mathrm{S}_1 = \mathrm{S}_4$.

The equality $\mathrm{S}_2 = \mathrm{S}_3$ follows from the discussion in §3.2.9.

Let L be a valued extension of K, let $z \in (\mathrm{L}^\times)^n$ be a point such that $f(z) = 0$ and let $x = \lambda(z)$. One has $\sum c_m z^m = 0$. Since the absolute value is nonarchimedean, the supremum of all $|c_m z^m|$ must be attained twice. Since $\lambda(c_m z^m) = \log(|c_m|) + \langle m, x \rangle$, this shows that there exist two distinct elements $m, q \in \mathbf{Z}^n$ such that $\log(|c_m|) + \langle m, x \rangle = \tau_f(x)$. In other words, x belongs to the hypersurface $\mathscr{T}_f$. This proves that one has $\mathrm{S}_2 \subset \mathrm{S}_1$.

By definition, the set S_5 is the closure of a subset of S_2; since S_1 is closed, one also has $\mathrm{S}_5 \subset \mathrm{S}_1$.

By the corollary to the lifting proposition below, one has $\mathscr{T}_f \cap \Gamma^n \in S_5$. Since K is algebraically closed and its valuation is nontrivial, the group Γ is a non zero $\mathbf{Q}$-subspace of $\mathbf{R}$; in particular, it is dense in $\mathbf{R}$. On the other hand, since $\mathscr{T}_f$ is a Γ-strict polyhedral subspace of $\mathbf{R}^n$, its subset $\mathscr{T}_f \cap \Gamma^n$ is dense in $\mathscr{T}_f$. Since $\mathscr{T}_f$ is closed in $\mathbf{R}^n$, this implies that $S_1 = \mathscr{T}_f \subset S_5$.

Using Gauss absolute values (proposition 3.1.10) and example 3.1.12, there exists an algebraically closed valued extension L of K whose value group Γ_L contains the coordinates of x. By the corollary of the lifting proposition, there exists $z \in (L^\times)^n$ such that $f(z) = 0$ and $\lambda(z) = x$; in other words, one has $x \in S_2$. Consequently, $S_1 \subset S_2$. This concludes the proof of the theorem. □

Proposition **(3.3.7)** (Lifting). — *Assume that* K *is an algebraically closed valued field with residue field* k. *Let* $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *be a Laurent polynomial. We assume that the coefficients of* f *belong to the valuation ring of* K *and that its reduction* $\varphi \in k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *is nonzero.*

For every $\alpha \in (k^\times)^n$ *such that* $\varphi(f) = 0$, *there exists* $a \in (R^\times)^n$ *such that* $\rho(a) = \alpha$ *and* $f(a) = 0$. *Moreover, if* f *is irreducible, then the set of such* a *is Zariski dense in the closed subscheme* $V(f)$ *of* $\mathbf{G}_{m_K}^n$.

Proof. — We do the proof by induction on n.

Let us first assume that $n = 1$. Since K is algebraically closed, we may write $f = cT^m \prod_{j=1}^q (T - a_j)$, for some $c \in K^\times$, $m \in \mathbf{Z}$, $q \in \mathbf{N}$ and $a_1, \dots, a_q \in K^\times$. If $|a_j| > 1$, we write $T - a_j = -a_j(1 - a_j^{-1}T)$, so that

$$f = c \prod_{|a_j|>1} (-a_j) T^m \prod_{|a_j|>1} (1 - a_j^{-1}T) \prod_{|a_j|\leqslant 1} (T - a_j).$$

Let $c' = c \prod_{|a_j|>1}(-a_j)$. If $|c'| < 1$, then this formula shows that f reduces to 0 in $k[T^{\pm 1}]$, contradicting the stated hypothesis that $\varphi \neq 0$. If $|c'| > 1$, the coefficients of $(c')^{-1}f$ belong to the maximal ideal of the valuation ring of K, so that the reduction of $(c')^{-1}f$ is zero; on the other hand, we see that this reduction is equal to $T^m \prod_{|a_j|\leqslant 1}(T - \overline{a_j})$. then the relation $(c')^{-1}f$ and the hypothesis Consequently, $|c'| = 1$ and $\varphi = \rho(c')T^m(\prod_{|a_j|<1} T) \prod_{|a_j|=1}(T - \rho(a_j))$. Since φ vanishes at α, there must exist $j \in \{1, \dots, q\}$ such that $|a_j| = 1$ and $\rho(a_j) = \alpha$. This proves the proposition in this case.

To do the induction step, we first perform a multiplicative Noether normalization theorem to reduce to the case where the map $m \mapsto m_1$ from the support $S(f)$ of f to $\mathbf{Z}$ is injective. To see that it is possible, we make an invertible monomial change of variables $T_1 \to T_1$, $T_2 \to T_2T_1^q, \dots, T_{n-1} \to T_{n-1}T_1^{q^{n-2}}, T_n \to T_nT_1^{q^{n-1}}$ for some integer q, chosen to be large enough so that $q > |m_j - m'_j|$ for all $m, m' \in S(f)$ and all $j \in \{1, \dots, n\}$. This change of variables transforms the Laurent polynomial f into the polynomial

$$f_q = \sum_{m \in S(f)} c_m T_1^{\varphi(m)} T_2^{m_2} \dots T_n^{m_n},$$

where

$$\varphi(m) = m_1 + qm_2 + q^2m_3 + \dots + q^{n-1}m_n.$$

Let $m, m' \in S(f)$ be such that $m \neq m'$; let $k \in \{1, \dots, n\}$ be such that $m_j = m'_j$ for $j > k$ and $m_k \neq m'_k$; then one has

$$\varphi(m') - \varphi(m) = \sum_{j=1}^{n} q^{j-1}(m'_j - m_j) = \sum_{j=1}^{k-1} q^{j-1}(m'_j - m_j) + q^{k-1}(m'_k - m_k).$$

In absolute value, the last term is at least q^{k-1}, because $m'_k \neq m_k$; on the other hand, the first one is bounded from the above by

$$\sum_{j=1}^{k-1} q^{j-1}(q-1) = (q-1)\frac{q^{k-1}-1}{q-1} = q^{k-1} - 1,$$

hence $|\varphi(m') - \varphi(m)| \geqslant 1$.

Assume that this property holds. In other words, if f is written as a Laurent polynomial in T_1, with coefficients Laurent polynomials in $T_2, \dots, T_n$, then all of these coefficients are monomials.

Then fix any lifting $a' = (a_2, \dots, a_n) \in (R^\times)^{n-1}$ of $\alpha' = (\alpha_2, \dots, \alpha_n)$. The polynomial f is not a monomial; otherwise φ would be a monomial and would not vanish at α. Thanks to the property imposed on the exponents of f, the one-variable Laurent polynomial $f(T, a')$ is not a monomial either; its reduction is $\varphi(T, \alpha')$ and vanishes at α_1. By the $n = 1$ case, there exists $a_1 \in R^\times$ such that $\rho(a_1) = \alpha_1$ and $f(a_1, a') = 0$.

To prove the density, we let Z be the Zariski closure in $\mathbf{G}_{\mathrm{m}\,\mathrm{K}}^{n}$ of the set of these elements $a \in (\mathrm{R}^\times)^n$ such that $f(a) = 0$ and $\rho(a) = \alpha$. By definition, the ideal $\mathscr{I}(\mathrm{Z})$ of Z is the set of all Laurent polynomials $h \in \mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ such that $h(a) = 0$ for all these a. One has $f \in \mathscr{I}(\mathrm{Z})$ by construction, hence $(f) \subset \mathscr{I}(\mathrm{Z})$. To prove that $\mathrm{Z} = \mathscr{V}(f)$, it suffices to prove that $\mathscr{I}(\mathrm{Z}) = (f)$.

Let $g \in \mathscr{I}(\mathrm{Z}) - (f)$. Since $\mathrm{K}[\mathrm{T}_2^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$ is a unique factorization domain and f is irreducible in $\mathrm{K}[\mathrm{T}_1^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$, Gauss's theorem shows that f is either a unit, or irreducible in the one-variable polynomial ring $\mathrm{K}(\mathrm{T}_2, \dots, \mathrm{T}_n)[\mathrm{T}_1, \mathrm{T}_1^{\pm1}]$. Since g does not belong to (f), the polynomials f and g are coprime in this principal ideal domain and there exist polynomials $u, v \in \mathrm{K}(\mathrm{T}_2, \dots, \mathrm{T}_n)[\mathrm{T}_1, \mathrm{T}_1^{\pm1}]$ such that $uf + vg$ is a nonzero element of $\mathrm{K}(\mathrm{T}_2, \dots, \mathrm{T}_n)$. Multiplying by a common denominator, this furnishes a nonzero element h of $(f, g) \cap \mathrm{K}[\mathrm{T}_2^{\pm1}, \dots, \mathrm{T}_n^{\pm1}]$. Let $a' \in (\mathrm{R}^\times)^{n-1}$ be such that $\rho(a') = \alpha'$. By what precedes, there exists $a \in \mathrm{R}^n$ of the form $a = (t, a')$ such that $f(t, a') = 0$ and $\rho(a) = \alpha$; by assumption, $g(a) = 0$, hence $h(a') = 0$. This contradicts the fact that these elements a' are Zariski-dense in $\mathbf{G}_{\mathrm{m}\,\mathrm{K}}^{n-1}$ (lemma 3.3.8 below). □

Lemma **(3.3.8)**. — *Let* K *be a field, let* $\mathrm{A}_1, \dots, \mathrm{A}_n$ *be subsets of* K *and let* $\mathrm{A} = \mathrm{A}_1 \times \dots \times \mathrm{A}_n$. *Let* $f \in \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. *If* $\mathrm{Card}(\mathrm{A}_j) > \deg_{\mathrm{T}_j}(f)$ *for all* j, *then there exists* $a \in \mathrm{A}$ *such that* $f(a) \neq 0$.

In particular, if $\mathrm{A}_1, \dots, \mathrm{A}_n$ *are infinite, then* A *is Zariski dense in* $\mathbf{A}^n$.

Proof. — If $n = 1$, this amounts to the fact that a polynomial in one variable has no more roots than its degree. We then prove the result by induction on n, writing $f = f_0 + f_1\mathrm{T}_1 + \dots + f_m\mathrm{T}_1^m$, for $f_0, \dots, f_m \in \mathrm{K}[\mathrm{T}_2, \dots, \mathrm{T}_n]$, where $m = \deg_{\mathrm{T}_1}(f)$, hence $f_m \neq 0$. By induction, there exists $a_2 \in \mathrm{A}_2, \dots, a_n \in \mathrm{A}_n$ such that $f_m(a_2, \dots, a_n) \neq 0$. This implies that the polynomial $f(\mathrm{T}, a_2, \dots, a_n)$ has degree m. Since $\mathrm{Card}(\mathrm{A}_1) > \deg_{\mathrm{T}_1}(f) = m$, there exists $a_1 \in \mathrm{A}_1$ such that $f(a_1, a_2, \dots, a_n) \neq 0$, as was to be shown. □

Corollary **(3.3.9)**. — *Assume that* K *is an algebraically closed split valued field and let* $x \in \Gamma^n$. *Then, for every* $\alpha \in (k^\times)^n$ *such that* $\mathrm{in}_x(f)(\alpha) = 0$, *there exists* $a \in (\mathrm{K}^\times)^n$ *such that* $\lambda(a) = x$ *and* $f(a) = 0$. *Moreover, if* f *is irreducible, then the set of such* a *is Zariski dense in* $\mathscr{V}(f)$.

Proof. — By assumption, there exists $b \in (\mathrm{K}^\times)^n$ such that $\lambda(b) = x$. Let $c \in (\mathrm{R}^\times)^n$ be such that $\rho(c) = \rho(b)$. Since $\lambda(c) = 1$, we then have $\lambda(bc^{-1}) = \lambda(b) = x$ and $\rho(bc^{-1}) = \rho(b)\rho(c)^{-1} = 1$. Replacing b by bc^{-1}, we now assume that $\lambda(b) = x$ and $\rho(b) = 1$. Let $g(\mathrm{T}) = f(b_1\mathrm{T}_1, \dots, b_n\mathrm{T}_n)$; writing $f = \sum_{m\in \mathrm{S}(f)} c_m \mathrm{T}^m$, we have $g = \sum_{m\in\mathrm{S}(f)} c_m b^m \mathrm{T}^m$. Consequently, for every $z \in \mathbf{R}^n$, one has

$$\tau_g(z) = \sup(\log(|c_m|) + \langle m, x+z\rangle) = \tau_f(x+z).$$

This also shows that $\mathrm{S}_z(g) = \mathrm{S}_{x+z}(f)$ and that

$$\mathrm{in}_z(g) = \sum_{m\in\mathrm{S}_z(g)} \rho(c_m b^m)\mathrm{T}^m = \sum_{m\in\mathrm{S}_{x+z}(f)} \rho(c_m)\mathrm{T}^m = \mathrm{in}_{x+z}(f).$$

In particular, $x \in \mathscr{T}_f$ if and only if $0 \in \mathscr{T}_g$, $\mathrm{in}_x(f)(\alpha) = 0$ if and only if $\mathrm{in}_0(g)(\alpha) = 0$, and $g(a) = 0$ if and only if $f(ab) = 0$, where $ab = (a_1b_1, \dots, a_nb_n)$.

By the lifting proposition, there exists $a \in (\mathrm{R}^\times)^n$ such that $\rho(a) = \alpha$ and $g(a) = 0$; then $ab \in (\mathrm{K}^\times)^n$ satisfies $\rho(ab) = \alpha$ and $f(ab) = 0$.

Moreover, if f is irreducible, then g is irreducible as well, the set of such elements a is Zariski dense in $\mathscr{V}(g)$, hence the set of such elements ab is Zariski dense in $\mathscr{V}(f)$. □

3.4. Monomial ideals

Definition **(3.4.1)**. — *An ideal of* $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ *is said to be* monomial *if it is generated by a set of monomials.*

Observe that if an ideal I is generated by a family (f_i) of monomials, then a monomial f belongs to I if and only if it is divisible by some f_i.

Lemma **(3.4.2)**. — *Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. *The following properties are equivalent:*

(i) *The ideal* I *is monomial;*

(ii) *For every polynomial* $f \in \mathrm{I}$, *every monomial that appears in* f *belongs to* I.

If I is an ideal of $K[T_1, \dots, T_n]$, we shall sometimes consider the ideal J generated by all monomials which belong to I; it is the largest monomial ideal contained in I.

Proof. — (i)⇒(ii). Assume that I is monomial. Let $f \in I$; we may write $f = \sum_{i=1}^m f_i g_i$, where f_i is a monomial in a given generating family of I and $g_i \in K[T_1, \dots, T_n]$. Let cT^m be a (nonzero) monomial that appears in f. There exists $i \in \{1, \dots, m\}$ such that m belongs to the support of $f_i g_i$; since every monomial of $f_i g_i$ is divisible by the monomial f_i, this implies that f_i divides T^m, hence $cT^m \in (I)$.

(ii)⇒(i). Let (f_i) be a generating family of I. By assumption, all the monomials of the f_i belong to I. The family consisting of all of these monomials generates an ideal which is contained in I by assumption, and which contains I since it contains all of the f_i. □

Example **(3.4.3)**. — The ideal generated by a subfamily $(T_i)_{i \in S}$ of the indeterminates is a monomial ideal. It is also prime, since the quotient ring, isomorphic to the polynomial ring $K[(T_i)_{i \notin S}]$ in the other indeterminates, is an integral domain.

Conversely, all prime monomial ideals are of this form. Let indeed I be a prime monomial ideal of $K[T_1, \dots, T_n]$ and let S be the set of all $i \in \{1, \dots, n\}$ such that $T_i \in I$; let us prove that $I = ((T_i)_{i \in S})$. The inclusion $((T_i)_{i \in S}) \subset I$ is obvious. Conversely, let $f \in I$ and let us prove that $f \in ((T_i)_{i \in S})$. Since all monomials of f belong to I, we may assume that f is a monomial; write $f = cT^m = cT_1^{m_1} \dots T_n^{m_n}$. If none of the indeterminates that appear in f belong to I, then neither does their product, by definition of a prime ideal. Consequently, there exists $i \in S$ such that $m_i \geqslant 1$, and $f \in (T_i) \subset ((T_i)_{i \in S})$.

Proposition **(3.4.4)**. — a) *The sum and the intersection of a family of monomial ideals is a monomial ideal.*

b) *The radical of a monomial ideal is a monomial ideal.*

c) *Every monomial ideal has a primary decomposition which consists of monomial ideals. In particular, the prime ideals associated with a monomial ideal are monomial ideals.*

Proof. — a) The case of a sum follows directly from the definition. Let (I_j) be a family of monomial ideals and let $I = \bigcap_j I_j$. Let $f \in I$ and let cT^m be a monomial that appears in f. Fix an index j; since $f \in I_j$ and I_j is a monomial ideal, we have $cT^m \in I_j$. Consequently, $cT^m \in I$. This proves that I is a monomial ideal.

b) Let I be a monomial ideal and let $J = \sqrt{I}$; let us prove that J is a monomial ideal. Let $f \in J$ and let us prove that every monomial of f belongs to J. Subtracting from f its monomials that belong to J, we may assume that no monomial of f belongs to J; assume, arguing by contradiction, that $f \neq 0$ and write $f = \sum c_m T^m$. Let $m \in \mathbf{N}^m$ be a vertex of the Newton polytope of f, so that $c_m \neq 0$ and $T^m \notin J$. Then for every integer $s \geqslant 1$, the exponent sm is a vertex of the Newton polytope of f^s, because $NP_{f^s} = sNP_f$, and the coefficient of T^{sm} in f^s is equal to c_m^s. Since I is a monomial ideal, one has $c_m^s T^{sm} \in I$; by the definition of the radical, one has $T^m \in J$, a contradiction.

c) Let I be a monomial ideal and let us consider a primary decomposition $I = \bigcap_\alpha I_\alpha$ of I. For every α, let P_α be the radical of I_α, let J_α be the largest monomial ideal in I_α.

Let Q_α be the radical of J_α. It is the largest monomial ideal contained in P_α. Indeed, if a monomial T^m belongs to P_α, then there exists $s \geqslant 1$ such that $T^{sm} \in I_\alpha$, hence $T^{sm} \in J_\alpha$, hence $T^m \in Q_\alpha$.

Let us prove that Q_α is a prime ideal. It is contained in P_α, hence is not equal to (1). Let $f, g \in K[T_1, \dots, T_n]$ be such that $fg \in Q_\alpha$; subtracting from f and g all of their monomials that belong to Q_α, we may assume that they have no monomial in Q_α; assuming that $f \neq 0$, we need to prove that g belongs to Q_α. We may assume that $g \neq 0$. The Newton polytope of fg is equal to the Minkowski sum of the Newton polytopes and f and g. Considering a vertex of the Newton polytope of fg, we get two monomials $c_m T^m$ of f, and $d_q T^q$ of g, such that their product $c_m d_q T^{m+q}$ is a monomial of fg, and their power $(c_m d_q)^s T^{s(m+q)}$ is a monomial of $(fg)^s$, for every integer $s \geqslant 1$. Since Q_α is the radical of J_α, there exists s such that $(fg)^s \in J_\alpha$; since J_α is a monomial ideal, one then has $T^{s(m+q)} \in J_\alpha \subset I_\alpha$, hence $T^{m+q} \in P_\alpha$. The monomial T^m does not belong to P_α, hence $T^q \in P_\alpha$, hence $T^q \in Q_\alpha$.

We now prove that J_α is a Q_α-primary ideal. Similarly, we consider $f, g \in K[T_1, \dots, T_n]$ such that $fg \in J_\alpha$ and $f \notin Q_\alpha$, and prove that $g \in J_\alpha$. Subtracting from f and g all monomials that belong to Q_α and J_α respectively, we reduce ourselves to the case where no monomial of f belongs to Q_α, and no monomial of g belongs to J_α. Assume that $f, g \neq 0$; as above, there are monomials $c_m T^m$ of f and $d_q T^q$ of g such that $c_m d_q T^{m+q}$ is a monomial of fg. Since J_α is a monomial ideal, one has $T^{m+q} \in J_\alpha \subset I_\alpha$. Since $T^m \notin Q_\alpha$ and T^m is a monomial, one has $T^m \notin P_\alpha$. Since I_α is P_α-primary, one then has $T^q \in I_\alpha$, hence $T^q \in J_\alpha$, a contradiction.

Let us now prove that $I = \bigcap_\alpha J_\alpha$. One has $J_\alpha \subset I_\alpha$ for all α, hence $\bigcap_\alpha J_\alpha \subset \bigcap_\alpha I_\alpha = I$. To prove the other inclusion, let $f \in I$ and let us prove that $f \in J_\alpha$ for all α. Since I is a monomial ideal, it suffices to treat the case where f is a monomial. Then for every α, one has $f \in I_\alpha$, hence $f \in J_\alpha$ since f is a monomial. Consequently, $f \in \bigcap_\alpha J_\alpha$.

□

Theorem **(3.4.5)** (Maclagan, 2001). — *Let* K *be a field and let* $\mathscr{F}$ *be an infinite set of monomial ideals in* $K[T_1, \dots, T_n]$. *There exists a strictly decreasing sequence of elements of* $\mathscr{F}$.

Proof. — The set of monomial prime ideals is finite. Considering minimal primary decompositions consisting of monomial ideals and successively extracting infinite subsets, we reduce to the case where all ideals in $\mathscr{F}$ are primary with respect to the same prime ideal, $(T_1, \dots, T_m)$. Replacing K by the field $K(T_{m+1}, \dots, T_n)$, we are reduced to the case where all ideals in $\mathscr{F}$ are primary with respect to the maximal ideal $(T_1, \dots, T_n)$.

For every monomial ideal I, let M(I) be the set of $m \in \mathbf{N}^n$ such that $T^m \notin I$.

If $I \in \mathscr{F}$, there exists an integer $N \geqslant 1$ such that $(T_1^N, \dots, T_n^N) \subset I$, so that the set M(I) is contained in $[0; N]^n$; in particular, M(I) is finite.

Observe that the inclusion $I \subset J$ is equivalent to the inclusion $M(J) \subset M(I)$. We will first prove by contradiction that there are ideals $I, J \in \mathscr{F}$ such that $I \subsetneq J$. Assume otherwise.

Let J_0 be the intersection of all ideals in $\mathscr{F}$ and choose $I_1 \in \mathscr{F}$. For every $I \in \mathscr{F}$ such that $I \neq I_1$, one has $I_1 \not\subset I$, so that there exists $m \in M(I_1)$ such that $T^m \in I$. Since $\mathscr{F}$ is infinite and $M(I_1)$ is finite, there exists an infinite subset $\mathscr{F}_1$ of $\mathscr{F}$ and a nonempty subset M_1 of $M(I_1)$ such that for all $I \in \mathscr{F}_1$ and all $m \in \mathbf{N}^n$, $m \in M_1$ if and only if $m \in M(I_1)$ and $T^m \in I$; let then J_1 be the intersection of all ideals I, for $I \in \mathscr{F}_1$. One has $J_0 \subset J_1$, by construction. On the other hand, if $m \in M_1$, then $T^m \in I$ for every $I \in \mathscr{F}_1$, but $T^m \notin I_1$, so that $T^m \in J_1$ and $T^m \notin J_0$, so that $J_0 \subsetneq J_1$.

Iterating this construction, we construct a strictly increasing sequence (J_k) of ideals in $K[T_1, \dots, T_n]$. This contradicts the fact that this ring is noetherian.

Consequently, in any infinite set of monomial ideals which are primary with respect to the maximal ideal, we can find two ideals which are contained one in another.

Let us now construct a strictly decreasing sequence of ideals in such a set $\mathscr{F}$. Since the ring $K[T_1, \dots, T_n]$ is noetherian, the set $\mathscr{F}$ has finitely many maximal elements; for one of them, say I_1, the set $\mathscr{F}_1$ of ideals $I \in \mathscr{F}$ such that $I \subsetneq I_1$ is infinite. Applying this construction with $\mathscr{F}_1$ instead of $\mathscr{F}$, we obtain an ideal $I_2 \in \mathscr{F}_1$ such that $I_1 \subsetneq I_2$ and an infinite subset of $\mathscr{F}_2$ consisting of ideals contained in $\mathscr{F}$. Iterating this construction, we obtain the desired decreasing sequence. □

3.5. Initial ideals and Gröbner bases

Let K be a valued field, let R be its valuation ring and let k be its residue field. It will be important below to admit the case where the valuation of K is trivial; in fact, we will apply the theory to polynomials with coefficients in k, when viewed as a trivially valued field.

Let $\Gamma = \log(|K^\times|)$ be the value group of K; it is a subgroup of $\mathbf{R}$.

We assume implicitly that *the valued field* K *is split*, denoting by $\gamma \mapsto t^\gamma$ a morphism of groups from Γ to $K^\times$; one has $\log(|t^\gamma|) = \gamma$ for all $\gamma \in \Gamma$. We also write $\rho \colon K^\times \to k^\times$ for the group morphism given by $a \mapsto \operatorname{red}(at^{-\log(|a|)})$.

3.5.1. — With a polynomial $f \in K[T_0, \dots, T_n]$, we have associated a tropical polynomial $\tau_f \colon \mathbf{R}^{n+1} \to \mathbf{R}$ as well as, for every $x \in \mathbf{R}^{n+1}$, an initial form $\mathrm{in}_x(f) \in k[\mathrm{T}_0, \dots, \mathrm{T}_n]$. The exponents of the monomials of $\mathrm{in}_x(f)$ are exponents of monomials of f; in particular, if f is homogeneous of degree d, then so is $\mathrm{in}_x(f)$.

Definition **(3.5.2)**. — *Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *The* initial ideal *of* I *at* x *is the ideal of* $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *generated by all initial forms* $\mathrm{in}_x(f)$, *for* $f \in \mathrm{I}$. *It is denoted by* $\mathrm{in}_x(\mathrm{I})$.

Lemma **(3.5.3)**. — *Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *If* I *is a homogeneous ideal, then* $\mathrm{in}_x(\mathrm{I})$ *is a homogeneous ideal.*

Proof. — Let J be the ideal of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ generated by the initial forms $\mathrm{in}_x(f)$, for all *homogeneous* polynomials $f \in \mathrm{I}$; one has $\mathrm{J} \subset \mathrm{in}_x(\mathrm{I})$, and J is a homogeneous ideal. Let $f \in \mathrm{I}$ and let $f = \sum_{d \in \mathbf{N}} f_d$ be its decomposition as a sum of homogeneous polynomials, f_d being of degree d. Since I is a homogeneous ideal, one has $f_d \in \mathrm{I}$. By definition of the tropical polynomial, one has

$$\tau_f(x) = \sup_{d \in \mathbf{N}} (\tau_{f_d}(x)).$$

Let D be the set of all $d \in \mathbf{N}$ such that $f_d \neq 0$ and $\tau_f(x) = \tau_{f_d}(x)$. By definition of $\mathrm{in}_x(f)$, one then has

$$\mathrm{in}_x(f) = \sum_{d \in \mathrm{D}} \mathrm{in}_x(f_d),$$

because of the exponents of the monomials appearing in the polynomials f_d are pairwise distinct. In particular, $\mathrm{in}_x(f) \in \mathrm{J}$. This proves that $\mathrm{in}_x(\mathrm{I}) = \mathrm{J}$ is a homogeneous ideal. □

3.5.4. — The initial ideal at 0, $\mathrm{in}_0(\mathrm{I})$, is the image in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ of the ideal $\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ by the reduction morphism. Let indeed J be this ideal. For every $f \in \mathrm{I}$, written as $f = \sum c_m \mathrm{T}^m$, one has $\tau_f(0) = \sup_m \log(|c_m|)$ and $\mathrm{in}_0(f)$ is the image of the element $f t^{-\tau_f(0)} \in \mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, so that $\mathrm{in}_0(f) \in \mathrm{J}$. On the other hand, if $f \in \mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, then either $\tau_f(0) < 0$, in which case the image of f in $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ is zero, or $\tau_f(0) = 0$, in which case $\mathrm{in}_0(f)$ is the image of f. This proves that $\mathrm{J} = \mathrm{in}_0(\mathrm{I})$.

Moreover, $\mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n]/(\mathrm{I} \cap \mathrm{R}[\mathrm{T}_0, \dots, \mathrm{T}_n])$ is a torsion free R-module, hence is *flat*, because R is a valuation ring. In the case where I is a homogeneous ideal, this says that the family $\operatorname{Proj}(\mathrm{R}[\mathrm{T}_0,\dots,\mathrm{T}_n]/(\mathrm{I}\cap \mathrm{R}[\mathrm{T}_0,\dots,\mathrm{T}_n])) \to \operatorname{Spec}(\mathrm{R})$ is a flat morphism of projective schemes; its generic fiber is $\operatorname{Proj}(\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I}) = \mathrm{V}(\mathrm{I})$, and its closed fiber is $\operatorname{Proj}(k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_0(\mathrm{I})) = \mathrm{V}(\mathrm{in}_0(\mathrm{I}))$. This flatness has the following important consequences:[2]

– The Hilbert functions of I and $\mathrm{in}_0(\mathrm{I})$ coincide. Explicitly, for every integer d, one has

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_0(\mathrm{I}))_d);$$

– If $\mathrm{V}(\mathrm{I})$ is integral, then $\mathrm{V}(\mathrm{in}_0(\mathrm{I}))$ is equidimensional, of the same dimension.

3.5.5. — Let $x \in \mathbf{R}^{n+1}$; let us assume that the coordinates of x belong to the value group Γ. For every $j \in \{0,\dots,n\}$, fix $a_j \in \mathrm{K}^\times$ such that $\log(|a_j|) = x_j$; let also $\alpha_j = \rho(a_j)$ for every j.

For every $f = \sum c_m \mathrm{T}^m \in \mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$, one has $f(a\mathrm{T}) = \sum c_m a^m \mathrm{T}^m$, so that

$$\tau_{f(a\mathrm{T})}(0) = \sup_m(\log(|c_m|) + \langle m, x\rangle) = \tau_f(x),$$

as well as

$$\mathrm{in}_0(f(a\mathrm{T})) = \sum_{m\in \mathrm{S}_f(x)} \rho(c_m a^m)\mathrm{T}^m = \sum_{m\in \mathrm{S}_f(x)} \rho(c_m)\alpha^m \mathrm{T}^m = \mathrm{in}_x(f)(\alpha\mathrm{T}).$$

Let φ_a be the K-algebra automorphism of $\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$ given by $\varphi_a(f) = f(a_0\mathrm{T}_0,\dots,a_n\mathrm{T}_n)$ and let ψ_α be the k-algebra automorphism of $k[\mathrm{T}_0,\dots,\mathrm{T}_n]$ given by $\psi_\alpha(f) = f(\alpha_0\mathrm{T},\dots,\alpha_n\mathrm{T})$. By the preceding computation, we have $\psi_\alpha(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_0(\varphi_a(\mathrm{I}))$ is the image of the ideal $\varphi_a(\mathrm{I})\cap \mathrm{R}[\mathrm{T}_0,\dots,\mathrm{T}_n]$ in $k[\mathrm{T}_0,\dots,\mathrm{T}_n]$.

This change of variables will allow to reduce properties of the initial ideal $\mathrm{in}_x(\mathrm{I})$ to the case of $x=0$. In particular, it immediately implies the following lemma.

Lemma **(3.5.6)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$ *and let* $x\in\mathbf{R}^{n+1}$ *be such that its coordinates belong to the value group of* K.

a) *The initial ideal* $\mathrm{in}_x(\mathrm{I})$ *is the set of all* $\mathrm{in}_x(f)$, *for* $f\in\mathrm{I}$;

[2]Maybe write an appendix with material from commutative algebra and algebraic geometry that is used in the notes.

b) *If* $V(I)$ *is integral, then* $V(\mathrm{in}_x(I))$ *is equidimensional, of the same dimension;*

c) *The Hilbert functions of* I *and* $\mathrm{in}_x(I)$ *coincide. Explicitly, for every integer* d*, one has*

$$\dim_K((K[T_0,\dots,T_n]/I)_d) = \dim_k((k[T_0,\dots,T_n]/\mathrm{in}_x(I))_d).$$

One of the goals of the theory that we develop now is to extend these properties to an arbitrary $x \in \mathbf{R}^{n+1}$.

Remark **(3.5.7)**. — Let $x_1, x_2 \in \mathbf{R}$ be $\mathbf{Q}$-linearly independent real numbers such that $(\mathbf{Q}x_1 + \mathbf{Q}x_2) \cap \log(|K^\times|) = 0$. Let $I = (T_1, T_2)$. One has $\mathrm{in}_x(I) \subset (T_1, T_2)$, and the relations $\mathrm{in}_x(T_1) = T_1$ and $\mathrm{in}_x(T_2) = T_2$ imply that $\mathrm{in}_x(I) = (T_1, T_2)$.

On the other hand, let $f \in K[T_1, T_2]$, written $\sum c_m T^m$, and let $m, n \in \mathbf{N}^2$ be elements such that $\log(|c_m|) + \langle m, x\rangle = \log(|c_n|) + \langle x, n\rangle = \tau_f(x)$. Then $\log(|c_m/c_n|) + x_1(m_1 - n_1) + x_2(m_2 - n_2) = 0$, so that $c_m/c_n \in R^\times$, $m_1 = n_1$ and $m_2 = n_2$; this proves that $\mathrm{in}_x(f)$ is a monomial. In that case, the set of polynomials of the form $\mathrm{in}_x(f)$, for $f \in I$, is not an ideal of I. In particular, the statement of Lemma 2.4.2 in Maclagan & Sturmfels (2015) is incorrect (this is signaled in the *errata* of that reference).

The next lemma is a weakening of the expected property.

Lemma **(3.5.8)**. — *Let* I *be a homogeneous ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$.

a) *Every element of* $\mathrm{in}_x(I)$ *is a sum of polynomials of the form* $\mathrm{in}_x(f)$*, for* $f \in I$.

b) *Let* $f, g \in I$*. If the supports of* $\mathrm{in}_x(f)$ *and* $\mathrm{in}_x(g)$ *are not disjoint, then there exists* $h \in I$ *such that* $\mathrm{in}_x(h) = \mathrm{in}_x(f) + \mathrm{in}_x(g)$*. If* $\tau_f(x) = \tau_g(x)$ *and* $\mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0$*, then one may even take* $h = f + g$.

c) *Let* $m \in \mathbf{N}^{n+1}$*; if* $T^m \in \mathrm{in}_x(I)$*, then there exists* $f \in I$ *such that* $T^m = \mathrm{in}_x(f)$.

Proof. — a) Let $f \in I$, let $\alpha \in k^\times$ and let $m \in \mathbf{N}^{n+1}$. Let $a \in R^\times$ be such that $\rho(a) = \alpha$. One has $\tau_{aT^m f}(x) = \tau_f(x) + \langle m, x\rangle$ and $\mathrm{in}_x(aT^m f) = \alpha T^m \mathrm{in}_x(f)$ (this is an elementary instance of lemma 3.3.4). This proves that the set of initial forms is stable under multiplication by a monomial.

In particular, the additive monoid it generates in $k[T_0, \dots, T_n]$ is an ideal of $k[T_0, \dots, T_n]$.

b) Write $f = \sum c_m T^m$, $g = \sum d_m T^m$ and let $\mu \in \mathbf{N}^{n+1}$ be a common point of the supports of $\mathrm{in}_x(f)$ and of $\mathrm{in}_x(g)$. This means that $\log(|c_\mu|) + \langle \mu, x\rangle = \tau_f(x) = \sup_m(\log(|c_m|) + \langle m, x\rangle)$ and $\log(|d_\mu|) + \langle \mu, x\rangle = \tau_g(x) = \sup_m(\log(|d_m|) + \langle m, x\rangle)$. In particular, $\tau_f(x) - \tau_g(x) = \log(|c_\mu/d_\mu|)$. Replacing f by $f t^{-\log(|c_\mu|)}$ and g by $g t^{-\log(|d_\mu|)}$ does not change $\mathrm{in}_x(f)$ and $\mathrm{in}_x(g)$ and allows us to assume that $|c_\mu| = |d_\mu| = 1$ and $\tau_f(x) = \tau_g(x) = \langle \mu, x\rangle$.

If $\mathrm{in}_x(f) + \mathrm{in}_x(g) = 0$, then we take $h = 0$.

Let us now assume that $\mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0$ and let $h = f + g = \sum (c_m + d_m) T^m$. For all m, one has $\log(|c_m|) + \langle m, x\rangle \leqslant \tau_f(x)$ and $\log(|d_m|) + \langle m, x\rangle \leqslant \tau_g(x) = \tau_f(x)$, so that $\log(|c_m + d_m|) + \langle m, x\rangle \leqslant \tau_f(x)$ and $\tau_h(x) \leqslant \tau_f(x)$.

Let $m \in \mathbf{N}^{m+1}$.

Let us assume that $m \in S_x(f) - S_x(g)$. Then $\log(|c_m|) + \langle m, x\rangle = \tau_f(x)$ but $\log(|d_m|) + \langle m, x\rangle < \tau_g(x) = \tau_f(x)$; we then have $|d_m| < |c_m|$, hence $|c_m + d_m| = |c_m|$ and $\log(|c_m + d_m|) + \langle m, x\rangle = \tau_f(x)$. This implies that $\tau_h(x) = \tau_f(x)$. Moreover, $\rho(c_m + d_m) = \rho(c_m)$ is the coefficient of T^m in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ and in $\mathrm{in}_x(h)$.

Similarly, if $m \in S_x(g) - S_x(f)$, then $|c_m + d_m| = |d_m| > |c_m|$, $\tau_h(x) = \tau_f(x)$, and $\rho(c_m + d_m) = \rho(d_m)$ is the coefficient of T^m in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ and in $\mathrm{in}_x(h)$.

If $m \in S_x(f) \cap S_x(g)$ and $\rho(c_m) + \rho(d_m) \neq 0$, then $\log(|c_m|) + \langle m, x\rangle = \tau_f(x) = \tau_g(x) = \log(|d_m|) + \langle m, x\rangle$, so that $|c_m| = |d_m|$ and

$$\begin{aligned} 0 &\neq \rho(c_m) + \rho(d_m) \\ &= \mathrm{red}(c_m t^{-\log(|c_m|)}) + \mathrm{red}(d_m t^{-\log(|c_m|)}) \\ &= \mathrm{red}((c_m + d_m) t^{-\log(|c_m|)}), \end{aligned}$$

so that $|c_m + d_m| = |c_m|$. Then $\tau_h(x) = \tau_f(x)$ and $\rho(c_m + d_m) = \rho(c_m) + \rho(d_m)$ is the coefficient of T^m in $\mathrm{in}_x(f) + \mathrm{in}_x(g)$ and in $\mathrm{in}_x(h)$.

Since $\mathrm{in}_x(f) + \mathrm{in}_x(g) \neq 0$, by assumption, at least one of these three cases appears. This already proves that $\tau_h(x) = \tau_f(x)$.

Two possibilities remain for $m \in \mathbf{N}^{m+1}$.

If $m \notin S_x(f) \cup S_x(g)$, then $\log(|c_m|) + \langle m, x\rangle < \tau_f(x)$ and $\log(|d_m|) + \langle m, x\rangle < \tau_g(x)$, so that $\log(|c_m + d_m|) + \langle m, x\rangle < \tau_f(x) = \tau_h(x)$. Then m does not appear in $\operatorname{in}_x(f)$, $\operatorname{in}_x(g)$ or $\operatorname{in}_x(h)$.

Let us finally assume that $m \in S_x(f) \cap S_x(g)$ and $\rho(c_m) + \rho(d_m) = 0$. As above, one has $|c_m| = |d_m|$ and $\rho(c_m) + \rho(d_m)$ is the reduction of $(c_m + d_m)t^{-\log(|c_m|)}$. This implies that $|c_m + d_m| < |c_m|$, hence T^m does not appear in $\operatorname{in}_x(h)$, and neither does it appear in $\operatorname{in}_x(f) + \operatorname{in}_x(g)$.

c) Let $\varphi \in \operatorname{in}_x(I)$ and let $(f_i)_{1\leqslant i\leqslant p}$ be a finite family of minimal cardinality of elements of I such that $\varphi = \sum_{i=1}^p \operatorname{in}_x(f_i)$. By minimality of p, one has $\operatorname{in}_x(f_i) \neq 0$ for all i. Applying b), we deduce from the minimality of p that for all $i \neq j$, the supports of $\operatorname{in}_x(f_i)$ and $\operatorname{in}_x(f_j)$ are disjoint. The support of their sum, $\sum_i \operatorname{in}_x(f_i) = \varphi$, is then the union of their supports, hence it has at least p elements.

If φ is a monomial, this implies that $p = 1$, so that there exists $f \in I$ such that $\varphi = \operatorname{in}_x(f)$. □

Definition **(3.5.9)**. — *Let* I *be an ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *A finite family* $(f_1, \dots, f_m)$ *of elements of* I *is called a* Gröbner basis *for* I *at* x *if the initial forms* $\operatorname{in}_x(f_j)$ *at* x *generate the initial ideal* $\operatorname{in}_x(I)$ *of* I *at* x.

Since the initial ideal $\operatorname{in}_x(I)$ is generated by the polynomials of the form $\operatorname{in}_x(f)$, for $f \in I$, the existence of a Gröbner basis follows from the noetherian property of the ring $k[T_0, \dots, T_n]$ (aka, Hilbert's finite basis theorem).

Assume, moreover, that I is homogeneous and let $(f_1, \dots, f_m)$ be a Gröbner basis for I at a point x. For every j, $\operatorname{in}_x(f_j)$ is the sum of the initial forms of the homogeneous components of f_j, as we saw in the proof of lemma 3.5.3. This implies that the homogeneous components of the f_j constitute a Gröbner basis for I at x.

In the next lemma, we consider initial forms of polynomials of $k[T_0, \dots, T_n]$; this means that the field k is considered as a valued field, for the trivial absolute value.

Lemma **(3.5.10)**. — *Let* $x \in \mathbf{R}^{n+1}$ *and let* $f \in K[T_0, \dots, T_n]$. *There exists a strictly positive real number* δ *such that for every* $y \in \mathbf{R}^{n+1}$ *such that* $\|y\| < \delta$, *one has* $\operatorname{in}_{x+y}(f) = \operatorname{in}_y(\operatorname{in}_x(f))$ *and* $\tau_f(x+y) = \tau_{\operatorname{in}_x(f)}(y)$.

Proof. — Write $f = \sum c_m \mathrm{T}^m$; let $\mathrm{S}(f)$ be its support and let $\mathrm{S}_x(f)$ be the set of all $m \in \mathrm{S}(f)$ such that

$$\log(|c_m|) + \langle m, x\rangle = \tau_f(x) = \sup_m(\log(|c_m|) + \langle m, x\rangle),$$

so that one has $\mathrm{in}_x(f) = \sum_{m\in\mathrm{S}_x(f)} \rho(c_m)\mathrm{T}^m$. Let then $\mathrm{S}_{x,y}(f)$ be the set of all $m \in \mathrm{S}_x(f)$ such that

$$\langle m, y\rangle = \sup_{m\in\mathrm{S}_x(f)} \langle m, y\rangle = \tau_{\mathrm{in}_x(f)}(y),$$

so that $\mathrm{in}_y(\mathrm{in}_x(f)) = \sum_{m\in\mathrm{S}_{x,y}(f)} \rho(c_m)\mathrm{T}^m$.

Let ε be a strictly positive real number such that $\log(|c_m|) + \langle m, x\rangle < \tau_f(x) - \varepsilon$ for $m \in \mathrm{S}(f) - \mathrm{S}_x(f)$. Let also $\delta > 0$ be such $|\langle m, y\rangle| < \varepsilon/2$ for every $m \in \mathrm{S}(f)$ and every $y \in \mathbf{R}^{n+1}$ such that $\|y\| < \delta$. For every such y and every $m \in \mathrm{S}_{x,y}(f)$, one then has

$$\log(|c_m|) + \langle m, x + y\rangle = (\log(|c_m|) + \langle m, x\rangle) + \langle m, y\rangle = \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y).$$

In particular, one has $\tau_f(x + y) \geqslant \tau_{\mathrm{in}_x(f)}(y)$. If $m \in \mathrm{S}(f) - \mathrm{S}_x(f)$, one has

$$\begin{aligned}\log(|c_m|) + \langle x + m, y\rangle &= (\log(|c_m|) + \langle m, x\rangle) + \langle m, y\rangle \\ &< \tau_f(x) - \varepsilon + \langle m, y\rangle \\ &< \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y).\end{aligned}$$

(Indeed, $\tau_{\mathrm{in}_x(f)}(y) > -\varepsilon/2$ and $\langle m, y\rangle < \varepsilon/2$.) On the other hand, if $m \in \mathrm{S}_x(f) - \mathrm{S}_{x,y}(f)$, then

$$\begin{aligned}\log(|c_m|) + \langle x + m, y\rangle &= (\log(|c_m|) + \langle m, x\rangle) + \langle m, y\rangle \\ &= \tau_f(x) + \langle m, y\rangle \\ &< \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y).\end{aligned}$$

This proves that $\tau_f(x + y) = \tau_f(x) + \tau_{\mathrm{in}_x(f)}(y)$ and that

$$\mathrm{in}_{x+y}(f) = \sum_{m\in\mathrm{S}_{x,y}(f)} \rho(c_m)\mathrm{T}^m = \mathrm{in}_y(\mathrm{in}_x(f)).$$

□

Proposition **(3.5.11)**. — *Let* I *be a homogeneous ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. *For any* $y \in \mathbf{R}^{n+1}$, *let* M_y *be the largest monomial ideal contained in* $\mathrm{in}_y(\mathrm{I})$.

a) *Let* $y \in \mathbf{R}^{n+1}$ *be such that* M_y *is maximal among the ideals of this form. Then* $\mathrm{in}_y(\mathrm{I}) = \mathrm{M}_y$ *— in particular,* $\mathrm{in}_y(\mathrm{I})$ *is a monomial ideal.*

b) *Assume, moreover that the valuation of* K *is trivial. Then there exists* $\delta > 0$ *such that for every* $z \in \mathbf{R}^{n+1}$ *such that* $\|z\| < \delta$*, one has one has* $\mathrm{in}_{y+z}(\mathrm{I}) = \mathrm{in}_y(\mathrm{I}) = \mathrm{M}_y$*;*

c) *Let* $x \in \mathbf{R}^{n+1}$*. Let* $y \in \mathbf{R}^{n+1}$ *be such that* $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ *is maximal among the ideals of this form. Then there exists a finite family* (f_i) *in* I *such that the polynomials* $\mathrm{in}_y(\mathrm{in}_x(f_i))$ *generate the ideal* $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$*. Moreover, there exists* $\delta > 0$ *such that for every* $\varepsilon \in \mathbf{R}$ *such that* $0 < \varepsilon < \delta$*, one has* $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$*, and this ideal is monomial.*

Proof. — a) By construction, the ideal M_y is generated by a family (T^{m_i}) of monomials belonging to $\mathrm{in}_y(\mathrm{I})$; by Hilbert's basis theorem, this family can be assumed to be finite. By lemma 3.5.8, there exists, for every i a polynomial $f_i \in \mathrm{I}$ such that $\mathrm{T}^{m_i} = \mathrm{in}_y(f_i)$.

We now argue by contradiction and consider $f \in \mathrm{I}$ such that $\mathrm{in}_y(f) \notin \mathrm{M}_y$. If a monomial appearing in f belongs to M_y, we choose $g \in \mathrm{I}$ such that $\mathrm{in}_y(g)$ is that monomial; by lemma 3.5.8, there exists $h \in \mathrm{I}$ such that $\mathrm{in}_y(f) - \mathrm{in}_y(g) = \mathrm{in}_y(h)$, and that monomial does not appear in $\mathrm{in}_y(h)$; moroever, $\mathrm{in}_y(h) \notin \mathrm{M}_y$. Repeating this argument, we assume that no monomial of $\mathrm{in}_y(f)$ belongs to M_y.

Let now μ be a vertex of the Newton polytope of $\mathrm{in}_y(f)$ and let $z \in \mathbf{R}^{n+1}$ be the coefficients of a linear form defining μ. In other words, μ belongs to the support of $\mathrm{in}_y(f)$, and for every other m in this support, one has $\langle m, z\rangle < \langle \mu, z\rangle$. Then $\mathrm{in}_z(\mathrm{in}_y(f))$ is the monomial of exponent μ in $\mathrm{in}_y(f)$. By lemma 3.5.10, for $z \in \mathbf{R}^{n+1}$ such that $\|z\|$ is small enough, one has $\mathrm{in}_{y+z}(f) = \mathrm{in}_z(\mathrm{in}_y(f))$. Similarly, if $\|z\|$ is small enough, then for every i, one has $\mathrm{in}_{y+z}(f_i) = \mathrm{in}_z(\mathrm{in}_y(f_i)) = \mathrm{in}_y(f_i)$ since $\mathrm{in}_y(f_i)$ is a monomial. This implies that M_{y+z} contains M_y. On the other hand, the monomial T^{μ} belongs to M_{y+z} but not to M_y. This contradicts the hypothesis that M_y is maximal among the ideals of this form.

b) The ideal $\mathrm{in}_y(\mathrm{I})$ is generated by the monomials $\mathrm{in}_y(f_i)$. For $z \in \mathbf{R}^{n+1}$ such that $\|z\|$ is small enough, one has $\mathrm{in}_{y+z}(f_i) = \mathrm{in}_z(\mathrm{in}_y(f_i)) = \mathrm{in}_y(f_i)$ since $\mathrm{in}_y(f_i)$ is a monomial and the valuation of K is trivial. Consequently, $\mathrm{in}_{y+z}(\mathrm{I})$ contains the monomial ideal $\mathrm{in}_y(\mathrm{I})$. By maximality, the equality follows.

c) Let us apply the first part of the proposition to the ideal $\mathrm{in}_x(\mathrm{I})$ of $k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ and choose $y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ is maximal for this property — it is then a monomial ideal, by *a*). We shall prove that $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ for $\varepsilon > 0$ small enough.

Let $(g_1, \dots, g_m)$ be a finite family of elements of $\mathrm{in}_x(\mathrm{I})$ such that $\mathrm{in}_y(g_i)$ is a monomial, for every i, and such these monomials generate $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$. Fix i. As in the proof of lemma 3.5.8, *c*), there exists a finite family $(f_{i,j})_j$ of elements of I, with pairwise disjoint supports, such that $g_i = \sum_j \mathrm{in}_x(f_{i,j})$. Then the polynomials $\mathrm{in}_y(\mathrm{in}_x(f_{i,j}))$ have pairwise disjoint supports, and there exists a unique j such that the monomial $\mathrm{in}_y(g_i)$ appears in $\mathrm{in}_y(\mathrm{in}_x(f_{i,j}))$, in which case $\mathrm{in}_y(g_i) = \mathrm{in}_y(\mathrm{in}_x(f_{i,j}))$. This shows that there exists a finite family (f_i) in I such that $\mathrm{in}_y(\mathrm{in}_x(f_i))$ is a monomial for each i, and such that these monomials generate the ideal $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$.

Let then $\delta > 0$ be such that $\mathrm{in}_{x+\varepsilon y}(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$ for every i and every $\varepsilon \in \mathbf{R}$ such that $0 < \varepsilon < \delta$; in particular, $\mathrm{in}_y(\mathrm{in}_x(f_i)) \in \mathrm{in}_{x+\varepsilon y}(\mathrm{I})$, hence $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) \subset \mathrm{in}_{x+\varepsilon y}(\mathrm{I})$. Let us assume that the inclusion is strict. Then, there exists $f \in \mathrm{I}$ such that $\mathrm{in}_{x+\varepsilon y}(f)$ does not belong to the monomial ideal $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$. Subtracting from f an adequate linear combination of the the f_i, we may moreover assume that no monomial of $\mathrm{in}_{x+\varepsilon y}(f)$ belongs to $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$.

Let $z \in \mathbf{R}^{n+1}$ be such that $\mathrm{in}_z(\mathrm{in}_{x+\varepsilon y}(f))$ is a monomial. (In other words, z does not belong to the tropical hypersurface associated with $\mathrm{in}_{x+\varepsilon y}(f)$.) For $\delta > 0$ small enough (depending on y, ε, f), one then has $\mathrm{in}_{x+\varepsilon y+\delta z}(f) = \mathrm{in}_z(\mathrm{in}_{x+\varepsilon y}(f))$, hence is a nonzero monomial. However, applying *b*), we observe that if $\varepsilon y + \delta z$ is small enough (depending on x and I uniquely), then that monomial belongs to $\mathrm{in}_{\varepsilon y+\delta z}(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$, a contradiction which concludes the proof that $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$. ☐

Theorem **(3.5.12)**. — *Let* I *be a* homogeneous *ideal of* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. *For every* $x \in \mathbf{R}^{n+1}$, *the Hilbert functions of* I *and* $\mathrm{in}_x(\mathrm{I})$ *are equal: for every integer* d, *one has*

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{I})_d) = \dim_k((k[\mathrm{T}_0, \dots, \mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d).$$

Lemma **(3.5.13)**. — *The conclusion of theorem 3.5.12 holds if* $\mathrm{in}_x(\mathrm{I})$ *is a monomial ideal.*

Proof. — Fix $d \in \mathbf{N}$.

Let M be the set of $m \in \mathbf{N}^{n+1}$ such that $|m| = d$ and $\mathrm{T}^m \notin \mathrm{in}_x(\mathrm{I})$. Let us prove that the family $(\mathrm{T}^m)_{m\in\mathrm{M}}$ is free in $(\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d$. Let $(c_m)_{m\in\mathrm{M}}$ be a family in K such that $\sum_{m\in\mathrm{M}} c_m\mathrm{T}^m \in \mathrm{I}$. Then there exist a family $(\tilde{c}_m)_{m\in\mathrm{M}}$ in k such that $\mathrm{in}_x(f) = \sum \tilde{c}_m\mathrm{T}^m$. By definition, one has $\mathrm{in}_x(f) \in \mathrm{in}_x(\mathrm{I})$, and since $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal, one has $\tilde{c}_m\mathrm{T}^m \in \mathrm{in}_x(\mathrm{I})$ for every $m \in \mathrm{M}$. Since $\mathrm{T}^m \notin \mathrm{in}_x(\mathrm{I})$ for $m \in \mathrm{M}$, this implies $\tilde{c}_m = 0$, hence $\mathrm{in}_x(f) = 0$ and $f = 0$. As a consequence, one has

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) \geqslant \mathrm{Card}(\mathrm{M}).$$

On the other hand, since the homogeneous ideal $\mathrm{in}_x(\mathrm{I})$ is generated by monomials, one has

$$\mathrm{Card}(\mathrm{M}) = \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d),$$

so that

$$\dim_{\mathrm{K}}((\mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{I})_d) \geqslant \mathrm{Card}(\mathrm{M}) \geqslant \dim_k((k[\mathrm{T}_0,\dots,\mathrm{T}_n]/\mathrm{in}_x(\mathrm{I}))_d).$$

In the other direction, let now M′ be the set of $m \in \mathbf{N}^{n+1}$ such that $|m| = d$ and $\mathrm{T}^m \in \mathrm{in}_x(\mathrm{I})$. For every $m \in \mathrm{M}'$, there exists $f_m \in \mathrm{I}$ such that $\mathrm{in}_x(f_m) = \mathrm{T}^m$. Since I is a homogeneous ideal, we may also assume that f_m is homogeneous of degree d. Multiplying f_m by an element of $\mathrm{R}^\times$, we may assume that $f_m = \mathrm{T}^m + \sum_{p\neq m} a_{m,p}\mathrm{T}^p$. Let us prove that the family $(f_m)_{m\in\mathrm{M}'}$ is free. Let $(c_m)_{m\in\mathrm{M}'}$ be a family in K such that $\sum c_m f_m = 0$.

Let $\mu \in \mathrm{M}'$ such that $\log(|c_\mu|) + \langle \mu, x\rangle$ is maximal. Considering the coefficient of T^μ in $\sum c_m f_m$, one has

$$c_\mu + \sum_{m\neq\mu} c_m a_{m,\mu} = 0.$$

By ultrametricity, there exists $m \neq \mu$ such that $|c_\mu| \leqslant |c_m a_{m,\mu}|$, and then

$$\log(|c_m|) + \langle m, x\rangle \leqslant \log(|c_\mu| + \langle \mu, x\rangle \leqslant \log(|c_m|) + \log(|a_{m,\mu}|) + \langle \mu, x\rangle,$$

so that

$$\langle m, x\rangle \leqslant \log(|a_{m,\mu}|) + \langle \mu, x\rangle,$$

contradicting the hypothesis that $\mathrm{in}_x(f_m)$ is the monomial T^m.

Consequently,

$$\dim_K(K[T_0,\dots,T_n]_d\cap I)\geqslant \operatorname{Card}(M')=\dim_K(k[T_0,\dots,T_n]_d\cap \operatorname{in}_x(I)).$$

Since I is a homogeneous ideal, one has

$$\begin{aligned}\dim_K((K[T_0,\dots,T_n]/I)_d)&\\ &=\dim_K(K[T_0,\dots,T_n]_d)-\dim(K[T_0,\dots,T_n]_d\cap I)\\ &\leqslant \dim_k(k[T_0,\dots,T_n]_d)-\dim(k[T_0,\dots,T_n]_d\cap I)\\ &=\dim_k((k[T_0,\dots,T_n]/\operatorname{in}_x(I))_d).\end{aligned}$$

This concludes the proof of the lemma. □

Proof of theorem 3.5.12. — We fix $y\in\mathbf{R}^{n+1}$ and $\varepsilon>0$ such that $\operatorname{in}_y(\operatorname{in}_x(I))=\operatorname{in}_{x+\varepsilon y}(I)$ is a monomial ideal.

Applying lemma 3.5.13 to the ideal I of $K[T_0,\dots,T_n]$ and the point $x+\varepsilon y$, we have

$$\dim_K((K[T_0,\dots,T_n]/I)_d)=\dim_k((k[T_0,\dots,T_n]/\operatorname{in}_{x+\varepsilon y}(I))_d).$$

Applying that lemma to the ideal $\operatorname{in}_x(I)$ of $k[T_0,\dots,T_n]$ and the point y, we have

$$\begin{aligned}\dim_k((k[T_0,\dots,T_n]/\operatorname{in}_x(I))_d)&=\dim_k((k[T_0,\dots,T_n]/\operatorname{in}_y(\operatorname{in}_x(I)))_d)\\ &=\dim_k((k[T_0,\dots,T_n]/\operatorname{in}_{x+\varepsilon y}(I))_d).\end{aligned}$$

This shows that

$$\dim_K((K[T_0,\dots,T_n]/I)_d)=\dim_k((k[T_0,\dots,T_n]/\operatorname{in}_x(I))_d),$$

as claimed. □

Corollary **(3.5.14)**. — *Let* I *be a* homogeneous *ideal of* $K[T_0,\dots,T_n]$, *let* $x\in\mathbf{R}^{n+1}$ *and let* $(f_1,\dots,f_m)$ *be a Gröbner basis of* I *at* x. *Then* $I=(f_1,\dots,f_m)$.

Proof. — Let J be the homogeneous ideal of $K[T_0,\dots,T_n]$ generated by the homogeneous components of $f_1,\dots,f_m$. One has $J\subset I$, because these homogeneous components belong to I. Moreover, for every j, the initial form $\operatorname{in}_x(f_j)$ is the sum of the initial forms of the homogeneous components of f_j, so that $\operatorname{in}_x(f_j)\subset\operatorname{in}_x(J)$. As a consequence, $\operatorname{in}_x(I)\subset\operatorname{in}_x(J)$, hence the equality $\operatorname{in}_x(J)=\operatorname{in}_x(I)$. By theorem 3.5.12, the homogeneous ideals I and J have the same Hilbert functions. Since $J\subset I$, this implies $J=I$. □

3.6. The Gröbner polyhedral decomposition associated with an ideal

3.6.1. — Let I be a homogeneous ideal in $K[T_0,\dots,T_n]$. For $x \in \mathbf{R}^{n+1}$, let $C'_x(I)$ be the set of $y \in \mathbf{R}^{n+1}$ such that $\operatorname{in}_y(I) = \operatorname{in}_x(I)$ and let $C_x(I)$ be its closure in $\mathbf{R}^{n+1}$. Let $e = (1,\dots,1) \in \mathbf{R}^{n+1}$.

Here is the main theorem

Theorem **(3.6.2)**. — *Let* I *be a homogeneous ideal in* $K[T_0,\dots,T_n]$. *The sets* $C_x(I)$ *form a* Γ*-strict and* $\mathbf{R}e$*-invariant polyhedral decomposition of* $\mathbf{R}^{n+1}$.

Proposition **(3.6.3)**. — *Let* $x \in \mathbf{R}^{n+1}$.

a) *The set* $C_x(I)$ *is a closed* Γ*-strict and* $\mathbf{R}e$*-invariant polyhedron in* $\mathbf{R}^{n+1}$;

b) *If* $\operatorname{in}_x(I)$ *is a monomial ideal, then* $C'_x(I)$ *is the interior of* $C_x(I)$;

c) *If* $\operatorname{in}_x(I)$ *is not a monomial ideal, then there exists* $y \in \mathbf{R}^{n+1}$ *such that* $\operatorname{in}_y(\operatorname{in}_x(I))$ *is a monomial ideal; for every such* y, *the polyhedron* $C_x(I)$ *is a face of* $C_y(I)$.

Proof. — Fix $y \in \mathbf{R}^{n+1}$ satisfying the conditions of proposition 3.5.11, *c*), small enough so that $\operatorname{in}_{x+y}(I) = \operatorname{in}_y(\operatorname{in}_x(I))$ is a monomial ideal. As a consequence of *b*), it will be enough to assume that $\operatorname{in}_y(\operatorname{in}_x(I))$ is a monomial ideal and y is small enough.

Let $z = x + y$. Fix a finite family $(f_1,\dots,f_r)$ in I such that the polynomials $\operatorname{in}_z(f_i)$ are monomials and generate $\operatorname{in}_z(I)$; we may also assume that $\operatorname{in}_z(f_i) = \operatorname{in}_y(\operatorname{in}_x(f_i))$ for all i.

For each i, let $m_i \in \mathbf{N}^{n+1}$ be such that $\operatorname{in}_z(f_i) = T^{m_i}$. By the argument explained in the proof of lemma 3.5.13, there exists a unique polynomial $g_i \in K[T_0,\dots,T_n]$, homogeneous of degree $|m_i|$, such that $T^{m_i} - g_i \in I$, and such that no monomial appearing in g_i belongs to $\operatorname{in}_z(I)$; write $g_i = \sum c_{i,m} T^m$ and set $f_i = T^{m_i} - g_i$. Since T^{m_i} is the only monomial appearing in f_i that belongs to the monomial ideal $\operatorname{in}_y(\operatorname{in}_x(I)) = \operatorname{in}_z(I)$, one has $T^{m_i} = \operatorname{in}_y(\operatorname{in}_x(f_i))$. The family (f_i) is thus a Gröbner basis for I at z.

Lemma **(3.6.4)**. — *With the preceding notation, the set* $C'_z(I)$ *is defined by the strict inequalities*

$$\langle m - m_i, \cdot \rangle + \log(|c_{i,m}|) < 0,$$

for all $i \in \{1, \dots, r\}$ *and all* $m \in \mathbf{N}^{m+1}$ *in the support of* f_i. *The set* $\mathrm{C}_z(\mathrm{I})$ *is the* Γ*-strict polyhedron defined by the inequalities*

$$\langle m - m_i, \cdot \rangle + \log(|c_{i,m}|) \leqslant 0,$$

for $i \in \{1, \dots, r\}$ *and* $m \in \mathbf{N}^{m+1}$ *in the support of* f_i.

Proof. — Let $w \in \mathrm{C}'_z(\mathrm{I})$. By definition of $\mathrm{C}'_z(\mathrm{I})$, one has $\mathrm{in}_w(f_i) \in \mathrm{in}_w(\mathrm{I}) = \mathrm{in}_z(\mathrm{I})$, so that the only monomial that can appear in $\mathrm{in}_w(f_i)$ is T^{m_i}, hence $\log(|c_{i,m}|) + \langle m, w\rangle < \langle m_i, w\rangle$ for all i and all m such that $m \neq m_i$. In the other direction, if w statisfies these inequalities, then $\mathrm{in}_w(f_i) = \mathrm{T}^{m_i}$ for all i, hence $\mathrm{in}_w(\mathrm{I})$ contains $\mathrm{in}_z(\mathrm{I})$. Since both of these ideals have the same Hilbert function, they have to be equal and $w \in \mathrm{C}'_z(\mathrm{I})$.

Let P be the closed convex polyhedron in $\mathbf{R}^{n+1}$ defined by the inequalities $\langle m - m_i, \cdot \rangle + \log(|c_{i,m}|) \leqslant 0$, for all i and $m \neq m_i$. By what precedes, one has $\mathring{\mathrm{P}} = \mathrm{C}'_z(\mathrm{I})$. Since $\mathring{\mathrm{P}}$ is nonempty (it contains z), one has $\mathrm{P} = \mathrm{C}_z(\mathrm{I})$. □

Lemma **(3.6.5)**. — *The set* $\mathrm{C}_x(\mathrm{I})$ *is the smallest face of the polyhedron* $\mathrm{C}_z(\mathrm{I})$ *that contains* x.

Proof. — By the choice of y, one has $\mathrm{in}_{x+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$ for all ε such that $0 < \varepsilon < 1$. In particular, $x + \varepsilon y \in \mathrm{C}'_z(\mathrm{I})$. If we let ε go to 0, we obtain $x \in \mathrm{C}_z(\mathrm{I})$.

Let $x' \in \mathrm{C}'_x(\mathrm{I})$; since $\mathrm{in}_{x'}(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$, the preceding analysis still applies when one replaces the point x with x', so that

$$\mathrm{in}_{x'+\varepsilon y}(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_{x'}(\mathrm{I})) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_z(\mathrm{I})$$

for $\varepsilon > 0$ small enough. Then $x' + \varepsilon y \in \mathrm{C}'_z(\mathrm{I})$ and $x' \in \mathrm{C}_z(\mathrm{I})$. Taking the closure, we obtain $\mathrm{C}_x(\mathrm{I}) \subset \mathrm{C}_z(\mathrm{I})$.

Moreover, T^{m_i} is the only monomial in the support of f_i that belongs to $\mathrm{in}_y(\mathrm{in}_{x'}(f_i))$; this implies that $\mathrm{in}_y(\mathrm{in}_{x'}(f_i)) = \mathrm{T}^{m_i}$. On the other hand, the polynomial $\mathrm{in}_{x'}(f_i) - \mathrm{in}_x(f_i)$ belongs to $\mathrm{in}_x(\mathrm{I})$, and none of its monomials belongs to $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$, by the definition of f_i. Its initial form at y must vanish, which implies that $\mathrm{in}_{x'}(f_i) = \mathrm{in}_x(f_i)$. Since T^{m_i} appears in $\mathrm{in}_{x'}(f_i)$, this shows that $\tau_{f_i}(x') = \langle m_i, x'\rangle$, so that $\log(|c_{i,m}|) + \langle m, x'\rangle = \langle m_i, x'\rangle$ for every m such that T^m is in the support of $\mathrm{in}_x(f_i)$; on the other hand, if T^m is not in that support, then

$\log(|c_{i,m}|) + \langle m, x' \rangle < \langle m_i, x' \rangle$. Conversely, these inequalities imply that $\mathrm{in}_{x'}(f_i) = \mathrm{in}_x(f_i)$ for all i, so that $\mathrm{in}_{x'}(\mathrm{I}) \supset \langle (\mathrm{in}_x(f_i))_i \rangle = \mathrm{in}_x(\mathrm{I})$. Since both ideals $\mathrm{in}_x(\mathrm{I})$ and $\mathrm{in}_{x'}(\mathrm{I})$ have the same Hilbert function, we obtain the equality $\mathrm{in}_{x'}(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$.

This proves that $\mathrm{C}_x(\mathrm{I})$ is contained in the face of $\mathrm{C}_z(\mathrm{I})$ defined by the equalities $\log(|c_{i,m}|) + \langle m - m_i, x' \rangle = 0$ for all i and all m such that $m \neq m_i$ and $\log(|c_{i,m}|) + \langle m - m_i, x \rangle = 0$. Conversely, if w is a point of this face, then every point of the open segment $]x; w[$ belongs to $\mathrm{C}'_x(\mathrm{I})$, hence w belongs to $\mathrm{C}_x(\mathrm{I})$.

Finally, a face of $\mathrm{C}_z(\mathrm{I})$ containing a point x is obtained by replacing, in the system of affine inequalities defining this polyhedron, by the corresponding equalities some of those inequalities which are equalities at x. The smallest such face is obtained in replacing all possible such inequalities. By the previous description, this is exactly $\mathrm{C}_x(\mathrm{I})$. □

Lemma 3.6.5 proves part *c*) of proposition 3.6.3. The formulas of lemma 3.6.4 prove that $\mathrm{C}_z(\mathrm{I})$ is a closed Γ-strict polyhedron in $\mathbf{R}^{n+1}$. Moreover, since f_i is homogeneous, one has $\langle m_i - m, w + te \rangle = \langle m_i - m, w \rangle$ for every $w \in \mathbf{R}^{n+1}$, every $t \in \mathbf{R}$ and every $m \in \mathbf{N}^{m+1}$ such that $c_{i,m} \neq 0$, so that e belongs to the lineality space of $\mathrm{C}_z(\mathrm{I})$. Since $\mathrm{C}_x(\mathrm{I})$ is a face of $\mathrm{C}_z(\mathrm{I})$, the same properties hold for $\mathrm{C}_x(\mathrm{I})$.

Let us finally assume that $\mathrm{in}_x(\mathrm{I})$ is a monomial ideal. For every monomial f belonging to $\mathrm{in}_x(\mathrm{I})$, one has $\mathrm{in}_y(f) = f$, so that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ contains $\mathrm{in}_x(\mathrm{I})$. This implies that $\mathrm{in}_z(\mathrm{I}) = \mathrm{in}_y(\mathrm{in}_x(\mathrm{I}))$ contains $\mathrm{in}_x(\mathrm{I})$, hence $\mathrm{in}_z(\mathrm{I}) = \mathrm{in}_x(\mathrm{I})$ since both ideals have the same Hilbert function. As a consequence, $\mathrm{C}'_x(\mathrm{I}) = \mathrm{C}'_z(\mathrm{I})$, hence $\mathrm{C}_x(\mathrm{I}) = \mathrm{C}_z(\mathrm{I})$. By the formulas of lemma 3.6.4, $\mathrm{C}'_z(\mathrm{I})$ is a the closure of a nonempty convex open subset of $\mathbf{R}^{n+1}$, and every point of $\mathrm{C}_z(\mathrm{I}) - \mathrm{C}'_z(\mathrm{I})$ belongs to a face of $\mathrm{C}_z(\mathrm{I})$. This proves that $\mathrm{C}'_z(\mathrm{I})$ is the interior of $\mathrm{C}_z(\mathrm{I})$. □

Lemma **(3.6.6)** (Maclagan). — *The set of monomial ideals in* $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ *which are of the form* $\mathrm{in}_x(\mathrm{I})$*, for some* $x \in \mathbf{R}^{n+1}$*, is finite.*

Proof. — Let $\mathscr{F}$ be this set of ideals. If $\mathscr{F}$ were infinite, there would exist, by theorem 3.4.5, two elements $x, y \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_x(\mathrm{I})$ and $\mathrm{in}_y(\mathrm{I})$ are monomial ideals and $\mathrm{in}_x(\mathrm{I}) \subsetneq \mathrm{in}_y(\mathrm{I})$. This contradicts the fact that these two ideals have the same Hilbert function. □

Proof of theorem 3.6.2. — Let $\mathscr{C}$ be the set of all subsets of $\mathbf{R}^{n+1}$ the form $C_x(I)$, for some $x \in \mathbf{R}^{n+1}$. The sets $C_x(I)$ are Γ-strict convex polyhedra in $\mathbf{R}^{n+1}$. Since $x \in C_x(I)$ for all x, their union is equal to $\mathbf{R}^{n+1}$. If $\operatorname{in}_x(I)$ is a monomial ideal, then $C_x(I)$ has dimension $n+1$; otherwise, $C_x(I)$ is a face of a polyhedron of the form $C_w(I)$. By lemma 3.6.6, the set $\mathscr{C}$ is finite. Consequently, the set of initial ideals $\operatorname{in}_x(I)$ is finite, when x varies in $\mathbf{R}^{n+1}$.

Let $x, y \in \mathbf{R}^{n+1}$. The preceding description shows that $x \in C_y(I)$ if and only if $C_x(I) \subset C_y(I)$. In this case, $C_x(I)$ and $C_y(I)$ are faces of a common $(n+1)$-dimensional polyhedron of the form $C_z(I)$; in particular, $C_x(I)$ is a face of $C_y(I)$.

If F is a face of $C_y(I)$, choose x in the relative interior of F; then F and $C_x(I)$ are faces of $C_y(I)$ which both have the point x in their relative interiors; necessarily, $F = C_x(I)$.

Let $x, y \in \mathbf{R}^{n+1}$. For every point $z \in C_x(I) \cap C_y(I)$, one has $C_z(I) \subset C_x(I)$, since $z \in C_x(I)$, and $C_z(I) \subset C_y(I)$, since $z \in C_y(I)$, so that $C_z(I) \subset C_x(I) \cap C_y(I)$. This proves that $C_x(I) \cap C_y(I)$ is a union of faces of $C_x(I)$. However, a union of faces of a polyhedron is convex if and only it has a unique maximal element — so that one of these faces contains all of them. As a consequence, $C_x(I) \cap C_y(I)$ is a face of $C_x(I)$, and it belongs to $\mathscr{C}$. □

Proposition **(3.6.7)**. — *Let* I *be a homogeneous ideal of* $K[T_0, \dots, T_n]$ *and let* $x \in \mathbf{R}^{n+1}$. *Let* $L = \operatorname{affsp}(C_x(I)) - x$ *be the minimal vector subspace of* $\mathbf{R}^{n+1}$ *such that* $C_x(I) \subset x + L$. *One has* $\operatorname{in}_y(\operatorname{in}_x(I)) = \operatorname{in}_x(I)$ *for every* $y \in L$.

Proof. — Let $g \in \operatorname{in}_x(I)$; let $f_1, \dots, f_r \in I$ be such that g decomposes as a sum $\sum_{i=1}^r \operatorname{in}_x(f_i)$ of initial forms with pairwise disjoint supports. For all i and all $\varepsilon > 0$ small enough, one has $\operatorname{in}_{x+\varepsilon y}(f_i) = \operatorname{in}_y(\operatorname{in}_x(f_i))$. Let J be the subset of $\{1, \dots, r\}$ consisting of all i where $\tau_{\operatorname{in}_x(f_i)}(y)$ is maximal; by the disjointness of the supports of the polynomials $\operatorname{in}_x(f_i)$, one has

$$\operatorname{in}_y(g) = \sum_{i \in J} \operatorname{in}_y(\operatorname{in}_x(f_i)) = \sum_{i \in J} \operatorname{in}_{x+\varepsilon y}(f_i).$$

If ε is small enough, one has $x + \varepsilon y \in C'_x(I)$, hence $\operatorname{in}_{x+\varepsilon y}(f_i) \in \operatorname{in}_x(I)$ for all i, so that $\operatorname{in}_y(g) \in \operatorname{in}_x(I)$. This implies the inclusion $\operatorname{in}_y(\operatorname{in}_x(I)) \subset$

$\mathrm{in}_x(\mathrm{I})$. Since these two homogeneous ideals have the same Hilbert functions, one has equality. □

3.7. Tropicalization of algebraic varieties

The goal of this section is to generalize theorem 3.3.6 to all ideals of $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. We first recall how to pass from ideals of this ring to homogeneous ideals of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$, and back.

3.7.1. — Let $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. The support in $\mathbf{Z}^n$ of the homogeneous Laurent polynomial $f(\mathrm{T}_1/\mathrm{T}_0, \dots, \mathrm{T}_n/\mathrm{T}_0)$ have an infimum, say $p = (p_0, \dots, p_n)$. Explicitly, if $\mathrm{S}(f)$ is the support of f and $f = \sum_{m \in \mathrm{S}(f)} c_m \mathrm{T}^m$, then

$$f(\mathrm{T}_1/\mathrm{T}_0, \dots, \mathrm{T}_n/\mathrm{T}_0) = \sum_{m \in \mathrm{S}(f)} c_m \mathrm{T}_0^{-m_1 - \dots - m_n} \mathrm{T}_1^{m_1} \dots \mathrm{T}_n^{m_n},$$

so that $p_0 = -\deg(f)$ and $p_j = \mathrm{ord}_{\mathrm{T}_j}(f)$ for $j \in \{1, \dots, n\}$. Let then f^{h} be the polynomial $\mathrm{T}^{-p} f(\mathrm{T}_1/\mathrm{T}_0, \dots, \mathrm{T}_n/\mathrm{T}_0)$; it is the unique homogeneous polynomial in $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ such that $\mathrm{ord}_{\mathrm{T}_j}(f^{\mathrm{h}}) = 0$ for every $j \in \{0, \dots, n\}$ and $f = \mathrm{T}_1^{p_1} \dots \mathrm{T}_n^{p_n} f^{\mathrm{h}}(1, \mathrm{T}_1, \dots, \mathrm{T}_n)$.

3.7.2. — Let I be an ideal in $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. The ideal I^{h} generated by all polynomials f^{h}, for $f \in \mathrm{I}$, is a homogeneous ideal of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$. The ring morphism $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n] \to \mathrm{K}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ with kernel $(\mathrm{T}_0 - 1)$ corresponds to setting to 1 the homogeneous coordinate T_0, it identifies the invertibility locus of T_0 in $\mathbf{P}_{\mathrm{K}}^n$ with the affine space $\mathbf{A}_{\mathrm{K}}^n$. The locus of invertibility of $\mathrm{T}_0 \dots \mathrm{T}_n$ is defined by requiring further that the other homogeneous coordinates are invertible too: this is an open subscheme of $\mathbf{P}_{\mathrm{K}}^n$ which is is naturally isomorphic to $\mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^n$ and corresponds to the ring morphism $f \mapsto f(1, \mathrm{T}_1, \dots, \mathrm{T}_n)$ from $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ to $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$.

Ideals of $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ correspond to closed subschemes of $\mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^n = \mathrm{Spec}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$. Homogeneous ideals of $\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n]$ correspond to closed subschemes of $\mathbf{P}_{\mathrm{K}}^n = \mathrm{Proj}(\mathrm{K}[\mathrm{T}_0, \dots, \mathrm{T}_n])$. Then $\mathrm{V}(\mathrm{I}^{\mathrm{h}})$ is the Zariski closure of $\mathrm{V}(\mathrm{I})$.

As a consequence, several geometric properties of $V(I)$ are preserved when passing to $V(I^h)$:

– If $V(I)$ is irreducible, then so is $V(I^h)$;
– If $V(I)$ is integral, then so is $V(I^h)$;
– One has $\dim(V(I^h)) = \dim(V(I))$;
– If $V(I)$ is equidimensional, then so is $V(I^h)$.

3.7.3. — Let $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$; let $p \in \mathbf{Z}^n$ be such that

$$f = T_1^{p_1} \dots T_n^{p_n} f^h(1, T_1, \dots, T_n).$$

Let $x' \in \mathbf{R}^n$ and let $x = (0, x') \in \mathbf{R}^{n+1}$; then the definitions of the tropical polynomials and of the initial forms imply that $\tau_f(x) = \langle p, x\rangle + \tau_{f^h}(x')$ and $\mathrm{in}_x(f) = T_1^{p_1} \dots T_n^{p_n} \mathrm{in}_{x'}(f^h)$. In particular, $\mathrm{in}_{x'}(f^h) \in \mathrm{in}_x(I)^h$. Every homogeneous element of I^h is of the form $T^m f^h$ for some elements $f \in I$ and $m \in \mathbf{Z}^n$, one then has $\mathrm{in}_{x'}(T^m f^h) = T^m \mathrm{in}_{x'}(f^h) \in \mathrm{in}_x(I)^h$, hence $\mathrm{in}_{x'}(I^h) \subset \mathrm{in}_x(I)^h$. Conversely, if $f \in I$, then there exists $m \in \mathbf{Z}^{n+1}$ such that $T^m \mathrm{in}_x(f)^h = \mathrm{in}_{x'}(f^h)$, hence $T^m \mathrm{in}_x(f)^h \in \mathrm{in}_{x'}(I^h)$. This proves the relation

$$\mathrm{in}_x(I)^h = (\mathrm{in}_{x'}(I^h) : (T_0 \dots T_n)^\infty) = K[T_0, \dots, T_n] \cap \mathrm{in}_{x'}(I^h)_{T_0 \dots T_n}.$$

In any case, identifying $\mathbf{R}^n$ with $\{0\} \times \mathbf{R}^n$, the Gröbner decomposition Σ_{I^h} of $\mathbf{R}^{n+1}$ associated with the ideal I^h furnishes a similar decomposition Σ_I of $\mathbf{R}^n$. When x varies in an open cell of this decomposition, and $x' = (0, x)$, the initial ideal $\mathrm{in}_{x'}(I^h)$ is constant, hence the initial ideal $\mathrm{in}_x(I)$ is constant. The reader shall be cautious not to state an indue converse assertion: for example, $\mathrm{in}_x(I) = (1)$ only means that $\mathrm{in}_{x'}(I^h)$ contains a monomial, but the different initial ideals $\mathrm{in}_{x'}(I^h)$ can be very different.

Definition **(3.7.4)**. — *Let* K *be a valued field, let* I *be an ideal of* $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *and let* X *be the closed subscheme* $V(I)$ *of* $\mathbf{G}_m{}^n$.

a) *The* tropical variety $\mathscr{T}_X$ *of* X *is the intersection, for all* $f \in I$, *of the tropical hypersurfaces* $\mathscr{T}_f$.

b) *A* tropical basis *of* I *is a finite family* $(f_1, \dots, f_m)$ *in* I *such that* $\mathscr{T}_X = \bigcap_{i=1}^m \mathscr{T}_{f_i}$.

By definition, for $x \in\in \mathbf{R}^n$, one has $x \notin \mathscr{T}_X$ if and only if there exists $f \in I$ such that the supremum defining $\tau_f(x)$ is achieved at a single monomial $c_m T^m$ of f. This also means that $NP_{f,x}$ is reduced to a point, or, if K is split, that the initial form $\mathrm{in}_x(f)$ is a monomial.

Replacing f by $c_m^{-1} T^{-m} f$, we may assume that $\tau_f(x) = 0$, which is achieved uniquely at the monomial 1, that is $NP_{f,x} = \{0\}$. From the point of view of initial forms, this means that $\mathrm{in}_x(f) = 1$.

Proposition **(3.7.5)**. — *Let* I *be an ideal of* $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *and let* X *be the closed subscheme* V(I). *Let* (X_j) *be the family of its irreducible components; for every* j, *let* $I_j = I(X_j)$ *be the prime ideal defining* X_j. *One has* $\mathscr{T}_X = \bigcup_j \mathscr{T}_{X_j}$.

Proof. — The ideals I_j are the minimal prime ideals of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ containing I; as a consequence, their intersection $J = \bigcap_j I_j$ is the radical of I, the set of all elements $f \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ such that there exists $m \geqslant 1$ such that $f^m \in I$.

For every j, one has $I \subset I_j$, hence $\mathscr{T}_{X_j} \subset \mathscr{T}_X$. Consequently, $\bigcup \mathscr{T}_{X_j} \subset \mathscr{T}_X$. Conversely, let $x \in \mathbf{R}^n - \bigcup \mathscr{T}_{X_j}$. For every j, there exists $f_j \in I_j$ such that $\mathrm{in}_x(f_j) = 1$. Let $f = \prod f_j$; one has $f \in \bigcap I_j = J$, hence there exists $m \in \mathbf{N}$ such that $f^m \in I$. Then $\mathrm{in}_x(f^m) = \prod_j \mathrm{in}_x(f_j)^m = 1$, hence $\mathrm{in}_x(I) = 1$ and $x \notin \mathscr{T}_X$. □

Proposition **(3.7.6)**. — *Let* K *be a valued field, let* I *be an ideal of* $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ *and let* X *be the closed subscheme* V(I) *of* $\mathbf{G}_m{}^n$.

a) *The ideal* I *admits a tropical basis.*
b) *The tropical variety* $\mathscr{T}_X$ *is a* Γ*-strict polyhedral subspace of* $\mathbf{R}^n$.
c) *For every valued extension* L *of* K, *one has* $\mathscr{T}_{X_L} = \mathscr{T}_X$.

Proof. — We first prove assertion *a*) under the assumption that there is a splitting of the valuation $K^\times \to \Gamma$.

Let I^h be the homogeneous ideal of $K[T_0, \dots, T_n]$ associated with I. Let $\mathscr{T}_X^h$ be the set of all $x \in \mathbf{R}^{n+1}$ such that $\mathrm{in}_x(I^h)$ does not contain any monomial.

If $x \notin \mathscr{T}_X^h$, then $\mathrm{in}_x(I^h)$ contains a monomial, say T^m, hence there exists $f \in I^h$ such that $\mathrm{in}_x(f) = T^m$. Then $\mathrm{in}_y(f) = T^m$ for every $y \in \mathbf{R}^{n+1}$ close enough to x, so that $\mathscr{T}_X^h$ is closed in $\mathbf{R}^{n+1}$.

If $x \in \mathscr{T}_X^h$, then the open cell $C'_x(I^h)$ is contained in $\mathscr{T}_X^h$ as well, and its closure $C_x(I^h)$ too. Consequently, $\mathscr{T}_X^h$ is a union of some cells of the Gröbner polyhedral decomposition Σ_{I^h}. In particular, it is a Γ-strict polyhedral subset of $\mathbf{R}^{n+1}$.

Let $x \in \mathbf{R}^n$ and let $x' = (0, x)$. Then $\mathrm{in}_x(I) = (1)$ if and only if there exists $f \in I$ such that $\mathrm{in}_x(f) = 1$; then $\mathrm{in}_{x'}(f^h)$ is a monomial in $T_1, \dots, T_n$ multiplied by a polynomial in T_0. By homogeneity, $\mathrm{in}_{x'}(f^h)$ is a monomial. Conversely, if $\mathrm{in}_{x'}(f^h)$ is a monomial, then $\mathrm{in}_x(f)$ is a monomial as well. This proves that $\mathscr{T}_X$ is the set of $x \in \mathbf{R}^n$ such that $(0, x) \in \mathscr{T}_X^h$.

Moreover, for every cell $C'_x(I^h)$ such that the corresponding initial ideal $\mathrm{in}_x(I^h)$ contains a monomial, we may choose $f \in I$ such that $\mathrm{in}_x(f^h)$ is a monomial. The family (f_i) of these polynomials satisfies the required condition.

To prove c), we may assume that the valuation of the field L has a splitting, so that assertion *a*) holds for $\mathscr{T}_{X_L}$.

The inclusion $\mathscr{T}_{X_L} \subset \mathscr{T}_X$ follows from the definition. Indeed, if $x \in \mathbf{R}^n - \mathscr{T}_X$, there exists $f \in I$ such that the supremum defining $\tau_f(x)$ is reached for one monomial only, and the same property holds for f viewed as an element of I_L, so that $x \notin \mathscr{T}_{X_L}$.

Conversely, let $x \in \mathbf{R}^n - \mathscr{T}_{X_L}$ and let $f = \sum c_m T^m \in I_L$ be such that the supremum defining $\tau_f(x)$ is reached at only one monomial. Let us consider an expression $f = \sum_{j=1}^r a_j f_j$, where $a_j \in L$ and $f_j \in I$, and the integer r is minimal. Let $S \subset \mathbf{Z}^n$ be the union of the supports of the f_j and let us consider the $r \times S$ matrix A given by the coefficients of these Laurent polynomials. Among all finite families $R = (m_1, \dots, m_r)$ in S, let us choose one such that the quantity

$$\log(|\det(A^R)|) + \sum_{j=1}^r \langle m_j, x \rangle$$

is maximal, where A^R is the $r \times r$ submatrix of A with columns $m_1, \dots, m_r$. Since A has rank r, the matrix A^R is invertible and there exists a matrix $U \in \mathrm{GL}(r,\mathrm{K})$ such that $(UA)^{\mathrm{R}}UA^{\mathrm{R}} = I_r$. Then

$$\log(|\det((UA)^{\mathrm{R}})|)+\sum_{j=1}^{r}\langle m_j, x\rangle = \log(|\det(U)|)+\log(|\det(A^{\mathrm{R}})|)+\sum_{j=1}^{r}\langle m_j, x\rangle$$

is maximal. For $i \in \{1,\dots,r\}$ and $m \in \mathrm{S}\smallsetminus\mathrm{R}$, exchanging the columns m and m_i replaces the above quantity by

$$\log(|(UA)_{i,m}|) + \sum_{j=1}^{r}\langle m_j, x\rangle + \langle m, x\rangle - \langle m_i, x\rangle,$$

so that

$$\log(|(UA)_{i,m}|) + \langle m, x\rangle \geqslant \langle m_i, x\rangle.$$

Replacing the polynomials $f_1,\dots,f_r$ by the polynomials whose coefficients are given by the matrix UA, we may assume that there are Laurent polynomials $g_j = \sum_{m\in\mathrm{S}\smallsetminus\mathrm{R}} c_{j,m}\mathrm{T}^m$ with support contained in $\mathrm{S}\smallsetminus\mathrm{R}$ (for $j \in \{1,\dots,r\}$) such that $f_j = \mathrm{T}^{m_j} + g_j$ and such that $\log(|c_{j,m}|) + \langle m, x\rangle \leqslant \langle m_j, x\rangle$. Then

$$f = \sum_{j=1}^{r} a_j\mathrm{T}^{m_j} + \sum_{m\in\mathrm{S}\smallsetminus\mathrm{R}}\Big(\sum_{j=1}^{r} a_j c_{j,m}\Big)\mathrm{T}^m,$$

so that

$$\tau_f(x) \geqslant \sup_j(\log(|a_j|) + \langle m_j, x\rangle).$$

Then, for every $m \in \mathrm{S}\smallsetminus\mathrm{R}$ and every $j \in \{1,\dots,r\}$, one has the inequality

$$\log(|a_j c_{j,m}|) + \langle m, x\rangle \leqslant \log(|a_j|) + \langle m_j, x\rangle,$$

so that $\log(|c_{j,m}|)+\langle m, x\rangle \leqslant \langle m_j, x\rangle$ and $\tau_f(x) = \sup_j(\log(|a_j|)+\langle m_j, x\rangle)$.

By the assumption $x \notin \mathscr{T}_f$, there exists a unique $j \in \{1,\dots,r\}$ such that $\tau_f(x) = \log(|a_j|) + \langle m_j, x\rangle$, and $\log(|c_m|) + \langle m, x\rangle < \tau_f(x)$ for every $m \in \mathrm{S}\smallsetminus\mathrm{R}$. For $i \in \{1,\dots,r\}$ such that $i \neq j$, one thus has $\log(|a_i|) + \langle m_i, x\rangle < \log(|a_j|) + \langle m_j, x\rangle$. Then for every $m \in \mathrm{S}\smallsetminus\mathrm{R}$, one has

$$\log(|a_i c_{i,m}|) + \langle m, x\rangle \leqslant \log(|a_i|) + \langle m_i, x\rangle < \log(|a_j|) + \langle m_j, x\rangle,$$

so that $\log(|c_m|) = \log(|a_j c_{j,m}|)$. Since $\log(|c_m|) + \langle m, x\rangle < \tau_f(x) = \log(|a_j|) + \langle m_j, x\rangle$, one has $\log(|c_{j,m}|) + \langle m, x\rangle < \langle m_j, x\rangle$. This proves that the supremum defining $\tau_{f_j}(x)$ is reached for the monomial m_j only. Since $f_j \in \mathrm{I}$, we have proved that $x \notin \mathscr{T}_{\mathrm{X}}$. This proves assertion *c*).

In fact, the same argument also allows to deduce assertion *a*) in full. We may indeed apply it to every element f of a tropical basis of I_{L}, and every point $x \in \mathbf{R}^n$. For a given f, there are only finitely many possible families $(m_1, \dots, m_r)$ as above, so that when x varies, the procedure furnishes finitely Laurent polynomials in I. The collection (f_i) of these Laurent polynomials is a tropical basis of I, as sought for.

For every i, $\mathscr{T}_{f_i}$ is a Γ-strict polyhedral subspace of $\mathbf{R}^n$, hence so is their intersection $\mathscr{T}_{\mathrm{X}}$. This proves *b*) and concludes the proof of the proposition. □

Remark **(3.7.7)**. — Let V be a closed subvariety of $(\mathbf{C}^*)^n$ and let I be its ideal in $\mathbf{C}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. Let us endow the field $\mathbf{C}$ with the trivial valuation. Then $\mathscr{T}_{\mathrm{V(I)}}$ coincides with the tropical variety $\mathscr{T}_{\mathrm{V}}$ of definition 2.6.3. This proves that $\mathscr{T}_{\mathrm{V}}$ is a $\mathbf{Q}$-rational polyhedral set. In particular, theorem 2.6.6 applies to V, and this concludes the proof of the Bieri–Groves theorem (theorem 2.6.5).

Theorem **(3.7.8)** (Einsiedler, Kapranov & Lind, 2006)

Let I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* X *be the closed subscheme* V(I) *of* ${\mathbf{G}_{\mathrm{m}}}^n$*. The following three subsets of* $\mathbf{R}^n$ *coincide:*

(i) *The tropical variety* $\mathscr{T}_{\mathrm{X}}$*;*

(ii) *The set of all* $x \in \mathbf{R}^n$ *such that there exists a valued extension* L *of* K *and a point* $z \in \mathrm{X(L)} \subset (\mathrm{L}^\times)^n$ *such that* $x = \lambda(z)$*;*

(iii) *The image of* $\mathrm{X}^{\mathrm{an}} = \mathscr{V}(\mathrm{I}) \subset ({\mathbf{G}_{\mathrm{m}}}^n)^{\mathrm{an}}$ *by the tropicalization map* $p \mapsto (\log(p(\mathrm{T}_1)), \dots, \log(p(\mathrm{T}_n)))$.

If the valuation of K *admits a splitting, they also coincide with:*

(iv) *The set of all* $x \in \mathbf{R}^n$ *such that* $\mathrm{in}_x(\mathrm{I}) \neq \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$*;*

For any algebraically closed extension L *of* K*, endowed with an absolute value extending that of* K *which is nontrivial, they also coincide with:*

(v) *The closure of the set of all* $x \in \mathbf{R}^n$ *such that there exists a point* $z \in \mathrm{X(L)} \subset (\mathrm{L}^\times)^n$ *such that* $x = \lambda(z)$.

Proof. — Let us denote these subsets of $\mathbf{R}^n$ by $\mathrm{S}_1 = \mathscr{T}_{\mathrm{X}}, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5^{\mathrm{L}}$. As for the proof of theorem 3.3.6, some inclusions are essentially formal. The equality $\mathrm{S}_2 = \mathrm{S}_3$ has been proved in §3.2.9. If the valuation has a splitting, the equality $\mathrm{in}_x(\mathrm{I}) = \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ is equivalent to the existence of $f \in \mathrm{I}$ such that $\mathrm{in}_x(\mathrm{I})$ is invertible, that is, a monomial. This proves that $S_1 = S_4$. By definition, S_5^L is the closure of a subset of S_2; since S_3 is closed, one has $S_5^L \subset S_3$. Finally, for every $f \in I$, one has $S_2 \subset \mathscr{T}_f$, hence $S_2 \subset \mathscr{T}_X = S_1$.

The rest of the proof follows from the results proved below. We first establish (lemma 3.7.9) that the dimension of $\mathscr{T}_X$ is at most that of V(I). Under the assumption that K is algebraically closed and its valuation is nontrivial, this is then used to prove that for every point $x \in \mathscr{T}_X \cap \Gamma^n$, there exists $z \in X(K)$ such that $\lambda(z) = x$ (proposition 3.7.10). In this case, this implies the inclusion $\mathscr{T}_X \subset S_5^K$, hence the equality of all five sets.

In the general case, let us consider an algebraically closed valued extension K′ of K whose value group is nontrivial; in particular, the valuation admits a splitting. By the case already proved, the subsets $S'_1, \dots, S'_4, S'_5 = S_5^{K'}$ of $\mathbf{R}^n$ corresponding to the ideal $I_{K'}$ of $K'[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ deduced from I satisfy the equalities $S'_1 = S'_2 = S'_3 = S'_4 = S'_5$. The inclusions $S'_1 \subset S_1$ and $S'_5 \subset S'_2 \subset S_2 = S_3$ follow from the definitions, and the equality $S'_1 = S_1$ has been proved in proposition 3.7.6, *c*). One then obtains the missing inclusion $S_1 = S'_1 \subset S'_5 \subset S_3$, and that will conclude the proof of the theorem. □

Lemma **(3.7.9)**. — *The polyhedral set* $\mathscr{T}_X$ *has dimension at most* dim(X)*.*

Using theorem 3.7.8, we shall prove later (theorem 3.8.4) that the dimension of $\mathscr{T}_X$ is equal to dim(X).

Proof. — Thanks to proposition 3.7.6, we may assume that the valuation of K has a splitting and its image Γ is dense in **R**. Then a point $x \in \mathbf{R}^n$ belongs to $\mathscr{T}_X$ if and only if $\mathrm{in}_x(I) = (1)$, if and only if there exists $f \in I$ such that $\mathrm{in}_x(f) = 1$.

Let then C be a maximal cell of the Gröbner polyhedral decomposition of $\mathscr{T}_X$ and let $m = \dim(C)$. Since C is a Γ-strict polyhedron, and since Γ is dense in **R**, there exists a point x in the relative interior of C whose coordinates belong to Γ. Up to a monomial change of variables, we may assume that the affine span of C is $x + (\mathbf{Z}^m \times \{(0, \dots, 0)\})$. Fix a finite generating family $(f_1, \dots, f_r)$ of $\mathrm{in}_x(I)$ such that no nontrivial subpolynomial of the f_j belongs to $\mathrm{in}_x(I)$.

Let $y \in \mathbf{R}^n$ such that $y_{m+1} = \cdots = y_n = 0$. It follows from proposition 3.6.7 and a homogeneization-dehomogeneization argument that $\mathrm{in}_y(\mathrm{in}_x(\mathrm{I})) = \mathrm{in}_x(\mathrm{I})$. Since $\mathrm{in}_y(f_j)$ is a subpolynomial of f_j, nonzero if $f_j \neq 0$, this implies that $\mathrm{in}_y(f_j) = f_j$ for all j. Apply this remark when y is one of the first m vectors $e_1, \dots, e_m$ of the canonical basis of $\mathbf{R}^n$. Writing $f_j = \sum c_m \mathrm{T}^m$, one has $\tau_{f_j}(e_i) = \sup_{m \in \mathrm{S}(f)} m_j = \deg_{\mathrm{T}_1}(f_j)$ (recall that the residue field k is endowed with the trivial absolute value). The relation $\mathrm{in}_{e_i}(f_j) = f_j$ implies that f_j is a power of T_i multiplied by a polynomial in the other variables. In other words, there exists a Laurent polynomial $g_j \in \mathrm{K}[\mathrm{T}_{m+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ and $p \in \mathbf{Z}^m$ such that $f_j = \mathrm{T}_1^{p_1} \dots \mathrm{T}_m^{p_m} g_j$. Letting J be the ideal of $\mathrm{K}[\mathrm{T}_{m+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ generated by $g_1, \dots, g_r$, one has $\mathrm{V}(\mathrm{in}_x(\mathrm{I})) = \mathbf{G}_{\mathrm{m}}{}_k^m \times \mathrm{V}(\mathrm{J})$. Since $\mathrm{in}_x(\mathrm{I}) \neq (1)$, one has $\mathrm{J} \neq (1)$ and $\dim(\mathrm{V}(\mathrm{in}_x(\mathrm{I}))) = m + \dim(\mathrm{V}(\mathrm{J})) \geqslant m$. On the other hand, $\dim(\mathrm{V}(\mathrm{in}_x(\mathrm{I}))) = \dim(\mathrm{V}(\mathrm{I}))$. This concludes the proof. ☐

Proposition **(3.7.10)**. — *Assume that the field* K *is algebraically closed and that its value group* Γ *is nontrivial. Let* $x \in \mathscr{T}_{\mathrm{X}} \cap \Gamma^n$. *Then there exists* $z \in \mathrm{X}(\mathrm{K})$ *such that* $\lambda(z) = x$. *If, moreover,* X *is irreducible, then the set of such* z *is Zariski-dense in* X.

Proof. — It suffices to treat the case where X is irreducible. Replacing the ideal I of X by its radical $\sqrt{\mathrm{I}}$ does not change $\mathscr{T}_{\mathrm{X}}$, nor the set X(K). We may thus assume that I is a prime ideal of $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$.

The proof of this proposition is by induction on n; we will make use of the case of hypersurfaces, already proved in theorem 3.3.6. The proposition is obvious if I = (0).

Assume that $\dim(\mathrm{X}) = n - 1$. We first recall that there exists $f \in \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ such that $\mathrm{I} = (f)$. Indeed, let f be a nonzero element of I; it is a product of irreducible elements, and one of them belongs to I, since I is prime. We can thus assume that f is irreducible; since $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ is a unique factorization domain, the ideal (f) is then prime. The inclusion $(f) \subset \mathrm{I}$ implies an inclusion $\mathrm{X} \subset \mathrm{V}(f) \subsetneq \mathbf{G}_{\mathrm{m}}{}_{\mathrm{K}}^n$ of irreducible sets. Since $\dim(\mathrm{X}) = n - 1$, this implies $\mathrm{X} = \mathrm{V}(f)$, hence $\mathrm{I} = (f)$. Consequently, the proposition follows from corollary 3.3.9 in this case.

We now assume that $\dim(X) < n-1$. Let $x \in \mathscr{T}_X \cap \Gamma^n$. To prove the existence of $z \in X(K)$ such that $\lambda(z) = x$, we shall project X to $\mathbf{G}_{m_K}^{n-1}$. Take a nonzero element $f \in I$. Up to a permutation of the variables, we may assume that f is not a monomial in T_n. We then make a monomial change of variables given by $T_1 \to T_1$, $T_2 \to T_2 T_1^q, \dots, T_n \to T_n T_1^{q^{n-1}}$, as in the proof of proposition 3.3.7, so as assuming that, when written as a polynomial in T_1, every coefficient of f is a monomial in the other variables. This implies that the projection morphism p from $\mathbf{G}_{m_K}^n$ to $\mathbf{G}_{m_K}^{n-1}$ (forgetting the first coordinate) induces an integral, hence finite, morphism from X to its image. This image X′ is then a closed integral subscheme of $\mathbf{G}_{m_K}^{n-1}$. One has $p(x) \in \mathscr{T}_{X'}$, so that there exists $z' \in X'(K)$ such that $\lambda(z') = p(x)$. By finiteness, the point z' lifts to a point $z \in X(K)$, but not all lifts will satisfy $\lambda(z) = x$. We force this property by making use of the diversity of possible projections, using multiple change of variables as above. Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be a linear map of the form $(x_1, \dots, x_n) \mapsto (x_2 + qx_1, \dots, x_n + q^{n-1}x_1)$; one has $\operatorname{Ker}(\pi) = \mathbf{R}(-1, q, \dots, q^{n-1})$. We shall choose the integer q so that $\pi^{-1}(\pi(x)) \cap \mathscr{T}_X = \{x\}$. Let $x' \in \pi^{-1}(\pi(x)) \cap \mathscr{T}_X$ be such that $x' \neq x$; let C be polyhedron of a given polyhedral decomposition of $\mathscr{T}_X$ such that $x' \in \mathscr{T}_X$; then $x' - x \in \operatorname{Ker}(\pi)$, so that there exists $t \in \mathbf{R}$ such that $x' - x = t(-1, q, \dots, q^{n-1})$, and the line $\mathbf{R}(-1, q, q^2, \dots, q^{n-1}, 1)$ meets $C - x$ in a nonzero point. On the other hand, since $\dim(\mathbf{R}C + \mathbf{R}x) \leqslant n-1$, it is contained in a nontrivial affine hyperplane with equation, say $a_1x_1 + \dots + a_nx_n = b$, and for all but finitely many $q \in \mathbf{Z}$, one has $-a_1 + a_2a + \dots + a_nq^{n-1} \neq 0$. We may thus impose that $\pi^{-1}(\pi(x)) \cap \mathscr{T}_X = \{x\}$.

By induction, the set V′ of elements $z' \in X'(K)$ such that $\lambda(z') = \pi(x)$ is Zariski-dense in X′. Since $p : X \to X'$ is a surjective morphism of irreducible schemes and K is algebraically closed, the inverse image $V = p^{-1}(V')$ of V′ is Zariski-dense in X(K). For every $z \in V$, one has $\lambda(z) \in \mathscr{T}_X$ and $\pi(\lambda(z)) = \lambda(p(z)) = \pi(x)$ since $p(z) \in V'$, so that $\lambda(z) = x$. This concludes the proof. □

3.8. Dimension of tropical varieties

Proposition **(3.8.1)**. — *Let* K *be a valued field and let* X *be a closed subscheme of* $\mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n}$. *Let* $p\colon \mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n} \to \mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{m}$ *be a monomial morphism of tori, let* $\pi\colon \mathbf{R}^n \to \mathbf{R}^m$ *be the corresponding linear map and let* $\mathrm{Y} = \overline{p(\mathrm{X})}$ *be the schematic image of* X *under* p. *One has* $\mathscr{T}_{\mathrm{Y}} = \pi(\mathscr{T}_{\mathrm{X}})$.

Proof. — Write

$$\mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n} = \operatorname{Spec}(\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]) \quad \text{and} \quad \mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{m} = \operatorname{Spec}(\mathrm{K}[\mathrm{S}_1^{\pm 1}, \dots, \mathrm{S}_m^{\pm 1}]);$$

the morphism p corresponds to a morphism of K-algebras

$$p^*\colon \mathrm{K}[\mathrm{S}_1^{\pm 1}, \dots, \mathrm{S}_m^{\pm 1}] \to \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}].$$

By assumption, $p^*(\mathrm{S}_j)$ is a monomial, for every j. Let I be the ideal of X and let $\mathrm{J} = (p^*)^{-1}(\mathrm{I})$, so that the morphism p^* induces an injective morphism of K-algebras, still denoted by p^*:

$$\mathrm{K}[\mathrm{S}_1^{\pm 1}, \dots, \mathrm{S}_m^{\pm 1}]/\mathrm{J} \to \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]/\mathrm{I}.$$

Then p maps $\mathrm{X} = \mathrm{V}(\mathrm{I})$ into a Zariski-dense subset of $\mathrm{Y} = \mathrm{V}(\mathrm{J})$; moreover, by Chevalley's theorem (Mumford (1994), corollary 2, p. 51)[3], the pointwise image $p(\mathrm{X})$ of X contains a dense open subscheme Y′ of Y.

For every valued extension L of K and every $z \in \mathrm{X}(\mathrm{L})$, one has $\lambda(p(z)) = \pi(\lambda(z))$; this implies that $\pi(\mathscr{T}_{\mathrm{X}}) \subset \mathscr{T}_{\mathrm{Y}}$. Conversely, let $y \in \mathscr{T}_{\mathrm{Y}}$. Fix an algebraically closed valued extension L of K which is non trivially valued. By proposition 3.7.10, the set of points $t \in \mathrm{Y}(\mathrm{L})$ such that $\lambda(t) = y$ is Zariski-dense in Y. Consequently, it meets the dense open subscheme Y′ of Y; let thus choose $t \in \mathrm{Y}'(\mathrm{L})$ such that $\lambda(t) = y$. Since L is algebraically closed, there exists $z \in \mathrm{X}(\mathrm{L})$ such that $p(z) = t$. Then $\lambda(z) \in \mathscr{T}_{\mathrm{X}}$ and $\pi(\lambda(z)) = \lambda(p(z)) = \lambda(t) = y$, which proves that $\mathscr{T}_{\mathrm{Y}} \subset \pi(\mathscr{T}_{\mathrm{X}})$. □

Proposition **(3.8.2)**. — *Let* X *be a closed subscheme of* $\mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n}$ *such that* $\mathscr{T}_{\mathrm{X}}$ *is finite. Then* X *is finite.*

Proof. — We argue by induction on n. The result is obvious if $n = 0$. One has $\mathrm{X} \neq \mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n}$ for, otherwise, one would have $\mathscr{T}_{\mathrm{X}} = \mathbf{R}^n$; consequently, $\mathrm{I}(\mathrm{X}) \neq 0$. Choosing a nonzero Laurent polynomial $f \in \mathrm{I}(\mathrm{X})$,

[3] Add a reference in the yet-to-be-written appendix

we may find an adequate monomial projection $p\colon \mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n} \to \mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n-1}$ that induces a finite morphism from X to $\mathbf{G}_{\mathrm{m}_{\mathrm{K}}}^{n-1}$, and let Y be its image. By proposition 3.8.1, the tropical variety $\mathscr{T}_{\mathrm{Y}}$ is finite. By induction this implies that Y is finite. Since $p\colon \mathrm{X} \to \mathrm{Y}$ is finite, this implies that X is finite as well. □

Lemma **(3.8.3)**. — *Let* K *be a split valued field. Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* $\mathrm{X} = \mathrm{V}(\mathrm{I})$. *Let* $x \in \mathscr{T}_{\mathrm{X}}$. *Then* $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) = \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$.

Proof. — Fix a tropical basis $(f_1, \dots, f_r)$ of I. Recall that the polyhedral set $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}})$ is the set of $y \in \mathbf{R}^n$ such that $x + \varepsilon y \in \mathscr{T}_{\mathrm{X}}$ for $\varepsilon > 0$ small enough.

Let $y \in \mathbf{R}^n$ be such that $y \notin \mathrm{Star}_x(\mathscr{T}_{\mathrm{X}})$. Then, for every $\varepsilon > 0$ small enough, one has $x + \varepsilon y \notin \mathscr{T}_{\mathrm{X}}$, hence there exists i such that $\mathrm{in}_{x+\varepsilon y}(f_i)$ is a monomial. On the other hand, for all $\varepsilon > 0$ small enough, one has $\mathrm{in}_{x+\varepsilon y}(f_i) = \mathrm{in}_y(\mathrm{in}_x(f_i))$. Consequently, $\mathrm{in}_y(\mathrm{in}_x(f_i))$ is a monomial and $y \notin \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$.

Conversely, let $y \in \mathbf{R}^n$ be such that $y \notin \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$. By definition, there exists $g \in \mathrm{in}_x(\mathrm{I})$ such that $\mathrm{in}_y(g)$ is a monomial. There is a finite family $(f_1, \dots, f_r)$ in I such that the initial forms $\mathrm{in}_x(f_1), \dots, \mathrm{in}_x(f_r)$ have disjoint supports and $g = \sum \mathrm{in}_x(f_j)$. Since $\mathrm{in}_y(g)$ is a monomial, there exists $j \in \{1, \dots, r\}$ such that $\mathrm{in}_y(\mathrm{in}_x(f_j))$ contains this monomial, and by the disjointness property of the supports, $\mathrm{in}_y(\mathrm{in}_x(f_j))$ is a monomial. For $\varepsilon > 0$ small enough, one has $\mathrm{in}_y(\mathrm{in}_x(f_j)) = \mathrm{in}_{x+\varepsilon y}(f_j)$, hence $x + \varepsilon y \notin \mathscr{T}_{\mathrm{X}}$ for $\varepsilon > 0$ small enough and $y \notin \mathrm{Star}_x(\mathscr{T}_{\mathrm{X}})$. This proves the other inclusion $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) \subset \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$. □

Theorem **(3.8.4)**. — *Let* K *be a valued field, let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* $\mathrm{X} = \mathrm{V}(\mathrm{I})$. *One has* $\dim(\mathscr{T}_{\mathrm{X}}) = \dim(\mathrm{X})$. *More precisely, if* X *is nonempty and every irreducible component of* X *has dimension p, then the tropical variety* $\mathscr{T}_{\mathrm{X}}$ *is a purely p-dimensional polyhedral set.*

Proof. — We start by copying the proof of lemma 3.7.9. We may assume that the valuation of K has a splitting and that its image Γ is dense in **R**. We consider a maximal cell C in the Gröbner polyhedral decomposition of $\mathscr{T}_{\mathrm{X}}$ and a point x which belongs to the relative interior of C. By a monomial change of coordinates, we may assume that the affine span of C is $x + (\mathbf{R}^m \times \{(0,\dots,0)\})$. If I is the ideal of X, there exists an ideal J of $k[\mathrm{T}_{m+1}^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]$ such that $\mathrm{in}_x(\mathrm{I}) = \langle \mathrm{J}\rangle$ so that $\mathrm{V}(\mathrm{in}_x(\mathrm{I})) = \mathbf{G}_{\mathrm{m}}{}_k^m \times \mathrm{V}(\mathrm{J})$. Moreover, if $p\colon \mathbf{G}_{\mathrm{m}}{}_{\mathrm{K}}^n \to \mathbf{G}_{\mathrm{m}}{}_{\mathrm{K}}^{n-m}$ is the projection $(z_1,\dots,z_n) \mapsto (z_{m+1},\dots,z_n)$, and $\pi\colon \mathbf{R}^n \to \mathbf{R}^{n-m}$, $(x_1,\dots,x_n) \mapsto (x_{m+1},\dots,x_n)$ is the corresponding linear projection, one has $\mathrm{V}(\mathrm{J}) = p(\mathrm{V}(\mathrm{in}_x(\mathrm{I})))$, hence $\mathscr{T}_{\mathrm{V}(\mathrm{J})} = \pi(\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))})$. Since x belongs to the relative interior of C, one has

$$\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))} = \mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) = \mathrm{affsp}(\mathrm{C}) - x.$$

Its image under π is equal to 0, hence $\mathscr{T}_{\mathrm{V}(\mathrm{J})} = \{0\}$. By proposition 3.8.2, this implies that V(J) is finite, so that $\mathrm{V}(\mathrm{in}_x(\mathrm{I})) = \mathbf{G}_{\mathrm{m}}{}_k^m \times \mathrm{V}(\mathrm{J})$ has dimension m. Since V(I) is irreducible, one then has $\dim(\mathrm{V}(\mathrm{I})) = \dim(\mathrm{V}(\mathrm{in}_x(\mathrm{I}))) = m$. ☐

Remark **(3.8.5)**. — Let I be an ideal of $\mathbf{Q}[\mathrm{T}_1^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]$ and $\mathrm{X} = \mathrm{V}(\mathrm{I})$. Every absolute value v of $\mathbf{Q}$ gives rise to a corresponding tropical variety $\mathscr{T}_{\mathrm{X},v}$ in $\mathbf{R}^n$. Let us prove that for all but finitely many prime numbers p, the tropical variety $\mathscr{T}_{\mathrm{X},p}$ associated with the p-adic absolute value coincides with the tropical variety $\mathscr{T}_{\mathrm{X},0}$ associated with the trivial absolute value. Also recall from example 3.1.7 that $\mathscr{T}_{\mathrm{X},0}$, the non-archimedean amoeba of X associated with the trivial valuation on X, is the logarithmic limit set of the complex (archimedean) amoeaba of X.

The case where $\mathrm{X} = \mathrm{V}(f)$ is a hypersurface, where $f \in \mathbf{Q}[\mathrm{T}_1^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]$ is a nonzero Laurent polynomial, follows from the description of $\mathscr{T}_{\mathrm{X}}$ as the non-smooth locus of the tropical polynomial τ_f. Indeed, if $f = \sum_{m\in \mathrm{S}(f)} c_m \mathrm{T}^m$, one has

$$\tau_{f,p}(x) = \sup_{m\in\mathrm{S}(f)} \log(|c_m|_p) + \langle m, x\rangle,$$

for every prime number p. For al but finitely many primes p, one has $|c_m|_p = 1 = |c_m|_0$ for every $m \in \mathrm{S}(f)$. Consequently, $\tau_{f,p} = \tau_{f,0}$ for all but finitely many prime numbers f, whence the equality $\mathscr{T}_{\mathrm{X},p} = \mathscr{T}_{\mathrm{X},0}$.

Let us now prove the general case.

Let $x \in \mathbf{R}^n$ such that $x \notin \mathscr{T}_{\mathrm{X},0}$. By definition, there exists $f \in \mathrm{I}$ such that $\mathrm{in}_{x,|\cdot|_0}(f) = 1$. Write $f = \sum c_m \mathrm{T}^m$. For $m \in \mathrm{S}(f)$, the set of prime numbers p such that $|c_m|_p \neq 1$ is finite. For any prime number p outside of the union of these finite sets, one has $\mathrm{in}_{x,|\cdot|_p}(f) = \mathrm{in}_{x,|\cdot|_0}(f) = 1$, hence

$x \notin \mathscr{T}_{X,p}$. This proves the existence of a finite set S of prime numbers such that for every prime number p such that $p \notin S$, one has $\mathscr{T}_{X,p} \subset \mathscr{T}_{X,0}$.

To prove the other inclusion, we argue by induction on n. The result is obvious if $\dim(X) = n$, and it corresponds to the case of hypersurfaces if $\dim(X) = n-1$; let us now assume that $\dim(X) < n-1$. Since the polyhedral sets $\mathscr{T}_{X,p}$ and $\mathscr{T}_{X,0}$ have the same dimension, namely $\dim(X)$, As in the proof of theorem 3.7.8, there exists a monomial morphism $q \colon \mathbf{G_m}^n \to \mathbf{G_m}^{n-1}$, whose associated linear map $\chi \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ is surjective and is such that $\chi^{-1}(y) \cap \mathscr{T}_{X,0}$ has at most one point, for every $y \in \mathbf{R}^{n-1}$. Let $Y = \overline{q(X)}$. One has $\chi(\mathscr{T}_{X,p}) = \mathscr{T}_{Y,p}$ for every prime number p, and $\chi(\mathscr{T}_{X,0}) = \mathscr{T}_{Y,0}$. By induction, up to enlarging the finite set S, we may assume that $\mathscr{T}_{Y,p} = \mathscr{T}_{Y,0}$ for all prime numbers p such that $p \notin S$. This implies that $\mathscr{T}_{X,p} = \mathscr{T}_{X,0}$ for all such prime numbers p. Let indeed $x \in \mathscr{T}_{X,0}$ and let $y = q(x) \in \mathscr{T}_{Y,0}$. By what precedes, one has $y \in \mathscr{T}_{Y,p} = \chi(\mathscr{T}_{X,p})$, so that there exists $x' \in \mathscr{T}_{X,p}$ such that $y = q(x')$. Since $\mathscr{T}_{X,p} \subset \mathscr{T}_{X,0}$, one has $x' \in \mathscr{T}_{X,0}$. By the choice of the linear map q, this implies that $x' = x$, hence $x \in \mathscr{T}_{X,p}$, as was to be shown.

3.9. Multiplicities

3.9.1. — Let I be an ideal of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ and let $X = V(I) \subset \mathbf{G}_{m_K}^n$. A crucial notion of tropical geometry is that the tropical variety $\mathscr{T}_X$ carries an additional information, of algebraic nature: positive integers, called *multiplicities*, attached to polyhedra of maximal dimension of the Gröbner decomposition of $\mathscr{T}_X$.

The definition of these multiplicities below requires that the valuation of K admits a splitting and that the value group is dense in $\mathbf{R}$. This allows the Gröbner theory to function and garantees that the points with coordinates in the value group are dense in every cell of the Gröbner decomposition of $\mathscr{T}_X$. It also requires that the residue field k be algebraically closed. Note that both conditions are achieved, in particular, when K is algebraically closed. On the other hand, the definition of multiplicities will then be invariant under further extensions of the valued field K.

3.9.2. — Let $d = \dim(X)$ and let C be a polyhedron of dimension d in the Gröbner decomposition of $\mathscr{T}_X$. By definition of C, the initial ideal $\mathrm{in}_x(I) \subset k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ does not depend on the choice of a point x in the relative interior of C.

Let Z be a d-dimensional irreducible components of $V(\mathrm{in}_x(I))$. It corresponds to prime ideals of $k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ containing $\mathrm{in}_x(I)$, and the local ring of $V(\mathrm{in}_x(I))$ at the generic point of Z is equal to

$$(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I)_P \simeq k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]_P/I_P \simeq (k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/Q)_P,$$

where

$$Q = I \cap k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]_Q$$

is the P-primary component of I associated with the prime ideal P. This local ring is noetherian and has dimension zero, hence has finite length; this length is called the *multiplicity* of Z in $V(\mathrm{in}_x(I))$.

Definition **(3.9.3)**. — *(We assume that k is algebraically closed.) The* multiplicity *of the polyhedron* C *in the tropical variety* $\mathscr{T}_X$ *is the sum, for all d-dimensional irreducible components* Z *of* $V(\mathrm{in}_x(I))$, *of the multiplicity of* Z *in* $V(\mathrm{in}_x(I))$. *We denote it by* $\mathrm{mult}_{\mathscr{T}_X}(C)$.

In general, if $\bar{k}$ is an algebraically closed extension of k, the multiplicity of C is defined by applying this recipe to the scheme $V(\mathrm{in}_x(I)_{\bar{k}})$.

We start by giving an alternative formula for this multiplicity which, in fact, works even if k is not algebraically closed.

Lemma **(3.9.4)**. — *Assume that* X *is equidimensional of dimension d that the affine span of* C *is* $x + \mathbf{R}^d \times \{0\}$.

a) *The scheme* $V(\mathrm{in}_x(I))$ *is invariant under the action of* $\mathbf{G}_{\mathrm{m}}{}_k^d \times \{1\}$.

b) *The initial ideal* $\mathrm{in}_x(I)$ *has a basis consisting of elements of* $k[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]$.

c) *Let* $J_x = \mathrm{in}_x(I) \cap k[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]$. *Then the multiplicity of* C *in* $\mathscr{T}_X$ *is given by*

$$\mathrm{mult}_{\mathscr{T}_X}(C) = \dim_k(k[T_{d+1}^{\pm 1}, \dots, T_n^{\pm 1}]/J_x).$$

Up to a monomial change of variables, the second part of the assumption is not restrictive. Geometrically, this lemma then says that the d-dimensional "initial scheme" $V(\mathrm{in}_x(I))$ is invariant under the action of a d-dimensional subtorus of $\mathbf{G}_{\mathrm{m}}{}_k^n$, and the multiplicity is the "number" of orbits, appropriately counted.

Proof. — If k is replaced by an algebraically closed extension, then the assertions of the theorem are equivalent with their replacement. We thus assume that k is algebraically closed.

Assertions *a*) and *b*) have already been proved in the proof of theorem 3.8.4. As for the third one, the minimal prime ideals of $k[T_{d+1}^{\pm1},\dots,T_n^{\pm1}]$ containing J_x are maximal ideals, and furnish, by extension, the minimal prime ideals of $k[T_1^{\pm1},\dots,T_n^{\pm1}]$ containing $\mathrm{in}_x(I)$, and their lengths coincide (because $k[T_1^{\pm1},\dots,T_d^{\pm1}]$ has a maximal ideal with codimension 1). In fact, k being algebraically closed, all maximal ideals of $k[T_{d+1}^{\pm1},\dots,T_n^{\pm1}]$ are of the form $(T_{d+1}-a_{d+1},\dots,T_n-a_n)$, for some $(a_{d+1},\dots,a_n)\in(k^\times)^{n-d}$, by Hilbert's Nullstellensatz. Consequently,

$$\mathrm{length}(k[T_{d+1}^{\pm1},\dots,T_n^{\pm1}]/J_x)=\dim_k(k[T_{d+1}^{\pm1},\dots,T_n^{\pm1}]/J_x),$$

hence the lemma. □

Example **(3.9.5)** (Multiplicities for hypersurfaces). — Let I be the ideal generated by a Laurent polynomial $f\in K[T_1^{\pm1},\dots,T_n^{\pm1}]$. The open cells of the Gröbner decomposition of $\mathscr{T}_X$ corresponds to given values for the initial ideal $\mathrm{in}_{(0,x)}(f^{\mathrm{h}})$ in $k[T_0,\dots,T_n]$, that is, for different values of the initial form $\mathrm{in}_x(f)$.

Let $S(f)$ be the support of f; write $f=\sum_{m\in S(f)}c_mT^m$. For $x\in\mathbf{R}^n$, let $S_x(f)$ be the support of $\mathrm{in}_x(f)$. The open cell C′ containing x is then defined by the relations

$$\log(|c_m|)+\langle m,x\rangle=\log(|c_p|)+\langle p,x\rangle,$$

for $m,p\in S_x(f)$, and the inequalities

$$\log(|c_m|)+\langle m,x\rangle<\log(|c_p|)+\langle p,x\rangle,$$

for $m\in S_x(f)$ and $p\in S(f)-S_x(f)$.

The Gröbner decomposition of $\mathbf{R}^n$ is thus "dual" to the regular polyhedral decomposition of the Newton polytope NP_f which is associated with the function $m \mapsto \log(|c_m|)$ on $S(f)$ as in §1.9.10. Each polyhedron F in this decomposition is a polytope, and its vertices form a subset S_F of $S(f)$; to this polyhedron Q corresponds the polyhedron C_F in $\mathbf{R}^n$ consisting of those $x \in \mathbf{R}^n$ such that $S_x(f)$ contains S_F. The polyhedra F of dimension n correspond to points in S, the polyhedra F of dimension $n-1$ to edges; more generally, one has $\dim(C_F)+\dim(F) = n$.

Let C be an $(n-1)$-dimensional polyhedron of the Gröbner decomposition, corresponding to an edge $F = [a; b]$ with endpoints in $S(f)$. By a monomial change of variables, we may assume that C is parallel to $\mathbf{R}^{n-1} \times \{0\}$; as in the lemma 3.9.4, there exists $m \in \mathbf{Z}^n$ and $g \in k[T_n, T_n^{-1}]$ such that $f = T^m g$. We may moreover assume that $g \in k[T_n]$ is a polynomial which does not vanish at 0. In these coordinates, one thus has $F = [m, m + de_n]$, where e_n is the last basis vector and $d = \deg(g)$. By lemma 3.9.4, one has

$$\mathrm{mult}_{\mathscr{T}_X}(C) = \dim_k(k[T_n^{\pm1}]/(g)) = \dim_k(k[T_n]/(g)) = \deg(g),$$

because $g \notin (T_n)$.

We also see that the edge F contains exactly $(d + 1)$ integer points, namely the points $m + pe_n$, for $p \in \{0, \dots, d\}$. This shows that the integer d can already be computed in the initial system of coordinates where $F = [a; b]$: it is the gcd of the coordinates of $b - a$. This integer is also called the "lattice length" of the segment $[a; b]$: if $d = \gcd(b_1 - a_1, \dots, b_n - a_d)$, the line (a, b) is directed by the primitive vector $v = (b - a)/d$ and $b - a = d \cdot v$.

This description also explains how to compute $\mathscr{T}_f$ explicitly, by first computing the regular polyhedral decomposition of NP_f described above. Its edges furnish the polyhedral set $\mathscr{T}_f$.

Proposition **(3.9.6)**. — *Assume that* $\dim(X) = 0$, *so that* $\mathscr{T}_X$ *is a finite set.*

a) *Assume that* $\mathscr{T}_X = \{x\}$, *for some* $x \in \mathbf{R}^n$. *The multiplicity of* $\{x\}$ *in* $\mathscr{T}_X$ *is then given by*

$$\mathrm{mult}_{\mathscr{T}_X}(\{x\}) = \dim_k(K[T_1^{\pm1}, \dots, T_n^{\pm1}]/I).$$

b) *For every* $x \in \mathscr{T}_X$, *there exists a smallest ideal* I_x *of* $k[T_1^{\pm1}, \dots, T_n^{\pm1}]$ *such that* $I \subset I_x$ *and* $\mathscr{T}_{V(I_x)} = \{x\}$. *One has* $I = \bigcap_{x \in \mathscr{T}_X} I_x$ *and for every* $x \in \mathbf{R}^n$, *the* *multiplicity of* $\{x\}$ *is given by*

$$\mathrm{mult}_{\mathcal{T}_X}(\{x\}) = \dim_K(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I_x).$$

Proof. — For the proof, we assume that K (hence k) is algebraically closed.

To establish *a*), we need prove that

$$\dim_K(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I) = \dim_k(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\mathrm{in}_x(I)).$$

While multiplicities are defined using initial ideals of $k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, the proposition involves the ideal I in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. The relation between both sides rests on the comparison of Hilbert functions, but this holds for homogeneous ideals only; we thus have to compare the ideals $\mathrm{in}_{(0,x)}(I^h)$ and $\mathrm{in}_x(I)^h$.

By 3.7.3, one has

$$\mathrm{in}_x(I)^h = (\mathrm{in}_{(0,x)}(I^h) : (T_0 \dots T_n)^\infty).$$

The schemes V(I) and $V(\mathrm{in}_x(I))$ are zero-dimensional. By homogeneization-dehomogeneization, one has equalities, for $d \to +\infty$,

$$\dim_K(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I) = \dim_K((K[T_0, \dots, T_n]/I^h)_d)$$

and

$$\dim_k(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\mathrm{in}_x(I)) = \dim_k((k[T_0, \dots, T_n]/\mathrm{in}_x(I)^h)_d).$$

Moreover, by comparison of Hilbert functions, one has

$$\dim_K((K[T_0, \dots, T_n]/I^h)_d) = \dim_k((K[T_0, \dots, T_n]/\mathrm{in}_{(0,x)}(I^h))_d).$$

By what precedes, one then has, for $d \to +\infty$,

$$\begin{aligned} \mathrm{mult}_{\mathcal{T}_X}(\{x\}) &= \dim_k(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\mathrm{in}_x(I)) \\ &= \dim_k((k[T_0, \dots, T_n]/\mathrm{in}_x(I)^h)_d) \\ &= \dim_k((k[T_0, \dots, T_n]/(\mathrm{in}_{(0,x)}(I^h) : (T_0 \dots T_n)^\infty))_d). \end{aligned}$$

Let us now prove that $(\mathrm{in}_{(0,x)}(I^h) : (T_0 \dots T_n)^\infty)_d = \mathrm{in}_{(0,x)}(I^h)_d$ for d large enough. To that aim, let us consider a primary decomposition of $\mathrm{in}_{(0,x)}(I^h)$ in $k[T_0, \dots, T_n]$. Since the scheme $V(I^h)$ in $\mathbf{P}^n_K$ has dimension 0, the same holds for the scheme $V(\mathrm{in}_{(0,x)}(I^h))$ and the prime ideals associated with I^h are of two forms, either the irrelevant ideal $M = (T_0, \dots, T_n)$, or homogeneous prime ideals of the form $P_a = (a_i T_j - a_j T_i)$ defining points $a \in \mathbf{P}^n(k)$. We thus write $\mathrm{in}_{(0,x)}(I^h) = Q_0 \cap \bigcap_a Q_a$, a finite intersection where the ideal Q_0 is M-primary and, for each a, the ideal Q_a is P_a-primary.

Let $g \in (\mathrm{in}_{(0,x)}(I^h) : (T_0 \dots T_n)^\infty)$, and let $m \in \mathbf{N}$ be such that $T_0^m \dots T_n^m g \in \mathrm{in}_{(0,x)}(I^h)$. Fix $a \in \mathbf{P}^n(k)$.

Let $j \in \{1, \dots, n\}$ and let us consider the polynomial $f = \prod_{z \in V(I)(K)} (T_j - z_j T_0)$. Since $f(z) = 0$ for all $z \in V(I)(K)$ and K is algebraically closed, Hilbert's Nullstellensatz implies that there exists an integer m such that $f^m \in I$. Since $\log(|z_j|) = x_j$ for every $z \in V(I)(K)$, one has $\mathrm{in}_{(0,x)}(f) = \prod_{z \in V(I)(K)} (T_j - \rho(z_j) T_0)$ and $\prod_{z \in V(I)(K)} (T_j - \rho(z_j) T_0)^m \in \mathrm{in}_{(0,x)}(I)$. Consequently, there exists z such that $T_j - \rho(z_j) T_0 \in P_a$. As a consequence, $T_0 \in P_a$ if and only if $T_j \in P_a$.

It follows that $T_0 \dots T_n \notin P_a$, for, otherwise, $(T_0, \dots, T_n) \subset P_a$, a contradiction. By the definition of a P_a-primary ideal, one then has $g \in Q_a$.

This proves the relation

$$(\mathrm{in}_{(0,x)}(I^h) : (T_0 \dots T_n)^\infty) = \mathrm{in}_x(I^h) \cap (Q_0 : (T_0 \dots T_n)^\infty).$$

Since Q_0 contains a power of the irrelevant ideal M, we have shown

$$\mathrm{in}_x(I)^h_d = (\mathrm{in}_{(0,x)}(I^h) : (T_0 \dots T_n)^\infty)_d = \mathrm{in}_{(0,x)}(I^h)_d$$

for all large enough integers d. It then follows from the constancy of Hilbert functions that

$$\mathrm{in}_x(I)^h_d = \mathrm{in}_{(0,x)}(I^h)_d = I^h_d,$$

and

$$\mathrm{mult}_{\mathscr{T}_X}(\{x\}) = \dim(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\mathrm{in}_x(I)) = \dim(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I),$$

which established part a) of the proposition.

Let $x \in \mathscr{T}_X$. Since K is algebraically closed, irreducible components of X correspond to point $z \in X(K)$, hence there is such a point with $\lambda(z) = x$. Consequently, there are ideals J containing I and such that $\mathscr{T}_{V(J)} = \{x\}$. Since $\mathscr{T}_X$ is finite, the ring $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I$ is artinian, so that every nonempty set of ideals containing I admits minimal elements. Therefore, there are minimal ideals J such that $I \subset J$ and $\mathscr{T}_{V(J)} = \{x\}$. If

J and J′ are such ideals, then $I \subset J \cap J'$ and $\mathscr{T}_{V(J\cap J')} \subset \mathscr{T}_{V(J)} \cup \mathscr{T}_{V(J')} = \{x\}$; necessarily $\mathscr{T}_{V(J\cap J')} = \{x\}$ since $V(J \cap J') \neq \varnothing$. By minimality, one has $J = J'$. This proves the existence of a unique minimal ideal I_x containing I such that $\mathscr{T}_{V(I_x)} = \{x\}$.

Let us prove that $I = \bigcap_{x\in\mathscr{T}_X} I_x$. To that aim, let us consider a minimal primary decoposition of I. Since V(I) is zero-dimensional, the associated prime ideals of I are maximal ideals and such a decomposition takes the form $I = \bigcap_{z\in V(I)(K)} Q_z$, where, for each z, Q_z is an M_z-primary idea, $M_z = (T_j - z_j)$ being the maximal ideal of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ such that $V(M_z) = \{z\}$. Since M_z is minimal among the associated prime ideals of I, the primary ideal Q_z is given by $I_{M_z} \cap K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

Let $x \in \mathscr{T}_X$. Since $I \subset \bigcap_{\lambda(z)=x} Q_z$ and the tropicalization of $V(\bigcap_{\lambda(z)=x} Q_z)$ is equal to $\{x\}$, one has $I_x \subset \bigcap_{\lambda(z)=x} Q_z$. Consequently, $I \subset \bigcap_x I_x \subset \bigcap_z Q_z = I$, hence the equality $I = \bigcap_x I_x$.

For every $y \in \mathscr{T}_X$ such that $y \neq x$, one has $x \notin \mathscr{T}_{V(I_y)}$, hence there exists $f_y \in I_y$ such that $\mathrm{in}_x(f_y) = 1$. Letting $f_x = \prod_{y\neq x} f_y$, one has $\mathrm{in}_x(f_x) = 1$. Moreover, for every $f \in I_x$, one has $f f_x \in \bigcap_{y\neq x} I_y \cap I_x = I$, hence $\mathrm{in}_x(f) = \mathrm{in}_x(f f_x) \in \mathrm{in}_x(I)$; this implies that $\mathrm{in}_x(I_x) \subset \mathrm{in}_x(I)$. Since $I \subset I_x$, one also has $\mathrm{in}_x(I) \subset \mathrm{in}_x(I_x)$, hence the equality. It then follows from part *a*) that

$$\begin{aligned}
\mathrm{mult}_{\mathscr{T}_X}(\{x\}) &= \dim_k(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\mathrm{in}_x(I)) \\
&= \dim_k(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\mathrm{in}_x(I_x)) \\
&= \dim_k(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/I_x),
\end{aligned}$$

as claimed.[4] □

Beware:

The definition of the nonarchimedean amoebas has been modified so as to be more consistent with the definition in the archimedean case. I made the necessary corrections up to here, but there are certainly inconsistencies below.

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[4] Is it true that I_x coincides with $J_x = \bigcap_{\lambda(z)=x} Q_z$? In any case, one also has $\mathrm{in}_x(J_x) = \mathrm{in}_x(I)$, by the same argument. The ideal J_x being maybe more explicit, maybe statement *b*) should be phrased in terms of it.

3.10. The balancing condition

3.10.1. — Let I be an ideal of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]X$ and let $X = V(I)$ be the closed subscheme it defines. Let $d = \dim(X)$. The tropical variety $\mathscr{T}_X$ of X is a polyhedral subset of $\mathbf{R}^n$ of dimension d. The Gröbner polyhedral decomposition of $\mathbf{R}^{n+1}$ associated with the homogeneous ideal I^h induces a rational polyhedral decomposition of $\mathscr{T}_X$, and each polyhedron C of dimension d of that decomposition has been assigned a multiplicity $m_{\mathscr{T}_X}(C)$. In fact, the specific polyhedral decomposition of $\mathscr{T}_X$ is not that relevant: for every point $x \in \mathscr{T}_X$, one can define the multiplicity m_x of $V(\mathrm{in}_x(I))$ and what the construction shows is that the function $x \mapsto m_x$ on $\mathscr{T}_X$ is locally constant outside of a polyhedral subset of smaller dimension.

In the sequel we fix a rational polyhedral decomposition $\mathscr{C}$ of $\mathscr{T}_X$ such that the multiplicity function is constant on the relative interior of every polyhedron of dimension d belonging to $\mathscr{C}$.

For any $C \in \mathscr{C}$, the affine space $\mathrm{affsp}(C)$ is directed by a rational vector subspace V_C of $\mathbf{R}^n$. The intersection $L_C = V_C \cap \mathbf{Z}^n$ is then a free finitely generated abelian group of rank $\dim(V_C) = \dim(C)$. Moreover, if D is a face of C, then V_D is a subspace of V_C, and L_D is a saturated subgroup of L_C; in particular, there exists a basis of L_C that contains a basis of L_D.

3.10.2. — Let $D \in \mathscr{C}$ be a polyhedron of dimension $d-1$ and let $\mathscr{C}_D$ be the set of all polyhedra $C \in \mathscr{C}$ of dimension d of which D is a face.

For every $C \in \mathscr{C}_D$, there exists a vector $v_C \in L_C$ that induces a basis of L_C/L_D and that v_C belongs to the image of C modulo D: precisely, for every x in the relative interior D' of D and every small enough real number $t > 0$, one has $x + tv_C \in C$. Such a vector v_C is unique modulo a vector of L_D.

The *balancing condition* around the polyhedron D is the following relation:

$$\sum_{C \in \mathscr{C}_D} \mathrm{mult}_{\mathscr{T}_X}(C) v_C \in L_D.$$

Example **(3.10.3)** (Balancing condition for hypersurfaces)

CHAPTER 4

TORIC VARIETIES

4.1. Tori, characters and graduations

4.1.1. Tori. — Let k be a field. The n-dimensional torus

$$\mathbf{G}_{\mathrm{m}}{}_k^n = \operatorname{Spec}(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}])$$

is already an important object of this course, as an algebraic k-variety (synonymous, say, for separated, integral scheme of finite type), or, if $k = \mathbf{C}$, as the complex manifold $(\mathbf{C}^*)^n$. We now consider its structure of an algebraic group. On the complex manifold side, it corresponds to the product of coordinates, which is indeed given by polynomials, the identity element being the point $(1, \dots, 1)$.

There are two ways to understand the scheme theoretic side of algebraic groups. Maybe the most natural one consists in viewing a k-scheme X as its *functor of points*, associating with any k-algebra R the set $h_X(R) = \operatorname{Hom}(\operatorname{Spec}(R), X)$, and with any morphism $f : R \to S$ of k-algebras, the map $h_X(f) : X(R) \to X(S)$ induced by composition with the morphism of schemes $f^* : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$. In this respect, the functor of points of $X = \mathbf{G}_{\mathrm{m}}{}_k^n$ is simply given by $h_X(R) = (R^\times)^n$, for every k-algebra R, and for a morphism $f : R \to S$, by the map $h_X(f) : (R^\times)^n \to (S^\times)^n$ which is induced by f coordinate-wise.

That X be an algebraic group means that these sets $h_X(R) = (R^\times)^n$ are endowed (by coordinate-wise multiplication) with a structure of a group, and the maps $h_X(f)$ are morphisms of groups. The unit element is the point $(1, \dots, 1)$ of $X(k)$.

The other way to view algebraic groups consists in interpreting the notion of group in the category of k-schemes. The group law is then the morphism of k-schemes

$$m \colon \mathbf{G}_{\mathrm{m}}{}_k^n \times_k \mathbf{G}_{\mathrm{m}}{}_k^n \to \mathbf{G}_{\mathrm{m}}{}_k^n$$

that corresponds, at the level of k-algebras, to the morphism of k-algebras

$$m^* \colon k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}] \to k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}] \otimes_k k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$$

such that $m^*(\mathrm{T}_j) = \mathrm{T}_j \otimes \mathrm{T}_j$. The identity element is the morphism $e \colon \mathrm{Spec}(k) \to \mathbf{G}_{\mathrm{m}}{}_k^n$ that corresponds to the k-point $(1, \dots, 1)$. The inverse is the morphism $i \colon \mathbf{G}_{\mathrm{m}}{}_k^n \to \mathbf{G}_{\mathrm{m}}{}_k^n$ induced by the morphism

$$i^* \colon k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}] \to k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$$

of k-algebras such that $i^*(\mathrm{T}_j) = \mathrm{T}_j^{-1}$ for every $j \in \{1, \dots, n\}$.

In the sequel, it shall be useful to have a coordinate-free version of tori.

Definition **(4.1.2)**. — *Let k be a field. A k-*torus *is an algebraic group over k which is isomorphic (as an algebraic group) to the n-dimensional torus* $\mathrm{Spec}(k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$*, for some integer $n \geqslant 0$. A morphism of tori is a morphism of algebraic groups.*

Note that the definition could be enlarged so as to allow any ring k. On the other hand, it only corresponds to what is called a *split torus* in the classical litterature, but the difference is not relevant for tropical geometry (at least, not yet).

Definition **(4.1.3)**. — *Let* T *be a k-torus. A* character *of* T *is a morphism from* T *to* $\mathbf{G}_{\mathrm{m}k}$*; a* cocharacter *of* T *is a morphism from* $\mathbf{G}_{\mathrm{m}k}$ *to* T*.*

Let T be a k-torus. Any morphism from T to $\mathbf{G}_{\mathrm{m}}{}_k^n$ is of the form $f = (f_1, \dots, f_n)$, where $f_1, \dots, f_n$ are characters of T.

Let $f, g \colon \mathrm{T} \to \mathbf{G}_{\mathrm{m}k}$ be characters of a torus T. Let $m(f, g)$ be the morphism given by composing $(f, g) \colon \mathrm{T} \to \mathbf{G}_{\mathrm{m}}{}_k^2$ with the group law $m \colon \mathbf{G}_{\mathrm{m}}{}_k^2 \to \mathbf{G}_{\mathrm{m}}$ of $\mathbf{G}_{\mathrm{m}k}$. Since $\mathbf{G}_{\mathrm{m}k}$ is commutative, this is a morphism of algebraic groups, hence a character of T, that we simply denote by fg. Similarly, the morphism $i \circ f$ obtained by composing f with the inverse morphism of $\mathbf{G}_{\mathrm{m}}$ is a character of T, that we denote by f^{-1}. The constant map with image e, composition of the projection morphism

$\mathrm{T} \to \mathrm{Spec}(k)$ and the unit element $\mathrm{Spec}(k) \to \mathbf{G}_{\mathrm{m}k}$ is also a character of T, called the trivial character. The set of characters of T is an abelian group, we denote it by $\mathscr{X}^*(\mathrm{T})$.

Similarly, the set of cocharacters of T is an abelian group, and we denote it by $\mathscr{X}_*(\mathrm{T})$.

Note that $\mathscr{X}^*(\mathrm{T})$ and $\mathscr{X}_*(\mathrm{T})$ are functorial in T. In particular, given a morphism of tori $\varphi : \mathrm{T} \to \mathrm{T}'$, composition with φ furnishes morphisms of abelian groups $\varphi^* : \mathscr{X}^*(\mathrm{T}') \to \mathscr{X}^*(\mathrm{T})$ and $\varphi_* : \mathscr{X}_*(\mathrm{T}) \to \mathscr{X}_*(\mathrm{T}')$. If $\psi : \mathrm{T}' \to \mathrm{T}''$ is a second morphism of tori, then one has $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ and $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. Moreover, the identity morphism id_{T} induces the identity morphism on $\mathscr{X}^*(\mathrm{T})$ and $\mathscr{X}_*(\mathrm{T})$.

Proposition **(4.1.4)**. — a) *There is a unique morphism of abelian groups from* $\mathbf{Z}$ *to* $\mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, \mathbf{G}_{\mathrm{m}})$ *that maps* 1 *to the identity map; this morphism is an isomorphism. The character* f_m *associated with an integer* $m \in \mathbf{Z}$ *is induced by the morphism* $f_m^* : k[\mathrm{T}, \mathrm{T}^{-1}] \to k[\mathrm{T}, \mathrm{T}^{-1}]$ *of k-algebras such that* $f_m^*(\mathrm{T}) = \mathrm{T}^m$.

b) *Let* T, T' *be tori. Let* $\varphi \in \mathscr{X}_*(\mathrm{T})$ *and* $\psi \in \mathscr{X}_*(\mathrm{T}')$*; the map* $(\varphi, \psi) : \mathbf{G}_{\mathrm{m}} \to \mathrm{T} \times \mathrm{T}'$ *is a cocharacter of* $\mathrm{T} \times \mathrm{T}'$. *The corresponding mapping* $\mathscr{X}_*(\mathrm{T}) \times \mathscr{X}_*(\mathrm{T}') \to \mathscr{X}_*(\mathrm{T}, \mathrm{T}')$ *is an isomorphism.*

c) *Let* T, T' *be tori. Let* $f \in \mathscr{X}^*(\mathrm{T})$ *and* $g \in \mathscr{X}^*(\mathrm{T}')$*; the composition* $(f, g) : \mathrm{T} \times \mathrm{T}' \to \mathbf{G}_{\mathrm{m}k} \times \mathbf{G}_{\mathrm{m}k} \xrightarrow{m} \mathbf{G}_{\mathrm{m}k}$ *is a character of* $\mathrm{T} \times \mathrm{T}'$. *The corresponding mapping* $\mathscr{X}^*(\mathrm{T}) \times \mathscr{X}^*(\mathrm{T}') \to \mathscr{X}^*(\mathrm{T}, \mathrm{T}')$ *is an isomorphism.*

d) *Let* T *be an n-dimensional torus. Then the abelian groups* $\mathscr{X}_*(\mathrm{T})$ *and* $\mathscr{X}^*(\mathrm{T})$ *are isomorphic to* $\mathbf{Z}^n$. *Moreover, the map* $\mathscr{X}^*(\mathrm{T}) \times \mathscr{X}_*(\mathrm{T}) \to \mathscr{X}_*(\mathbf{G}_{\mathrm{m}}) \simeq \mathbf{Z}$ *given by* $(f, \varphi) \mapsto f \circ \varphi$ *is a perfect duality abelian groups.*

Proof. — *a*) The identity morphism $\mathrm{id}_{\mathbf{G}_{\mathrm{m}k}}$ is a morphism of algebraic groups, hence the existence and uniqueness of a morphism $\mathbf{Z} \to \mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, \mathbf{G}_{\mathrm{m}})$ that maps 1 to $\mathrm{id}_{\mathbf{G}_{\mathrm{m}k}}$. Let f_m be the image of $m \in \mathbf{Z}$. The constant morphism with image e is the unit element of $\mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, \mathbf{G}_{\mathrm{m}})$, hence is equal to f_0. Since $\mathbf{G}_{\mathrm{m}k}$ is commutative, the inverse i is a morphism of algebraic groups; one checks that $m(i, \mathrm{id}_{\mathbf{G}_{\mathrm{m}k}}) = f_0$. By induction on $|m|$, the morphism of k-algebras f_m^* is given by $f_m^*(\mathrm{T}) = \mathrm{T}^m$. This proves in particular that the morphism $m \mapsto f_m$ is injective.

Conversely, let $f\colon \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ be a morphism and let $\varphi = f^*(\mathrm{T}) \in k[\mathrm{T},\mathrm{T}^{-1}]$. Since it is invertible, with inverse $f^*(\mathrm{T}^{-1})$, there exist $c \in k^\times$ and $m \in \mathbf{Z}$ such that $\varphi = c\mathrm{T}^m$. Since $f(e) = e$, one has $\varphi(1) = 1$, hence $\varphi = \mathrm{T}^m$ and $f = f_m$.

Assertions *b*) and *c*) follow from the definitions.

Let us finally prove *d*). We first start with the case $\mathrm{T} = \mathbf{G}_{\mathrm{m}}{}^n$. Let $m, \mu \in \mathbf{Z}^n$; let φ_μ be the corresponding cocharacter of T and f_m be the corresponding character of T. The morphism $\varphi_\mu\colon \mathbf{G}_{\mathrm{m}k} \to \mathbf{G}_{\mathrm{m}k}^n$ corresponds to the morphism of k-algebras $\varphi_\mu^*\colon k[\mathrm{T}_1^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}] \to k[\mathrm{T},\mathrm{T}^{-1}]$ such that $\varphi_\mu^*(\mathrm{T}_j) = \mathrm{T}_j^{\mu_j}$, for every $j \in \{1,\dots,n\}$; the morphism $f_m\colon \mathbf{G}_{\mathrm{m}k}^n \to \mathbf{G}_{\mathrm{m}k}$ corresponds to the morphism of k-algebras $f_m^*\colon k[\mathrm{T},\mathrm{T}^{-1}] \to k[\mathrm{T}_1^{\pm 1},\dots,\mathrm{T}_n^{\pm 1}]$ such that $f_m^*(\mathrm{T}) = \mathrm{T}_1^{m_1}\dots\mathrm{T}_n^{m_n} = \mathrm{T}^m$. Their composition $f_m \circ \varphi_\mu\colon \mathbf{G}_{\mathrm{m}k} \to \mathbf{G}_{\mathrm{m}k}$ thus corresponds to the morphism of k-algebras from $k[\mathrm{T},\mathrm{T}^{-1}]$ to itself that maps T to

$$\begin{aligned}\varphi_\mu^*(f_m^*(\mathrm{T})) = \varphi_\mu^*(\mathrm{T}_1^{m_1}\dots\mathrm{T}_n^{m_n}) &= \varphi_\mu^*(\mathrm{T}_1)^{m_1}\dots\varphi_\mu^*(\mathrm{T}_n)^{m_n}\\ &= \mathrm{T}^{\mu_1 m_1}\dots\mathrm{T}^{\mu_n m_n} = \mathrm{T}^{\langle m,\mu\rangle}.\end{aligned}$$

This proves the claim in this case since the bilinear map $\mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}$ given by $(m,\mu) \mapsto \langle m,\mu\rangle$ is a perfect duality.

To establish the general case, we choose an isomorphism $j\colon \mathrm{T} \simeq \mathbf{G}_{\mathrm{m}k}^n$. Then j induces isomorphisms $j^*\colon \mathbf{Z}^n = \mathscr{X}^*(\mathbf{G}_{\mathrm{m}k}^n) \to \mathscr{X}^*(\mathrm{T})$ and $j_*\colon \mathscr{X}_*(\mathrm{T}) \to \mathscr{X}_*(\mathbf{G}_{\mathrm{m}k}^n) = \mathbf{Z}^n$. Let $f \in \mathscr{X}^*(\mathrm{T})$ and $\varphi \in \mathscr{X}_*(\mathrm{T})$; Let $m, \mu \in \mathbf{Z}^n$ be such that $f = j^*(f_m)$ and $\varphi_\mu = j_*(\varphi)$; then $f \circ \varphi = f_m \circ j^{-1} \circ j \circ \varphi_\mu = f_m \circ \varphi_\mu$ is the character of $\mathbf{G}_{\mathrm{m}}$ associated with $\langle m,\mu\rangle$. This concludes the proof. □

Corollary **(4.1.5)**. — *The functors of cocharacters and characters respectively induce a covariant and a contravariant equivalence of categories from the category of free finitely generated abelian groups to the category of k-tori. A quasi-inverse of the functor of characters is given by* $\mathrm{M} \mapsto \operatorname{Spec}(k^{(\mathrm{M})})$.

4.1.6. — Let k be a field. Actions of an algebraic k-group G on a k-variety X can either be defined by translating the classical definition into the language of schemes, or by referring to the functors of points. In the latter way, this means that one is given, in a functorial way, an action of the group G(R) on the set X(R), for every k-algebra R. In the language of schemes, this corresponds to a morphism $\mu : G \times_k X \to X$ that satisfies the following two properties:

a) The two morphisms $\mu \circ (m \times \mathrm{id}_X)$ and $\mu \circ (\mathrm{id}_G \times \mu)$ from $G \times_k G \times_k X$ to X coincide;

b) The morphism $\mu \circ (e, \mathrm{id}_X)$ from X to X is the identity.

4.1.7. — Let T be a k-torus and let M be its group of characters. Let A be a k-algebra and let $X = \mathrm{Spec}(A)$ be the corresponding affine scheme, let μ be an action of T on X. It corresponds to μ a morphism $\mu^* : A \to k^{(M)} \otimes_k A$ of k-algebras. For every $a \in A$, let us write $\mu^*(a) = \sum_{m \in M} T^m \otimes a_m$. The maps $\mu^*_m : A \to A$ are k-linear. That μ is an action corresponds to the following two relations:

a) For every $a \in A$, the equality $\sum_{n \in M} \mu^*_n(\mu^*_m(a)) T^m S^n = \sum_n \mu^*_n(a)(ST)^n$ in $k^{(M)} \otimes k^{(M)} \otimes A$. In other words,

$$\mu^*_n(\mu^*_m(a)) = \begin{cases} \mu^*_m(a) & \text{if } n = m; \\ 0 & \text{otherwise;} \end{cases}$$

b) For every $a \in A$, the equality

$$a = \sum_{m \in M} \mu^*_m(a).$$

This implies that $A = \bigoplus_{m \in M} A_m$. Indeed, the second relation shows that $A = \sum_{m \in M} A_m$. On the other hand, let us consider a family (a_m) with finite support, with $a_m \in A_m$ for every $m \in M$, and $\sum_m a_m = 0$. Then $\mu_n * (a_n) = a_n$, and $\mu^*_n(a_m) = 0$ if $n \neq m$, so that $0 = a_n$. This proves that the family (A_m) is in direct sum. Finally, the fact that μ^* is a morphism of k-algebras translates as

$$A_m \cdot A_n \subset A_{m+n}$$

for $m, n \in M$.

In other words, the family $(A_m)_{m \in M}$ is an M-*graduation* of the k-algebra A. Moreover, the map μ^*_m is the projector from A to A_m wich vanishes on the other factors.

Conversely, the given formulas show that every such graduation of A gives rise to an action of T on Spec(A).

4.2. Toric varieties

Definition **(4.2.1)**. — *Let k be a field. A toric variety is a k-variety* X *(separated, integral scheme of finite type) endowed with the action of a torus* T *and with a point* $x \in \mathrm{X}(k)$ *whose* T*-orbit is dense.*

The point x is called the base-point of the toric variety X.

By Chevalley's theorem (Mumford (1994), corollary 2, p. 51), the orbit of x is a constructible subset of X; since it is dense, it contains an open subscheme of X, which implies that this orbit is itself open in X. Let T_x be the stabilizer of x in T; this is a closed subgroup scheme of T. Since X is separated and T_x acts trivially on a dense open subscheme of X, it acts trivially on X. On the other hand, there is a quotient algebraic group T/T_x, and it is a torus; its character group is the subgroup of $\mathscr{X}^*(\mathrm{T})$ consisting of characters f such that $f|_{\mathrm{T}_x} = 1$. Replacing T by T/T_x allows to assume that the stabilizer of x is trivial.

In this case, the morphism $t \mapsto t \cdot x$ from T to X is an open immersion. It is then inoccuous to identify T with its image in X. The variety T then appears as an equivariant (partial) compactification of T, "equivariant" meaning that the action of T on itself given by the group law extends to an action of T on X.

Example **(4.2.2)**. — Let $\mathrm{T} = \mathbf{G}_{\mathrm{m}}{}^n = \mathrm{Spec}(k[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ and $\mathrm{X} = \mathbf{P}_n = \mathrm{Proj}(k[\mathrm{X}_0, \dots, \mathrm{X}_n])$ be the n-dimensional projective space. There is an action of T on X given by

$$(t_1, \dots, t_n) \cdot [x_0 : x_1 : \dots : x_n] = [x_0 : t_1 x_1 : \dots : t_n x_n].$$

The orbit of the point $x = [1 : \dots : 1]$ is the principal open subset $\mathrm{D}(\mathrm{T}_0 \dots \mathrm{T}_n)$ of $\mathbf{P}_n$.

The open subset $\mathrm{D}(\mathrm{T}_0)$ of $\mathbf{P}_n$ is the affine space $\mathbf{A}^n$. Since it contains x and is invariant under T, it is a toric variety as well.

We can also consider $\mathbf{P}_{n-1}$ as a toric variety with underlying torus $\mathbf{G}_{\mathrm{m}}{}^n$, with action given by

$$(t_1, \dots, t_n) \cdot [x_1 : \dots : x_n] = [t_1 x_1 : \dots : t_n x_n],$$

and base-point $[1 : \dots : 1]$. The stabilizer of the base-point is the diagonal subgroup of $\mathbf{G}_{\mathrm{m}}{}^n$ and the quotient is isomorphic to $\mathbf{G}_{\mathrm{m}}{}^{n-1}$.

4.2.3. — Let X be a toric variety with torus T and base-point x. For every point $y \in X(k)$, the closure $X_y = \overline{T \cdot y}$ of its orbit is a toric variety, with base-point and underlying torus the quotient T/T_y.

More generally, let S be a torus and $f : S \to T$ be a morphism of tori. Then the torus S acts on X, by the formula $s \cdot x = f(s) \cdot x$, and the closure of the S-orbit of a point y is a toric variety with underlying torus S/S_y and base-point y.

Example **(4.2.4)**. — a) Let us consider the n-dimensional affine space $\mathbf{A}^n$ as a toric variety with underlying torus ${\mathbf{G}_m}^n$, the action of ${\mathbf{G}_m}^n$ on $\mathbf{A}^n$ being given by

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1 x_1, \dots, t_n x_n),$$

the orbit of the base-point $(1, \dots, 1)$ being dense in $\mathbf{A}^n$.

Let T be a torus, let $M = \mathscr{X}^*(T)$ be its character group and let $\mathscr{A} = (m_1, \dots, m_n)$ be a finite family in M. It induces a morphism $f_{\mathscr{A}} : T \to {\mathbf{G}_m}^n$. The Zariski closure $Y_{\mathscr{A}}$ of the orbit of the base-point $(1, \dots, 1)$ is a toric variety in $\mathbf{A}^n$. Its underlying torus is the quotient of T by the kernel of $f_{\mathscr{A}}$, its character group is identified with the abelian group $\langle m_1, \dots, m_n \rangle$ generated by $\mathscr{A}$. In particular, the dimension of $Y_{\mathscr{A}}$ is that of the linear span of $\mathscr{A}$ in $M_{\mathbf{R}}$.

b) Similarly, starting from $\mathbf{P}_{n-1}$, viewed as a toric variety with torus ${\mathbf{G}_m}^n$ and base-point $[1 : \dots : 1]$, we can restrict the action to T, and the Zariski closure of the base-point is a toric variety $X_{\mathscr{A}}$. The diagonal torus $\mathbf{G}_m$ in ${\mathbf{G}_m}^n$ acts trivially on $\mathbf{P}_{n-1}$, and the underlying torus of $X_{\mathscr{A}}$ is its inverse by $f_{\mathscr{A}}$; its character group identifies with the subgroup of M generated by the elements $m_i - m_j$. Consequently, the dimension of $X_{\mathscr{A}}$ is that of the affine span of $\mathscr{A}$ in $M_{\mathbf{R}}$.

Proposition **(4.2.5)**. — *Let* T *be a torus, let* $\mathscr{A} = (m_1, \dots, m_n)$ *be a finite family of characters of* T *and let* $X_{\mathscr{A}} \subset \mathbf{P}_{n-1}$ *and* $Y_{\mathscr{A}} \subset \mathbf{A}^n$ *be the associated toric varieties.*

Let $X_1, \dots, X_n$ *be the homogeneous (resp. affine) coordinates of* $\mathbf{P}_{n-1}$ *(resp.* $\mathbf{A}^n$*).*

a) *The ideal of* $Y_{\mathscr{A}}$ *in* $k[X_1, \dots, X_n]$ *is generated by the binomials* $X^p - X^q$*, for all* $p, q \in \mathbf{N}^n$ *such that* $\sum p_j m_j = \sum q_j m_j$ *in* $\mathscr{X}^*(T)$.

b) *The homogeneous ideal of* $X_{\mathscr{A}}$ *is the ideal generated by those binomials which are homogeneous. In particular, if the ideal of* $Y_{\mathscr{A}}$ *is homogeneous, then it is the ideal of* $X_{\mathscr{A}}$*, and the variety* $Y_{\mathscr{A}}$ *is the affine cone over* $X_{\mathscr{A}}$*.*

Proof. — *a*) Let M be the character group of T and $\varphi \colon T \to \mathbf{A}^n$ be the morphism given by $\varphi(t) = t \cdot (1, \dots, 1)$. It corresponds to the morphism of k-algebras $\varphi^* \colon k[X_1, \dots, X_n] \to k^{(M)}$ such that $\varphi^*(X_j) = T^{m_j}$ for every $j \in \{1, \dots, n\}$. By definition of the Zariski closure of the image of φ, the ideal I of $Y_{\mathscr{A}}$ is the kernel of φ^*.

If $p, q \in \mathbf{N}^n$ satisfy $\sum p_j m_j = \sum q_j m_j$, one has

$$\varphi^*(X^p - X^q) = \prod T^{m_j p_j} - \prod T^{m_j q_j} = T^{\sum p_j m_j} - T^{\sum q_j m_j} = 0,$$

which shows that the indicated binomials belong to the ideal I. We need to prove the opposite inclusion. On the other hand, let $f = \sum c_p X^p \in I$ and let S be the support of f; we argue by induction on S that f belongs to the ideal J generated by these binomials. Let $q \in S$ and let $\mu = \sum q_j m_j$. One has

$$0 = \varphi^*(f) = \sum c_p T^{\sum p_j m_j} = \sum_{m \in M} \left(\sum_{\sum p_j m_j = m} c_p \right) T^m.$$

The sum of all c_p, for $p \in S$ such that $\sum p_j m_j = \mu$, is the coefficient of T^μ in $\varphi^*(f)$, hence vanishes. Since this sum contains c_q, there exists $p \in S - \{q\}$ such that $\sum p_j m_j = m$. The polynomial $g = f - c_q X^q + c_q X^p$ satisfies $\varphi^*(g) = \varphi^*(f) - c_q \varphi^*(X^q - X^p) = 0$; its support is contained in $S - \{q\}$. By induction, g belongs to J, hence so does f.

b) Let $f = X^p - X^q$ be an homogeneous binomial in I, where $p, q \in \mathbf{N}^n$ satisfy $\sum p_j m_j = \sum q_j m_j$ and $\sum p_j = \sum q_j$. It vanishes on $X_{\mathscr{A}}$, so that the homogenous ideal of $X_{\mathscr{A}}$ contains the ideal J generated by those homogeneous binomials. Conversely, the same argument as in *a*) proves that an homogeneous polynomial that vanishes on $X_{\mathscr{A}}$ belongs to the ideal generated by these elements.

The ideal of $X_{\mathscr{A}}$ is contained in the ideal of $Y_{\mathscr{A}}$, and they coincide if the latter is already homogeneous. □

Remark **(4.2.6)**. — The map $\varphi^* \colon \mathbf{N}^n \to M$ given by $\varphi^*(p_1, \dots, p_n) = \sum p_i m_i$ is a morphism of monoids. Let K be its kernel, namely, the set of pairs $(p,q) \in \mathbf{N}^n$ such that $\varphi^*(p) = \varphi^*(q)$; it is a submonoid of $\mathbf{N}^n \times \mathbf{N}^n$ that contains the diagonal Δ (corresponding to pairs with $p = q$). Let $S \subset K$ be a subset such that the smallest submonoid of $\mathbf{N}^n \times \mathbf{N}^n$ that contains $S \cup S' \cup \Delta$ is equal to K, where S' is the symmetric of S, that is, the set of pairs (q,p) for $(p,q) \in S$. (It will follow from proposition 4.3.1 below that there exists a *finite* such set S.) Let $J'_{\mathscr{A}}$ be the ideal of $k[X_1,\dots,X_n]$ generated by the polynomials $X^p - X^q$, for $(p,q) \in S$. For $s \in S$, write (p_s,q_s) for the pair corresponding to s.

Let $p,q \in \mathbf{N}^n$ be such that $\varphi^*(p) = \varphi^*(q)$, that is, $(p,q) \in K$. By the assumption on S, there are elements $m_s, m'_s \in \mathbf{N}$ (for $s \in S$) and an element $r \in \mathbf{N}^n$ such that

$$(p,q) = \sum m_s(p_s,q_s) + \sum m'_s(q_s,p_s) + (r,r).$$

One has $X^{p_s} \equiv X^{q_s} \pmod{J'_{\mathscr{A}}}$, for every $s \in S$, so that $\prod X^{m_s p_s} \equiv \prod X^{m_s q_s}$, and $\prod X^{m'_s q_s} \equiv \prod X^{m'_s p_s}$, as well as $X^r \equiv X^r$. Consequently, $X^p \equiv X^q \pmod{J'_{\mathscr{A}}}$, so that $X^p - X^q \in J'_{\mathscr{A}}$.

This proves that the ideal $J'_{\mathscr{A}}$ is the ideal of $Y_{\mathscr{A}}$ in $k[X_1,\dots,X_n]$.

A similar description holds for the ideal of $X_{\mathscr{A}}$. Let $K_h \subset K$ be the set of pairs $(p,q) \in K$ such that $|p| = |q|$ and let S_h be a subset of K_h such that K_h is the smallest submonoid of $\mathbf{N}^n \times \mathbf{N}^n$ containing S_h, S'_h and the diagonal. Then the polynomials $X^p - X^q$, for $(p,q) \in S_h$, generate the homogeneous ideal of $X_{\mathscr{A}}$.

Example **(4.2.7)**. — Let us consider $T = \mathbf{G}_m$ with character group identified with $\mathbf{Z}$, and the family $\mathscr{A} = (2,3)$. Let I be the ideal of $Y_{\mathscr{A}}$ in $k[X,Y]$; let us prove that $I = (X^3 - Y^2)$.

Let us compute the kernel K of the morphism of monoids $\mathbf{N}^2 \to \mathbf{Z}$ given by $(p,p') \mapsto 2p+3p'$. Let $(p,p'),(q,q') \in \mathbf{N}^2$ be such that $2p+3p' = 2q+3q'$; then there exists $a \in \mathbf{Z}$ such that $p - q = 3a$ and $p' - q' = -2a$, so that $(p,p') = (q,q') + a(3,2)$. By the preceding remark, one may take $S = \{(3,2)\}$. Consequently, the ideal I of $Y_{\mathscr{A}}$ is generated by the polynomial $X^3 - Y^2$.

Since this polynomial $X^3 - Y^2$ is not homogeneous, the toric variety $Y_{\mathscr{A}} \subset \mathbf{A}^2$ is not the cone over $X_{\mathscr{A}}$. In fact, one has $X_{\mathscr{A}} = \mathbf{P}_1$ and the homogeneous ideal of $X_{\mathscr{A}}$ is zero, because the homogeneous part K_h of K is reduced to 0.

Remark **(4.2.8)**. — Replacing the torus T by the torus $T' = T \times \mathbf{G}_m$ changes M into $M' = M \oplus \mathbf{Z}$. Let us set $m'_j = (m_j, 1)$ for every $j \in \{1, \dots, n\}$ and let $\mathscr{A}' = (m'_1, \dots, m'_n)$. This gives rise to toric varieties $Y_{\mathscr{A}'} \subset \mathbf{A}^n$ and $X_{\mathscr{A}'} \subset \mathbf{P}_{n-1}$. Now, $Y_{\mathscr{A}'}$ is the cone over $X_{\mathscr{A}'}$ which itself coincides with $X_{\mathscr{A}}$.

Remark **(4.2.9)** (To be done). — Let T, T' be tori and let $f \colon T' \to T$ be a morphism of tori; let $f^* \colon M \to M'$ be the morphism of abelian groups that it induces on character groups. Let $\mathscr{A}$ and $\mathscr{A}'$ be finite families in M and M′ respectively.

a) Assume that $\mathscr{A} = (m_1, \dots, m_n)$ and $\mathscr{A} = (f^*(m_1), \dots, f^*(m_n))$. Explain that $X_{\mathscr{A}}$ and $X_{\mathscr{A}'}$ coincide, as well as $Y_{\mathscr{A}}$ and $Y_{\mathscr{A}'}$.

b) Find more general conditions on $\mathscr{A}$ and $\mathscr{A}'$ that allow to compare these varieties.

4.2.10. — Let S be a commutative monoid and let $M = S^{gp}$ be the abelian group associated with S; we assume that S is finitely generated, that the canonical map from S to M is injective and that $M \simeq \mathbf{Z}^n$. In the language used by logarithmic geometry, see for example Ogus (2018), such monoids are called *fine*; we will also mainly consider the case where S is *saturated*, in the sense that for any $s \in M$ such that there exists $n \geqslant 1$ with $ns \in S$, then s belongs to S (rather, to the image of S in M).

Then the monoid algebra $k^{(S)}$ is a finitely generated k-algebra, and is a subalgebra of $k^{(M)}$. Let $X_S = \operatorname{Spec}(k^{(S)})$ and let T be the torus with character group M. Since $k^{(S)}$ is a finitely generated integral k-algebra, the scheme X_S is an integral k-variety. The obvious M-graduation of $k^{(S)}$ endowes X_S with a T-action. Let $\mathscr{A} = (m_1, \dots, m_n)$ be a finite generating family of the monoid S. Then the k-algebra corresponding to the principal open subset $D(T^{m_1+\dots+m_n})$ of X_S is $k^{(M)}$, so that this open subset identifies with T. We thus have constructed an affine toric variety with torus T.

The elements $T^{m_1}, \dots, T^{m_n}$ of $k^{(S)} = \Gamma(X_S, \mathscr{O}_{X_S})$ define a morphism $\varphi \colon X_S \to \mathbf{A}^n$. One has $\varphi(e) = (1, \dots, 1)$. By construction, φ factors through $Y_{\mathscr{A}}$. In fact, φ induces an isomorphism of toric varieties $X_S \simeq Y_{\mathscr{A}}$. Indeed, it corresponds to φ the morphism of monoid algebras

$\varphi^* : k[T_1, \dots, T_n] \to k^{(S)}$ such that $\varphi^*(T_j) = T^{m_j}$ for every $j \in \{1, \dots, n\}$. This morphism is surjective, because $\mathscr{A}$ generates S. Composing with the embedding of $k^{(S)}$ into $k^{(M)}$, we see that the kernel of φ^* is the ideal I of $Y_{\mathscr{A}}$. This proves that φ induces an isomorphism from X_S into $Y_{\mathscr{A}}$, as claimed.

4.3. Affine toric varieties and cones

Proposition **(4.3.1)** (Gordan's lemma). — *Let* C *be a rational polyhedral convex cone in* $\mathbf{R}^n$. *Then* $C \cap \mathbf{Z}^n$ *is a finitely generated monoid.*

Proof. — It is clear that $C \cap \mathbf{Z}^n$ is a monoid; what needs to be proved is that it is finitely generated.

By the general theory, there exists a finite family $(v_1, \dots, v_m)$ of vectors in $\mathbf{Q}^n$ such that $C = \operatorname{cone}(v_1, \dots, v_m)$. We can then assume that these vectors v_j belong to $\mathbf{Z}^n$. Let $K = [0;1]v_1 + \dots + [0;1]v_m$; this is a compact subset of $\mathbf{R}^n$, hence $S = K \cap \mathbf{Z}^n$ is finite. Let us prove that $C \cap \mathbf{Z}^n$ is generated by S.

Let then $v \in C \cap \mathbf{Z}^n$; we can write $v = \sum a_j v_j$, with $(a_1, \dots, a_m) \in \mathbf{R}_+^m$. Let $v' = \sum \lfloor a_j \rfloor v_j$; since $v_j \in S$ for all j, one has $v' \in \langle S \rangle$. Moreover $v - v' = \sum (a_j - \lfloor a_j \rfloor) v_j \in K$, by definition of K, and $v - v' \in \mathbf{Z}^n$, because both v and v' belong to $\mathbf{Z}^n$; consequently, $v - v' \in S \subset \langle S \rangle$. Finitely, $v = v' + (v - v') \in \langle S \rangle$, as was to be shown. □

4.3.2. — Let T be a torus and n its dimension. let M be its group of characters and N be its group of cocharacters; they are isomorphic to $\mathbf{Z}^n$. Then $M_{\mathbf{R}}$ are $N_{\mathbf{R}}$ are $\mathbf{R}$-vector spaces isomorphic to $\mathbf{R}^n$ with $\mathbf{Q}$-structures respectively given by $M_{\mathbf{Q}}$ and $N_{\mathbf{Q}}$.

Let σ be a rational polyhedral convex cone in $N_{\mathbf{R}}$ and let σ° be its polar cone in $M_{\mathbf{R}}$, namely the set of all $f \in M_{\mathbf{R}} = (N_{\mathbf{R}})^*$ such that that $f(x) \leqslant 0$ for all $x \in \sigma$; it is a rational polyhedral convex cone. According to proposition 4.3.1, $\sigma^\circ \cap M$ is a finitely generated monoid and one defines the toric variety X_σ by the formula

$$X_\sigma = \operatorname{Spec}(k^{(\sigma^\circ \cap M)}).$$

Its associated torus is the quotient of T with character group $(\sigma^\circ - \sigma^\circ) \cap M$.

The cone σ° generates $M_{\mathbf{R}}$ if and only if if $\sigma \cap (-\sigma) = 0$; in this case, the associated torus of X_σ is T.

In the sequel, we will always assume that this property is satisfied and mention it by saying that σ is *strongly* convex.

4.3.3. — The functor of points of the affine toric variety X_σ has an elementary description.

Let R be a k-algebra. Then k-morphisms of Spec(R) to X_σ correspond to morphisms of k-algebras from $k^{(\sigma^\circ \cap M)}$ to R, hence, by the universal property of the monoid algebras, to morphisms of monoids from $\sigma^\circ \cap M$ to $(R, \cdot)$. This description is functorial: if $f \colon R \to S$ is a morphism of k-algebras, the natural map $X_\sigma(R) \to X_\sigma(S)$ induced by composition with the morphism ${}^a f \colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ corresponds to composition with f.

Analogously, k-morphisms from Spec(R) to T correspond functorially to morphisms of k-algebras from $k^{(M)}$ to R; by definition of the group algebra, they correspond to morphisms of groups from M to $R^\times$. (Note that since M is a group, the image of a morphism of monoids from M to $(R, \cdot)$ is contained in $R^\times$.)

In these descriptions, the action of T(R) on $X_\sigma(R)$ is simply given as follows: if $t \in T(R)$ and $x \in X_\sigma(R)$ respectively correspond to morphisms $\tau \colon M \to R^\times$ and $\xi \colon \sigma^\circ \cap M \to R$, then the point $t \cdot x \in X_\sigma(R)$ corresponds to the morphism of monoids $m \mapsto \tau(m)\xi(m)$ from $\sigma^\circ \cap M \to R$.

Example **(4.3.4)**. — Here are some examples, borrowed from §1.2 of Cox et al (2011). Let $T = \mathbf{G}_{m_k}^n$, so that M and N are both identified with $\mathbf{Z}^n$. Let $(e_1, \dots, e_n)$ be the canonical basis of N.

a) Let $r \in \{0, \dots, n\}$ and let $\sigma = \operatorname{cone}(-e_1, \dots, -e_r)$. Then σ° is the set of $m \in \mathbf{R}^n$ such that $m_1 \geqslant 0, \dots, m_r \geqslant 0$. Consequently, $\sigma^\circ \cap M = \mathbf{N}^r \times \mathbf{Z}^{n-r}$ and $X_\sigma = \mathbf{A}_k^r \times \mathbf{G}_{m_k}^{n-r}$.

b) Assume that $n = 2$. Let d be a strongly positive integer and let $\sigma = \operatorname{cone}(-de_1 + e_2, -e_2)$. Then σ° is the set of $m \in \mathbf{R}^2$ such that $dm_1 - m_2 \geqslant 0$ and $m_2 \geqslant 0$, that is, $0 \leqslant m_2 \leqslant dm_1$.

To compute a generating set of $\sigma^\circ \cap \mathbf{Z}^2$, we can use the argument of the proof of proposition 4.3.1. We can also argue directly: the points $(1, m)$, for $0 \leqslant e \leqslant d$ belong to $\sigma^\circ \cap \mathbf{Z}^2$; moreover, if $(m_1, m_2) \in \mathbf{N}^2$

satisfies $0 \leqslant m_2 \leqslant dm_1$ then, choosing $e \in \{0, \dots, d-1\}$ such that $em_1 \leqslant m_2 \leqslant (e+1)m_1$, one has

$$(m_1, m_2) = ((e+1)m_1 - m_2)(1, e) + (m_2 - em_1)(1, e+1),$$

so that (m_1, m_2) belongs to the monoid generated by the $d+1$ points $(1, 0), \dots, (1, d)$.

c) Assume that $n = 3$ and let $\sigma = \operatorname{cone}(-e_1, -e_2, -e_1 - e_3, -e_2 - e_3)$. Its polar cone σ° is the set of $m \in \mathbf{R}^3$ such that $m_1 \geqslant 0$, $m_2 \geqslant 0$, $m_1 + m_3 \geqslant 0$ and $m_2 + m_3 \geqslant 0$. This cone has 4 extremal rays, generated by the integral vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, -1)$. Moreover, for $(m_1, m_2, m_3) \in \sigma^\circ$, the expressions

$$(m_1, m_2, m_3) = m_1(1, 0, 0) + m_2(0, 1, 0) + m_3(0, 0, 1)$$

if $m_3 \geqslant 0$, and

$$(m_1, m_2, m_3) = (m_1 + m_3)(1, 0, 0) + (m_2 + m_3)(0, 1, 0) - m_3(1, 1, -1)$$

if $m_3 \leqslant 0$, show that $\sigma^\circ \cap \mathbf{Z}^3$ is generated by these 4 vectors. Consequently, $k^{(\sigma^\circ \cap \mathbf{Z}^3)}$ is the subalgebra $k[\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_1\mathrm{T}_2\mathrm{T}_3^{-1}]$ of $k[\mathrm{T}_1^{\pm 1}, \mathrm{T}_2^{\pm 1}, \mathrm{T}_3^{\pm 1}]$. The morphism $\varphi \colon k[\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{T}] \to k[\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_1\mathrm{T}_2\mathrm{T}_3^{-1}]$ mapping the indeterminates X, Y, Z, T to the monomials $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_1\mathrm{T}_3\mathrm{T}_3^{-1}$ is surjective. The morphism φ identifies X_σ with the toric variety $\mathrm{Y}_{\mathscr{A}}$ in $\mathbf{A}^4$ defined by $\mathscr{A} = ((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, -1))$. The corresponding morphism of monoids from $\mathbf{N}^4$ to $\mathbf{Z}^3$ is generated by the pair $((1, 1, 0, 0), (0, 0, 1, 1))$, its symmetric, and the diagonal. Consequently, the ideal of $\mathrm{Y}_{\mathscr{A}}$ is generated by the polynomial $\mathrm{XY} - \mathrm{ZT}$.

Proposition **(4.3.5)**. — *Let* T, T' *be two tori, with cocharacter groups respectively* N *and* N′ *and let* $v \colon \mathrm{N} \to \mathrm{N}'$ *be a morphism of abelian groups, with corresponding morphism of tori* $\varphi \colon \mathrm{T} \to \mathrm{T}'$. *Let* σ *and* σ' *be rational polyhedral strongly convex cones in* $\mathrm{N}_{\mathbf{R}}$ *and* $\mathrm{N}'_{\mathbf{R}}$ *respectively. Then* φ *extends to a morphism of toric varieties* $\mathrm{X}_\sigma \to \mathrm{X}_{\sigma'}$ *if and only if* $v(\sigma) \subset \sigma'$; *such an extension is then unique.*

Proof. — Let M, M′ be the character groups of T, T′ respectively, and let $\mathrm{S} = \sigma^\circ \cap \mathrm{M}$ and $\mathrm{S}' = (\sigma')^\circ \cap \mathrm{M}'$ be the corresponding monoids, so that $\mathrm{X}_\sigma = \operatorname{Spec}(k^{(\mathrm{S})})$ and $\mathrm{T} = \operatorname{Spec}(k^{(\mathrm{M})})$ on the one side, and $\mathrm{X}_{\sigma'} = \operatorname{Spec}(k^{(\mathrm{S}')}$ and $\mathrm{T}' = \operatorname{Spec}(k^{(\mathrm{M}')})$ on the other side. By duality, the morphism of abelian groups $v\colon \mathrm{N}\to \mathrm{N}'$ induces a morphism $u\colon \mathrm{M}'\to \mathrm{M}$ which, in turn, gives rise to the morphism of k-algebras $k^{(\mathrm{M}')}\to k^{(\mathrm{M})}$ and to a morphism of tori $\varphi\colon \mathrm{T}\to \mathrm{T}'$ such that $\varphi^*(\mathrm{T}^m)=\mathrm{T}^{u(m)}$ for every $m\in \mathrm{M}'$.

If $v(\sigma)\subset\sigma'$, then $u((\sigma')^\circ)\subset\sigma^\circ$: indeed, for every $m\in(\sigma')^\circ$ and every $\mu\in\sigma$, one has $\langle u(m),\mu\rangle=\langle m,v(\mu)\rangle\geqslant 0$ since $v(\mu)\in\sigma'$ and $m\in(\sigma')^\circ$. Consequently, $u(\mathrm{S}')\subset \mathrm{S}$, so that u induces a morphism of monoids $\mathrm{S}'\to \mathrm{S}$, hence a morphism of k-algebras $k^{(\mathrm{S}')}\to k^{(\mathrm{S})}$ and then a morphism of toric varieties $\varphi\colon \mathrm{X}_\sigma\to \mathrm{X}_{\sigma'}$ that extends φ.

The necessity of the condition is proved similarly, as well as the uniqueness of an extension. Indeed, if $\varphi\colon \mathrm{X}_\sigma\to \mathrm{X}_{\sigma'}$ extends the morphism of tori $\varphi\colon \mathrm{T}\to \mathrm{T}'$, one has $\mathrm{T}^m\in\Gamma(\mathrm{X}_{\sigma'},\mathscr{O}_{\mathrm{X}_{\sigma'}})$ for every $m\in \mathrm{S}'$, hence $\varphi^*(\mathrm{T}^m)\in\Gamma(\mathrm{X}_\sigma,\mathscr{O}_{\mathrm{X}_\sigma})$. On the other hand, the restriction of $\varphi^*(\mathrm{T}^m)$ to the torus T is the element $\mathrm{T}^{u(m)}$ of $k^{(\mathrm{M})}$, so that $u(m)\in \mathrm{S}$. We thus have $u(\mathrm{S}')\subset \mathrm{S}$. Since the polyhedral convex cone $(\sigma')^\circ$ is rational, it has a generating subset in M′, hence in S′. By what precedes, $u((\sigma')^\circ\subset\sigma^\circ$. □

Example **(4.3.6)**. — In $\mathbf{R}^2$, let us take $\sigma=-\operatorname{cone}((1,0),(1,1))$ and $\sigma'=-\operatorname{cone}((1,0),(0,1))$, so that $\sigma\subset\sigma'$. One has $\sigma^\circ=\{(m_1,m_2)\,;\,m_1\geqslant 0,\ m_1+m_2\geqslant 0\}=\operatorname{cone}((0,1),(1,-1))$, and $\mathrm{X}_\sigma=\operatorname{Spec}(k[\mathrm{T}_2,\mathrm{T}_1\mathrm{T}_2^{-1})\simeq\mathbf{A}^2$. Similarly, $(\sigma')^\circ=\mathbf{R}_+^2$ and $\mathrm{X}_{\sigma'}=\operatorname{Spec}(k[\mathrm{T}_1,\mathrm{T}_2])=\mathbf{A}^2$. The identity morphism from ${\mathbf{G}_\mathrm{m}}^2$ to itself extends a morphism $f\colon \mathrm{X}_\sigma\to \mathrm{X}_{\sigma'}$. At the level of rings, it corresponds to the inclusion $f^*\colon k[\mathrm{T}_1,\mathrm{T}_2]\to k[\mathrm{T}_1\mathrm{T}_2^{-1},\mathrm{T}_2]$. Composing with the isomorphism $k[\mathrm{T}_1\mathrm{T}_2^{-1},\mathrm{T}_2]\simeq k[\mathrm{X},\mathrm{Y}]$ that maps $\mathrm{T}_1\mathrm{T}_2^{-1}$ to X and T_2 to Y, we get the morphism $j\circ f^*\colon k[\mathrm{T}_1,\mathrm{T}_2]\to k[\mathrm{X},\mathrm{Y}]$ such that $\mathrm{T}_1\mapsto \mathrm{XY}$ and $\mathrm{T}_2\mapsto \mathrm{Y}$. Identifying X_σ and $\mathrm{X}_{\sigma'}$ with $\mathbf{A}^2$, the morphism $f\colon \mathrm{X}_\sigma\to \mathrm{X}_{\sigma'}$ which is induced from the identity map of $\mathbf{Z}^2$ is thus given by $f(x,y)=(xy,y)$.

Proposition **(4.3.7)**. — *Let σ be a rational polyhedral strongly convex cone in $\mathrm{N}_\mathbf{R}$. For every face τ of σ, the morphism of toric varieties $\mathrm{X}_\tau\to \mathrm{X}_\sigma$ which corresponds to the identity map of* M *is an open immersion.*

Proof. — Since σ is a rational polyhedral convex cone in $\mathrm{N}_\mathbf{R}$ and τ is a face of σ, there exists a primitive element $m\in \mathrm{M}$ satisfying the following two properties, for $x\in\sigma$:

(i) One has $\langle x,m\rangle\leqslant 0$;
(ii) One has $x\in\tau$ if and only if $\langle x,m\rangle=0$.

Then $m \in \sigma^\circ \subset \tau^\circ$, and $-m \in \tau^\circ$. Let us prove that $\tau^\circ = \sigma^\circ + \mathbf{R} + (-m)$ and $(\tau^\circ \cap \mathrm{M}) = (\sigma^\circ \cap \mathrm{M}) + \mathbf{N}(-m)$. The inclusions $\sigma^\circ + \mathbf{R} + (-m) \subset \tau^\circ$ and $(\sigma^\circ \cap \mathrm{M}) + \mathbf{N}(-m) \subset (\tau^\circ \cap \mathrm{M})$ are clear. Conversely, let $w \in \tau^\circ$. Let $(x_1, \dots, x_s)$ be a finite family in N such that $\sigma = \operatorname{cone}(x_1, \dots, x_s)$ and let $c = \sup_j(|\langle x_j, w\rangle|)$. Let $j \in \{1, \dots, s\}$. If $\langle x_j, m\rangle = 0$, then $x_j \in \tau$ hence $\langle x_j, w\rangle \geqslant 0$ and

$$\langle x_j, w + cm\rangle = \langle x_j, w\rangle + c\langle x_j, m\rangle \geqslant 0.$$

Otherwise, since $\langle x_j, m\rangle \in \mathbf{N}$, one has $\langle x_j, m\rangle \geqslant 1$ and

$$\langle x_j, w + cm\rangle = \langle x_j, w\rangle + c\langle x_j, m\rangle \geqslant -c + c = 0.$$

This proves that $w + cm \in \sigma^\circ$, hence $w \in \sigma^\circ + \mathbf{R}_+(-m)$. In the case where $w \in \tau^\circ \cap \mathrm{M}$, we have $c \in \mathbf{N}$ because $\langle x_j, w\rangle \in \mathbf{Z}$ for all j, hence $w \in \sigma^\circ + \mathbf{N}(-m)$.

Then $\mathrm{T}^m \in k^{(\sigma^\circ \cap \mathrm{M})}$ and $k^{(\tau^\circ \cap \mathrm{M})} = k^{(\sigma^\circ \cap \mathrm{M})}[\mathrm{T}^{-m}]$ is the localization of $k^{(\tau^\circ \cap \mathrm{M})}$ by the multiplicative subset generated by T^m. This proves that X_τ is the principal open subscheme $\mathrm{D}(\mathrm{T}^m)$ of X_σ defined by T^m. ☐

Proposition **(4.3.8)**. — *Let* T *be a torus with group of characters* M *and group of cocharacters* N *and let* X *be an affine integral toric variety with torus* T*; assume that the canonical morphism* $\mathrm{T} \to \mathrm{X}$ *is an open immersion.*

a) *There exists a unique submonoid* $\mathrm{S} \subset \mathrm{M}$ *such that* $\mathrm{X} = \operatorname{Spec}(k^{(\mathrm{S})})$*, the immersion of* T *into* X *being given by the inclusion of* $k^{(\mathrm{S})}$ *in* $k^{(\mathrm{M})}$.

b) *The monoid* S *is finitely generated and generates* M *as an abelian group.*

c) *The variety* X *is normal if and only if the monoid* S *is saturated; in this case, there exists a rational polyhedral convex cone* σ *in* $\mathrm{N}_{\mathbf{R}}$ *such that* $\mathrm{S} = \sigma^\circ \cap \mathrm{M}$.

Proof. — a) Let $\mathrm{A} = \mathscr{O}(\mathrm{X})$ be the algebra of X; since X is integral, A is a subalgebra of $\mathscr{O}(\mathrm{T}) = k^{(\mathrm{M})}$. Let S be the set of all $m \in \mathrm{M}$ such that $\mathrm{T}^m \in \mathrm{A}$; it is a submonoid of M and $\mathrm{A} = k^{(\mathrm{S})}$.

b) Let M′ be the subgroup of M generated by S; there exists a basis $(e_1, \dots, e_n)$ of M and positive integers $d_1, \dots, d_n$ such that $(d_1 e_1, \dots, d_n e_n)$ generates M′. Then the fraction field of A coincides with $k(\mathrm{T}_1^{d_1}, \dots, \mathrm{T}_n^{d_n})$, but is also equal to $k(\mathrm{T}_1, \dots, \mathrm{T}_n)$ since T is dense in X. Necessarily, $d_1 = \dots = d_n = 1$ and $\mathrm{M}' = \mathrm{M}$.

Because we assume that X is an affine *variety*, the k-algebra A is finitely generated. Let $f_1, \dots, f_r \in A$ be such that $A = k[f_1, \dots, f_r]$. Necessarily, the supports of the f_j are contained in the monoid S, and conversely, S is contained in the submonoid generated by the union of these supports; in particular, S is a finitely generated monoid.

c) Let us assume that X is normal and let us show that S is a saturated submonoid of M. Let $m \in M$ and let $d \geqslant 1$ bu such $dm \in S$. Then $f = T^m$ is an element of the fraction field of A such that $f^d \in A$; in particular, f is integral over A. Since X is normal, A is integrally closed in its field of fractions, so that $f \in A$ and $m \in S$.

Conversely, let us assume that S is a saturated submonoid of M. Let τ be the rational polyhedral convex cone generated by S; one has $S \subset \tau \cap M$ by construction. On the other hand, any extremal ray of τ is generated by a vector $s \in S$; conversely, if $v \in M$ belongs to that ray, one has $v \in \mathbf{Q}_+ s$, hence there exists an integer $d \geqslant 1$ such that $dv \in S$; since S is saturated, this implies $v \in S$. As a consequence, $S = \tau \cap M$ and we see that $X = X_\sigma$, with $\sigma = \tau^\circ$.

Finally, let us assume that $S = \sigma^\circ \cap M$ and let us prove that $X = X_\sigma$ is normal. Let f be an element of the field of fractions of A which is integral over A. In particular, f is integral over $\mathscr{O}(T) \simeq k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. Since this ring is a unique factorization domain, f belongs to $\mathscr{O}(T)$. It remains to prove that A is integrally closed in $\mathscr{O}(T)$. Let $f \in \mathscr{O}(T)$ which is integral over A. Let $x \in \sigma$; let us show that for every element $m \in S(f)$, one has $\langle x, m\rangle \leqslant 0$. Otherwise, let us consider a vertex $m \in S(f)$ of the Newton polytope of f such that $c = \langle x, m\rangle > 0$ is maximal. Then dm is a vertex of the Newton polytope of f^d, so that T^{dm} appears in f^d. However, every element $p \in M$ of the support of f^{d-j} satisfies $\langle x, p\rangle \leqslant (d-j)c$, every element p of the support of u_j satisfies $\langle x, p\rangle \leqslant 0$, so that every element p of the support of $-\sum_{j=0}^{d-1} u_j f^{d-j}$ satisfies $\langle x, p\rangle \leqslant (d-1)c$. Since dm belongs to the support of f^d and $\langle x, dm\rangle = dc > (d-1)c$, this contradicts the relation $f^d + \sum_{j=0}^{d-1} u_j f^{d-j} = 0$. ☐

4.4. Normal toric varieties and fans

4.4.1. — Let T be a torus, let M be its group of characters and N be its group of cocharaters. Recall that a (rational) fan in $N_{\mathbf{R}}$ is a nonempty finite set Σ of rational polyhedral convex cones in $N_{\mathbf{R}}$ such that:

(i) Every face of a cone in Σ belongs to Σ;
(ii) The intersection of two cones of Σ is a face of both of them.

In the sequel, we will also assume that the cones of a fan are strongly convex, that is, do not contain any line; equivalently, we assume that the punctual cone $\{0\}$ belongs to Σ.

We recall that a fan is determined by its set of maximal cones.

4.4.2. — Given a fan Σ, one can define a toric variety X_Σ with underlying torus T by glueing the affine toric varieties X_σ, for $\sigma \in \Sigma$.

Precisely, for $\sigma, \tau \in \Sigma$, $\sigma \cap \tau$ is a face of both σ and τ, and the identity map of M furnishes an open immersion $j_{\sigma\tau} \colon X_{\sigma\cap\tau} \to X_\sigma$; let $V_{\sigma\tau}$ be its image. Exchanging the roles of σ and τ, we see that the open subscheme $V_{\tau\sigma}$ of X_τ is the image of the open immersion $j_{\tau\sigma}$. Let $\varphi_{\sigma\tau} = j_{\sigma\tau} \circ j_{\tau\sigma}^{-1}$ be the isomorphism of schemes from $V_{\tau\sigma}$ to $V_{\sigma\tau}$.

The *glueing condition* takes three cones $\sigma, \tau, \psi \in \Sigma$ and asks that $\varphi_{\tau\sigma}$ defines by restriction an isomorphism $\varphi'_{\tau\sigma}$ from $V_{\sigma\tau} \cap V_{\sigma\psi}$ to $V_{\tau\psi} \cap V_{\tau\sigma}$, and that $\varphi'_{\sigma\psi} = \varphi'_{\sigma\tau} \circ \varphi'_{\tau\psi}$ (*cocycle relation*). The first property holds, in fact, $V_{\sigma\tau} \cap V_{\sigma\psi}$ is the toric variety associated with the cone $\sigma \cap \tau \cap \psi$ which is a face of $\sigma \cap \tau$, $\sigma \cap \psi$ and $\tau \cap \psi$. The cocycle relation follows from the fact that these isomorphisms $\varphi'_{\sigma\psi}$ all correspond to the identity map on M.

Lemma **(4.4.3)**. — *For every point $x \in X_\Sigma$, there exists a smallest cone $\sigma \in \Sigma$ such that $x \in X_\sigma$.*

In other words, the cone σ is such that for every cone $\tau \in \Sigma$, the assertions $x \in X_\tau$ and $\sigma \subset \tau$ are equivalent.

Proof. — Let $x \in X_\Sigma$ and let $\sigma \in \Sigma$ be a minimal cone such that $x \in X_\sigma$; such a cone exists because Σ is finite. Let now $\tau \in \Sigma$. If $\sigma \subset \tau$, then σ is a face of τ, and X_σ is an open subscheme of X_τ, hence $x \in X_\tau$. In the other direction, assume that $x \in X_\tau$. By construction, the intersection

$X_\sigma \cap X_\tau$ is the open subscheme $X_{\sigma\cap\tau}$ of X_Σ. By minimality of σ, one has $\sigma \cap \tau = \sigma$, hence $\sigma \subset \tau$. □

Example **(4.4.4)**. — Let $T = \mathbf{G}_m$, so that $M = N = \mathbf{Z}$, and let Σ be the fan consisting of the three cones $\sigma = \mathbf{R}_-$, $\sigma' = \mathbf{R}_+$ and $\tau = \{0\}$ in $N_\mathbf{R} = \mathbf{R}$. Then $\sigma^\circ = \mathbf{R}_+$, $(\sigma')^\circ = \mathbf{R}_-$ and $\tau^\circ = \mathbf{R}$, so that the three corresponding affine toric varieties are $X_\sigma = \operatorname{Spec}(k[T])$, $X_{\sigma'} = \operatorname{Spec}(k[T^{-1}])$ and $X_\tau = \operatorname{Spec}(k[T, T^{-1}])$. Both X_σ and $X_{\sigma'}$ are isomorphic to $\mathbf{A}^1_k$, and X_τ is identified with the open set $\mathbf{G}_{mk}$, their glueing being done via the automorphism $t \mapsto t^{-1}$ of $\mathbf{G}_m$. One thus gets the projective line $\mathbf{P}^1_k$.

Example **(4.4.5)**. — Let T, T' be two tori, with character groups M, M' and cocharacter groups N, N', and let Σ, Σ' be two fans in $N_\mathbf{R}, N'_\mathbf{R}$ respectively. A cocharacter of the character group of the product torus $T \times T'$ is of the form (λ, λ'), where λ, λ' are cocharacters of T, T' respectively, so that $\mathscr{X}_*(T \times T')$ is identified with $N \times N'$. The set of cones $\sigma \times \sigma'$ in $N_\mathbf{R} \times N'_\mathbf{R}$, for $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$, is a fan of $N_\mathbf{R} \times N'_\mathbf{R}$ which we denote by $\Sigma \times \Sigma'$. One has $(\sigma \times \sigma')^\circ = \sigma^\circ \times (\sigma')^\circ$, so that

$$(\sigma \times \sigma')^\circ \cap (M \times M') = (\sigma^\circ \cap M) \times ((\sigma')^\circ \cap M'),$$

and the monoid algebra of $(\sigma \times \sigma')^\circ \cap (M \times M')$ identifies with the tensor product of the monoid algebras of $\sigma^\circ \cap M$ and of $(\sigma')^\circ \cap M'$. This identifies $X_{\sigma\times\sigma'}$ with the product $X_\sigma \times_k X_{\sigma'}$.

Glueing these affine toric varieties, we identify the toric variety $X_{\Sigma\times\Sigma'}$ with the product $X_\Sigma \times_k X_{\Sigma'}$.

Example **(4.4.6)**. — Let $T = {\mathbf{G}_m}^n$, with character and cocharacter groups identified with $\mathbf{Z}^n$. Let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbf{Z}^n$ and let $e_0 = e_1 + \dots + e_n$. For every $j \in \{0, \dots, n\}$, we let $\sigma_j = \operatorname{cone}(e_0, -e_1, \dots, \widehat{-e_j}, \dots, -e_n\}$, where $\widehat{-e_j}$ means that this vector is omitted from the list. The cones $\sigma_0, \dots, \sigma_n$ and their faces form a fan in $\mathbf{R}^n$. Let us check that this fan defines the projective space $\mathbf{P}_n$.

Let us set $U_j = X_{\sigma_j}$; it corresponds to an open subscheme of X_Σ. Since faces of σ_j give rise to open subsets of X_{σ_j}, the open sets $U_0, \dots, U_n$ cover X_Σ. One has $\sigma_0^\circ = \mathbf{R}^n_+$, so that $\sigma_0^\circ \cap \mathbf{Z}^n = \mathbf{N}^n$ and $X_0 = \operatorname{Spec}(k[T_1, \dots, T_n]) = \mathbf{A}^n_k$. Fix $j \in \{1, \dots, n\}$. An element $m \in \mathbf{R}^n$ belongs to σ_j° if and only if $m_i \geqslant 0$ for $i \neq j$ and $m_1 + \dots + m_n \leqslant 0$, that

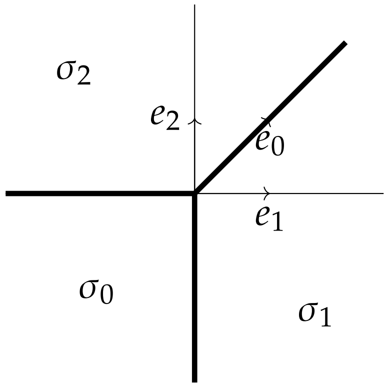


FIGURE 1. The fan of $\mathbf{P}_2$

is, $m_j \leqslant -\sum_{i\neq j} m_i$. Writing $m = \sum m_i e_i = \sum_{i\neq j} m_i(e_i - e_j) + (\sum_i m_i)e_j$, we see that the cone σ_j° is generated by the vectors $e_i - e_j$, for $i \neq j$, and by the vector $-e_j$. These vectors forming a basis of $\mathbf{Z}^n$, we have

$$\sigma_j^\circ \cap \mathbf{Z}^n = \mathbf{N}(e_1 - e_j) + \cdots + \mathbf{N}(e_n - e_j) + \mathbf{N}(-e_j),$$

which identifies U_j with $\mathrm{Spec}(k[\mathrm{T}_1\mathrm{T}_j^{-1}, \dots, \mathrm{T}_n\mathrm{T}_j^{-1}, \mathrm{T}_j^{-1}])$; in particular, it is isomorphic to $\mathbf{A}_k^n$.

Written in this way, the coordinates furnish an explicit isomorphism between the principal open subscheme $\mathrm{D}(\mathrm{T}_j)$ of U_0 and the open subscheme $\mathrm{D}(\mathrm{T}_j^{-1})$ of U_j, and an explicit isomorphism between the principal open subscheme $\mathrm{D}(\mathrm{T}_i\mathrm{T}_j^{-1})$ of U_j and the open subscheme $\mathrm{D}(\mathrm{T}_j\mathrm{T}_i^{-1})$ of U_i. This is the standard definition of the n-dimensional projective space by glueing $n+1$ affine spaces of dimension n.

Proposition **(4.4.7)**. — *Let* T *be a torus,* N *its group of cocharacters and let* Σ *be a fan in* $\mathrm{N}_{\mathbf{R}}$.

a) *The toric variety* X_Σ *is separated.*
b) *It is proper if and only if the support of* Σ *is equal to* $\mathrm{N}_{\mathbf{R}}$.

Proof. — The proof of the proposition relies on the valuative criteria for separation and properness (GROTHENDIECK, 1961, proposition 7.2.3 and théorème 7.3.8). Namely, a k-scheme X of finite type is separated (resp. proper) if for every k-algebra R which is a discrete valuation ring, with field of fractions K, every k-morphism $f : \mathrm{Spec}(\mathrm{K}) \to \mathrm{X}$ has at most one (resp. has exactly one) extension to a k-morphism $\varphi : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$. We will also need to make three observations:

– Affine schemes are separated, so that if U is an affine open subscheme such that f factors through U, then there exists at most one extension φ : Spec(R) $\to$ U;

– If a morphism φ : Spec(R) $\to$ X maps the closed point of Spec(R) to an open subscheme U of X, then it factors through U, so that φ is the unique extension of f that factors through U;

– It suffices to treat the case of a morphism f that factors through a given dense open subscheme of X (Grothendieck, 1961, corollaire 7.3.10).

We apply this with $X = X_\Sigma$ and its dense open subscheme T.

a) Since X_Σ is described by glueing open subschemes X_σ, to prove that it is separated, it will suffice to consider, for each cone $\sigma \in \Sigma$, the only possible extension φ_σ of f through X_σ, and to show that all of these extensions coincide as a morphism from Spec(R) to X_Σ. The morphism f is a K-point of T, hence corresponds to a morphism of abelian groups $u \colon M \to K^\times$. The morphism f extends to a morphism $\varphi_\sigma \colon \operatorname{Spec}(R) \to X_\sigma$ if and only if $u(\sigma^\circ \cap M) \subset R$.

Assume, thus, that f extends to morphisms $\varphi_\sigma \colon \operatorname{Spec}(R) \to X_\sigma$ and $\varphi_\tau \colon \operatorname{Spec}(R) \to X_\tau$. This means that $u(\sigma^\circ \cap M) \subset R$ and $u(\tau^\circ \cap M) \subset R$. Let then $\psi = \sigma \cap \tau$ is a cone of Σ with dual $\psi^\circ = \sigma^\circ + \tau^\circ$, and $u(\psi^\circ \cap M) \subset R$. Then f extends to a morphism $\varphi_\psi \colon \operatorname{Spec}(R) \to X_\psi$. However, ψ being a face of σ, X_ψ is an open subscheme of X_σ; since X_σ is affine, it is separated, hence $\varphi_\psi = \varphi_\sigma$. Similarly, one has $\varphi_\psi = \varphi_\sigma$.

This proves that X_Σ is separated.

b) First assume that $|\Sigma| \neq N_{\mathbf{R}}$. Since $|\Sigma|$ is closed, its complement is open and nonempty, and it contains a nonempty cube C. Since Σ is a cone, one has $tC \cap |\Sigma| = \varnothing$ for all $t > 0$. If t is large enough, then tC has size > 1, so that it contains a point $v \in N - \{0\}$. Let $\lambda \colon \mathbf{G}_{mk} \to T$ be the inverse of the corresponding cocharacter; the associated morphism of groups $u \colon M \to k(T)^\times$ is given by $m \mapsto T^{-\langle v,m\rangle}$. I claim that the morphism λ does not extend to a morphism $\overline{\lambda} \colon \mathbf{A}^1_k \to X_\Sigma$. Assume otherwise and let $\sigma \in \Sigma$ be the minimal cone such that $\overline{\lambda}(0) \in X_\sigma$. Since $\overline{\lambda}(\mathbf{G}_m) \subset X_\sigma$, the morphism $\overline{\lambda}$ factors through X_σ, so that the morphism of groups $M \to k(T)^\times$ corresponding to λ maps $\sigma^\circ \cap M$ to $k[T]$. In other words, $\langle v, m\rangle \leqslant 0$ for all $m \in \sigma^\circ \cap M$. Since σ° is a rational cone, it is generated by vectors in M, so that $\langle v, m\rangle \leqslant 0$ for all $m \in \sigma^\circ$. By duality, we have $v \in \sigma^{\circ\circ} = \sigma$, a contradiction. This proves that if $|\Sigma| \neq N_{\mathbf{R}}$, then X_Σ is not proper.

Let us now prove that if $|\Sigma| = N_{\mathbf{R}}$, then X_Σ is proper. Let R be a discrete valuation ring, let K be its field of fractions and let $f\colon \mathrm{Spec}(K) \to T$ be a morphism, corresponding to a morphism of abelian groups $u\colon M \to K^\times$. Composing with the valuation of K, we obtain a linear form on M, hence an element $x \in N$ such that $v(u(m)) = -\langle x, m\rangle$ for every $m \in M$. Let $\sigma \in \Sigma$ be such that $x \in \sigma$. For every $m \in \sigma^\circ \cap M$, one has $v(u(m)) = -\langle x, m\rangle \geqslant 0$, hence $u(m) \in R$. Consequently, f extends to a morphism $\varphi\colon \mathrm{Spec}(R) \to X_\sigma$, and this concludes the proof of the proposition. □

Proposition **(4.4.8)**. — *We consider two tori* T, T', *with character groups* M, M' *and cocharacter groups* N, N'. *Let* Σ, Σ' *be fans in* $N_{\mathbf{R}}, N'_{\mathbf{R}}$ *respectively. Let* $f\colon T \to T'$ *be a morphism of tori, let* $v\colon N \to N'$ *be the morphism that it induces on cocharacters.*

There is at most one morphism of toric varieties $\varphi\colon X_\Sigma \to X_{\Sigma'}$ *that extends* f. *For such a morphism to exist, it is necessary and sufficient that for every cone* $\sigma \in \Sigma$, *there exists a cone* $\sigma' \in \Sigma'$ *such that* $v(\sigma) \subset \sigma'$.

Proof. — That there is at most one morphism from X_Σ to $X_{\Sigma'}$ that extends f follows from the fact that the torus T is (schematically) dense in X_Σ, and that $X_{\Sigma'}$ is separated. Indeed, let φ, φ' be two such morphisms, and let us consider the morphism $(\varphi, \varphi')\colon X_\Sigma \to X_{\Sigma'} \times_k X_{\Sigma'}$. Since $X_{\Sigma'}$ is separated, its diagonal is a closed subscheme of $X_{\Sigma'} \times_k X_{\Sigma'}$, hence so is its inverse image in X_Σ. This inverse image contains T, hence is equal to X_Σ itself, because T is dense in X_Σ.

To prove the existence of such a morphism φ, we first observe that f extends to a morphism $\varphi_\sigma\colon X_\sigma \to X_{\Sigma}$, for every $\sigma \in \Sigma$. Indeed, we choose a cone $\sigma' \in \Sigma'$ such that $v(\sigma) \subset \sigma'$ and define φ_σ as the morphism from X_σ to $X_{\sigma'}$ that extends f which is given by proposition 4.3.5 composed with the open immersion of $X_{\sigma'}$ into $X_{\Sigma'}$. If τ is a face of σ, then X_τ is an open subscheme of X_σ, and $\varphi_\sigma|_{X_\tau}$ extends f, hence is equal to φ_τ, by the uniqueness property applied to the fan with τ as a unique maximal cone, □

Remark **(4.4.9)**. — Let T be a torus, let M be its group of characters and let N be its group of cocharacters.

Let Σ be a fan in $N_{\mathbf{R}}$. Since the monoids $\sigma^\circ \cap M$ are saturated, for $\sigma \in \Sigma$, the varieties X_σ are normal, hence X_Σ is a normal toric variety.

Conversely, if X is a normal toric variety with underlying torus T, there exists a fan Σ in $N_{\mathbf{R}}$ such that X is isomorphic to Σ. Indeed, by a theorem of Sumihiro (1974), the variety X has an affine cover (U_j) by T-invariant open subschemes. For each j, there exists a cone σ_j in $N_{\mathbf{R}}$ such that $U_j \simeq X_{\sigma_j}$. There exists a fan Σ in $N_{\mathbf{R}}$ containing the cones σ_j and such that $|\Sigma|$ is the union of the ones σ_j. Then X is isomorphic to X_Σ; in fact, there exists a unique isomorphism $X \xrightarrow{\sim} X_\Sigma$ which extends a given identification of the underlying tori.

Example **(4.4.10)**. — Let T be a torus, let M be its group of characters and let N be its group of cocharacters. Let $P \subset M_{\mathbf{R}}$ be a rational polyhedron of dimension rank(M) and let Σ_P be its normal fan (definition 1.10.7): this is the set of all cones $N_F(P) = (P - F)^\circ = P^\circ \cap F^\perp$, where F ranges over all faces of P. These cones are rational and, since P has dimension rank(M), their lineality spaces are reduced to 0. Let us denote by X_P the toric variety associated with this fan Σ_P.

Every face F of P gives rise to a torus-invariant open subset X_F in X_P, given by $X_P = \operatorname{Spec}(k^{((P-F)\cap M)})$. The ordering relation between faces gives rise to the opposite ordering on normal cones; in particular, the maximal cones of Σ_P correspond to the vertices of P.

The support $|\Sigma_P| \subset N_{\mathbf{R}}$ of Σ_P is the set of all linear forms on $M_{\mathbf{R}}$ which are bounded from above on P. In particular, if P is a polytope, then $|\Sigma_P| = N_{\mathbf{R}}$. Consequently, proposition 4.4.7 asserts that X_P is a proper variety if and only if P is a polytope.

Let us assume that P is a polytope with vertices in M and let $S = P \cap M$. Every element $s \in M$ corresponds to a character of T which we denote by $\chi_s \colon T \to \mathbf{G}_m$. Let $\mathbf{P}_S$ be the projective space of dimension $\operatorname{Card}(S) - 1$ with homogeneous coordinates indexed by S. For Let $f \colon T \to \mathbf{P}_S$ be the morphism $t \mapsto [\chi_s(t)]_{s\in S}$; by definition, the Zariski closure of its image is the toric variety X_S. Let us show that f extends uniquely to a morphism from X_P to X_S.

Every face F of P gives rise to a torus-invariant open subset X_F in X_P, given by $X_P = \mathrm{Spec}(k^{((P-F)\cap M)})$. It suffices to prove that f extends uniquely on each of them. It suffices to treat the case of a face reduced to a vertex v of P. For every $s \in S$, one has $s - v \in P - F$, so that the character $\chi_s\chi_v^{-1}$ on T extends to a regular function on X_v. By definition of homogeneous coordinates, one has $[(\chi_s(t)]_{s\in S} = [(\chi_s(t)\chi_v(t)^{-1}]_{s\in S}$, and in the latter expression, all homogeneous coordinates extend to regular functions on X_v, and one of them is identically equal to 1. Consequently, f extends uniquely to a from X_v to $\mathbf{P}_S$, and these extensions glue to furnish the desired morphism from X_P to X_S.

4.5. Toric orbits and cones

When a k-torus T (more generally, a k-group scheme of finite type) acts on a k-scheme of finite type X, the scheme X admits a finite partition in *orbits*, minimal locally closed subsets which are invariant under T. In the case of a toric variety with underlying torus T, one of those orbits is the torus itself. We will show that the other orbits are, in a natural way, quotients of the torus T, and that their closures in X are themselves toric varieties.

4.5.1. — Let T be a torus, let M be its group of characters and let N be its group of cocharacters; let Σ be a fan in $N_{\mathbf{R}}$.

Let $\sigma \in \Sigma$.

Let $\sigma^\perp = \sigma^\circ \cap (-\sigma^\circ)$ be the set of elements $m \in M_{\mathbf{R}}$ such that $\langle x, m\rangle = 0$ for all $x \in \sigma$. This is a rational vector subspace in $M_{\mathbf{R}}$ of codimension $\dim(\sigma)$, it is the lineality space of the cone σ°. Observe that $\sigma^\perp \cap M$ is a submonoid of $\sigma^\circ \cap M$. Let then $\varphi_\sigma \colon \sigma^\circ \cap M \to k$ be the map such that $\varphi_\sigma(m) = 1$ if $m \in \sigma^\perp$ and $\varphi_\sigma(m) = 0$ otherwise.

One has $\varphi_\sigma(0) = 1$.

Let $m, m' \in \sigma^\circ \cap M$. If $m + m' \in \sigma^\perp$, then $\langle x, m\rangle + \langle x, m'\rangle = 0$ for all $x \in \sigma^\circ$, hence $\langle x, m\rangle = \langle x, m'\rangle = 0$ since both of them are positive real numbers, such that $m, m' \in \sigma^\perp$ and $\varphi_\sigma(m + m') = 1 = \varphi_\sigma(m)\varphi_\sigma(m')$. On the other hand, if $m + m' \notin \sigma^\perp$, then either m, or m' does not belong to $\sigma^\perp$ and $\varphi_\sigma(m + m') = 0 = \varphi_\sigma(m)\varphi_\sigma(m')$.

This proves that φ_σ is a morphism of monoids. It corresponds to φ_σ a morphism of k-algebras $k^{(\sigma^\circ \cap \mathrm{M})} \to k$, hence a point $x_\sigma \in \mathrm{X}_\sigma(k)$.

Let us compute the stabilizer of x_σ. Let R be a k-algebra and let $t \in \mathrm{T}(\mathrm{R})$; let $\varphi \colon \mathrm{M} \to \mathrm{R}^\times$ be the corresponding morphism of groups. Since X_σ is a T-stable open subscheme of X_Σ and since $x_\sigma \in \mathrm{X}_\sigma(k)$, the point $t \cdot x_\sigma$ belongs to X_σ and corresponds to the morphism of monoids $\varphi \cdot \varphi_\sigma \colon \sigma^\circ \cap \mathrm{M} \to (\mathrm{R}, \cdot)$ that maps $m \in \sigma^\perp$ to $\varphi(m)$, and the rest to $0 = \varphi_\sigma(m)$. In other words, $t \cdot x_\sigma = x_\sigma$ if and only if $\varphi(m) = 1$ for every $m \in \sigma^\perp \cap \mathrm{M}$. This proves that the stabilizer of x_σ is the group subscheme of T defined by the equations $t^m = 1$ for all $m \in \sigma^\perp \cap \mathrm{M}$.

The orbit O_σ of x_σ is then identified to the quotient torus T_σ, with character group $\sigma^\perp \cap \mathrm{M}$. One has $\dim(\mathrm{T}_\sigma) = \dim(\sigma^\perp \cap \mathrm{M}) = \dim(\sigma^\perp) = \dim(\mathrm{T}) - \dim(\sigma)$.

Lemma **(4.5.2)**. — *Let $n \in \mathrm{N}$ and let $\lambda \colon \mathbf{G}_\mathrm{m} \to \mathrm{T}$ be the corresponding morphism of group schemes. The morphism λ extends to a morphism λ' from $\mathbf{A}^1_k$ to X_Σ if and only if $-n \in |\Sigma|$; one then has $\lambda'(0) = x_\sigma$, where σ is the smallest cone of Σ containing $-n$.*

Proof. — The morphism λ corresponds to the morphisms of groups $\varphi \colon \mathrm{M} \to k[\mathrm{T}, \mathrm{T}^{-1}]^\times$ given by $m \mapsto \mathrm{T}^{\langle m,n\rangle}$.

Assume that $-n \in |\Sigma|$ and let σ be the smallest cone of Σ containing $-n$; in other words, $-n$ belongs to the relative interior of σ. For every $m \in \sigma^\circ \cap \mathrm{M}$, one has $\langle m, n\rangle \leqslant 0$, hence $\mathrm{T}^{-\langle m,n\rangle} \in k[\mathrm{T}]$, so that the morphism λ induces a morphism of monoids $\varphi' \colon \sigma^\circ \cap \mathrm{M} \to (k[\mathrm{T}], \cdot)$. Therefore, λ extends to a morphism of schemes $\lambda' \colon \mathbf{A}^1_k \to \mathrm{X}_\sigma$, hence to a morphism from $\mathbf{A}^1_k$ to X_Σ.

The point $\lambda'(0) \in \mathrm{X}_\sigma(k)$ corresponds to the morphism of monoids $\varphi'_0 \colon \sigma^\circ \cap \mathrm{M} \to (k, \cdot)$ such that $\varphi'_0(m) = 1$ for $m \in \sigma^\circ \cap \mathrm{M}$ such that $\langle m, n\rangle = 0$, and $\varphi'_0(m) = 0$ otherwise. Since n belongs to the relative interior of σ, the condition $\langle m, n\rangle = 0$ for $m \in \sigma^\circ \cap \mathrm{M}$ implies that $m \in \sigma^\perp$. We thus see that $\lambda'(0) = x_\sigma$.

Conversely, let us assume that λ extends to a morphism of schemes $\lambda' \colon \mathbf{A}^1_k \to \mathrm{X}_\Sigma$. Let $\sigma \in \Sigma$ be a cone such that $\lambda'(0) \in \mathrm{X}_\sigma$. Since a morphism of schemes is continuous, λ' maps a neighborhood of 0 into X_σ;

it maps the complement of 0 into T, which is contained in X_σ, hence λ' factors through a morphism, still denoted by λ', from $\mathbf{A}^1_k$ to X_σ.

Let $\varphi' \colon \sigma^\circ \cap M \to (k[T], \cdot)$ be the corresponding morphism of monoids. Since $\varphi'(m) = T^{-\langle m,n\rangle}$, one has $\langle m, n\rangle \leqslant 0$ for every $m \in \sigma^\circ \cap M$; by duality, this proves that $n \in \sigma$, hence $n \in |\Sigma|$. ☐

Theorem **(4.5.3)**. — a) *A point $x \in X_\Sigma$ belongs to the torus orbit O_σ if and only if σ is the minimal cone of Σ such that $x \in X_\sigma$. In particular, the torus orbits O_σ, for $\sigma \in \Sigma$, form a partition of X_Σ in locally closed subsets;*

b) *For $\sigma, \tau \in \Sigma$, one has $\overline{O_\sigma} \subset \overline{O_\tau}$ if and only if $\tau \subset \sigma$.*

c) *For every cone $\sigma \in \Sigma$, one has*

$$X_\sigma = \bigcup_{\substack{\tau\in\Sigma \\ \tau\subset\sigma}} O_\tau \quad \textit{and} \quad \overline{O_\sigma} = \bigcup_{\substack{\tau\in\Sigma \\ \tau\supset\sigma}} O_\tau.$$

Proof. — a) Let $x \in X_\Sigma$ and let $\sigma \in \Sigma$ be the smallest cone such that $x \in X_\sigma$ (lemma 4.4.3). Let us prove that $x \in O_\sigma$. Let $K = \kappa(x)$ be the residue field of x and let $\varphi \colon \sigma^\circ \cap M \to (K, \cdot)$ be the morphism of monoids corresponding to x.

Let $m \in \sigma^\circ \cap M$ be such that $\varphi(m) \neq 0$. Then $D(T^m)$ is a principal open subscheme of X_σ, stable under the action of T, with associated monoid $\mathbf{N}(-m) + \sigma^\circ \cap M$; it corresponds to a face τ of σ and one has $x \in X_\tau$. By minimality, one has $\tau = \sigma$, hence $-m \in \sigma^\circ \cap M$ and $m \in \sigma^\perp$. Conversely, if $m \in \sigma^\perp \cap M$, then $-m \in \sigma^\circ \cap M$, hence $\varphi(m)\varphi(-m) = \varphi(0) = 1$, so that $\varphi(m) \in K^\times$. Then the morphism of groups $\varphi|_{\sigma^\perp\cap M} \colon \sigma^\perp \cap M \to K^\times$ corresponds to a point $t \in T_\sigma(K)$ and $x = t \cdot x_\sigma$. In particular, $x \in O_\sigma$.

Conversely, let τ be the smallest cone of Σ such that $x_\sigma \in X_\tau$; one has $\tau \subset \sigma$, since $x_\sigma \in X_\sigma$ by construction. The points x_τ and x_σ both belong to X_σ and correspond to morphisms of monoids $\varphi_\tau, \varphi_\sigma \colon \sigma^\circ \cap M \to (k, \cdot)$. By construction, $\varphi_\sigma(m) = 1$ if $m \in \sigma^\perp \cap M$, and $\varphi_\sigma(m) = 0$ otherwise; on the other hand, using that X_τ is an open subscheme of X_σ and the definition of x_τ, we see that $\varphi_\tau(m) = 1$ if $m \in \sigma^\circ \cap \tau^\perp \cap M$, and $\varphi_\tau(m) = 0$ otherwise. By what precedes, there exists $t \in T(k)$ such that $x_\sigma = t \cdot x_\tau$; the point t corresponds to a morphism of groups $\varphi \colon M \to k^\times$, and one has $\varphi_\sigma(m) = \varphi(m)\varphi_\tau(m)$ for all $m \in \sigma^\circ \cap M$. Consequently, $\sigma^\circ \cap \tau^\perp \cap M = \sigma^\perp \cap M$; since $\sigma^\circ \cap \tau^\perp$ and σ° are rational polyedral cones, this implies $\sigma^\circ \cap \tau^\perp = \sigma^\perp$; by duality, we then have $\sigma + (\tau - \tau) = \sigma - \sigma$.

Let $x \in \sigma$; then there exists $y \in \sigma$ and $z \in \tau$ such that $-x = y - z$, hence $x + y = z$; since τ is a face of σ, one has $x \in \tau$. This proves that $\sigma = \tau$.

b) Applied to the fan generated by a cone σ, part *a*) proves that X_σ is the union of the orbits O_τ, for all faces τ of σ.

As a consequence, if τ is not a face of σ, then O_τ does not meet X_σ, hence is contained in the closed subscheme $X_\Sigma - X_\sigma$. Taking the closure, we have $\overline{O_\tau} \cap X_\sigma = \varnothing$; in particular, $O_\sigma \cap \overline{O_\tau} = \varnothing$. This proves that if O_σ meets $\overline{O_\tau}$, then τ is a face of σ.

Conversely, let τ be a face of σ and let us prove that $O_\sigma \subset \overline{O_\tau}$. This is obvious if $\sigma = \tau$, hence assume that $\tau \subsetneq \sigma$. Let then $n \in N$ such that $n \in \sigma - \tau$; view n as a cocharacter $\nu : \mathbf{G}_m \to T$ and let us consider the morphism $\lambda : t \mapsto \nu(t)^{-1} \cdot x_\tau$, from $\mathbf{G}_m$ to X_τ. It corresponds to the morphism of monoids $\varphi : \tau^\circ \cap M \to (k[T, T^{-1}], \cdot)$ given by

$$m \mapsto T^{-\langle m,n\rangle}\varphi_\tau(m) = \begin{cases} T^{-\langle m,n\rangle} & \text{if } m \in \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Since $n \in \sigma$, one has $\langle m, n\rangle \leqslant 0$ for all $m \in \sigma^\circ \cap N$ and this morphism of monoids extends to a morphism of monoids from $\tau^\circ \cap M$ to $(k[T], \cdot)$, hence λ extends to a morphism $\lambda' : \mathbf{A}^1_k \to X_\tau$. The point $\lambda(0)$ corresponds to the morphism of monoids from $\tau^\circ \cap M$ to $(k, \cdot)$ which maps $m \in \tau^\circ \cap M$ to 1 if $m \in \tau^\perp \cap \sigma^\perp = \sigma^\perp$ and to 0 otherwise, that is, $\lambda(0) = x_\sigma$. One thus has $x_\sigma \in \overline{O_\tau}$, hence the inclusion $O_\sigma \subset \overline{O_\tau}$, and then $\overline{O_\sigma} \subset \overline{O_\tau}$.

c) Let $\sigma \in \Sigma$. Since X_σ is stable under T, it is the union of the torus orbits O_τ, for those $\tau \in \Sigma$ such that $x_\tau \in X_\sigma$; by *a*), this is equivalent to the relation $\tau \subset \sigma$.

Similarly, the closure of a torus orbit is stable under T, so that $\overline{O_\sigma}$ is the union of the torus orbits O_τ, for those τ such that $O_\tau \subset \overline{O_\tau}$ or, equivalently, such that $\overline{O_\tau} \subset \overline{O_\tau}$. By *b*), this is equivalent to the relation $\tau \supset \sigma$. □

4.5.4. — Let $\tau \in \Sigma$. The closure $\overline{O_\tau}$ of the torus orbit O_τ in X_Σ is a toric variety with underlying torus T_τ. Let M_τ and N_τ be the group of characters and cocharacters of T_τ. The quotient morphism $T \to T_\tau$ corresponds to an injection $M_\tau \to M$ (with torsion-free quotient) and a quotient $N \to N_\tau$. In fact, one has $M_\tau = \tau^\perp \cap M$ and $M_{\tau,\mathbf{R}} = \tau^\perp$; then, $N_{\tau,\mathbf{R}} = M^\circ_{\tau,\mathbf{R}} = (\tau^\perp)^\circ = N_\mathbf{R}/(\tau^\perp)^\perp = N_\mathbf{R}/\langle\tau\rangle$ and $N_\tau = N/(\langle\tau\rangle \cap N)$.

Let us show that $\overline{O_\tau}$ is a normal toric variety and let us compute its associated fan in $N_{\tau,\mathbf{R}}$.

The intersection $\overline{O_\tau} \cap X_\sigma$ is the union of all torus orbits O_ψ, for $\psi \in \Sigma$ such that $\tau \subset \psi \subset \sigma$. In particular, $\overline{O_\tau} \cap X_\sigma$ is nonempty if and only if $\tau \subset \sigma$; in this case, O_τ is contained in X_σ and $\overline{O_\tau} \cap X_\sigma$ is the closure of O_τ in this affine toric variety X_σ.

We first identify its R-points, for every k-algebra R. Elements $x \in X_\sigma(R)$ correspond to morphisms of monoids $\varphi \colon \sigma^\circ \cap M \to (R, \cdot)$. Let $\psi \in \Sigma$ be a cone such that $\tau \subset \psi \subset \sigma$. The point x_ψ corresponds to the morphism φ_ψ that maps $\sigma^\circ \cap \psi^\perp \cap M$ to 1, and the rest to 0; consequently, if $x \in O_\psi$, then one has $\varphi(m) = 0$ for all $m \in \sigma^\circ \cap M$ such that $m \notin \psi^\perp$. The inclusion $\tau \subset \psi \subset \sigma$ implies $\sigma^\perp \subset \psi^\perp \subset \tau^\perp$, so that the union of the subspaces $\psi^\perp$, for those cones ψ, is equal to $\tau^\perp$. This proves that $x \in \overline{O_\tau}$ if and only if $\varphi(m) = 0$ for all $m \in \sigma^\circ \cap M$ such that $m \notin \tau^\perp$.

Equivalently, the ideal I_τ of $\overline{O_\tau} \cap X_\sigma$ in $k^{(\sigma^\circ \cap M)}$ is generated by the monomials T^m, for $m \in \sigma^\circ \cap M$ such that $m \notin \tau^\perp$. Since τ is a face of σ, the quotient identifies with the monoid algebra of $\sigma^\circ \cap \tau^\perp \cap M$.

The intersection $\sigma^\circ \cap \tau^\perp$ in $\tau^\perp = M_{\tau,\mathbf{R}}$ is the polar of the cone σ_τ in $N_{\tau,\mathbf{R}}$, image of the cone σ (or of the cone $\sigma + \langle\tau\rangle$) by the projection $N_\mathbf{R} \to N_{\tau,\mathbf{R}}$. This identifies $\overline{O_\tau} \cap X_\sigma$ with the affine toric variety with cone σ_τ.

When σ runs among the cones of Σ that contain τ, these cones σ_τ form a fan Σ_τ in $N_{\tau,\mathbf{R}}$ and the associated toric variety X_{Σ_τ} identifies with $\overline{O_\sigma}$.

4.5.5. Missing. — – The fan of a projective toric variety given by a polytope

– Bernstein's theorem, mixed volumes...

– Lattice points and sections of line bundles, and Riemann–Roch

– The fan of a toric variety as a subset of its analytic space.

4.6. The extended tropicalization associated with a toric variety

If R is a k-algebra, we have seen that the R-points of a toric variety X_Σ associated with a fan Σ are described by morphisms of monoids to the multiplicative monoid of R. We first amplify this fact and define, functorially, the points of a toric variety with values in a monoid.

4.6.1. — Let M, N be free finitely generated abelian groups, endowed with a bilinear map $M \times N \to \mathbf{Z}$ that identifies each of them with the dual of the other. Let Σ be a (rational) fan in $N_{\mathbf{R}}$.

4.6.2. — Let S be a commutative monoid.

For every $\sigma \in \Sigma$, one sets $X_\sigma(S)$ to the set of all morphisms of monoids from $\sigma^\circ \cap M$ to S. Let τ be a face of σ and let $m \in \sigma^\circ \cap M$ that defines τ in σ: for every $x \in \sigma$, one has $x \in \tau$ if and only if $\langle x, m\rangle = 0$. Then $\tau^\circ = \sigma^\circ + \mathbf{R}_+(-m)$, $\tau^\circ \cap M = \sigma^\circ \cap M + \mathbf{N}(-m)$. Let then $j_{\sigma\tau} \colon X_\tau(S) \to X_\sigma(S)$ be the map such that $j_{\sigma\tau}(\varphi) = \varphi|_{\sigma^\circ \cap M}$. It is injective and its image is the set of all $\varphi \in X_\sigma(S)$ such that $\varphi(m) \in S^\times$.

In the same way as we glued the affine varieties X_σ into a toric variety X_Σ, we define a set $X_\Sigma(S)$.

For $\Sigma = \{0\}$, we get an abelian group $T(S) = \mathrm{Hom}(M, S^\times)$. This group acts on $X_\Sigma(S)$: for $\tau \in \mathrm{Hom}(M, S^\times)$ and $\varphi \in \mathrm{Hom}(\sigma^\circ \cap M, S)$, $\tau \cdot \varphi$ is just the map $m \mapsto \tau(m)\varphi(m)$.

4.6.3. — Let $f \colon R \to S$ be a morphism of commutative monoids. Composition with f induces maps $X_\sigma(R) \to X_\sigma(S)$ which glue to a map $f_* \colon X_\Sigma(R) \to X_\Sigma(R)$.

This map is compatible with the actions of T(R) and T(S): one has $f_*(\tau \cdot \varphi) = f_*(\tau) \cdot f_*(\varphi)$.

Example **(4.6.4)**. — Let us consider the multiplicative monoid $S = \{0, 1\}$. For every $\sigma \in \Sigma$, let $\varphi_\sigma \colon \sigma^\circ \cap M$ be the map that maps $\sigma^\perp \cap M$ to 1 and the rest to 0. One has $\varphi_\sigma \in X_\sigma(S)$; more precisely, $\varphi_\sigma \in X_\tau(S)$ if and only if $\tau \subset \sigma$. The map $\sigma \mapsto \varphi_\sigma$ is a bijection from Σ to $X_\Sigma(S)$.

The injection $S \to k$ is a morphism of multiplicative monoids. It induces an injective map $X_\Sigma(S) \to X_\Sigma(k)$ which maps φ_σ to the point x_σ, for every $\sigma \in \Sigma$.

Conversely, for every field K, the surjection $K \to S$ that maps 0 to 0 and the rest to 1 is a morphism of monoids. It induces a surjective map $X_\Sigma(K)$ to $X_\Sigma(S)$; the image of a point x is the unique point x_σ such that $x \in O_\sigma$ — equivalently, σ is the smallest cone of Σ such that $x \in X_\sigma$.

4.6.5. — Let us assume that S is a Hausdorff topological commutative monoid (the composition law $S \times S \to S$ is continuous).

For $\sigma \in \Sigma$, we endow the set $X_\sigma(S)$ with the topology of pointwise convergence.

In fact, if $(m_1, \dots, m_n)$ is a finite family in $\sigma^\circ \cap M$ that generates $\sigma^\circ \cap M$, then the map from $X_\sigma(S)$ to S^n given by $\varphi \mapsto (\varphi(m_1), \dots, \varphi(m_n))$ is a homeomorphism onto its image which is the closed subset of S^n defined by the relations $\prod s_i^{a_i} = \prod s_i^{b_i}$, for all (a, b) in the kernel of the morphism of monoids from $\mathbf{N}^n$ to $\sigma^\circ \cap M$ that maps $(a_1, \dots, a_n)$ to $\prod s_i^{a_i}$.

The set $X_\Sigma(S)$, which is a quotient of the sum of the family $(X_\sigma(S))_{\sigma \in \Sigma}$ is then endowed with the quotient topology.

Assume, moreover, that $S^\times$ is open in S. If τ is a face of σ, then $X_\tau(S)$ is an open subset of $X_\sigma(S)$. This implies that the topology of $X_\Sigma(S)$ induces on $X_\sigma(S)$ its initial topology.

4.6.6. — Let $f : R \to S$ be a continuous morphism of topological commutative monoids. The map $f_* : X_\Sigma(R) \to X_\Sigma(S)$ is continuous.

Let $f : R \to S$ be a morphism of commutative monoids.

Definition **(4.6.7)**. — *The multiplicative monoid* $\mathbf{R}_+$ *is called the* tropical monoid, *and* $X_\Sigma(\mathbf{R}_+)$ *is called the* extended tropicalization *of the toric variety* X_Σ.

Note that the logarithm map is an isomorphism of topological monoids from the multiplicative $\mathbf{R}_+$ to the additive monoid $\mathbf{R} \cup \{-\infty\}$.

Let k be a valued field. The map $a \mapsto |a|$ from k to $\mathbf{R}_+$ is a morphism of topological monoids, the map $a \mapsto \log(|a|)$ from k to $\mathbf{R} \cup \{-\infty\}$ is a morphism of topological monoids.

Definition **(4.6.8)**. — *The map* $\tau : X_\Sigma(\mathbf{C}) \to X_\Sigma(\mathbf{R}_+)$ *associated with the absolute value* $\mathbf{C} \to \mathbf{R}_+$ *is called the* extended tropicalization map. *The image of a subvariety* V *of* $X_\Sigma(\mathbf{C})$ *is called its* extended amoeba *and is denoted by* $\overline{\mathscr{A}}_V$.

Example **(4.6.9)**. — Let us take $M = N = \mathbf{Z}$; let $\sigma = \mathbf{R}_+$, $\sigma' = \mathbf{R}_+$ and $\tau = 0$, and let us consider the fan $\Sigma = \{\mathbf{R}_-, \mathbf{R}_+, 0\}$ in $\mathbf{R}$.

One has $\sigma^\circ \cap M = \mathbf{N}$, and the map $\varphi \mapsto \varphi(1)$ is a bijection from $X_\sigma(\mathbf{R}_+)$ to $\mathbf{R}_+$. One has $(\sigma')^\circ \cap M = -\mathbf{N}$, and the map $\varphi \mapsto \varphi(-1)$ is a bijection from $X_{\sigma'}(\mathbf{R}_+)$ to $\mathbf{R}_-$. One has $\tau^\circ \cap M = \mathbf{Z}$; the map $\varphi \mapsto \varphi(1)$ is a bijection from $X_\tau(\mathbf{R}_+)$ to $]0;+\infty[$, and the map $\varphi \mapsto \varphi(-1)$ is a bijection from $X_\tau(\mathbf{R}_+)$ to $]0;+\infty[$.

This identifies $X_\Sigma(\mathbf{R}_+)$ with $[0;+\infty]$, where $X_\sigma(\mathbf{R}_+) = [0;+\infty[$ and $X_{\sigma'}(\mathbf{R}_+) =]0;+\infty]$. The action of $T(\mathbf{R}_+) = \mathbf{R}_+^*$ is just by multiplication.

Taking log, we get $[-\infty;+\infty]$, $T(\mathbf{R}_+)$ then corresponds to $\mathbf{R}$ acting on $[-\infty;+\infty]$ by addition.

4.6.10. — Let us now assume that k is a non-archimedean valued field. The Berkovich spaces X_σ^{an} associated with the k-varieties X_σ, for $\sigma \in \Sigma$, are naturally glued into a topological space X_Σ^{an}. Its points are equivalence classes of pairs (K, x), where K is a valued extension of k and $x \in X_\Sigma(K)$.

The absolute value induces continuous maps $X_\Sigma(K) \to X_\Sigma(\mathbf{R}_+)$, which are combined to a continuous map $\lambda : X_\Sigma^{an} \to X_\Sigma(\mathbf{R}_+)$, the *extended tropicalization map.*

CHAPTER 5

MATROIDS AND TROPICAL GEOMETRY

5.1. Hyperplane arrangements

5.1.1. — An *hyperplane arrangement* in a projective space P is a finite sequence $(V_0, \dots, V_n)$ of hyperplanes in P.

There are obvious variants for affine or vector spaces, they can be reduced to the case of an hyperplane arrangement in a projective space. Indeed, if L is an affine space, it can be viewed as the complement of the hyperplane at infinity in the projective compactification of L. Moreover, an arrangement of linear hyperplanes in a vector space is the particular case of an arrangement of hyperplane in an affine spaces where all hyperplanes meet in one point. Conversely, the case of hyperplane arrangements in projective spaces essentially reduces to the vector case.

In this setting, P can be either a "classical" projective space (that is, the set of lines in a k-vector space), and this is the point of view used in classical complex geometry or topology. We will rather consider that P is the scheme theoretic version of a projective space, and the V_j are subschemes. Of course, one recovers the classical case by taking k-points. While these two points of view are really equivalent if k is infinite, the one we adopt has the advantage of allowing to consider the case of a finite field k.

If $\mathscr{A} = (V_0, \dots, V_n)$ is an hyperplane arrangement in P, the complement $X_{\mathscr{A}} = P - \bigcup_{j=0}^{n} V_j$ is a k-variety. Classically, in the theory of hyperplane arrangements, the goal is to relate the geometry of $X_{\mathscr{A}}$ to combinatorial properties of the arrangement.

5.1.2. — Let $\mathscr{A} = (V_0, \dots, V_n)$ be an hyperplane arrangement in a projective space P. One says that this arrangement is *essential* if one has $\bigcap_{j=0}^n V_j = \varnothing$.

If the arrangement is not essential, then $Q = \bigcap_{j=0}^n V_j$ is a projective space, and $X_{\mathscr{A}}$ is isomorphic to the product of an affine space by the complement of an essential arrangement.

5.1.3. — Let $\mathscr{A} = (V_0, \dots, V_n)$ be an essential hyperplane arrangement

For every $j \in \{0, \dots, n\}$, let us choose a linear form $f_j \in \Gamma(P, \mathscr{O}_P(1))$ defining V_j in P. Since the arrangement is essential, the f_j have no common zero and we obtain a morphism from P to $\mathbf{P}^n_k$. The morphism $f_{\mathscr{A}}$ is a closed immersion from P to a projective subspace of $\mathbf{P}^n_k$; moreover, for every j, the hyperplane V_j is the inverse image of the coordinate hyperplane $H_j = V(T_j)$.

In classical terms, $P = \mathbf{P}(V)$, for a k-vector space V, and the f_j are linear forms on V such that $V_j = \mathbf{P}(L_j)$, with $L_j = \operatorname{Ker}(f_j)$. The hypothesis that the arrangement is essential means that $\bigcap_{j=0}^n \operatorname{Ker}(f_j) = 0$; equivalently, the f_j generate V^*. For every extension K of k, a point $x \in X_{\mathscr{A}}(K)$ is the line generated by a vector $v \in V \otimes_k K$ such that $f_j(v) \neq 0$ for all j, and one has $f_{\mathscr{A}}(x) = [f_0(v) : \dots : f_n(v)]$.

Conversely, let P be a linear subscheme of $\mathbf{P}^n_k$, that is, a subscheme defined by linear forms, which is not contained in any of the coordinate hyperplanes H_j. The family $(H_0 \cap P, \dots, H_n \cap P)$ is then an essential hyperplane arrangement in P. Moreover, if $f_j = T_j|_P$ is the restriction to P of the global section $T_j \in \Gamma(\mathbf{P}^n_k, \mathscr{O}(1))$, then $(f_0, \dots, f_n)$ defines the embedding of P in $\mathbf{P}^n_k$.

5.1.4. — We fix an hyperplane arrangement $\mathscr{A}$ as above and keep the preceding notation. We also introduce some vocabulary from matroid theory.

We say that a subset J of $\{0, \dots, n\}$ is $\mathscr{A}$-free (resp. $\mathscr{A}$-dependent) if the family of linear forms $(f_j)_{j \in J}$ is linearly independent (resp. is linearly dependent). By definition, the empty set is linearly independent. Moreover, the full set $\{0, \dots, n\}$ is linearly independent if and only if $P = \mathbf{P}^n_k$.

We say that it is an $\mathscr{A}$*-circuit* if it is minimal among all linearly dependent sets: it is linearly dependent, and for every $j \in \mathrm{J}$, the set $\mathrm{J} - \{j\}$ is linearly independent.

The empty set is not a circuit.

For every linearly independent set F and every linearly dependent set D, and every $j \in \mathrm{D}-\mathrm{F}$, a minimal subset C such that $\mathrm{F} \cup \{j\} \subset \mathrm{C} \subset \mathrm{D}$ is a circuit. In particular, if $\mathrm{P} \neq \mathbf{P}^n_k$, then for every element $j \in \{0, \dots, n\}$ a circuit containing j.

Lemma **(5.1.5)**. — a) *Let* C *be an* $\mathscr{A}$*-circuit in* $\{0, \dots, n\}$*. There exists, up to scalar multiple, a unique nonzero linear form* $f_{\mathrm{C}} = \sum_{j\in\mathrm{C}} a_j \mathrm{T}_j$ *with support contained in* C *such that* $f_{\mathrm{C}}|_{\mathrm{P}} = \sum a_j f_j = 0$*. One has* $\mathrm{S}(f_{\mathrm{C}}) = \mathrm{C}$*.*

b) *Every linear form* $f \in \Gamma(\mathbf{P}^n_k, \mathscr{O}_{\mathrm{P}}(1))$ *which vanishes on* P *is a linear combination of these forms* f_{C}*, where* C *ranges over the set of* $\mathscr{A}$*-circuits in* $\{0, \dots, n\}$*. In particular, these forms generate the ideal* I(P) *of* P*.*

c) *The Hilbert function of* I(P) *is the Hilbert function of* $\mathbf{P}^m_k$*, where* $m = \dim(\mathrm{P})$*.*

Proof. — a) By the definition of a circuit, there exists such a linear form. Let $f_{\mathrm{C}} = \sum_{j\in\mathrm{C}} a_j \mathrm{T}_j$ and $f'_{\mathrm{C}} = \sum_{j\in\mathrm{C}} a'_j \mathrm{T}_j$ be two nonzero linear forms such that $f_{\mathrm{C}}|_{\mathrm{P}} = f'_{\mathrm{C}}|_{\mathrm{P}} = 0$. Since C is a circuit, one has $a_j \neq 0$ for all $j \in \mathrm{C}$. Since the empty set is free, one has $\mathrm{C} \neq \varnothing$. Fix $j \in \mathrm{C}$ and consider the linear form $a'_j f_{\mathrm{C}} - a_j f'_{\mathrm{C}}$; it vanishes on P and its support is contained in $\mathrm{C} - \{j\}$. Consequently, it is 0, which shows that f_{C} and f'_{C} are proportional.

b) Let $f = \sum_{j=0}^n a_j \mathrm{T}_j$ be a linear form such that $f|_{\mathrm{P}} = 0$. Let us prove by induction on the cardinality of $\mathrm{S}(f)$ that f is a linear combination of forms f_{C}. This holds obviously if $\mathrm{S}(f) = \varnothing$, that is, if $f = 0$. Otherwise, $f \neq 0$, hence $\mathrm{S}(f)$ is linearly dependent; fix $j \in \mathrm{S}(f)$ and choose a circuit C such that $\{j\} \subset \mathrm{C} \subset \mathrm{S}(f)$. For $\lambda \in k$, one has $\mathrm{S}(f - \lambda f_{\mathrm{C}}) \subset \mathrm{S}(f)$; moreover, if λ is the quotient of the euclidean division of f by f_{C} with respect to the variable T_j, then one has $\mathrm{S}(f - \lambda f_{\mathrm{C}}) \subset \mathrm{S}(f) - \{j\}$. By induction, $f - \lambda f_{\mathrm{C}}$ is a linear combination of forms associated with circuits, as was to be shown.

c) By a linear change of variables on $\mathbf{P}^n_k$, we may assume that $\mathrm{P} = \mathrm{V}(\mathrm{T}_{m+1},\dots,\mathrm{T}_n)$. As a graded k-algebra, $k[\mathrm{T}_0,\dots,\mathrm{T}_n]$ is then isomorphic to $k[\mathrm{T}_0,\dots,\mathrm{T}_m]$, whence the assertion. ☐

Theorem **(5.1.6)**. — *Let* K *be a valued field. Let* $\mathrm{P} \subset \mathbf{P}^n_{\mathrm{K}}$ *be a projective subspace and let* $\mathscr{A}$ *be the associated hyperplane arrangement of* P*. Let* $\mathrm{X}_{\mathscr{A}}$ *be its complement in* $\mathbf{G}_{\mathrm{m}\mathrm{K}}^{n}$ *and let* $\mathrm{I} \subset \mathrm{K}[\mathrm{T}_1^{\pm1},\dots,\mathrm{T}_n^{\pm1}]$ *be its ideal and let* $\mathrm{I}^{\mathrm{h}} \subset \mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]$ *be the associate homogeneous ideal.*

a) *The family of forms* $f_{\mathrm{C}}(1,\mathrm{T}_1,\dots,\mathrm{T}_n)$*, where* C *ranges over the set of* $\mathscr{A}$*-circuits, is a tropical basis of* I.

b) *A point* $x \in \mathbf{R}^n$ *belongs to* $\mathscr{T}_{\mathrm{X}_{\mathscr{A}}}$ *if and only if for every* $\mathscr{A}$*-circuit* C*, the initial form* $\mathrm{in}_x(f_{\mathrm{C}})$ *is not an indeterminate.*

c) *For every* $x \in \mathbf{R}^{n+1}$*, the family of forms* f_{C}*, where* C *ranges over the set of* $\mathscr{A}$*-circuits, is a Gröbner basis for* I *at* x*: the initial forms* $\mathrm{in}_x(f_{\mathrm{C}})$ *generate the initial ideal* $\mathrm{in}_x(\mathrm{I})$*.*

Proof. — The first two assertions are equivalent, by the definition of a tropical basis. Setting $x_0 = 0$, we will identify a point $x \in \mathbf{R}^n$ with the point $(0,x)$ of $\mathbf{R}^{n+1}$.

Let $x \in \mathscr{T}_{\mathrm{X}_{\mathscr{A}}}$. For every $\mathscr{A}$-circuit C, the dehomogenized form $f_{\mathrm{C}}(1,\mathrm{T}_1,\dots,\mathrm{T}_n)$ belongs to I, so that $\mathrm{in}_x(f_{\mathrm{C}}(1,\mathrm{T}_1,\dots,\mathrm{T}_n))$ is not a monomial.

Conversely, let $x \in \mathbf{R}^n - \mathscr{T}_{\mathrm{X}_{\mathscr{A}}}$ and let us prove that there exists a circuit C such that $\mathrm{in}_x(f_{\mathrm{C}}(1,\mathrm{T}_1,\dots,\mathrm{T}_n))$ is a monomial.

Let $m = \dim(\mathrm{P})$. To start with, we recall that the initial ideal $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ admits the same Hilbert function as I. The subspace $\mathrm{I}^{\mathrm{h}} \cap \mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]_1$ has dimension $n-m$, hence the same property holds for $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}}) \cap k[\mathrm{T}_0,\dots,\mathrm{T}_n]_1$. We thus can find linear forms $f_1,\dots,f_{n-m} \in \mathrm{I}^{\mathrm{h}} \cap \mathrm{K}[\mathrm{T}_0,\dots,\mathrm{T}_n]_1$ such that $\mathrm{in}_x(f_1),\dots,\mathrm{in}_x(f_{n-m})$ are linearly independent. The ideal $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ thus contains the ideal $\mathrm{I}(\mathrm{P}_x)$ of the projective space P_x of dimension m in $\mathbf{P}^n_k$ which is defined by these linear forms $\mathrm{in}_x(f_j)$. Since the homogeneous ideals I^{h}, $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ and $\mathrm{I}(\mathrm{P}_x)$ have the same Hilbert function, the inclusion $\mathrm{I}(\mathrm{P}_x) \subset \mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ implies that $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}}) = \mathrm{I}(\mathrm{P}_x)$, and $\mathrm{V}(\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})) = \mathrm{P}_x$. In particular, $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ is a homogeneous prime ideal.

Since $x \notin \mathscr{T}_{\mathrm{X}_{\mathscr{A}}}$, this ideal $\mathrm{in}_x(\mathrm{I}^{\mathrm{h}})$ contains a monomial and, being prime, it contains an indeterminate. By lemma 3.5.8, there exists $f \in$

$I^h \cap K[T_0, \dots, T_n]_1$ such that $\mathrm{in}_x(f)$ is an indeterminate, and we choose f so that $\mathrm{Card}(S(f))$ is minimal.

For simplicity of notation, we assume that $\mathrm{in}_x(f) = T_0$, so that $\tau_f(x) = 0$. Replacing f by a multiple, we may write $f = T_0 + \sum_{j=1}^n a_j T_j$, hence $v(a_j) + x_j > 0$ for all $j \in \{1, \dots, n\}$.

Let C be an $\mathscr{A}$-circuit such that $C \subset S(f)$ and normalize the form $f_C = \sum_{j \in C} b_j T_j$ so that $\tau_{f_C}(0) = \inf j \in C(v(b_j) + x_j) = 0$.

Let $j \in C \smallsetminus \{0\}$ be such that $v(b_j) + x_j = 0$ and let us set $f' = f - a_j b_j^{-1} f_C$; this is an element of $I^h \cap K[T_0, \dots, T_n]_1$ such that $S(f') \subset S(f) \smallsetminus \{j\}$; moreover $\tau_{f'}(x) = \tau_f(x) = 0$, and $\mathrm{in}_x(f') = \rho(1 - a_j b_j^{-1} b_0) T_j$ is a multiple of an indeterminate, contradicting the minimality of $\mathrm{Card}(f)$. Consequently, $v(b_j) + x_j > 0$ for all $j \in C \smallsetminus \{0\}$, hence $\mathrm{in}_x(f_C) = \rho(b_0) T_0$. □

5.1.7. — It follows from the proof that for any $x \in \mathbf{R}^n$, the initial ideal $\mathrm{in}_{(0,x)}(I^h)$ associated with the homogeneous ideal I^h of P in P^n_K defines a linear subspace P_x of $\mathbf{P}^n_k$. One has $x \in \mathscr{T}_{X_{\mathscr{A}}}$ if and only if P_x is not contained in one of the coordinate hyperplanes. In this case, P_x defines an hyperplane arrangement over the residue field k. We will call it the *reduction* of the hyperplane arrangement $\mathscr{A}$ at x.

5.1.8. — At least when the valuation of K is trivial, one can give an alternate, possibly more explicit, description of $\mathscr{T}_{X_{\mathscr{A}}}$. It involves another notion from matroid theory.

We say that a subset I of $\{0, \dots, n\}$ is an *$\mathscr{A}$-flat* if there exists a (possibly empty) projective subspace Q of P such that for $i \in \{0, \dots, n\}$, the conditions $i \in I$ and $\varphi_i|_Q = 0$ are equivalent. By definition, $\{0, \dots, n\}$ is a flat, corresponding to $Q = P$. The intersection of two flats is a flat, given by the intersection of the corresponding subspaces. For every subset J of $\{0, \dots, n\}$, there exists a smallest flat $\langle J \rangle$ containing J; it is given by the projective subspace Q of P generated by the subspaces $V(\varphi_j)$, for $j \in J$.

A *flag* of $\mathscr{A}$-flats is a strictly increasing sequence $F = (F_0, \dots, F_{s+1})$ of $\mathscr{A}$-flats such that $F_0 = \varnothing$ and $F_{s+1} = \{0, \dots, n\}$.

Let F and F′ be two flags; one says that the flag F′ refines the flag F if every flat of F appears in F′.

5.1.9. — Let $(e_0, \dots, e_n)$ be the canonical basis of $\mathbf{R}^{n+1}$. With each subset I of $\{0, \dots, n\}$, we associate the vector $e_{\mathrm{I}} = \sum_{i \in \mathrm{I}} e_i$ of $\mathbf{R}^{n+1}$; we also let $\mathbf{1} = (1, \dots, 1) = e_{\{0,\dots,n\}}$. With each flag $\mathrm{F} = (\mathrm{F}_0, \mathrm{F}_1, \dots, \mathrm{F}_s, \mathrm{F}_{s+1})$ we associate the cone $\mathrm{C}_{\mathrm{F}} = \operatorname{cone}(e_{\mathrm{F}_1}, \dots, e_{\mathrm{F}_s}) + \mathbf{R1}$ in $\mathbf{R}^{n+1}$; its dimension is $s+1$ and its lineality space is $\mathbf{R1}$.

Let $\mathrm{F} = (\mathrm{F}_0, \mathrm{F}_1, \dots, \mathrm{F}_{s+1})$ be a flag of $\mathscr{A}$-flats. A point $x \in \mathbf{R}^{n+1}$ belongs to the cone C_{F} if and only if for all $r, r' \in \{1, \dots, s+1\}$, every $i \in \mathrm{F}_r$ and every $i' \in \mathrm{F}_{r'}$, the inequalities $x_i \leqslant x_{i'}$ and $r \leqslant r'$ are equivalent; this means that x_i takes a constant value c_r on each $\mathrm{F}_r - \mathrm{F}_{r-1}$, for $r \in \{1, \dots, s\}$, and that one has $c_1 \leqslant \dots \leqslant c_{s+1}$. The relative interior of the cone C_{F} is described by the equalities $x_i = x_{i'}$ for $r \in \{1, \dots, s+1\}$, $i, i' \in \mathrm{F}_r - \mathrm{F}_{r-1}$, and the strict inequalities $x_i < x_{i'}$ for $r \in \{1, \dots, s+1\}$, $i \in \mathrm{F}_r$ and $i' \notin \mathrm{F}_r$; with the above notation, it corresponds to the strict inequalities $c_1 < \dots < c_{s+1}$.

Lemma **(5.1.10)**. — *Let* F *and* F′ *be two flags of* $\mathscr{A}$*-flats.*

a) *One has* $\mathrm{C}_{\mathrm{F}} \subset \mathrm{C}_{\mathrm{F}'}$ *is and only if the flag* F′ *refines the flag* F*, which means that every flat of* F *appears in* F′*.*

b) *There exists a finest flag* F″ *which is coarser both than* F *and* F′*, and one has* $\mathrm{C}_{\mathrm{F}} \cap \mathrm{C}_{\mathrm{F}'} = \mathrm{C}_{\mathrm{F}''}$*.*

Proof. — We start with a remark. Let $x \in \mathbf{R}^{n+1}$. Define a sequence $(\mathrm{I}_0, \dots, \mathrm{I}_{r+1})$ by induction, letting $\mathrm{I}_0 = \varnothing$, and, if $\mathrm{I}_0, \dots, \mathrm{I}_m$ are defined and $\mathrm{I}_m \neq \{0, \dots, n\}$, I_{m+1} being the set of $i \in \{0, \dots, n\} - \mathrm{I}_m$ such that x_i is minimal; one has $\mathrm{I}_{r+1} = \{0, \dots, n\}$. By construction, for every $m \in \{1, \dots, r+1\}$, x_i takes a constant value c_i for $i \in \mathrm{I}_m - \mathrm{I}_{m-1}$, and one has $c_1 < c_2 < \dots < c_{r+1}$. Let F be a flag of $\mathscr{A}$-flats; by definition of the cone C_{F}, one has $x \in \mathrm{C}_{\mathrm{F}}$ if and only if there exists a strictly increasing sequence $(j_1, \dots, j_r)$ such that $\mathrm{I}_m = \mathrm{F}_{j_m}$ for $m \in \{1, \dots, r\}$.

a) Write $\mathrm{F} = (\mathrm{F}_0, \dots, \mathrm{F}_{s+1})$ and $\mathrm{F}' = (\mathrm{F}'_0, \dots, \mathrm{F}'_{s'+1})$. Let x be a point of the relative interior of C_{F}; For $r \in \{1, s+1\}$, let c_r be the common value of x_i for $i \in \mathrm{F}_r - \mathrm{F}_{r-1}$, so that $c_1 < \dots < c_{s+1}$. One has $x \in \mathrm{C}_{\mathrm{F}'}$ if and only if there exists a strictly increasing sequence $(j_1, \dots, j_{s+1})$ such that $\mathrm{F}_r = \mathrm{F}'_{j_r}$ for every $r \in \{1, \dots, s\}$. This means that the flag F′ refines F.

Assume that this holds and choose, for every $m \in \{1, \dots, s+1\}$, an element $i_m \in \mathrm{F}'_{j_m} - \mathrm{F}'_{j_{m-1}}$. Let f be the linear form on $\mathbf{R}^{n+1}$ defined by $f(x) = \sum_{m=1}^{r}(x_{i_{m+1}} - x_{i_m})$. It is positive on $C_{F'}$ and one has $C_F = C_{F'} \cap \operatorname{Ker}(f)$, so that C_F is a face of $C_{F'}$.

b) Extracting from F the flats that appear in F′ defines a flag of flats $F'' = (F''_0, \dots, F''_{s''+1})$ that is coarser both than F and F′; and it is the finest such flag. One has $C_{F''} \subset C_F \cap C_{F'}$. Conversely, let $x \in C_F \cap C_{F'}$ and let us prove that $x \in C_{F''}$. Define $(I_0, \dots, I_r)$ as in the preamble of the proof. By assumption, there are integers $j_1, \dots, j_r$ in $\{1, \dots, s+1\}$ and $j'_1, \dots, j'_r$ in $\{1, \dots, s'+1\}$ such that $I_m = F_{j_m} = F'_{j'_m}$ for every $m \in \{1, \dots, r\}$. By construction of F″, the set I_m is thus a flat in the flag F″, for every m, hence $x \in C_{F''}$. □

Theorem **(5.1.11)**. — *Assume that the valuation of* K *is trivial. The family of cones* C_F*, where* F *runs over the set of flags of* $\mathscr{A}$*-flats, is a fan* $\Sigma(\mathscr{A})$ *in* $\mathbf{R}^{n+1}$*. A point* $x \in \mathbf{R}^n$ *belongs to* $\mathscr{T}_{X_{\mathscr{A}}}$ *if and only if the point* $x' = (0, x)$ *belongs to* $|\Sigma(\mathscr{A})|$.

If the valuation of K is not trivial, I expect that the support of this fan computes similarly the recession cone of the polyhedral set $\mathscr{T}_{X_{\mathscr{A}}}$, but I can only prove one inclusion. A fuller description is still possible, but it involves further matroid theory.

Proof. — Let $x \in \mathbf{R}^n$ and set $x_0 = 0$. for every $r \in \{0, \dots, n\}$, let $F_r = \{i \in \{0, \dots, n\} ; x_i \geqslant x_r\}$.

Assume that $F_0, \dots, F_n$ are $\mathscr{A}$-flats. Let $r, s \in \{0, \dots, n\}$ be such that $F_r \not\subset F_s$ and let $i \in F_r - F_s$: this means that $x_s > x_i \geqslant x_r$; then, for all $j \in F_s$, one has $x_j \geqslant x_s > x_i \geqslant x_r$, hence $j \in F_r$, so that $F_s \subset F_r$. This proves that the set $\{F_0, \dots, F_n\}$ is totally ordered. If $x_r = \inf_{i \in \{0,\dots,n\}}(x_i)$, one has $F_r = \{0, \dots, n\}$. Let us index the elements F_r in increasing order: there exists an integer $r \in \{0, \dots, n-1\}$ and a family $(i_1, \dots, i_r, i_{r+1})$ of elements of $\{0, \dots, n\}$ such that $F_{i_1} \subsetneq \cdots \subsetneq F_{i_r} \subsetneq F_{i_{r+1}} = \{0, \dots, n\}$ and $\{F_0, \dots, F_n\} = \{F_{i_1}, \dots, F_{i_{r+1}}\}$. Then $F = (\varnothing, F_{i_1}, \dots, F_{i_{r+1}})$ is a flag of $\mathscr{A}$-flats and x belongs to the cone C_F.

Now assume that x does not belong to $|\Sigma(\mathscr{A})|$. By what precedes, there exists an integer $r \in \{0, \dots, n\}$ such that F_r is not a flat. Let $i \in \langle F_r \rangle - F_r$. By definition of F_r, one has $x_i < x_r \leqslant x_j$ for all $j \in F_r$. By definition of $\langle F_r \rangle$, there exists a family $(a_j)_{j \in F_r}$ in K such that $\varphi_i = \sum_{j \in F_r} a_j \varphi_j$. Let then $f = T_i - \sum_{j \in F_r} a_j T_j$; this is an homogeneous element of the homogeneous ideal I^h. Since the valuation of K is trivial, the initial form $\mathrm{in}_x(f)$ is equal to T_i, hence $x \notin \mathscr{T}_{X_{\mathscr{A}}}$.

Conversely, let us prove that $|\Sigma(\mathscr{A})| \subset \mathscr{T}_{X_{\mathscr{A}}}$. Let $x \in |\Sigma(\mathscr{A})|$ and let $F = (F_0, F_1, \dots, F_{r+1})$ be a flat of $\mathscr{A}$-flats such that x belongs to the cone C_F.

Let C be an $\mathscr{A}$-circuit; write $f_C = \sum_{i \in C} a_i T_i$ the linear form associated with the circuit C. Since $C \subset F_{r+1}$, there exists a smallest integer $s \in \{0, \dots, r\}$ be such that $C \subset F_{s+1}$. If $s = 0$, then $C \not\subset F_s = \varnothing$; if $s \geqslant 1$, then $s - 1 \in \{0, \dots, r\}$ and the minimality of s implies that $C \not\subset F_s$.

Let $C' = C - F_s$; one has $C' \neq \varnothing$; let us prove that $\mathrm{Card}(C') \geqslant 2$. Otherwise, there exists $j \in C$ such that $C' = \{j\}$. Since $\sum_{i \in C} a_i \varphi_i = 0$, the linear form φ_j is a linear combination of the forms φ_i, for $i \in C \cap F_s$; since F_s is a flat, this implies that $j \in F_s$, a contradiction that proves $\mathrm{Card}(C') \geqslant 2$.

Let then j, k be distinct elements of C'. One has $j, k \in F_{s+1} - F_s$, hence $x_j = x_k$, because x belongs to the cone C_F. Moreover, $x_i \geqslant x_j$ for all $i \in F_s$. Since the valuation of K is trivial, this implies that T_i, T_j belong to the support of $\mathrm{in}_x(f_C)$, so that $\mathrm{in}_x(f_C)$ is not a monomial. Consequently, $x \in \mathscr{T}_{f_C}$.

By theorem 5.1.6, one has $x \in \mathscr{T}_{X_{\mathscr{A}}}$. □

5.2. Matroids

Matroids were invented by Whitney (1935) to express the abstract combinatorial properties of linear independence in vector spaces by developing an axiomatic treatment of free resp. dependent families, of rank, of flats, of generating families, etc. It has soon be observed that they allow a large number of equivalent formalizations. In this section, we present the definitions and the vocabulary of matroid theory, referring the reader to a later appendix for some proofs. Our main reference was (Oxley, 1992).

Definition **(5.2.1)**. — *Let* M *be a finite set. A* matroid *structure on* M *is the datum of a subset* $\mathscr{I}_M$ *of* $\mathfrak{P}(M)$ *satisfying the following properties:*

(I_1) *The empty set belongs to* $\mathscr{I}_M$*;*

(I_2) *If* A, B *are subsets of* M *such that* $A \subset B$ *and* $B \in \mathscr{I}_M$*, then* $A \in \mathscr{I}_M$*;*

(I_3) *If* A, B *are elements of* $\mathscr{I}_M$ *such that* $\mathrm{Card}(A) < \mathrm{Card}(B)$, *there exists* $b \in B - A$ *such that* $A \cup \{b\} \in \mathscr{I}_M$.

Let M be a matroid given by a subset $\mathscr{I}_M$ of $\mathfrak{P}(M)$ and let A be a subset of M. One says that A is *independent*, or *free*, if it belongs to $\mathscr{I}_M$, and that it is *dependent* otherwise.

Example **(5.2.2)**. — Let M be a matroid and let A be a subset of M.
The intersection of $\mathscr{I}_M$ with $\mathfrak{P}(A)$ is a matroid structure on M. The corresponding matroid is denoted by $M \mid A$.

A *basis* of M is a maximal free subset; a *circuit* of M is a minimal dependent subset.
Since M is assumed to be finite, we observe that a subset is free if and only if it is contained in a basis, and a subset is dependent if and only if it contains a circuit.
A *loop* is an element $i \in M$ such that $\{i\}$ is not free.

Lemma **(5.2.3)**. — *Let* M *be a matroid. Every free subset of* M *is contained in a basis. All bases have the same cardinality.*

Proof. — Let L be a free subset of M. Since M is finite, there exists among all free subsets of M that contain L, one which has maximal cardinality. It is a basis.
Let B, B' be two bases of M. If $\mathrm{Card}(B) < \mathrm{Card}(B')$, there exists an element $a \in B' - B$ such that $B \cup \{a\}$ is free; this contradicts the hypothesis that B is a maximal free subset. Consequently, $\mathrm{Card}(B) \geqslant \mathrm{Card}(B')$. By symmetry, this implies that $\mathrm{Card}(B) = \mathrm{Card}(B')$. □

One defines the *rank* of a matroid M to be the cardinality of any of its bases; it is denoted by rank(M). For any subset A of M, one defines $\mathrm{rank}_M(A)$ to be the rank of the matroid $M \mid A$.

Lemma **(5.2.4)**. — *Let* M *be a matroid and let* A *be a subset of* M*. Then* A *is a basis of* M *if and only if* $\mathrm{Card}(A) = \mathrm{rank}_M(A) = \mathrm{rank}_M(M)$; *it is free if and only if* $\mathrm{Card}(A) = \mathrm{rank}_M(A)$.

Proof. — One has $\mathrm{rank}_M(A) \leqslant \mathrm{Card}(A)$, by construction. If equality holds, then A is a basis of $M \mid A$, hence A is free in $M \mid A$, so that

A is free. Conversely, if A is free, then A is a basis of M | A and $\mathrm{Card}(A) = \mathrm{rank}_M(A)$.

If A is a basis of M, then $\mathrm{Card}(A) = \mathrm{rank}_M(A)$ by what precedes, and $\mathrm{Card}(A) = \mathrm{rank}_M(M)$, by the definition of the rank, hence the equality $\mathrm{Card}(A) = \mathrm{rank}_M(A) = \mathrm{rank}_M(M)$.

Let us assume, conversely, that this equality holds. First of all, A is free. Let B be a basis of M such that $A \subset B$. Then $\mathrm{Card}(A) = \mathrm{rank}_M(M) = \mathrm{Card}(B)$. Consequently, $A = B$ and A is a basis of M. □

Example **(5.2.5)**. — 1) Let K be a field and let V be a K-vector space. Let $(v_i)_{i\in M}$ be a finite family in V and let W be the vector subspace it generates. The set $\mathscr{I}$ of all subsets A of M such that the family $(v_i)_{i\in A}$ is linearly independent defines a structure of matroid on M. For this structure, a subset A is a basis if and only if $(v_i)_{i\in A}$ is a basis of W.

The axiom (I_1) holds because the empty family is linearly independent, and the axiom (I_2) follows from the fact that a subfamily of a linearly family is linearly independent. To prove the axiom (I_3), we now by dimension theory that the subspace W_A spanned by $(v_i)_{i\in A}$ has dimension Card(A), while the subspace W_B spanned by $(v_i)_{i\in B}$ has dimension Card(B). If $v_b \in W_A$ for all $b \in B$, then $W_B \subset W_A$, which contradicts the inequality $\dim(W_B) = \mathrm{Card}(B) > \mathrm{Card}(A) = \dim(W_A)$. Consequently, there exists $b \in B$ such that $v_b \notin W_A$ and the family $(v_i)_{i\in A\cup\{b\}}$ is linearly independent.

An element $i \in M$ is a loop if and only if $v_i = 0$.

A basis is a subset I of M such that $(v_i)_{i\in I}$ is a basis of the subspace spanned by the family $(v_i)_{i\in M}$.

2) Let K be a field and let V be an affine space over K. Let $(v_i)_{i\in M}$ be a finite family in V and let W be the affine subspace it generates. The set $\mathscr{I}$ of all subsets A of M such that the family $(v_i)_{i\in A}$ is affinely independent defines a structure of matroid on M. Its bases are the affine bases of the affine subspace of V spanned by the points v_i.

3) Let K be a field and let P be an affine space over K. Let $(x_i)_{i\in M}$ be a finite family in P and let Q be the projective linear subspace it generates. The set $\mathscr{I}$ of all subsets A of M such that the family $(x_i)_{i\in A}$ is projectively independent defines a structure of matroid on M. Its bases

are the projective frames of the projective subspace of P spanend by the points x_i.

Such matroids are called *representable* over K.

Proposition **(5.2.6)**. — *Let* M *be a matroid. The set* $\mathscr{C}_M$ *of all circuits of* M *satisfies the following properties:*

(C_1) *The empty set does not belong to* $\mathscr{C}_M$*;*

(C_2) *If* C, C' *are distinct elements of* $\mathscr{C}_M$*, then* $C \not\subset C'$*;*

(C_3) *If* C, C' *are distinct elements of* $\mathscr{C}_M$ *and* $e \in C \cap C'$*, there exists* $D \in \mathscr{C}_M$ *such that* $D \subset (C \cup C') - \{e\}$.

Conversely, if M *is a finite set and* $\mathscr{C}$ *is a subset of* $\mathfrak{P}(M)$ *satisfying the properties* (C_1), (C_2), (C_3)*, there exists a unique structure of matroid on* M *of which* $\mathscr{C}$ *is the set of circuits.*

For the proof, see §A.1.1.

Example **(5.2.7)**. — Let G be a finite graph. That is, G is given by finite sets V (vertices) and A (arrows), by two maps $s, t : A \to V$ (source and target), and by a fixed-point-free involution of V, $a \mapsto \overline{a}$, such that $s(\overline{a}) = t(a)$ for every $a \in A$. In this formalism, the arrows are oriented, $\overline{a}$ is the opposite of the arrow a, and one defines an edge of G as a pair $[a] = \{a, \overline{a}\}$ of opposite arrows. A path in G is a sequence of arrows $(e_1, \ldots, e_n)$ such that the target of e_i is the origin of e_{i+1} for $1 \leqslant i < n$; a path is a closed if the target of e_n is the origin of e_1; a closed path is a circuit if $e_{i+1} \neq \overline{e_i}$ for $1 \leqslant i < n$ and $e_1 \neq \overline{e_n}$, and if $t(e_i) \neq s(e_1)$ for $1 \leqslant i < n$.

Given a circuit $(e_1, \ldots, e_n)$, consider the set $\{[e_1], \ldots, [e_n]\}$ of edges. These sets constitute the circuits of a matroid structure on the set E of edges of G. In this context, the axiom (C_3) is the elimination rule for two circuits that share a common edge: if $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$ are circuits such that $e_1 = f_1$, then $(e_2, \ldots, e_m, \overline{f_n}, \ldots, \overline{f_2})$ is a closed path in G, and one can build a circuit from it by eliminating consecutive opposite arrows.

The independent sets of this matroid are the forests of the graph G.

Lemma **(5.2.8)**. — *Let* M *be a matroid, let* L *be a free subset of* M *and let* $e \in M$ *be such that* $L \cup \{e\}$ *is dependent.*

a) *There exists a unique circuit* C *of* M *such that* $C \subset L \cup \{e\}$.
b) *One has* $e \in C$.
c) *For every* $f \in M$, *the set* $(L - \{f\}) \cup \{e\}$ *is free if and only if* $f \in C$.

Proof. — Since $L \cup \{e\}$ is dependent, it contains a circuit C. Since L is free, one has $C \not\subset L$, so that $e \in C$. Let C, C' be distinct circuits contained in $L \cup \{e\}$; they both contain e. Let then D be a circuit such that $D \subset (C \cup C') - \{e\}$. One has $D \subset L$, a contradiction. This proves *a*) and *b*).

Let us prove *c*). Let $f \in M$. If $f \notin C$, then $(L - \{f\}) \cup \{e\}$ contains C, hence is dependent. Conversely, if this subset is dependent, it contains a circuit, say C'. One has $C' \subset L \cup \{e\}$, hence $C' = C$, by *a*). Since $f \notin C'$, this proves that $f \notin C$. ☐

Proposition **(5.2.9)**. — *Let* M *be a matroid. The set* $\mathscr{B}_M$ *of all bases of* M *satisfies the following properties:*

(B_1) *The set* $\mathscr{B}_M$ *is not empty;*

(B_2) *If* B, B' *belong to* $\mathscr{B}_M$ *and* $x \in B - B'$, *there exists* $y \in B' - B$ *such that* $(B - \{x\}) \cup \{y\}$ *belongs to* $\mathscr{B}_M$.

Conversely, if M *is a finite set and* $\mathscr{B}$ *is a subset of* $\mathfrak{P}(M)$ *satisfying the properties* (B_1) *and* (B_2), *there exists a unique structure of matroid on* M *of which* $\mathscr{B}$ *is the set of bases.*

For the proof, see §A.1.3

Proposition **(5.2.10)**. — *Let* M *be a matroid. The function* $\operatorname{rank}_M : \mathfrak{P}(M) \to \mathbf{N}$ *satisfies the following properties:*

(R_1) *For every subset* A *of* M, *one has* $0 \leqslant \operatorname{rank}_M(A) \leqslant \operatorname{Card}(A)$;

(R_2) *If* A, B *are subsets of* M *such that* $A \subset B$, *one has* $\operatorname{rank}_M(A) \leqslant \operatorname{rank}_M(B)$;

(R_3) *If* A, B *are subsets of* M, *one has* (submodular inequality)

$$\operatorname{rank}_M(A \cup B) + \operatorname{rank}_M(A \cap B) \leqslant \operatorname{rank}_M(A) + \operatorname{rank}_M(B).$$

Conversely, if M *is a finite set and* $r : \mathfrak{P}(M) \to \mathbf{N}$ *is a function satisfying the properties* (R_1), (R_2), (R_3), *there exists a unique structure of matroid on* M *such that* $r(A) = \operatorname{rank}_M(A)$ *for every subset* A *of* M.

For the proof, see §A.1.4.

Definition **(5.2.11)**. — *Let* M *be a matroid. For every subset* A *of* M*, let* $\langle A\rangle$ *be the set of all* $x \in M$ *such that* $\operatorname{rank}_M(A \cup \{x\}) = \operatorname{rank}_M(A)$.

The set $\langle A\rangle$ is called subset of M *generated* by A, or the *closure* of A with respect to the matroid structure of M. The loops of M are the elements of $\langle\varnothing\rangle$.

One says that A generates M if one has $\langle A\rangle = M$.

Lemma **(5.2.12)**. — *Let* M *be a matroid and let* A *be a subset of* M*. The set* $\langle A\rangle$ *is the largest subset of* M *containing* A *such that* $\operatorname{rank}_M(\langle A\rangle) = \operatorname{rank}_M(A)$. *In particuliar,* A *is generating if and only if* $\operatorname{rank}_M(A) = \operatorname{rank}_M(M)$.

Proof. — Let us assume that $\operatorname{rank}_M(A) = \operatorname{rank}_M(M)$ and let us prove that $\langle A\rangle = M$. Then $\operatorname{rank}_M(A \cup \{x\}) = \operatorname{rank}_M(M)$ for every $x \in M$, so that $x \in \langle A\rangle$; this proves that A is generating. Conversely, let us assume that $\langle A\rangle = M$, and let us prove that $\operatorname{rank}_M(A) = \operatorname{rank}_M(M)$. Let B be a basis of $M \mid A$, and let B′ be a basis of M containing M. Let $x \in B' - B$. Then $B \cup \{x\}$ is free. Since B is a basis of $M \mid A$, this implies $x \notin A$. Then $\operatorname{rank}_M(A\cup\{x\}) \geqslant \operatorname{Card}(B\cup\{x\}) = \operatorname{Card}(B)+1 > \operatorname{rank}_M(A)$. This contradicts the assumption that A is generating, and concludes the proof of the second assertion.

The first assertion follows from this, applied to the matroids $M \mid B$, for all subsets B of M that contain A. ☐

Lemma **(5.2.13)**. — *Let* M *be a matroid and let* A *be a subset of* M*. The following properties are equivalent:*

(i) *The set* A *is a basis of* M*;*
(ii) *The set* A *is a minimal generating subset of* M*;*
(iii) *The set* A *is a maximal independent subset of* M*;*
(iv) *The set* A *is generating and independent.*

Proof. — (i)⇒(iv). Let A be a basis of M. Then A is independent, by definition, and $\operatorname{rank}_M(A) = \operatorname{Card}(A) = \operatorname{rank}_M(M)$. Moreover, for every $x \in M$, the inclusions $A \subset A \cup \{x\} \subset M$ imply that $\operatorname{rank}_M(A) \leqslant \operatorname{rank}_M(A\cup\{x\}) \leqslant \operatorname{rank}_M(M)$; consequently, $\operatorname{rank}_M(A\cup\{x\}) = \operatorname{rank}_M(A)$, so that $x \in \langle A\rangle$. This proves that A is generating.

(iv)⇒(iii). Let A be a generating and independent subset of M. Let $x \in M - A$. Then $\operatorname{rank}_M(A \cup \{x\}) = \operatorname{rank}_M(M) = \operatorname{rank}_M(A)$, because A is generating. Moreover, $\mathrm{Card}(A) = \mathrm{rank}_M(A)$ so that $\mathrm{rank}_M(A \cup \{x\}) < \mathrm{Card}(A \cup \{x\})$. This proves that $A \cup \{x\}$ is dependent. Consequently, A is a maximal independent subset.

The equivalence (iii)⇔(i) is the definition of a basis.

(iv)⇒(ii). Let A be a generating and independent subset of M, so that one has $\mathrm{Card}(A) = \mathrm{rank}_M(A) = \mathrm{rank}_M(M)$. For any subset B of M such that $A \subsetneq B$, one thus has $\mathrm{rank}_M(B) = \mathrm{rank}_M(M) = \mathrm{rank}_M(A) < \mathrm{Card}(B)$, hence B is not free. This proves that A is generating and maximal.

(ii)⇒(iv). Let A be a minimal generating subset of M and let B be a basis of $M \mid A$. One has $\mathrm{Card}(B) = \mathrm{rank}_M(A)$. If A is not free, then $B \neq A$, hence B is not generating, by assumption, which means that there exists $x \in M$ such that $\mathrm{rank}_M(B \cup \{x\}) \neq \mathrm{rank}_M(B)$. It then follows from inequality (R_3) that $\mathrm{rank}_M(B\cup\{x\}) = \mathrm{rank}_M(B)+1 = \mathrm{Card}(B\cup\{x\})$, so that $B \cup \{x\}$ is free. In particular, $x \notin A$.

$$\mathrm{rank}_M(A) + \mathrm{rank}_M(B \cup \{x\}) \leqslant \mathrm{rank}_M(A \cup \{x\}) + \mathrm{rank}_M(B),$$

so that $\mathrm{Card}(B)+1 \leqslant \mathrm{rank}_M(A\cup\{x\})$. Since $\mathrm{rank}_M(A\cup\{x\}) \leqslant \mathrm{rank}_M(A)+1$, this implies that $\mathrm{rank}_M(A \cup \{x\}) = \mathrm{rank}_M(A) + 1$, contradicting the hypothesis that A is generating. □

Proposition **(5.2.14)**. — *Let M be a matroid. The function $A \mapsto \langle A\rangle$ satisfies the following properties:*

(c_1) *For every subset A of M, one has $A \subset \langle A\rangle$;*
(c_2) *For every subsets A, B of M such that $A \subset B$, one has $\langle A\rangle \subset \langle B\rangle$;*
(c_3) *For every subset A of M, one has $\big\langle\langle A\rangle\big\rangle = \langle A\rangle$;*
(c_4) *If A is a subset of M, $a \in M$ and $b \in \langle A \cup \{a\}\rangle - \langle A\rangle$, then $a \in \langle A \cup \{b\}\rangle$.*

Conversely, if M is a finite set and $c \colon \mathfrak{P}(M) \to \mathfrak{P}(M)$ is a function satisfying the properties (c_1), (c_2), (c_3) and (c_4), there exists a unique structure of matroid on M such that $c(A)$ is the subset generated by A, for every subset A of M.

For the proof, see §A.1.6.

Definition **(5.2.15)**. — *Let M be a matroid. One says that a subset A of M is a* flat *if one has $A = \langle A\rangle$.*

Proposition **(5.2.16)**. — *Let* M *be a matroid. The set* $\mathscr{F}_M$ *of flats in* M *satisfies the following properties:*

(F$_1$) *The set* $\mathscr{F}_M$ *is stable under intersection. (Equivalently,* $M \in \mathscr{F}_M$*, and if* $A, B \in \mathscr{F}_M$*, then* $A \cap B \in \mathscr{F}_M$*.)*

(F$_2$) *For every* $A \in \mathscr{F}_M$ *such that* $A \neq M$*, the set of elements of* $\mathscr{F}_M$ *which strictly contain* A *and are minimal for this property cover* M.

Conversely, if M *is a finite set and* $\mathscr{F}_M$ *is a subset of* $\mathfrak{P}(M)$ *satisfying* (F$_1$) *and* (F$_2$)*, there exists a unique structure of matroid on* M *of which* $\mathscr{F}_M$ *is the set of flats.*

For the proof, see §A.1.7.

5.2.17. — Let us endow the set $\mathscr{F}_M$ of flats of M with the order given by inclusion. If A, B are flats, then $A \cap B = \inf(A, B)$ and $\langle A \cup B \rangle = \sup(A, B)$. In particular, $\mathscr{F}_M$ is a lattice.

5.2.18. — Let $(X, \preceq)$ be a finite nonempty ordered set.

A *chain* in X is a strictly increasing nonempty sequence $(x_0, \dots, x_m)$ in X; its floor is x_0, its roof is x_m and its length is m. One says that a chain $(x_0, \dots, x_m)$ refines a chain $(y_0, \dots, y_n)$ if for every $j \in \{0, \dots, n\}$, there exists $i \in \{0, \dots, m\}$ such that $y_j = x_i$; intuitively, the chain $(x_0, \dots, x_m)$ is obtained by inserting new elements.

For $x \in X$, the height of x is the supremum $\mathrm{ht}_X(x)$ of all lengths of chains with roof x, and its coheight $\mathrm{coht}_X(x)$ is the supremum of lengths of chains with floor x.

The ordered set X is said to be catenary if $\mathrm{ht}_X(x) + \mathrm{coht}_X(x)$ is independent of x. In this case, for every $x, y \in X$ such that $x \preceq y$, all maximal chains in X with floor x and roof y have length $\mathrm{ht}(y) - \mathrm{ht}(x) = \mathrm{coht}(x) - \mathrm{coht}(y)$.

Assume that X is a lattice. An atom in X is an element of height 1.

Lemma **(5.2.19)**. — *Let* M *be a matroid and let* $\mathscr{F}_M$ *be its lattice of flats. For every flat* A*, one has* $\mathrm{ht}(A) = \mathrm{rank}_M(A)$ *and* $\mathrm{coht}(A) = \mathrm{rank}_M(M) - \mathrm{rank}_M(A)$*. The atoms of* $\mathscr{F}_M$ *are the flats of the form* $\langle a \rangle$*, for* $a \in M \smallsetminus \langle \varnothing \rangle$.

Proof. — If P is the minimal flat of M, then $\mathrm{ht}(P) = 0$; the maximal flat of M is M itself and $\mathrm{coht}(M) = 0$. In a chain $(A_0, \dots, A_n)$ of flats, the ranks strictly increase; moreover this chain cannot be refined if and only if $\mathrm{rank}_{\mathrm{M}}(\mathrm{A}_j) = \mathrm{rank}_{\mathrm{M}}(\mathrm{A}_0) + j$ for every j. Taking $\mathrm{A}_n = \mathrm{M}$, this implies that $\mathrm{coht}(\mathrm{A}) = \mathrm{rank}_{\mathrm{M}}(\mathrm{M}) - \mathrm{rank}_{\mathrm{M}}(\mathrm{A})$ for every flat A; taking $\mathrm{A}_0 = \mathrm{P}$, we get $\mathrm{ht}(\mathrm{A}) = \mathrm{rank}_{\mathrm{M}}(\mathrm{A})$. ☐

Proposition **(5.2.20)**. — *Let* L *be a finite lattice. Then* L *is isomorphic to the lattice of flats of a matroid if and only if the following properties hold:*

(L_1) *The lattice* L *is catenary;*
(L_2) *For every* $x, y \in \mathrm{L}$, *one has*

$$\mathrm{ht}(x) + \mathrm{ht}(y) \geqslant \mathrm{ht}(\inf(x, y)) + \mathrm{ht}(\sup(x, y));$$

(L_3) *For every* $x \in \mathrm{L}$, *there exist an integer* $m \in \mathbf{N}$ *and a sequence* $(x_1, \dots, x_m)$ *of atoms of* L *such that* $x = \sup(x_1, \dots, x_m)$.

For the proof, see §A.1.8.

5.3. Matroids and polytopes

The following theorem is due to Edmonds (2003) (written in 1970) and has been rediscovered by Gelfand *et al* (1987).

5.3.1. The greedy algorithm. — Let E be a finite set and let $w \colon \mathrm{E} \to \mathbf{R}$ be a function ("weight"). For every finite family $\mathrm{A} = (e_i)_{i \in \mathrm{I}}$ in E, one sets $w(\mathrm{A}) = \sum_{i \in \mathrm{I}} w(e_i)$ — this is the weight of A.

Let $\mathscr{I}$ be a subset of $\mathfrak{P}(\mathrm{E})$ which is nonempty and stable by inclusion (these are properties (I_1) and (I_2) of independent families in matroids). A sequence $(e_1, \dots, e_m)$ in E is *w-admissible* with respect to $\mathscr{I}$ if it satisfies the following properties:

(A_1) The set $\{e_1, \dots, e_m\}$ belongs to $\mathscr{I}$ and has cardinality m;
(A_2) For every $n \in \{0, \dots, m-1\}$, one has

$$w(e_{n+1}) = \inf_{\substack{e \notin \{e_1, \dots, e_n\} \\ \{e_1, \dots, e_{n-1}, e\} \in \mathscr{I}}} w(e).$$

(A_3) For every $e \in \mathrm{E} - \{e_1, \dots, e_m\}$, one has $\{e_1, \dots, e_m, e\} \notin \mathscr{I}$.

Lemma **(5.3.2)**. — *Let* M *be a set, let* $w \colon \mathrm{M} \to \mathbf{R}$ *be a function and let* $\mathscr{I}$ *be a subset of* $\mathfrak{P}(\mathrm{M})$ *satisfying the properties* (I_1) *and* (I_2).

a) *There are admissible sequences.*

b) *Let* $(e_1, \dots, e_n)$ *be an admissible sequence in* M. *One has* $w(e_1) \leqslant \dots \leqslant w(e_n)$ *and the set* $\{e_1, \dots, e_n\}$ *is a maximal element of* M.

Proof. — One constructs admissible sequences by induction: starting from a sequence $(e_1, \dots, e_m)$ satisfying properties (A1) and (A2), Let E′ be the set of all elements $e \in \mathrm{E} - \{e_1, \dots, e_m\}$ such that $\{e_1, \dots, e_m, e\} \in \mathscr{I}$ is nonempty, then one chooses for e_{m+1} an element e of that set such that $w(e)$ is minimal.

Let then $(e_1, \dots, e_n)$ be an admissible sequence. For every integer m such that $1 \leqslant m < n$, the set $\{e_1, \dots, e_{m-1}, e_{m+1}\}$ belongs to $\mathscr{I}$; by property (A2), this implies that $w(e_{m+1}) \geqslant w(e_m)$. Consequently, one has $w(e_1) \leqslant \dots \leqslant w(e_n)$.

By property (A3), the set $\{e_1, \dots, e_n\}$ is a maximal element of $\mathscr{I}$. □

Theorem **(5.3.3)**. — *Let* M *be a matroid and let* $w : \mathrm{M} \to \mathbf{R}$ *be a function. For every subset* A *of* M, *set* $w(\mathrm{A}) = \sum_{a \in \mathrm{A}} w(a)$. *Let* $(e_1, \dots, e_n)$ *be a sequence of distinct elements in* M *such that* $w(e_1) \leqslant \dots \leqslant w(e_n)$ *and* $\{e_1, \dots, e_n\}$ *is a basis of* M.

Then $(e_1, \dots, e_n)$ *is w-admissible with respect to the set* $\mathscr{I}_{\mathrm{M}}$ *of independent subsets of* M *if and only if* $\sum_{j=1}^{n} w(e_j) \leqslant w(\mathrm{B})$ *for every basis* B *of* M.

In other words, up to ordering, the greedy algorithm constructing admissible sequences produces exactly all bases B (enumerated by increasing weights) such that $w(\mathrm{B})$ is minimal. The terminology "greedy" refers to the fact that at each step of the construction, one chooses an element of minimal weight that can be added to the current sequence while keeping it independent.

Proof. — Let B′ be a basis of M. Since $\{e_1, \dots, e_n\}$ is a basis of M, one has $\mathrm{Card}(\mathrm{B}') = n$. Enumerate the elements of B′ as $(f_1, \dots, f_n)$ in such a way that $w(f_1) \leqslant \dots \leqslant w(f_n)$. To prove that $w(\mathrm{B}) \leqslant w(\mathrm{B}')$, it suffices to prove that $w(e_m) \leqslant w(f_m)$ for every $m \in \{1, \dots, n\}$.

We argue by induction on m. Let $m \in \{1, \dots, n\}$ be such that $w(e_i) \leqslant w(f_i)$ for all $i < m$ and let us prove that $w(e_m) \leqslant w(f_m)$. Let $\mathrm{A} = \{e_1, \dots, e_{m-1}\}$ and $\mathrm{B} = \{f_1, \dots, f_m\}$; these are free subsets of M. By the property (I3) of independent subsets, there exists $j \in \{1, \dots, m\}$ such that $f_j \notin \mathrm{A}$ and $\mathrm{A} \cup \{f_j\} \in \mathscr{I}_{\mathrm{M}}$. By definition of an admissible sequence, one has $w(f_j) \geqslant w(e_m)$. Then, $w(f_m) \geqslant w(f_j) \geqslant w(e_m)$, as was to be shown. □

Remark **(5.3.4)**. — Let M be a set, let $\mathscr{I}$ be a subset of $\mathfrak{P}(\mathrm{M})$ satisfying (I_1) and (I_2). Assume that for every function $w : \mathrm{E} \to \mathbf{R}$, the greedy algorithm only produces maximal subsets A of $\mathscr{I}$ which make $w(\mathrm{A})$ minimal. Then $\mathscr{I}$ is the set of independent subsets of a matroid on M.

By assumption, the set $\mathscr{I}$ satisfies the axioms (I_1) and (I_2) of independent subsets, so that we just have to prove axiom (I_3). Let us argue by contradiction, considering sets $\mathrm{A}, \mathrm{B} \in \mathscr{I}$ such that $\mathrm{Card}(\mathrm{A}) < \mathrm{Card}(\mathrm{B})$ and such that $\mathrm{A} \cup \{e\} \notin \mathscr{I}$ for every $e \in \mathrm{B} - \mathrm{A}$.

Let α and β be real numbers such that $0 < \alpha < \beta < 1$; let us define a weight w on M by setting $w(e) = -1$ for $e \in \mathrm{A} \cap \mathrm{B}$, $w(e) = -\alpha$ for $e \in \mathrm{A} - \mathrm{B}$, $w(e) = -\beta$ for $e \in \mathrm{B} - \mathrm{A}$ and $w(e) = 0$ otherwise. By definition, the greedy algorithm will first select the elements of $\mathrm{A} \cap \mathrm{B}$, and then the elements of $\mathrm{A} - \mathrm{B}$; at that point, it has selected all of A, because $\mathrm{A} \in \mathscr{I}$. Then, the algorithm will select elements of $\mathrm{B} - \mathrm{A}$, if possible, but it can't select any since we have assumed that $\mathrm{A} \cup \{e\} \notin \mathscr{I}$ for all $e \in \mathrm{B} - \mathrm{A}$. So, it will select a subset A' of $\mathrm{M} - (\mathrm{A} \cup \mathrm{B})$ such that $\mathrm{A} \cup \mathrm{A}'$ is a maximal element of $\mathscr{I}$; its weight is equal to

$$w(\mathrm{A}) + w(\mathrm{A}') = w(\mathrm{A}) = -\mathrm{Card}(\mathrm{A} \cap \mathrm{B}) - \alpha\,\mathrm{Card}(\mathrm{A} - \mathrm{B}).$$

On the other hand, let B' be a subset of $\mathrm{M} - \mathrm{B}$ such that $\mathrm{B} \cup \mathrm{B}'$ is a maximal element of $\mathscr{I}$; its weight is

$$w(\mathrm{B}) + w(\mathrm{B}') = -\mathrm{Card}(\mathrm{A} \cap \mathrm{B}) - \beta\,\mathrm{Card}(\mathrm{B} - \mathrm{A}) - \mathrm{Card}(\mathrm{A} \cap \mathrm{B}').$$

By the hypothesis about the outcome of the greedy algorithm, we obtain

$$\mathrm{Card}(\mathrm{A} \cap \mathrm{B}') + \beta\,\mathrm{Card}(\mathrm{B} - \mathrm{A}) \leqslant \alpha\,\mathrm{Card}(\mathrm{A} - \mathrm{B}).$$

If α and β are close to 0, this implies $\mathrm{A} \cap \mathrm{B}' = \varnothing$. We then get

$$\beta\,\mathrm{Card}(\mathrm{B} - \mathrm{A}) \leqslant \alpha\,\mathrm{Card}(\mathrm{A} - \mathrm{B}),$$

which is impossible if β is close to α, since the inequality $\mathrm{Card}(\mathrm{A}) < \mathrm{Card}(\mathrm{B})$ implies that $\mathrm{Card}(\mathrm{B} - \mathrm{A}) > \mathrm{Card}(\mathrm{A} - \mathrm{A})$.

Example **(5.3.5)**. — Let M be the matroid associated with a finite graph G; the underlying set of this matroid is the set of edges of G. Let $w\colon \mathrm{M} \to \mathbf{R}$ be a weight function. The application of the greedy algorithm to this matroid furnishes a maximal independent subset of M, and one recovers Kruskal's algorithm for obtaining a maximal forest of G.

5.3.6. — Let M be a matroid and let $w \in \mathbf{R}^{\mathrm{M}}$. We associate with w an increasing filtration $(\mathrm{F}_t\mathrm{M})_{t\in\mathbf{R}}$ of the matroid $\mathbf{R}$, indexed by the real numbers, where, for every $t \in \mathbf{R}$, $\mathrm{F}_t\mathrm{M} = \{e \in \mathrm{M}\,;\, w_e \leqslant t\}$.

The filtration is separated (one has $\mathrm{F}_t\mathrm{M} = \varnothing$ when $t \to -\infty$) and exhaustive (one has $\mathrm{F}_t\mathrm{M} = \mathrm{M}$ for $t \to +\infty$). Conversely, such a filtration $(\mathrm{F}_t\mathrm{M})$ is defined by a unique weight function $w \in \mathbf{R}^{\mathrm{M}}$: it is defined by $w_e = t$ if and only if $e \in \mathrm{F}_t\mathrm{M}$ and $e \notin \mathrm{F}_s\mathrm{M}$ for $s < t$.

One says that the filtration $(\mathrm{F}_t\mathrm{M})$ of the matroid M is *flat* if $\mathrm{F}_t\mathrm{M}$ is a flat of M, for every $t \in \mathbf{R}$.

Corollary **(5.3.7)**. — *Let* M *be a matroid, let* $w \in \mathbf{R}^{\mathrm{M}}$ *and let* $(\mathrm{F}_t\mathrm{M})$ *be the real filtration of* M *associated with* w. *A basis* B *of* M *is minimal with respect to* w *if and only if one has* $\mathrm{Card}(\mathrm{B} \cap \mathrm{F}_t\mathrm{M}) = \mathrm{rank}_{\mathrm{M}}(\mathrm{F}_t\mathrm{M})$ *for every* $t \in \mathbf{R}$.

Proof. — The bases which are produced by the greedy algorithm are exactly those bases B for which, for every $t \in \mathbf{R}$, $\mathrm{B} \cap \mathrm{F}_t\mathrm{M}$ is a maximal independent subset of $\mathrm{F}_t\mathrm{M}$, that is, a basis of $\mathrm{F}_t\mathrm{M}$. ☐

5.3.8. — Let M be a matroid. Let $(e_i)_{i\in\mathrm{M}}$ be the canonical basis of the vector space $\mathbf{R}^{\mathrm{M}}$. For every subset A of M, let $e_{\mathrm{A}} = \sum_{i\in\mathrm{A}} e_i$. Following Gelfand, Goresky, MacPherson & Serganova (1987), we associate with the matroid M the *polytope* P_{M} which is the convex hull of the vectors e_{B}, for all bases B of M.

It is a subset of $[0;1]^{\mathrm{M}}$. Moreover, for every $x \in \mathrm{P}_{\mathrm{M}}$, one has $\sum_{i\in\mathrm{M}} x_i = \mathrm{rank}(\mathrm{M})$.

Theorem **(5.3.9)**. — a) *Let* M *be a matroid. For every edge* $[x;y]$ *of the matroid polytope* P_{M}, *there exist* $i, j \in \mathrm{M}$ *such that* $y - x = e_j - e_i$.

b) *Conversely, let* M *be a finite set, let* $P \in [0;1]^M$ *be a nonempty polytope and let* r *be an integer such that* $\sum_{i\in M} x_i = r$ *for every* $x \in P$. *We make the following assumptions:*

(i) *The vertices of* P *are of the form* $\sum_{i\in A} e_i$, *for some subset* A *of* M.

(ii) *For every edge* $[x;y]$ *of* P, *there exist* $i, j \in M$ *such that* $y - x = e_j - e_i$.

Then there exists a unique structure of a matroid on M *such that* $P = P_M$.

Proof. — a) The vertices of P_M are vectors e_B, where B is a basis of M, hence edges of P_M are segments of the form $[e_B; e_{B'}]$. If $\mathrm{Card}(B \cap B') = \mathrm{Card}(B) - 1$, then let $i, j \in M$ be such that $B - B' = \{i\}$ and $B' - B = \{j\}$; one has $e_{B'} - e_B = e_j - e_i$, as claimed. Let us now assume that $\mathrm{Card}(B \cap B') < \mathrm{Card}(B) - 1$ and let us prove that $[e_B; e_{B'}]$ is not an edge of P_M. Let $p = \mathrm{Card}(B) - \mathrm{Card}(B \cap B')$. Let us construct sequences (i_k) in $B - B'$, and sequences (j_k) in $B' - B$ as follows:

– We fix any element $i_1 \in B - B'$;

– Assume that $i_1, \dots, i_k$ and $j_1, \dots, j_{k-1}$ are defined. Applying axiom (B_2) of bases, let $j_k \in B' - B$ be such that $(B - \{i_k\}) \cup \{j_k\}$ is a basis of M;

– Assume that $i_1, \dots, i_k$ and $j_1, \dots, j_k$ are defined. Applying axiom (B_2) of bases, let $i_{k+1} \in B - B'$ be such that $(B' - \{j_k\}) \cup \{i_{k+1}\}$ is a basis of M.

For every integer $k \geqslant 1$, we let $B_k = (B - \{i_k\}) \cup \{j_k\}$ and $B'_k = (B' - \{j_k\}) \cup \{i_{k+1}\}$. These are bases of M; one has $B_k \neq B$ and $B'_k \neq B'$ for all k. Since B and B′ differ by at least two elements, one also has $B_k \neq B'$ and $B'_k \neq B$ for all k. Also note that one has

$$e_{B_k} + e_{B'_k} = e_B + e_{B'} - e_{i_k} + e_{j_k} - e_{j_k} + e_{i_{k+1}} = e_B + e_{B'} - e_{i_k} + e_{i_{k+1}}$$

while

$$e_{B_{k+1}} + e_{B'_k} = e_B + e_{B'} - e_{i_{k+1}} + e_{j_{k+1}} - e_{j_k} + e_{i_{k+1}} = e_B + e_{B'} + e_{j_{k+1}} - e_{j_k}.$$

Since $B - B'$ is finite, there exists a smallest integer k such that either

– The elements $i_1, \dots, i_k$ are pairwise distinct, the elements $j_1, \dots, j_{k-1}$ are pairwise distinct, but $j_k \in \{j_1, \dots, j_{k-1}\}$; item The elements $i_1, \dots, i_k$ are pairwise distinct, the elements $j_1, \dots, j_k$ are pairwise distinct, but $i_{k+1} \in \{i_1, \dots, i_k\}$.

In the first case, let $m \in \{1, \dots, k-1\}$ be such that $j_k = j_m$; one has

$$\sum_{s=m}^{k-1} (e_{\mathrm{B}_{s+1}} + e_{\mathrm{B}'_s}) = \sum_{s=m}^{k} (e_{\mathrm{B}} + e_{\mathrm{B}'} + e_{j_{s+1}} - e_{j_s}) = (k-m)(e_{\mathrm{B}} + e_{\mathrm{B}'}).$$

Since $k > m$, this proves that the midpoint of the segment $[e_{\mathrm{B}}; e_{\mathrm{B}'}]$ belongs to the convex hull of the vectors e_{C}, for all bases C of M which are distinct from B and B′. In particular, $[e_{\mathrm{B}}; e_{\mathrm{B}'}]$ is not an edge of P_{M}. In the second case, let $m \in \{1, \dots, k\}$ such that $i_{k+1} = i_m$. Then

$$\sum_{s=m}^{k} (e_{\mathrm{B}_s} + e_{\mathrm{B}'_s}) = \sum_{s=m}^{k} (e_{\mathrm{B}} + e_{\mathrm{B}'} + e_{i_{s+1}} - e_{i_s}) = (k+1-m)(e_{\mathrm{B}} + e_{\mathrm{B}'}).$$

Since $m \leqslant k$, we obtain as precedently that $[e_{\mathrm{B}}; e_{\mathrm{B}'}]$ is not an edge of P_{M}.

b) Consider an integer r and a polytope P as in the statement. Let $\mathscr{B}$ be the set of all subsets B of M such that $e_{\mathrm{B}} = \sum_{i \in \mathrm{B}} e_i$ is a vertex of P. By property (i), P is the convex hull of the vectors e_{B}, for $\mathrm{B} \in \mathscr{B}$. Since P is nonempty, $\mathscr{B}$ is nonempty as well. To prove that P is the matroid polytope for a unique structure of matroid on M, it suffices to prove that the echange property (B_2) holds.

Let B, B′ be two elements of $\mathscr{B}$ and let $a \in \mathrm{B} - \mathrm{B}'$. Let $\mathrm{B}_1, \dots, \mathrm{B}_m$ be the distinct elements of B such that the edges of P out of e_{B} are $[e_{\mathrm{B}}; e_{\mathrm{B}_j}]$, for $1 \leqslant j \leqslant m$. By assumption, $e_{\mathrm{B}_i} - e_{\mathrm{B}}$ is of the form $e_{b_i} - e_{a_i}$, for $a_i, b_i \in \mathrm{M}$ and $a_i \neq b_i$. Rewriting this equality as $e_{\mathrm{B}_i} + e_{a_i} = e_{\mathrm{B}} + e_{b_i}$, we deduce that $a_i \in \mathrm{B} - \mathrm{B}_i$, $b_i \in \mathrm{B}_i - \mathrm{B}$ and $\mathrm{B}_i \cup \{a_i\} = \mathrm{B} \cup \{b_i\}$. Since P is contained in the cone with apex P and rays the vectors $e_{\mathrm{B}_i} - e_{\mathrm{B}}$[1], there exists a family $(c_1, \dots, c_m)$ of positive real numbers such that

$$e_{\mathrm{B}'} - e_{\mathrm{B}} = \sum_{i=1}^{m} c_i (e_{\mathrm{B}_i} - e_{\mathrm{B}}) = \sum_{i=1}^{m} c_i e_{b_i} - \sum_{i=1}^{m} c_i e_{a_i}.$$

By assumption, $a \in \mathrm{B} - \mathrm{B}'$, so that the coefficient of e_a on the left hand side is equal to -1. No b_i is equal to a, hence the coefficient of e_a on the right hand side is equal to $-\sum_{a_i = a} c_i$. In particular, there exists $j \in \{1, \dots, m\}$ such that $a_j = a$ and $c_j > 0$. Let $b = b_j$. By construction, $\mathrm{B}_j = (\mathrm{B} - \{a\}) \cup \{b\}$ is an element of $\mathscr{B}$. It remains to prove that $b \in \mathrm{B}' - \mathrm{B}$. We know that $b \notin \mathrm{B}$. For every i, one has $a_i \in \mathrm{B}$ and $b \notin \mathrm{B}$,

[1]This is visually obvious, but it needs an explanation.

hence $a_i \neq b$; consequently, the coefficient of e_b on the right hand side is equal to $\sum_{b_i=b} c_i \geqslant c_j > 0$. Consequently, it is strictly positive in the left hand side as well, which implies that $b \in B'$. This establishes the property (B_2) and concludes the proof that $\mathscr{B}$ is the set of bases for a matroid structure on M. By construction, $P = P_M$. □

Proposition **(5.3.10)**. — *Let* M *be a matroid and let* $w : M \to \mathbf{R}$ *be a function. The set of all* w*-minimal bases of* M *is the set of bases of a matroid* M_w *on* M. *The associated polytope* P_{M_w} *is a face of the polytope* P_M.

Proof. — Let c_w be the minimal weight of a basis of M. By construction, the polytope P_M is contained in the half-space defined by the inequality $\sum_{i\in M} x_i \geqslant c_w$. Consequently, the intersection of P_M with the hyperplane H_w with equation $\sum x_i = c_w$ is a face of P_M, and is the convex hull of the vectors e_B, for all w-minimal bases of M. Being edges of P_M, its edges are of the form $e_b - e_a$, for $a, b \in M$. By theorem 5.3.9, $P_M \cap H$ is a matroid polytope, which means that the set of w-minimal bases of M is the set of bases for some matroid structure M_w on M. □

Remark **(5.3.11)**. — One can also check directly the axiom (B_2) of bases, but this requires a strengthening of that axiom that we prove in proposition A.1.5. Let B, B′ be w-minimal bases and let $a \in B - B'$; let $b \in B' - B$ such that $(B - \{a\}) \cup \{b\}$ and $(B' - \{b\}) \cup \{a\}$ are bases of M. By minimality of $w(B)$, one has $w(b) \geqslant w(a)$; by minimality of $w(B')$, one has $w(a) \geqslant w(b)$. Consequently, $w(a) = w(b)$ and these two constructed bases are w-minimal.

Theorem **(5.3.12)** (Ardila & Klivans, 2006). — *Let* M *be a matroid and let* $w \in \mathbf{R}^n$. *The matroid* M_w *has no loop if and only if the filtration* $(F_t M) = (F_0, \dots, F_{m+1})$ *of* M *associated with* w *is flat.*

5.4. Grassmann variety

5.4.1. — Let k be a field, let V be a finite dimensional k-vector space and let p be an integer such that $0 \leqslant p \leqslant \dim(V)$. The Grassmann variety of V, denoted by $G_p(V)$, is an algebraic variety over k that parameterizes p-dimensional vector subspaces of V. In fact, functorial considerations lead to the "correct" definition in Grothendieck & Dieudonné (1971) that parameterizes p-dimensional *quotients* of V. In the present context, that just amounts to replace V by its dual V^*.

In this section, we present the construction of this variety, a natural projective embedding (Plücker coordinates) and state some results about its tropicalization.

5.4.2. — Let $n = \dim(V)$ and let $(e_1, \dots, e_n)$ be a basis of V.

A p-dimensional vector subspace W of V is the image of an injective linear map from k^p to V, itself represented by a matrix $A \in M_{n,p}(k)$. Saying that A has rank p means that there exists a minor of size p which is invertible, in other words, a subset $I \subset \{1, \dots, n\}$ of cardinality p such that the determinant $\det(A_I)$ of the extracted matrix is nonzero. We can multiply A on the right by the inverse of the matrix A_I, this does not change the range of A and makes the extracted matrix $A_I = I_p$. Conversely, if I is fixed and W is a p-dimensional vector subspace of V, there exists exactly at most one matrix A such that $A_I = I_p$ and W = range(A), and this identifies the subset of all matrices $A \in M_{n,p}(k)$ such that $A_I = I_p$. with a subset U_I of $G_p(V)(k)$. When I varies along all cardinality-p subsets of $\{1, \dots, n\}$, the corresponding subsets U_I cover $G_p(V)(k)$.

A matrix A such that $A_I = I_p$ is determined by $(n-p)p$ coefficients: those $a_{i,j}$ with $i \in \{1, \dots, n\} - I$ and $j \in \{1, \dots, p\}$. This endowes U_I with the structure of an algebraic variety, namely the affine space $\mathbf{A}^{(n-p)p}$ of dimension $(n-p)p$.

Let us compare these structures. Let I and J be subsets of $\{1, \dots, n\}$ such that $\mathrm{Card}(I) = \mathrm{Card}(J) = p$. Let W be a p-dimensional subspace that belongs to $U_I(k) \cap U_J(k)$, and let $A \in M_{n,p}(k)$ be a matrix such that $W = \mathrm{range}(A)$. Then W is represented by the matrix $\varphi_I(A) = A(A_I)^{-1}$ in $U_I(k)$, and by the matrix $\varphi_J(A) = A(A_J)^{-1}$ in $U_J(k)$. One thus has $\varphi_I(A) = \varphi_J(A)\varphi_{JI}(A)$, where $\varphi_{JI}(A) = A_J(A_I)^{-1}$.

Let U_{IJ} be the principal open subset of U_I defined by the non-vanishing of the polynomial representing $\det(A_J)$, and define U_{JI} by symmetry. By Cramer's formulas, the matrix $A_I(A_J)^{-1}$ is represented by a rational function φ_{IJ} whose denominator is $\det(A_J)$, so has no poles on U_{IJ}, The map $A \to A\varphi_{\mathrm{IJ}}(A)$ induces an isomorphism of algebraic varieties from $\mathrm{U_{IJ}}$ to $\mathrm{U_{JI}}$; its inverse is given by $A \to A\varphi_{\mathrm{JI}}(A)$.

The cocycle relation holds: if K is a cardinality-p subset of $\{1,\dots,n\}$, one has $\varphi_{\mathrm{JK}}|_{\mathrm{U_{JK}}\cap\mathrm{U_{JI}}} \circ \varphi_{\mathrm{IJ}}|_{\mathrm{U_{IJ}}\cap\mathrm{U_{IK}}} = \varphi_{\mathrm{IK}}|_{\mathrm{U_{IJ}}\cap\mathrm{U_{IK}}}$. Indeed, both maps are given by right-multiplying a matrix A by the matrix representing $A_{\mathrm{I}}(A_{\mathrm{K}})^{-1}$.

By glueing, we obtain an algebraic variety $\mathrm{G}_{p,n}$ over k which we call the *Grassmann variety* of p-planes in k^n.

Proposition **(5.4.3)**. — *The Grassmann variety $\mathrm{G}_{p,n}$ is a proper, smooth and connected k-variety of dimension $(n-p)p$.*

Proof. — The smoothness and the dimension of $\mathrm{G}_{p,n}$ follows from its construction by glueing affine spaces of dimension $(n-p)p$. It is irreducible because the $\mathrm{U_I}$ are irreducible, and the open subschemes $\mathrm{U_{IJ}}$ are dense in them. To prove that it is proper, we check the valuative criterion of properness.

Let K be a field and let R be a valuation ring of K, and let $\mathrm{W} \in \mathrm{G}_{p,n}(\mathrm{K})$. We want to prove that the morphism $f\colon \mathrm{Spec}(\mathrm{K}) \to \mathrm{G}_{p,n}$ representing W extends uniquely to a morphism $\varphi\colon \mathrm{Spec}(\mathrm{R}) \to \mathrm{G}_{p,n}$.

Let $A \in \mathrm{M}_{n,p}(\mathrm{K})$ be a matrix such that $\mathrm{W} = \mathrm{range}(A)$. Among all subset I of $\{1,\dots,n\}$ such that $\mathrm{Card}(\mathrm{I}) = p$, let us choose one such that the valuation of $\det(A_{\mathrm{I}})$ is minimal. In particular, $\det(A_{\mathrm{I}}) \neq 0$. Multiplying A on the right by $(A_{\mathrm{I}})^{-1}$, replaces A_{I} by I_p. For every subset J of $\{1,\dots,n\}$ such that $\mathrm{Card}(\mathrm{J}) = p$, this replaces A_{J} by $A_{\mathrm{J}}(A_{\mathrm{I}})^{-1}$, and one has

$$\begin{aligned} v(\det(A_{\mathrm{J}}(A_{\mathrm{I}})^{-1})) &= v(\det(A_{\mathrm{J}})) - v(\det(A_{\mathrm{I}})) \\ &\geqslant 0 = v(I_p) = v(\det(A_{\mathrm{I}}(A_{\mathrm{I}})^{-1})), \end{aligned}$$

so that the minimal property still holds.

I then claim that $A \in \mathrm{M}_{n,p}(\mathrm{R})$. Let $i \in \{1, ;n\}$ and $j \in \{1,\dots,p\}$. If $i \in \mathrm{I}$, then $A_{i,j} \in \{0,1\}$. Otherwise, let us enumerate the elements of I in increasing order, say $i_1,\dots,i_p$, let us replace I by $\mathrm{I}' = \mathrm{I} - \{i_j\} \cup \{i\}$ and let us replaces the j-th row of A_{I} by the i-th row of A. One has $\det(A_{\mathrm{I}'}) = a_{i,j}$, hence $v(a_{i,j}) \geqslant 0$ and $a_{i,j} \in \mathrm{R}$.

The matrix A then defines a morphism φ from Spec(R) to U_I, hence to $G_{p,n}$.

On the other hand, let $\varphi' \colon \operatorname{Spec}(R) \to G_{p,n}$ be a morphism extending f. Let J be a cardinality-p subset of $\{1,\dots,n\}$ such that φ' maps the closed point s of Spec(R) into U_J. Since R is a local ring, Spec(R) factors through U_J, and there exists a matrix $A' \in M_{p,n}(R)$ describing φ'. Then A and A' define the same subspace of K^n, hence there exists a matrix $P \in M_p(K)$ such that $A' = AP$. One has $A'_I = A_I P = P$, hence $P \in M_p(R)$. Similarly, $A_J = A'_J P^{-1} = P^{-1}$, hence $P^{-1} \in M_p(R)$. Consequently, $P \in GL_p(R)$ and φ' factors through $U_I(R)$. Since U_I is affine, one has $\varphi = \varphi'$, hence the uniqueness of an extension of f. Let J be a cardinality-p subset of $\{1,\dots,n\}$ □

5.4.4. — It is important to understand the functoriality of this construction. Let thus V′ a finite dimensional k-vector space, let m be its dimension, let $(e'_1,\dots,e'_m)$ be a basis of V′, and let $f \colon V \to V'$ be a linear map. Let us show how f gives rise to a *rational map* from $G_{p,n}$ to $G_{p,m}$. Let (U'_J) be the family of Zariski open subschemes, isomorphic to affine spaces $\mathbf{A}^{(m-p)p}$.

Let $P \in M_{m,n}(k)$ be the matrix of f. Let W be a p-dimensional vector subspace of V. Then $f(W)$ is a vector subspace of V′, but its dimension is p if and only if $\operatorname{Ker}(f) \cap W = 0$. If a matrix $A \in M_{n,p}(k)$ represents W, namely, $W = \operatorname{range}(A)$, then $f(W) = \operatorname{range}(PA)$. The matrix PA has rank p if and only there exists a cardinality-p subset J of $\{1,\dots,m\}$ such that $\det((PA)_J) \neq 0$. For every cardinality-p subset I of $\{1,\dots,n\}$, the non-vanishing of this polynomial defines a principal open subscheme U_I^J of U_I, and multiplication by P defines a morphism of algebraic varieties $\varphi_I \colon U_I^J \to U'_J$.

Theses morphisms glue and define a morphism $\varphi \colon G'_{p,n} \to G_{p,m}$, defined on the Zariski open subscheme $G'_{p,n} = \bigcup_{I,J} U_I^J$ of $G_{p,n}$. If f is injective, then $G'_{p,n} = G_{p,n}$. Note that $G'_{p,n} = \varnothing$ if and only if $\operatorname{rank}(f) < p$; otherwise, $G'_{p,n}$ is dense.

If $f = \mathrm{id}_{k^n}$, then $G'_{p,n} = G_{p,n}$ and $\varphi = \mathrm{id}$.

Let $g \colon V' \to V''$ is a linear map to a finite dimensional k-vector space, with corresponding rational morphism $\gamma \colon G'_{p,m} \to G_{p,q}$, where $q =$

$\dim(\mathrm{V}'')$. Then $\varphi^{-1}(\mathrm{G}'_{p,m}) \cap \mathrm{G}'_{p,n}$ is an open subscheme of $\mathrm{G}_{p,n}$ on which one has $\gamma \circ \varphi$ is defined and corresponds to the morphism associated to $g \circ f$.

In particular, if f is an isomorphism, then φ is an isomorphism.

5.4.5. — The Grassmann variety admits a natural embedding into a projective space. For every matrix $A \in \mathrm{M}_{n,p}(k)$ of rank p, let $\pi'(A)$ be the family $(\det(A_{\mathrm{I}}))$, where I ranges over the cardinality-p subsets of $\{1, \dots, n\}$. This subset of $\mathfrak{P}(\{1, \dots, n\})$ has cardinality $\binom{n}{p}$; once it is ordered, we view the point $\pi'(A)$ as an element of $k^{\binom{n}{p}}$. The coordinates of this point are called the *Plücker coordinates* of range(A).

Let us show that these coordinates only depend on range(A) by multiplication by a common scalar. Since A has rank p, one of the minors of size p is invertible, hence $\pi'(A) \neq 0$. If A' is another matrix representing the same subspace of k^n, namely, range(A') = range(A), there exists $P \in \mathrm{GL}_n(k)$ such that $A' = AP$. Then $A'_{\mathrm{I}} = A_{\mathrm{I}}P$, hence $\det(A'_{\mathrm{I}}) = \det(A_{\mathrm{I}})\det(P)$ and $\pi'(A') = \pi'(A)\det(P)$.

This proves that the image $\pi(A)$ of $\pi'(A)$ in $\mathbf{P}_{\binom{n}{p}-1}(k)$ only depends on range(A), but not on the choice of A.

On a given affine chart U_{J} of $\mathrm{G}_{p,n}$, the map π' is obviously defined by polynomials, and the J-th coordinate of $\pi'(A)$ is equal to 1 for every matrix A in U_{J}. We thus have defined a morphism of algebraic varieties $\pi \colon \mathrm{G}_{p,n} \to \mathbf{P}_{\binom{n}{p}-1}$, the *Plücker embedding*, and the following proposition justifies the term "embedding".

Proposition **(5.4.6)**. — *The Plücker morphism* $\pi \colon \mathrm{G}_{p,n} \to \mathbf{P}_{\binom{n}{p}-1}$ *is a closed immersion.*

Proof. — We have seen that on the affine open subscheme U_{I} of $\mathrm{G}_{p,n}$, identified with the affine space $\mathbf{A}^{(n-p)p}$, the Plücker morphism is defined by a morphism $\pi' \colon \mathbf{A}^{(n-p)p} \to \mathbf{A}^{\binom{n}{p}} - \{0\}$, such that the I-th coordinate of $\pi'(A)$ is equal to 1 for every $A \in \mathrm{U}_{\mathrm{I}}$. To prove the proposition, it suffices to prove that $\pi'|_{\mathrm{U}_{\mathrm{I}}}$ defines a closed immersion from U_{I} to a closed subscheme of $\mathbf{A}^{\binom{n}{p}}$. In fact, as we already observed, one can recover a matrix $A \in \mathrm{U}_{\mathrm{I}}$ from $\pi(A)$. Let $(i_1, \dots, i_p)$ be the unique increasing sequence such that $I = \{i_1; \dots; i_p\}$. One already has $A_{i_k,j} = \delta_{j,k}$ for $j, k \in \{1; \dots; p\}$. Then, if $i \in \{1, \dots, n\} - I$ and $j \in \{1, \dots, p\}$, one has $a_{i,j} = \det(A_{I'}) = \pi'(A)_{I'}$, where $I' = I - \{i_j\} \cup \{i\}$, up to a sign, equal to the signature of the permutation of $(i_1; \dots; i_{j-1}; i; i_{j+1}; \dots; i_p)$ that reorders this sequence. This furnishes a morphism $\varphi_I \colon \mathbf{A}^{\binom{n}{p}} \to U_I$ which is a retraction of $\pi'|_{U_I}$. This implies the claim. □

Example **(5.4.7)**. — There are four trivial cases, two of them being even more trivial.

a) If $p = 0$, then $W = 0$ is the only zero-dimensional subspace of k^n. Since $\binom{n}{p} = 1$, there is only one Plücker coordinate, corresponding to the empty sequence, and it is equal to 1. One has $G_{0,n} = \operatorname{Spec}(k) = \mathbf{P}_0$, and the Plücker morphism is an isomorphism.

b) If $p = n$, then $W = V$ is the only n-dimensional subspace of k^n, so that $G_{n,n} = \operatorname{Spec}(k)$. One has $\binom{n}{p} = 1$ and the Plücker morphism is an isomorphism.

c) Assume that $p = 1$. A 1-dimensional subspace W of k^n is a line, generated by a vector $v \in k^n$. Its Plücker coordinates are the coordinates of v. Then $G_{1,n} = \mathbf{P}_{n-1}$ and the Plücker morphism is an isomorphism.

d) Assume finally that $p = n - 1$. A $(n-1)$-dimensional subspace W of k^n is a hyperplane, the kernel of a nonzero linear form $f \in V^*$. Write $f(x) = a_1x_1 + \cdots + a_nx_n$. Assume for simplicity that $a_n = 1$; then the family $(e_i - a_ie_n)_{i<n}$ is a basis of $\operatorname{Ker}(f)$. Its corresponding Plücker coordinates are $(-1)^{n+1}a_1, (-1)^n a_2, \dots, -a_{n-1}, 1$, so that we recover, up to signs, the coefficients of f. In this case, $G_{n-1,n}$ is the dual projective space of V and the Plücker morphism is an isomorphism.

5.4.8. — From now on, we assume that $2 \leqslant p \leqslant n-2$. Then, the homogeneous ideal of the image of $G_{p,n}$ in $\mathbf{P}_{(n-p)p}$ has a classical description: it is generated by a family of quadratic polynomials — the Grassmann relations. The following presentation is inspired by the treatement of Bourbaki (2007); the paper of Kleiman & Laksov (1972) provides a more elementary, "determinant-only", approach.

We first revisit the Plücker embedding in an intrinsic way. Recall that we have chosen a basis $(e_1, \dots, e_n)$ of V. It induces a basis $(e_I)_I$ of the p-th exterior power $\bigwedge^p V$ indexed by the set of all strictly increasing sequences $\mathrm{I} = (i_1;\dots;i_p)$ in $\{1,\dots,n\}$, were $e_{\mathrm{I}} = e_{i_1} \wedge \cdots \wedge e_{i_p}$. If W is a vector subspace of V of dimension p, with basis $(v_1,\dots,v_p)$, the Plücker coordinates of W are just the coordinates of the p-vector $v_1 \wedge \cdots \wedge v_p \in \bigwedge^p \mathrm{V}$ in the basis $(e_{\mathrm{I}})_{\mathrm{I}}$.

Recall that there exists a unique bilinear map $\bigwedge^p \mathrm{V}^* \times \bigwedge^p \mathrm{V} \to k$ which maps $(f_1 \wedge \cdots \wedge f_p, v_1 \wedge \cdots \wedge v_p)$ to $\det(f_i(v_j))$. It identifies each space to the dual of the other. Let $f \in \mathrm{V}^*$. The map $\varphi \mapsto \varphi \wedge f$ from $\bigwedge^{p-1} \mathrm{V}^*$ to $\bigwedge^p \mathrm{V}^*$ admits an adjoint, $\bigwedge^p \mathrm{V} \to \bigwedge^{p-1} \mathrm{V}$; the image of $\alpha \in \bigwedge^p \mathrm{V}$ is denoted by $f \lrcorner \alpha$. Explicitly, one has $\langle \varphi \wedge f, \alpha \rangle = \langle \varphi, f \lrcorner \alpha \rangle$ for every $\varphi \in \bigwedge^{p-1} \mathrm{V}^*$, every $f \in \mathrm{V}^*$ and every $\alpha \in \bigwedge^p \mathrm{V}$. The element $f \lrcorner \alpha$ is sometimes called the *interior product* of f and α. It can be described explicitly as follows. Let $(f_1,\dots,f_n)$ be the basis of V^* dual to the basis $(e_1,\dots,e_n)$ of V; for every sequence $\mathrm{J} = (j_1;\dots;j_m)$, let $f_{\mathrm{J}} = f_{j_1} \wedge \cdots \wedge f_{j_m}$; when J runs along all strictly increasing sequences of length m, the elements f_{J} form a basis of $\bigwedge^m \mathrm{V}^*$. If J and J′ have the same support, then $f_{\mathrm{J}'} = \varepsilon_{\mathrm{J}}^{\mathrm{J}'} f_{\mathrm{J}}$, where $\varepsilon_{\mathrm{J}}^{\mathrm{J}'}$ is the signature of the permutation that reorders J to J′.

Let $\mathrm{J} = (j_1,\dots,j_m)$, $\mathrm{K} = (k_1,\dots,k_q)$ and $\mathrm{I} = (i_1,\dots,i_p)$ be strictly increasing sequences with $p = m + q$. We set $\varepsilon_{\mathrm{JK}}^{\mathrm{I}} = 0$ if $\mathrm{I} \neq \mathrm{J} \cup \mathrm{K}$; otherwise, let $\varepsilon_{\mathrm{JK}}^{\mathrm{I}}$ be the signature of the permutation that maps to sequence (J, K) to I. Since

$$\langle f_{\mathrm{J}}, f_{\mathrm{K}} \lrcorner e_{\mathrm{I}} \rangle = \langle f_{\mathrm{J}} \wedge f_{\mathrm{K}}, e_{\mathrm{I}} \rangle = \varepsilon_{\mathrm{JK}}^{\mathrm{I}}$$

for all such I, J, K, it follows that

$$f_{\mathrm{K}} \lrcorner e_{\mathrm{I}} = \sum_{\mathrm{J}} \varepsilon_{\mathrm{JK}}^{\mathrm{I}} e_{\mathrm{J}}.$$

Proposition **(5.4.9)**. — *Let $\alpha \in \bigwedge^p \mathrm{V}$.*

a) *There exists a smallest vector subspace V_α in V such that $\alpha \in \bigwedge^p \mathrm{V}_\alpha$.*

b) *Its orthogonal $\mathrm{V}_\alpha^\perp$ in V^* is the set of $f \in \mathrm{V}^*$ such that $f \lrcorner \alpha = 0$.*

c) *It is the image of the linear map from $\bigwedge^{p-1} \mathrm{V}^*$ to V given by $\varphi \mapsto \varphi \lrcorner \alpha$.*

d) *If $\alpha = 0$, then $\mathrm{V}_\alpha = 0$. Otherwise, the following properties are equivalent: $\dim(\mathrm{V}_\alpha) = p$; one has $x \wedge \alpha = 0$ for every $x \in \mathrm{V}_\alpha$; one has $(\varphi \lrcorner \alpha) \wedge \alpha = 0$ for every $\varphi \in \bigwedge^{p-1} \mathrm{V}^*$.*

Proof. — a) Let W_1 and W_2 be subspaces of minimal dimension of V such that α belongs to $\bigwedge^p \mathrm{W}_1$ and to $\bigwedge^p \mathrm{W}_2$. Let $\mathrm{W} = \mathrm{W}_1 \cap \mathrm{W}_2$. Choose a basis of V containing of a basis of W, then extended so as to contain a basis of W_1 and a basis of W_2. In the basis of $\bigwedge^p V$ associated with this basis, we see that $\bigwedge^p W = \bigwedge^p W_1 \cap \bigwedge^p W_2$, hence $\alpha \in \bigwedge^p W$. By minimality of the dimensions of W_1 and W_2, we have $W_1 = W = W_2$. This implies the first assertion.

b) Let W be a complementary subspace to V_α. This gives a direct sum decomposition

$$\bigwedge^p V = \bigwedge^p V_\alpha \oplus W \otimes \bigwedge^{p-1} V_\alpha \oplus \cdots \oplus \bigwedge^p W.$$

For every $f \in V^*$ such that $V_\alpha \subset \operatorname{Ker}(f)$ and every $\varphi \in \bigwedge^{p-1} V^*$, one has $\langle \varphi, f \lrcorner \alpha \rangle = \langle \varphi \wedge f, \alpha \rangle = 0$. In other words, $V_\alpha^\perp$ is contained in the kernel L of the morphism $f \mapsto f \lrcorner \alpha$ from V^* to $\bigwedge^{p-1} V$.

Conversely, let $f \in V^*$ be such that $f|_{V_\alpha} \neq 0$. Choose a basis $(e_1, \dots, e_m)$ of V_α such that $f(e_1) = 1$ and $f(e_j) = 0$ for $j \geqslant 2$; extend this basis to a basis $(e_i)_{1 \leqslant i \leqslant n}$ of V, and let (f_i) be the dual basis, so that $f = f_1$. Write $\alpha = \sum_I a_I e_I$; one then has $f \lrcorner \alpha = \sum_I a_I f_1 \lrcorner e_I = \sum_J (-1)^{p-1} a_{(1,J)} e_J$. There exists J such that $a_{(1,J)} \neq 0$; otherwise, we would have $\alpha \in \bigwedge^p \langle e_2, \dots, e_m \rangle$, which contradicts the minimality of V_α. Consequently, $f \lrcorner \alpha \neq 0$.

c) Let us consider the linear mapping $\lambda : f \mapsto f \lrcorner \alpha$ from V^* to $\bigwedge^{p-1} V$. By *b*), its kernel is the orthogonal of V_α. The transpose of λ is a linear mapping μ from $\bigwedge^{p-1} V^*$ to V; by duality, $V_\alpha = \operatorname{range}(\mu)$. On the other hand, for every $f \in V^*$ and every $\varphi \in \bigwedge^{p-1} V^*$, one has the following equalities

$$\begin{aligned} \langle f, \mu(\varphi) \rangle = \langle \varphi, \lambda(f) \rangle = \langle \varphi, f \lrcorner \alpha \rangle = \langle \varphi \wedge f, \alpha \rangle \\ = (-1)^{p-1} \langle f \wedge \varphi, \alpha \rangle = (-1)^{p-1} \langle f, \varphi \lrcorner \alpha \rangle, \end{aligned}$$

so that $\mu(\varphi) = (-1)^{p-1} \varphi \lrcorner \alpha$. This proves *c*).

d) Let $m = \dim(V_\alpha)$. If $m < p$, then $\bigwedge^p V_\alpha = 0$, hence $\alpha = 0$; in this case, one has $V_\alpha = 0$ and $m = 0$. Assume that $\alpha \neq 0$, so that $m \geqslant p$. If $m = p$, then $\bigwedge^{p+1} V_\alpha = 0$, hence $x \wedge \alpha = 0$ for every $x \in V_\alpha$. Finally assume that $m > p$. Then $\bigwedge^m V_\alpha \simeq k$, the bilinear form $\bigwedge^{m-p} V_\alpha \times \bigwedge^p V_\alpha \to \bigwedge^m V_\alpha$ is nondegenerate and identifies each space with the dual of the other. In particular, there exists $y \in \bigwedge^{m-p} V_\alpha$ such that $y \wedge \alpha \neq 0$. Since $m > p$, there exists $x \in V_\alpha$ such that $x \wedge \alpha \neq 0$.

This proves that $\dim(V_\alpha) = p$ if and only if $x \wedge \alpha = 0$ for every $x \in V_\alpha$. The final characterization then follows from c). □

5.4.10. — Let α be a nonzero element of $\bigwedge^p V$. When φ runs among a basis of $\bigwedge^{p-1} V^*$, the relations $(\varphi \lrcorner \alpha) \wedge \alpha = 0$ give a necessary and sufficient condition for the equality $\dim(V_\alpha) = p$. On the other hand, this equality is equivalent to the existence of vectors $x_1, \dots, x_p \in V$ such that $\alpha = x_1 \wedge \dots \wedge x_p$. This, written in a basis of V, is then equivalent to the fact that α belongs to the image of π'. In other words, we have given a family of $\binom{n}{p+1}\binom{n}{p-1}$ homogeneous quadratic equations for the image of the Plücker embedding.

Let $(e_1, \dots, e_n)$ be a basis of V, let $(f_1, \dots, f_n)$ be the dual basis of V^*. If J is a stricly increasing sequence of length $p-1$ in $\{1, \dots, n\}$ and $k \in \{1, \dots, n\}$, let $\varepsilon_{J,k}$ be 0 if k appears in J, and (-1) to the number of elements in J which are greater than k. Similarly, if K a stricly increasing sequence of length $p+1$ in $\{1, \dots, n\}$ and $k \in \{1, \dots, n\}$, let ε_K^k be 0 if k does not appear in K, and (-1) to the number of elements in K which are strictly smaller than k. One thus has $e_k \wedge e_J = \varepsilon_{J,k} e_{J \cup \{k\}}$ and $e_k \wedge e_{K - \{k\}} = \varepsilon_K^k e_K$.

Writing $\alpha = \sum_I a_I e_I$ and taking $\varphi = f_J$, Let $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_{p-1})$ be strictly increasing sequences, one has

$$f_J \lrcorner e_I = \sum_{k=1}^{n} \varepsilon_{Jk}^{I} e_k.$$

Write $\alpha = \sum_I a_I e_I$; one then has

$$f_J \lrcorner \alpha = \sum_{k=1}^{n} \varepsilon_{J,k} a_{Jk} e_k,$$

where by Jk, we denote the strictly increasing sequence with image $J \cup \{k\}$. Consequently,

$$(f_J \lrcorner \alpha) \wedge \alpha = \sum_I \sum_{k=1}^{n} \varepsilon_{J,k} a_{Jk} a_I e_k \wedge e_I.$$

Fix a stricly increasing sequence K in $\{1,\dots,n\}$ of length $p+1$. The coefficient of e_{K} in the preceding expression is equal to

$$\sum_{k\in \mathrm{K}\cap \complement \mathrm{J}} \varepsilon_{\mathrm{J},k}\varepsilon_{\mathrm{K}}^{k} a_{\mathrm{J}\cup\{k\}} a_{\mathrm{K}-\{k\}}.$$

For $k\in \mathrm{K}\cap\complement\mathrm{J}$, set $\varepsilon_{\mathrm{K},\mathrm{J}}^{k}=\varepsilon_{\mathrm{J},k}\varepsilon_{\mathrm{K}}^{k}$.

These expressions are called the Grassmann relations Their vanishing, when J and K run among the strictly increasing sequences of lengths $p-1$ and $p+1$ in $\{1,\dots,n\}$, define the image of the Plücker embedding in $\mathbf{P}_{\binom{n}{p}-1}$.

Theorem **(5.4.11)**. — *Index the homogeneous polynomial ring of* $\mathbf{P}_{\binom{n}{p}-1}$ *by the cardinality-p subsets of* $\{1,\dots,n\}$. *The ideal of* $\pi(\mathrm{G}_{p,n})$ *in* $\mathbf{P}_{\binom{n}{p}-1}$ *is generated by the* $\binom{n}{p-1}\binom{n}{p+1}$ *homogeneous quadratic polynomials*

$$\sum_{k\in \mathrm{K}\cap \complement \mathrm{J}} \varepsilon_{\mathrm{K},\mathrm{J}}^{k} \mathrm{T}_{\mathrm{J}\cup\{k\}} \mathrm{T}_{\mathrm{K}-\{k\}},$$

where J *and* K *run along the subsets of* $\{1,\dots,n\}$ *with cardinality* $p-1$ *and* $p+1$ *respectively.*

Proof. — Let I be the indicated ideal. The previous discussion shows that for every field k, a point $a\in \mathbf{P}_{\binom{n}{p}-1}(k)$ belongs to $\pi(\mathrm{G}_{p,n})$ if and only if it belongs to $\mathrm{V}(\mathrm{I})(k)$. This proves that the radical of I coincides with the ideal of $\pi(\mathrm{G}_{p,n})$.

Proving that I is actually a prime ideal is more difficult, and classically requires representation theory of the linear group, or at least the introduction of *Young tableaux*. I refer to (Fulton, 1997, §8.4) or to (Sturmfels, 2008, §3.1) for a Gröbner-oriented presentation. ☐

The Grassmann relation associated with a pair (J,K) is a sum of $\mathrm{Card}(\mathrm{K}\cap\complement\mathrm{J})$ quadratic monomials. Since $\mathrm{Card}(\mathrm{K})=\mathrm{Card}(\mathrm{J})+2$, one has $\mathrm{Card}(\mathrm{K}\cap\complement\mathrm{J})\geqslant 2$. If $\mathrm{Card}(\mathrm{K}\cap\complement\mathrm{J})=2$, then this relation is trivial.

Example **(5.4.12)**. — Let us take $p=2$ and $n=4$ — this is the first nontrivial case of a Grassmann variety. In this case, there is, up to sign, only one Grassmann relation, which in fact had been exhibited earlier by Plücker.

Assume $J = \{1\}$. If 1 is in K then, up to permutation, we may assume that $K = \{1, 2, 3\}$, and the Plücker relation for (J, K) vanishes. Otherwise, $K = \{2, 3, 4\}$, and the Plücker relation for (J, K) writes

$$T_{12}T_{34} - T_{13}T_{24} + T_{14}T_{23} = 0.$$

All other Grassmann relations are either zero, or equal to this one up to sign. We also observe that it is irreducible, for example because each indeterminate appears only once. This identifies the Grassmann variety $G_{2,4}$ with a hypersurface in $\mathbf{P}_5$.

Example **(5.4.13)**. — The three-term Plücker relation for $G_{2,4}$ can be generalized in all Grassmann varieties $G_{p,n}$ if $p \geqslant 2$.

Let S be a subset of $\{1, \dots, n\}$ of cardinality $p - 2$; let a, b, c, d be elements not in S, where $a < b < c < d$, and set $J = S \cup \{a\}$ and $K = S \cup \{b, c, d\}$. Up to sign, the associated Grassmann relation is

$$T_{abS}T_{cdS} - T_{acS}T_{bdS} + T_{adS}T_{bcS} = 0.$$

If $p = 2$, then these three-term Grassmann relations are the only (non zero) ones. However, if $n-1 > p > 2$, then there are pairs (J, K) of subsets of $[\![1, n]\!]$ with $\mathrm{Card}(J) = p - 1$, $\mathrm{Card}(K) = p + 1$ and $\mathrm{Card}(K \cap \complement J) > 3$.

5.5. Tropicalizing the Grassmannian manifold

I can't write this down in a correct order...

5.5.1. — Let n and p be integers such that $2 \leqslant p \leqslant n - 2$. The Plücker embedding π maps $G_{p,n}$ to $\mathbf{P}_{\binom{n}{p}-1}$; inside this projective space, we identify the open subscheme defined by the nonvanishing of the standard homogeneous coordinates with the torus $\mathbf{G}_{\mathrm{m}}{}^{\binom{n}{p}-1}$. Let $G'_{p,n}$ be the inverse image of this torus in $G_{p,n}$. This is the open subscheme of the Grassmann manifold $G_{p,n}$ that parameterizes p-spaces W of $V = k^n$ such that $W + \langle e_{i_1}, \dots, e_{i_{n-p}} \rangle = V$ for every strictly increasing sequence $(i_1, \dots, i_{n-p})$.

Although the homogeneous ideal of $\pi(G_{p,n})$ in $\mathbf{P}_{\binom{n}{p}-1}$ is not generated by the three-term Grassmann relations, this holds after intersecting with the torus.

Proposition **(5.5.2)**. — *Assume that* $2 \leqslant p \leqslant n-2$. *The ideal of* $k[(\mathrm{T}_{\mathrm{I}}^{\pm 1})_{\mathrm{I}}]$ *generated by the homogeneous ideal of* $\pi(\mathrm{G}_{p,n})$ *is generated by the three-term Grassmann relations.*

Proof. — A Grassmann relation is associated with a pair (J, K) of subsets of $[\![1, n]\!]$, where $\mathrm{Card}(\mathrm{J}) = p-1$ and $\mathrm{Card}(\mathrm{J}) = p+1$; it has $\mathrm{Card}(\mathrm{K} \cap \complement \mathrm{J})$ terms. We will show that those with more than 3 terms belon to the ideal generated by the three-term Grassmann relations, provided the indeterminates T_{I} are inverted.

To simplify notation, we write $(\mathrm{I}) = \mathrm{T}_{\mathrm{I}}$ for any subset I of $[\![1, n]\!]$; also, if k is an element, we write $(\mathrm{I}k)$ for $(\mathrm{I} \cup \{k\})$ and $(\mathrm{I} - k)$ for $(\mathrm{I} - \{k\})$.

Set $\mathrm{A} = \mathrm{K} \cap \complement \mathrm{J}$, $\mathrm{B} = \mathrm{J} \cap \complement \mathrm{K}$ and $\mathrm{S} = \mathrm{K} \cap \mathrm{J}$. One has $\mathrm{Card}(\mathrm{K}) = p+1 = \mathrm{Card}(\mathrm{A}) + \mathrm{Card}(\mathrm{S})$, $\mathrm{Card}(\mathrm{J}) = p-1 = \mathrm{Card}(\mathrm{B}) + \mathrm{Card}(\mathrm{S})$, hence $\mathrm{Card}(\mathrm{A}) = \mathrm{Card}(\mathrm{B}) + 2 = p+1-\mathrm{Card}(\mathrm{S})$.

For $k \in \mathrm{A}$, let $\varepsilon_k = (-1)^{\mathrm{Card}([1;k[\cap \mathrm{A})}(-1)^{\mathrm{Card}([1;k[\cap \mathrm{B})}$; the Grassmann relation for (K, J) is then written

$$\mathrm{R} = \mathrm{R}(\mathrm{A}, \mathrm{B}) = \sum_{k \in \mathrm{A}} \varepsilon_k (\mathrm{A} - k)(\mathrm{B}k).$$

Let $a = \inf(\mathrm{A})$, $a' = \sup(\mathrm{A})$ and $b = \sup(\mathrm{B})$. Let us modify K, J by removing a and adding b to A, so that $\mathrm{A}' = \mathrm{A} \cup \{b\} - \{a\}$; this replaces K with $\mathrm{K}' = \mathrm{K} \cup \{b\} - \{a\}$ and does not change J, nor S. The Grassmann relation for (K′, J) is then written

$$\mathrm{R}' = \mathrm{R}(\mathrm{A}', \mathrm{B}) = \sum_{k \in \mathrm{A} - a} \varepsilon'_k (\mathrm{A}b - ak)(\mathrm{B}k),$$

where ε'_k is $(-1)^{\mathrm{Card}([1;k[\cap \mathrm{A}')}(-1)^{\mathrm{Card}([1;k[\cap \mathrm{B})}$. Note that it has one less term than the relation R; by induction, one thus has,

$$(\mathrm{A}b - aa')(\mathrm{B}a') \equiv - \sum_{a<k<a'} \varepsilon'_{a'} \varepsilon'_k (\mathrm{A}b - ak)(\mathrm{B}k)$$

modulo the ideal generated by the three-term Grassmann relations. Consequently, modulo this ideal, one has the following congruences:

$$\begin{aligned}
(\mathrm{A}b - aa')\mathrm{R} &= \sum_{k\in \mathrm{A}} \varepsilon_k(\mathrm{A}-k)(\mathrm{B}k)(\mathrm{A}b-aa') \\
&= \sum_{k<a'} \varepsilon_k(\mathrm{A}-k)(\mathrm{B}k)(\mathrm{A}b-aa') + \varepsilon_{a'}(\mathrm{A}-a')(\mathrm{B}a')(\mathrm{A}b-aa') \\
&\equiv \sum_{k<a'} \varepsilon_k(\mathrm{A}-k)(\mathrm{B}k)(\mathrm{A}b-aa') \\
&\qquad - \varepsilon_{a'}(\mathrm{A}-a') \sum_{a<k<a'} \varepsilon'_{a'}\varepsilon'_k(\mathrm{A}b-ak)(\mathrm{B}k) \\
&\equiv \varepsilon_a(\mathrm{A}-a)(\mathrm{B}a)(\mathrm{A}b-aa') + \sum_{a<k<a'} (\mathrm{B}k)\varepsilon_k\mathrm{C}_k,
\end{aligned}$$

where we have set

$$\mathrm{C}_k = (\mathrm{A}-k)(\mathrm{A}b-aa') - \eta_{a'}\eta_k(\mathrm{A}b-ak)(\mathrm{A}-a'),$$

and $\eta_k = \varepsilon_k\varepsilon'_k$ for any $k \in \mathrm{A}$. Observe that $\varepsilon'_k = \varepsilon_k$ if $b > k$ and $\varepsilon'_k = -\varepsilon_k$ if $b < k$, so that $\eta_k = \varepsilon_k\varepsilon'_k = 1$ if $b > k$ and -1 otherwise.

For a given index k, the term C_k can be rewritten $(aa'\mathrm{T})(kb\mathrm{T}) - \eta_{a'}\eta_k(a'b\mathrm{T})(ak\mathrm{T})$, with $\mathrm{T} = \mathrm{A}-\{a, a', k\}$. let us now write the three-term relation associated with the pair $(\mathrm{T}\cup\{a, a', k\}, \mathrm{T}\cup\{k\})$. There are three terms $(aa'\mathrm{T})(kb\mathrm{T})$, $(a'b\mathrm{T})(ak\mathrm{T})$ and $(ab\mathrm{T})(ka'\mathrm{T})$, and we will prove that C_k is, up to a sign, equal to $(ab\mathrm{T})(a'k\mathrm{T})$. Note that $a < k < a'$, and there are four possibilities for the position of b with respect to these:

– If $a' < b$, then $\eta_k = \eta_{a'} = 1$, the three-term relation is $(ak\mathrm{T})(a'b\mathrm{T}) - (aa'\mathrm{T})(kb\mathrm{T}) + (ab\mathrm{T})(ka'\mathrm{T})$, so that $\mathrm{C}_k \equiv (ab\mathrm{T})(ka'\mathrm{T})$;

– If $k < b < a'$, then $\eta_k = 1$, $\eta_{a'} = -1$, the three-term relation is $(ak\mathrm{T})(a'b\mathrm{T}) - (ab\mathrm{T})(ka'\mathrm{T}) + (aa'\mathrm{T})(kb\mathrm{T})$, and $\mathrm{C}_k \equiv (ab\mathrm{T})(ka'\mathrm{T})$;

– If $a < b < k$, then $\eta_k = \eta_{a'} = -1$, the three-term relation is $(ab\mathrm{T})(a'k\mathrm{T}) - (ak\mathrm{T})(ba'\mathrm{T}) + (aa'\mathrm{T})(kb\mathrm{T})$, and $\mathrm{C}_k \equiv -(ab\mathrm{T})(a'k\mathrm{T})$.

– Finally, if $b < a$, then $\eta_k = \eta_{a'} = -1$, the three-term relation is $(ab\mathrm{T})(a'k\mathrm{T}) - (ak\mathrm{T})(ba'\mathrm{T}) + (aa'\mathrm{T})(kb\mathrm{T})$, and $\mathrm{C}_k \equiv -(ab\mathrm{T})(a'k\mathrm{T})$.

In other words, $\mathrm{C}_k \equiv \theta_k(ab\mathrm{T})(ka'\mathrm{T})$, where $\theta_k = 1$ if $k < b$, and $\theta_k = -1$ if $k > b$. With the above notation, $(ab\mathrm{T}) = (\mathrm{A}b-a'k)$ and $(a'k\mathrm{T}) = (\mathrm{A}-a)$, so that

$$(\mathrm{A}b - aa')\mathrm{R} \equiv (\mathrm{A}-a)(\mathrm{B}a)(\mathrm{A}b - aa') + \sum_{a<k<a'} (\mathrm{B}k)\varepsilon_k\theta_k(\mathrm{A}b - a'k)(\mathrm{A}-a)$$
$$\equiv (\mathrm{A}-a)\sum_{k<a'} \varepsilon_k\theta_k(\mathrm{B}k)(\mathrm{A}b - a'k).$$

Observe that $\varepsilon_k\theta_k = \varepsilon_k$ if $k < b$, and $-\varepsilon_k$ if $k > b$. Consequently, $\varepsilon_k\theta_k = -(-1)^{\mathrm{Card}((\mathrm{A}b - a')\cap[1,k[)}(-1)^{\mathrm{Card}(\mathrm{B}\cap[1,k[)}$. We thus recognize the opposite of the Grassmann relation for the pair $(\mathrm{A}b - a', \mathrm{B})$. This proves that $(\mathrm{A}b - aa')\mathrm{R}$ belongs to the ideal generated by the three-term Grassmann relations; since $(\mathrm{A}b - aa')$ is a monomial, the relation R also belongs to that ideal, as was to be shown. □

5.5.3. — The affine space k^n has an action of the group $(k^\times)^n$, acting diagonally: $(t_1,\dots,t_n)\cdot(x_1,\dots,x_n) = (t_1x_1,\dots,t_nx_n)$. In the language of schemes, this corresponds to an action of $\mathbf{G}_{\mathrm{m}}{}^n$ on $\mathbf{A}^n$.

This action gives rise to an action of $\mathbf{G}_{\mathrm{m}}{}^n$ on the Grassmann variety $\mathrm{G}_{p,n}$: for $t \in (k^\times)^n$, a point $w \in \mathrm{G}_{p,n}(k)$ corresponding to a subspace W, the point $t\cdot w$ is the subspace $t\cdot \mathrm{W}$ of V. By definition of a vector subspace, if $t = (u,\dots,u)$ is an element of the diagonal torus, then $t\cdot\mathrm{W} = \mathrm{W}$, so that the diagonal torus $\mathbf{G}_{\mathrm{m}}$ acts trivially.

Let $f\colon \mathbf{G}_{\mathrm{m}}{}^n \to \mathbf{G}_{\mathrm{m}}{}^{\binom{n}{p}}$ be the morphism of tori given by $f(t_1,\dots,t_n) = (\prod_{i\in\mathrm{I}} t_i)_{\mathrm{I}}$, where I ranges over the $\binom{n}{p}$ p-element subsets of $[\![1,n]\!]$. It follows from the definition of the Plücker coordinates that $\pi(t\cdot\mathrm{V}) = f(t)\cdot\pi(\mathrm{V})$. Let $\varphi\colon \mathbf{R}^n \to \mathbf{R}^{\binom{n}{p}}$ be the corresponding linear map, given by $f(t_1,\dots,t_n) = (\sum_{i\in\mathrm{I}} x_i)_{\mathrm{I}}$.

Proposition **(5.5.4)**. — *The tropical variety of $\mathrm{G}'_{p,n}$ is a purely $(n-p)p$-dimensional polyhedral subspace of $\mathbf{R}^{\binom{n}{p}}/\mathbf{R1}$. Its lineality space is $\varphi(\mathbf{R}^n)/\mathbf{R1}$; it has dimension $n-1$.*

Example **(5.5.5)**. — The case of the Grassmann variety $\mathrm{G}_{2,4}$ is particularly simple. Indeed $\pi(\mathrm{G}_{2,4})$ is the hypersurface with equation $\mathrm{T}_{12}\mathrm{T}_{34} - \mathrm{T}_{13}\mathrm{T}_{24} + \mathrm{T}_{14}\mathrm{T}_{23}$. The action of $\mathbf{G}_{\mathrm{m}}{}^4$ on $\mathbf{A}^6$ is given by $(t_1,t_2,t_3,t_4)\cdot(x_{12},\dots,x_{34}) = (t_1t_2x_{12},\dots,t_3t_4x_{34})$.

This implies that the tropical variety of $\pi(G_{2,4})$ in $\mathbf{R}^6/\mathbf{R1}$ admits a lineality space containing the vectors $(1,1,1,0,0,0)$, $(1,0,0,1,1,0)$, $(0,1,0,1,0,1)$ and $(0,0,1,0,1,1)$. (Their sum is $2(1,\dots,1) = 2\cdot\mathbf{1}$.)

If we quotient $\mathbf{R}^6$ by this 4-dimensional vector space, we obtain $\mathbf{R}^2$. This corresponds to consider the subspaces $W = \mathrm{range}(A)$ of V such that only 2 Plücker coordinates are different from 1. If we let $(t,1,1,1)$ act, we can assume that $x_{12} = 1$; letting $(1,1,1,t)$ act, we also assume that $x_{34} = 1$. We then let $(t,t^{-1},1,1)$ act; this does not change x_{12} and x_{34} but multiplies x_{13} by t and x_{23} by t^{-1}; in this sort, we reduce to the case where $x_{13} = 1$. Finally, we let (t,t^{-1},t^{-1},t) act. This does not change x_{12}, x_{34}, x_{13} but multiplies x_{23} by t^{-2}. This allows to assume that $x_{23} = 1$ (over an algebraically closed field). Then, only two coordinates remain, x_{14} and x_{24}, and one has $f = 1 - T_{24} + T_{14}$. The tropicalization of f is a tropical line.

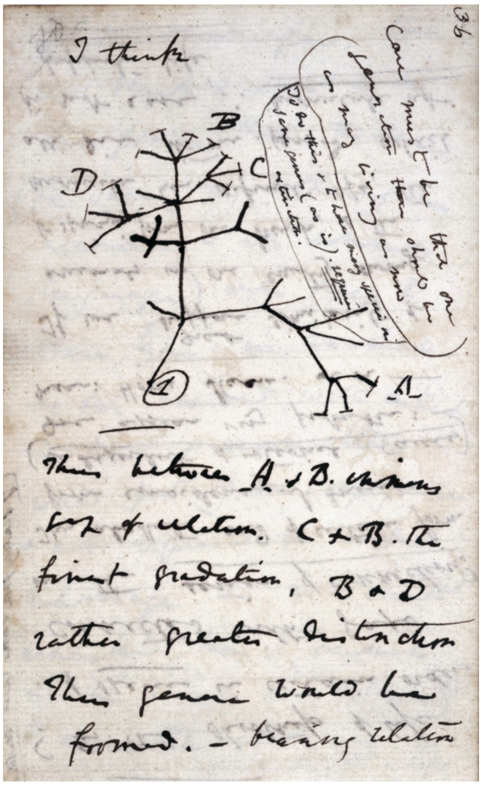
It follows that the tropicalization of $G'_{2,4}$ is the union of 3 cones of dimension 4.

5.5.6. — For $p = 2$ and $n \geqslant 4$ arbitrary, the tropicalization of $G'_{p,n}$ has a combinatorial description as the space of *phylogenetic trees*, which is the kind of datum evolutionary biologists use to represent the evolution of a species (a virus, say) along time. This model originates from a passage of Charles Darwin's book *On the Origin of Species* (1859):

> The affinities of all the beings of the same class have sometimes been represented by a great tree. I believe this simile largely speaks the truth.

Figure 1 can be found on a notebook of Darwin dated 1837, it is the first sketch of such a tree; figure 2 is the phylogenetic tree of the Sars-CoV2 (Severe Acute Respiratory Syndrome COronaVirus 2) that is presently (Spring 2020) confining at home half of the humankind in all continents.

5.5.7. — Let n be an integer such that $n \geqslant 1$; the *bouquet* with n *stems* is the quotient B_n of $\{1,\dots,n\}\times[0;1]$ by the finest equivalence relation for which all points $(i,0)$ are identified, for $i \in \{1,\dots,n\}$. We write $[i,t]$ for the image of the point $(i,t) \in \{1,\dots,n\}\times[0;1]$. This bouquet is endowed with the distance such that $d([i,t],[j,s]) = t+s$ if $i \neq j$, and



The handwritten text reads:

"I think case must be that one generation should have as many living as now. To do this and to have as many species in same genus (as is) requires extinction . Thus between A + B the immense gap of relation. C + B the finest gradation. B+D rather greater distinction. Thus genera would be formed. Bearing relation [to ancient types with several extinct forms]"

FIGURE 1. Evolutionary tree, Charles Darwin, 1837. (Picture taken from Wikipedia, *Tree of life*).

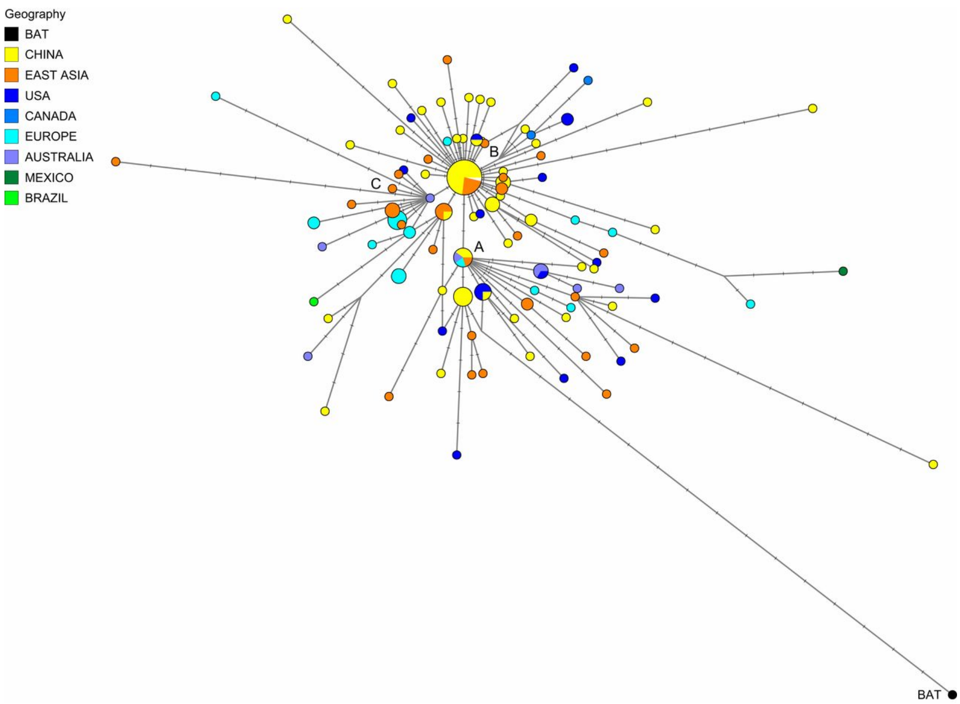


FIGURE 2. This picture, taken from FORSTER *ET AL* (2020), represent the phylogenetic tree of 160 SARS-CoV2 genomes. The initial "bat" virus is on the bottom right.

$d([i,t],[i,s]) = |t-s|$. The point $b = [i,0]$ is called the *base-point* of the bouquet; the points $[i,1]$ are called its *leaves*.

Let $x \in B_n$ and let V be a connected neighborhood of x in B_n. If x is a leaf of B_n, then $V - \{x\}$ is connected; if x is the base of B_n, then $V - \{x\}$ has exactly n connected components; otherwise, $V - \{b\}$ has exactly 2 connected components.

A compact metric space is called a *metrized graph.* if for every point $p \in T$, there exists an integer $n \geqslant 1$, a neighborhood V of p and an isometry from V to a neighborhood of b in the bouquet B_n that maps p to b. The integer n is called the *degree* of the point p.

All but finitely many points of a metrized graph have degree 2. Points of degree $\neq 2$ are called vertices; vertices of valency 1 are called leaves.

A bouquet with n stems is a metrized graph. If $n = 1$ or $n = 2$, then it has 2 leaves, and no other vertex. If $n \geqslant 3$, then it has n leaves and the base is its other vertex, with degree n.

A *metrized tree* is a metrized graph G which is simply connected (in particular, connected). Equivalently, for every two points p, q of G, there exists a unique continuous map $f : [0; d(p,q)] \to G$ such that $d(p, f(t) = t$ and $d(q, f(t)) = d(p,q) - t$ for every $t \in [0; d(p,q)]$. This map is called the *geodesic* linking p to q.

A metrized tree has at least two leaves.

The following lemma characterizes the restriction to the leaves of the distance function of a metrized tree.

Lemma **(5.5.8)**. — a) *Let* T *be a metrized tree and let* A *be its set of leaves. For every* $x, y, z, t \in A$, *the supremum of the three real numbers*

$$d(x,y) + d(z,t), \quad d(x,z) + d(y,t), \quad d(x,t) + d(y,z)$$

*is attained at least twice (*four-point condition*).*

b) *Conversely, let* A *be a finite set of cardinality at least 2 and let* $\delta : A \times A \to \mathbf{R}_+$ *be a distance satisfying the four-point condition. Then there exists a unique metrized tree* T *of which* A *is the set of leaves such that the distance of* T *coincides with the given distance on* A.

Distances on a finite set A satisfying this property are called *tree distances*.

Proof. — a) Let $x, y, z, t \in A$. The union of the geodesics linking x, y, z, t is a subtree of T with set of leaves A. Let u be the point

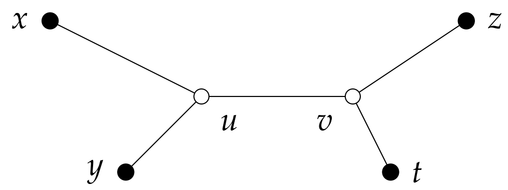


FIGURE 3. A representation of a metrized tree with four leaves x, y, z, t

closest to z on the geodesic $[x; y]$. This isolates the subtree of T with leaves x, y, z as a kind of bouquet with u for its only vertex of degree 3. Let then v be the point of T which is closest to t. There are three possibilities.

– Either $v \in [u; z]$ (as on the picture); in this case,

$$\begin{aligned} d(x,t) + d(y,z) &= d(x,u) + d(u,v) + d(v,t) + d(y,u) + d(u,v) + d(v,z) \\ &= d(x,z) + d(y,t), \end{aligned}$$

and

$$\begin{aligned} d(x,y) + d(z,t) &= d(x,u) + d(u,y) + d(y,v) + d(v,t) \\ &\leqslant d(x,t) + d(y,z). \end{aligned}$$

– If $v \in [u; x]$, one exchanges the roles of x and z, hence $d(z,t) + d(y,x) = d(x,z) + d(y,t) \geqslant d(z,y) + d(x,t)$.

– Finally, if $v \in [u; y]$, then one exchanges the roles of y and z, so that $d(x,t) + d(y,z) = d(y,z) + d(x,t) \geqslant d(x,z) + d(y,t)$.

b) We argue by induction on the cardinality of A. If $A = \{a, b\}$, then we take for tree T a segment of length $\delta(a,b)$ with endpoints a and b. If $A = \{a, b, c\}$ has three elements, the construction is similar: the tree T will be the union of three segments $[u; a]$, $[u; b]$, $[u; c]$ with lengths $\frac{1}{2}(\delta(a,c) + \delta(a,c) - \delta(b,c))$, $\frac{1}{2}(\delta(b,a) + \delta(b,c) - \delta(a,c))$ and $\frac{1}{2}(\delta(c,a) + \delta(c,b) - \delta(a,b))$. We now assume that $\mathrm{Card}(A) \geqslant 4$.

Let a, b, c be three points of A such that $\delta(a,c) + \delta(b,c) - \delta(a,b)$ is maximal and let $A' = A - \{a\}$. By induction, there exists a metrized tree T′ with leaves A′ that induces the given distance on A′. Set $u = \frac{1}{2}(\delta(a,b) + \delta(a,c) - \delta(b,c))$ and $v = \frac{1}{2}(\delta(a,b) + \delta(b,c) - \delta(a,c))$. The direct analysis predicts the common point p of the geodesics $[a;b]$, $[a;c]$ and $[b;c]$: it should be at distance u of a on $[a;c]$, and at distance v of b on $[b;c]$. Since d is a distance, one already has $u, v \geqslant 0$. Let $x \in \mathrm{A}' - \{b,c\}$ be one of the remaining leaves. By the choice of a, b, c, one has $\delta(a,c)+\delta(b,c)-\delta(a,b) \geqslant \delta(a,c)+\delta(x,c)-\delta(a,x)$ and $\delta(a,c)+\delta(b,c)-\delta(a,b) \geqslant \delta(x,c)+\delta(b,c)-\delta(x,b)$. Consequently, $\delta(b,c)+\delta(a,x)$ and $\delta(a,c)+\delta(x,b)$ are both greated than $\delta(x,c)+\delta(a,b)$, so that the hypothesis implies their equality:

$$\delta(b,c)+\delta(a,x) = \delta(a,c)+\delta(x,b) \geqslant \delta(x,c)+\delta(a,b).$$

In particular,

$$\begin{aligned}\delta(a,c)-u &= \delta(a,c) - \frac{1}{2}(\delta(a,b)+\delta(a,c)-\delta(b,c)) \\ &= \frac{1}{2}(\delta(b,c)+\delta(a,c)-\delta(a,b)) \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\delta(b,c)-v &= \delta(b,c) - \frac{1}{2}(\delta(a,b)+\delta(b,c)-\delta(a,c)) \\ &= \frac{1}{2}(\delta(a,c)+\delta(b,c)-\delta(a,b)) \geqslant 0.\end{aligned}$$

Let us thus extend the tree T' to a tree T by attaching a segment of length u at the point p which is at distance v of b on $[b;c]$, with other endpoint a.

It remains to check that this metrized tree satisfies the distance conditions for two points (x,y). This is obvious is $x=y=a$ and follows from the choice of the tree T' if neither x nor y is equal to a. Let us compute $d(a,x)$ for any $x \in \mathrm{A}'$. By construction, one has

$$\begin{aligned}d(a,c) &= d(a,p)+d(p,c) = u + (d(b,c)-d(b,p)) = u + \delta(b,c) - v \\ &= \frac{1}{2}(\delta(a,b)+\delta(a,c)-\delta(b,c)) + \delta(b,c) \\ &\qquad - \frac{1}{2}(\delta(a,b)+\delta(b,c)-\delta(a,c)) \\ &= \delta(a,c).\end{aligned}$$

Let then assume that $x \in \mathrm{A}' - \{b,c\}$. The common point q to the geodesics $[b;c]$, $[b;x]$ and $[x;c]$ is at distance $w = \frac{1}{2}(\delta(b,x)+\delta(b,c)-$

$\delta(x,c))$ of the point b. One has

$$\begin{aligned}2(w-v) &= (\delta(b,x)+\delta(b,c)-\delta(x,c)) - (\delta(a,b)+\delta(b,c)-\delta(a,c)) \\ &= \delta(b,x)-\delta(x,c)-\delta(a,b)+\delta(a,c) \\ &\geqslant 0,\end{aligned}$$

so that p belongs to the geodesic $[q;x]$. This implies that

$$\begin{aligned}d(a,x) &= d(a,p)+d(p,x) = u+(d(b,x)-d(b,p)) = u-v+\delta(b,x) \\ &= \frac{1}{2}(\delta(a,b)+\delta(a,c)-\delta(b,c)) \\ &\qquad -\frac{1}{2}(\delta(a,b)+\delta(b,c)-\delta(a,c))+\delta(b,x) \\ &= \delta(a,c)+\delta(b,x)-\delta(b,c) \\ &= \delta(a,x),\end{aligned}$$

as was to be shown. □

5.5.9. — One says that a metrized tree T is *equidistant* if there exists a point $p \in \mathrm{T}$ such that the distances $d(p,x)$ are equal, for all leaves x of T.

Corollary **(5.5.10)**. — a) *Let* T *be a metrized tree and let* A *be its set of leaves. The restriction to* A *of the distance of* T *is ultrametric if and only if the tree* T *is equidistant.*

b) *Let* A *be a finite set of cardinality* $\geqslant 2$ *and let* δ *be a ultrametric distance on* A*. There exists a unique metrized tree* T *with set of leaves* A *such that the distance of* T *induces on* A *the given distance. In particular,* T *is equidistant.*

Proof. — a) Let x,y,z be leaves of T and let a be the common point to the three geodesics $[x;y]$, $[x;z]$, $[y;z]$.

Let p be such a point. Assume, by symmetry, that $p \in [a;z]$. Then $d(x,y) = d(x,a)+d(y,a) \leqslant 2d(x,p)$. On the other hand, $p \in [x;z]$ and $p \in [y;z]$ so that $d(x,z) = d(x,p)+d(p,z) = 2d(x,p)$, and similarly, $d(y,z) = 2d(x,p)$. This proves that the restriction to A of the distance of T is ultrametric.

Conversely, assume that this distance is ultrametric. By symmetry, we may assume that $d(a,x) = d(a,y) \geqslant d(a,z)$. Let then p be the midpoint of the geodesic $[x;z]$; by the previous inequality, it is as well the midpoint of the geodesic $[y;z]$, and it is equidistant of x, y, z. This implies that the midpoints of all geodesics linking two leaves coincide, and the tree is equidistant.

b) It suffices to prove that an ultrametric distance on A satisfies the four-point condition. So let x, y, z, t be four elements of A. Up to permutation, we may assume that $d(x, y)$ is the smallest of the 6 mutual distances, and that $d(x, z) \leqslant d(x, t)$. By the ultrametric property, one then has $d(x, y) \leqslant d(z, x) = d(z, y)$ and $d(x, y) \leqslant d(t, x) = d(t, y)$. This already implies that $d(x, z) + d(y, t) = d(x, t) + d(y, z)$. Moreover, $d(z, t) \leqslant \sup(d(z, x), d(x, t)) = d(x, z)$; consequently, $d(x, z) + d(y, t) \geqslant d(z, t) + d(y, t) = d(z, t) + d(y, x)$. This establishes the four-point condition for the family (x, y, z, t). □

Theorem **(5.5.11)**. — *A point* $x = (x_{i,j})_{i<j} \in \mathbf{R}^{\binom{n}{2}}$ *belongs to the tropicalisation of* $G'_{2,n}$ *if and only if there exist a tree distance* δ *on* $[\![1, n]\!]$ *and a sequence* $(t_1, \dots, t_n)$ *of real numbers such that* $\delta(i, j) = -x_{ij} + t_i + t_j$ *for every* $i, j \in [\![1, n]\!]$ *such that* $i < j$.

Proof. — We write the pairs (i, j) of elements of $[\![1, n]\!]$ such that $i < j$ in lexographic order. In the proof, we identify a function $d\colon [\![1, n]\!]^2 \to \mathbf{R}$ which is symmetric and satisfies $d(i, i) = 0$ for all i with the element $(d(i, j))_{1\leqslant i<j\leqslant n}$ of $\mathbf{R}^{\binom{n}{2}}$.

Let $x \in \mathbf{R}^{\binom{n}{2}}$ be such that its image modulo $\mathbf{R1}$ belongs to the tropicalization of $G'_{2,n}$. Adding a multiple of the vector $\mathbf{1}$, we may assume that $-x_{ij} > 0$ for all i, j, and that $-x_{ij} - x_{jk} > -x_{ik}$ for all i, j, k. Then, $\delta(i, j) = -x_{ij}$ defines a distance on $[\![1, n]\!]$.

By assumption, there exists a nonarchimedean valued field K, a 2-dimensional subspace W of $V = K^n$ such that $\tau(\pi(W)) = x$. Let (p_{ij}) be the family of Plücker coordinates of W; one has $v(p_{ij}) = -x_{ij}$. Let $i, j, k, \ell \in [\![1, n]\!]$; the associated 3-term Grassmann relation writes

$$f = T_{ij}T_{k\ell} - T_{ik}T_{j\ell} + T_{i\ell}T_{jk} = 0.$$

The condition that $-x$ belongs to $\mathscr{T}_f$ is precisely equivalent to the four-point condition of lemma 5.5.8. Consequently, δ is a tree distance.

Conversely, let us consider a tree distance δ on $[\![1; n]\!]$. Let T be a metrized tree with set of leaves $A = [\![1, n]\!]$ and distance d that induces the tree distance δ on A. Let us choose a point $p \in \mathrm{T}$ which is not a leaf. Let R be a real number such that $\mathrm{R} > \sup_x d(p,x)$, the supremum being over the set of leaves. We then attach to each leaf x a segment of length $\mathrm{R} - d(p,x)$, so that the new tree is equidistant. The new distances satisfy $d'(x,y) = d(x,y) + (\mathrm{R} - d(p,x)) + (\mathrm{R} - d(p,y))$ for all leaves x, y. Since the lineality space of the tropicalization of $\mathrm{G}'_{2,n}$ contains the image of the linear map $\mathbf{R}^n \to \mathbf{R}^{\binom{n}{2}}$ given by $(t_i) \mapsto (t_i + t_j)_{i<j}$, the point $-d'$ belongs to the tropicalization of $\mathrm{G}'_{2,n}$ if and only if $-d$ belongs to it. This allows to assume that the restriction to A of the distance δ is ultrametric.

Let $\mathrm{R} = \sup_{x,y\in \mathrm{A}} \delta(x,y)$. By the ultrametric property, the relation "$\delta(x,y) < \mathrm{R}$" in A is an equivalence relation on A. If x, y are not equivalent, then $\delta(x,y) = \mathrm{R}$, so that there are at least two equivalence classes. By induction, for each equivalence class B, there exists elements $(t_x)_{x\in \mathrm{B}}$ in K such that $v(t_x - t_y) = -d(x,y)$ for $x, y \in \mathrm{B}$. Since the residue field of K is infinite and R is in the value group of K, there exists a family $(u_\mathrm{B})_\mathrm{B}$ of elements of K such that $v(u_\mathrm{B}) = v(u_\mathrm{B} - u_{\mathrm{B}'}) = \mathrm{R}$ for every equivalence classes B, B′. For an equivalence class B and $x \in \mathrm{B}$, we set $z_x = t_x + u_\mathrm{B}$. The plane W spanned by $(1, \dots, 1)$ and $(z_x)_{1\leqslant x\leqslant n}$ admits the family $(z_y - z_x)_{x,y}$ for Plücker coordinates. If x and y are equivalent, then $v(z_y - z_x) = v(t_y - t_x) = -d(x,y)$. Otherwise, the equivalence classes B of x and B′ of y are distinct, and $v(z_y - z_x) = v(t_y - t_x + u_\mathrm{B} - u_{\mathrm{B}'}) = \mathrm{R}$ since $v(u_\mathrm{B} - u_{\mathrm{B}'}) = \mathrm{R}$ and $v(t_y - t_x) > \mathrm{R}$. This proves that $v(\pi(\mathrm{W})) = -(\delta(i,j))_{i<j}$ and concludes the proof of the theorem. □

5.5.12. — The above discussion also furnishes a description of the tropicalization of $\mathrm{G}'_{2,n}$ as a union of some cones. For this, we analyse the combinatorial of metrized trees with n leafs, numbered $1, \dots, n$. To such a tree T, we attach a combinatorial tree, whose vertices are the vertices of T, and whose edges are those geodesics linking two vertices which do not contain another vertex. Each leaf of T is the endpoint of exactly one edge, those edges will be called terminal; the remaining edges are called inner edges; let E′ be the set of inner edges, E″ be the set of terminal edges, and $\mathrm{E} = \mathrm{E}' \cup \mathrm{E}''$. The only additional information which is be needed to reconstruct the metrized tree T is the family of lengths of its edges.

The case of *binary* trees will be important below, it is the case where every vertex of T has degree 1 or 2. For $n = 2$ or $n = 3$, there is only one combinatorial tree with n leaves, and it is binary. However, for $n = 4$, there are three binary such trees, the figure 4 only shows the one where the geodesics $[1;2]$ and $[3;4]$ are disjoint, and one non-binary tree.

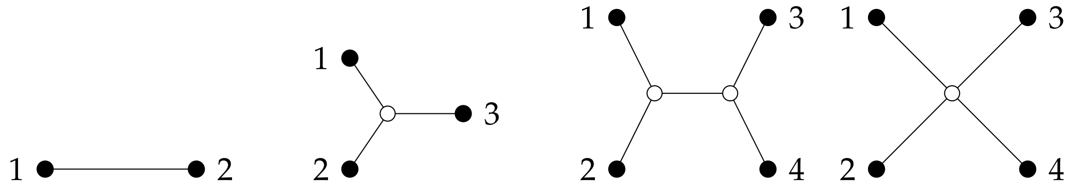


FIGURE 4. Metrized binary tree with two, three and four leaves, and a non-binary tree with four leaves.

5.5.13. — Let us fix a combinatorial tree T with n leaves $\{1,\dots,n\}$; let V be the set of its vertices and let E be the set of its edges, E′ be the set of inner edges and E″ be the set of terminal edges. For any family of real numbers $x = (x_e)_{e\in \mathrm{E}}$, let us define a function $d_{\mathrm{T}} \colon [\![1,n]\!]^2 \to \mathbf{R}$ as follows: for $a,b \in [\![1,n]\!]$, the geodesic $[a;b]$ is a union of some edges, and we let $d_{\mathrm{T},x}(a,b)$ be the sum of the corresponding real numbers.

If $x_e > 0$ for every edge e, then the family x induces a metric on the realization of the combinatorial tree, and its restriction to the leaves is given by $d_{\mathrm{T},x}$. The corresponding metrized tree T_x is thus the one associated with $d_{\mathrm{T},x}$ by lemma 5.5.8. If one lets the length x_e of an inner edge e, the corresponding edge collapses in the initial graph, two inner vertices being identified. In this way, we see that metrized trees with n leaves appear as limits of metrized binary trees with n leaves.

Since $n \geqslant 2$, we identify E″ with $[\![1,n]\!]$ — a terminal edge has only one endpoint which is a leaf. Let then $\varphi \colon \mathbf{R}^n \to \mathbf{R}^{\binom{n}{2}}$ be the linear map given by $\varphi(x_i) = (x_i + x_j)$; let L be its image.

Let also $\varphi_{\mathrm{T}} \colon \mathbf{R}^{\mathrm{E}'} \to \mathbf{R}^{\binom{n}{2}}$ be the map given by $x \mapsto (d_{\mathrm{T},x}(a,b))_{a<b}$. It is linear and the image of the polyhedral convex cone $\mathbf{R}_+^{\mathrm{E}'}$ Let $\mathrm{X}_{\mathrm{T}} = \mathbf{R}_+^{\mathrm{E}'} \times \mathbf{R}^{\mathrm{E}''} \subset \mathbf{R}^{\mathrm{E}}$ be the set of vectors x such that $x_e \geqslant 0$ for all $e \in \mathrm{E}'$; this is a polyhedral convex cone. Let $\mathrm{C}_{\mathrm{T}} \subset \mathbf{R}^{\binom{n}{2}}$ be the image of the polyhedral convex cone in $\mathbf{R}^{\mathrm{E}}$ consisting of all (x_e) such that $x_e \geqslant 0$ if $e \in \mathrm{E}'$. For $e \in \mathrm{E}'$, the image v_e of the vector $\mathbf{1}_e$ is as follows: if one deletes the edge e

from T, this disconnects the tree T into two disjoint connected trees, and leaves I_e, I'_e respectively, so that I_e and I'_e are disjoint, nonempty, and $[\![1, n]\!] = I_e \cup I'_e$. If $x, y \in I_e$ or $x, y \in I'_e$, then $d_{T,\mathbf{1}_e}(x, y) = 0$; otherwise, $d_{T,\mathbf{1}_e}(x, y) = 1$. These vectors v_e have nonnegative coordinates, and one of them is strictly positive. This proves that $C_T = \operatorname{cone}((v_e)_{e\in E'})$ is a polyhedral convex cone of dimension $\operatorname{Card}(E')$.

By theorem 5.5.11, the tropicalization $v(G'_{2,n})$ is equal to the union of all cones $C_T + L$. One has $\dim(C_T + L) = n + \operatorname{Card}(E') = \operatorname{Card}(E)$. It is also sufficient to consider only the cones C_T for those combinatorial trees which are binary.

5.5.14. — A combinatorial binary tree with n leaves is constructed by induction, from a combinatorial tree with $(n - 1)$ leaves by attaching the terminal edge of the nth leaf to one of the edges. This adds one vertex of degree 2 and one edge. By induction, we conclude that such a combinatorial tree has $n - 2$ non-leaf vertices and $2n - 3$ edges, n of them being terminal. Still by induction, this analysis also shows that the number of combinatorial binary trees with n leaves (up to isomorphism preserving the numbering of the leaves) is equal to: $1 \cdot 3 \dots (2n - 5)$.

Corollary **(5.5.15)**. — *The tropicalization of* $G'_{2,n}$ *in* $\mathbf{R}^{\binom{n}{2}}$ *is the union of* $(1 \cdot 3 \dots (2n - 5))$ *polyhedral convex cones of dimension* $(2n - 3)$ *with lineality space* L*, each of them generated by* $n-3$ *linearly independent vectors modulo* L.

5.6. Valuated matroids, tropical linear spaces

Definition **(5.6.1)**. — *Let* M *be a matroid on a set* E *and let* $\mathscr{B}_M$ *be its set of bases. An* absolute value p *on* M *is function* $p : \mathscr{B}_M \to \mathbf{R}_+^*$ *satisfying the following properties:*

(V$_1$) *If* B, B' *belong to* $\mathscr{B}_M$ *and* $x \in B - B'$*, there exists* $y \in B' - B$ *such that* $(B - \{x\}) \cup \{y\}$*, and* $(B' - \{y\}) \cup \{x\}$ *are bases of* M*, and*

$$p(B)p(B') \leqslant p((B - \{x\}) \cup \{y\})p((B' - \{y\}) \cup \{x\}).$$

There is a similar notion of *valuation* on a matroid, replacing the group $\mathbf{R}_+^*$ with a totally ordered abelian group Γ, and reversing the inequality. We shall only be interested here in the case where $\Gamma = \mathbf{R}$ and freely pass from a valued matroid (M, p) to a valuated matroid (M, v) by setting $v = -\log(p(B))$ and $p = e^{-v(B)}$ for every basis B of M.

Example **(5.6.2)**. — Let M be a matroid; set $p(B) = 1$ for every basis B of M. Then p is an absolute value on M. Indeed, axiom (V_1) is then the *strong exchange property* for bases, which is known to hold. We call it the trivial absolute value on M.

Example **(5.6.3)**. — Let K be a field endowed with a nonarchimedean absolute value. Let E be a finite set and let $(v_e)_{e\in E}$ be a family of vectors of K^n. Let M be the associated matroid on E: its independent subsets are the subsets F of E such that $(v_e)_{e\in F}$ is linearly independent. Let W be the space generated by the v_e, let m be its dimension and let ε be a basis of $\bigwedge^m W$. Let us also endow E with a total ordering.

Then to every subset F of E, one can attach the exterior product v_F of the v_e, for $e \in F$, written in increasing order. One has $v_F = 0$ if and only if F is dependent. If B is a basis of M, there exists a unique element $c_B \in K^\times$ such that $v_B = c_B\varepsilon$; set $p(B) = |c_B|$.

Let us show that p is an absolute value on the matroid M. Let B, B′ be bases of M and let $y \in B' - B$. The identity

$$\sum_{x\in B\cup\{y\}} \varepsilon_x c_{B\cup\{y\}-\{x\}} c_{B'-\{y\}\cup\{x\}} c_{B\cup\{y\}-\{x\}} c_{B'-\{y\}\cup\{x\}} = 0$$

is a rewriting of the Grassmann relation, ε_x being a sign depending on whether the number of elements of $B - B' \cup \{y\}$ is even or odd. The term of this identity corresponding to $x = y$ is $c_B c_{B'}$. The terms corresponding to $x \in B' \cap B$ and $x \neq y$ vanish, because $B' - \{y\} \cup \{x\}$ has cardinality $\mathrm{Card}(B') - 1$. Consequently, the ultrametric property of K implies that there exists $x \in B - B'$ such that

$$|c_{B\cup y-\{y\}} c_{B'-\{y\}\cup\{x\}}| \geqslant |c_B c_{B'}|.$$

Consequently, $p(B)p(B') \leqslant p(B \cup y - \{y\})p(B' - \{y\} \cup \{x\})$, as claimed.

Let (a_e) be a family in $K^\times$; for every e, set $v'_e = a_e v_e$. Then the family (v'_e) defines the same matroid M. Let also ε' be another basis of $\bigwedge^m W$; let $c \in K^\times$ be such that $\varepsilon = c\varepsilon'$. These choices give rise to another valuation p' on the matroid M, and one has

$$p'(\mathrm{B}) = |c|p(\mathrm{B})\prod_{e\in\mathrm{B}}|a_e|.$$

The end of the preceding example justifies the introduction of the following definition, whose proof is an immediate verification.

Lemma **(5.6.4)**. — *Let* M *be a matroid on a finite set* E *and let p be an absolute value on* E. *Let $c \in \mathbf{R}_+^*$ and let $(a_e)_{e\in\mathrm{E}}$ be a family of strictly positive real numbers. For every basis* B, *set $p'(\mathrm{B}) = cp(\mathrm{B})\prod_{e\in\mathrm{B}} a_e$. Then p' is an absolute value on* E.

Such absolute values p' are said to similar to p; if, moreover, $a_e = 1$ for all e, then one says that p and p' are equivalent. Equivalence and similarity of valuations are equivalence relations.

Definition **(5.6.5)**. — *Let* M *be a finite matroid. The* Dressian *of* M *is the subspace* Dr_M *of* $\mathbf{R}^{\mathscr{B}_\mathrm{M}}$ *consisting of families* $(v(\mathrm{B}))_{\mathrm{B}\in\mathscr{B}_\mathrm{M}}$ *for all valuations p on* M.

Proposition **(5.6.6)**. — *Let* M *be a finite matroid, let $p = \mathrm{rank}(\mathrm{M})$ and let $n = \mathrm{Card}(\mathscr{B}_\mathrm{M})$. The Dressian of a finite matroid* M *is the support of a fan in* $\mathbf{R}^{\mathscr{B}_\mathrm{M}}$ *(dimension ?). Its lineality space contains the image of the (not necessarily injective) linear map $f : \mathbf{R}^\mathrm{M} \to \mathbf{R}^{\mathscr{B}_\mathrm{M}}$ given by $(x_e) \mapsto (\sum_{e\in\mathrm{B}} x_e)_{\mathrm{B}\in\mathscr{B}_\mathrm{M}}$.*

Proof. — Let B, B' be bases of M and let $x \in \mathrm{B} - \mathrm{B}'$; for every $y \in \mathrm{B} - \mathrm{B}'$ such that $\mathrm{B} - \{x\} \cup \{y\}$ and $\mathrm{B}' - \{y\} \cup \{x\}$ are bases, let $\mathrm{C}^y_{\mathrm{B},\mathrm{B}',x}$ be the half-space of $\mathbf{R}^{\mathscr{B}_\mathrm{M}}$ defined by the inequality

$$v_\mathrm{B} + v_{\mathrm{B}'} \geqslant v_{\mathrm{B}-\{x\}\cup\{y\}} + v_{\mathrm{B}'-\{y\}\cup\{x\}}.$$

Otherwise, set $\mathrm{C}^y_{\mathrm{B},\mathrm{B}',x} = \varnothing$. By definition, the Dressian of M is the subset

$$\mathrm{Dr}_\mathrm{M} = \bigcap_{\mathrm{B},\mathrm{B}'\in\mathscr{B}_\mathrm{M}} \bigcap_{x\in\mathrm{B}-\mathrm{B}'} \bigcup_{y\in\mathrm{B}'-\mathrm{B}} \mathrm{C}^y_{\mathrm{B},\mathrm{B}',x}.$$

It is thus a polyhedral subspace of $\mathbf{R}^{\mathscr{B}_\mathrm{M}}$. Since it is also a cone, it is the support of a fan.

Moreover, if $f : \mathbf{R}^\mathrm{M} \to \mathbf{R}^{\mathscr{B}_\mathrm{M}}$ is the linear map $(a_e)_{e\in\mathrm{M}} \mapsto (\sum_{e\in\mathrm{B}} a_e)_\mathrm{B}$, then for every $x \in \mathbf{R}^{\mathscr{B}_\mathrm{M}}$, one has $x \in \mathrm{Dr}_\mathrm{M}$ if and only if $f(a) + x \in \mathrm{Dr}_\mathrm{M}$. In particular, the lineality space of Dr_M contains the image of f.

The rest, I don't know... □

5.6.7. — Let $G_M \subset G_{p,n}$ be the closed subscheme that parameterizes p-dimensional subspaces W of k^n such that the associated hyperplane arrangement is of type M. Its ideal I_M under the Plücker embedding is obtained by adding to the ideal I of $G_{p,n}$ the indeterminates T_B, where B is a p-element subset of M which is not a basis of M.

Let $G'_M \subset G_M$ be the open subscheme obtained by imposing the non-vanishing of the indeterminates T_B, when B is a basis of M. Taking valuations, the restriction of the Plücker embedding $\pi : G_M \to \mathbf{P}_{\binom{n}{p}-1}$ induces a continuous map $(G'_M)^{an} \to \mathbf{R}^{\mathscr{B}_M}/\mathbf{R1}$, $W \mapsto (v(p_B(W)))$. Its image is contained in the Dressian tropical variety of M.

On the other hand, we have seen that the ideal of the linear space W in k^n was generated the linear forms associated with the circuits of the matroid M, and that these linear forms even constituted a tropical basis of W. Let $(\pi_B(W))_B$ be a choice of Plücker coordinates. Let B be a basis of M and let $a \in M - B$; let C be the unique circuit of M such that $C \subset B \cup \{a\}$. The linear form associated with C can we written

$$f_C = \sum_{x \in B \cup \{a\}} \pi_{B \cup \{a\} - \{x\}}(W) T_k$$

(up to signs). Moreover, every circuit of M is obtained in this way. This justifies the following definition.

Definition **(5.6.8)**. — *Let* (M, v) *be a matroid of rank d endowed with a valuation. For every subset* K *of* M *such that* $\operatorname{rank}_M(K) = d$ *and* $\operatorname{Card}(K) = d + 1$, *let* τ_K *be the tropical polynomial on* $\mathbf{R}^M$ *defined by*

$$\tau_K(x) = \inf_{\substack{k \in K \\ K - \{k\} \in \mathscr{B}_M}} (v(K - \{k\}) + x_k).$$

The tropical linear space *defined by* (M, v) *is the intersection of the corresponding tropical hypersurfaces. It is denoted by* $L(M, v)$.

Explicitly, this means that a point $x \in \mathbf{R}^M$ belongs to the tropical for every such subset K, the set of $k \in K$ such that $\tau_K(x) = v(K - \{k\}) + x_k$ has cardinality at least 2.

In the case (M, v) is the valuated matroid associated with a linear subspace W of K^n, where K is an ultrametric valued field, the tropical linear space (M, v) coincides with the tropicalization of $W \cap \mathbf{G}_m{}^n$.

Lemma **(5.6.9)**. — *Let* (M, v) *be a valuated matroid and let* $x \in \mathbf{R}^M$. *The bases* B *of* M *such that* $v(B) - \sum_{e \in B} x_e$ *is minimal are the bases of a matroid on* M.

This matroid will be denoted by $(M, v)_x$.

Proof. — For every basis B of M, set $v_x(B) = v(B) - \sum_{e \in B} x_e$ and let $\mathscr{B}'$ be the set of bases of M that minimize v_x. We need prove that this set $\mathscr{B}'$ satisfies the axioms of bases of a matroid. It is nonempty, because $\mathscr{B}_M$ is not empty. Let then $B, B' \in \mathscr{B}'$ and let $b \in B' - B$. By the axiom (V_1) of valuated matroids, there exists $a \in B - B'$ such that $B - \{a\} \cup \{b\}$ and $B - \{b\} \cup \{a\}$ are bases of M and such that

$$v(B) + v(B') \geqslant v(B - \{a\} \cup \{b\}) + v(B - \{b\} \cup \{a\}).$$

Adding $-\sum_{e \in B} x_e - \sum_{e \in B'} x_e$ on both sides, we get

$$v_x(B) + v_x(B') \geqslant v_x(B - \{a\} \cup \{b\}) + v_x(B - \{b\} \cup \{a\}).$$

Since $v_x(B)$ and $v_x(B')$ are minimal, this implies that the preceding inequality is an equality and that both $B - \{a\} \cup \{b\}$ and $B - \{b\} \cup \{a\}$ belong to $\mathscr{B}'$. This establishes the exchange property (B_2) for the bases of a matroid, hence the lemma. □

Proposition **(5.6.10)**. — *Let* (M, v) *be a valuated matroid and let* $x \in \mathbf{R}^M$. *Then* x *belongs to the tropical linear space* $L(M, v)$ *if and only if the matroid* $(M, v)_x$ *has no loop.*

A *loop* of a matroid is an element e such that $\{e\}$ is dependent; equivalently, it is a circuit of cardinality 1.

Proof. — Assume that $x \in L(M, v)$ and let C be a circuit of $(M, v)_x$. Let $a \in C$; then $C - \{a\}$ is an independent subset of $(M, v)_x$, so that there exists a basis B of $(M, v)_x$ such that $C - \{a\} \subset B$. In particular, B is a basis of M and $C \subset B \cup \{a\}$; more precisely, C is the unique circuit which is contained in $B \cup \{a\}$. By assumption, x belongs to the tropical hypersurface defined by the tropical polynomial $\tau_{B \cup \{a\}}$. By the choice of B, one has $\tau_{B\cup\{a\}}(x) = v(B) + x_a$; let then $b \in B$ such that $B \cup \{a\} - \{b\}$ is a basis of M and $\tau_{B\cup\{a\}}(x) = v(B \cup \{a\} - \{b\}) + x_b$. In particular,

$$v_x(B \cup \{a\} - \{b\}) = v(B) + x_a - x_b = v_x(B),$$

so that $B \cup \{a\} - \{b\}$ is a basis of the matroid $(M, v)_x$. Since C is dependent, one has $C \not\subset B \cup \{a\} - \{b\}$; since $C \subset B \cup \{a\}$, this shows that $b \in C$. In particular, C contains the two distinct elements a, b, hence $\operatorname{Card}(C) \geqslant 2$ and $(M, v)_x$ has no loop.

Conversely, assume that $x \notin L(M, v)$. By the definition of $L(M, v)$, there exists a subset K of M such that $\operatorname{rank}_M(K) = p = \operatorname{Card}(K) - 1$ such that x does not belong to the tropical hypersurface defined by the tropical polynomial τ_K. There exists then $a \in K$ such that $K - \{a\}$ is a basis of M and such that $v(K - \{k\}) + x_k > v(K - \{a\}) + x_a$ for every $k \in K$ such that $K - \{k\}$ is a basis of M. Let us show that a is a loop of $(M, v)_x$. Otherwise, there would exist a basis B of $(M, v)_x$ such that $a \in B$. Let us apply the exchange property for the two bases $K - \{a\}$ and B, and the element a of B. There exists $k \in B$ such that $K - \{k\}$ and $B - \{a\} \cup \{k\}$ are bases of M and such that

$$v(K - \{a\}) + v(B) \geqslant v(K - \{k\}) + v(B - \{a\} \cup \{k\}).$$

Adding $-\sum_{e\in B} x_b$ on both sides, we get

$$v(K - \{a\}) + v_x(B) \geqslant v(K - \{k\}) + v_x(B - \{a\} \cup \{k\}) - x_a + x_k.$$

Since $v_x(B)$ is minimal, this implies $v(K - \{a\}) + x_a \geqslant v(K - \{k\} + x_k$ and contradicts the definition of a. Consequently, a is a loop of $(M, v)_x$. □

Theorem **(5.6.11)**. — *Let* (M, v) *be a valuated matroid.*

a) *The associated tropical linear space* $L(M, v)$ *is a polyhedral subspace of* $\mathbf{R}^M$ *of rank* $\operatorname{rank}_M(M)$.

b) *Its recession fan is equal to* $L(M)$*, where* M *is endowed with the trivial valuation. Its lineality space contains the vector* $\mathbf{1}$.

c) *For every* $x \in L(M, v)$*, one has* $\operatorname{Star}_x(L(M, v)) = L((M, v)_x)$*, where the matroid* $(M, v)_x$ *is endowed with the trivial valuation.*

Proof. — As an intersection of finitely many tropical hypersurfaces, $L(M, v)$ is a polyhedral subspace of $\mathbf{R}^M$. For every subset K of $\mathscr{B}_M$ such that $\operatorname{rank}_M(K) = \operatorname{rank}_M(M) = \operatorname{Card}(K) - 1$, every $x \in \mathbf{R}^M$ and every

$t \in \mathbf{R}$, one has $\tau_K(x + t\mathbf{1}) = \tau_K(x) + t$; this implies that the lineality space of $L(M, v)$ contains the line $\mathbf{R1}$.

Let $x \in L(M, v)$ and let $y \in \mathbf{R}^M$. For every positive real number ε and every basis B of M, one has

$$v_{x+\varepsilon y}(B) = v(B) - \sum_{j \in B}(x_j + \varepsilon y_j) = v_x(B) - \varepsilon \sum_{j \in B} y_j.$$

Let c be the minimum value of $v_x(B)$, when B runs over all bases of M, and let $\alpha > 0$ be such that $v_x(B) > c + \alpha$ if $v_x(B) \neq c$. Assume that $0 < \varepsilon < \alpha/2\|y\|$. If $v_x(B) \neq c$, then $v_{x+\varepsilon y}(B) > c + \frac{1}{2}\alpha$; on the other hand, if $v_x(B) = c$, then $v_{x+\varepsilon y}(B) \leqslant c - \varepsilon \sum j \in By_j < c + \frac{1}{2}\alpha$. This proves that B is a basis of $(M, v)_{x+\varepsilon y}$ if and only if B is a basis of $((M, v)_x)_y$, where $(M, v)_x$ is viewed as a trivially valued matroid. By definition of the link, one has $y \in \mathrm{Star}_x(L(M, v))$ if and only if $x + \varepsilon y \in L(M, v)$ for all small enough real numbers ε. This is equivalent to the fact that matroid $L(M, v)_{x+\varepsilon y}$ has no look, hence to the fact that the matroid $(L(M, v)_x)_y$ has no loop, hence to the fact that $y \in L((M, v)_x)$.

Since $(M, v)_x$ is a trivially valued matroid, the tropical linear space $L((M, v)_x)$ is its Bergman fan; it is purely of dimension $\mathrm{rank}_M(M)$.

The computation of the recession fan is supposed to be analogous but I dont't understand it. □

CHAPTER 6

TROPICAL INTERSECTIONS

6.1. Minkowski weights

All polyhedra are implicitly assumed to be rational.

6.1.1. — Let $L \simeq \mathbf{Z}^n$ be a free finitely generated $\mathbf{Z}$-module and let $V = L_{\mathbf{R}} \simeq \mathbf{R}^n$ be the associated $\mathbf{R}$-vector space.

Let p be an integer such that $0 \leqslant p \leqslant n$. We define as follows the group $F_p(V)$ of p-dimensional *weighted polyhedral subspaces* of V: it is generated by closed polyhedra of dimension $\leqslant p$ in V with the following relations:

(i) $[P] = 0$ for every polyhedron P such that $\dim(P) < p$;

(ii) $[P] + [P \cap H] = [P \cap V_+] + [P \cap V_-]$ whenever P is a p-dimensional polyhedron in V and V_+, V_- are half-spaces such that $V_+ \cap V_-$ is a hyperplane H and $V = V_+ \cup V_-$.

Note that this second relation is trivial when $P \subset H$; on the other hand, if $P \not\subset H$, then $\dim(P \cap H) < \dim(P) \leqslant p$, so that the first relation implies $[P \cap H] = 0$ and that second one relation can be rewritten as $[P] = [P \cap V_+] + [P \cap V_-]$.[1]

The submonoid of $F_p(V)$ generated by the classes [P] of polyhedral subspaces is denoted by $F_p^+(V)$. Its elements are said to be *effective*.

The group $F_0(V)$ identifies with $\mathbf{Z}^{(V)}$, the free abelian group on V. We denote by $\deg \colon F_0(V) \to \mathbf{Z}$ the unique morphism of groups such that $\deg([x]) = 1$ for every $x \in V$.

[1] Ajouter un dessin avec P, H, V_+, V_-.

6.1.2. — As for any group defined by generators and relations, one defines a morphism λ from $F_p(V)$ to a given abelian group A by prescribing $\lambda(P)$ for every polyhedron P of V such that $\dim(V) \leqslant p$ such that $\lambda(P) = 0$ if $\dim(P) < p$ and $\lambda(P) + \lambda(P \cap H) = \lambda(P \cap V_+) + \lambda(P \cap V_-)$ for every hyperplane H of V dividing V into two closed half-spaces V_+ and V_-.

The simplest example of such a morphism is given by the Lebesgue measure μ_W on a subspace W of V such that $\dim(W) = p$. Let indeed C be a compact polyhedron of W; for every polyhedron P of V such that $\dim(P) \leqslant p$, set $\lambda_C(P) = \mu_W(C \cap P)$. If $\dim(P) < p$, then $\dim(C \cap P) < p$ hence $\lambda_C(P) = 0$; on the other hand, if H is a hyperplane of V dividing V into two closed half-spaces V_+ and V_-, then the additivity of measure implies that $\lambda_C(P) + \lambda_C(P \cap H) = \lambda_C(P \cap V_+) + \lambda_C(P \cap V_-)$. Consequently, there exists a unique morphism of abelian groups $\lambda_C : F_p(V) \to \mathbf{R}$ such that $\lambda_C([P]) = \mu_W(P \cap C)$ for every closed polyhedron P of V such that $\dim(P) \leqslant p$.

Observe that $\lambda_C(S) \geqslant 0$ for every effective class $S \in F_p^+(V)$.

6.1.3. — Every closed polyhedral subspace P of V such that $\dim(P) \leqslant p$ has a class [P] in $F_p(V)$: it is the sum of all polyhedra of any polyhedral decomposition of V. This class is effective and vanishes if and only if $\dim(P) < p$.

For every element S of $F_p(V)$, there exists a polyhedral decomposition $\mathscr{C}$ of V and a family $(w_C)_{C \in \mathscr{C}_p}$, where $\mathscr{C}_p$ is the set of all polyhedra $C \in \mathscr{C}$ such that $\dim(C) = p$, such that

$$S = \sum_{C \in \mathscr{C}_p} w_C[C].$$

One then says that $\mathscr{C}$ is *adapted* to S.

Let K be a convex compact polyhedron of dimension p contained in a polyhedron $C \in \mathscr{C}_p$; then one has $\lambda_K(S) = w_C \lambda_K(C \cap K)$. This shows that the family (w_C) is uniquely determined by S and the given polyhedral decomposition. Moreover, S is effective if and only $w_C \geqslant 0$ for every $C \in \mathscr{C}_p$. The element w_C is called the *weight* of C in S.

More generally, if $S' = \sum_{C' \in \mathscr{C}'_p} w'_{C'}[C']$ is another class $S' \in F_p(V)$ adapted to a polyhedral decomposition $\mathscr{C}'$, then the equality $S = S'$

is equivalent to the equalities $w_C = w'_{C'}$ for every pair of polyhedra $(C, C') \in \mathscr{C}_p \times \mathscr{C}'_p$ such that $\dim(C \cap C') = p$.

The union of all polyhedra $C \in \mathscr{C}$ such that $w_C \neq 0$ is called the *support* of S, and is denoted by $|S|$. It is a polyhedral subspace of V, and is everywhere of dimension p.

One has $|S+S'| \subset |S| \cup |S|'$ and $|mS| = |S|$ for every non-zero integer m.

Let A be an abelian group. A similar definition allows to define the group $F_p(V; A)$ of polyhedra with coefficients in A.

6.1.4. — Let us recast the balancing condition in this context. Let $S \in F_p(V)$ be a weighted polyhedral subspace of dimension $\leqslant p$.

Let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S, and let $S = \sum_{C \in \mathscr{C}_p} w_C[C]$.

Let $D \in \mathscr{C}$ be a polyhedron of dimension $p-1$. Let $\mathscr{C}_D$ be the set of all polyhedra $C \in \mathscr{C}$ of which D is a face and such $\dim(C) = p$.

For every $C \in \mathscr{C}$, let V_C be the lineality space of $\langle C \rangle$; since the polyhedron C is rational, the intersection $L_C = V_C \cap L$ is a free finitely generated submodule of L of rank dim(C). For every $C \in \mathscr{C}_D$, there exists a vector vector $v_C \in L_C \cap C$ which generates the quotient abelian group L_C/L_D; such a vector is unique modulo L_D. We say that S satisfies the balancing condition along D if one has

$$\sum_{C \in \mathscr{C}_D} w_C v_C \in L_D.$$

We say that S is balanced (in dimension p) if it satisfies the balancing condition along all $(p-1)$-dimensional polyhedra of $\mathscr{C}$.

This condition is independent of the choice of the polyhedral decomposition which is adapted to S.

If $S, S' \in F_p(V)$ are balanced weighted polyhedral subspace, then so are $S + S'$ and mS, for every $m \in \mathbf{Z}$.

6.1.5. — Let $S \in F_p(V)$ and $x \in V$. One says that S is a fan with apex x if there exists a polyhedral decomposition of V adapted to S of which every polyhedron is a cone with apex x.

Let $S \in F_p(V)$ let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S; write $S = \sum w_C[C]$. Let $x \in V$ and let $\mathscr{C}_x$ be the set of polyhedra in $\mathscr{C}$ which contain x; their union is a neighborhood of x in V. For every $C \in \mathscr{C}_x$, let $\lambda_x(C) = \mathbf{R}_+(C - x)$ be the cone with apex x generated by C; the set of all $\lambda_x(C)$, for $C \in \mathscr{C}_x$ is a fan of V. Then $\lambda_x(S) = \sum_{C \in \mathscr{C}_x} w_C[\lambda_x(C)]$ is a fan with apex x.

Moreover, S satisfies the balancing condition along a polyhedron $D \in \mathscr{C}_x$ if and only if $\lambda_x(S)$ satisfies the balancing condition along $\lambda_x(D)$. In particular, if S is balanced, then so is $\lambda_x(S)$.

Definition **(6.1.6)**. — *A balanced p-dimensional weighted polyhedral subspace is called a p-dimensional* Minkowski weight*, or a p-dimensional* tropical cycle.

They form a subgroup $MW_p(V)$ of $F_p(V)$.
[2]

Example **(6.1.7)**. — Let K be a nonarchimedean valued field, let X be a subvariety of $\mathbf{G}_{m_K}^n$ and let $p = \dim(X)$. The tropicalization $\mathscr{T}_X$ of X is a polyhedral subspace of $\mathbf{R}^n$ of dimension p. There exists a polyhedral decomposition $\mathscr{C}$ of $\mathbf{R}^n$ such that the set $\mathscr{C}_X$ of all polyhedra in $\mathscr{C}$ that meet $\mathscr{T}_X$ is a polyhedral decomposition of $\mathscr{T}_X$. For $C \in \mathscr{C}_X$ with $\dim(C) = p$, we have defined a multiplicity $\operatorname{mult}_{\mathscr{T}_X}(C)$; Then $S = \sum_{C \in \mathscr{C}_X} \operatorname{mult}_{\mathscr{T}_X}(C)[C]$ is a weighted polyhedral subspace of V of dimension p with support $\mathscr{T}_X$. It satisfies the balancing condition, hence defines a Minkowski weight in $MW_p(\mathbf{R}^n)$. By abuse of language, this Minkowski weight is still denoted by $\mathscr{T}_X$.

Example **(6.1.8)**. — The Bergman fan $\Sigma(M)$ of a matroid, more generally, the tropical linear space associated with a valuated matroid, is the support of a Minkowski weight (all weights are equal to 1).

Example **(6.1.9)**. — Let $n = \dim(V)$; the class $[V] \in F_n(V)$ is balanced. The morphism $\mathbf{Z} \to MW_n(V)$ given by $a \mapsto a[V]$ is injective; let us show that it is an isomorphism

Let $S \in MW_n(V)$ and let $\mathscr{C}$ be polyhedral decomposition of V which is adapted to S; write $S = \sum_C w_C[C]$. Let $D \in \mathscr{C}$ be a polyhedron of dimension $n-1$. There are exactly two polyhedra $C, C' \in \mathscr{C}$ containing D

[2]Define $F_p(V; A)$ and $MW(V; A)$ for any abelian group A?

such that $\dim(C) = \dim(C') = n$: the affine space V_D generated by D is a hyperplane that delimits V in two half-spaces, one containing C, the other C′. The vectors v_C and $v_{C'}$ that appear in the formulation of the balancing condition can then be chosen opposite, hence $w_C = w_{C'}$.

Let then C, C' be arbitrary polyhedra of dimension n in $\mathscr{C}$. There exists a sequence $(C_0, \dots, C_m)$ of polyhedra in $\mathscr{C}$ such that $C_0 = C$, $C_m = C'$, and such that for each $k \in \{1, \dots, m\}$, C_{k-1} and C_k share a face of dimension $n-1$; By what precedes, one then has $w_{C_{k-1}} = w_{C_k}$. Consequently, $w_C = w_{C_0} = w_{C_1} = \dots = w_{C_m} = w_{C'}$. Let a be this common value.

Finally, one has $S = \sum_C a[C] = a[V]$.

Remark **(6.1.10)**. — One can amplify the previous example for Minkowski weights of arbitrary dimension. Let indeed $S \in F_p(V)$ be a weighted polyhedral subspace. The support of S, $|S|$, is a polyhedral subspace, and the weight of S can be viewed as a function from $|S|$ to $\mathbf{Z}$ which is defined and locally constant outside of a $(p-1)$-dimensional polyhedral subspace of $|S|$, the union of the polyhedra of dimension $< p$ contained in $|S|$ in a polyhedral decomposition of V which is adapted to S.

Let P be a polyhedron of dimension p which is contained in $|S|$ and such that $|S|$ is a submanifold at every point of $\mathring{P}$. In other words, $\mathring{P}$ is open in $|S|$.

If S is balanced, then its weight is constant on P.

Example **(6.1.11)**. — Let L, L' be free finitely generated abelian groups, let $V = L_{\mathbf{R}}$ and $V' = L'_{\mathbf{R}}$. There exists a unique bilinear map

$$F_p(V) \times F_q(V') \to F_{p+q}(V \times V')$$

such that $([C], [C']) \to [C \times C']$ for every p-dimensional polyhedron C in V and every q-dimensional polyhedron C′ in V′. If $S \in F_p(V)$ and $S' \in F_q(V')$ are weighted polyhedral subspaces, the image of (S, S') is denoted by $S \times S'$.

Choose polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ which are respectively adapted to S and S′. The family $(C \times C')$, for $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$, is a polyhedral decomposition which is adapted to $S \times S'$: one has

$$S \times S' = \sum_{C \in \mathscr{C}_p} \sum_{C' \in \mathscr{C}'_q} w_C w'_{C'} [C \times C']$$

if, for every (C, C'), w_C is the weight of C in S and $w'_{C'}$ is the weight of C′ in S′.

If S *and* S′ *are balanced, then so is* $S \times S'$. Indeed, let us consider a polyhedron E of dimension $p+q-1$ belonging to the polyhedral decomposition $\mathscr{C} \times \mathscr{C}'$. Let us write $E = D \times D'$, where $D \in \mathscr{C}$ and $D' \in \mathscr{C}'$.

Let $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$ be polyhedra such that E is a face of $C \times C'$. Then $D \subset C$ and $D' \subset C'$, so that D is a face of C and D′ is a face of C′. Since $\dim(D) + \dim(D') = \dim(C) + \dim(C') - 1$, there are two possibilities: either $\dim(D) = \dim(C) - 1$ and $D' = C'$, or $\dim(D') = \dim(C') - 1$ and $D = C$.

This already shows that the balancing condition along E is trivial if $\dim(D) \neq p$ and $\dim(D') \neq q$.

Let us now assume that $\dim(D) = p$ (hence $\dim(D') = q - 1$). By what precedes, the polyhedra of the form $C \times C'$, where $C \in \mathscr{C}_p$ and $C' \in \mathscr{C}'_q$ of which E is a face are of the form $D \times C'$, where $D' \subset C' \in \mathscr{C}'_q$. The balancing condition along E for $S \times S'$ follows from the balancing condition for S′ along D′.

Similarly, if $\dim(D') = q$ and $\dim(D) = p - 1$, then the balancing condition along E for $S \times S'$ follows from the balancing condition for S along D.

6.1.12. — A Minkowski weight is said to be *effective* if the corresponding weighted polyhedral subspace is effective. Effective Minkowski weights form a submonoid $MW_p^+(V)$ of $MW_p(V)$.

Proposition **(6.1.13)**. — *Every Minkowski weight is the difference of two effective Minkoswski weights.*

Proof. — Let $S \in MW_p(V)$ be a Minkowski weight and let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S; for $C \in \mathscr{C}_p$, let w_C be the weight of C in S. Let $\mathscr{N}$ be the set of all $C \in \mathscr{C}_p$ such that $w_C < 0$; for $C \in \mathscr{N}$, let $S_C = [\langle C \rangle]$ be the weighted polyhedral subspace associated with the affine space generated by C; it is balanced. Set $S' = \sum_{C\in\mathscr{N}}(-w_C)S_C$; is is an effective Minkowski weight. then one has

$$\begin{aligned} S + S' &= \sum_{C\in\mathscr{C}_p} w_C[C] + \sum_{C\in\mathscr{N}} w_C[\langle C\rangle] \\ &= \sum_{\substack{C\in\mathscr{C}_p \\ w_C>0}} w_C[C] + \sum_{C\in\mathscr{N}} (-w_C)\,([\langle C\rangle] - [C])\,. \end{aligned}$$

Since $C \subset \langle C\rangle$, the weighted polyhedral subspace $[\langle C\rangle]-[C]$ is effective. Consequently, $S+S'$ is effective; it is also balanced. Then $S = (S+S') - S'$ is the difference of two effective Minkowski weights, as was to be shown. ◻

6.2. Stable intersection

6.2.1. — Let L, L' be free finitely generated abelian groups, let $V = L_{\mathbf{R}}$, $V' = L'_{\mathbf{R}}$ and let $f\colon V \to V'$ be a linear map such that $f(L) \subset L'$.

There exists a unique linear map $f_*\colon F_p(V) \to F_p(V')$ satisfying the following properties, for every p-dimensional polyhedron C of V:

(i) If $\dim(f(C)) < p$, then $f_*([C]) = 0$;

(ii) If $\dim(f(C)) = p$, then $f(L_C)$ is subgroup of rank p of $L_{f(C)}$, so that the index $[L_{f(C)} : f(L_C)]$ is finite, and $f_*([C]) = [L_{f(C)} : f(L_C)]\,[C]$.

For every $S \in F_p(V)$, one has $|f_*(S)| \subset f(|S|)$.

Proposition **(6.2.2)**. — *If S is balanced, then $f_*(S)$ is balanced. In other words, one has $f_*(MW_p(V)) \subset MW_p(V')$.*

Proof. — Replacing V' be its image, we may assume that f is surjective. There is a polyhedral decomposition $\mathscr{C}$ of V such that the polyhedra $f(C)$, for $C \in \mathscr{C}$, form a polyhedral decomposition $\mathscr{C}'$ of V' (corollary 1.8.10).

Let D' be polyhedron of dimension $p-1$ in $\mathscr{C}'$. Let $\mathscr{C}_{D'}$ be the set of all polyhedra C' in $\mathscr{C}'$ such that $\dim(C') = p$ and $D' \subset C'$. For $C' \in \mathscr{C}_{D'}$, define $v_{C'/D'} \in L'_{C'}$ which generates $L'_{C'}/L'_{D'}$ and is such that $x + tv_{C'} \in C'$ for every $x \in \mathring{D}'$ and every small enough positive real number t.

Let $\mathscr{D}_{D'}$ be the set of all polyhedra D of dimension $p-1$ of $\mathscr{C}$ such that $f(D) = D'$. For every $D \in \mathscr{D}_{D'}$, let $\mathscr{C}_D$ be the set of all polyhedra $C \in \mathscr{C}$ such that $\dim(C) = p$ and $D \subset C$. For every $D \in \mathscr{D}_{D'}$ and every $C \in \mathscr{C}_D$, let $v_{C/D} \in L_C$ be a vector that maps to a generator of L_C/L_D and is such that $x + tv_C \in C$ for every $x \in \mathring{D}$ and every small enough positive real number t. The balancing condition at D for S writes

$$\sum_{D\in\mathscr{C}_D} w_C v_{C/D} \in L_D.$$

Since $f(C)$ contains $f(D) = D'$, the image $f(C)$ of C is either equal to D', or it belongs to $\mathscr{C}_{D'}$. In the latter case, set $C' = f(C)$. There exists $k_C \in \mathbf{N}^*$ such that $f(v_{C/D}) = k_C v_{C'}$; one has

$$k_C = [L'_{C'} : (L'_{D'} + \mathbf{Z}f(v_{C/D}))].$$

Then

$$\begin{aligned}[L'_{C'} : f(L_C)] &= [L'_{C'} : f(L_D + \mathbf{Z}v_{C/D})] \\ &= [L'_{C'} : (f(L_D) + \mathbf{Z}f(v_{C/D}))] \\ &= [L'_{C'} : (L'_{D'} + \mathbf{Z}f(v_{C/D}))]\,[L'_{D'} : f(L_D)] \\ &= k_C\,[L'_{D'} : f(L_D)],\end{aligned}$$

so that

$$k_C = [L'_{C'} : f(L_C)]/[L'_{D'} : f(L_D)].$$

Modulo $L'_{D'}$, the vector of L′ responsible for the balancing condition along D′ is equal to

$$\begin{aligned}&\sum_{C'\in\mathscr{C}_{D'}}\left(\sum_{D\in\mathscr{D}_{D'}}\sum_{\substack{C\in\mathscr{C}_D\\ f(C)=C'}} w_C[L'_{C'} : f(L_C)]\right)v_{C'} \\ &\qquad = \sum_{C'\in\mathscr{C}_{D'}}\left(\sum_{D\in\mathscr{D}_{D'}}\sum_{\substack{C\in\mathscr{C}_D\\ f(C)=C'}} w_C[L'_{D'} : f(L_D)]f(v_{C/D})\right) \\ &\qquad = \sum_{D\in\mathscr{D}_{D'}}[L'_{D'} : L_D]\sum_{\substack{C\in\mathscr{C}_D\\ \dim(f(C))=p}} w_C f(v_{C/D}),\end{aligned}$$

hence it belongs to $L'_{D'}$. Indeed, for every $D \in \mathscr{D}_{D'}$, the balancing condition of S along D asserts that $\sum_{C\in\mathscr{C}_D} w_C v_{C/D} \in L_D$; applying f, we get $\sum_{C\in\mathscr{C}_D} w_C f(v_{C/D}) \in L'_{D'}$; on the other hand, if $\dim(f(C)) < p$, then $f(C) \subset D'$ and $f(v_{C/D}) \in L'_{D'}$.

Consequently, $f_*(S)$ is balanced along D′, as was to be shown. □

6.2.3. — Let p, q be two integers, let $S \in MW_p(V)$ and $S' \in MW_q(V)$. Choose polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ of V which are respectively adapted to S and S′; write $S = \sum_{C\in\mathscr{C}_p} w_C[C]$ and $S' = \sum_{C\in\mathscr{C}'_q} w'_C[C]$. The polyhedra $(C\cap C')$, for $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$ form a polyhedral decomposition of V which is simultaneously adapted to S and S′, in particular to the intersection $|S| \cap |S'|$.

Let P be a polyhedron in V. One says that S and S′ *intersect transversally along* P if there exist $C \in \mathscr{C}_p$ and $C' \in \mathscr{C}'_q$ such that $\mathring{P} \subset \mathring{C} \cap \mathring{C}'$ and $\dim(P) = p + q - n$. This implies that $\dim(C + C') = n$.

For $v \in V$, define

$$\mu(P, v) = \sum_{D,D'} w_D w'_{D'} [L : L_D + L'_{D'}],$$

where the sum is over all pairs (D, D') of polyhedra such that $D \in \mathscr{C}_p$, $D' \in \mathscr{C}'_q$, $P \subset D \cap D'$, $\dim(D + D') = n$ and $D \cap (v + D') \neq \varnothing$.

This formula implies that for every $x \in (C \cap C')^\circ$, one has $\mu(\mathrm{Star}_x(P), v) = \mu(P, v)$. Indeed, the pairs of polyhedra that appear in the formula for $\mu(\mathrm{Star}_x(P), v)$ are precisely of the form $(\mathrm{Star}_x(D), \mathrm{Star}_x(D'))$ where (D, D') appear in the formula for $\mu(P, v)$, and the weights are the same.

Lemma **(6.2.4)**. — a) *If* S *and* S′ *intersect transversally along* P, *then* $v \mapsto \mu(P, v)$ *is constant in a neighborhood of* 0 *in* V.

b) *There exists a strictly positive real number* δ *and a polyhedral subspace* B *of* V *of dimension* $< \dim(V)$, *and an integer* $\mu(P)$ *such that* $\mu(P, v) = \mu(P)$ *for all* $v \in V - B$ *such that* $\|v\| < \delta$.

Proof. — We may assume that $0 \in P^\circ$ and replace S, S′ by the associated conic Minkowski weights with apex at 0. In particular, all polyhedra in $\mathscr{C}$ are cones. Moreover, P is a vector subspace, and is contained in the lineality spaces of all cones involved. To check the lemma, we also may mod out by P, which reduces us to the case where $P = \{0\}$.

a) Assume that S and S′ intersect transversally along P; as in the definition, let $C \in \mathscr{C}_p$, $C' \in \mathscr{C}'_q$ be such that $\mathring{P} \subset \mathring{C} \cap \mathring{C}'$ and $\dim(P) = p + q - n$. Since $\mathring{C} \cap \mathring{C}'$ is non-empty, by assumption, it is equal to $(C \cap C')^\circ$, hence it contains 0, so that both C and C′ are linear subspaces.

Let $v \in V$ and (D, D') be a pair of polyhedra that appear in the definition of $\mu(C, C', v)$. Since $0 \in \mathring{C}$, and $0 \in C \cap C' \subset D$, one has $C \subset D$; since $\dim(D) = p$, this implies $D = C$. Similarly, $D' = C'$. Then the sum defining $\mu(C, C', v)$ reduces to $w_C w'_{C'}[L : L_C + L'_{C'}]$; in particular, it is constant.

b) Let $S \times S'$ be the $(p+q)$-dimensional weighted polyhedral subspace of $V \times V$ defined by

$$S \times S' = \sum_{C \in \mathscr{C}_p} \sum_{C' \in \mathscr{C}'_q} w_C w'_{C'}[C \times C'].$$

It is balanced (example 6.1.11).

Let $f : V \times V \to V$ be the linear map given by $f(x, y) = x - y$. Let us consider polyhedral decompositions $\mathscr{C}_1$ of V and $\mathscr{C}_2$ of $V \times V$ that respectively refine $\mathscr{C}$ and $\mathscr{C}'$, and $\mathscr{C} \times \mathscr{C}'$, and such that $f(C \times C')$ is a union of cones in $\mathscr{C}$ for every $C, C' \in \mathscr{C}$ (corollary 1.8.10). The expression $\mu(C, C', v)$ is the coefficient of the cone $[C - C'] = f(C \times C')$ in the Minkowski weight $f_*(S \times S')$. Since this is a Minkowski weight of dimension n, there exists $a \in \mathbf{Z}$ such that $f_*(S \times S') = a[V]$. It follows that $\mu(C, C', v) = a$ for every vector v which does not belong to a polyhedron of $\mathscr{C}$ of dimension $< n$. □

6.2.5. — Let $S \in MW_p(V)$ and $S' \in MW_q(W)$ be Minkowski weights; let $\mathscr{C}$ and $\mathscr{C}'$ be polyhedral decompositions of V which are adapted to S and S′ respectively. Let $\mathscr{D}$ be the polyhedral decomposition of V consisting of the polyhedra $C \cap C'$, for $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$. For $P \in \mathscr{D}$, denote by $\mu(P)$ the common value $\mu(P, v)$ where $v \in V$ is a generic vector; note that $\dim(P) = p + q - n$ if $\mu(P) \neq 0$. We then define an element of $F_{p+q-n}(V)$ by

$$S \cap_{st} S' = \sum_{P \in \mathscr{D}} \mu(P)[P].$$

In particular, it is 0 if $p + q < n$. Moreover, one has $|S \cap_{st} S'| \subset |S| \cap |S'|$.

This element is called the *stable intersection* of S and S′. It does not depend on the chosen polyhedral decomposition $\mathscr{C}$ and is bilinear in S and S′.

Since multiplicities $\mu(P)$ can be computed after passing to links, one also has $\mathrm{Star}_x(S \cap_{st} S') = \mathrm{Star}_x(S) \cap_{st} \mathrm{Star}_x(S')$ for every $x \in V$.

At this point, it is not so clear that $S \cap_{st} S'$ *belongs to* $F_{p+q-n}(V)$*, because we have not yet proved that the polyhedra* $[P]$ *involved in its definition have dimension* $p + q - n$*, if* $\mu(P) \neq 0$*.*

6.2.6. — Let $S \in MW_p(V)$ and $S' \in MW_q(V)$. According to Mikhalkin & Rau (2018), one says that $|S|$ and $|S'|$ *intersect transversally* if $\dim(|S| \cap |S'|) = p+q-n$ and if there exist polyhedral decompositions $\mathscr{C}$ of $|S|$, and $\mathscr{C}'$ of $|S'|$, such that for every polyhedron P satisfying $\dim(P) = p+q-n$ and $P \subset |S| \cap |S'|$, there exists a unique pair (C, C') of polyhedra, with $C \in \mathscr{C}$ and $\mathscr{C}'$, such that $\dim(C) = p$, $\dim(C') = q$ and $P \subset C \cap C'$.

Proposition **(6.2.7)**. — *If* S *and* S′ *intersect transversally, then* $S \cap_{st} S' \in MW_{p+q-n}(V)$ *and* $|S \cap_{st} S'| = |S| \cap |S'|$.

Proof. — Fix polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ adapted to S and S′ that attest of their transversal intersection; let (w_C), resp. $(w'_{C'})$ be the weights of S, resp. of S′. For every pair (C, C'), where $C \in \mathscr{C}_p$ and $C' \in \mathscr{C}_q$ are such that $w_C \neq 0$, $w'_{C'} \neq 0$ and $C \cap C' \neq \varnothing$, one has $\dim(C \cap C') = p + q - n$, and the definition of $\mu(C, C')$ shows that $\mu(C, C') = w_C w'_{C'}$. In fact, the sum defining $\mu(C, C', v)$ is reduced to (C, C'), for every small enough $v \in V$. This already proves that $S \cap_{st} S'$ belongs to $F_{p+q-n}(V)$ and that $|S \cap_{st} S'| = |S| \cap |S'|$.

Let us prove the balancing condition. By construction, $|S \cap_{st} S'|$ is a union of polyhedra of dimension $p + q - n$ of the form $C \cap C'$, for $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$, and they only meet along faces which are of the form $D \times C'$, or $C \times D'$, where D is a codimension 1 face of C, or D′ is a codimension 1 face of C′. Consequently, the balancing condition needs only be checked along such faces. We thus assume that $E = D \cap C'$, where $D \in \mathscr{C}_{p-1}$ and $C' \in \mathscr{C}_q$, the other case being similar by symmetry. The polyhedra of $S \cap_{st} S'$ that border E are of the form $C \cap C'$, where $C \in \mathscr{C}_p$ contains D.

For every such C, fix a vector $v_{C/D} \in L_C$ which generates L_C/L_D and which is such that $x + tv_{C/D} \in C$ for every $x \in \mathring{D}$ and every small enough positive real number t. The balancing condition for S along D writes $\sum_C w_C v_{C/D} \in L_D$.

Let us fix a normal vector $v'_{C\cap C'/D\cap C'} \in L_{C\cap C'}$ associated with the face $D \times C'$ of $C \times C'$. There exists a unique integer $p_C \in \mathbf{N}^*$ such that $v'_{C\cap C'/D\cap C'} = p_C v_{C/D} \pmod{L)_D}$, so that the balancing condition for $S \cap_{st} S'$ along $D \times C'$ writes $\sum_C \mu(C, C') p_C v_{C/D} \in L_D$. To conclude the proof, since $\mu(C, C') = w_C w'_{C'}[L : L_C + L_{C'}]$, it now suffices to prove that $p_C[L_C + L_{C'}]$ is independent of C.

One has

$$L_C \cap L_{C'} = L_{C\cap C'} = L_{D\cap C'} + \mathbf{Z}v_{C\cap C'/D\cap C'},$$

hence

$$[(L_C \cap L_{C'}) + L_D = L_D + \mathbf{Z}v_{C\cap C'/D\cap C'} = L_D + \mathbf{Z}p_C v_{C/D}.$$

Since $L_C = L_D + \mathbf{Z}v_{C/D}$, it follows that

$$p_C = [L_C : (L_C \cap L_{C'}) + L_D] = [L_C + L_{C'} : L_{C'} + L_D]$$

and

$$p_C[L : L_C + L_{C'}] = [L : L_{C'} + L_D].$$

□

Proposition **(6.2.8)**. — a) *There exists a polyhedral subspace* B *of* V *such that* dim(B) < *n and such that for every* $v \in V - B$*, the Minkowski weights* S *and* $S' + v$ *intersect transversally.*

b) *If* $n = p + q$*, then* $\deg(S \cap_{st} (S' + v))$ *is independent of* $v \in V - B$.

Proof. — We fix polyhedral decompositions $\mathscr{C}$ and $\mathscr{C}'$ of V respectively adapted to S and S'.

Let $\mathscr{I}$ be the set of all pairs (C, C') such that $C \in \mathscr{C}_p$, $C' \in \mathscr{C}'_q$, $w_C \neq 0$, $w'_{C'} \neq 0$. Let $(C, C') \in \mathscr{I}$. For $v \in V$, one has $C \cap (v + C') \neq \varnothing$ if and only if $v \in C - C'$. Let B_1 be the union of all $\partial(C - C')$, for $(C, C') \in \mathscr{I}$ such that $\dim(C - C') < n$. Let $(C, C') \in \mathscr{I}$ be such that $\dim(C - C') = n$ and let $\partial(C - C') = (C - C') - (C - C')^\circ$; it is a polyhedron of dimension $< n$. If $v \notin (C - C')$, then $C \cap (v + C') = \varnothing$; if $v \in (C - C')^\circ$, then $v \in \mathring{C} - \mathring{C}'$, hence $\mathring{C} \cap (v + \mathring{C}') \neq \varnothing$. Let B_2 be the union of all $\partial(C - C')$, for $(C, C') \in \mathscr{I}$ such that $\dim(C - C') = n$. Let $B = B_1 \cup B_2$. This is a polyhedral subspace of V of dimension $< n$.

Let $v \in V - B$. By construction, S and $S' + v$ intersect transversally along $C \cap (C' + v)$, for every pair (C, C') such that $C \cap (C' + v) \neq \varnothing$. This proves that S and $S' + v$ intersect transversally.

Assume that $p + q = n$. Let U be a connected component of $V - B$ such that $0 \in \overline{U}$. Fix $(C, C') \in \mathscr{I}$. When $v \in U$, the pairs $(D, D') \in \mathscr{I}$ such that $v \in \mathring{D} \cap (v + \mathring{D}')$ remain the same, and in fact, v is their unique point of intersection. This gives

$$\begin{aligned}
\deg(S \cap_{st} (v + S')) &= \sum_{(D,D')} w_D w'_{D'} [L : L_D + L_{D'}] \\
&= \sum_{(C,C')} \sum_{\substack{(D,D') \\ D \cap D' = C \cap C'}} w_D w'_{D'} [L : L_D + L_{D'}] \\
&= \sum_{(C,C')} \mu(C, C') \\
&= \deg(S \cap_{st} S').
\end{aligned}$$

This implies the claim. □

Theorem **(6.2.9)**. — *Let p, q be integers such that $p + q \geqslant n$. For any $S \in MW_p(V)$ and $S' \in MW_q(V)$, one has $S \cap_{st} S' \in MW_{p+q-n}(V)$.*

Proof. — Let E be a polyhedron of dimension $p + q - n - 1$ along which we wish to check the balancing condition for $S \cap_{st} S'$. Choosing an origin in $\mathring{E}$ and replacing S and S' by the fan-like Minkowski weights, we can assume that there are polyhedral decompositions of V adapted to S and S', all polyhedra of which are cones. We may also quotient by E and reduce to the case where $E = \{0\}$; then $p + q = n + 1$.

We will first prove that $S \cap_{st} S' = \operatorname{recc}(S \cap_{st} (v + S'))$ for all $v \in V$. *It suffices to prove this when* S *and* $v + S'$ *intersect transversally.* If C and C' are cones such that $C \cap (C' + v) \neq \varnothing$, then one has $\operatorname{recc}(C \cap (C' + v)) =$

$C \cap C'$. (Let $x \in C \cap (C' + v)$; then for every $u \in C \cap C'$, one has $x + u \in C \cap (C' + v)$. On the other hand, if $x + tu \in C \cap (C' + v)$ for every $t \in \mathbf{R}_+$, then $u \in C \cap C'$, as one sees letting $t \to \infty$.) By transversality, $\dim(C \cap C') = \dim(C \cap (C' + v)) = 1$. Multiplicities add up as well. This implies the equality $\operatorname{recc}(S \cap_{st} (S' + v)) = S \cap_{st} S'$. Since S and $v + S'$ intersect transversally, one has $S \cap_{st} (S' + v) \in MW_1(V)$. To conclude the proof of the theorem, it thus follows to establish the following lemma. □

Lemma **(6.2.10)**. — *Let* $S \in MW_1(V)$. *Then* $\operatorname{recc}(S) \in MW_1(V)$.

Proof. — Let $\mathscr{C}$ be a polyhedral decomposition of V which is adapted to S; for $C \in \mathscr{C}_1$, let w_C be the weight of C in S.

Let $C \in \mathscr{C}_1$, so that $L_C \simeq \mathbf{Z}$; we fix arbitrarily one generator v_C of L_C. There are three possibilities.

– Either there exist $x_C, y_C \in C$ such that $C = [x_C; y_C]$, chosen such that $y_C \in x_C + \mathbf{R}_+ v_C$. Then its recession cone is 0;

– Or there exists $x \in V$ such that $C = x_C + \mathbf{R}_+ v_C$ or $C = x_C - \mathbf{R}_+ v_C$. Up to changing v_C into $-v_C$, we assume that we are in the former case. Then $\operatorname{recc}(C) = \mathbf{R}_+ v_C$;

– Or there exists $x_C \in V$ such that $C = x_C + \mathbf{R} v_C$; then $\operatorname{recc}(C) = \mathbf{R} v_C$.

Let $\mathscr{C}_1^2, \mathscr{C}_1^1, \mathscr{C}_1^0$ be the corresponding subsets of $\mathscr{C}_1$. The recession fan of S is given by the sum

$$\operatorname{recc}(S) = \sum_{C \in \mathscr{C}_1^1} w_C [\mathbf{R}_+ v_C] + \sum_{C \in \mathscr{C}_1^0} w_C [\mathbf{R} v_C].$$

The balancing condition at the origin for recc(S) is thus the relation

$$\sum_{C \in \mathscr{C}_1^1} w_C v_C = 0.$$

We now write the balancing condition for S at a point $p \in \mathscr{C}_0$. Let $\mathscr{C}_p$ be the set of $C \in \mathscr{C}_1$ such that $p \in C$. If $C \in \mathscr{C}_1^1$, then $p = x_C$; moreover, v_C is an admissible normal vector for (p, C). Otherwise, $C \in \mathscr{C}_1^2$ and there are two possibilities:

– Either $p = x_C$; then v_C is an admissible normal vector for (p, C);

– Or $p = y_C$ and then $-v_C$ is an admissible normal vector for (p, C).

The balancing condition at p thus writes

$$\sum_{\substack{C\in\mathscr{C}_1^1\\ x_C=p}} w_C v_C + \sum_{\substack{C\in\mathscr{C}_1^2\\ x_C=p}} w_C v_C - \sum_{\substack{C\in\mathscr{C}_1^2\\ y_C=p}} w_C v_C = 0.$$

Adding all of these relations, for all $p\in\mathscr{C}_0$, we obtain

$$0 = \sum_{C\in\mathscr{C}_1^1} w_C v_C + \sum_{C\in\mathscr{C}_1^2} w_C v_C - \sum_{C\in\mathscr{C}_1^2} w_C v_C = \sum_{C\in\mathscr{C}_1^1} w_C v_C,$$

as was to be shown. □

Proposition **(6.2.11)**. — *The stable intersection product endowes the abelian group* $\mathrm{MW}(V) = \bigoplus_p \mathrm{MW}_p(V)$ *with a ring structure. The neutral element is* $[V]$.

Proof. — It follows from the definitions that the stable intersection product is commutative and bilinear. It also follows from the definitions that $S\cap[V]=S$.

Let us check associativity. Let S, S', S'' be three Minkowski weights of dimensions p, q, r and let us prove that $(S\cap_{st} S')\cap_{st} S'' = S\cap_{st}(S'\cap_{st} S'')$. Let us first treat the case where these Minkowski weights intersect transversally, in the sense that $\mathring{C}\cap\mathring{C}'\mathring{C}''\neq\varnothing$ for every $C\in\mathscr{C}_p$, $C'\in\mathscr{C}'_q$, $C''\in\mathscr{C}''_r$ such that $w_C, w'_{C'}, w''_{C''}\neq 0$ and $C\cap C'\cap C''\neq\varnothing$. If this holds, then S' and S'' intersect transversally and

$$S'\cap_{st} S'' = \sum_{C',C''} w'_{C'} w''_{C''}[L : L_{C'} + L_{C''}][C'\cap C''].$$

Moreover, S and $S'\cap_{st} S''$ intersect transversally and

$$S\cap_{st}(S'\cap_{st} S'') \\ = \sum_{C,C',C''} w_C w'_{C'} w''_{C''}[L : L_{C'} + L_{C''}][L : L_C + (L_{C'}\cap L_{C''})]\,[C\cap C'\cap C''].$$

By symmetry, one also has

$$(S\cap_{st} S')\cap_{st} S'') \\ = \sum_{C,C',C''} w_C w'_{C'} w''_{C''}[L : L_C + L_{C'}][L : (L_C\cap L_{C'}) + L_{C''}]\,[C\cap C'\cap C''].$$

It thus suffices to prove the following equality of indices:

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] = [L : L_C + L_{C'}][L : (L_C \cap L_{C'}) + L_{C''}].$$

On the other hand, one has

$$\begin{aligned}[L : L_C + (L_{C'} \cap L_{C''})] &= [L : L_C + L_{C'}][L_C + L_{C'} : L_C + (L_{C'} \cap L_{C''})] \\ &= [L : L_C + L_{C'}][L_{C'} : (L_C \cap L_{C'}) + (L_{C'} \cap L_{C''})],\end{aligned}$$

so that

$$\begin{aligned}&[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] \\ &\qquad = [L : L_{C'} + L_C]\,[L : L_{C'} + L_{C''}]\,[L_{C'} : (L_C \cap L_{C'}) + (L_{C'} \cap L_{C''})],\end{aligned}$$

an expression which is invariant when one exchanges the roles of C and C″. Therefore,

$$[L : L_{C'} + L_{C''}][L : L_C + (L_{C'} \cap L_{C''})] = [L : L_{C'} + L_C][L : L_{C''} + (L_{C'} \cap L_C)],$$

as was to be shown.

In the general case, we consider arbitrarily small vectors $v \in V$, $w \in V$ such that S, $S' + v$ and $S'' + w$ intersect transversally. If C, C', C'' are polyhedra of dimensions p, q, r, the multiplicity $\mu(C, C', C'')$ of $[C \cap C' \cap C'']$ in $(S \cap_{st} S') \cap_{st} S''$ is a sum of multiplicities $\mu(D, D', D''; v, w)$, where $C \cap C' \cap C'' = D \cap D' \cap D''$ and $D, D' + v, D'' + w$ intersect transversally, associated with $(S \cap_{st} (S' + v)) \cap_{st} (S'' + w)$. By the case of transverse intersections, they coincide with the multiplicity of $[C \cap C' \cap C'']$ in $S \cap_{st} ((S' + v) \cap_{st} (S'' + w))$. □

Example **(6.2.12)** (Unfinished). — Assume that $L = \mathbf{Z}^n$ and let $(e_1, \dots, e_n)$ be its canonical basis; set also $e_0 = -e_1 - \cdots - e_n$. For $I \subsetneq \{0, \dots, n\}$, let C_I be the cone generated by the vectors e_i, for $i \in I$; one has $\dim(C_I) = \operatorname{Card}(I)$. Note that $C_I \cap C_J = C_{I\cap J}$ for $I, J \subsetneq \{0, \dots, n\}$, so that the set of cones $(C_I)_{I\subsetneq\{0,\dots,n\}}$ is a fan in $\mathbf{R}^n$.

For $p \in \{0, \dots, n\}$, we define an effective weighted polyhedral subspace of dimension p by

$$S_p = \sum_{\operatorname{Card}(I)=p} [C_I].$$

(This is a tropical linear space of dimension p.) One has $L_{C_I} = \sum_{i\in I} \mathbf{Z} e_i$. It is balanced. The only polyhedra along which the balancing condition is not obvious are of the form C_J, where $\mathrm{Card}(J) = p-1$, and its adjacent polyhedra are of the form $C_{J\cup\{i\}}$, for $i \in \{0,\dots,n\} - J$; one may take e_i as a normal vector. The balancing condition along C_J then writes

$$\sum_{i\in\{0,\dots,n\}-J} e_i = \sum_{i\in\{0,\dots,n\}} e_i - \sum_{j\in J} e_j \in L_{C_J}$$

since $\sum_{i=0}^{n} e_i = 0$.

Let us prove that $S_p \cap_{st} S_q = S_{p+q-n}$.

Proposition **(6.2.13)**. — *Let* $S \in MW_p(V)$ *and let* $S' \in MW_q(V)$. *If* $\Delta \in MW_n(V\times V)$ *is the diagonal, then one has*

$$\Delta \cap_{st} (S \boxtimes S') = S \cap_{st} S'.$$

6.3. The tropical hypersurface associated with a piecewise linear function

6.3.1. — Let $f: \mathbf{R}^n \to \mathbf{R}$ be a continuous piecewise affine function and let $\mathscr{C}$ be a polyhedral decomposition of $\mathbf{R}^n$ which is adapted to f. We assume that *f has integral slopes,* in the sense that for every $C \in \mathscr{C}$, there exists an linear function $\varphi_C \in L^\vee$ such that $f(y) - f(x) = \varphi_C(y-x)$ for every $x, y \in C$.

Let $x \in \mathbf{R}^n$ and let D be the unique polyhedron of $\mathscr{C}$ such that $x \in \mathring{D}$. If $\dim(D) \neq n-1$, set $w_f(D) = 0$. Otherwise, if $\dim(D) = n-1$, then D is a face of exactly two n-dimensional polyhedra C^+, C^- in $\mathscr{C}$; one has $D = C^+ \cap C^-$.

Fix a point $x \in \mathring{D}$.

The quotient group $\mathbf{Z}^n/L_D$ is isomorphic to $\mathbf{Z}$, and it admits a unique generator which is the image of an element v^+ such that $x + tv^+ \in C^+$ for every small enough $t \in \mathbf{R}_+$.

Define v^- similarly. In fact, one has $v^- = -v^+$.

By assumption, f is affine with integral slopes on C^+; let $\varphi^+ : V \to \mathbf{R}$ be the unique linear map such that $f(y) - f(x) = \varphi^+(y-x)$ if $x, y \in C^+$. We define similarly φ^-.

We then set

$$w_D = \varphi^+(v^+) + \varphi^-(v^-).$$

and define

$$\partial(f) = \sum_{D\in\mathscr{C}_{n-1}} w_D[D].$$

Proposition **(6.3.2)**. — *Let f be a piecewise linear function f with integral slopes on* V.

a) *The weighted polyhedral subspace $\partial(f)$ is a Minkowski weight of dimension $n-1$ adapted to the polyhedral decomposition $\mathscr{C}$.*
b) *Its support $|\partial(f)|$ is the non-linearity locus of f.*
c) *If f is convex, then $\partial(f)$ is effective.*

Proof. — We have to prove that $\partial(f)$ statisfies the balancing condition.

Let $E \in \mathscr{C}$ be a polyhedron of dimension $n-2$. Fix a point $x \in \mathring{E}$ and consider a 2-dimensional plane through x which is transverse to E. We get a fan in $\mathbf{R}^2$ which reduces the verification of the balancing condition to the case $n = 2$, for $E = \{0\}$.

The 1-dimensional polyhedra that contain the origin are (chunks of) rays $D_1 = \mathbf{R}_+u_1, \dots, D_n = \mathbf{R}_+u_n$, where $u_1, \dots, u_n \in \mathbf{Z}^2$ are primitive vectors.[3] The balancing condition at 0 is the equation

$$\sum_{j=1}^{n} w_{D_j}u_j = 0.$$

Up to a reordering of the u_j, unique modulo cyclic permutations, the 2-dimensional polyhedra that contain the origin are (chunks) of sectors $C_1 = \operatorname{cone}(u_1, u_2), \dots, C_{n-1} = \operatorname{cone}(u_{n_1}, u_n), C_n = \operatorname{cone}(u_n, u_1)$. Set $\varphi(x) = f(x) - f(0)$; for every j, let φ_j be the linear function on $\mathbf{R}^2$ such that $f(x) = f(0) + \varphi_j(x)$ for every point $x \in C_j$ which is close to 0.

If ρ is the rotation of angle $\pi/2$, we then may take $D_j^+ = C_j$ and $D_j^- = C_{j-1}$, $v_j^+ = \rho(u_j)$ and $v_j^- = \rho^{-1}(u_j) = -v_j^+$. Then $w_{D_j} = \varphi_j(\rho(u_j)) - \varphi_{j-1}(\rho(u_j))$ for all $j \in \{1, \dots, n\}$.

[3]Picture?

We thus have

$$\sum_{j=1}^{n} w_{\mathrm{D}_j} u_j = \sum_{j=1}^{n} \varphi_j(\rho(u_j))u_j - \sum_{j=1}^{n} \varphi_{j-1}(\rho(u_j))u_j$$
$$= \sum_{j=1}^{n} \varphi_j(\rho(u_j))u_j - \sum_{j=1}^{n} \varphi_j(\rho(u_{j+1}))u_{j+1}.$$

The continuity of f along the ray u_j writes $\varphi_j(u_j) = \varphi(u_j) = \varphi_{j-1}(u_j)$. Let $a_j, b_j \in \mathbf{R}$ be such that $\rho(u_j) = a_j u_j + b_j u_{j+1}$. Then

$$\varphi_j(\rho(u_j)) = a_j\varphi_j(u_j) + b_j\varphi_j(u_{j+1}) = a_j\varphi(u_j) + b_j\varphi(u_{j+1}).$$

Similarly, $\rho(u_{j+1}) = a_{j-1}u_{j-1} + b_{j-1}u_j$, hence

$$\varphi_j(\rho(uj+1) = a_{j-1}\varphi_j(u_{j-1}) + b_{j-1}\varphi_j(u_j) = a_{j-1}\varphi(u_{j-1}) + b_{j-1}\varphi(u_j).$$

Finally,

$$\sum_{j=1}^{n} w_{\mathrm{D}_j} u_j = \sum_{j=1}^{n} \left(a_j\varphi(u_j) + b_j\varphi(u_{j+1})\right) - \left(a_{j-1}\varphi(u_{j-1}) + b_{j-1}\varphi(u_j)\right) = 0.$$

This proves that ∂f belongs to $\mathrm{MW}_{n-1}(\mathrm{V})$.

By construction, f is locally differentiable on $\mathrm{V} - \bigcup_{\mathrm{D}\in\mathscr{C}_{n-1}} \mathrm{D}$. For $\mathrm{D} \in \mathscr{C}_{n-1}$ and $x \in \mathring{\mathrm{D}}$, observe that f is differentiable on a neighborhood of x if and only if $w_{\mathrm{D}} = 0$. Consequently, the open non-differentiability locus of f is equal to $|\partial(f)|$.

[4]

a) b) With the previously introduced notation, it suffices to prove that $w_{\mathrm{D}} \geqslant 0$ for every $\mathrm{D} \in \mathscr{C}_{n-1}$.

For every positive real number t, one has

$$tw_{\mathrm{D}} = \varphi^+(tv^+) + \varphi^-(tv^-) = (f(x+tv^+) - f(x)) + (f(x-tv^+) - f(x))$$

if t is small enough. By convexity, one has

$$f(x) = \frac{1}{2}\left(f(x+tv^+) + f(x-tv^+)\right),$$

so that $tw_{\mathrm{D}} \geqslant 0$; if $t > 0$, this implies $w_{\mathrm{D}} \geqslant 0$. □

[4]Some points to check. . .

Proposition **(6.3.3)**. — *The map* $f \mapsto \partial(f)$ *from the abelian group* $\mathrm{PL}(\mathrm{V})$ *of piecewise linear functions on* V *with integral slopes to the group* $\mathrm{MW}_{n-1}(\mathrm{V})$ *of* $(n-1)$*-dimensional Minkowski weights is a surjective morphism of groups. Its kernel is the subgroup of affine functions with integral slopes on* V.

Proof. — ☐

Theorem **(6.3.4)**. — *Let* f *be a piecewise linear function with integer slopes and let* $\mathrm{S} \in \mathrm{MW}_p(\mathrm{V})$. *The Minkowski weight* $\partial(f) \cap_{\mathrm{st}} \mathrm{S}$ *can be computed explicitly as follows. Let* $\mathscr{C}$ *be a polyhedral decomposition of* V *which is adapted to* S *and such that* $f|_{\mathrm{C}}$ *is affine, for every* $\mathrm{C} \in \mathscr{C}$. *For every* $\mathrm{D} \in \mathscr{C}_{p-1}$, *let* $\mathscr{C}_{\mathrm{D}}$ *be the set of* $\mathrm{C} \in \mathscr{C}_p$ *such that* $\mathrm{D} \subset \mathrm{C}$. *For* $\mathrm{C} \in \mathscr{C}_{\mathrm{D}}$, *let* $v_{\mathrm{C/D}} \in \mathrm{L}_{\mathrm{C}}$ *be a vector that generates* $\mathrm{L}_{\mathrm{C}}/\mathrm{L}_{\mathrm{D}}$ *and such that* $x + tv_{\mathrm{C/D}} \in \mathrm{C}$ *for every* $x \in \mathring{\mathrm{D}}$ *and every small enough positive real number* t. *Set*

$$w'_{\mathrm{D}} = \sum_{\mathrm{C} \in \mathscr{C}_{\mathrm{D}}} w_{\mathrm{C}} \left(\lim_{t \to 0^+} \frac{f(x + tv_{\mathrm{C/D}}) - f(x)}{t} \right).$$

Then $\partial(f) \cap_{\mathrm{st}} \mathrm{S} = \sum w'_{\mathrm{D}}[\mathrm{D}]$.

Theorem **(6.3.5)** (Projection formula). — *Let* $u \colon \mathrm{L} \to \mathrm{L}'$ *be a morphism of free finitely generated abelian groups, let* $\mathrm{V} = \mathrm{L}_{\mathbf{R}}$ *and* $\mathrm{V}' = \mathrm{L}'_{\mathbf{R}}$. *Still write* u *for* $u_{\mathbf{R}} \colon \mathrm{V} \to \mathrm{V}'$. *Let* S *be a Minkowski weight on* V *and let* f *be a piecewise linear function on* V′. *One has*

$$u_*(u^*(f) \cap_{\mathrm{st}} \mathrm{S}) = f \cap_{\mathrm{st}} u_*(\mathrm{S}).$$

Remark **(6.3.6)**. — There should be a projection formula of the form

$$u_*(\mathrm{S} \cap_{\mathrm{st}} u^*(\mathrm{S}')) = u_*(\mathrm{S}) \cap_{\mathrm{st}} \mathrm{S}'$$

if $u \colon \mathrm{L} \to \mathrm{L}'$ is a morphism of free finitely generated abelian groups.

If u is surjective, then $\mathrm{L} \simeq \mathrm{L}' \times \mathrm{L}''$, and $u^*(\mathrm{S}') = \mathrm{S}' \boxtimes \mathrm{L}''$.

Otherwise, one can/needs to define u^* by stable intersection, say $u^*(\mathrm{S}') = p_*(\Gamma_u \cap_{\mathrm{st}} (\mathrm{L} \boxtimes \mathrm{S}'))$, where $\Gamma_u = (\mathrm{id} \times u)_*(\mathrm{V})$ is the graph of u and $p \colon \mathrm{V} \times \mathrm{V}' \to \mathrm{V}$ is the first projection.

6.4. Comparing algebraic and tropical intersections

6.4.1. — Let X and Y be subvarieties of ${\mathbf{G}_{\mathrm{m}}}^n$, respectively defined by ideals I and J of $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}$. Their intersection $X \cap Y$ is the subvariety of ${\mathbf{G}_{\mathrm{m}}}^n$ with ideal $I + J$.

Note that in general, $X \cap Y$ might not be integral. It may have multiple component. It also may be non-reduced, for example if Y is a hyperplane tangent to X at some point a: the tangency will then be reflected by the fact that the local ring $\mathscr{O}_{X\cap Y,a}$ contains non-trivial nilpotent elements.

By a general inequality in algebraic geometry, one has

$$\dim_a(X \cap Y) \geqslant \dim_a(X) + \dim_a(Y) - n,$$

for every $a \in X \cap Y$. This inequality is an equality in certain cases, for example when X and Y are smooth at a, and $T_aX + T_aY = T_a{\mathbf{G}_{\mathrm{m}}}^n$ (then, we say that the intersection is *transverse* around a). But the strict inequality may hold, for example in the trivial case where $X = Y$, but also in less obvious cases.

We are interested in computing the tropicalization of $X \cap Y$. How does it compare to the intersection $\mathscr{T}_X \cap \mathscr{T}_Y$, beyond the obvious inclusion? This guess is however often too large, for example if $\mathscr{T}_X = \mathscr{T}_Y$? Then how does it compare to the stable intersection $\mathscr{T}_X \cap_{\mathrm{st}} \mathscr{T}_Y$? While that second guess is often too small, it is indubitably better, since we will show that it suffices to translate "generically" Y in ${\mathbf{G}_{\mathrm{m}}}^n$, without changing its tropicalization, to make it correct.

We start with the case of transversal tropical intersections, where the picture is particularly nice.

Lemma **(6.4.2)**. — *Let* X, Y *be subvarieties of* ${\mathbf{G}_{\mathrm{m}}}^n$.

a) *Let* $x \in \mathbf{R}^n$. *If* $\mathscr{T}_X$ *and* $\mathscr{T}_Y$ *meet transversally at* x, *then*

$$\mathrm{Star}_x(\mathscr{T}_{X\cap Y}) = \mathrm{Star}_x(\mathscr{T}_X \cap_{\mathrm{st}} \mathscr{T}_Y).$$

b) *If* $\mathscr{T}_X$ *and* $\mathscr{T}_Y$ *intersect transversally everywhere, then*

$$\mathscr{T}_{X\cap Y} = \mathscr{T}_X \cap_{\mathrm{st}} \mathscr{T}_Y.$$

Proof. — Let I, J be the ideals of X, Y in $K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. By assumption, there exists polyhedra C and C′ of the Gröbner polyhedral decompositions of $\mathscr{T}_X$ and $\mathscr{T}_Y$ respectively such that $x \in \mathring{C} \cap \mathring{C}'$; moreover,

$\dim(C + C') = n$. In particular, $W = \mathrm{Star}_x(\mathscr{T}_X)$ and $W' = \mathrm{Star}_y(\mathscr{T}_Y)$ are vector spaces, with a constant multiplicity is constant, and $W + W' = \mathbf{R}^n$. Let $p = \dim(W)$, $q = \dim(W')$; let $W'' = W \cap W'$, so that $r = \dim(W'') = p + q - n$. Choose a rational basis of $\mathbf{R}^n$ as follows, starting from a basis of W'', and extending it to rational bases of W and W'. This shows that there exists a rational isomorphism $\varphi \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\varphi(W'') = \mathbf{R}^r \times \{0\} \times \{0\}$, $\varphi(W) = \mathbf{R}^r \times \mathbf{R}^{p-r} \times \{0\}$ and $\varphi(W') = \mathbf{R}^r \times \{0\} \times \mathbf{R}^{q-r}$. We may also assume that $\varphi(\mathbf{Z}^n) \subset \mathbf{Z}^n$. Let then $f \colon \mathbf{G_m}^n \to \mathbf{G_m}^n$ be the morphism of tori whose action on cocharacters is given by φ. It is finite and surjective.

Let $X' = f(X)$ and $Y' = f(Y)$; by proposition 3.8.1, one has $\mathscr{T}_{X'} = \varphi_*(\mathscr{T}_X)$, $\mathscr{T}_{Y'} = \varphi_*(\mathscr{T}_Y)$ and $\mathscr{T}_{X'\cap Y'} = \varphi_*(\mathscr{T}_{X\cap Y})$. Since φ_* is a linear isomorphism, we may assume, for proving the lemma, that φ is the identity. [5]

Let $I_x = I \cap k[T_{p+1}^{\pm 1}, \dots, T_n^{\pm 1}]$ and $J_x = J \cap k[T_{r+1}^{\pm 1}, \dots, T_p^{\pm 1}]$. By lemma 3.9.4, one has $I = I_j \cdot k[T_1^{\pm 1}, \dots]$ and $\mathrm{mult}_{\mathscr{T}_Y}(C') = \mathrm{codim}(J_x)$; similarly, $J = J_x \cdot k[T_1^{\pm 1}, \dots]$ and $\mathrm{mult}_{\mathscr{T}_Y}(C') = \mathrm{codim}(J_x)$.

We now observe that

$$\mathrm{in}_x(I + J) = \mathrm{in}_x(I) + \mathrm{in}_x(J),$$

and that

$$\mathrm{in}_x(I + J) \cap k[T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1}] = I_x + J_x,$$

so that

$$k[T_{r+1}^{\pm 1}, \dots, T_n^{\pm 1}]/(I_x + J_x) \simeq (k[T_{p+1}^{\pm 1}, \dots, T_n^{\pm 1}]/I_x) \otimes_k (k[T_{r+1}^{\pm 1}, \dots, T_p^{\pm 1}]/J_x)$$

has dimension $\mathrm{mult}_{\mathscr{T}_X}(C)\,\mathrm{mult}_{\mathscr{T}_Y}(C')$. The same result holds for every other point in $\mathring{C} \cap \mathring{C}'$. This shows that $C \cap C' \subset \mathscr{T}_{X\cap Y}$ contains a polyhedron of the Gröbner decomposition of $X \cap Y$, and that its multiplicity is the product of the multiplcities of C and C′. This concludes the proof of the first assertion of the lemma, and the second follows directly from it. □

[5] Oops! That proposition says nothing about multiplicities. . .

6.4.3. — Let K be a valued field. Let $L = K(s)$ be the field of rational functions in one indeterminate s with coefficients in K, endowed with the Gauss absolute value. Let $I \subset L[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be an ideal and let $X = V(I)$. Assume that X is equidimensional and let $d = \dim(X)$.

Consider $K(s)$ as the field of functions of the affine line $\mathbf{A}^1$. The Zariski closure $\mathscr{X}$ of X in ${\mathbf{G}_m}^n_{\mathbf{A}^1}$ is defined by the ideal $\mathscr{I} = K[s][T^{\pm 1}] \cap I$. For every point $a \in K$, or rather in a valued extension K′ of K, we can then consider the ideal $\mathscr{I}_a$ of $K'[T^{\pm 1}]$ deduced from I by setting $s = a$ and the subscheme $\mathscr{X}_a = V(\mathscr{I}_a)$ of ${\mathbf{G}_m}^n_{K'}$.

The relations between X and the schemes $\mathscr{X}_a$, its specializations, are well-studied in algebraic geometry. In fact, $\mathscr{X}$ is a flat $\mathbf{A}^1$-scheme, and $\mathscr{X}_a$ is its fiber. In particular, the schemes $\mathscr{X}_a$ are equidimensional if $\mathscr{X}$ is, with the same dimension.

We first prove that, up to finitely many obstructions, the schemes $\mathscr{X}_a$ have the same tropicalization as X provided $v(a) = v(s) = 0$.

All this should be rewritten replacing $\mathbf{A}^1$ *with* $\mathbf{A}^n$*, possibly even any integral variety* V*; the outcome is an analytic domain containing a given Zariski-dense point of* V^{an}*.*

Proposition **(6.4.4)**. — *There exists a finite subset* B *of* $\bar{K}$*, a finite subset* C *of* $\bar{k}$ *such that for every* a *in a valued extension* K′ *of* K *(with residue field* k'*) such that* $a \notin B$, $v(a) = 0$, *and* $\overline{a} \notin C$, *the variety* $\mathscr{X}_a$ *has the same tropicalization than* X*: one has an equality of weighted polyhedra* $\mathscr{T}_X = \mathscr{T}_{\mathscr{X}_a}$.

Proof. — Let us consider the homogeneization $I^h \subset K(s)[T_0, \dots, T_n]$ of I. Let $(f_1, \dots, f_m)$ be a finite set of homogeneous polynomials in I^h which is a universal Gröbner basis, *i.e.*, at every $x \in \mathbf{R}^{n+1}$; we may assume that it is contained in $\mathscr{I}^h = K[s][T_0, \dots, T_n] \cap I^h$.

a) The family $(f_1, \dots, f_m)$ generates the ideal I^h in $K(s)[T_0, \dots, T_n]$. Since $K[s][T_0, \dots, T_n]$ is a noetherian ring, the homogeneous ideal $\mathscr{I}^h$ has a finite basis $(g_1, \dots, g_p)$ consisting of homogeneous polynomials. For every $j \in \{1, \dots, p\}$, there exist homogeneous polynomials $k_{j,1}, \dots, k_{j,m} \in K(s)[T_0, \dots, T_n]$ such that $g_j = \sum_{i=1}^m k_{j,i} f_i$. Let $h \in K[s]$ be a non-zero polynomial such that $hk_{j,i} \in K[s][T_0, \dots, T_n]$ for all i, j. We then obtain inclusions $h\mathscr{I}^h \subset (f_1, \dots, f_m) \subset \mathscr{I}^h$ of homogeneous ideals of $K[s][T_0,\dots,T_n]$. In particular, for every a in a valued extension K' of K such that $h(a)\neq 0$, the ideal $\mathscr{I}_a$ of $K'[T_0,\dots,T_n]$ coincides with the ideal generated by $f_1(a;T),\dots,f_m(a;T)$. We define $B\subset \bar{K}$ as the set of roots of h.

b) Let $f\in K[s][T_0,\dots,T_n]$; write $f=\sum_{m\in S(f)} f_m(s)c_m T^m$, where $c_m\in K^\times$ and $f_m\in K[s]$ is a polynomial of Gauss-norm 1. The reductions $\overline{f}_m$ of the polynomials f_m are non-zero polynomials in $k[s]$. Let h_f be their product. By construction, for every a in a valued extension K' of k such that $v(a)=0$ and $h_f(\overline{a})\neq 0$, one has $v(f_m(a))=0$ for all $m\in S(f)$. It follows that for every such a, one has $\tau_x(f)=\tau_x(f(a;T))$ and $\mathrm{in}_x(f)(\bar{a};T)=\mathrm{in}_x(f(a;T))$ for all $x\in\mathbf{R}^{n+1}$.

c) Let h' be the product of the polynomials h_{f_j} and let $C_1\subset\bar{k}$ be the set of roots of h'.

For $x\in\mathbf{R}^{n+1}$, set $J_x=\mathrm{in}_x(I^h)$; note that there are only finitely many ideals of the form J_x, when $x\in\mathbf{R}^{n+1}$. Let $\mathscr{J}_x=J_x\cap k[s][T_0,\dots,T_n]$; for b in an extension k' of k, let $\mathscr{J}_{x,b}$ be the image of $\mathscr{J}_x$ in $k'[T_0,\dots,T_n]$.

For $x\in\mathbf{R}^{n+1}$, the ideal J_x is generated by $(\mathrm{in}_x(f_1),\dots,\mathrm{in}_x(f_m))$, by definition of a universal Gröbner basis. It follows that there exists a finite subset C_2 of $\bar{k}$ such that for every b in an extension k' of k such that $b\notin C_2$, one has $\mathscr{J}_{x,b}=(\mathrm{in}_x(f_1)(b;T),\dots,\mathrm{in}_x(f_m)(b;T))$.

Let a be an element of a valued extension K' of K such that $a\notin B$, $v(a)=0$ and $\bar{a}\notin C_1\cup C_2$. Then one has $\mathrm{in}_x(f_j(a;T))=\mathrm{in}_x(f_j)(\bar{a};T)$, so that $\mathscr{J}_{x,\bar{a}}=(\mathrm{in}_x(f_1)(\bar{a};T),\dots,\mathrm{in}_x(f_m)(\bar{a};T))\subset\mathrm{in}_x(\mathscr{I}_a)$.

By flatness of $K[s][T_0,\dots,T_n]/\mathscr{I}$ over $K[s]$, the homogeneous ideals $\mathscr{I}_a$ and I have the same Hilbert function. Similarly, the homogeneous ideals $\mathscr{J}_{x,\bar{a}}$ and J_x have the same Hilbert function. Moreover, by theorem 3.5.12, the homogeneous ideals $I^h\subset K(s)[T_0,\dots,T_n]$ and $J_x=\mathrm{in}_x(I^h)\subset k(s)[T_0,\dots,T_n]$ have the same Hilbert function; similarly, the homogenous ideals $\mathscr{I}_a^h\subset K'[T_0,\dots,T_n]$ and $\mathrm{in}_x(\mathscr{I}_a^h)\subset k'[T_0,\dots,T_n]$ have the same Hilbert function. It follows that the inclusion $\mathscr{J}_{x,\bar{a}}\subset\mathrm{in}_x(\mathscr{I}_a^h)$ is an equality: $\mathscr{J}_{x,\bar{a}}=\mathrm{in}_x(\mathscr{I}_a^h)$.

d) These equalities imply that the Gröbner decompositions of $\mathbf{R}^{n+1}$ associated with the homogeneous ideals I^h and $\mathscr{I}_a^h$ coincide, for every such a. Let $x\in\mathbf{R}^n$ and let $x'=(0,x)\in\mathbf{R}^{n+1}$; we know that $x\in\mathscr{T}_X$ if and only if $\mathrm{in}_x(I)\neq(1)$, if and only if $\mathrm{in}_x(I^h)$ contains no monomials.

Similarly, $x \in \mathscr{T}_{\mathscr{X}_a}$ if and only if $\mathrm{in}_x(\mathscr{I}_a) \neq (1)$, if and only if $\mathrm{in}_x(\mathscr{I}_a^{\mathrm{h}})$ contains no monomial.

For good a as above, this already implies that $\mathscr{T}_{\mathscr{X}_a} \subset \mathscr{T}_{\mathrm{X}}$. Let indeed $x \in \mathbf{R}^n - \mathscr{T}_{\mathrm{X}}$ and let $x' = (0, x)$. Then $\mathrm{J}_{x'} = \mathrm{in}_{x'}(\mathrm{I}^{\mathrm{h}})$ contains a monomial; it then belongs to $\mathscr{J}_{x'}$, so that $\mathrm{in}_{x'}(\mathscr{I}_a^{\mathrm{h}}) = \mathscr{J}_{x',\bar{a}}$ contains a monomial as well. Consequently, $x \notin \mathscr{T}_{\mathscr{X}_a}$.

The converse inclusion will require to put an additional restriction on the set of good a. Let $\mathscr{Y}_x \subset \mathbf{G_m}^n_{k[s]}$ be the closed subscheme defined by the ideal $\mathscr{J}_x$. Its image V_x in $\mathbf{A}^1_k = \mathrm{Spec}(k[s])$ is the set of points α of $\mathbf{A}^1$ such that $\mathscr{Y}_{x,\alpha} \neq (1)$. By a theorem of Chevalley, it is a constructible subset of $\mathbf{A}^1_k$. Since $\mathbf{A}^1_k$ has dimension 1, there are only two possibilities: either V_x is a strict closed subset, or V_x is a dense open subset and its complement is finite. The first case happens if and only if the generic point of $\mathbf{A}^1_k$ does not belong to V_x, *i.e.*, if J_x contains 1, that is, if and only if $x \notin \mathscr{T}_{\mathrm{X}}$. Let C_3 be the set of points in $\bar{k}$ which do not belong to those V_x, for $x \in \mathscr{T}_{\mathrm{X}}$. Since there are only finitely many ideals of the form J_x, the set C_3 is finite.

Let a be an element of a valued extension K' of K such that $a \notin \mathrm{B}$, $v(a) = 0$ and $\bar{a} \notin \mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{C}_3$. By construction, if a point $x \in \mathbf{R}^n$ belongs to $\mathscr{T}_{\mathrm{X}}$, then $\mathscr{Y}_{x,\bar{a}} \neq \varnothing$, hence $\mathrm{in}_x(\mathscr{I}_a) \neq (1)$ and $x \in \mathscr{T}_{\mathscr{X}_a}$.

This proves the equality $\mathscr{T}_{\mathrm{X}} = \mathscr{T}_{\mathscr{X}_a}$ for all such a. We also saw above the coincidence of the Gröbner polyhedral decompositions of this polyhedral subset of $\mathbf{R}^n$ respectively associated with the ideals I and $\mathscr{I}_a$.

e) It remains to prove the equality of multiplicities. Let $x \in \mathbf{R}^n$ and let C be a polyhedron of these Gröbner decompositions. Up to a monomial change of variable, we may assume that the affine span of C is $x + \mathbf{R}^d \times \{0\}$. Then one has

$$\mathrm{mult}_{\mathscr{T}_{\mathrm{X}}}(\mathrm{C}) = \dim(k(s)[\mathrm{T}_{d+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]/\mathrm{J}_x \cap k(s)[\mathrm{T}_{d+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}])$$

and

$$\mathrm{mult}_{\mathscr{T}_{\mathscr{X}_a}}(\mathrm{C}) = \dim(k'[\mathrm{T}_{d+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]/\mathscr{J}_{x,\bar{a}} \cap k'[\mathrm{T}_{d+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]).$$

Let $\mathscr{A}$ be the finitely generated $k[s]$-algebra $k[s][\mathrm{T}_{d+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]/\mathscr{J}_x \cap k[s][\mathrm{T}_{d+1}^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. It is flat, by construction, and its generic fiber $\mathscr{A} \otimes_{k[s]} k(s)$ is a finite $k(s)$-algebra of rank $\mathrm{mult}_{\mathscr{T}_{\mathrm{X}}}(\mathrm{C})$. Consequently, $\mathscr{A}$ is finite over $k[s]$, of constant rank. In particular,

$$\mathrm{mult}_{\mathscr{T}_{\mathscr{X}_a}}(\mathrm{C}) = \dim_{k'}(\mathscr{A} \otimes_{k[s]} k') = \mathrm{mult}_{\mathscr{T}_\mathrm{X}}(\mathrm{C}).$$

This concludes the proof. □

Lemma **(6.4.5)**. — *Let* $\mathrm{I} \subset \mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* $x \in \mathbf{R}^{n-1} \times \{0\}$. *One has the following equality of ideals in* $k(s)[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$:

$$\mathrm{in}_x(\mathrm{I}_{\mathrm{K}(s)} + (\mathrm{T}_n - s)) = \mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s).$$

Recall that the field $\mathrm{K}(s)$ is endowed with the Gauss absolute value; in particular, $v(s) = 0$.

Proof. — One has $\mathrm{in}_x(\mathrm{I})_{k(s)} = \mathrm{in}_x(\mathrm{I}_{\mathrm{K}(s)})$ and $\mathrm{in}_x(\mathrm{T}_n - s) = \mathrm{T}_n - s$ since $x_n = 0$. This implies the inclusion

$$\mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s) \subset \mathrm{in}_x(\mathrm{I}_{\mathrm{K}(s)} + (\mathrm{T}_n - s)).$$

Conversely, let $h \in \mathrm{I}_{\mathrm{K}(s)} + (\mathrm{T}_n - s)$ and let us prove that $\mathrm{in}_x(h) \in \mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s)$. Up to multiplying h by a non-zero element of $\mathrm{K}[s]$, we may assume that there exist $p \in \mathrm{K}[s]$, $f \in \mathrm{I}$ and $g \in \mathrm{K}[s][\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ such that $h = pf + (\mathrm{T}_n - s)g$. Writing $s = \mathrm{T}_n - (\mathrm{T}_n - s)$, there exists a polynomial $q \in k[s][\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ such that $p = p(\mathrm{T}_n) + (\mathrm{T}_n - s)q$. We then write $h = pf + (\mathrm{T}_n - s)g = p(\mathrm{T}_n)f + (\mathrm{T}_n - s)(g + q)$. This allows to assume that $p = 1$.

Observe that $\tau_x((\mathrm{T}_n - s)g) = \tau_x(\mathrm{T}_n - s) + \tau_x(g) = \tau_x(g)$ since $x_n = 0$ and $v(s) = 0$; moreover, $\mathrm{in}_x((\mathrm{T}_n - s)g) = (\mathrm{T}_n - s)\mathrm{in}_x(g)$.

If $\tau_x(f) < \tau_x((\mathrm{T}_n - s)g)$, then $\tau_x(h) = \tau_x(f + (\mathrm{T}_n - s)g) = \tau_x(f)$ and $\mathrm{in}_x(h) = \mathrm{in}_x(f)$.

Similarly, if $\tau_x(f) > \tau_x((\mathrm{T}_n - s)g)$, then $\tau_x(h) = \tau_x((\mathrm{T}_n - s)g) = \tau_x(g)$ and $\mathrm{in}_x(h) = \mathrm{in}_x((\mathrm{T}_n - s)g) = (\mathrm{T}_n - s)\mathrm{in}_x(g)$.

Assume finally that $\tau_x(f) = \tau_x((\mathrm{T}_n - s)g)$. Since $\deg_s(\mathrm{in}_x(f)) = 0$ and $\deg_s(\mathrm{in}_x((\mathrm{T}_n - s)g)) \geqslant 1$, one has $\mathrm{in}_x(f) + \mathrm{in}_x((\mathrm{T}_n - s)g) \neq 0$. Consequently, $\tau_x(h) = \tau_x(f)$ and $\mathrm{in}_x(h) = \mathrm{in}_x(f) + \mathrm{in}_x((\mathrm{T}_n - s)g) = \mathrm{in}_x(f) + (\mathrm{T}_n - s)\mathrm{in}_x(g)$.

In these three cases, this proves that $\mathrm{in}_x(h) \in \mathrm{I}_{k(s)} + (\mathrm{T}_n - s)$. This concludes the proof of the lemma. □

Proposition **(6.4.6)** (Jensen & Yu (2016)). — *Let* I *be an ideal of* $\mathrm{K}[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$, *let* $\mathrm{X} = \mathrm{V}(\mathrm{I})$. *Let* $\mathrm{H} = \partial(\sup(x_n, 0)) \subset \mathbf{R}^n$ — *the* *hyperplane defined by* $x_n = 0$ *with multiplicity* 1. *Let* $\mathrm{J} = \mathrm{I}_{\mathrm{K}(s)} + (\mathrm{T}_n - s) \subset \mathrm{K}(s)[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$ *and let* $\mathrm{Y} = \mathrm{V}(\mathrm{J})$. *One has the equality of tropicalizations*

$$\mathscr{T}_{\mathrm{Y}} = \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}.$$

Proof. — Let us prove that the following five assertions, for $x \in \mathbf{R}^n$, are equivalent.

(i) One has $x \in \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}$;
(ii) One has $\mathscr{T}_{\mathrm{in}_x(\mathrm{I})} \not\subset \mathrm{H}$;
(iii) One has $\mathrm{in}_x(\mathrm{I}) \cap k[\mathrm{T}_n, \mathrm{T}_n^{-1}] = (0)$;
(iv) One has $\mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s) \neq (1)$;
(v) One has $x \in \mathscr{T}_{\mathrm{Y}}$.

(i)⇔(ii). One has $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) = \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$.

If $\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})} \subset \mathrm{H}$, then a generic deplacement by a vector v such that $v_n \neq 0$ shows that the stable intersection is empty; in particular, $x \notin \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})} \cap_{\mathrm{st}} \mathrm{H}$, hence $x \notin \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}$.

Otherwise, there exists a polyhedral convex cone $\mathrm{Q} \subset \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})}$ such that that $x \in \mathrm{Q}$ and $\mathrm{Q} \not\subset \mathrm{H}$. The polyhedral convex cone $\mathrm{Q} + \mathrm{H}$ has dimension n. If we perform a generic deplacement by a vector $v \in \mathring{\mathrm{Q}} + \mathrm{H}$ such that $v_n > 0$, we obtain a strictly positive contribution of (Q, H) to the intersection $\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I})} \cap_{\mathrm{st}} \mathrm{H}$. In particular, $x \in \mathscr{T}_{\mathrm{X}} \cap_{\mathrm{st}} \mathrm{H}$.

(ii)⇔(iii). Let $p \colon {\mathbf{G}_{\mathrm{m}}}^n \to \mathbf{G}_{\mathrm{m}}$ be the projection to the last factor; similarly, let $\pi \colon \mathbf{R}^n \to \mathbf{R}$ be the projection to the last factor. One has $\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}) = \mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}$ and $\pi(\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}})) = \mathscr{T}_{\mathrm{V}(\mathrm{I}_n)}$, where $\mathrm{I}_n = \mathrm{in}_x(\mathrm{I}) \cap \mathrm{K}[\mathrm{T}_n^{\pm 1}]$, since $\overline{p(\mathrm{V}(\mathrm{in}_x(\mathrm{I})))} = \mathrm{V}(\mathrm{I}_n)$. The inclusion $\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))} \subset \mathrm{H}$ is equivalent to $\pi(\mathscr{T}_{\mathrm{V}(\mathrm{in}_x(\mathrm{I}))}) = \{0\}$, hence to $\mathscr{T}_{\mathrm{V}(\mathrm{I}_n)} = \{0\}$. It implies that $\mathrm{I}_n \neq (0)$ (otherwise, $\mathrm{V}(\mathrm{I}_n) = \mathbf{G}_{\mathrm{m}k}$ and $\mathscr{T}_{\mathrm{V}(\mathrm{I}_n)} = \mathbf{R}$). Conversely, if $\mathrm{I}_n \neq (0)$, then $\mathrm{V}(\mathrm{I}_n)$ is a finite subscheme of $\mathbf{G}_{\mathrm{m}}$, $\pi(\mathrm{Star}_x(\mathscr{T}_{\mathrm{X}}))$ is finite; since it is a fan, it is then reduced to 0.

(iii)⇔(iv). — Let $f \in k[\mathrm{T}_n^{\pm 1}]$ be a non-zero Laurent polynomial. Since s is transcendental, one has $f(s) \neq 0$ and the ideal $(f, \mathrm{T}_n - s)$ of $k(s)[\mathrm{T}_n^{\pm 1}]$ contains 1. If, moreover, $f \in \mathrm{in}_x(\mathrm{I})$, this implies that $\mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s) = (1)$. Assume conversely that $\mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n) = (1)$ and let us consider a relation of the form $1 = \sum g_j \mathrm{in}_x(f_j) + (\mathrm{T}_n - s)h$, where $f_j \in \mathrm{I}$, $g_j \in k(s)$ and $h \in k(s)[\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. Let $p \in k[s]$ be a non-zero polynomial such that $pg_j \in k[s]$ for all j, and $ph \in k[s][\mathrm{T}_1^{\pm 1}, \dots, \mathrm{T}_n^{\pm 1}]$. In the relation $p = \sum pg_j \mathrm{in}_x(f_j) + (\mathrm{T}_n - s)ph$ we substitute T_n to s. We obtain $p(\mathrm{T}_n) = \sum_j (pg_j)(\mathrm{T}_n)\mathrm{in}_x(f_j)$, which proves that $p(\mathrm{T}_n) \in \mathrm{in}_x(\mathrm{I}) \cap k[\mathrm{T}_n^{\pm 1}]$.

The equivalence (iv)⇒(v) follows from the preceding lemma. Indeed, $x \in \mathscr{T}_{\mathrm{Y}}$ if and only if $\mathrm{in}_x(\mathrm{J}) \neq (1)$, which is then equivalent to $\mathrm{in}_x(\mathrm{I})_{k(s)} + (\mathrm{T}_n - s) \neq (1)$.

It remains to explain compare the multiplicities. □

6.5. A tropical version of Bernstein's theorem

APPENDIX A

APPENDIX

A.1. Matroids

A.1.1. Circuits of a matroid. — We first prove that the circuits of a matroid M satisfy the axioms given in proposition 5.2.6.

The axiom (C_1) follows from (I_1): since the empty set is free, it is not a circuit.

Let C, C' be distinct dependent subsets of M such that $C \subset C'$. Then C' is not minimal, hence C is not a circuit. This establishes the axiom (C_2).

Let us C, C' be distinct cricuits and let $e \in C \cap C'$. Let us assume that $(C \cup C') - \{e\}$ does not contain any circuit. Then it is free. By (C_2), one has $C \not\subset C'$, hence we may choose $f \in C' - C$; by the definition of a circuit, the set $C' - \{f\}$ is free. Let L be a maximal free subset of $C \cup C'$ containing $C' - \{f\}$. Since C and C' are not free, neither C nor C' is contained in L, so that $f \notin L$ and there exists $g \in C - L$; since $f \in C' - C$, one has $f \neq g$. Then $L \subset (C \cup C') - \{f, g\}$, hence $\mathrm{Card}(L) \leqslant \mathrm{Card}(C \cup C') - 2 < \mathrm{Card}((C \cup C') - \{e\})$. By the axiom ($I_3$) of free subsets, there exists an element x of $(C \cup C') - \{e\}$ such that $x \notin L$ and such that $L \cup \{x\}$ is free; this contradicts the maximality of L.

Conversely, let us prove that the axioms of circuits give rise to a matroid structure. We consider a subset $\mathscr{C}_M$ of $\mathfrak{P}(M)$ satisfying the axioms (C_1), (C_2) and (C_3) and prove that it is the set of circuits of a unique matroid structure on the set M. Necessarily, its set $\mathscr{I}_M$ of independent subsets consists of all subsets of M that do not contain any circuit.

Since circuits are nonempty (axiom (C_1)), the empty set is independent; this establishes axiom (I_1).

The proof of axiom (I_2) is an argument of ordered sets: let A, B be subsets of M such that $A \subset B$ and $A \notin \mathscr{I}_M$. Then A contains a circuit $C \in \mathscr{C}_M$, hence $B \supset C$ and $B \notin \mathscr{I}_M$.

Let us now establish axiom (I_3). Let A, B be independent subsets of M such that $\mathrm{Card}(A) < \mathrm{Card}(B)$; let us prove that there exists $e \in B - A$ such that $A \cup \{e\}$ is independent. Among all free subsets of $A \cup B$, let us choose one L which has maximal cardinality, and, if there are more than one of them, such that $\mathrm{Card}(L - A)$ is minimal. One has $\mathrm{Card}(L) \geqslant \mathrm{Card}(B) > \mathrm{Card}(A)$. If $A \subset L$, then there exists $b \in L - A$; since $A \cup \{b\} \subset L$, this implies that $A \cup \{b\}$ is free. Otherwise, let us choose $a \in A - L$.

For every $f \in L - A$, let $L_f = (L \cup \{e\}) - \{f\}$, so that $\mathrm{Card}(L_f) = \mathrm{Card}(L)$ and $\mathrm{Card}(L_f - A) = \mathrm{Card}(L - A) - 1$. By the choice of L, the set L_f is not free, hence it contains a circuit C_f. One has $f \notin C_f$, because $f \notin L$; one has $e \in C_f$, because $L - \{f\}$ is free. Moreover, $C_f \cap (L - A) \neq \varnothing$, for, otherwise, one would have $C_f \subset A \cap A \cap (L \cup \{e\}) \subset A$, which contradicts the hypothesis that A is free.

Let then $g \in L - A$ and let us choose $h \in C_g - A$. One has $h \in C_g$ but $h \notin C_h$, hence $C_g \neq C_h$. One also has $e \in C_g \cap C_h$. By the axiom (C_3), there exists a circuit D such that $D \subset (C_g \cup C_h) - \{e\}$. Since $C_g, C_h \subset L \cup \{e\}$, one has $D \subset L \cup \{e\}$. Consequently, $D \subset L$, which contradicts the hypothesis that L is free.

Proposition **(A.1.2)**. — *Let* M *be a matroid, let* C, C' *be distinct circuits in* M, *let* $e \in C \cap C'$ *and let* $f \in C - C'$. *Then there exists a circuit* D *in* M *such that* $f \in D$ *and* $D \subset (C \cup C') - \{e\}$.

Proof. — We argue by induction on $\mathrm{Card}(C \cup C')$.

By the axiom (C_3) of circuits, there exists a circuit D_1 in M such that $D_1 \subset (C \cup C') - \{e\}$. If $f \in D_1$, then we are done. Assume that $f \notin D_1$. Since D_1 is a circuit and $e \in C - D_1$, one has $D_1 \not\subset C$ and there exists $g \in D_1 - C$; in particular, one has $g \in C'$.

Since $C' \cup D_1 \subset (C \cup C') - \{f\}$, one has $\mathrm{Card}(C' \cup D_1) < \mathrm{Card}(C \cup C')$; apply the induction hypothesis to the circuits C' and D_1, to the element $g \in D_1 \cap C'$ and to the element $e \in C' - D_1$: there exists a circuit D_2 such that $e \in D_2$ and $D_2 \subset (C' \cup D_1) - \{g\}$. Note that $f \notin D_2$.

One has $g \notin C \cup D_2$, hence $C \cup D_2 \subset (C \cup C') - \{g\}$ and $\mathrm{Card}(C \cup D_2) < \mathrm{Card}(C \cup C')$. By the induction hypothesis, applied to the circuits C and D_2, to the element $e \in C \cap D_2$ and to the element $f \in C - D_2$, there exists a circuit D in M such that $f \in D$ and $D \subset (C \cup D_2) - \{e\}$. This concludes the proof, since $C \cup D_2 \subset C \cup C'$. □

A.1.3. Bases of a matroid. — Let us prove that the bases of a matroid M satisfy the properties (B_1) and (B_2) of proposition 5.2.9.

Existence of bases ((B_1)) is proposition 5.2.3.

Let then B, B' be bases and let $x \in B - B'$. Then $B - \{x\}$ and B' are free subsets of M, and one has $\mathrm{Card}(B - \{x\}) = \mathrm{Card}(B) - 1 < \mathrm{Card}(B')$, by proposition 5.2.3. By axiom (I_3) of independent subsets, there exists $y \in B' - (B - \{x\})$ such that $(B - \{x\}) \cup \{y\}$ is free, hence is contained in some basis B^*. Since $\mathrm{Card}(B^*) \geqslant \mathrm{Card}((B - \{x\}) \cup \{y\}) = \mathrm{Card}(B)$, one has $B^* = (B - \{x\}) \cup \{y\}$. This establishes the axiom (B_2) of bases.

Conversely, let $\mathscr{B}_M$ be a subset of $\mathfrak{P}(M)$ satisfying the properties (B_1) and (B_2), and let us prove that it is the set of bases for a unique structure of matroid on M.

We first prove that *all elements of $\mathscr{B}_M$ have the same cardinality*. Otherwise, let B, B' be elements of $\mathscr{B}_M$ such that $\mathrm{Card}(B) < \mathrm{Card}(B')$ and such that $\mathrm{Card}(B' - B)$ is minimal. In particular, $B' - B$ is not empty and there exists $x \in B' - B$. By the axiom (B_2) of bases, there exists $y \in B - B'$ such that $B^* = (B' - \{x\}) \cup \{y\}$ is a basis. Since $B^* - B = B' - (B \cup \{x\})$ has cardinality less than $B' - B$, this contradicts the minimality assumption on the pair (B, B').

Necessarily, the independent subsets for some matroid structure on M with bases $\mathscr{B}_M$ must be defined as the subsets of some element of $\mathscr{B}_M$ and we need to prove this set $\mathscr{I}_M$ satisfies the axioms (I_1), (I_2), (I_3). Since $\mathscr{B}_M$ is not empty, $\varnothing$ is contained in some basis, hence $\varnothing \in \mathscr{I}_M$; this proves ($I_1$).

Let A, B be subsets of M such that $B \in \mathscr{I}_M$. Let $B' \in \mathscr{B}_M$ be a basis of M containing B. Then $A \subset B'$, hence $A \in \mathscr{I}_M$, establishing (I_2).

Let L, L' be elements of $\mathscr{I}_M$ such that $\mathrm{Card}(L) \subset \mathrm{Card}(L')$. Let us choose bases B, B' with $L \subset B$ and $L \subset B'$ and such that $\mathrm{Card}(B' - (L' \cup B))$ is minimal.

One has $L' - B \subset L' - L$. Assume that the inclusion is strict; then there exists $x \in L' \cap B$ such that $x \notin L$. Since $L \cup \{x\} \subset B$, the set $L \cup \{x\}$ is free, and $x \in L'$, so that the axiom (I_3) holds.

Otherwise, one has $L' - B = L' - L$. Let us prove that $B' \subset L' \cup B$. Otherwise, let $x \in B' - B$ such that $x \notin L'$. By the axiom (B_2), there exists $y \in B - B'$ such that $B'_1 = (B' - \{x\}) \cup \{y\}$ belongs to $\mathscr{B}_M$. One has $L' \subset B'_1$ by construction. Moreover, $B'_1 - (L' \cup B) = B' - (L' \cup B \cup \{y\})$, and this contradicts the minimality assumption on $\operatorname{Card}(B' - (L' \cup B))$. In conclusion, one has $B' - B = L' - B = L' - L$.

Let us prove that $B \subset L \cup B'$. Otherwise, let $x \in B$ such that $x \notin L \cup B'$. By the axiom (B_2), there exists $y \in B' - B$ such that $B_1 = (B - \{x\}) \cup \{y\}$ belongs to $\mathscr{B}_M$. Since $L \cup \{y\} \subset B_1$, this implies that $L \cup \{y\}$ is free. Since $y \in B' - B = L' - L$, one has $y \in L' - L$. This concludes the proof of axiom (I_3).

A.1.4. Matroids and the rank function. — Let us prove that the rank function of a matroid M satisfies the axioms (R_1), (R_2) and (R_3).

If X is a a basis of $M \mid A$, then $X \subset A$, hence $0 \leqslant \operatorname{Card}(X) \leqslant \operatorname{Card}(A)$; since $\operatorname{rank}_M(A) = \operatorname{Card}(X)$, this establishes axiom (R_1).

Similarly, if X is a basis of $M \mid A$ and $A \subset B$, then X is a free subset of $M \mid B$, so that $\operatorname{rank}_M(A) = \operatorname{rank}_{M|A}(M \mid A) = \operatorname{Card}(X) \leqslant \operatorname{rank}_{M|A}(M \mid B) = \operatorname{rank}_M(B)$.

Let A, B be subsets of M. Let X be a basis of $M \mid (A \cap B)$. As a subset of $M \mid (A \cup B)$, the set X is still free, hence there exists a basis Y of $M \mid (A \cup B)$ such that $X \subset Y$. Then, $Y \cap A$ is free in $M \mid A$, hence $\operatorname{rank}_M(A) \geqslant \operatorname{Card}(Y \cap A)$; similarly, $\operatorname{rank}_M(B) \geqslant \operatorname{Card}(Y \cap B)$. Consequently,

$$\begin{aligned}\operatorname{rank}_M(A) + \operatorname{rank}_M(B) &\geqslant \operatorname{Card}(Y \cap A) + \operatorname{Card}(Y \cap B) \\ &= \operatorname{Card}(Y \cap A \cap B) + \operatorname{Card}(Y \cap (A \cup B)) \\ &= \operatorname{Card}(Y \cap A \cap B) + \operatorname{Card}(Y).\end{aligned}$$

One has $\operatorname{Card}(Y) = \operatorname{rank}_M(A \cup B)$, by definition. Moreover, $Y \cap A \cap B$ is a free subset of $A \cap B$ that contains X; since X is a basis of $M \mid (A \cap B)$, one has $Y \cap A \cap B = X$ and $\operatorname{Card}(Y \cap A \cap B) = \operatorname{rank}_M(A \cap B)$. This proves the inequality

$$\operatorname{rank}_M(A) + \operatorname{rank}_M(B) \geqslant \operatorname{rank}_M(A \cap B) + \operatorname{rank}_M(A \cup B)$$

and establishes axiom (R_3).

Conversely, let us assume that r is a function on $\mathfrak{P}(M)$ which satisfies the axioms (R_1), (R_2) and (R_3), and let us prove that it is the rank of a unique structure of matroid on M. Necessarily, the free subsets of M would be the subsets A such that $r(A) = \operatorname{Card}(A)$, so that it suffices to prove that the set $\mathscr{I}_M$ of such subsets A satisfies the axioms of independents subsets of a matroid.

Let us first prove the following property: *Let* $A, B \subset M$ *be such that* $r(A \cup \{b\}) = r(A)$ *for all* $b \in B$; *then* $r(A \cup B) = r(A)$. We argue by induction on Card(B); the property holds if $B \subset A$; let then $b \in B - A$ and $B' = B - \{b\}$. One has $r(A \cup B') = r(A)$ by the induction hypothesis, and $r(A \cup \{b\}) = r(A)$ by assumption. Using (R_3), one has

$$2r(A) = r(A \cup B') + r(A \cup \{b\}) \geqslant r(A \cup B) + r(A),$$

hence $r(A) \geqslant r(A \cup B)$. This implies the equality $r(A \cup B) = r(A)$, as claimed.

Using (R_1), one has $0 \leqslant r(\varnothing) \leqslant \operatorname{Card}(\varnothing)$, hence $r(\varnothing) = 0 = \operatorname{Card}(\varnothing)$; consequently, $\varnothing$ is free, establishing axiom (I_1).

Let A, B be subsets of M such that $A \subset B$ and $B \in \mathscr{I}_M$. Then $r(B) = \operatorname{Card}(B)$, by assumption, hence

$$\begin{aligned}\operatorname{Card}(B) = r(B) &= r(A \cup (B - A)) + r(A \cap (B - A)) \\ &\leqslant r(A) + r(B - A) \leqslant \operatorname{Card}(A) + \operatorname{Card}(B - A) = \operatorname{Card}(B).\end{aligned}$$

This implies that $r(A) = \operatorname{Card}(A)$, hence $A \in \mathscr{I}_M$. Axiom (I_2) is proved.

Let A, B be subsets of M belonging to $\mathscr{I}_M$ such that $\operatorname{Card}(A) < \operatorname{Card}(B)$. Let us assume that $A \cup \{b\}$ does not belong to $\mathscr{I}_M$, for all $b \in B - A$. Then $r(A \cup \{b\}) = r(A)$ for all $b \in B$, hence $r(A \cup B) = r(A)$, by the property established above. On the other hand, $B \subset A \cup B$, hence $r(A \cup B) \geqslant r(B) = \operatorname{Card}(B)$, contradicting the inququality $\operatorname{Card}(A) < \operatorname{Card}(B)$. Consequently, there exists $b \in B - A$ such that $A \cup \{b\}$ belongs to $\mathscr{I}_M$; this proves axiom (I_3).

Finally, it remains to prove that the rank function rank_M associated with this matroid structure coincides with the function r. let A be a subset of M. If A is free, then $\mathrm{rank}_{\mathrm{M}}(\mathrm{A}) = \mathrm{Card}(\mathrm{A}) = r(\mathrm{A})$. Otherwise, let B be a maximal free subset of A, that is, a basis of $\mathrm{M} \mid \mathrm{A}$; one has $\mathrm{rank}_{\mathrm{M}}(\mathrm{A}) = \mathrm{Card}(\mathrm{B}) = r(\mathrm{B})$ by definition. For every $a \in \mathrm{B} - \mathrm{A}$, the set $\mathrm{B} \cup \{a\}$ is not free, hence $r(\mathrm{B} \cup \{a\}) \neq \mathrm{Card}(\mathrm{B} \cup \{a\})$. Since $\mathrm{Card}(\mathrm{B}) = r(\mathrm{B}) \leqslant r(\mathrm{B} \cup \{a\}) \leqslant \mathrm{Card}(\mathrm{B}) + 1$, this implies that $r(\mathrm{B} \cup \{a\}) = r(\mathrm{B})$. By the property above, one thus has $r(\mathrm{A}) = r(\mathrm{B} \cup (\mathrm{A} - \mathrm{B})) = r(\mathrm{B})$. In other words, $\mathrm{rank}_{\mathrm{M}}(\mathrm{A}) = r(\mathrm{A})$, as was to be shown.

Proposition **(A.1.5)** (Brualdi, 1969). — *Let* M *be a matroid, let* B, B′ *be bases of* M *and let* $a \in \mathrm{B} - \mathrm{B}'$. *There exists* $b \in \mathrm{B}' - \mathrm{B}$ *such that* $(\mathrm{B} - \{a\}) \cup \{b'\}$ *and* $(\mathrm{B}' - \{b\}) \cup \{a\}$ *are bases of* M.

Proof. — Let C′ be the unique circuit contained in $\mathrm{B}' \cup \{a\}$ (lemma 5.2.8); one has $a \in \mathrm{C}'$ and $\mathrm{C}' - \mathrm{B} \subset \mathrm{B}'$. For $x \in \mathrm{C}'$, the set $(\mathrm{B}' - \{x\}) \cup \{a\}$ is a basis of M, since it is free (lemma 5.2.8, *c*) and its cardinality is equal to $\mathrm{Card}(\mathrm{B}')$.

Since B is free, one has $\mathrm{C}' \not\subset \mathrm{B}$; since $\mathrm{C}' - \mathrm{B}$ is nonempty. For any $x \in \mathrm{C}' - \mathrm{B}$, let then C_x be the unique circuit such that $\mathrm{C}_x \subset \mathrm{B} \cup \{x\}$ (lemma 5.2.8).

First assume that there exists $b \in \mathrm{C}' - \mathrm{B}$ such that $a \in \mathrm{C}_b$. In this case, lemma 5.2.8, *c*), applied to B and *b*, implies that $(\mathrm{B} - \{a\}) \cup \{b\}$ is free, hence is a basis of M since it has the same cardinality as B. As we have seen above, $(\mathrm{B}' - \{b\}) \cup \{a\}$ is a basis of M. This establishes the proposition in this case.

Let us thus argue by contradiction and suppose que $a \notin \mathrm{C}_x$ for all $x \in \mathrm{C}' - \mathrm{B}$. Let us define sequences (x_n) and (D_n), where D_n is a circuit of M such that $a \in \mathrm{D}_n$, $x_n \in \mathrm{D}_n - \mathrm{B}$ and $\mathrm{D}_{n+1} \subset (\mathrm{D}_n \cup \mathrm{C}_{x_n}) - \{x_n\}$ for all n. We set $\mathrm{D}_0 = \mathrm{C}$. Assume that $(\mathrm{D}_0, \dots, \mathrm{D}_n)$ and $(x_0, \dots, x_{n-1})$ are defined. One has $a \in \mathrm{D}_n$ by construction, and $\mathrm{D}_n \not\subset \mathrm{B}$; let $x_n \in \mathrm{D}_n - \mathrm{B}$. By proposition A.1.2, applied to the circuits D_n and C_{x_n} and to the points $x_n \in \mathrm{D}_n \cap \mathrm{C}_{x_n}$ and $a \in \mathrm{D}_n - \mathrm{C}_{x_n}$, there exists a circuit D_{n+1} such that $\mathrm{D}_{n+1} \subset (\mathrm{D}_n \cup \mathrm{C}_{x_n}) - \{x_n\}$ and $a \in \mathrm{D}_{n+1}$. This concludes the desired construction.

Since $\mathrm{C}_{x_n} \subset \mathrm{B} \cup \{x_n\}$, one has $\mathrm{D}_{n+1} - \mathrm{B} \subset \mathrm{D}_n$. Moreover, $x_n \in \mathrm{D}_n - \mathrm{B}$ and $x_n \notin \mathrm{D}_{n+1}$. This implies that the sequence $\mathrm{D}_n - \mathrm{B}$ is strictly decreasing, in contradiction with the fact that is is a sequence of finite sets. This concludes the proof. □

A.1.6. Matroids and the closure operator. — Let us prove properties (c_1) to (c_4).

Property (c_1) is obvious: for $x \in A$, one has $A \cup \{x\} = A$, hence $\operatorname{rank}_M(A) = \operatorname{rank}_M(A \cup \{x\})$.

Let A, B be subsets of M such that $A \subset B$. Let $a \in \langle A \rangle$. If $a \in B$, then $a \in \langle B \rangle$; let us assume that $a \notin B$. Let X be a basis of $M \mid A$; then X is a basis of $M \mid (A \cup \{a\})$ because $\operatorname{rank}_M(A \cup \{a\}) = \operatorname{rank}_M(A) = \operatorname{Card}(X)$. Let then Y be a basis of $M \mid (B \cup \{a\})$ that contains X. One has $a \notin Y$; otherwise, $X \cup \{a\}$ would be a free subset of $A \cup \{a\}$ of cardinality $\operatorname{Card}(X) + 1$, contradicting the hypothesis that X is a basis of $A \cup \{a\}$. Consequently, Y is a basis of $M \mid B$, and $\operatorname{rank}_M(B \cup \{a\}) = \operatorname{Card}(Y) = \operatorname{rank}_M(B)$, so that $a \in \langle B \rangle$. This proves property (c_2).

Let A be a subset of M. The subset $\langle\langle A \rangle\rangle$ is the largest subset of M containing $\langle A \rangle$ with rank equal to $\operatorname{rank}_M(\langle A \rangle) = \operatorname{rank}_M(A)$. Since $\langle A \rangle$ contains A, lemma 5.2.12 implies that $\langle\langle A \rangle\rangle = \langle A \rangle$, hence ($c_3$).

Let A be a subset of M, let $a \in M$ and let $b \in \langle A \cup \{a\} \rangle - \langle A \rangle$. Then $\operatorname{rank}_M(A \cup \{b\}) = \operatorname{rank}_M(A) + 1$ and $\operatorname{rank}_M(A \cup \{a, b\}) = \operatorname{rank}_M(A \cup \{a\})$. The inequalities

$$\operatorname{rank}_M(A)+1 \leqslant \operatorname{rank}_M(A \cup \{b\}) \leqslant \operatorname{rank}_M(A \cup \{a, b\}) = \operatorname{rank}_M(A \cup \{a\}) \leqslant \operatorname{rank}_M(A)+1$$

imply that

$$\operatorname{rank}_M(A \cup \{a, b\}) = \operatorname{rank}_M(A \cup \{b\}) = \operatorname{rank}_M(A) + 1,$$

hence $a \in \langle A \cup \{b\} \rangle$, proving ($c_4$).

Conversely, let us assume that $c : \mathfrak{P}(M) \to \mathfrak{P}(M)$ is a map satisfying the properties (c_1) to (c_4), and let us show that there is a unique matroid structure on M such that $c(A) = \langle A \rangle$ for every subset A of M. Let us say that a subset A of M is free if $x \notin c(A - \{x\})$ for all $x \in A$, and let $\mathscr{I}_M$ be the set of all free subsets of M. We now prove that $\mathscr{I}_M$ satisfies the axioms (I_1), (I_2) and (I_3) of a matroid structure.

By definition, the emptyset is free, establishing (I_1).

Let A, B be subsets of M such that $A \subset B$ and B is free; let us prove that A is free. Let $x \in A$. Then $c(A - \{x\}) \subset c(B - \{x\})$; since $x \notin c(B - \{x\})$, this implies that $x \notin c(A - \{x\})$. This establishes (I_2).

Let us now prove an intermediate result: *Let* A *be a free subset of* M *and* $x \in M$; *then* $A \cup \{x\}$ *is not free if and only if* $x \in c(A) - A$. First assume that $A \cup \{x\}$ is not free; since A is free, one has $x \notin A$. Let then $y \in A \cup \{x\}$ be such that $y \in c((A \cup \{x\}) - \{y\})$. Si If $y = x$, then we get $x = y \in c(A - \{x\}) \subset c(A)$. Otherwise, one has $y \in A$, hence $y \notin c(A - \{y\})$, because A is free, and $y \in c((A - \{y\}) \cup \{x\})$. By the axiom ($c_4$), this implies $x \in c((A - \{y\}) \cup \{y\}) = c(A)$. Conversely, let $x \in c(A) - A$. Since $x \notin A$, one has $(A \cup \{x\}) - \{x\} = A$, hence $x \in c((A \cup \{x\}) - \{x\})$; this proves that $A \cup \{x\}$ is not free.

We prove axiom (I_3) by induction on $\mathrm{Card}(A \cap B)$. Let A, B be free subsets of M such that $\mathrm{Card}(A) < \mathrm{Card}(B)$. Let $b \in B - A$. If $A \subset c(B - \{b\})$, then $c(A) \subset c(B - \{b\})$, hence $b \notin c(A)$, because B is free. By the intermediate result, $B \cup \{b\}$ is free. Otherwise, let us consider $a \in A$ such that $a \notin c(B - \{b\})$. By the intermediate result, $B' = (B - \{b\}) \cup \{a\}$ is free, and $A \cap B' = (A \cap B) \cup \{a\}$ has cardinality $\mathrm{Card}(A \cap B) + 1$. By the induction hypothesis, there exists $c \in B' - A$ such that $A \cup \{c\}$ is free. Since $a \in A$, one has $c \in B$, and this concludes the proof.

Finally, we prove that $c(A) = \langle A \rangle$ for every subset A of M. Let B be a basis of $M \mid A$, so that B is a free subset of A such that $\mathrm{Card}(B) = \mathrm{rank}_M(A)$. For every $a \in A - B$, the set $B \cup \{a\}$ is not free, hence $A \subset c(B) \subset c(A)$, by the indermediate result. As a consequence, $c(B) = c(A)$.

Let $a \in \langle A \rangle$ and let us prove that $a \in c(A)$. By assumption, $\mathrm{rank}_M(A \cup \{a\}) = \mathrm{rank}_M(A)$, hence $B \cup \{a\}$ is not free. By the intermediate result, this implies that $a \in c(B)$, hence $a \in c(A)$; this proves that $\langle A \rangle \subset c(A)$.

On the other hand, let $a \in c(A) - A$. By the intermediate result, $A \cup \{a\}$ is not free, so that $a \in \langle A \rangle$.

A.1.7. Matroid and flats. — Let us prove that the set $\mathscr{F}_M$ of flats of a matroid satisfies the axioms (F_1) and (F_2).

Since $M \subset \langle M \rangle \subset M$, one has $M = \langle M \rangle$ and M is a flat.

Let A, B be flats in M. One has $\langle A \cap B \rangle \subset \langle A \rangle = A$ and $\langle A \cap B \rangle \subset \langle B \rangle B$, hence $\langle A \cap B \rangle \subset A \cap B$. This implies that $A \cap B$ is flat.

This establishes axiom (F_1).

Let $A \in \mathscr{F}_M$ be such that $A \neq M$. Let $x \in M - A$ and let $F = \langle A \cup \{x\} \rangle$; this is the smallest flat of M that contains $A \cup \{x\}$. Let $y \in F - A$. By the axiom (c_4), one has $x \in \langle A \cup \{y\} \rangle$, hence $x \in \langle A \cup \{y\} \rangle$. This implies that F is a minimal flat of M among those which strictly contain A. Since $x \in F$, this establishes axiom (F_2).

Conversely, let $\mathscr{F}$ be a subset of $\mathfrak{P}(M)$ satisfying the properties (F_1) and (F_2), and let us prove that it is the set of flats of M for a unique matroid structure on M. We define a closure operator c on $\mathfrak{P}(M)$ by setting $c(A)$ to be the intersection of all elements of $\mathscr{F}$ that contain A. This is the smallest subset of $\mathscr{F}$ containing A. If a matroid structure on M has $\mathscr{F}$ for flats, it has the map c for its closure operator, and conversely since a set A is flat if and only if $A = c(A)$. Consequently, it suffices to prove that the map c satisfies the axioms (c_1) to (c_4).

Properties (c_1), (c_2)and (c_3)follow from the definition of c. Let now A be a subset of M, $a \in M$ and $b \in c(A \cup \{a\}) - c(A)$. Then $c(A \cup \{a\})$ is the minimal element of $\mathscr{F}$ that contains $A \cup \{a\}$. Since $b \in c(A \cup \{a\})$, one has $c(A) \subset c(A \cup \{b\}) \subset c(A \cup \{a\})$. By property ($F_2$), the only elements F of $\mathscr{F}$ such that $c(A) \subset F \subset c(A \cup \{a\})$ are $c(A)$ and $c(A \cup \{a\})$. Since $b \notin c(A)$, one has $c(A \cup \{b\}) = c(A \cup \{a\}$, establishing axiom ($c_4$).

A.1.8. The lattice of flats of a matroid. — We first prove that the lattice $\mathscr{F}_M$ of flats of a matroid satisfies these properties. Axiom (L_1) is lemma 5.2.19. Granted the relation between height and rank provided by that lemma, axiom (L_2) is exactly the axiom (R_3) of ranks. Let A be a flat. The set of loops in M is the smallest flat $\langle \varnothing \rangle$ of M; in particular, it is contained in A. For any $a \in A$ which is not a loop of M, the flat $A_a = \langle a \rangle$ is an atom of $\mathscr{F}_M$ which is contained in A and contains a. Moreover, the supremum of these atoms is contained in A, contains each element of A which is not a loop, and contains each loop of A; it is thus equal to A.

Let now L be a finite lattice satisfying the axioms of proposition 5.2.20, and let us show that L is isomorphic to the lattice of flats of some matroid. Let E be the set of atoms of L; for any subset A of E, let $\langle A \rangle = \sup(A)$ and let $r(A) = \operatorname{ht}(\langle A \rangle)$.

Since $\langle \varnothing \rangle$ is the smallest element of L, one has $r(\varnothing) = 0$.

Let A, B be subsets of E such that $A \subset B$; one has $\sup(A) \preceq \sup(B)$, hence $r(A) \leqslant r(B)$.

Let A be a subset of E and let $e \in E$. Then

$$\begin{aligned} r(A \cup \{e\}) &= \operatorname{ht}(\sup(\sup(A), e)) \\ &\leqslant \operatorname{ht}(\sup(A)) + \operatorname{ht}(e) - \operatorname{ht}(\inf(\sup(A), e)) \\ &\leqslant r(A) + 1. \end{aligned}$$

Assume that $r(A \cup \{e\}) = r(A \cup \{f\}) = r(A)$. Then,

$$\begin{aligned} r(A \cup \{e, f\}) &= \operatorname{ht}(\sup(\langle A \cup \{e\}\rangle, \langle A \cup \{f\}\rangle) \\ &\leqslant \operatorname{ht}(\langle A \cup \{e\}\rangle) + \operatorname{ht}(\langle A \cup \{f\}\rangle) - \operatorname{ht}(\langle A \cup \{e\}\rangle \cap \langle A \cup \{f\}\rangle) \\ &\leqslant r(A \cup \{e\}) + r(A \cup \{f\}) - r(A), \end{aligned}$$

because $\langle A\rangle \preceq \inf(\langle A \cup \{e\}\rangle, \langle A \cup \{f\}\rangle)$. Consequently, $r(A \cup \{e, f\}) \leqslant r(A)$, and one has equality $r(A \cup \{e, f\}) = r(A)$.

These properties are the so-called *local axioms* for the rank function $r \colon \mathfrak{P}(E) \to \mathbf{N}$; they imply the axioms (R_1), (R_2) and (R_3). The inequalities of (R_1) and (R_2) hold, by induction. To prove the submodular inequality, we first remark that the intermediate result proved §A.1.4 still holds in this context: *Let* $x, y_1, \dots, y_n \in L$ *be such that* $r(\sup(x, y_i)) = r(x)$ *for all* i*; then* $r(\sup(x, y_1, \dots, y_n)) = r(x)$. It by induction, it suffices to treat the case $n = 2$, which is precisely the last of the local axioms. From that point on, one proves in the same way that the subsets $X \subset E$ such that $r(X - \{x\}) < r(X)$ for all $x \in X$ are the free sets for a unique matroid structure on E.

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