ALGEBRAIC GEOMETRY OF SCHEMES

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Version of February 7, 2024, 10h39

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CHAPTER 1

COMMUTATIVE ALGEBRA

1.1. Recollections (Uptempo)

1.1.1. Basic algebraic structures. — The concepts of *groups*, *rings*, *fields*, *modules* are assumed to be known, as well as the notion of morphisms of groups, rings, fields, modules, etc.

In this course, rings are always commutative and possess a unit element, generally denoted by 1. The multiplicative group of invertible elements of a ring A will be denoted by A^* or $A^\times$.

1.1.2. Algebras. — Let k be a ring. A *k-algebra* is a ring A endowed with a morphism of rings $f\colon k \to A$. When this morphism is injective, we will often understate the morphism f and consider that A is an overring of k, or that k is a subring of A... Let $(A, f\colon k \to A)$ and $(B, g\colon k \to B)$ we two k-algebras; a morphism of k-algebras is a ring morphism $\varphi\colon A \to B$ such that $g = \varphi \circ f$.

1.1.3. Polynomial algebras. — Let I be a set. One defines a k-algebra $k[(X_i)_{i\in I}]$ of polynomials with coefficients in k in a family $(X_i)_{i\in I}$ of indeterminates indexed by I. This algebra satisfies the following universal property: for every family $(a_i)_{i\in I}$ of elements of A, there exists a unique morphism $\varphi\colon k[(X_i)_{i\in I}] \to A$ of k-algebras such that $\varphi(X_i) = a_i$ for every $i \in I$. In other words, for every k-algebra A, the canonical map

$$\mathrm{Hom}_{k-\boldsymbol{Algebras}}(k[X_i], A) \to \mathrm{Hom}_{\boldsymbol{Ens}}(I, A), \qquad \varphi \mapsto (i \mapsto \varphi(X_i))$$

is a bijection.

When I has one, two, three,... elements, the indeterminates are often denoted by individual letters, say X, Y, Z,...

Let J be a subset of I, and let K be its complementary subset. The polynomial algebra $k[(X_i)_{i\in I}]$ is isomorphic to the polynomial algebra $k[(X_i)_{i\in J}][(X_i)_{i\in K}]$

in the indeterminates X_i (for $i \in K$) with coefficients in the polynomial algebra $k[(X_i)_{i\in J}]$ with coefficients in k in the indeterminates X_i (for $i \in J$).

We do not detail the notion of *degree* in one of the indeterminates (of degree, if I is a singleton).

There is a notion of euclidean division in polynomial rings. Let A be a ring, let $f, g \in A[X]$ be polynomials in one indeterminate X with coefficients in A. If the leading coefficient of g is invertible in A, there exist a unique pair (q, r) of polynomials in $A[X]$ such that $f = gq + r$ and $\deg(r) < \deg(g)$.

1.1.4. Ideals. — An *ideal* of a ring A is a non-empty subset I which is stable under addition, and such that $ab \in I$ for every $a \in A$ and every $b \in I$. In other words, this is a A-submodule of A.

The subsets $\{0\}$ and A are ideals. The intersection of a family of ideals of A is an ideal. If S is a subset of A, the ideal generated by S is the smallest ideal of A containing S (it is the intersection of all ideals of A which contain S). Let I and J be ideals of A; the ideal $I + J$ (resp. the ideal $I \cdot J$, also denoted by IJ) is the ideal generated by the set of sums $a + b$ (resp. the set of products ab) for $a \in I$ and $b \in J$. The ideal generated by a family of elements of A is often denoted by $((a_i)_{i\in I})$; for example (a), (a, b), (a_1, a_2, a_3)...

The image $\varphi(I)$ of an ideal I of A under a morphism of rings $\varphi\colon A \to B$ is generally not an ideal of B; the ideal it generates is often denoted by IB. However, the inverse image of an ideal J of B by such a morphism of rings is always an ideal of A. In particular, the *kernel* $\ker(\varphi) = \varphi^{-1}(0)$ of a morphism of rings is an ideal of A.

Let I be an ideal of A. The relation $x \sim y$ defined by $x - y \in I$ is an equivalence relation. The quotient set $A/\sim$, denoted by A/I, admits a unique ring structure such that the canonical surjection $\pi\colon A \to A/\sim$ is a morphism of rings. The so-called *quotient ring* A/I possesses the following universal property: for every ring B and every morphism of rings $f\colon A \to B$ such that $f(I) = \{0\}$, there exists a unique morphism of rings $\varphi\colon A/I \to B$ such that $f = \varphi \circ \pi$.

The kernel of the canonical morphism π is the ideal I itself. More generally, the map associating with an ideal J of A/I the ideal $\pi^{-1}(J)$ of A is a bijection between the (partially ordered) set of ideals of A/I and the (partially ordered) set of ideals of A which contain I.

1.1.5. Domains. — Let A be a ring. One says that an element $a \in A$ is a *zero-divisor* if there exists $b \in A$, such that $ab = 0$ and $b \neq 0$. One says that A is an *integral domain*, or a *domain*, if $A \neq \{0\}$ and if 0 is its only zero-divisor. Fields are integral domains.

1.1.6. Prime and maximal ideals. — One says that an ideal I of A is *prime* if the quotient ring A/I is an integral domain. This means that $I \neq A$ and that for every $a, b \in A$ such that $ab \in I$, either $a \in I$, or $b \in I$.

One says that an ideal I of A is *maximal* if the quotient ring A/I is a field. This means that I is a maximal element of the partially ordered set of ideals of A which are not equal to A. A maximal ideal is a prime ideal.

One deduces from Zorn's theorem that every ideal of A which is distinct from A is contained in some maximal ideal. (Indeed, if I is an ideal of A such that $I \neq A$, the set of ideals J of A such that $I \subseteq J \subsetneq A$, ordered by inclusion, is inductive—every totally ordered subset admits an upper-bound) In particular, every non-zero ring contains maximal ideals.

Hilbert's Nullstellensatz (theorem 1.7.1 below) gives a description of the maximal ideals of polynomials rings over algebraically closed fields.

1.1.7. — If a ring admits exactly one maximal ideal, one says that it is a *local ring*. A ring is local if and only if its set of non-invertible elements is an ideal (*exercise!*).

Let A and B be local rings; let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be their maximal ideals; let $\kappa(A) = A/\mathfrak{m}_A$ and $\kappa(B) = B/\mathfrak{m}_B$ be their residue fields. A morphism $f\colon A \to B$ is said to be *local* if $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ or, equivalently, if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. Observe that a local morphism $f\colon A \to B$ passes to the quotient and induces a morphism $\kappa(A) \to \kappa(B)$ between their residue fields.

1.1.8. — The intersection J of all maximal ideals of a ring A is called its Jacobson radical. It admits the following characterization: one has $a \in J$ if and only if $1 + ab$ is invertible in A for every $b \in A$ (*exercise!*).

1.1.9. — Let A be an integral domain and let K be its field of fractions. One says that A is a *valuation ring* if, for every non-zero element a of K, either $a \in R$, or $1/a \in R$ (or both).

Assume that A is a valuation ring. Let a, b be element of A which are not invertible. If $a = 0$, then $a + b = b$ is not invertible; assume that $a \neq 0$ and let

$x = b/a$. If $x \in R$, then $a + b = a(1 + x)$ is not invertible; otherwise, $x \neq 0$, hence $1/x \in R$ and $a + b = b(1 + 1/x)$ is not invertible as well. This implies that the set $A - A^{\times}$ of non-invertible elements of A is an ideal, hence *a valuation ring is a local ring.*

1.1.10. — Let A be an integral domain. One says that an element $a \in A$ is *irreducible* if it is not invertible and if the equality $a = bc$ for $b, c \in A$ implies that b or c is invertible. An element a is said to be prime if the principal ideal (a) is prime; this implies that a is irreducible but the converse does not hold (*exercise!*; show, for example, that the element $1 + i\sqrt{5}$ of the ring $\mathbf{Z}[i\sqrt{5}]$ is irreducible but not prime).

One says that the ring A is a *unique factorization domain* (UFD, in short) if the following two properties hold:

(a) Every strictly increasing sequence of principal ideals of A is finite;
(b) Every irreducible element of A generates a prime ideal.

Indeed, these two properties are equivalent to the fact that every non-zero element of A can be written as the product of an invertible element and of finitely many prime elements of A, in a unique way up to the order of the factors and to multiplication of the factors by units.

Condition (ii) is sometimes stated under the name of "Gauss's lemma": If A is a UFD, then *every irreducible element a which divides a product bc must divide one of the factors b or c.* Condition (i) obviously holds when A is noetherian. Consequently, a noetherian ring for which Gauss's lemma holds is a UFD.

Principal ideal rings are unique factorization domains, as well as polynomial rings over fields. In fact, if A is a UFD, then so is A[X] (a theorem proved by Gauss for $A = \mathbf{Z}$).

1.2. Localization (Medium up)

Let A be a ring.

1.2.1. Nilpotent elements. — One says that an element $a \in A$ is *nilpotent* if there exists an integer $n \geqslant 1$ such that $a^n = 0$. The set of nilpotent elements of A is an ideal of A, called its *nilradical*. When 0 is the only nilpotent element of A, one says that A is *reduced*. More generally, when I is an ideal of A, one defines the *radical* of I, denoted by $\sqrt{I}$, as the set of all $a \in I$ for which there exists an integer $n \geqslant 1$ such that $a^n \in I$; it is an ideal of A which contains I. An ideal which is equal to its radical is called a radical ideal.

1.2.2. Multiplicative subsets. — A *multiplicative subset* of A is a subset $S \subseteq A$ which contains 1 and such that $ab \in S$ for every $a, b \in S$.

1.2.3. — Let M be an A-module. The *fraction module* $S^{-1}M$ (sometimes also denoted by M_S) is the quotient of the set $M \times S$ by the equivalence relation $\sim$ such that $(m, s) \sim (m', s')$ if and only if there exists $t \in S$ such that $t(sm' - s'm) = 0$. Let us denote by m/s the class in $S^{-1}M$ of the pair $(m, s) \in M \times S$. The addition of the abelian group $S^{-1}M$ is given by the familiar formulas

$$(m/s) + (m'/s') = (s'm + sm')/ss',$$

for $m, m' \in M$, $s, s' \in S$; its zero is the element $0/1$. Its structure of an A-module is given by $a \cdot (m/s) = (am)/s$, for $a \in A$, $m \in M$ and $s \in S$.

For every $s \in S$, the multiplication by s is an isomorphism on $S^{-1}M$—one says that S acts by automorphisms on $S^{-1}M$. The map $\theta \colon M \to S^{-1}M$ given by $\theta(m) = m/1$ is a morphism of A-modules; it satisfies the following universal property: For every morphism of A-modules $f \colon M \to N$ such that S acts by automorphisms on N, there exists a unique morphism of A-modules $\varphi \colon S^{-1}M \to N$ such that $f = \varphi \circ \theta$ (explicitly: $f(m) = \varphi(m/1))$ for every $m \in M$).

1.2.4. — Let B be an A-algebra. Then the module of fractions $S^{-1}B$ has a natural structure of an A-algebra for which the multiplication is given by the familiar formulas

$$(b/s) \cdot (b'/s') = (bb')/(ss'),$$

for $b, b' \in B$ and $s, s' \in S$; its zero and unit are the elements $0/1$ and $1/1$. The canonical map $\theta \colon B \to S^{-1}B$ is a morphism of A-algebras, and the images of the elements of S are invertible in $S^{-1}B$. In fact, this morphism satisfies the following universal property: For every morphism of A-algebras $f \colon B \to B'$ such that the images of elements of S are units of B′, there exists a unique morphism of A-Algebras $\varphi \colon S^{-1}B \to B'$ such that $f = \varphi \circ \theta$.

In particular, $S^{-1}A$ itself is an A-algebra.. Moreover, for every A-module M, the A-module $S^{-1}M$ has a natural structure of a $S^{-1}A$-module.

The ring $S^{-1}A$ is the zero ring if and only if $0 \in S$.

1.2.5. Examples. — Let us give examples of multiplicative subsets and let us describe the corresponding ring of fractions.

a) Let $a \in A$; the set $S = \{1, a, a^2, \dots\}$ is a multiplicative subset which contains 0 if and only if a is nilpotent. The corresponding fraction ring is often denoted by A_a. Let $\varphi_1 \colon A[T] \to A_a$ be the morphism of rings given by $\varphi_1(P) = P(1/a)$; it is surjective and its kernel contains the polynomial $1 - aT$. Let $\varphi \colon A[T]/(1 - aT) \to A_a$ be the morphism of rings which is deduced from φ_1 by passing to the quotient; let us show that φ is an isomorphism by constructing its inverse.

The obvious morphism $\psi_1 \colon A \to A[T]/(1-aT)$ maps a to an invertible element of $A[T]/(1 - aT)$; by the universel property of the localization, there exists a unique morphism of rings $\psi \colon A_a \to A[T]/(1 - aT)$ such that $\psi(b) = b$ for every $b \in A$; one has $\psi(b/a^n) = b\,\mathrm{cl}(T)^n$ for every $b \in A$ and every integer $n \geqslant 0$. Moreover, $\varphi \circ \psi(b/a^n) = b/a^n$, so that $\varphi \circ \psi = \mathrm{id}$. In the other direction, $\psi \circ \varphi(b) = b$ for every $b \in A$ and $\psi \circ \varphi_1(T) = \psi(1/a) = \mathrm{cl}(T)$; consequently, $\psi \circ \varphi_1(P) = \mathrm{cl}(P)$ for every polynomial $P \in A[T]$, hence $\psi \circ \varphi = \mathrm{id}$. This shows that φ is an isomorphism, with inverse ψ, as claimed.

b) Let I be an ideal of A. The set $S = 1 + I = \{a \in I\,;\ a - 1 \in I\}$ is a multiplicative subset of A.

c) Let $f \colon A \to B$ be a morphism of rings, let T be a multiplicative subset of B and let $S = f^{-1}(T)$. Then S is a multiplicative subset of A and there is a unique morphism of rings $\varphi \colon S^{-1}A \to T^{-1}B$ such that $\varphi(a/1) = f(a)/1$ for every $a \in A$.

d) If A is an integral domain, then $S = A \setminus \{0\}$ is a multiplicative subset of A; the fraction ring $S^{-1}A$ is a field, called the *field of fractions of* A.

e) Let $\mathfrak{p}$ be an ideal of A and let $S = A \setminus \mathfrak{p}$. Then S is a multiplicative subset of A if and only if $\mathfrak{p}$ is a prime ideal of A; the fraction ring is denoted $A_{\mathfrak{p}}$.

1.2.6. — Let A be a ring, let S be a multiplicative subset of A. For every ideal I of A, the ideal $\theta(I)(S^{-1}A)$ generated by the image of I in $S^{-1}A$ is denoted by $S^{-1}I$. It is equal to $S^{-1}A$ if and only if $S \cap I \neq \varnothing$. Moreover, every ideal of $S^{-1}A$ is of this form.

Finally, the map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection from the set of prime ideals of A which do not meet S to the set of prime ideals of $S^{-1}A$.

In particular, for every prime ideal $\mathfrak{p}$ of A, the ring $A_{\mathfrak{p}}$ is a local ring, called the localization of A at $\mathfrak{p}$, and $\mathfrak{p}A_{\mathfrak{p}}$ is its maximal ideal.

Lemma (**1.2.7**). — *Let* A *be a ring, let* S *be a multiplicative subset of* A *and let* I *be an ideal of* A. *If* I *does not meet* S, *then there exists a prime ideal* $\mathfrak{p}$ *of* A *which contains* I *and does not meet* S.

Proof. — Since $I \cap S = \varnothing$, the ideal $S^{-1}I$ is distinct from $S^{-1}A$, hence is contained in some maximal ideal of $S^{-1}A$, of the form $S^{-1}\mathfrak{p}$, for some prime (but non necessarily maximal) ideal $\mathfrak{p}$ of A. One then checks that $I \subseteq \mathfrak{p}$. Let indeed $a \in I$. Since one has $a/1 \in S^{-1}I \subseteq S^{-1}\mathfrak{p}$, there exists $b \in \mathfrak{p}$ and $s \in S$ such that $a/1 = b/s$. By definition of the ring $S^{-1}A$, there exists $t \in S$ such that $t(as - b) = 0$. In particular, $sta = tb \in \mathfrak{p}$. Since $st \in S$ and $S \cap \mathfrak{p} = \varnothing$, the definition of a prime ideal implies that $a \in \mathfrak{p}$, as was to be shown. □

Proposition (**1.2.8**). — *The radical of an ideal is the intersection of the prime ideals which contain it. In particular, the nilradical of a ring is the intersection of its prime ideals.*

Proof. — Let A be a ring. Nilpotent elements are contained in every prime ideal of A. Conversely, let $a \in A$ be a non-nilpotent element. By definition, the multiplicative subset $S = \{1, a, a^2, \dots\}$ is disjoint from the ideal $\{0\}$, hence there exists a prime ideal $\mathfrak{p}$ of A which does not meet S; in particular, $a \notin \mathfrak{p}$. □

Lemma (**1.2.9**). — *Let* A *be a ring and let* M *be an* A*-module. The following properties are equivalent:*

(i) *One has* $M = 0$*;*
(ii) *One has* $M_{\mathfrak{p}}$ *for every prime ideal* $\mathfrak{p}$ *of* A*;*
(iii) *One has* $M_{\mathfrak{m}} = 0$ *for every maximal ideal* $\mathfrak{m}$ *of* A.

Proof. — The implications (i)⇒(ii) and (ii)⇒(iii) are obvious. Let us assume that (iii) holds and let us show that $M = 0$. Let $x \in M$ and let I be the set of all elements $a \in A$ such that $ax = 0$; then I is an ideal of A. By assumption, for every $\mathfrak{m} \in \operatorname{Spm}(A)$, there exists an element $a \in A \smallsetminus \mathfrak{m}$ such that $ax = 0$; in other words, I is not contained in any maximal ideal of A. This implies that $I = A$, hence $1 \in A$ and $x = 0$. Consequently, $M = 0$, as claimed. □

Definition (**1.2.10**). — *Let* A *be a ring and let* M *be an* A*-module. The* support *of* M *is the set of all prime ideals* $\mathfrak{p}$ *of* A *such that* $M_{\mathfrak{p}} \neq 0$.

Proposition (**1.2.11**). — *Let* M *be an* A*-module.*

a) *One has* $\operatorname{Supp}_A(M) = \varnothing$ *if and only if* $M = 0$.

b) *If* M *is generated by one element* m*, then* $\mathrm{Supp}_A(M) = V(\mathrm{Ann}_A(M))$.

c) *If* M *is finitely generated, then* $\mathrm{Supp}_A(M)$ *is a closed subset of* $\mathrm{Spec}(A)$ *contained in* $V(\mathrm{Ann}_A(M))$.

Proof. — a) If $M = 0$, then $M_{\mathfrak p} = 0$ for every $\mathfrak p \in \mathrm{Spec}(A)$, so that $\mathrm{Supp}_A(M)$ is empty. Conversely, if $\mathrm{Supp}_A(M)$ is empty, then $M = 0$ by lemma 1.2.9.

b) For a prime ideal $\mathfrak p \in \mathrm{Spec}(A)$, one has $m/1 = 0$ in $M_{\mathfrak p}$ if and only if there exists $s \in A - \mathfrak p$ such that $sm = 0$, that is $\mathrm{Ann}_A(M) \notin \mathfrak p$. Equivalently, $m/1 \neq 0$ in $M_{\mathfrak p}$ if and only if $\mathrm{Ann}_A(M) \subseteq \mathfrak p$, so that $\mathrm{Supp}_A(M) = V(\mathrm{Ann}_A(M))$.

c) Let S be a finite generating subset of M. For every $\mathfrak p \in \mathrm{Spec}(A)$, the set S generates $M_{\mathfrak p}$. Consequently, $\mathrm{Supp}_A(M)$ is the union of the sets $\mathrm{Supp}_A(m)$, for $m \in S$. In particular, this is a closed subset of $\mathrm{Spec}(A)$.

For every $m \in S$, one has $\mathrm{Ann}_A(M) \subseteq \mathrm{Ann}_A(m)$, so that $V(\mathrm{Ann}_A(m)) \subseteq V(\mathrm{Ann}_A(M))$. Consequently, $\mathrm{Supp}_A(M) \subseteq V(\mathrm{Ann}_A(M))$. □

1.3. Nakayama's lemma

Theorem **(1.3.1)** ("Cayley–Hamilton"). — *Let* A *be a ring and let* J *be an ideal of* A. *Let* M *be an* A*-module which is generated by* n *elements and let* u *be an endomorphism of* M *such that* $u(M) \subseteq JM$. *Then there exists elements* $a_1 \in J$, $a_2 \in J^2, \dots, a_n \in J^n$ *such that*

$$u^n + a_1 u^{n-1} + \cdots + a_{n-1} u + a_n \mathrm{Id}_M = 0.$$

Proof. — Let $(m_1, \dots, m_n)$ be a finite family which generates M. For every $i \in \{1, \dots, n\}$, there exist elements $a_{ij} \in J$ such that $u(m_i) = \sum_{j=1}^n a_{ij} m_j$; let P be the matrix (a_{ij}). We consider M as an A[T]-module, where T acts by u; we then let $n \times n$ matrices with coefficients in A[T] act on M^n by the usual formulas. Let I_n be the identity matrix; then the matrix $TI_n - P$ annihilates the vector $(m_1, \dots, m_n) \in M^n$. Let Q be the adjunct matrix of the matrix $TI_n - P$; one has $Q \cdot (TI_n - P) = \det(TI_n - P) I_n$. Consequently, the element $\det(TI_n - P)$ of A[T] annihilates the vector $(m_1, \dots, m_n)$ as well, that is, $\det(TI_n - P) \cdot m_i = 0$ for every i. Since $(m_1, \dots, m_n)$ generates M as an A-module, it follows that $\det(TI_n - P) \cdot m = 0$ for every $m \in M$.

Expanding the determinant, we have

$$\det(TI_n - P) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n (TI_n - P)_{i\sigma(i)}.$$

Fix $\sigma \in \mathfrak{S}_n$ and let p be the number of fixed points of σ; then $\prod_{i=1}^{n}(\mathrm{T}I_n - P)_{i\sigma(i)}$ is the product of p factors of the form $\mathrm{T} - a$, with $a \in \mathrm{J}$, and $n-p$ factors of the form $a \in \mathrm{J}$; by induction on p, this is a monic polynomial of degree p and the coefficient of T^m belongs to J^{n-p}. Consequently, there are elements $a_0, a_1, \dots, a_n \in \mathrm{A}$, with $a_i \in \mathrm{J}^i$ for every i, such that $\det(\mathrm{T}I_n - P) = a_0\mathrm{T}^n + a_1\mathrm{T}^{n-1} + \dots + a_n$; moreover, $a_0 = 1$. By the definition of the structure of A[T]-module on M, we conclude that $u^n + a_1u^{n-1} + \dots + a_n\mathrm{Id}_\mathrm{M} = 0$. □

Corollary **(1.3.2)** (Nakayama's lemma). — *Let* A *be a ring, let* J *be an ideal of* A *and let* M *be a finitely generated* A*-module such that* $\mathrm{M} = \mathrm{JM}$. *There exists* $a \in \mathrm{J}$ *such that* $(1+a)\mathrm{M} = 0$.

In particular, if J *is contained in the Jacobson radical of* A *(which happens, for example, if* A *is local and* J *is its maximal ideal), then* $\mathrm{M} = 0$.

Proof. — Let us apply theorem 1.3.1 to the endomorphism $u = \mathrm{Id}_\mathrm{M}$ of M. With the notation of that theorem, there exist an integer $n \geqslant 1$ and elements $a_1, \dots, a_n \in \mathrm{J}$ such that $(1 + a_1 + \dots + a_n)\mathrm{Id}_\mathrm{M} = 0$. It thus suffices to set $a = a_1 + \dots + a_n$.

If J is contained in the Jacobson radical of A, one has $1 + a \in \mathrm{A}^\times$; the relation $(1+a)\mathrm{M} = 0$ then implies that $\mathrm{M} = 0$. □

Corollary **(1.3.3)**. — *Let* A *be a ring, let* J *be its Jacobson radical. Let* P *be an* A*-module, let* M *and* N *be submodules of* P *such that* $\mathrm{JM} + \mathrm{N} = \mathrm{M} + \mathrm{N}$. *If* M *is finitely generated, then* $\mathrm{M} \subseteq \mathrm{N}$.

Proof. — Let $\mathrm{M}' = (\mathrm{M}+\mathrm{N})/\mathrm{N} = \mathrm{M}/(\mathrm{M}\cap\mathrm{N})$; it is a finitely generated A-module. Moreover, one has $\mathrm{JM}' = (\mathrm{JM}+\mathrm{N})/\mathrm{N} = (\mathrm{M}+\mathrm{N})/\mathrm{N} = \mathrm{M}'$. By corollary 1.3.2, one has $\mathrm{M}' = 0$, hence $\mathrm{M} = \mathrm{M} \cap \mathrm{N}$, that is, $\mathrm{M} \subseteq \mathrm{N}$. □

1.4. Integral and algebraic dependence relations

1.4.1. — Let $f\colon \mathrm{A} \to \mathrm{B}$ be a morphism of rings. One says that an element $x \in \mathrm{B}$ is *integral* over A if there exists an integer $n \geqslant 1$, and elements $a_1, \dots, \dots, a_n \in \mathrm{A}$ such that

$$x^n + f(a_1)x^{n-1} + \dots + f(a_{n-1})x + f(a_n) = 0.$$

Such an equation is called an *integral dependence relation*. Very often, the morphism f is understated and the previous relation is written simply $x^n + a_1x^{n-1} + \dots + a_n = 0$.

Proposition (**1.4.2**). — *An element $x \in \mathrm{B}$ is integral over* A *if and only if there exists a subring* R *of* B *which contains* A$[x]$ *and which is finitely generated as an* A*-module.*

Proof. — Let us assume that x possesses an integral dependence relation as above; then, the A-subalgebra A$[x]$ generated by x in B is generated as an A-module by the elements $1, x, \dots, x^{n-1}$. It suffices to set R = A$[x]$.

Conversely, let R be an A-subalgebra of B which contains x and which is finitely generated as an A-module. By theorem 1.3.1, applied to the endomorphism u of R given by multiplication by x and to the ideal J = A, there exist an integer n and elements $a_1, \dots, a_n \in \mathrm{A}$ such that $u^n + a_1 u^{n-1} + \dots + a_n = 0$ as an endomorphism of R. Considering the image of 1, we obtain an integral dependence relation for x, as required. □

Corollary (**1.4.3**). — *Let $f \colon \mathrm{A} \to \mathrm{B}$ be a morphism of rings. The set of all elements $x \in \mathrm{B}$ which are integral over* A *is an* A*-subalgebra of* B*, called the* integral closure *of* A *in* B.

Proof. — Let $\widetilde{\mathrm{A}}$ be this subset of B. Let x, y be elements of $\widetilde{\mathrm{A}}$. Let m and n be the degrees of integral dependence relations for x and y respectively, and let R be the A-submodule of B generated by the finite family $(x^i y^j)$, for $0 \leqslant i < m$ and $0 \leqslant j < n$; it is a subring of B. Since it contains $x + y$ and xy, this shows that these elements are integral over A, hence belong to $\widetilde{\mathrm{A}}$. Moreover, every element of $f(\mathrm{A})$ is integral over A; in particular, 0 and 1 are integral over A. This shows that $\widetilde{\mathrm{A}}$ is a subring of B; since it contains $f(\mathrm{A})$, it is an A-subalgebra of B. □

1.4.4. — One says that a morphism of rings $f \colon \mathrm{A} \to \mathrm{B}$ is *integral*, or that B is integral over A, or also that B is an integral A-algebra, if every element of B is integral over A.

If B is finitely generated as an A-module, then B is integral over A. Conversely, if B is finitely generated as an A-algebra, and if it is integral over A, then it is finitely generated as an A-module. We say that B is a *finite* A-algebra.

Lemma (**1.4.5**). — *Let* B *be an integral domain and let* A *be a subring of* B *such that* B *is integral over* A*. Then* A *is a field if and only if* B *is a field.*

Proof. — Let us assume that A is a field. Let $b \in \mathrm{B}$ be a non-zero element and let $b^n + a_1 b^{n-1} + \dots + a_{n-1} b + a_n = 0$ be an integral dependence relation of *minimal* *degree* for b. Let $c = b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}$, so that $bc + a_n = 0$. If $a_n = 0$, one would have $bc = 0$, hence, since $b \neq 0$ and B is an integral domain, $c = 0$, which is an integral dependence relation of degree $n-1$ for b. This contradicts the definition of n, so that $a_n \neq 0$. Since A is a field, a_n is invertible in A; let $d \in A$ be such that $a_n d = 1$. Then $bcd = -a_n d = -1$; consequently, b is invertible in B, with inverse $-cd$. This shows that B is a field.

Let us now assume that B is a field. Let $a \in A$ be any non-zero element and let b be its inverse in B. By assumption, b is integral over A; let $b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$ be an integral dependence relation. Since $ab = 1$, one has

$$b = a^{n-1} b^n = -a^{n-1}(a_1 b^{n-1} + \cdots + a_n) = -(a_1 + a_2 a + \cdots + a_n a^{n-1}).$$

In particular, $b \in A$, so that a is invertible in A. □

1.4.6. — It is crucial that the leading coefficient of an integral dependence relation be equal to 1 (it could be a unit). When A and B are fields, this becomes pointless; in this setting, one usually replaces the adjective *integral* by the adjective *algebraic*. One thus speaks of *algebraic dependence relation*, of an *algebraic* element, of the *algebraic closure* of A in B, etc.

Let $f : K \to L$ be an extension of fields. Elements of L which are not algebraic over K are said to be *transcendental*. A field K is said to be algebraically closed if it is algebraically closed in every extension L of K.

Every field K possesses an *algebraic closure*: this is an algebraic extension $K \to \overline{K}$ which is algebraic and algebraically closed. Any two algebraic closures of a field K are isomorphic (as K-algebras).

1.4.7. — Let $f : K \to L$ be an extension of fields. One says that a family $(a_i)_{i \in I}$ of elements of L is *algebraically independent* if there does not exist a non-zero polynomial $P \in K[(X_i)_{i \in I}]$ such that $P((a_i)) = 0$, in other words if the canonical morphism of K-algebras $K[(X_i)_{i \in I}] \to L$ which, for every i, maps X_i to a_i is injective.

A *transcendence basis* of L over K is an algebraically independent family (a_i) such that L be algebraic over the subextension of L generated by the a_i.

Transcendence basis exist. More precisely, the following analogue of the incomplete basis theorem holds: *Let* $A \subseteq C$ *be two subsets of* L, *where* A *is algebraically independent over* K, *and* L *is algebraic over the subextension generated by* C; *then there exists a transcendence basis* B *such that* $A \subseteq B \subseteq C$. Two transcendence basis have the same cardinality, called the *transcendence degree* of L over K and denoted $\operatorname{tr.deg}_K(L)$, or even $\operatorname{tr.deg}(L)$ if the field K is clear from the context. Finally, let $K \to L$ and $L \to M$ be two field extensions. One has the relation

$$\operatorname{tr.deg}_K(L) + \operatorname{tr.deg}_L(M) = \operatorname{tr.deg}_K(M).$$

By abuse of language, we will sometimes make use of the words algebraic, algebraically independent, transcendence degree, in the context of a K-algebra A which is an integral domain, to speak of the corresponding notions of the field of fractions of A.

1.5. The spectrum of a ring

1.5.1. — Let A be a ring. The set of all prime ideals of A is called the *spectrum* (or the prime spectrum) of A and denoted by $\operatorname{Spec}(A)$; the subset $\operatorname{Spm}(A)$ of all maximal ideals of A is called its maximal spectrum.

Every non-zero ring possesses maximal ideals. Consequently, the following assertions are equivalent:

(i) A is the zero ring;

(ii) Its spectrum $\operatorname{Spec}(A)$ is empty;

(iii) Its maximal spectrum $\operatorname{Spm}(A)$ is empty.

For every subset E of A, let $V(E)$ be the set of prime ideals $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $E \subseteq \mathfrak{p}$. One also writes $V(a, b, \dots)$ for $V(\{a, b, \dots\})$.

The following properties essentially follow from the definitions.

Lemma (**1.5.2**). — a) *One has* $V(\varnothing) = \operatorname{Spec}(A)$ *and* $V(1) = \varnothing$;

b) *If* E *and* E′ *are subsets of* A *such that* $E \subseteq E'$, *one has* $V(E') \subseteq V(E)$;

c) *For every family* $(E_\lambda)_{\lambda \in L}$ *of subsets of* A, *one has* $V(\bigcup_{\lambda \in L} E_\lambda) = \bigcap_{\lambda \in L} V(E_\lambda)$;

d) *Let* E, E′ *be two subsets of* A *and let* EE′ *be the set of all products ab, for $a \in E$ and $b \in E'$; then one has* $V(EE') = V(E) \cup V(E')$;

e) *Let* E *be a subset of* A *and let* I *be the ideal of* A *generated by* E; *then one has* $V(E) = V(I)$.

Proof. — a) The first property is obvious, and the second follows from the fact that A is not a prime ideal of itself.

b) Let $\mathfrak{p} \in V(E')$; then $\mathfrak{p}$ is a prime ideal of A such that $E' \subseteq \mathfrak{p}$; it follows that $E \subseteq \mathfrak{p}$, hence $\mathfrak{p} \in V(E)$.

c) Let $\mathfrak{p}$ be a prime ideal of A. One has $\mathfrak{p} \in V(\bigcup E_\lambda)$ if and only if $\mathfrak{p}$ contains E_λ for every λ, which means that $\mathfrak{p}$ belongs to $V(E_\lambda)$ for every λ.

d) Let $\mathfrak{p} \in V(E)$. Let $a \in E$ and $b \in E'$; one has $a \in \mathfrak{p}$, hence $ab \in \mathfrak{p}$, so that $\mathfrak{p} \in V(EE')$. This shows that $V(E) \subseteq V(EE')$, and the inclusion $V(E') \subseteq V(EE')$ follows by symmetry. Conversely, let $\mathfrak{p} \in V(EE')$. Assume that $\mathfrak{p} \notin V(E')$ and let us show that $\mathfrak{p} \in V(E)$; let $b \in E'$ be such that $b \notin \mathfrak{p}$. For every $a \in E$, one has $ab \in EE'$, hence $ab \in \mathfrak{p}$; Since $\mathfrak{p}$ is a prime ideal, this implies that $a \in \mathfrak{p}$. Consequently, $\mathfrak{p} \in V(E)$, as was to be shown. □

1.5.3. The spectral topology. — Let us decree that a subset of Spec(A) is *closed* if it is of the form V(E) for some subset E of A. By property *d*) of lemma 1.5.2, we may even assume that E is an ideal.

By property *a*) of that lemma, the empty set and Spec(A) are closed subsets. According to property *c*), the intersection of a family of closed subsets is closed; by property *d*), the union of two closed subsets is closed.

The sets V(E), where E runs among all subsets of A, are the closed subsets of a topology on the spectrum Spec(A). We call it the *spectral topology*, or the *Zariski topology*

1.5.4. — For every subset Z of Spec(A), let $\mathrm{j}(Z)$ be the set of $a \in A$ such that $Z \subseteq V(a)$. One thus has $\mathrm{j}(Z) = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$; in particular, $\mathrm{j}(Z)$ is a radical ideal of A.

Lemma **(1.5.5)**. — a) *If* Z *and* Z′ *are subsets of* Spec(A) *such that* $Z \subseteq Z'$, *then* $\mathrm{j}(Z') \subseteq \mathrm{j}(Z)$*;*

b) *If* $(Z_\lambda)_{\lambda \in L}$ *is a family of subsets of* Spec(A), *then* $\mathrm{j}(\bigcup_{\lambda \in L} Z_\lambda) = \bigcap_{\lambda \in L} \mathrm{j}(Z_\lambda)$*;*

c) *For every subset* Z *of* Spec(A), *one has the inclusion* $Z \subseteq V(\mathrm{j}(Z))$, *with equality if and only if* Z *is of the form* V(E) *for some subset* E *of* A.

d) *For every subset* E *of* A, *one has the inclusion* $E \subseteq \mathrm{j}(V(E))$, *with equality if and only if* E *is of the form* $\mathrm{j}(Z)$, *for some subset* Z *of* Spec(A).

Proof. — Only the cases of equality in assertions *c*) and *d*) do not follow directly from the definitions.

For *c*), it suffices to prove that $V(E) = V(\mathrm{j}(V(E)))$. We already know that $V(E) \subseteq V(\mathrm{j}(V(E))$; by the inclusion *d*), one has $E \subseteq \mathrm{j}(V(E))$; applying the map V, we conclude that $V(\mathrm{j}(V(E))) \subseteq V(E)$.

Similarly, we prove *d*) by establishing that $j(Z) = j(V(j(Z)))$. We know the inclusion $j(Z) \subseteq j(V(j(Z)))$. According to the general inclusion *c*), we have $Z \subseteq V(j(Z))$; applying the map j, we conclude that $j(V(j(Z))) \subseteq j(Z)$. □

Proposition **(1.5.6)**. — a) *For every ideal* I *of* A, *one has* $j(V(I)) = \sqrt{I}$.

b) *For every subset* Z *of* Spec(A), *one has* $V(j(Z)) = \overline{Z}$, *the closure of* Z *for the spectral topology.*

c) *The maps* $E \mapsto V(E)$ *and* $Z \mapsto j(Z)$ *induce bijections, inverse one of the other, between the set of radical ideals of* A *and the set of closed subsets of* Spec(A).

Proof. — a) By definition, V(I) is the set of prime ideals containing I, so that $j(V(I))$ is the intersection of all prime ideals containing I. By proposition 1.2.8, one has $j(V(I)) = \sqrt{I}$.

b) Since $V(j(Z))$ is closed and contains Z, it contains its closure $\overline{Z}$ for the spectral topology. Conversely, let Z′ be a closed subset of Spec(A) containing Z and let us show that $Z' \supseteq V(j(Z))$. Applying the map $V \circ j$ to the inclusion $Z \subseteq Z'$, we obtain $V(j(Z)) \subseteq V(j(Z'))$. Since Z′ is of the form V(E), one has $V(j(Z')) = Z'$, hence $V(j(Z)) \subseteq Z'$, as was to be shown.

c) This follows directly from properties *a*) and *b*). □

Exercise **(1.5.7)**. — Let A be a ring and let X be the topological space Spec(A). An idempotent element of A is an element *e* such that $e^2 = e$. Show that the map $a \mapsto V(a)$ defines a bijection between the set of idempotents of A and the set of open and closed subsets of Spec(A). (If *e* is idempotent, observe that $X = V(e) \cup V(1 - e)$.) In particular, X is connected if and only if the only idempotent elements of A are 0 and 1.

1.5.8. Basic open sets. — For every $a \in A$, one defines $D(a) = \operatorname{Spec}(A) \setminus V(a)$. It is an open subset of Spec(A). One has $D(1) = \operatorname{Spec}(A)$ and $D(a) = \varnothing$ if *a* is nilpotent.

Let E be a subset of A. Since $V(E) = \bigcap_{a \in E} V(a)$, we have $\operatorname{Spec}(A) \setminus V(E) = \bigcup_{a \in E} D(a)$. This shows that the open sets of the form $D(a)$, for $a \in A$, form a basis of the topology of Spec(A).

Exercise **(1.5.9)**. — a) Let *x* be a point of Spec(A) and let $\mathfrak{p} = j(\{x\})$ be the corresponding prime ideal of A. Prove that the point $\{x\}$ is closed in Spec(A) if and only if $\mathfrak{p}$ is a maximal ideal.

b) Let x, y be two points of Spec(A) such that $x \neq y$. Prove that $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. (This says that Spec(A) is a Kolmogorov topological space, aka T_0.)

c) Describe the topological space Spec($\mathbf{Z}$). Show in particular that it is not Hausdorff.

d) Prove that every open cover of Spec(A) has a finite subcover (one says that it is quasi-compact).

Proposition (**1.5.10**). — a) *Let* $\varphi\colon A \to B$ *be a morphism of rings. For every prime ideal* $\mathfrak{q}$ *of* B, *the ideal* $\varphi^{-1}(\mathfrak{q})$ *is a prime ideal of* A. *The associated map* ${}^a\varphi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *given by* ${}^a\varphi(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ *is continuous.*

b) *Let* I *be an ideal of* A *and let* $\varphi\colon A \to A/I$ *be the canonical morphism. The associated map* ${}^a\varphi$ *is a homeomorphism from* Spec(A/I) *to the subspace* V(I) *of* Spec(A).

c) *Let* S *be a multiplicative subset of* A *and let* $\theta\colon A \to S^{-1}A$ *be the canonical morphism. The associated map* ${}^a\theta$ *is a homeomorphism from* $\mathrm{Spec}(S^{-1}A)$ *to its image in* Spec(A), *which is the set of prime ideals of* A *disjoint from* S.

If $S = \{1, a, a^2, \dots\}$, *then* ${}^a\theta$ *identifies* $\mathrm{Spec}(S^{-1}A)$ *with the open subset* $D(a)$ *of* Spec(A).

Proof. — a) Since $\mathfrak{q} \neq B$, one has $1 \notin \mathfrak{q}$, hence $1 = \varphi(1) \notin \varphi^{-1}(\mathfrak{q})$; consequently, $\varphi^{-1}(\mathfrak{q}) \neq A$. Moreover, let $a, b \in A$ be such that $ab \in \varphi^{-1}(\mathfrak{q})$; then $\varphi(ab) = \varphi(a)\varphi(b) \in \mathfrak{q}$, hence $\varphi(a) \in \mathfrak{q}$ or $\varphi(b) \in \mathfrak{q}$, by definition of a prime ideal. This implies that a or b belongs to $\varphi^{-1}(\mathfrak{q})$, proving that $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of A.

To prove that the map ${}^a\varphi$ is continuous, we need to show that the inverse image of a closed subset is closed. So let E be a subset of A. A prime ideal $\mathfrak{q}$ of B belongs to $({}^a\varphi)^{-1}(V(E))$ if and only if ${}^a\varphi(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ belongs to V(E), which means that $E \subseteq \varphi^{-1}(\mathfrak{q})$, and is equivalent to the inclusion $\varphi(E) \subseteq \mathfrak{q}$. In other words, we have $({}^a\varphi)^{-1}(V(E)) = V(\varphi(E))$; this is a closed subset of Spec(B).

b) We known that the map $J \mapsto \varphi^{-1}(J)$ is a bijection from the set of ideals of A/I to the set of ideals of A which contain J. Moreover, for every ideal J of A/I, the morphism φ induces an isomorphism from $A/\varphi^{-1}(J)$ to $(A/I)/J$. In particular, an ideal J of B is prime if and only if the associated ideal $\varphi^{-1}(J)$ is prime, and the prime ideals of A of this form are exactly those containing I. This shows that the map ${}^a\varphi$ is a bijection from Spec(A/I) to the closed subset V(I) of Spec(A).

Moreover, for every ideal J of A/I, one has ${}^a\varphi(V(J)) = V(\varphi^{-1}(J))$, so that ${}^a\varphi$ is a closed map. Since it is a continuous bijection, it is a homeomorphism.

c) We know that the continous $J \mapsto \theta^{-1}(J)$ induces a continous bijection from the set $\operatorname{Spec}(S^{-1}A)$ of prime ideals of $S^{-1}A$ to the subset X of Spec(A) consisting of prime ideals of A which do not meet S.

Let us show that this bijection is closed. Let E be a subset of $S^{-1}A$; let E′ be the set of elements $a \in A$ such that there exists $s \in S$ with $a/s \in E$, and let us show that ${}^a\theta(V(E)) = V(E')$. Let $\mathfrak{p}$ be a prime ideal of A which does not meet S, let $\mathfrak{q} = S^{-1}\mathfrak{p}$, so that $\mathfrak{p} = \theta^{-1}(\mathfrak{q})$. Then $\mathfrak{p}$ belongs to ${}^a\varphi(V(E))$ if and only if $S^{-1}\mathfrak{p} \in V(E)$, that is if and only if $E \subseteq S^{-1}\mathfrak{p}$; on the other hand, $\mathfrak{p}$ belongs to $V(E')$ if and only if $E' \subseteq \mathfrak{p}$. It thus remains to show that for a prime ideal $\mathfrak{p}$ of A which does not meet S, the conditions $E \subseteq S^{-1}\mathfrak{p}$ and $E' \subseteq \mathfrak{p}$ are equivalent. Let us assume that $E \subseteq S^{-1}\mathfrak{p}$; let $a \in E'$ and let $s \in S$ be such that $a/s \in E$; then $a/s \in S^{-1}\mathfrak{p}$, hence $\theta(a) \in S^{-1}\mathfrak{p}$, hence $a \in \mathfrak{p}$; this shows that $E' \subseteq \mathfrak{p}$. Conversely, let us assume that $E' \subseteq \mathfrak{p}$; let $b \in E$ and let $(a, s) \in A \times S$ be such that $b = a/s$; then $a \in E'$, hence $a \in \mathfrak{p}$; consequently, $b = a/s \in S^{-1}\mathfrak{p}$; we have shown that $E \subseteq S^{-1}\mathfrak{p}$.

□

Remark **(1.5.11)**. — Let $\varphi\colon A \to B$ be a morphism of rings. Classical algebraic geometry is essentially concerned with finitely generated algebras over a field. In that context, corollary 1.6.3 shows that ${}^a\varphi$ maps Spm(B) into Spm(A), as the simple example of the canonical morphism $\varphi\colon \mathbf{Z} \to \mathbf{Q}$ shows. This is an indication that the spectrum of a ring is a more natural object than its maximal spectrum. Indeed, spectra of rings were the basic block of Grothendieck's refoundation of algebraic geometry in the 1960s.

1.6. Finitely generated algebras over a field

Theorem **(1.6.1)** (Noether normalization lemma). — *Let* K *be a field and let* A *be a finitely generated* K*-algebra; we assume that* A ≠ 0. *Then there exist an integer* $n \geqslant 0$, *elements* $a_1, \dots, a_n \in A$ *such that the unique morphism of* K*-algebras* $\varphi\colon K[X_1, \dots, X_n] \to A$ *which maps* X_i *to* a_i *is injective and integral.*

Proof. — Let $(x_1, \dots, x_m)$ be a family of elements of A such that $A = K[x_1, \dots, x_m]$. Let us prove the result by induction on m. If $m = 0$, then $A = K$ and the result holds with $n = 0$. We thus assume that the result for any K-algebra which is finitely generated by at most $m - 1$ elements.

Let $\varphi\colon \mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_m] \to \mathrm{A}$ be the unique morphism of K-algebras such that $\varphi(\mathrm{X}_i) = x_i$. If φ is injective, the result holds, taking $n = m$ and $a_i = x_i$ for every i.

Let us assume that there is a non-zero polynomial $\mathrm{P} \in \mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_m]$ such that $\mathrm{P}(x_1,\dots,x_m) = 0$. We are going to show that there exist strictly positive integers $r_1,\dots,r_{m-1}$ such that A is integral over the subalgebra generated by $y_2,\dots,y_m$, where $y_i = x_i - x_1^{r_i}$ for $i \in \{2,\dots,m\}$. Let $\mathrm{B} = \mathrm{K}[y_2,\dots,y_m]$ be the subalgebra of A generated by $y_2,\dots,y_m$.

Let $(c_{\mathbf{n}})$ be the coefficients of P, so that

$$\mathrm{P} = \sum_{\mathbf{n}\in\mathbf{N}^m} c_{\mathbf{n}} \prod_{i=1}^{m} \mathrm{X}_i^{n_i}.$$

Let r be an integer strictly greater than the degree of P in each variable; in other words, $c_{\mathbf{n}} = 0$ if there exists i such that $n_i \geqslant r$; then set $r_i = r^{i-1}$ and $y_i = x_i - x_1^{r_i}$ for $i \in \{2,\dots,m\}$. We define a polynomial $\mathrm{Q} \in \mathrm{B}[\mathrm{T}]$ by

$$\begin{aligned}\mathrm{Q}(\mathrm{T}) &= \mathrm{P}(\mathrm{T}, y_2 + \mathrm{T}^{r_2},\dots,y_m + \mathrm{T}^{r_m})\\ &= \sum_{\mathbf{n}\in\mathbf{N}^m} c_{\mathbf{n}}\mathrm{T}^{n_1}(y_2+\mathrm{T}^{r_2})^{n_2}\dots(y_m+\mathrm{T}^{r_m})^{n_m}\\ &= \sum_{\mathbf{n}\in\mathbf{N}^m}\sum_{j_2=0}^{n_2}\cdots\sum_{j_m=0}^{n_m}\binom{n_2}{j_2}\dots\binom{n_m}{j_m} c_{\mathbf{n}} y_2^{n_2-j_2}\dots y_m^{n_m-j_m}\mathrm{T}^{n_1+\sum_{i=2}^m j_i r_i}\end{aligned}$$

and observe that $\mathrm{Q}(x_1) = \mathrm{P}(x_1,x_2,\dots,x_m) = 0$.

Order $\mathbf{N}^m$ with the “reverse lexicographic order”: $(n'_1,\dots,n'_m) < (n_1,\dots,n_m)$ if and only if $n'_m < n_m$, or $n'_m = n_m$ and $n'_{m-1} < n_{m-1}$, etc. Let $\mathbf{n}$ be the largest multi-index in $\mathbf{N}^m$ such that $c_{\mathbf{n}} \neq 0$. For any other $\mathbf{n}' \in \mathbf{N}^m$ such that $c_{\mathbf{n}'} \neq 0$, one has $n'_i < r$ for every i, so that for any $j_2 \in \{0,\dots,n'_2\}$, …, $j_m \in \{0,\dots,n_m\}$,

$$n'_1 + j_2r_2 + \dots + j_mr_m \leqslant n'_1 + n'_2r + \dots + n'_mr^{m-1} < n_1 + n_2r + \dots + n_mr^{m-1}.$$

This implies that the degree of Q is equal to $n_1+n_2r+\dots+n_mr^{m-1}$ and that only the term with $j_k = n_k$ for $k \in \{2,\dots,m\}$ contributes the leading coefficient, which thus equals $c_{\mathbf{n}}$. In particular, Q is a polynomial in $\mathrm{B}[\mathrm{T}]$ whose leading coefficient is a unit, so that x_1 is integral over B. Consequently, $\mathrm{B}[x_1]$ is integral over B. For every $i \in \{2,\dots,m\}$, one has $x_i = y_i - x_1^{r_i} \in \mathrm{B}[x_1]$. Since $\mathrm{A} = \mathrm{K}[x_1,\dots,x_m]$, we conclude that $\mathrm{A} = \mathrm{B}[x_1]$ and A is integral over B.

By induction, there exist an integer $n \leqslant m-1$ and elements $a_1,\dots,a_n \in \mathrm{B}$ such that the unique morphism $f\colon \mathrm{K}[\mathrm{T}_1,\dots,\mathrm{T}_n] \to \mathrm{B}$ of K-algebras such that $f(\mathrm{T}_i) = a_i$ for all i is injective and such B is integral over $\mathrm{K}[a_1,\dots,a_n]$. Then

A is integral over $K[a_1,\dots,a_n]$ as well, and this concludes the proof of the theorem. □

We now deduce from the Noether normalization lemma some important algebraic properties of rings which are finitely generated algebras over a field. The following result is the basis of everything that follows; due to Zariski, it is sometimes considered as the "algebraic version" of Hilbert's Nullstellensatz.

Theorem **(1.6.2)** (Zariski). — *Let* K *be a field and let* A *be a finitely generated* K*-algebra. If* A *is a field, then* A *is a finite algebraic extension of* K.

Proof. — By the Noether normalization lemma (theorem 1.6.1), there exist an integer $n \geqslant 0$ and an injective and integral morphism of K-algebras $f\colon K[X_1,\dots,X_n] \to A$. Since A is a field, lemma 1.4.5 implies that $K[X_1,\dots,X_n]$ is a field as well. For $n \geqslant 1$, the ring of polynomials in n indeterminates is not a field (consider the degree with respect to X_1, for example), so that $n = 0$. Consequently, A is integral over K. Since A is finitely generated as a K-algebra, it is a finite K-module, hence a finite extension of K. □

Corollary **(1.6.3)**. — *Let* K *be a field and let* $\varphi\colon A \to B$ *be a morphism of finitely generated* K*-algebras. For every maximal ideal* $\mathfrak m$ *of* B, $\varphi^{-1}(\mathfrak m)$ *is a maximal ideal of* A. *In other words, the continuous map* ${}^a\varphi\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *maps* $\operatorname{Spm}(B)$ *to* $\operatorname{Spm}(A)$.

Proof. — Let $\mathfrak n = \varphi^{-1}(\mathfrak m)$; it is a prime ideal of A. Passing to the quotients, the morphism φ induces an injective morphism $\varphi'\colon A/\mathfrak n \to B/\mathfrak m$ of finitely generated K-algebras. By assumption, $B/\mathfrak m$ is a field; by corollary 1.6.2, it is a finite extension of K, that is a finite dimensional K-vector space. A fortiori, $A/\mathfrak n$ is a finite dimensional K-vector space. This implies that $A/\mathfrak n$ is integral over K; since K is a field and $A/\mathfrak n$ is an integral domain, this implies that $A/\mathfrak n$ is a field, hence $\mathfrak n$ is a maximal ideal of A. □

Corollary **(1.6.4)**. — *Let* K *be a field and let* A *be a finitely generated* K*-algebra.*

a) *The nilradical of* A *coincides with its Jacobson radical;*

b) *For every ideal* I *of* A, *its radical* $\sqrt{I}$ *is the intersection of all maximal ideals of* A *which contain* I.

c) *For every closed subset* Z *of* $\operatorname{Spec}(A)$, *the intersection* $Z \cap \operatorname{Spm}(A)$ *is dense in* Z.

Proof. — *a*) We need to prove that an element $a \in A$ is nilpotent if and only if it belongs to every maximal ideal of I. One direction is clear: if a is nilpotent, it belongs to every prime ideal of I, hence to every maximal ideal of I. Conversely, let us assume that a is not nilpotent and let us show that there exists a maximal ideal $\mathfrak{m}$ of A such that $a \notin \mathfrak{m}$. Let S be the multiplicative subset $\{1, a, a^2, \dots\}$ and let B be the K-algebra given by $B = S^{-1}A$; it is non-zero and finitely generated. By the preceding corollary, the inverse image in A of a maximal ideal of B is a maximal ideal of A which does not contain a. This concludes the proof of assertion *a*).

b) Let $B = A/I$; it is a finitely generated K-algebra and its maximal ideals are of the form $\mathfrak{m}/I$, where $\mathfrak{m}$ is a maximal ideal of A containing I. By part *a*), the nilradical of B is the intersection of the maximal ideals of B. Since the class of a element $a \in A$ is nilpotent in B if and only if $a \in \sqrt{I}$, this implies that $\sqrt{I}$ is the intersection of all maximal ideals of A which contain I.

c) Let I be an ideal of A such that $Z = V(I)$, let U be an open subset of Spec(A) such that $U \cap Z$ is non-empty. We need to show that $U \cap Z \cap \operatorname{Spm}(A)$ is non-empty. We may moreover assume that U is of the form $D(a)$, for some $a \in A$; then the image $\bar{a}$ of a in A/I is not nilpotent (otherwise, $D(\bar{a}) = \varnothing$ in Spec(A/I), hence $V(I) \cap D(a) = \varnothing$). Consequently, there exists a maximal ideal $\mathfrak{m}$ of A such that $I \subseteq \mathfrak{m}$ and $a \notin \mathfrak{m}$. This maximal ideal is an element of $D(a) \cap Z \cap \operatorname{Spm}(A)$, hence $D(a) \cap Z \cap \operatorname{Spm}(A)$ is non-empty. □

Exercise **(1.6.5)**. — This exercise revisits the main technical step of the Noether normalization lemma in the case where K is an *infinite* field. Let A be a (non-zero) finitely generated algebra; assume that $A = K[x_1, \dots, x_m]$, and let $P \in K[T_1, \dots, T_m]$ be a non-zero polynomial such that $P(x_1, \dots, x_m) = 0$. Prove that there exist elements $a_2, \dots, a_m \in K$ such that, denoting $y_i = x_i - a_i x_1$, x_1 is integral over the subring generated by $y_2, \dots, y_m$.

1.7. Hilbert's Nullstellensatz

Theorem **(1.7.1)** (Nullstellensatz, 1). — *Let* K *be an algebraically closed field and let n be an integer such that $n \geqslant 0$. For every maximal ideal $\mathfrak{m}$ of the polynomial ring* $K[X_1, \dots, X_n]$*, there exists a unique element $(a_1, \dots, a_n) \in K^n$ such that $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$. Conversely, every ideal of this form is a maximal ideal.*

Proof. — Let $(a_1,\dots,a_n) \in \mathrm{K}^n$ and let $\mathfrak{m}$ be the ideal $(\mathrm{X}_1 - a_1,\dots,\mathrm{X}_n - a_n)$ of $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$. Let $\varphi\colon \mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n] \to \mathrm{K}$ be the morphism of rings given by $\varphi(\mathrm{P}) = \mathrm{P}(a_1,\dots,a_n)$. It is surjective and its kernel contains $\mathfrak{m}$. Conversely, let $\mathrm{P} \in \mathrm{Ker}(\varphi)$. By euclidean divisions, we may write

$$\mathrm{P} = (\mathrm{X}_1 - a_1)\mathrm{Q}_1(\mathrm{X}_1,\dots,\mathrm{X}_n) + (\mathrm{X}_2 - a_2)\mathrm{Q}_2(\mathrm{X}_2,\dots,\mathrm{X}_n) + \\ + \dots + (\mathrm{X}_n - a_n)\mathrm{Q}_n(\mathrm{X}_n) + \mathrm{P}(a_1,\dots,a_n).$$

Since $\mathrm{P} \in \mathrm{Ker}(\varphi)$, $\mathrm{P}(a_1,\dots,a_n) = 0$, so that $\mathrm{P} \in \mathfrak{m}$.

Let now $\mathfrak{m}$ be a maximal ideal of $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$ and let A be the quotient ring $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]/\mathfrak{m}$. Since A is a field, corollary 1.6.2 implies that A is a finite extension of K. Since K is algebraically closed, the canonical morphism $\mathrm{K} \to \mathrm{A}$ is an isomorphism. In particular, for every $i \in \{1,\dots,n\}$, there exists a unique $a_i \in \mathrm{K}$ such that $\mathrm{X}_i - a_i \in \mathfrak{m}$. Then $(\mathrm{X}_1 - a_1,\dots,\mathrm{X}_n - a_n)$ is contained in $\mathfrak{m}$. Necessarily, one has $\mathfrak{m} = (\mathrm{X}_1 - a_1,\dots,\mathrm{X}_n - a_n)$. This concludes the proof of the theorem. □

1.7.2. Algebraic sets. — Let K be a field and let n be an integer such that $n \geqslant 0$. Let E be a subset of $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$. The *algebraic set*defined by E is the subset

$$\mathscr{V}(\mathrm{E}) = \{(a_1,\dots,a_n) \in \mathrm{K}^n\,;\ \forall \mathrm{P} \in \mathrm{E},\quad \mathrm{P}(a_1,\dots,a_n) = 0\}.$$

Lemma (**1.7.3**). — *Let* K *be a field and let* $n \geqslant 0$ *be an integer.*

a) *One has* $\mathscr{V}(\varnothing) = \mathrm{K}^n$ *and* $\mathscr{V}(1) = \varnothing$;

b) *If* E *and* E′ *are subsets of* $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$ *such that* $\mathrm{E} \subseteq \mathrm{E}'$, *then* $\mathscr{V}(\mathrm{E}') \subseteq \mathscr{V}(\mathrm{E})$;

c) *For every family* $(\mathrm{E}_\lambda)_{\lambda\in\mathrm{L}}$ *of subsets of* $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$, *one has* $\mathscr{V}(\bigcup_{\lambda\in\mathrm{L}} \mathrm{E}_\lambda) = \bigcap_{\lambda\in\mathrm{L}} \mathscr{V}(\mathrm{E}_\lambda)$;

d) *Let* E, E′ *be two subsets of* $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$ *and let* EE′ *be the set of all products ab, for* $a \in \mathrm{E}$ *and* $b \in \mathrm{E}'$; *then one has* $\mathscr{V}(\mathrm{EE}') = \mathscr{V}(\mathrm{E}) \cup \mathscr{V}(\mathrm{E}')$;

e) *Let* E *be a subset of* $\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n]$ *and let* I *be the ideal generated by* E; *then* $\mathscr{V}(\mathrm{E}) = \mathscr{V}(\mathrm{I}) = \mathscr{V}(\sqrt{\mathrm{I}})$.

Proof. — This lemma is analogous to lemma 1.5.2 and one can prove it in the same way. One can in fact deduce it from that lemma as follows. Let us consider the map $\varphi\colon \mathrm{K}^n \to \mathrm{Spec}(\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n])$ given by $\varphi(a_1,\dots,a_n) = (\mathrm{X}_1 - a_1,\dots,\mathrm{X}_n - a_n)$, whose image is contained in the maximal spectrum $\mathrm{Spm}(\mathrm{K}[\mathrm{X}_1,\dots,\mathrm{X}_n])$, and is even equal to the maximal spectrum when K is algebraically closed. Then $\mathscr{V}(\mathrm{E}) = \varphi^{-1}(\mathrm{V}(\mathrm{E}))$. □

1.7.4. — Let Z be a subset of K^n; one defines a subset $\mathscr{I}(Z)$ of $K[X_1,\dots,X_n]$ by

$$\mathscr{I}(Z) = \{P \in K[X_1,\dots,X_n] ; \forall (a_1,\dots,a_n) \in Z, \quad P(a_1,\dots,a_n) = 0\}.$$

It is an ideal of $K[X_1,\dots,X_n]$; it is in fact the kernel of the morphism of rings from $K[X_1,\dots,X_n]$ to the ring K^Z given by $P \mapsto (a \mapsto P(a))$.

Lemma **(1.7.5)**. — *Let* K *be a field and let* $n \geqslant 0$ *be an integer.*

a) *One has* $\mathscr{I}(\varnothing) = K[X_1,\dots,X_n]$ *and* $\mathscr{I}(K^n) = \{0\}$*;*

b) *If* Z *and* Z′ *are subsets of* K^n *such that* $Z \subseteq Z'$*, then* $\mathscr{I}(Z') \subseteq \mathscr{I}(Z)$*;*

c) *If* $(Z_\lambda)_{\lambda\in L}$ *is a family of subsets of* K^n*, then* $\mathscr{I}(\bigcup_{\lambda\in L} Z_\lambda) = \bigcap_{\lambda\in L} \mathscr{I}(Z_\lambda)$*;*

d) *For every subset* Z *of* K^n*, one has the inclusion* $Z \subseteq \mathscr{V}(\mathscr{I}(Z))$*, with equality if and only if* Z *is an algebraic set;*

e) *For every subset* E *of* $K[X_1,\dots,X_n]$*, one has the inclusion* $E \subseteq \mathscr{I}(\mathscr{V}(E))$*.*

Proof. — Assertions *a*), *b*) and *c*) follow directly from the definitions, as well as the inclusions *d*) and *e*).

Let us terminate the proof of *d*). If $Z = \mathscr{V}(\mathscr{I}(Z))$, then Z is an algebraic set. Conversely, let us assume that Z is an algebraic set, let E be a subset of $K[X_1,\dots,X_n]$ such that $Z = \mathscr{V}(E)$. By definition, one has $E \subseteq \mathscr{I}(Z)$, so that $\mathscr{V}(\mathscr{I}(Z)) \subseteq \mathscr{V}(E) = Z$, hence the desired equality. □

Theorem **(1.7.6)** (Nullstellensatz, 2). — *Let* K *be an algebraically closed field, let* $n \geqslant 0$ *be an integer, let* E *be a subset of* $K[X_1,\dots,X_n]$ *and let* I *be the ideal it generates. One has* $\mathscr{I}(\mathscr{V}(E)) = \sqrt{I}$*. In particular, if* $\mathscr{V}(E) = \varnothing$*, then* $I = (1)$*.*

Proof. — We give two proofs of this result; both rely on theorem 1.7.1 which describes the maximal spectrum of $K[X_1,\dots,X_n]$. The first one combines it with properties of the operations V and j on the spectrum of a ring, as well as with corollary 1.6.4 which is specific to finitely generated algebras over a field. The second proof will be more elementary.

1) According to theorem 1.7.1, assigning the maximal ideal $\mathfrak{m}_a = (X_1 - a_1,\dots,X_n - a_n)$ of $K[X_1,\dots,X_n]$ to the point $a = (a_1,\dots,a_n)$ of K^n defines a one-to-one correspondence between K^n and $\mathrm{Spm}(K[X_1,\dots,X_n])$. For every subset E of $K[X_1,\dots,X_n]$, this correspondence identifies the subset $\mathscr{V}(E)$ of K^n with the subset $V(E) \cap \mathrm{Spm}(K[X_1,\dots,X_n])$ of $\mathrm{Spec}(K[X_1,\dots,X_n])$; for every subset Z of $\mathrm{Spm}(K[X_1,\dots,X_n])$, identified with a subset of K^n, one has

$\mathscr{I}(Z) = \mathfrak{j}(Z)$. In particular, for every subset E of $K[X_1, \dots, X_n]$, one has

$$\mathscr{I}(\mathscr{V}(E)) = \mathfrak{j}(V(E) \cap \operatorname{Spm}(K[X_1, \dots, X_n])).$$

Let J be this ideal; it is a radical ideal of $K[X_1, \dots, X_n]$; according to proposition 1.5.6, $V(J)$ is the closure of $V(E) \cap \operatorname{Spm}(K[X_1, \dots, X_n])$ in $\operatorname{Spec}(K[X_1, \dots, X_n])$. By part *c*) of corollary 1.6.4, one thus has $V(J) = V(E)$. By proposition 1.5.6, *a*), one thus has $J = \mathfrak{j}(V(J)) = \mathfrak{j}(V(E)) = \mathfrak{j}(V(I)) = \sqrt{I}$.

2) The second proof of theorem 1.7.6 begins by showing the second assertion: let us assume that $I \neq (1)$ and let us prove that $\mathscr{V}(E) \neq \varnothing$.

Since $I \neq (1)$, there exists a maximal ideal $\mathfrak{m}$ of $K[X_1, \dots, X_n]$ such that $I \subseteq \mathfrak{m}$; in particular, $E \subseteq \mathfrak{m}$, hence $\mathscr{V}(\mathfrak{m}) \subseteq \mathscr{V}(E)$. Let $(a_1, \dots, a_n) \in K^n$ be such that $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$; one thus has $\mathscr{V}(\mathfrak{m}) = \{(a_1, \dots, a_n)\}$. We conclude that $(a_1, \dots, a_n) \in \mathscr{V}(E)$; it is in particular non-empty.

The inclusion $\sqrt{I} \subseteq \mathscr{I}(\mathscr{V}(E))$ follows from the definitions. Let indeed $P \in \sqrt{I}$ and let e be an integer such that $e \geqslant 1$ and $P^e \in I$. For every $(a_1, \dots, a_n) \in \mathscr{V}(E)$, one thus has $P^e(a_1, \dots, a_n) = 0$, hence $P(a_1, \dots, a_n) = 0$. This shows that $P \in \mathscr{I}(\mathscr{V}(E))$.

Conversely, let $P \in \mathscr{I}(\mathscr{V}(E))$; we need to show that $P \in \sqrt{I}$. The following proof relies on the so-called "Rabinowitsch trick" (Rabinowitsch (1930)). Let E' be the subset of $K[X_1, \dots, X_n, T]$ given by $E' = E \cup \{1 - TP\}$. It follows from its definition that $\mathscr{V}(E') = \varnothing$: indeed, a tuple $(a_1, \dots, a_n, b)$ belongs to $\mathscr{V}(E')$ if and only if $Q(a_1, \dots, a_n) = 0$ for every $Q \in E$ and $1 = bP(a_1, \dots, a_n)$; the first conditions imply that $(a_1, \dots, a_n) \in \mathscr{V}(E)$, so that $P(a_1, \dots, a_n) = 0$ since $P \in \mathscr{I}(\mathscr{V}(E))$; the last condition $1 = bP(a_1, \dots, a_n)$ is then impossible. By the first case, the ideal of $K[X_1, \dots, X_n, T]$ generated by E' is equal to (1); in particular, there exist polynomials $Q_1, \dots, Q_m \in E$, $R_1, \dots, R_m, S \in K[X_1, \dots, X_n, T]$ such that

$$1 = Q_1R_1 + \dots + Q_mR_m + (1 - PT)S.$$

Let us substitute $T = 1/P(X_1, \dots, X_n)$ in this relation; it follows an equality of rational functions:

$$1 = \sum_{j=1}^{m} Q_j(X_1, \dots, X_n) R_j(X_1, \dots, X_n, 1/P).$$

Let e be an integer greater than the degrees of the polynomials $R_1, \dots, R_m, S$ with respect to the variable T; multiplying this relation by P^e, we obtain

$$P^e = \sum_{j=1}^{m} Q_j(X_1, \dots, X_n) P(X_1, \dots, X_n)^e R_j(X_1, \dots, X_n, 1/P).$$

By the choice of the integer e, the rational function $P^e R_j(X_1, \dots, X_n, 1/P)$ is a polynomial for every $j \in \{1, \dots, m\}$, so that P^e belongs to the ideal $(Q_1, \dots, Q_m)$. In particular, $P^e \in I$, which shows that $P \in \sqrt{I}$, as claimed. □

Corollary (**1.7.7**). — *Let* K *be an algebraically closed field and let* $n \geqslant 0$ *be an integer. The maps* $E \mapsto \mathscr{V}(E)$ *and* $Z \mapsto \mathscr{I}(Z)$ *induce bijections, inverse one of the other, from the set of radical ideals of* $K[X_1, \dots, X_n]$ *to the set of algebraic subsets of* K^n.

1.8. Tensor products (Medium up)

1.8.1. — Let k be a ring, let M and N be k-modules. Their *tensor product* $M \otimes_k N$ is a k-module endowed with a k-bilinear map $\varphi\colon M \times N \to M \otimes_k N$ which satisfies the following universal property: For every k-module P and every k-bilinear map $b\colon M \times N \to P$, there exists a unique k-linear map $\beta\colon M \otimes_k N \to P$ such that $b = \beta \circ \varphi$.

1.8.2. — It may be constructed as follows. Let $P_1 = k^{(M \times N)}$ the free k-module on $M \times N$. Its elements are maps with finite support from $M \times N$ to k. Let $\delta\colon M \times N \to P_1$ be the map which associates with $(m, n) \in M \times N$ the function which maps (m, n) to 1 and maps every other element of $M \times N$ to 0. Let P_2 be the submodule of P_1 generated by elements of the form

$$\begin{gathered} \delta(am, n) - a\delta(m, n), \\ \delta(m, an) - a\delta(m, n), \\ \delta(m + m', n) - \delta(m, n) - \delta(m', n), \\ \delta(m, n + n') - \delta(m, n) - \delta(m, n'), \end{gathered}$$

with $m, m' \in M$, $n, n' \in N$ and $a \in k$. Let $P = P_1/P_2$, let $\pi\colon P_1 \to P$ be the canonical surjective morphism and let $\varphi = \pi \circ \delta$.

The image $\varphi(m, n)$ is denoted $m \otimes n$ and called a split tensor.

1.8.3. — Let $u\colon M \to M'$ and $v\colon N \to N'$ be morphisms of k-modules. The map $M \times N \to M' \otimes_k N'$ given by $(m, n) \mapsto u(m) \otimes v(n)$ is k-bilinear. Consequently, there exists a unique morphism of k-modules, $w\colon M \otimes_k N \to M' \otimes_k N'$, such that $w(m \otimes n) = u(m) \otimes v(n)$ for every $m \in M$ and every $n \in N$. This morphism is often denoted by $u \otimes v$.

If u and v are surjective, then $u \otimes v$ is surjective.

If u and v are split injective, that is, if they admit retractions, then $u \otimes v$ is split injective. Indeed, let $u'\colon M' \to M$ and $v'\colon N' \to N$ be morphisms such that $u' \circ u = \mathrm{Id}_M$ and $v' \circ v = \mathrm{Id}_N$; then $(u' \otimes v') \circ (u \otimes v) = (u' \circ u) \otimes (v' \circ v) = \mathrm{Id}_M \otimes \mathrm{Id}_N = \mathrm{Id}_{M \otimes N}$. An important case where this happens is when k is a field.

1.8.4. — Let $(M_i)_{i \in I}$ be a family of k-modules, let $M = \bigoplus_{i \in I} M_i$ be their direct sum; for every $i \in I$, let $p_i\colon M \to M_i$ be the projection of index i. Let $(N_j)_{j \in J}$ be a family of k-modules and let $N = \bigoplus_{j \in J} N_j$ be their direct sum; for every $j \in J$, let $q_j\colon N \to N_j$ be the projection of index j. The map from $M \times N$ to $\bigoplus_{i,j}(M_i \otimes_k N_j)$ given by $(m, n) \mapsto \sum_{i,j} p_i(m) \otimes q_j(n)$ is k-bilinear; consequently, there exists a unique k-linear morphism $\pi\colon M \otimes_k N \to \bigoplus_{i,j}(M_i \otimes_k N_j)$ such that $\pi(m \otimes n) = \sum_{i,j} p_i(m) \otimes q_j(n)$. The morphism π is an isomorphism.

In particular, if M and N are free k-modules, their tensor product is a free k-module. More precisely, let $(m_i)_{i \in I}$ be a basis of M, let $(n_j)_{j \in J}$ be a basis of N; then the family $(m_i \otimes n_j)_{(i,j) \in I \times J}$ is a basis of $M \otimes_k N$.

1.8.5. Base change. — Let M be a k-module and let A be a k-algebra. The k-module $M \otimes_k A$ is naturally an A-module: the external multiplication being characterized by the relation $b(m \otimes a) = m \otimes ab$. It is called the A-module deduced from M by base change, and is sometimes denoted by M_A.

If $f\colon M \to M'$ is a morphism of k-modules, the morphism $f_A = f \otimes \mathrm{Id}_A\colon M_A \to M'_A$ is A-linear.

1.8.6. — Let A and B be k-algebras. Then the tensor product $A \otimes_k B$ has a unique structure of k-algebra for which $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$ for every $a, a' \in A$ and every $b, b' \in B$.

The map $A \to A \otimes_k B$ given by $a \mapsto a \otimes 1$ is a morphism of k-algebras; similarly, the map from B to $A \otimes_k B$ given by $b \mapsto 1 \otimes b$ is morphism of k-algebras.

1.8.7. — For example, let A and B be polynomial algebras in families of indeterminates $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ respectively. Then the tensor product $A \otimes_k B$ is isomorphic to the polynomial algebra in the family of indeterminates obtained by concatenation of the families (X_i) and (Y_j). Such an isomorphism is induced by the bilinear map from $k[(X_i)] \times k[(Y_j)]$ to $k[(X_i, Y_j)]$ which maps a pair $(P(X_i), Q(Y_j))$ to its product $P(X_i)Q(Y_j)$.

Lemma **(1.8.8)**. — *Let* I *be an ideal of* A*, let* J *be an ideal of* B*; let* (I, J) *denote the ideal of* $A \otimes_k B$ *which they generate. There exists a unique morphism of k-algebras*

$$(A \otimes_k B)/(I, J) \to (A/I) \otimes_k (B/J)$$

which maps the class of $a \otimes b$ to the tensor product $\bar{a} \otimes \bar{b}$ of the classes of a and b in A/I *and* B/J *respectively. This morphism is an isomorphism of rings.*

Theorem **(1.8.9)**. — *Let* K *be an algebraically closed field and let* A, B *be two* K*-algebras. If* A *and* B *are integral domains, then* $A \otimes_K B$ *is also an integral domain.*

Proof. — The tensor product of two non-zero K-vector spaces is a non-zero K-vector space; consequently, $A \otimes_K B \neq 0$ and it suffices to show that the product of two non-zero elements of $A \otimes_K B$ is non-zero.

Let f and g be two element of $A \otimes_K B$ such that $fg = 0$. We may decompose f as a sum $\sum_{i=1}^r a_i \otimes b_i$ of split tensors, where $b_1, \dots, b_r$ are linearly independent over K. Similarly, we write $g = \sum_{j=1}^s a'_j \otimes b'_j$, where $b'_1, \dots, b'_s$ are linearly independent over K.

Let A_1 be the subalgebra of A generated by $a_1, \dots, a_r, a'_1, \dots, a'_s$, let B_1 be the subalgebra of B generated by $b_1, \dots, b_r, b'_1, \dots, b'_s$. Let I and I′ be the ideals $(a_1, \dots, a_r)$ and $(a'_1, \dots, a'_s)$ of A_1. Since A_1 and B_1 have direct summands in A and B as K-modules, the canonical morphism from $A_1 \otimes_K B_1$ to $A \otimes_K B$ has a retraction, which allows to view $A_1 \otimes_K B_1$ as a subalgebra of $A \otimes_K B$. By construction, f and g belong to $A_1 \otimes_K B_1$, and $fg = 0$. Let us show that $I \cap I' = \{0\}$.

Let $\mathfrak{m}$ be a maximal ideal of A_1. The quotient ring $A_1/\mathfrak{m}$ is a finitely generated K-algebra, and is a field; consequently, it is an algebraic extension of K, hence is isomorphic to K since K is algebraically closed. Let $\mathrm{cl}_{\mathfrak{m}} \colon A_1 \to K$ be the corresponding morphism of K-algebras with kernel $\mathfrak{m}$. Let also $\theta_{\mathfrak{m}} \colon A_1 \otimes_K B_1 \to B_1$ be the morphism $\mathrm{cl}_{\mathfrak{m}} \otimes \mathrm{id}_{B_1}$; it is a morphism of K-algebras.

Since

$$\theta_{\mathfrak{m}}(f)\theta_{\mathfrak{m}}(g) = \theta_{\mathfrak{m}}(fg) = 0$$

and B_1 is an integral domain, either $\theta_{\mathfrak{m}}(f) = 0$ or $\theta_{\mathfrak{m}}(g) = 0$. Moreover, one has

$$\theta_{\mathfrak{m}}(f) = \sum_{i=1}^{r} \mathrm{cl}_{\mathfrak{m}}(a_i) b_i \quad \text{and} \quad \theta_{\mathfrak{m}}(g) = \sum_{j=1}^{s} \mathrm{cl}_{\mathfrak{m}}(a'_j) b'_j.$$

Assume that $\theta_{\mathfrak{m}}(g) = 0$. Since $b_1, \dots, b_r$ are linearly independent over K, we conclude that $\mathrm{cl}_{\mathfrak{m}}(a_i) = 0$ for every $i \in \{1, \dots, r\}$; in other words, the ideal $I = (a_1, \dots, a_r)$ is contained in $\mathfrak{m}$.

Similarly, if $\theta_{\mathfrak{m}}(f) = 0$, we obtain that the ideal $I' = (a'_1, \dots, a'_s)$ is contained in $\mathfrak{m}$.

In any case, one has $I \cap I' \subseteq \mathfrak{m}$.

This is valid for any maximal ideal $\mathfrak{m}$ of A_1. By corollary 1.6.4, every element of $I \cap I'$ is nilpotent. Since A_1 is an integral domain, one has $I \cap I' = \{0\}$.

Assume that $f \neq 0$. Then $I \neq 0$; let thus x be a non-zero element of I. For every $y \in I'$, $xy \in I \cap I'$, hence $xy = 0$. Since A_1 is an integral domain, this implies $y = 0$, hence $I' = 0$, hence $a'_1 = \cdots = a'_s = 0$ and $g = 0$. This concludes the proof that $A \otimes_K B$ is an integral domain. ☐

1.9. Noetherian rings

1.9.1. — Let k be a ring. One says that a k-module M is *noetherian* if one of the following equivalent properties holds:

(i) Every strictly increasing sequence of submodules of M is finite;
(ii) Every non-empty family of submodules of M has a maximal element;
(iii) Every submodule of M is finitely generated.

The equivalence of (i) and (ii) is elementary. Let us assume that they hold, let P be a submodule of M and let us prove that P is finitely generated. The set of all finitely generated submodules of P is non-empty, since $\{0\}$ is finitely generated, hence it has a maximal element, say P′. For every $m \in P$, $P' + Am$ is a finitely generated submodule of P which contains P′; by maximality of P′, one has $P' + Am = P'$, hence $m \in P'$; consequently, $P = P'$ and P is finitely generated, as was to be shown. Conversely, let us assume that every submodule of M is finitely generated and let us prove by contradiction that every stricly increasing sequence of submodules of M is finite. Let thus (P_n) be a stricly increasing infinite sequence of submodules of M and let P be their union. By assumption, P is finitely generated, hence there are elements $p_1, \dots, p_s \in P$ such that $P = Ap_1 + \cdots + Ap_s$. By definition of P, for each integer $i \in \{1, \dots, s\}$, there exists an integer n_i such that $p_i \in P_{n_i}$. If $n = \max(n_1, \dots, n_s)$, one has $p_i \in P_n$ for each i, hence $P \subseteq P_n$. Since $P_n \subseteq P_m \subseteq P$ for every integer $m \geqslant n$, this shows that $P_m = P_n$ and contradicts the hypothesis that the sequence (P_n) is strictly increasing.

One says that a ring A is *noetherian* if it is noetherian as a module over itself; since a submodule of A is an ideal of A, this means that one of the following equivalent properties holds:

(i) Every strictly increasing sequence of ideals of A is finite;
(ii) Every non-empty family of ideals of A has a maximal element;
(iii) Every idel of A is finitely generated.

In particular, principal ideal domains are noetherian.

1.9.2. — Let N be a submodule of M. Then M is noetherian if and only if both N and M/N are noetherian. In particular, finite direct sums of noetherian modules are noetherian.

If A is an noetherian ring, then an A-module is noetherian if and only if it is finitely generated.

Theorem **(1.9.3)** (Hilbert). — *For every noetherian ring* A, *the ring* A[X] *is noetherian. In particular, for every field* K *and every integer* $n \geqslant 0$, *the ring* $K[X_1, \dots, X_n]$ *is noetherian.*

Proof. — Let I be an ideal of A[X]. For every integer m, let J_m be the set of leading coefficients of elements of I whose degrees are equal to m (the leading coefficient of the zero polynomial being 0); one checks that is an ideal of A.

For every integer m, the ideal J_m is finitely generated. We may thus fix a finite set Q_m of polynomials belonging to I_m whose leading coefficients generate J_m. Moreover, the family $(J_m)_{m\geqslant 0}$ is increasing. Since A is noetherian, there exists an integer n such that $J_m = J_n$ for every integer m such that $m \geqslant n$.

Let Q be the finite set $Q = Q_0 \cup \dots \cup Q_n$ and let I′ be the ideal of A[X] it generates. One has $I' \subseteq I$; it suffices to prove that $I = I'$. Let thus $P \in I$ and let us prove by induction on deg(P) that $P \in I'$. Let $m = \deg(P)$, and let $a \in J_m$ be the leading coefficient of P. Let $p = \min(m, n)$; one has $a \in J_p$. By definition of Q_p, there exists a polynomial P′ of degree p which is a linear combination of polynomials in Q_p (hence an element of I′) whose leading coefficient is equal to a. The polynomial $P - T^{m-p}P'$ belongs to I and its degree is $< m$; by induction, it belongs to I'. Consequently, P belongs to I', as was to be shown. □

1.10. Irreducible components

Definition (**1.10.1**). — *Let* X *be a topological space. One says that* X *is* irreducible *if it is not empty and if it is not the union of two closed subsets of* X*, both non-empty and distinct from* X*. One says that a subspace of* X *is irreducible if the induced subspace is irreducible.*

In other words, a subset Z of X is irreducible if and only if it is non-empty and if for every two closed subsets Y_1 and Y_2 of X such that $Z \subseteq Y_1 \cup Y_2$, one has $Z \subseteq Y_1$ or $Z \subseteq Y_2$.

If Z is irreducible, then Z is connected.

This notion is very useful in the framework of algebraic geometry, where the Zariski topology plays a prominent rôle. However, it has little interest for the classical topological spaces; for example, the only irreducible subspaces of $\mathbf{R}^n$ are singletons.

Proposition (**1.10.2**). — *Let* A *be a ring.*

a) *The topological space* Spec(A) *is irreducible if and only if the nilradical of* A *is a prime ideal.*

b) *Let* I *be an ideal of* A*. The closed subset* V(I) *of* Spec(A) *is irreducible if and only if* $\sqrt{I}$ *is a prime ideal.*

Proof. — Assertion *a*) is the particular case of *b*) for $I = \{0\}$. Conversely, if I is an ideal of A, V(I) is homeomorphic to Spec(A/I) by proposition 1.5.10; moreover, the nilradical of A/I is equal to $\sqrt{I}/I$, hence is prime if and only if $\sqrt{I}$ is a prime ideal of A.

It thus suffices to treat part *a*). Since $\operatorname{Spec}(A/\mathfrak{n})$ is homeomorphic to Spec(A), we may even assume that A is reduced.

Let us assume that Spec(A) is reducible. Let Y_1 and Y_2 be closed subsets of Spec(A), distinct from Spec(A), such that $\operatorname{Spec}(A) = Y_1 \cup Y_2$; let I_1 and I_2 be radical ideals such that $Y_1 = V(I_1)$ and $Y_2 = V(I_2)$. Since Y_1 and Y_2 are not equal to Spec(A), one has $I_1 \neq \{0\}$ and $I_2 \neq \{0\}$; let then $a \in I_1$ and $b \in I_2$ be any two non-zero elements. Since $\operatorname{Spec}(A) = Y_1 \cup Y_2 = V(I_1 \cap I_2) = V(0)$, one has $I_1 \cap I_2 = \{0\}$. In particular, $ab = 0$, which shows that A is not an integral domain.

Conversely, let a, b be non-zero elements of A such that $ab = 0$. Then, $\operatorname{Spec}(A) = V(0) = V(ab) = V(a) \cup V(b)$. Since $a \neq 0$ and $\mathfrak{n} = 0$, there exists a prime ideal $\mathfrak{p}$ of A such that $a \notin \mathfrak{p}$; in particular, $V(a) \neq \operatorname{Spec}(A)$. The element a is not a unit (for, otherwise, $b = 0$, a contradiction); consequently, $V(a) \neq \varnothing$. Similarly, $V(b)$ is neither empty, nor equal to $\operatorname{Spec}(A)$. This implies that $\operatorname{Spec}(A)$ is not irreducible. □

Proposition **(1.10.3)**. — *Let* X *be an irreducible topological space and let* U *be a non-empty open subset of* X.

a) *The open subset* U *is dense in* X*, and is irreducible;*

b) *The map* $Z \mapsto Z \cap U$ *defines a bijection between the set of irreducible closed subsets of* X *which meet* U *and the set of irreducible closed subsets of* U*. Its inverse bijection is given by* $Z \mapsto \overline{Z}$.

Proof. — a) By definition of an irreducible topological space, the union of two closed subsets distinct from X is distinct from X. Considering the complementary subsets, the intersection of two non-empty open subsets of an irreducible topological space is non-empty. In particular, U meets every non-empty open subset of X, which means that U is dense.

Let us now prove that U is irreducible. Let Z_1 and Z_2 be closed subsets of X such that $U \subseteq Z_1 \cup Z_2$. It then follows that $X = \overline{U} \subseteq Z_1 \cup Z_2$, so that $X = Z_1$ or $X = Z_2$. In particular, $U \subseteq Z_1$ or $U \subseteq Z_2$.

b) Let Y be an irreducible closed subset of U and let Z be its closure in X; let us observe that $Y = Z \cap U$. Indeed, since Y is closed in U, there exists a closed subset Z' of X such that $Y = U \cap Z'$. By definition of the closure, we have $Z \subseteq Z'$. Then, $Y \subseteq Z \cap U \subseteq Z' \cap U = Y$, hence $Y = Z \cap U$.

Since Y is irreducible, it is non-empty, hence the set Z is not empty. Let Z_1 and Z_2 be closed subsets of X such that $Z \subseteq Z_1 \cup Z_2$. Then $Y = U \cap Z \subseteq (U \cap Z_1) \cup (U \cap Z_2)$. Since Y is irreducible, one has $Y \subseteq U \cap Z_1$ or $Y \subseteq U \cap Z_2$. In the first case, Z_1 is a closed subset of X containing Y, hence $Z \subseteq Z_1$; in the other case, $Z \subseteq Z_2$. This shows that Z is irreducible.

We may now conclude the proof of the proposition. By what precedes, setting $\alpha(Y) = \overline{Y}$ defines a map from the set of irreducible closed subsets of U to the set of irreducible closed subsets of X.

Applied to an irreducible subset Z of X and to its open subspace $Z \cap U$, part *a*) implies that if $Z \cap U \neq \varnothing$, then it is irreducible and $\overline{Z \cap U} = Z$. Consequently, one defines a map from the set of irreducible closed subsets of X which meet U to the set of irreducible closed subsets of U by setting $\beta(Z) = Z \cap U$. Moreover, if Z is a closed subset of X which meets U, then $\alpha \circ \beta(Z) = \overline{Z \cap U} = Z$; if Y is a closed subset of U, then we had already proved that $\beta \circ \alpha(Y) = \overline{Y} \cap U = Y$. This shows that α and β are bijections, inverse one of the other. □

Definition (**1.10.4**). — *An* irreducible component *of a topological space is a maximal irreducible subset.*

Lemma (**1.10.5**). — *Let* X *be a topological space.*

a) *The closure of an irreducible subset of* X *is irreducible. In particular, every irreducible component of* X *is closed.*

b) *Every irreducible subset of* X *is contained in some irreducible component. In particular,* X *is the union of its irreducible components.*

Proof. — a) Let A be an irreducible subset of X and let Z_1, Z_2 be closed subsets of X such that $\overline{A} \subseteq Z_1 \cup Z_2$. Consequently, $A \subseteq Z_1 \cup Z_2$, hence $A \subseteq Z_1$ or $A \subseteq Z_2$. Since Z_1 and Z_2 are closed, one thus has $\overline{A} \subseteq Z_1$ or $\overline{A} \subseteq Z_2$. This proves that $\overline{A}$ is irreducible.

b) Let $\mathscr{C}$ be the set of irreducible subsets of X which contain A. Let us show that the set $\mathscr{C}$, ordered by inclusion, is inductive. It is non-empty since $A \in \mathscr{C}$. Let $(Y_i)_{i \in I}$ be a non-empty totally ordered family of irreducible subsets of X containing A and let Y be its union. One has $A \subseteq Y$, because $I \neq \varnothing$. Let us show that Y is irreducible. First, $Y \neq \varnothing$. Let then Z_1 and Z_2 be closed subsets of X such that $Y \subseteq Z_1 \cup Z_2$. Let us assume that $Y \nsubseteq Z_2$, let $y \in Y$ be such that $y \notin Z_2$ and let $j \in I$ be such that $y \in Y_j$. Let $i \in I$ and let us show that $Y_i \subseteq Z_1$. If $Y_j \subseteq Y_i$, one has $Y_i \subseteq Y \subseteq Z_1 \cup Z_2$, and $Y_i \nsubseteq Z_2$ since Y_i contains Y_j; since Y_i is irreducible, one thus has $Y_i \subseteq Z_1$. In particular, $Y_j \subseteq Z_1$. On the other hand, if $Y_i \subseteq Y_j$, we have $Y_i \subseteq Y_j \subseteq Z_1$. Consequently, $Y = \bigcup_{i \in I} Y_i \subseteq Z_1$. This shows that Y is irreducible, hence that $\mathscr{C}$ is inductive. By Zorn's lemma, $\mathscr{C}$ has a maximal element; this is a maximal irreducible subset of X, hence an irreducible component of X; it contains A by construction.

For every $x \in X$, $\{x\}$ is irreducible. By what precedes, every point of X is contained in an irreducible component. This means exactly that X is the union of its irreducible components, as claimed. □

Example **(1.10.6)**. — An irreducible component of Spec(A) is a closed subset of the form $V(\mathfrak{p})$, where $\mathfrak{p}$ is a *minimal* prime ideal of A.

More generally, if I is an ideal of A such that $I \neq A$, the closed subset $V(I)$ is nonempty, and its irreducible components are of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ a prime ideal of A which is minimal among the prime ideals of A that contain I.

As a consequence of lemma 1.10.5, every prime ideal of A contains a minimal prime ideal of A.

Definition **(1.10.7)**. — *One says that a topological space is* noetherian *if every strictly decreasing sequence of closed subsets is finite.*

Equivalently, a topological space is noetherian if and only if every non-empty family of closed subsets has a minimal element.

Example **(1.10.8)**. — Indeed, the property for Spec(A) of being noetherian means that every non-empty family of *radical* ideals of A has a maximal element. In particular, we see that if A is a noetherian ring, then Spec(A) is a noetherian topological space.

Proposition **(1.10.9)**. — *Let* X *be a noetherian topological space.*

a) *Every subspace of* X *is noetherian;*
b) *The space* X *has finitely many irreducible components, and* X *is their union.*
c) *Every irreducible component of* X *contains a non-empty open subset of* X.

Proof. — a) Let A be a subspace of X and let (A_n) be a stricly decreasing sequence of closed subsets of A. By definition of the induced topology, there exists for each n a closed subset Y_n of X such that $A_n = A \cap Y_n$. Set $Z_n = Y_0 \cap Y_1 \cap \dots \cap Y_n$; the sequence (Z_n) is decreasing. Since one has $A_n = A \cap Z_n$ for each n, this sequence is in fact stricly decreasing, hence is finite because X is noetherian. This implies that the sequence (A_n) is finite, as was to be shown.

b) Since every subspace of X is noetherian, the assertion should hold for every subspace of X. We will thus prove the desired result by contradiction by considering a *minimal* subspace of X which is a counterexample.

Precisely, let $\mathfrak{C}$ be the set of closed subsets of X which cannot be written as a finite union of irreducible closed subspaces of X. Assume by contradiction that $\mathfrak{C}$ is non-empty. Since X is a noetherian topological space, the set $\mathfrak{C}$, ordered by inclusion, admits a minimal element W. By construction, W is a closed subset of X which is not a finite union of irreducible closed subspaces of X, but every closed subspace of W (distinct from W) is such a finite union.

The space W is not irreducible. Since the empty space is the union of the empty family, one has $W \neq \varnothing$. Consequently, there exist closed subsets W_1 and W_2 of W, non-empty and distinct from W, such that $W = W_1 \cup W_2$. By the minimality of W, W_1 and W_2 can be written as a finite union of irreducible closed subspaces of W; consequently, W is also a finite union of irreducible closed subspaces of W, a contradiction!

In particular, there exists a finite family $(X_1, \dots, X_n)$ of irreducible closed subsets of X such that $X = X_1 \cup \dots \cup X_n$. Up to removing X_i from this family if necessary, we may assume that for $j \neq i$, X_i is not a subspace of X_j.

Before we terminate the proof of *b*), let us prove that every irreducible subset Z of X is contained in one of the X_i. Since $Z = \bigcup_{i=1}^n (Z \cap X_i)$, there exists $i \in \{1, \dots, n\}$ such that $Z = Z \cap X_i$, this means that $Z \subseteq X_i$.

This implies in particular that every maximal element of the family $(X_1, \dots, X_n)$ is maximal among all closed irreducible subsets of X, so that $X_1, \dots, X_n$ are the irreducible components of X.

c) Let Y be an irreducible component of X, let Y' be the union of the other irreducible components, and let $U = Y - (Y \cap Y')$. Since X has finitely many irreducible components, Y' is closed, so that U is open. If, by contradiction, U is empty, then $Y \cap Y' = Y$, hence $Y \subseteq Y'$. By the argument used at the end of the proof of *b*), this implies that Y is contained in some other irreducible component of X, contradicting the definition of an irreducible component. So U is a non-empty open subset of X contained in Y. □

Corollary **(1.10.10)**. — *Let* A *be a reduced noetherian ring. Then* A *has finitely many minimal prime ideals. Their intersection is equal to* $\{0\}$ *and their union is the set of zero divisors of* A.

Proof. — Since A is noetherian, the topological space Spec(A) is noetherian. Consequently, A has finitely many minimal prime ideal, say $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, and every prime ideal contains a minimal prime ideal. In particular, the nilradical of A, which is the intersection of all prime ideals of A, is equal to the intersection of all minimal prime ideals. Since A is reduced, this intersection is equal to $\{0\}$.

It remains to show that an element $a \in A$ is a zero divisor if and only if it belongs to one of the $\mathfrak{p}_i$. One has $V(\bigcap_{j\neq i} \mathfrak{p}_j) = \bigcup_{j \neq i} V(\mathfrak{p}_j) \neq \operatorname{Spec}(A)$, hence the ideal $\bigcap_{j\neq i} \mathfrak{p}_j$ contains a non-zero element, say x. Then ax belongs to the intersection of all minimal prime ideals of A, hence $ax = 0$; this shows that a is a zero divisor. Conversely, let $a \in A$ be a zero divisor and let $x \in A - \{0\}$ be such that $ax = 0$. Since $x \neq 0$, there exists $i \in \{1, \dots, n\}$ such that $x \notin \mathfrak{p}_i$. The equality $ax = 0$ then implies that $a \in \mathfrak{p}_i$. □

1.11. Dimension

1.11.1. — Let E be a partially ordered set.

A *chain* in E is a stricly increasing family $x_0 < x_1 < \dots < x_n$. The *length* of that chain is equal to n, it starts at x_0 and ends at x_n.

The *dimension* of E, denoted by $\dim(E)$, is the supremum of the lengths of chains in E.

Let $x \in E$. The *height* (resp. the *coheight*) of x is the supremum of the length of chains ending (resp. starting) at x. They are denoted $\operatorname{ht}(x)$ and $\operatorname{coht}(x)$ respectively.

Definition (**1.11.2**). — *Let* X *be a topological space.*

The Krull dimension *of* X*, denoted* $\dim(X)$*, is the dimension of the set* $\mathfrak{C}(X)$ *of all irreducible closed subsets of* X*, ordered by inclusion.*

Let Z *be a closed irreducible subset of* X*. The* codimension *of* Z *in* X*, denoted* $\operatorname{codim}(Z)$*, is the coheight of* Z *in the partially ordered set* $\mathfrak{C}$*.*

The following facts follow directly from these definitions:

a) The dimension of X is the supremum of the dimensions of its irreducible components;

b) Each irreducible component of X has codimension 0;

c) For every closed irreducible subset Z of X, one has $\operatorname{codim}(Z) + \dim(Z) \leqslant \dim(X)$;

d) If Y and Z are irreducible closed subsets of X such that $Y \subseteq Z$, then $\dim(Y) \leqslant \dim(Z)$ and $\operatorname{codim}(Z) \leqslant \operatorname{codim}(Y)$.

In view of these facts, one may define the codimension of an arbitrary closed subset Z as the infimum of the codimensions of its irreducible components.

1.11.3. — Let A be a ring and let $X = \operatorname{Spec}(A)$. Every closed irreducible subset Z of X is of the form $Z = V(\mathfrak{p})$, for some unique prime ideal $\mathfrak{p}$ of A; in fact, $\mathfrak{p} = \mathfrak{j}(Z) = \mathfrak{j}(\{\mathfrak{p}\})$, so that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. Moreover, if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals, then $V(\mathfrak{q}) \subseteq V(\mathfrak{p})$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$. Consequently, the three following partially ordered sets are isomorphic:

- The set $\mathfrak{C}$ of closed irreducible subsets of X, ordered by inclusion;
- The set of all prime ideals of A, ordered by containment;
- The set Spec(A), ordered by the relation $x \prec y$ if and only if $x \in \overline{\{y\}}$.

It follows that the dimension of X is equal to the supremum of the lengths of chains of prime ideals of A, the *Krull dimension* dim(A) of the ring A.

For every prime ideal $\mathfrak{p}$ of A, the codimension of $V(\mathfrak{p})$ in Spec(A) is equal to the *height* $\mathrm{ht}(\mathfrak{p})$ of $\mathfrak{p}$, defined as the supremum of the lengths of chains of prime ideals of A ending at $\mathfrak{p}$. By the correspondence between prime ideals of the localized ring $A_{\mathfrak{p}}$ and prime ideals of A contained in $\mathfrak{p}$, one also has

$$\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}).$$

Moreover, one has $\dim(V(\mathfrak{p})) = \dim(A/\mathfrak{p})$, hence the inequality

$$\mathrm{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) \leqslant \dim(A).$$

Theorem **(1.11.4)** (First theorem of Cohen-Seidenberg)
Let B *be a ring and let* A *be a subring of* B. *Assume that* B *is integral over* A.

a) *Let* $\mathfrak{q}$ *be a prime ideal of* B *and let* $\mathfrak{p} = \mathfrak{q} \cap A$. *Then* $\mathfrak{p}$ *is a maximal ideal of* A *if and only if* $\mathfrak{q}$ *is a maximal ideal of* B.

b) *Let* $\mathfrak{q} \subseteq \mathfrak{q}'$ *be prime ideals of* B *such that* $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$. *Then* $\mathfrak{q} = \mathfrak{q}'$.

c) *The canonical map from* Spec(B) *to* Spec(A) *is surjective: for every prime ideal* $\mathfrak{p}$ *of* A, *there exists a prime ideal* $\mathfrak{q}$ *of* B *such that* $\mathfrak{q} \cap A = \mathfrak{p}$.

Proof. — a) Passing to the quotients, one gets an integral extension of integral domains $A/\mathfrak{p} \subseteq B/\mathfrak{q}$. By lemma 1.4.5, $A/\mathfrak{p}$ is a field if and only if $B/\mathfrak{q}$ is a field; in other words, $\mathfrak{p}$ is maximal in A if and only if $\mathfrak{q}$ is maximal in B.

b) Let $\mathfrak{p} = \mathfrak{q} \cap A$ and let us consider the integral extension of rings $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ induced by localization by the multiplicative subset $A - \mathfrak{p}$. (It is indeed injective: if a fraction a/s in $A_{\mathfrak{p}}$ maps to 0 in $B_{\mathfrak{p}}$, there exists $t \in A - \mathfrak{p}$ such that $at = 0$, hence $a = 0$.) Observe the obvious inclusion $\mathfrak{p}A_{\mathfrak{p}} \subseteq \mathfrak{q}B_{\mathfrak{p}}$. On the other hand, the ideal $\mathfrak{q}B_{\mathfrak{p}}$ does not contain 1, hence is contained in the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ of the local ring $A_{\mathfrak{p}}$. This shows that $\mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Similarly, $\mathfrak{q}'B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$.

Since $\mathfrak{p}A_{\mathfrak{p}}$ is maximal, $\mathfrak{q}B_{\mathfrak{p}}$ and $\mathfrak{q}'B_{\mathfrak{p}}$ are maximal ideals of $B_{\mathfrak{p}}$. However, the inclusion $\mathfrak{q} \subseteq \mathfrak{q}'$ implies $\mathfrak{q}B_{\mathfrak{q}} \subseteq \mathfrak{q}'B_{\mathfrak{q}}$. Necessarily, these two maximal ideals of $B_{\mathfrak{p}}$ are equal.

Since localization induces a bijection from the set of prime ideals of B disjoint from $A - \mathfrak{p}$ to the set of prime ideals of $B_\mathfrak{p}$, one gets $\mathfrak{q} = \mathfrak{q}'$.

c) Let $\mathfrak{p}$ be a prime ideal of A and let us consider the extension $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ obtained by localization with respect to the multiplicative subset $A - \mathfrak{p}$. Since $B_\mathfrak{p} \neq 0$, we may consider a maximal ideal $\mathfrak{m}$ of $B_\mathfrak{p}$. There exists a prime ideal $\mathfrak{q} \subseteq B$ disjoint from $A - \mathfrak{p}$ such that $\mathfrak{m} = \mathfrak{q}B_\mathfrak{p}$. Considering the integral extension $A_\mathfrak{p} \subseteq B_\mathfrak{p}$, part *a*) implies that $\mathfrak{m} \cap A_\mathfrak{p}$ is a maximal ideal of $A_\mathfrak{p}$, hence $\mathfrak{m} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$.

Let us show that $\mathfrak{q} \cap A = \mathfrak{p}$. Indeed, let $b \in \mathfrak{q} \cap A$; then $b/1 \in \mathfrak{q}B_\mathfrak{p} \cap A_\mathfrak{p}$, so that there exists $a \in A - \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, $b \in \mathfrak{p}$. Conversely, if $a \in \mathfrak{p}$, then $a/1 \in \mathfrak{p}A_\mathfrak{p}$ hence $a/1 \in qB_\mathfrak{p}$. Consequently, there exists $a' \in A - \mathfrak{p}$ such that $aa' \in \mathfrak{q}$. Observe that $a' \notin \mathfrak{q}$, for otherwise, one would have $a' \in \mathfrak{q} \cap A = \mathfrak{p}$, which does not hold. Since $\mathfrak{q}$ is a prime ideal, $a \in \mathfrak{q}$.

□

Corollary **(1.11.5)**. — *Let* B *be a ring, let* A *be a subring of* B. *If* B *is integral over* A, *then* $\dim(A) = \dim(B)$.

Proof. — Let $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ be a chain of prime ideals of B. Let us intersect these ideals with A; this gives an increasing family $(\mathfrak{q}_0 \cap A) \subseteq \cdots \subseteq (\mathfrak{q}_n \cap A)$ of prime ideals of A. By part *b*) of theorem 1.11.4, this is even a chain of prime ideals, so that $\dim(A) \geqslant \dim(B)$.

Conversely, let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals of A. For each $m \in \{0, \dots, n\}$, let us construct by induction a prime ideal $\mathfrak{q}_m$ of B such that $\mathfrak{q}_m \cap A = \mathfrak{p}_m$ and such that $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_n$. This will imply that $\dim(B) \geqslant \dim(A)$, hence the corollary.

By part *c*) of theorem 1.11.4, there exists a prime ideal $\mathfrak{q}_0$ of B such that $\mathfrak{q}_0 \cap A = \mathfrak{q}_0$. Assume $\mathfrak{q}_0, \dots, \mathfrak{q}_m$ are defined. Let us consider the integral extension $A/\mathfrak{p}_m \subseteq B/\mathfrak{q}_m$ of integral domains. By theorem 1.11.4, applied to the prime ideal $\mathfrak{p}_{m+1}/\mathfrak{p}_m$ of $A/\mathfrak{p}_m$, there exists a prime ideal $\mathfrak{q}$ of the ring $B/\mathfrak{q}_m$ such that $\mathfrak{q} \cap (A/\mathfrak{p}_m) = \mathfrak{p}_{m+1}/\mathfrak{p}_m$. Then, there exists a prime ideal $\mathfrak{q}_{m+1}$ containing $\mathfrak{q}_m$ such that $\mathfrak{q} = \mathfrak{q}_{m+1}/\mathfrak{q}_m$. Moreover, $\mathfrak{q}_{m+1} \cap A = \mathfrak{p}_{m+1}$.

This concludes the proof. □

The following theorem lies at the ground of dimension theory in algebraic geometry.

Theorem **(1.11.6)**. — *Let* K *be a field and let* A *be a finitely generated* K*-algebra. Assume that* A *is an integral domain and let* F *be its field of fractions. One has* $\dim(A) = \operatorname{tr.deg}_K(F)$.

Proof. — We prove the theorem by induction on the transcendence degree of F.

If $\operatorname{tr.deg}_K(F) = 0$, then A is algebraic over K. Consequently, $\dim(A) = \dim(K) = 0$.

Now assume that the theorem holds for finitely generated K-algebras which are integral domains and whose field of fractions has transcendence degree strictly less than $\operatorname{tr.deg}_K(F)$.

By the Noether normalization lemma (theorem 1.6.1), there exist an integer $n \geqslant 0$, elements $a_1, \dots, a_n$ of A such that the morphism $f : K[X_1, \dots, X_n] \to A$ such that $f(X_i) = a_i$ is injective, and such that A is integral over its subring $B = K[a_1, \dots, a_n] = f(K[X_1, \dots, X_n])$. Moreover, $n = \operatorname{tr.deg}_K(F)$. By corollary 1.11.5, it suffices to prove that the dimension of the polynomial ring $K[X_1, \dots, X_n]$ is equal to n.

Observe that

$$(0) \subseteq (X_1) \subseteq \cdots \subseteq (X_1, \dots, X_n)$$

is a chain of prime ideals of $K[X_1, \dots, X_n]$; since its length is equal to n, this shows that $\dim(K[X_1, \dots, X_n]) \geqslant n$. Conversely, let

$$(0) \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$$

be a chain of prime ideals of $K[X_1, \dots, X_n]$ and let us set $A' = K[X_1, \dots, X_n]/\mathfrak{p}_1$. Then A' is a finitely generated K-algebra and $\dim(A') \geqslant m - 1$. Since $\mathfrak{p}_1$ is a prime ideal, A' is an integral domain; let F' be its field of fractions. Any non-zero polynomial $P \in \mathfrak{p}_1$ furnishes gives a non-trivial algebraic dependence relation between the classes $x_1, \dots, x_n$ of $X_1, \dots, X_n$ in A'. Consequently, $\operatorname{tr.deg}_K(F') \leqslant n - 1$. By induction, $\operatorname{tr.deg}_K(F') = \dim(A')$, hence $m - 1 \leqslant n - 1$, and $m \leqslant n$. This concludes the proof. □

In the course of the proof of theorem 1.11.6, we established the following particular case.

Corollary **(1.11.7)**. — *For any field* K*, one has* $\dim(K[X_1, \dots, X_n]) = n$.

Proposition **(1.11.8)**. — *Let* K *be a field and let* A, B *be finitely generated* K*-algebras. One has* $\dim(A \otimes_K B) = \dim(A) + \dim(B)$.

Proof. — By the Noether normalization lemma (theorem 1.6.1), there exist integers $m, n \geqslant 0$ and injective integral morphisms $f\colon K[X_1,\dots,X_m] \to A$ and $g\colon K[Y_1,\dots,Y_n] \to B$. One has $m = \dim(A)$ and $n = \dim(B)$. Since A and B are finitely generated, these morphisms are even finite. It follows that the natural morphism

$$K[X_1,\dots,X_m,Y_1,\dots,Y_n] \simeq K[X_1,\dots,X_m] \otimes_K K[Y_1,\dots,Y_n] \to A \otimes_K B$$

is injective and finite. Consequently, $\dim(A \otimes_K B) = m + n = \dim(A) + \dim(B)$. □

Remark **(1.11.9)**. — Dimension theory of rings has a lot of subtleties which do not occur for finitely generated algebras over a field.

a) There are rings of infinite dimension, for example the ring $A = K[T_1, T_2, \dots]$ of polynomials in infinitely many indeterminates. Worse, while all strictly increasing sequences of ideals in a noetherian ring are finite, their lengths may not be bounded. In fact, Nagata has given the following example of a noetherian ring whose dimension is infinite. Let (m_n) be a stricly increasing sequence of positive integers such that $m_{n+1} - m_n$ is unbounded; for each n, let $\mathfrak{p}_n$ be the prime ideal of A generated by the elements T_i, for $m_n \leqslant i < m_{n+1}$. Let S be the intersection of the multiplicative subsets $S_n = A - \mathfrak{p}_n$. Then $S^{-1}A$ is noetherian, but $\dim(S^{-1}A) = +\infty$.

We shall prove below that noetherian local rings are finite dimensional.

b) There is a beautiful formula due to Grothendieck: let K be a field and let L and M be extensions of K. Then

$$\dim(L \otimes_K M) = \inf(\operatorname{tr.deg}_K(L), \operatorname{tr.deg}_L(M)).$$

This is proved in (Grothendieck, 1967, p. 349, remarque (4.2.1.4)).

c) If A is a finitely generated algebra over a field, proposition 1.11.8 asserts that $\dim(A[X]) = \dim(A) + 1$; in fact, this holds under the weaker assumption that A is noetherian, see (Serre, 1965, III, prop. 13). However, in the general case, it lies between $\dim(A) + 1$ and $2\dim(A) + 1$, and all possibilities appear!

1.12. Artinian rings

1.12.1. — Let k be a ring. One says that a k-module M is *artinian* if every strictly decreasing sequence of submodules of M is finite or, equivalently, if every non-empty family of submodules of M has a minimal element.

One says that a ring A is *artinian* if it is artinian as a module over itself; this means that every strictly decreasing sequence of ideals of A is finite. This also implies that every strictly increasing sequence of closed subsets of Spec(A) is finite.

1.12.2. — Let P be a submodule of M. Then M is artinian if and only if both P and M/P are artinian. In particular, finite direct sums of artinian modules are artinian.

1.12.3. — Let A be a ring. An A-module M is said to be *simple* if its only submodules are $\{0\}$ and M; this is equivalent to the existence of a maximal ideal $\mathfrak{m}$ of A such that $M \simeq A/\mathfrak{m}$.

The *length* of an A-module M is the dimension of the partially ordered set of its submodules. It is denoted by $\operatorname{length}_A(M)$, or even length(M) if the ring A is clear from the context.

Proposition (**1.12.4**). — *Let* M *be an* A*-module and let* N *be a submodule of* M*. If two of the modules* M, N *and* M/N *have finite length, then so does the third one, and one has the equality*

$$\operatorname{length}_A(M) = \operatorname{length}_A(N) + \operatorname{length}_A(M/N).$$

Proof. — Let $N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_a$ and $M_0/N \subsetneq M_1/N \subsetneq \cdots \subsetneq M_b/N$ be chains of submodules of N and M/N, then Then,

$$N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_a \subsetneq M_1 \subsetneq \cdots \subsetneq M_b$$

is a chain of length $a+b$ of submodules of M, hence the inequality $\operatorname{length}_A(M) \geqslant \operatorname{length}_A(N) + \operatorname{length}_A(M/N)$. In particular, if M has finite length, then so do N and M/N.

Conversely, let us assume that N and M/N have finite length; we want to prove that M has finite length and that $\operatorname{length}_A(M) = \operatorname{length}_A(N) + \operatorname{length}_A(M/N)$. Let thus $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_a$ be a chain of submodules of M. One observes that for every two submodules P′ and P″ of M such that $P' \subseteq P''$, $P' \cap N = P'' \cap N$ and $P' + N = P'' + N$, then $P' = P''$. It follows that for every integer $i \in \{0, \dots, a-1\}$, at least one of the two inclusions

$$M_i \cap N \subseteq M_{i+1} \cap N \quad \text{and} \quad M_i + N \subseteq M_{i+1} + N$$

is strict. This implies that $\mathrm{length}_A(N) + \mathrm{length}_A(M/N) \geqslant a$. It follows that $\mathrm{length}_A(N) + \mathrm{length}_A(M/N) \geqslant \mathrm{length}_A(M)$, whence the proposition. □

1.12.5. — An A-module has finite length if and only if it is artinian and noetherian. Moreover, every maximal chain of submodules of such an A-module M has length $\mathrm{length}_A(M)$.

Example **(1.12.6)**. — Let K be a field and let M be a K-vector space. Then the length of a M is nothing but its dimension. Moreover, the three following properties are equivalent: (i) dim(M) is finite; (ii) M is artinian; (iii) M is noetherian.

Lemma **(1.12.7)**. — *Let* A *be an artinian ring.*

a) *If* A *is an integral domain, then* A *is a field;*
b) *Every prime ideal of* A *is maximal;*
c) Spec(A) *is finite.*

Proof. — a) Let us assume that A is an integral domain. Let $x \in A - \{0\}$. The infinite decreasing sequence of ideals $A \supseteq (x) \supseteq (x^2) \supseteq \dots$ cannot be strictly decreasing, hence there exists an integer $n \geqslant 0$ such that $(x^n) = (x^{n+1})$. Let $a \in A$ be such that $x^n = ax^{n+1}$. Since $x \neq 0$ and A is an integral domain, we may simplify by x^n, hence $ax = 1$. This shows that x is invertible.

b) Let $\mathfrak{p}$ be a prime ideal of A. Then, $A/\mathfrak{p}$ is an artinian ring which is an integral domain. By part *a*), it is a field, hence $\mathfrak{p}$ is a maximal ideal.

c) Since every prime ideal of A is maximal, every point of Spec(A) is closed. If Spec(A) were infinite, there would exist an infinite sequence (x_n) of pairwise distinct points in Spec(A). The infinite sequence

$$\varnothing \subsetneq \{x_1\} \subsetneq \{x_1, x_2\} \subsetneq \dots$$

of closed subsets of Spec(A) is then strictly increasing, which contradicts the hypothesis that A is an artinian ring. □

Theorem **(1.12.8)** (Akizuki). — *Let* A *be a ring. The following properties are equivalent:*

(i) *The ring* A *is artinian;*
(ii) *The* A*-module* A *has finite length;*
(iii) *The ring* A *is noetherian and* $\dim(A) = 0$.

Proof. — Condition (ii) implies that every sequence of ideals of A which is either strictly increasing or strictly decreasing is finite, hence that A is artinian (condition (i)) and noetherian (the first half of condition (iii)).

Moreover, if A is artinian, then we have seen in lemma 1.12.7 that every prime ideal of A is maximal, hence $\dim(A) = 0$.

Let us assume that A is noetherian and that $\dim(A) = 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals of A, so that $V(\mathfrak{p}_1), \dots, V(\mathfrak{p}_n)$ are the irreducible components of $\operatorname{Spec}(A)$. Since $\dim(A) = 0$, $\mathfrak{p}_i$ is a maximal ideal and $V(\mathfrak{p}_i) = \{\mathfrak{p}_i\}$; in particular, $\operatorname{Spec}(A)$ is a finite and discrete topological space.

Let $\mathfrak{n}$ be the nilradical of A. One has $\mathfrak{n} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$, so that the A-module $A/\mathfrak{n}$ embeds into the finite product of the A-modules $A/\mathfrak{p}_i$, for $1 \leqslant i \leqslant n$. In particular, $\operatorname{length}_A(A/\mathfrak{n}) \leqslant n$. It follows from this that every $A/\mathfrak{n}$-module which is finitely generated has finite length.

For every integer $d \geqslant 0$, the ideal $\mathfrak{n}^d$ is finitely generated, because A is noetherian. This implies that $\mathfrak{n}^d/\mathfrak{n}^{d+1}$ is a finitely generated $A/\mathfrak{n}$-module, hence $\operatorname{length}(\mathfrak{n}^d/\mathfrak{n}^{d+1})$ is finite.

Every element of $\mathfrak{n}$ is nilpotent. Since A is noetherian, the ideal $\mathfrak{n}$ is finitely generated, hence there exists an integer $e \geqslant 0$ such that $\mathfrak{n}^e = 0$. Consequently,

$$\operatorname{length}_A(A) \leqslant \sum_{d=0}^{e-1} \operatorname{length}(\mathfrak{n}^d/\mathfrak{n}^{d+1})$$

is finite, which concludes the proof of implication (iii)⇒(ii).

It remains to show that an artinian ring has finite length. By lemma 1.12.7, we known that $\operatorname{Spec}(A)$ consists of finitely many maximal ideals, say $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Let J be their product; it is equal to the Jacobson radical of A. The decreasing infinite sequence of ideals $(A, J, J^2, \dots)$ cannot be strictly decreasing, so that there exists an integer $s \geqslant 0$ such that $J^s = J^{s+1}$. Let us prove that $J^s = 0$. Let thus $I = (0 : J^s)$ be the set of $a \in A$ such that $aJ^s = 0$; we will prove that $I = A$.

Assume otherwise. Since A is artinian and $A \neq I$, there exists an ideal I' of A such that $I \subsetneq I'$ and which is minimal for this property. Let now $a \in I' - I$. Observe that $aJ + I \subsetneq aA + I$; indeed, by corollary 1.3.3 to Nakayama's lemma, applied to the submodules aA and I of A, the relation $aJ + I = aA + I$ would imply that $a \in I$. Consequently, we have $I \subseteq aJ + I \subsetneq aA + I \subseteq I'$, hence $I = aJ + I$ by minimality of I'. We thus have shown that $aJ \subseteq I$. For every $b \in J$, we then have $ab \in I$, hence $abJ^s = 0$; this shows that $aJ^{s+1} = 0$. Since $J^s = J^{s+1}$, we have

$aJ^s = 0$, hence $a \in I$. This contradiction proves that $A = I$. Consequently, $J^s = 0$, as claimed.

Now consider the decreasing sequence of ideals

$$A \supseteq \mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_1 \ldots \mathfrak{p}_n = I \supseteq I\mathfrak{p}_1 \supseteq \cdots \supseteq I\mathfrak{p}_1 \ldots \mathfrak{p}_n = I^2 \supseteq I^2\mathfrak{p}_1 \supseteq \cdots \supseteq I^s = 0.$$

Each successive quotient is a noetherian A-module of the form $M/\mathfrak{m}M$, where $\mathfrak{m}$ is maximal ideal of A, hence a finite dimensional $A/\mathfrak{m}$-vector space; its length as an A-module is thus finite. Consequently, the length of A is finite, as was to be shown. □

1.13. Codimension

Lemma (**1.13.1**). — *Let* A *be a ring, let* $n \geqslant 1$ *be an integer and let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *be prime ideals of* A. *If* I *is an ideal of* A *such that* $I \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, *there exists an integer* i *such that* $I \subseteq \mathfrak{p}_i$.

Proof. — We prove the lemma by induction on n. The result is obvious if $n = 1$. Assume that I is contained in none of the ideals $\mathfrak{p}_i$. By induction, for every i, one has $I \notin \bigcup_{j \neq i} \mathfrak{p}_j$, hence there exists $x_i \in I$ such that $x_i \notin \mathfrak{p}_j$, if $j \neq i$. This implies that $x_i \in \mathfrak{p}_i$ for every i. Let $a = x_1 + x_2 \ldots x_n$. Since $x_1 \in \mathfrak{p}_1$ and $x_2, \ldots, x_n$ do not belong to $\mathfrak{p}_1$, one has $a \notin \mathfrak{p}_1$. Let $i \geqslant 2$; then $x_1 \notin \mathfrak{p}_i$ but $x_2 \ldots x_n \in \mathfrak{p}_i$, so that $a \notin \mathfrak{p}_i$. Consequently, a does not belong to the union of the ideals $\mathfrak{p}_i$, in contradiction with the fact that it belongs to I. □

Proposition (**1.13.2**). — *Let* $K \subseteq F$ *be a finite normal extension of fields. Let* A *be a subring of* K *which is integrally closed in* K *and let* B *be the integral closure of* A *in* F. *Let* G *be the group of automorphisms of* F *which restrict to identity on* K.

a) *For every* $\sigma \in G$, *one has* $\sigma(B) = B$*;*

b) *For every point* $x \in \operatorname{Spec}(A)$, *the group* G *acts transitively on the fiber* $({}^a\varphi)^{-1}(x)$ *in* Spec(B).

Proof. — a) Let $b \in B$. Then $\sigma(b)$ belongs to F and is integral over A. One thus has $\sigma(b) \in B$. This shows that $\sigma(B) \subseteq B$. Similarly, one has $\sigma^{-1}(B) \subseteq B$, hence $B \subseteq \sigma(B)$.

b) By the first theorem of Cohen-Seidenberg (theorem 1.11.4), the map ${}^a\varphi\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective, so that the fiber $({}^a\varphi)^{-1}(x)$ is non-empty. Let y, y' be two elements of this fiber; let $\mathfrak{q}, \mathfrak{q}'$ be the corresponding prime ideals of B. Let $b \in \mathfrak{q}'$. The product $a = \prod_{\sigma\in G} \sigma(b)$ is an element of F which is fixed by G. By Galois theory, it is radicial over K: there exists an integer $q \geqslant 1$ such that $a^q \in K$. (In fact, $q = 1$ if the extension F/K is separable, and otherwise q is a power of the caracteristic of K.) Since b is integral over A, each $\sigma(b)$ is integral over A, and a is integral over A, as well as a^q. Since A is integrally closed in K, one has $a^q \in A$. Moreover, $a^q \in \mathfrak{q}' \cap A = \mathfrak{p} = \mathfrak{q} \cap A$; in particular, $a^q \in \mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal, there exists $\sigma \in G$ such that $\sigma(b) \in \mathfrak{q}$. This shows that $b \in \sigma^{-1}(\mathfrak{q})$, hence $\mathfrak{q}' \subseteq \bigcup_{\sigma\in G} \sigma(\mathfrak{q})$.

By lemma 1.13.1, there exists $\sigma \in G$ such that $\mathfrak{q}' \subseteq \sigma(\mathfrak{q})$. Since $\sigma(\mathfrak{q}) \cap A = \sigma(\mathfrak{q} \cap A) = \sigma(\mathfrak{p}) = \mathfrak{q}' \cap A$, proposition 1.13.2 shows that $\mathfrak{q}' = \sigma(\mathfrak{q})$. This proves the proposition.

□

Theorem **(1.13.3)** (Second theorem of Cohen-Seidenberg)

Let B be an integral domain and let A be a subring of B. Assume that A is integrally closed in its field of fractions and that B is a finite A-module. Let $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n$ be a chain of prime ideals of A and let $\mathfrak{q}_n$ be a prime ideal of B such that $\mathfrak{q}_n \cap A = \mathfrak{p}_n$. There exists a chain of prime ideals $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_{n-1} \subseteq \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for every i.

Proof. — Let K be the field of fractions of A and let F be that of B. Let F′ be a finite extension of F which is normal over K, let B′ be the integral closure of A in F′. By the first Cohen-Seidenberg theorem (theorem 1.11.4), there exists a chain $\mathfrak{q}'_0 \subseteq \cdots \subseteq \mathfrak{q}'_n$ of prime ideals of B′ such that $\mathfrak{p}_i = \mathfrak{q}'_i \cap A$ for every i. Let $\widetilde{\mathfrak{q}}_n$ be a prime ideal of B′ such that $\widetilde{\mathfrak{q}}_n \cap B = \mathfrak{q}_n$.

By proposition 1.13.2, there exists an automorphism σ of F′ such that $\sigma|_K = \mathrm{id}$ and $\sigma(\mathfrak{q}'_n) = \widetilde{\mathfrak{q}}_n$. For every integer i such that $0 \leqslant i \leqslant n-1$, let $\mathfrak{q}_i = \sigma(\mathfrak{q}'_i) \cap B$. Then $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_n$ is a chain of prime ideals of B. For every integer i, one has

$$\mathfrak{q}_i \cap A = \sigma(\mathfrak{q}'_i) \cap B \cap A = \sigma(\mathfrak{q}'_i \cap A) = \sigma(\mathfrak{p}_i) = \mathfrak{p}_i,$$

hence the theorem. □

Corollary **(1.13.4)**. — *Let B be an integral domain, let A be a subring of B such that B is a finite A-module. Assume that A is integrally closed in its field of fractions. Then, for every prime ideal $\mathfrak{q}$ of B, one has*

$$\mathrm{ht}_B(\mathfrak{q}) = \mathrm{ht}_A(\mathfrak{q} \cap A).$$

Lemma **(1.13.5)**. — *Let* A *be a unique factorization domain and let* $\mathfrak{p}$ *be a prime ideal of* A. *If* $\operatorname{ht}(\mathfrak{p}) = 1$, *then there exists a prime element* $a \in A$ *such that* $\mathfrak{p} = (a)$.

Proof. — Let $a \in \mathfrak{p}$ be an arbitrary non-zero element. Since $\mathfrak{p}$ is a prime ideal, a is not a unit, hence it admits a decomposition $a = b_1 \dots b_n$ be a decomposition as a product of irreducible elements. Necessarily, $\mathfrak{p}$ contains one of these factors, so that we may assume that a is irreducible. Since A is a UFD, the ideal (a) is then a prime ideal. Since $\operatorname{ht}(\mathfrak{p}) = 1$, the inclusion $0 \subsetneq (a) \subseteq \mathfrak{p}$ implies that $\mathfrak{p} = (a)$. □

Theorem **(1.13.6)**. — *Let* K *be a field. Let* A *be a finitely generated* K*-algebra which is an integral domain. For every prime ideal* $\mathfrak{p}$ *of* A, *one has* $\dim(A) = \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p})$.

In other words, for every irreducible closed subset Z of $X = \operatorname{Spec}(A)$, one has the familiar relation $\dim(X) = \dim(Z) + \operatorname{codim}(Z)$. In spectra of finitely generated K-algebras, dimension and codimension behave as expected.

Proof. — We have already explained that $\dim(A) \geqslant \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p})$. On the other hand, by the Noether normalization lemma (theorem 1.6.1), there exists an integer $n \geqslant 0$ and an injective and integral morphism $f : K[X_1, \dots, X_n] \to A$. Let B be the image of f and let $\mathfrak{p} = \mathfrak{q} \cap B$. One thus has $\dim(A) = n$ and $\dim(A/\mathfrak{p}) = \dim(B/\mathfrak{q})$ (corollary 1.11.5), as well as $\operatorname{ht}_A(\mathfrak{q}) = \operatorname{ht}_B(\mathfrak{p})$ (corollary 1.13.4). It thus suffices to prove the result when $A = k[X_1, \dots, X_n]$. By induction on $\dim(A)$, it even suffices to prove the case when $\operatorname{ht}_A(\mathfrak{p}) = 1$.

In this case, lemma 1.13.5 asserts that there exists an irreducible polynomial $f \in A$ such that $\mathfrak{p} = (f)$. The transcendence degree of the field of fractions of $A/(f)$ is then at least $n - 1$: if the indeterminate X_n appears in f, then the images $x_1, \dots, x_{n-1}$ of $X_1, \dots, X_{n-1}$ are algebraically independent in $A/(f)$, since an algebraic dependence relation $P(x_1, \dots, x_{n-1}) = 0$ in $A/(f)$ means that $P \in (f)$, and this implies $P = 0$ if $\deg_{X_n}(f) \neq 0$. By theorem 1.11.6, one has $\dim(A/\mathfrak{p}) \geqslant n - 1$, hence the inequality $\operatorname{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) \geqslant n$, as was to be shown. □

Corollary **(1.13.7)**. — *Let* K *be a field and let* A *be a finitely generated* K*-algebra which is an integral domain. Every maximal chain of prime ideals of* A *has length* $\dim(A)$.

Proof. — Let $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ is a maximal chain of prime ideals of A. We argue by induction on n. One has $\mathfrak{p}_0 = (0)$, because A is an integral domain. If

$n = 0$, then A is a field, hence $\dim(A) = 0$. Let us assume that $n \geqslant 1$. The chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1$ of prime ideals is maximal among those ending at $\mathfrak{p}_1$. Since every maximal chain of prime ideals ending at $\mathfrak{p}_1$ begins at $(0) = \mathfrak{p}_0$, one has $\mathrm{ht}(\mathfrak{p}_1) = 1$. Moreover, the quotient ring $A/\mathfrak{p}_1$ is an integral domain and a finitely generated K-algebra. In this ring, the increasing sequence $\mathfrak{p}_1/\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n/\mathfrak{p}_1$ is a maximal chain of prime ideals. By induction, one has $\dim(A/\mathfrak{p}_1) = n - 1$. Consequently, $n = 1 + \dim(A/\mathfrak{p}_1) = 1 + \dim(A) - \mathrm{ht}(\mathfrak{p}_1) = \dim(A)$, as was to be shown. □

1.14. Krull's Hauptidealsatz and regular rings

Theorem **(1.14.1)** (Krull's Hauptidealsatz). — *Let* A *be a noetherian ring and let* f *be an element of* A. *The prime ideals of* A *which are minimal among those containing* f *have height at most* 1.

If f is not a zero-divisor, then f does not belong to any minimal prime ideal of A (see the proof of corollary 1.10.10, the hypothesis that A be reduced is not used for this assertion), so that the prime ideals of A which are minimal among those containing f have height exactly 1.

Proof. — Let $\mathfrak{p}$ be a prime ideal of A, minimal among those containing f; we need to prove that $\mathrm{ht}(\mathfrak{p}) \leqslant 1$, that is, that there does not exist a chain $\mathfrak{q}' \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$ of prime ideals of A. Let us argue by contradiction, considering such a chain. If we quotient by $\mathfrak{q}'$, we may moreover assume that $\mathfrak{q}' = \{0\}$, i.e., that A is an integral domain; we then have to prove that $\{0\}$ and $\mathfrak{p}$ are the only prime ideals of A which are contained in $\mathfrak{p}$. The ring of fractions $A_\mathfrak{p}$ is noetherian too, and its maximal ideal $\mathfrak{p}A_\mathfrak{p}$ is minimal among its prime ideals containing $f/1$. Replace the ring A by its fraction ring $A_\mathfrak{p}$ and f by its image in $A_\mathfrak{p}$, we may thus assume that A is a local, noetherian, integral domain, and that $\mathfrak{p}$ is its maximal ideal.

Let thus $\mathfrak{q}$ be a prime ideal of A, distinct from $\mathfrak{p}$, and let us show that $\mathfrak{q} = \{0\}$. Since $\mathfrak{p}$ is minimal among the prime ideals of A which contain f, one has $f \notin \mathfrak{q}$. For every integer $n \geqslant 0$, let $\mathfrak{q}_n = A \cap \mathfrak{q}^n A_\mathfrak{q}$; this ideal is called the nth symbolic power of $\mathfrak{q}$. It consists in the elements $a \in A$ for which there exists $b \notin \mathfrak{b}$ such that $ab \in \mathfrak{q}^n$. For every integer n such that $n \geqslant 0$, one has $\mathfrak{q}^{n+1} \subseteq \mathfrak{q}^n$, hence $\mathfrak{q}_{n+1} \subseteq \mathfrak{q}_n$.

By the correspondence between the prime ideals of the ring A/fA and the prime ideals of A which contain f, we see that $(\mathfrak{p} + fA)/fA$ is the only prime ideal of A/fA, so that $\dim(A/fA) = 0$. Since this ring A/fA is noetherian, it is then artinian, hence the sequence $(\mathfrak{q}_n + f\mathrm{A}/f\mathrm{A})_n$ of ideals of $\mathrm{A}/f\mathrm{A}$ is eventually constant. Let then n be an integer such that

$$\mathfrak{q}_n + f\mathrm{A} = \mathfrak{q}_{n+1} + f\mathrm{A}.$$

Let $x \in \mathfrak{q}_n$. By this relation, there exists $a \in \mathrm{A}$ such that $x + af \in \mathfrak{q}_{n+1}$; it follows in particular that $af \in \mathfrak{q}_n$, hence $a \in \mathfrak{q}_n$ since $f \notin \mathfrak{q}$. Consequently, $x \in \mathfrak{q}_{n+1} + f\mathfrak{q}_n$, whence the equality

$$\mathfrak{q}_n = \mathfrak{q}_{n+1} + f\mathfrak{q}_n.$$

Since $f \in \mathfrak{p}$, this implies

$$\mathfrak{q}_n = \mathfrak{q}_{n+1} + \mathfrak{p}\mathfrak{q}_n.$$

It now follows from Nakayama's lemma (corollary 1.3.3), that $\mathfrak{q}_n = \mathfrak{q}_{n+1}$. In particular, one has

$$\mathfrak{q}^n\mathrm{A}_\mathfrak{q} = \mathfrak{q}_n\mathrm{A}_\mathfrak{q} = \mathfrak{q}_{n+1}\mathrm{A}_\mathfrak{q} = \mathfrak{q}^{n+1}\mathrm{A}_\mathfrak{q} = \mathfrak{q} \cdot \mathfrak{q}^n\mathrm{A}_\mathfrak{q}.$$

By Nakayama's lemma again, one has $\mathfrak{q}^n\mathrm{A}_\mathfrak{q} = 0$. Since $\mathfrak{q}$ is a prime ideal, this implies $\mathfrak{q}\mathrm{A}_\mathfrak{q} = 0$, hence $\mathfrak{q} = 0$, as was to be shown. ☐

Corollary **(1.14.2)**. — *Let* A *be a noetherian ring, let n be an integer and let $f_1, \dots, f_n$ be elements of* A. *Let $\mathfrak{p}$ be a prime ideal of* A *which is minimal among those containing $(f_1, \dots, f_n)$; then* $\mathrm{ht}(\mathfrak{p}) \leqslant n$. *In particular, the height of $\mathfrak{p}$ is finite.*

Geometrically: for every irreducible component Z of $\mathrm{V}(f_1, \dots, f_n)$, one has $\mathrm{codim}(\mathrm{Z}) \leqslant n$.

Proof. — In the noetherian local ring $\mathrm{A}_\mathfrak{p}$, the maximal ideal $\mathfrak{p}\mathrm{A}_\mathfrak{p}$ is minimal among those containing the images of $f_1, \dots, f_n$. Moreover, the height of $\mathfrak{p}\mathrm{A}_\mathfrak{p}$ in $\mathrm{A}_\mathfrak{p}$ is equal to the height of $\mathfrak{p}$ in A. We may thus assume that A is local and that $\mathfrak{p}$ is its maximal ideal.

Let $\mathfrak{p}'$ be a prime ideal of A such that $\mathfrak{p}' \subsetneq \mathfrak{p}$; let us prove that $\mathrm{ht}(\mathfrak{p}') \leqslant n - 1$. Since A is noetherian, there exists a prime ideal $\mathfrak{p}'_1$ such that $\mathfrak{p}' \subseteq \mathfrak{p}'_1 \subsetneq \mathfrak{p}$ and which is maximal among these ideals. Since one has $\mathrm{ht}(\mathfrak{p}') \leqslant \mathrm{ht}(\mathfrak{p}'_1)$, it suffices to prove that $\mathrm{ht}(\mathfrak{p}'_1)$. We may thus assume that $\mathfrak{p}' = \mathfrak{p}'_1$.

Since $\mathfrak{p}' \neq \mathfrak{p}$, there exists $i \in \{1, \dots, n\}$ such that $f_i \notin \mathfrak{p}'$. By simplicity of notation, we assume that $i = 1$. Then $\mathfrak{p}' \subsetneq \mathfrak{p}' + (f_1) \subseteq \mathfrak{p}$, so that $\mathfrak{p}$ is the unique prime ideal of A which contains $\mathfrak{p}' + (f_1)$. Consequently, every element of $\mathfrak{p}$ is nilpotent modulo $\mathfrak{p}' + (f_1)$; let $m \in \mathbf{N}$, let $g_2, \dots, g_n \in \mathfrak{p}'$ and $a_2, \dots, a_n \in \mathrm{A}$ be such that $f_i^m = g_i + a_i f_1$ for every $i \in \{2, \dots, n\}$.

One thus has $\mathfrak{p} \supseteq (f_1, g_2, \dots, g_n)$; In fact, a prime ideal of A containing $(f_1, g_2, \dots, g_n)$ contains $f_2^m, \dots, f_n^m$, hence contains $f_2, \dots, f_n$, so that $\mathfrak{p}$ is the unique prime ideal of A containing $(f_1, g_2, \dots, g_n)$. Let $B = A/(g_2, \dots, g_n)$ and let $\mathfrak{q}$ be the image of $\mathfrak{p}$ in B. The prime ideal $\mathfrak{q}$ is the unique prime ideal which contains the image of f_1, hence $\mathrm{ht}_B(\mathfrak{q}) \leqslant 1$, by Krull's Hauptidealsatz (theorem 1.14.1). The inclusions $(g_2, \dots, g_n) \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$ then imply that the prime ideal $\mathfrak{p}'$ is minimal among those containing $(g_2, \dots, g_n)$.

By induction, one thus has $\mathrm{ht}(\mathfrak{p}') \leqslant n-1$, as claimed. Since $\mathfrak{p}'$ is maximal among the set of prime ideals of A distinct from $\mathfrak{p}$, one then has $\mathrm{ht}(\mathfrak{p}) \leqslant n$, as was to be shown.

Let now $\mathfrak{p}$ be any prime ideal of A. Since A is noetherian, there exists a finite family $(f_1, \dots, f_n)$ of elements of A such that $\mathfrak{m} = (f_1, \dots, f_n)$. Then $\mathfrak{p}$ is the smallest prime ideal of A containing $f_1, \dots, f_n$, hence $\mathrm{ht}(\mathfrak{p}) \leqslant n$. □

Corollary **(1.14.3)**. — *Let* A *be a noetherian ring and let* $\mathfrak{p}$ *be a prime ideal of* A. *The height of* $\mathfrak{p}$ *is the smallest integer n such that there exist elements* $f_1, \dots, f_n \in A$ *such that* $V(\mathfrak{p})$ *is an irreducible component of* $V(f_1, \dots, f_n)$.

Proof. — Since A is noetherian, there exists an integer n and elements $f_1, \dots, f_n$ of A such that $V(\mathfrak{p})$ is an irreducible component of $(f_1, \dots, f_n)$; it suffices, for example, that $\mathfrak{p} = (f_1, \dots, f_n)$. By the preceding corollary, we then have $\mathrm{ht}(\mathfrak{p}) \leqslant n$.

Conversely, let $n = \mathrm{ht}(\mathfrak{p})$, and let $\mathfrak{p}'$ be a prime ideal of A such that $\mathfrak{p}' \subsetneq \mathfrak{p}$ and $\mathrm{ht}(\mathfrak{p}') = n-1$. By induction, there exist elements $g_2, \dots, g_n \in A$ such that $V(\mathfrak{p}')$ is an irreducible component of $V(g_2, \dots, g_n)$. Let $(\mathfrak{p}'_i)$ be the family of minimal prime ideals of A containing $(g_2, \dots, g_n)$. Since A is noetherian, it is finite. Moreover, one has $\mathfrak{p} \not\subseteq \mathfrak{p}'_i$ for every i, since the inclusion $\mathfrak{p} \subseteq \mathfrak{p}'_i$ would imply that $\mathrm{ht}(\mathfrak{p}) \leqslant \mathrm{ht}(\mathfrak{p}'_i) \leqslant n-1$. By lemma 1.13.1, one has $\mathfrak{p} \not\subseteq \bigcup \mathfrak{p}'_i$, hence there exists an element $g_1 \in \mathfrak{p}$ such that $g_1 \notin \mathfrak{p}'_i$ for every i. Then $(g_1, \dots, g_n) \subseteq \mathfrak{p}$; moreover, any prime ideal $\mathfrak{q}$ which satisfies this relation and which is contained in $\mathfrak{p}$ contains $\mathfrak{p}'$, but cannot be equal to $\mathfrak{p}'$, hence is equal to $\mathfrak{p}$. This shows that $V(\mathfrak{p})$ is an irreducible component of $V(g_1, \dots, g_n)$. □

Corollary **(1.14.4)**. — *The dimension of a local noetherian ring is finite. More precisely, if* A *is a local noetherian ring, with maximal ideal* $\mathfrak{m}$, *then*

$$\dim(A) \leqslant \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2).$$

Proof. — One has $\dim(A) = \mathrm{ht}(\mathfrak{m})$; consequently, $\dim(A)$ is finite if A is noetherian. Let $f_1, \ldots, f_n$ be elements of $\mathfrak{m}$ which generate $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's lemma, one has $\mathfrak{m} = (f_1, \ldots, f_n)$. It then follows from corollary 1.14.2 that $\dim(A) = \mathrm{ht}(\mathfrak{m}) \leqslant n$. □

Definition **(1.14.5)**. — *Let* A *be local noetherian ring, let* $\mathfrak{m}$ *be its maximal ideal and let k be its residue field. One says that* A *is* regular *if* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A)$.

Let A be a noetherian ring. One says that A is regular if the local ring $A_{\mathfrak{m}}$ is regular, for every *maximal* ideal $\mathfrak{m}$ of A. If A is regular, an important theorem of Serre implies that $A_{\mathfrak{p}}$ is a regular local ring, for every prime ideal $\mathfrak{p}$ of A (see Serre (1965), chap. IV, prop. 23).

Proposition **(1.14.6)**. — *Let* A *be a local noetherian ring. If* A *is regular, then* A *is an integral domain.*

Proof. — We establish the result by induction on the dimension of A. Let $\mathfrak{m}$ be its residue field and let $k = A/\mathfrak{m}$ be its residue field.

If $\dim(A) = 0$, then $\mathfrak{m} = 0$, hence A is a field.

Let us now assume that $\dim(A) > 0$ and let $(\mathfrak{p}_1)$ be the family of minimal prime ideals of A. Since $\dim(A) > 0$, one has $\mathfrak{m} \neq \mathfrak{p}_i$ for every i; moreover, $\mathfrak{m} \neq \mathfrak{m}^2$. By the prime avoiding lemma, there exists an element $a \in \mathfrak{m}$ such that $a \notin \mathfrak{m}^2 \cup \bigcup \mathfrak{p}_i$. Let $B = A/(a)$; this is a noetherian local ring, because $a \in \mathfrak{m}$. Let $\mathfrak{n} = \mathfrak{m}B$ be its maximal ideal; the canonical map from $A/\mathfrak{m}$ to $B/\mathfrak{n}$ is an isomorphism.

Prime ideals of B correspond to prime ideals of A which contain (a); thus $\dim(A) - \dim(B)$ is the maximal height of a minimal prime ideal of A containing (a). Since a does not belong to any minimal prime ideal, the latter have height 1; one thus has $\dim(B) = \dim(A) - 1$. On the other hand, since $a \notin \mathfrak{m}^2$, the vector space $\mathfrak{n}/\mathfrak{n}^2$ is a strict quotient of $\mathfrak{m}/\mathfrak{m}^2$. Consequently,

$$\dim(A) - 1 = \dim(B) \leqslant \dim_k(\mathfrak{n}/\mathfrak{n}^2) < \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A).$$

It follows that $\dim(B) = \dim_k(\mathfrak{n}/\mathfrak{n}^2)$, hence B is a regular local ring. By induction, B is an integral domain, hence (a) is a prime ideal of A.

Let $\mathfrak{p}_i$ be a minimal prime ideal of A which is contained in (a). Since $a \notin \mathfrak{p}_i$, one has $\mathfrak{p}_i \subsetneq (a)$. Let then $x \in \mathfrak{p}_i$; there exists $y \in A$ such that $x = ay$, hence $y \in \mathfrak{p}_i$ since $\mathfrak{p}_i$ is a prime ideal which does not contain a. This shows that $\mathfrak{p}_i = a\mathfrak{p}_i \subseteq \mathfrak{m}\mathfrak{p}_i$, hence $\mathfrak{p}_i = \mathfrak{m}\mathfrak{p}_i$. By Nakayama's lemma, one then has $\mathfrak{p}_i = (0)$. In particular, (0) is a prime ideal, hence A is an integral domain. □

1.15. Associated ideals

Definition **(1.15.1)**. — *Let* A *be a ring and let* M *be an* A*-module. One says that a prime ideal* $\mathfrak{p}$ *of* A *is* associated *with* M *if there exists an element* $m \in \mathrm{M}$ *such that* $\mathfrak{p}$ *is minimal among the prime ideals of* A *that contain* $\mathrm{Ann}_{\mathrm{A}}(m)$.

The set of associated ideals of M is denoted by $\mathrm{Ass}_{\mathrm{A}}(\mathrm{M})$.

Example **(1.15.2)**. — Let A be a ring, let M be an A-module and let $m \in \mathrm{M}$.

If $m = 0$, then no prime ideal of A contains $\mathrm{Ann}_{\mathrm{A}}(m) = \mathrm{A}$; in particular, $\mathrm{Ass}_{\mathrm{A}}(0) = \varnothing$.

On the other hand, if $m \neq 0$, then $\mathrm{Ann}_{\mathrm{A}}(m) \neq \mathrm{A}$ and it follows from lemma 1.10.5 and example exem.minimal-prime that there exists a prime ideal of A which is minimal among those containing $\mathrm{Ann}_{\mathrm{A}}(m)$; such a prime ideal is associated with M.

This proves that $\mathrm{Ass}_{\mathrm{A}}(\mathrm{M}) = \varnothing$ if and only if $\mathrm{M} = 0$.

Example **(1.15.3)**. — Let $\mathfrak{p}$ be a prime ideal of A. Let us show that $\mathrm{Ass}_{\mathrm{A}}(\mathrm{A}/\mathfrak{p}) = \{\mathfrak{p}\}$.

Let $m \in \mathrm{A}/\mathfrak{p}$ be the class of an element $a \in \mathrm{A}$.

If $m = 0$, then $\mathrm{Ann}_{\mathrm{A}}(m) = \mathrm{A}$ and no prime ideal of A contains it.

Let us assume that $m \neq 0$ and let us prove that $\mathrm{Ann}_{\mathrm{A}}(m) = \mathfrak{p}$. The inclusion $\mathfrak{p} \subseteq \mathrm{Ann}_{\mathrm{A}}(m)$ is obvious. Conversely, let $b \in \mathrm{Ann}_{\mathrm{A}}(m)$; then bm is the class of ba, so that $ba \in \mathfrak{p}$. Since $m \neq 0$, we have $a \notin \mathfrak{p}$, hence $b \in \mathfrak{p}$ by definition of a prime ideal. As a consequence, $\mathfrak{p}$ is the only minimal prime ideal among those containing $\mathrm{Ann}_{\mathrm{A}}(m)$, hence $\{\mathfrak{p}\} = \mathrm{Ass}_{\mathrm{A}}(\mathrm{A}/\mathfrak{p})$.

Lemma **(1.15.4)**. — *Let* A *be a ring and let* M *be an* A*-module. Let* S *be a multiplicative subset of* A *and let* $j\colon \mathrm{Spec}(\mathrm{S}^{-1}\mathrm{A}) \to \mathrm{Spec}(\mathrm{A})$ *be the canonical injection. One has* $j^{-1}(\mathrm{Ass}_{\mathrm{A}}(\mathrm{M})) = \mathrm{Ass}_{\mathrm{S}^{-1}\mathrm{A}}(\mathrm{S}^{-1}\mathrm{M})$.

Proof. — Let $\mathfrak{p}$ be a prime ideal of A which is associated with M and is disjoint from S. We need to prove that $\mathfrak{p} \in \mathrm{Ass}_{\mathrm{A}}(\mathrm{M})$ if and only if $\mathrm{S}^{-1}\mathfrak{p} \in \mathrm{Ass}_{\mathrm{S}^{-1}\mathrm{A}}(\mathrm{M})$.

Let $m \in \mathrm{M}$ be such that $\mathfrak{p}$ is minimal among the prime ideals containing $\mathrm{Ann}_{\mathrm{A}}(m)$. Let us prove that $\mathrm{S}^{-1}\mathfrak{p}$ is a prime ideal of $\mathrm{S}^{-1}\mathrm{A}$ which is minimal among those containing $\mathrm{Ann}_{S^{-1}A}(m/1)$. Let thus $\mathfrak{q}'$ be a prime ideal of $S^{-1}A$ such that $\mathrm{Ann}_{S^{-1}A}(m/1) \subseteq \mathfrak{q}' \subseteq S^{-1}\mathfrak{p}$. There exists a prime ideal $\mathfrak{q}$ of A such that $\mathfrak{q}' = S^{-1}\mathfrak{q}$, and $\mathfrak{q} \subseteq \mathfrak{p}$. Let $a \in \mathrm{Ann}_A(m)$; since $a/1 \in \mathrm{Ann}_{S^{-1}A}(m/1)$, we have $a/1 \in \mathfrak{q}'$, so that there exists $s \in S$ such that $as \in \mathfrak{q}$. Since $s \notin \mathfrak{q}$, this implies $a \in \mathfrak{q}$. We thus have $\mathrm{Ann}_A(m/1) \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, and by minimality, this implies $\mathfrak{q} = \mathfrak{p}$. In particular, $S^{-1}\mathfrak{p} \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$.

Conversely, let $m \in M$ and $s \in S$ be such that $S^{-1}\mathfrak{p}$ is minimal among the prime ideals containing $\mathrm{Ann}_{S^{-1}A}(m/s)$. Since $\mathrm{Ann}(m/s) = \mathrm{Ann}(m/1)$, we may assume that $s = 1$. Let $\mathfrak{q}$ be a prime ideal of A which is contained in $\mathfrak{p}$, and is minimal among those containing $\mathrm{Ann}_A(m)$. Then $\mathfrak{q}$ is disjoint from S and, by what precedes, we have $\mathfrak{q} = \mathfrak{p}$. In other words, $\mathfrak{p} \in \mathrm{Ass}_A(M)$. □

Corollary **(1.15.5)**. — *Let* A *be a ring and let* M *be an* A*-module. A prime ideal of* A *belongs to* $\mathrm{Supp}_A(M)$ *if and only if it contains some element of* $\mathrm{Ass}_A(M)$.

Proof. — Let $\mathfrak{p} \in \mathrm{Spec}(A)$. By definition, $\mathfrak{p} \in \mathrm{Supp}_A(M)$ is equivalent to $M_\mathfrak{p} \neq 0$, hence to $\mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) \neq \varnothing$. Taking $S = A - \mathfrak{p}$ in the preceding lemma, we see that it is equivalent to $\mathrm{Ass}_A(M) \cap \mathrm{Spec}(A_\mathfrak{p}) \neq \varnothing$. The corollary then follows from the fact that $\mathrm{Spec}(A_\mathfrak{p})$ is the set of prime ideals which are contained in $\mathfrak{p}$. □

Corollary **(1.15.6)**. — *One has the inclusion* $\mathrm{Ass}_A(M) \subseteq \mathrm{Supp}_A(M)$*, and both sets have the same minimal elements.*

Proof. — This is essentially a reformulation of the preceding corollary.

Let $\mathfrak{p} \in \mathrm{Ass}_A(M)$. Then the inclusion $\mathfrak{p} \supseteq \mathfrak{p}$ shows that $\mathfrak{p} \in \mathrm{Supp}_A(M)$. Consequently, $\mathrm{Ass}_A(M) \subseteq \mathrm{Supp}_A(M)$.

In particular, a minimal element of $\mathrm{Ass}_A(M)$ is also a minimal element of $\mathrm{Supp}_A(M)$. Conversely, let $\mathfrak{q}$ be a minimal element of $\mathrm{Supp}_A(M)$. By the preceding corollary, there exists a prime ideal $\mathfrak{p} \in \mathrm{Ass}_A(M)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Then $\mathfrak{p} \in \mathrm{Supp}_A(M)$, hence $\mathfrak{p} = \mathfrak{q}$ by minimality. In particular, $\mathfrak{q} \in \mathrm{Ass}_A(M)$. □

Proposition **(1.15.7)**. — *Let* A *be a ring and let* M *be an* A*-module; let* $a \in A$.

a) *The homothety* $(a)_M$ *is injective if and only if* $\mathrm{Ass}_A(M) \cap V(a) = \varnothing$.
b) *One has* $M_a = 0$ *if and only if* $\mathrm{Ass}_A(M) \subseteq V(a)$.

Proof. — a) Let us assume that the homothety $(a)_M$ is not injective, and let $m \in M$ be such that $am = 0$ but $m \neq 0$. Let $\mathfrak{p}$ be an prime ideal which is minimal among those containing $\mathrm{Ann}_A(m)$. In particular, $a \in \mathfrak{p}$.

Conversely, let $\mathfrak{p} \in \mathrm{Ass}_A(M) \cap V(a)$ and let $m \in M$ be such that $\mathfrak{p}$ is minimal among the prime ideals containing $\mathrm{Ann}_A(m)$. Let S be the set of elements of the form $a^n b$, with $n \in \mathbf{N}$ and $b \in A - \mathfrak{p}$; it is a multiplicative subset of A that contains $A - \mathfrak{p}$. Assume $\mathrm{Ann}_A(m) \cap S = \varnothing$. Then the element $m/1$ of $S^{-1}M$ is not equal to zero, so that there exists a prime ideal $\mathfrak{q}'$ of $S^{-1}A$ containing $\mathrm{Ann}_{S^{-1}A}(m/1)$. This ideal corresponds to a prime ideal $\mathfrak{q}$ of A which is disjoint from S, and one has $\mathrm{Ann}_A(m) \subseteq \mathfrak{q}$. Since S contains $A - \mathfrak{p}$, one has $\mathfrak{q} \subseteq \mathfrak{p}$, and the minimality of $\mathfrak{p}$ implies that $\mathfrak{q} = \mathfrak{p}$. On the other hand, $a \in \mathfrak{p}$, hence $a \in \mathfrak{q}$, which contradicts the fact that $\mathfrak{q}$ is disoint from S, because $a \in S$. Consequently, $\mathrm{Ann}_A(m) \cap S$ is nonempty; let $n \in \mathbf{N}$ and $b \in A - \mathfrak{p}$ be such that $a^n bm = 0$, where n is chosen to be minimal.

If $n = 0$, then $bm = 0$, hence $b \in \mathrm{Ann}_A(m)$; since $b \notin \mathfrak{p}$, this contradicts the hypothesis that $\mathrm{Ann}_A(m) \subseteq \mathfrak{p}$. Consequently, $n \geqslant 1$, $a^{n-1}bm \neq 0$ by minimality, and the relation $a \cdot a^{n-1}bm = 0$ shows that the homothety $(a)_M$ is not injective.

b) We know that the relations $M_a = 0$ and $\mathrm{Ass}_{A_a}(M_a) = \varnothing$ are equivalent. Since $\mathrm{Ass}_{A_a}(M_a) = \mathrm{Ass}_A(M) \cap D(a)$, this implies that $M_a = 0$ is equivalent to the inclusion $\mathrm{Ass}_A(M) \subseteq V(a)$. □

Example **(1.15.8)**. — *Assume that* A *is a local ring with maximal ideal* $\mathfrak{p}$. *Then* $\mathfrak{p} \in \mathrm{Ass}_A(M)$ *if and only if there exists* $m \in M$ *such that* $\mathfrak{p} = \sqrt{\mathrm{Ann}_A(m)}$.

Let $m \in M$. We know that $\sqrt{\mathrm{Ann}_A(m)}$ is the intersection of all prime ideals of A which contain $\mathrm{Ann}_A(m)$. If this intersection is equal to the unique maximal ideal $\mathfrak{p}$ of A, this means that $\mathfrak{p}$ is the only prime ideal containing $\mathrm{Ann}_A(m)$. In particular, it is a minimal such ideal and $\mathfrak{p} \in \mathrm{Ass}_A(M)$.

Conversely, if $\mathfrak{p}$ is minimal among the prime ideals containing $\mathrm{Ann}_A(m)$, then it is the only such ideal, and $\mathfrak{p} = \sqrt{\mathrm{Ann}_A(m)}$.

Proposition **(1.15.9)**. — *Let* A *be a ring, let* M *be an* A*-module and let* N *be a submodule of* M. *One has the inclusions*

$$\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N).$$

Proof. — The inclusion $\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M)$ is elementary.

Let $m \in M$ and let $\mathfrak{p}$ be a prime ideal of A, minimal among those containing $\mathrm{Ann}_A(m)$. Let $m' \in M/N$ be the class of m; one has $\mathrm{Ann}_A(m) \subseteq \mathrm{Ann}_A(m')$. If $\mathrm{Ann}_A(m') \subseteq \mathfrak{p}$, then $\mathfrak{p}$ is also a minimal prime ideal among those containing $\mathrm{Ann}_A(m)$, hence $\mathfrak{p} \in \mathrm{Ass}_A(M/N)$.

Otherwise, one has $\mathrm{Ann}_A(m') \not\subseteq \mathfrak{p}$, and there exists $b \in A$ such that $bm' = 0$, hence $bm \in N$, and $b \notin \mathfrak{p}$.

Let us observe that $\mathrm{Ann}_A(m) \subseteq \mathrm{Ann}_A(bm) \subseteq \mathfrak{p}$. The first inclusion is elementary. Let then $a \in \mathrm{Ann}_A(bm)$; then $abm = 0$, hence $ab \in \mathrm{Ann}_A(m)$, so that $ab \in \mathfrak{p}$; since $b \notin \mathfrak{p}$, we thus have $a \in \mathfrak{p}$. Consequently, $\mathfrak{p}$ is a minimal prime ideal among those containing $\mathrm{Ann}_A(bm)$, so that $\mathfrak{p} \in \mathrm{Ass}_A(N)$. □

Theorem **(1.15.10)**. — *Let* A *be a noetherian ring and let* M *be an* A*-module. A prime ideal* $\mathfrak{p} \in \mathrm{Spec}(A)$ *belongs to* $\mathrm{Ass}_A(M)$ *if and only if there exists* $m \in M$ *such that* $\mathfrak{p} = \mathrm{Ann}_A(m)$.

Proof. — The sufficiency of this condition is obvious, so let $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and let us prove that there exists $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}_A(m)$.

We first treat the case where A is a local ring and $\mathfrak{p}$ is its maximal ideal. By example **??**, there exists $m \in M$ such that $\mathfrak{p} = \sqrt{\mathrm{Ann}_A(m)}$; in particular, $\mathrm{Ann}_A(m) \subseteq \mathfrak{p}$. Since A is noetherian, the ideal $\mathfrak{p}$ is finitely generated and there exists an integer n such that $\mathfrak{p}^n \subseteq \mathrm{Ann}_A(m)$. Let us consider a minimal such integer n. Since $\mathrm{Ann}_A(m) \subseteq \mathfrak{p}$, one has $n \geqslant 1$ and $\mathfrak{p}^{n-1} \not\subseteq \mathrm{Ann}_A(m)$. Let thus $b \in \mathfrak{p}^{n-1}$ be such that $bm \neq 0$. Then for any $a \in \mathfrak{p}$, one has $ab \in \mathfrak{p}^n \subseteq \mathrm{Ann}_A(m)$, so that $\mathfrak{p} \subseteq \mathrm{Ann}_A(m)$, and the converse inclusion is obvious. We thus have $\mathrm{Ann}_A(m) = \mathfrak{p}$, and this concludes the proof in this case.

Let us now consider the general case. Let $(a_1, \dots, a_r)$ be a finite family of elements of A such that $\mathfrak{p} = \langle a_1, \dots, a_r \rangle$. By the local case, we choose an element $m \in M$ such that $\mathrm{Ann}_{A_\mathfrak{p}}(m/1) = \mathfrak{p}A_\mathfrak{p}$. One has $\mathrm{Ann}_A(m) \subseteq \mathfrak{p}$.

For each $i \in \{1, \dots, r\}$, one has $a_i \in \mathfrak{p}$, hence $a_i m/1 = 0$ in $M_\mathfrak{p}$; consequently, there exists $b_i \in A - \mathfrak{p}$ such that $a_i b_i m = 0$. Set $b = b_1 \dots b_r$ and $m' = bm$; by construction, one has $b \notin \mathfrak{p}$. For every $i \in \{1, \dots, r\}$, one has $a_i m' = 0$, so that $\mathfrak{p} \subseteq \mathrm{Ann}_A(m')$. On the other hand, let $a \in \mathrm{Ann}_A(m')$; one then has $abm = 0$, hence $abm/1 = 0$ in $M_\mathfrak{p}$. By construction, this implies $ab \in \mathfrak{p}$, hence $a \in \mathfrak{p}$ because $b \notin \mathfrak{p}$. □

Corollary **(1.15.11)**. — *Let* A *be a noetherian ring and let* M *be a finitely generated* A*-module. There exists a finite sequence* $(M_0, \dots, M_n)$ *of submodules of* M *such that* $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$ *and prime ideals* $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ *of* A *such that* $M_r/M_{r-1} \simeq A/\mathfrak{p}_r$ *for every* $r \in \{1, \dots, n\}$.

Proof. — If $M = 0$, then we set $n = 0$ and $M_0 = 0$. Otherwise, we may consider an element $\mathfrak{p}_1 \in \mathrm{Ass}_A(M)$. By the preceding theorem, there exists $m_1 \in M$ such that $\mathfrak{p}_1 = \mathrm{Ann}_A(m_1)$. Set $M_1 = \langle m_1 \rangle$; observe that $M_1 \simeq A/\mathfrak{p}_1$. Applying the same argument to M/M_1, we construct a strictly increasing sequence $(M_0 = 0, M_1, \dots)$ of submodules of M, and a sequence of prime ideals $(\mathfrak{p}_1, \dots)$ of M such that $M_r/M_{r-1} \simeq A/\mathfrak{p}_r$. Since A is noetherian and M is finitely generated, the process has to stop, and this proves the corollary. □

Corollary **(1.15.12)**. — *Let* A *be a noetherian ring and let* M *be a finitely generated* A-*module. Then* $\mathrm{Ass}_A(M)$ *is a finite set.*

Proof. — We consider a finite sequence $(M_0, \dots, M_n)$ of submodules of M, and a finite sequence $(\mathfrak{p}_1, \dots, \mathfrak{p}_n)$ of prime ideals of A, as given by the preceding corollary. One has

$$\mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(M_1) \cup \mathrm{Ass}_A(M/M_1) \subseteq \dots \subseteq \bigcup_{r=1}^{n} \mathrm{Ass}_A(M_r/M_{r-1}).$$

Moreover, for every prime ideal $\mathfrak{p}$ of A, one has $\mathrm{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$. In other words,

$$\mathrm{Ass}_A(M) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

□

CHAPTER 2

CATEGORIES AND HOMOLOGICAL ALGEBRA

2.1. The language of categories

2.1.1. — A *category* $\mathbf{C}$ consists in the following data:

– A collection $\mathrm{ob}(\mathbf{C})$ of *objects*;
– For every two objects M, N, a set $\mathbf{C}(\mathrm{M},\mathrm{N})$ called *morphisms* from M to N;
– For every three objects M, N, P, a *composition map* $\mathbf{C}(\mathrm{M},\mathrm{N}) \times \mathbf{C}(\mathrm{N},\mathrm{P})$, $(f,g) \mapsto g \circ f$,

so that the following axioms are satisfied:

(i) For every object M, there is a distinguished morphism $\mathrm{id}_{\mathrm{M}} \in \mathbf{C}(\mathrm{M},\mathrm{M})$, called the identity;
(ii) One has $\mathrm{id}_{\mathrm{N}} \circ f = f$ for every $f \in \mathbf{C}(\mathrm{M},\mathrm{N})$;
(iii) One has $g \circ \mathrm{id}_{\mathrm{N}} = g$ for every $g \in \mathbf{C}(\mathrm{N},\mathrm{P})$;
(iv) For every four objects M, N, P, Q, and every three morphisms $f \in \mathbf{C}(\mathrm{M},\mathrm{N})$, $g \in \mathbf{C}(\mathrm{N},\mathrm{P})$, $h \in \mathbf{C}(\mathrm{P},\mathrm{Q})$, the two morphisms $h \circ (g \circ f)$ and $(h \circ g) \circ f$ in $\mathbf{C}(\mathrm{M},\mathrm{Q})$ are equal (associativity of composition).

A common notation for $\mathbf{C}(\mathrm{M},\mathrm{N})$ is also $\mathrm{Hom}_{\mathbf{C}}(\mathrm{M},\mathrm{N})$. Finally, instead of $f \in \mathbf{C}(\mathrm{M},\mathrm{N})$, one often writes $f\colon \mathrm{M} \to \mathrm{N}$.

2.1.2. — Let $f\colon \mathrm{M} \to \mathrm{N}$ be a morphism in a category $\mathbf{C}$. One says that f is left-invertible, resp. right-invertible, resp. invertible, if there exists a morphism $g\colon \mathrm{N} \to \mathrm{M}$ such that $g \circ f = \mathrm{id}_{\mathrm{M}}$, resp. $f \circ g = \mathrm{id}_{\mathrm{N}}$, resp. $g \circ f = \mathrm{id}_{\mathrm{M}}$ and $f \circ g = \mathrm{id}_{\mathrm{N}}$.

One proves in the usual way that if f is both left- and right-invertible, then it is invertible. An invertible morphism is also called an isomorphism.

Example (**2.1.3**). — The category $\mathbf{Set}$ of sets has for objects the sets, and for morphisms the usual maps between sets.

Example (**2.1.4**). — The category $\boldsymbol{Gr}$ of groups has for objects the groups and for morphisms the morphisms of groups. The category $\boldsymbol{Ab}$ of abelian groups has for objects the abelian groups and for morphisms the morphisms of groups. Observe that objects of $\boldsymbol{Ab}$ are objects of $\boldsymbol{Gr}$, and that morphisms in $\boldsymbol{Ab}$ coincide with those in $\boldsymbol{Gr}$; one says that $\boldsymbol{Ab}$ is a full subcategory of $\boldsymbol{Gr}$.

Example (**2.1.5**). — The category $\boldsymbol{Ring}$ of rings has for objects the rings and for morphisms the morphisms of rings.

Example (**2.1.6**). — Similarly, there is the category $\boldsymbol{Field}$ of fields and, if k is a field, the category $\boldsymbol{Vec}_k$ of k-vector spaces. More generally, for every ring A, there is a category $\boldsymbol{Mod}_{\mathrm{A}}$ of right A-modules, and a category ${}_{\mathrm{A}}\boldsymbol{Mod}$ of left A-modules.

Example (**2.1.7**). — Let $\boldsymbol{C}$ be a category; its *opposite category* $\boldsymbol{C}^{\mathrm{o}}$ has the same objects than $\boldsymbol{C}$, but the morphisms of $\boldsymbol{C}^{\mathrm{o}}$ are defined by $\boldsymbol{C}^{\mathrm{o}}(\mathrm{M},\mathrm{N}) = \boldsymbol{C}(\mathrm{N},\mathrm{M})$ and composed in the opposite direction.

It resembles the definition of an opposite group. However, a category is usually different from its opposite category.

Example (**2.1.8**). — Let I be a partially ordered set. One attaches to I a category $\boldsymbol{I}$ whose set of objects is I itself. Its morphisms are as follows: let $i, j \in \mathrm{I}$; if $i \leqslant j$, then $\boldsymbol{I}(i,j)$ has a single element, say the pair (i,j); otherwise, $\boldsymbol{I}(i,j)$ is empty. The composition of morphisms is the obvious one: $(j,k)\circ(i,j) = (i,k)$ if i, j, k are elements of I such that $i \leqslant j \leqslant k$.

Remark (**2.1.9**). — While, in this course, categories are mostly a *language* to state algebraic results of quite a formal nature, an adequate treatment of category theory involves set theoretical issues. Indeed, there does not exist a set containing all sets, nor a set containing all vector spaces, etc., so that the word *collection* in the above definition cannot be replaced by the word *set* (in the sense of Zermelo-Fraenkel's theory of sets). However, the theory of sets only considers sets! There are at least three ways to solve this issue:

a) The easiest one is to treat object of category theory as formulas, in the sense of first order logic. For example $\boldsymbol{Ring}$ is a formula $\varphi_{\boldsymbol{Ring}}$ with one free variable A that expresses that A is a ring. This requires to encode a ring A and all its laws as a tuple: for example, one may consider a ring to be a tuple $(\mathrm{A},\mathrm{S},\mathrm{P})$ where A is the ring, S is the graph of the addition law and P is the graph of the multiplication law. The formula $\varphi_{Ring}(x)$ then checks that x is a triplet of the form (A, S, P), where $S \subseteq A^3$ and $P \subseteq A^3$, that S is the graph of a map $A \times A \to A$ which is associative, commutative, has a neutral element, and for which every element has an opposite, etc.

Within such a framework, one can also consider functors (defined below), but only those which can be defined by a formula.

This treatment would be sufficient at the level of this course.

b) One can also use another theory of sets, such as the one of Bernays-Gödel-von Neumann, which allows for two kinds of collections: sets and classes. Sets, obey to the classical formalism of sets, but classes are more general, so that one can consider the class of all sets (but not the class of all classes). Functors are defined as classes.

This is a very convenient possibility at the level of this course. However, at a more advanced development of algebra, one is lead to consider the category of categories, or categories of functors. Then, this approach becomes unsufficient as well.

c) Within the classical Zermelo-Fraenkel theory of sets (with choice), Grothendieck introduced *universes* which are very large sets, so large than every usual construction of sets does not leave a given universe. One also needs to refine the axiom of choice, as well as to add the axiom that there is an universe, or, more generally, that every set belongs to some universe. This axiom is equivalent to the existence of *inaccessible cardinals*, an axiom which is well studied and often used in advanced set theory.

Remark **(2.1.10)**. — Let $\boldsymbol{C}$ be a category. One says that $\boldsymbol{C}$ is *small* if $\mathrm{ob}(\boldsymbol{C})$ is a set and if $\boldsymbol{C}(M, N)$ is a set for every pair (M, N) of objects of $\boldsymbol{C}$.

A category $\boldsymbol{C}$ such that the collection $\boldsymbol{C}(M, N)$ is a set for every pair (M, N) of objects is said to be *locally small*. In practice most categories considered in general mathematics, such as the categories of sets, of groups, abelian groups, of modules over a fixed ring, of vector spaces, etc., are locally small, but not small.

A locally small category $\boldsymbol{C}$ is said to be *essentially small* if the isomorphism classes of object of $\boldsymbol{C}$ form a set, that is, if there exists a set such that every object of $\boldsymbol{C}$ is isomorphic to one and only one member of this set.

For example, the category of finitely generated modules over a ring R is essentially small: for every finitely generated R-module M there is an integer $n \geqslant 0$ such that M is isomorphic to a quotient of R^n. The pairs (n, N) where $n \geqslant 0$

and N is a submodule of R^n form a set; if we take the quotient of this set by the equivalence relation for which $(n, N) \simeq (p, P)$ if $R^n/N \simeq R^p/P$, we get a set representing all isomorphism classes of finitely generated R-modules.

Definition **(2.1.11)**. — *Let* $\mathcal{C}$ *be a category, let* M, N *be objects of* $\mathcal{C}$ *and let* $f \in \mathcal{C}(M, N)$.

One says that f *is a* epimorphism *if for every object* P *of* $\mathcal{C}$ *and every morphisms* $g_1, g_2 \in \mathcal{C}(N, P)$ *such that* $g_1 \circ f = g_2 \circ f$, *one has* $g_1 = g_2$.

One says that f *is an* monomorphism *if for every object* L *of* $\mathcal{C}$ *and every morphisms* $g_1, g_2 \in \mathcal{C}(P, M)$ *such that* $f \circ g_1 = f \circ g_2$, *one has* $g_1 = g_2$.

Exercise **(2.1.12)**. — a) Prove that monomorphisms and epimorphisms in $\mathcal{S}et$ or in categories of modules are respectively injections and surjections.

b) Prove that in the category of rings, monomorphisms are the injective morphisms. However, show that the canonical morphism $f: \mathbf{Z} \to \mathbf{Q}$ is an epimorphism of rings.

2.2. Functors

Functors are to categories what maps are to sets.

2.2.1. — Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists in the following data:

– an object F(M) of $\mathcal{D}$ for every object M of $\mathcal{C}$;

– a morphism $F(f) \in \mathcal{D}(F(M), F(N))$ for every objets M, N of $\mathcal{C}$ and every morphism $f \in \mathcal{C}(M, N)$,

subject to the two following requirements:

(i) For every object M of $\mathcal{C}$, $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$;

(ii) For every objects M, N, P of $\mathcal{C}$ and every morphisms $f \in \mathcal{C}(M, N)$ and $g \in \mathcal{C}(N, P)$, one has

$$F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^o$ to $\mathcal{D}$. Explicitly, it consists in the following data

– an object F(M) of $\mathcal{D}$ for every object M of $\mathcal{C}$;

– a morphism $F(f) \in \mathcal{D}(F(N), F(M))$ for every objets M, N of $\mathcal{C}$ and every morphism $f \in \mathcal{C}(M, N)$, subject to the two following requirements:

(i) For every object M of $\boldsymbol{C}$, $\mathrm{F}(\mathrm{id}_{\mathrm{M}}) = \mathrm{id}_{\mathrm{F}(\mathrm{M})}$;

(ii) For every objects M, N, P of $\boldsymbol{C}$ and every morphisms $f \in \boldsymbol{C}(\mathrm{M},\mathrm{N})$ and $g \in \boldsymbol{C}(\mathrm{N},\mathrm{P})$, one has

$$\mathrm{F}(g \circ f) = \mathrm{F}(f) \circ \mathrm{F}(g).$$

2.2.2. — One says that such a functor F is *faithful*, resp. *full*, resp. *fully faithful* if for every objects M, N of $\boldsymbol{C}$, the map $f \mapsto \mathrm{F}(f)$ from $\boldsymbol{C}(\mathrm{M},\mathrm{N})$ to $\boldsymbol{C}(\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{N}))$ is injective, resp. surjective, resp. bijective. A similar definition applies for contravariant functors.

A functor F is *essentially surjective* if for every object P of $\boldsymbol{D}$, there exists an object M of $\boldsymbol{C}$ such that F(M) is isomorphic to P in the category $\boldsymbol{D}$.

Example **(2.2.3)** (Forgetful functors). — Many algebraic structures are defined by enriching other structures. Often, forgetting this enrichment gives rise to a functor, called a forgetful functor.

For example, a group is already a set, and a morphism of groups is a map. There is thus a functor that associates to every group its underlying set, thus forgetting the group structure. One gets a forgetful functor from $\boldsymbol{Gr}$ to $\boldsymbol{Set}$. It is faithful, because a group morphism is determined by the map between the underlying sets. It is however not full because there are maps between two (non-trivial) groups which are not morphism of groups.

Example **(2.2.4)**. — The construction of the spectrum of a ring defines a contravariant functor from the category $\boldsymbol{Ring}$ of rings to the category $\boldsymbol{Top}$ of topological spaces.

In the other direction, set $\mathscr{O}(\mathrm{X})$ to be the ring of continuous complex-valued functions on a topological space X. If $f\colon \mathrm{X} \to \mathrm{Y}$ is a continuous map of topological spaces, let $f^*\colon \mathscr{O}(\mathrm{Y}) \to \mathscr{O}(\mathrm{X})$ be the morphism of rings given by $f^*(u) = u \circ f$. This defines a contravariant functor from the category $\boldsymbol{Top}$ to the category of algebras over the field of complex numbers.

2.2.5. — Let F and G be two functors from a category $\boldsymbol{C}$ to a category $\boldsymbol{D}$. A *morphism of functors* α from F to G consists in the datum, for every object M of $\boldsymbol{C}$, of a morphism $\alpha_{\mathrm{M}}\colon \mathrm{F}(\mathrm{M}) \to \mathrm{G}(\mathrm{M})$ such that the following condition holds: For every morphism $f\colon \mathrm{M} \to \mathrm{N}$ in $\boldsymbol{C}$, one has $\alpha_{\mathrm{N}} \circ \mathrm{F}(f) = \mathrm{G}(f) \circ \alpha_{\mathrm{M}}$.

Morphisms of functors can be composed; for every functor F, one has an identity morphism from F to itself. Consequently, functors from $\mathcal{C}$ to $\mathcal{D}$ form themselves a category, denotes $\mathcal{F}(\mathcal{C},\mathcal{D})$.

2.2.6. — Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let F be a functor from $\mathcal{C}$ to $\mathcal{D}$ and let G be a functor from $\mathcal{D}$ to $\mathcal{C}$. One says that F and G are quasi-inverse functors if the functors $G \circ F$ and $F \circ G$ are isomorphic to the identity functors of the categories respectively $\mathcal{C}$ and $\mathcal{D}$.

One says that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that F and G are quasi-inverse functors.

Proposition (**2.2.7**). — *For a functor* $F\colon \mathcal{C} \to \mathcal{D}$ *to be an equivalence of categories, it is necessary and sufficient that it be fully faithful and essentially surjective.*

Proof. — Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor such that F and G are quasi-inverse. For every object P of $\mathcal{D}$, $F \circ G(P)$ is isomorphic to P, hence F is essentially surjective. Moreover, for every objects M, N of $\mathcal{C}$, the functor $G \circ F$, being isomorphic to $\mathrm{id}_{\mathcal{C}}$, induces a bijection from $\mathcal{C}(M,N)$ to itself. This bijection is the composition of the map $\Phi_F\colon \mathcal{C}(M,N) \to \mathcal{D}(M,N)$ induced by F and of the map $\Phi_G\colon \mathcal{D}(M,N) \to \mathcal{C}(M,N)$ induced by G. This implies that Φ_F is injective and Φ_G is surjective. By symmetry, Φ_F is surjective too, so that it is bijection. In other words, the functor F is fully faithful.

Let us now assume that F is fully faithful and essentially surjective. For every object M of $\mathcal{D}$, let us *choose* an object $G(M)$ of $\mathcal{C}$ and an isomorphism $\alpha_M\colon M \to F \circ G(M)$. Let M, N be objects of $\mathcal{D}$ and let $f \in \mathcal{D}(M,N)$; since F is fully faithful, there exists a unique morphism $f' \in \mathcal{C}(G(M),G(N))$ such that $F(f') = \alpha_N \circ f \circ \alpha_M^{-1}$; set $G(f) = f'$. Since $\alpha_M \circ \mathrm{id}_M \circ \alpha_M^{-1} = \mathrm{id}_{F\circ G(M)} = F(\mathrm{id}_{G(M)})$, one has $G(\mathrm{id}_M) = \mathrm{id}_{G(M)}$. Similarly, if M, N, P are objects of $\mathcal{D}$ and $f \in \mathcal{D}(M,N)$ and $g \in \mathcal{D}(N,P)$, one has

$$\begin{aligned}\alpha_P \circ g \circ f \circ \alpha_M^{-1} &= (\alpha_P \circ g \circ \alpha_N^{-1}) \circ (\alpha_N \circ f \circ \alpha_M^{-1}) \\ &= F(G(g)) \circ F(G(f)) \\ &= F(G(g) \circ G(f)),\end{aligned}$$

hence $G(g \circ f) = G(g) \circ G(f)$. Consequently, the assignment $M \mapsto G(M)$ and $f \mapsto G(f)$ is a functor from $\mathcal{D}$ to $\mathcal{C}$. Moreover, the maps $\alpha_M\colon M \to F \circ G(M)$ define an isomorphism of functors from the functor $\mathrm{Id}_{\mathcal{D}}$ to the functor $F \circ G$.

Let us now construct an isomorphism of functors from $\mathrm{Id}_{\mathcal{D}}$ to $\mathrm{G} \circ \mathrm{F}$. Let M be an object of $\mathcal{C}$. Since F is fully faithful, there exists a unique morphism $\beta_{\mathrm{M}} \in \mathcal{C}(\mathrm{M}, \mathrm{G} \circ \mathrm{F}(\mathrm{M}))$ such that $\mathrm{F}(\beta_{\mathrm{M}}) = \alpha_{\mathrm{F}(\mathrm{M})}$. Since $\alpha_{\mathrm{F}(\mathrm{M})}$ is an isomorphism, β_{M} is an isomorphism as well. Moreover, if M, N are objects of $\mathcal{C}$ and $f \colon \mathrm{M} \to \mathrm{N}$ is a morphism, then

$$\begin{aligned} \mathrm{F}(\mathrm{G} \circ \mathrm{F}(f) \circ \beta_{\mathrm{M}}) &= \alpha_{\mathrm{F}(\mathrm{N})} \circ \mathrm{F}(f) \circ \alpha_{\mathrm{F}(\mathrm{M})}^{-1} \circ \mathrm{F}(\beta_{\mathrm{M}}) \\ &= \alpha_{\mathrm{F}(\mathrm{N})} \circ \mathrm{F}(f) \\ &= \mathrm{F}(\beta_{\mathrm{N}} \circ f). \end{aligned}$$

Since F is fully faithful, one thus has $\beta_{\mathrm{N}} \circ f = \mathrm{G} \circ \mathrm{F}(f) \circ \beta_{\mathrm{M}}$. In other words, the isomorphisms β_{M}, for $\mathrm{M} \in \mathrm{ob}(\mathcal{C})$, define an isomorphism of functors from $\mathrm{Id}_{\mathcal{C}}$ to $\mathrm{G} \circ \mathrm{F}$.

As a consequence, the functor G is a quasi-inverse of the functor $\boldsymbol{F}$, hence F is an equivalence of categories. □

Example **(2.2.8)** (Linear algebra). — Let K be a field. Traditionally, undergraduate linear algebra only considers as vector spaces the subspaces of varying vector spaces K^n, and linear maps between them. This gives rise to a small category, because for every integer n, the subspaces of K^n form a set.

The obvious functor from this category to the category of finite dimensional K-vector spaces is an equivalence of categories. It is fully faithful (knowing that vector spaces lie in some K^n does not alter the linear maps between them). It is also essentially surjective: since vector spaces have bases, every finite dimensional K-vector space V is isomorphic to K^n, with $n = \dim(\mathrm{V})$. Consequently, the (small) "category of undergraduate linear algebra" is *equivalent* to the (large) category of finite dimensional vector spaces.

Example **(2.2.9)** (Covering theory). — Let X be a topological space, and let $x \in \mathrm{X}$. Let $\boldsymbol{Cov}_{\mathrm{X}}$ be the category of coverings of X. For every covering $p \colon \mathrm{E} \to \mathrm{X}$, the fundamental group $\pi_1(\mathrm{X}, x)$ acts on the fiber $p^{-1}(x)$. This defines a functor ("fiber functor") $\mathrm{F} \colon \mathrm{E} \mapsto \mathrm{F}(\mathrm{E}) = p^{-1}(x)$ from the category $\boldsymbol{Cov}_{\mathrm{X}}$ to the category of $\pi_1(\mathrm{X}, x)$-sets.

If X is connected and locally pathwise connected, then this functor is fully faithful. If, moreover, X has a simply connected cover (one says that X is "*délaçable*"; for example, locally contractible topological spaces are délaçable), then it is an equivalence of categories.

Example **(2.2.10)** (Galois theory). — Let K be a perfect field and let Ω be an algebraic closure of K; let G_K be the group of K-automorphisms of Ω. For every finite extension L of K, let $S(L) = \mathrm{Hom}_K(L, \Omega)$, the set of K-morphisms from L to Ω. This is a finite set, of cardinality $[L : K]$, and the group G_K acts on it by the formula $g \cdot \varphi = g \circ \varphi$, for every $\varphi \in S(L)$ and every $g \in G_K$; moreover, the action of G_K is transitive.

Every morphism of extensions $f\colon L \to L'$ induces a map $f^*\colon S(L') \to S(L)$ which is compatible with the actions of G_K. The assignments $L \mapsto S(L)$ and $f \mapsto f^*$ define a contravariant functor from the category of finite extensions of K to the category of finite sets endowed with a transitive action of G_K.

Galois theory can be summaried by saying that this functor is an equivalence of categories. An inverse functor F assigns to a set Φ endowed with an action of G_K the subfield $F(\Phi)$ of Ω which is fixed by the kernel of the action of G_K on Φ. Moreover, the automorphism group of the functor S is the group G_K.

By analogy with covering theory, it may look preferable to have a category equivalent to the full category of finite G_K-sets. To that aim, one just needs to replace in the previous definitions the category of finite extensions of K by the category of finitely dimensional reduced K-algebras (aka "finite étale K-algebras", which are nothing but finite products of finite extensions of K).

2.3. Limits and colimits

2.3.1. — A *quiver* Q is a tuple (V, E, s, t) where V and E are sets, and s, t are maps from E to V. Elements of V are called vertices; elements of E are called arrows; for an arrow $e \in E$, the vertices $s(e)$ and $t(e)$ are the source and the target of e.

Every small category $\boldsymbol{C}$ has an underlying quiver, whose set of vertices is the set of objects of $\boldsymbol{C}$, and whose set of arrows is the set of morphisms of $\boldsymbol{C}$.

2.3.2. Diagrams. — Let $Q = (V, E, s, t)$ be a quiver and let $\boldsymbol{C}$ be a category. A Q-diagram $\mathscr{A}$ in $\boldsymbol{C}$ consists in a family $(A_v)_{v \in V}$ of objects of $\boldsymbol{C}$ and in a family $(f_e)_{e \in E}$ of morphisms of $\boldsymbol{C}$ such that for every arrow $e \in E$, $f_e \in \boldsymbol{C}(A_{s(e)}, A_{t(e)})$.

2.3.3. Limits. — A *cone* on a diagram $\mathscr{A}$ is the datum of an object A of $\boldsymbol{C}$ and of morphisms $f_v\colon A \to A_v$, for every $v \in V$, such that $f_e \circ f_{s(e)} = f_{t(e)}$ for every $e \in E$. Such a cone is said to be a *limit* if for every cone $(B, (g_v)_{v \in V})$ of the diagram $\mathscr{A}$, there exists a unique morphism $g\colon B \to A$ in $\boldsymbol{C}$ such that $g_v = f_v \circ g$ for every $v \in V$.

Let $(A, (f_v))$ and $(A', (f'_v))$ be two limits of a diagram $\mathscr{A}$. Then there exists a unique morphism $\varphi\colon A' \to A$ such that $f'_v = f_v \circ \varphi$ for every $v \in V$; this morphism is an isomorphism. In other words, when they exist, limits of diagrams are unique up to a unique isomorphism.

A limit of a diagram $\mathscr{A}$ is sometimes denoted by $\varprojlim(\mathscr{A})$.

2.3.4. — Let $Q = (V, E, s, t)$ be a quiver, let $Q^o = (V, E, t, s)$ be the opposite quiver in which the source and target maps are exchanged. Every Q-diagram $\mathscr{A}$ in a category $\boldsymbol{C}$ is naturally a Q^o-diagram in the opposite category $\boldsymbol{C}^o$, which we denote by $\mathscr{A}^o$. A *colimit* of the diagram $\mathscr{A}$ is a limit of the diagram $\mathscr{A}^o$.

Explicitly, a colimit of the diagram $\mathscr{A} = ((A_v), (f_e))$ consists in an object A of $\boldsymbol{C}$, and in morphisms $f_v\colon A_v \to A$, for $v \in V$ such that $f_{t(e)} \circ f_e = f_{s(e)}$ for every $e \in E$ (such a family $(A, (f_v))$ can be called a *cocone* on the diagram $\mathscr{A}$), which satisfies the universal property: for every object B of $\boldsymbol{C}$ and every family $(g_v\colon B \to A_v)$ of morphisms such that $g_{t(e)} \circ f_e = g_{s(e)}$ for every $e \in E$, there exists a unique morphism $g\colon A \to B$ in $\boldsymbol{C}$ such that $g_v = g \circ f_v$ for every $v \in V$.

When they exists, colimits of a diagram $\mathscr{A}$ are unique up to a unique isomorphism. A colimit of a diagram $\mathscr{A}$ is sometimes denoted by $\varinjlim(\mathscr{A})$.

Example **(2.3.5)**. — a) Let Q be the empty quiver (no vertex, no arrow). Let us consider the unique Q-diagram; it consists in nothing. By definition, a cone on this diagram is just an object A of $\boldsymbol{C}$, and A is a limit if and only if there exists a unique morphism in $\boldsymbol{C}(B, A)$, for every object B of $\boldsymbol{C}$. Consequently, a limit of this diagram in the category $\boldsymbol{C}$ is called an *terminal object* of $\boldsymbol{C}$.

Dually, if A is a colimit of this diagram, it is an object such that, for every object B of $\boldsymbol{C}$, there exists a unique morphism $\boldsymbol{C}(A, B)$; it is called a *initial object.*

In the category of sets, the empty set is an initial object, while singletons are terminal objects. In the category of groups, or in the category of A-modules, the trivial group (with one element) is both an initial and a terminal object. In the category $\boldsymbol{Ring}$ of rings, the ring $\mathbf{Z}$ is an initial object (for any ring A, there is exactly one morphism from $\mathbf{Z}$ to A), and the ring 0 is a terminal object.

b) Let Q be the quiver $\bullet \quad \bullet$ (two vertices, no arrow). A Q-diagram $\mathscr{A}$ consists in a pair (A, A') of objects of $\mathbf{C}$. A colimit of this diagram $\mathscr{A}$ is called a *coproduct* of this diagram, and a limit is called a *product*.

This generalizes to quivers $Q = (V, \varnothing, s, t)$ whose set of arrows is empty: a Q-diagram is a family $\mathscr{A} = (A_v)_{v \in V}$ of objects indexed by V, a colimit of $\mathscr{A}$ is a coproduct, while a limit of $\mathscr{A}$ is a product.

A coproduct A is endowed with maps $f_v \colon A_v \to A$ and satisfies the following universal property: for every object B of $\mathbf{C}$ and every family $(g_v \colon A_v \to B)$ of morphisms, there exists a unique morphism $\varphi \colon A \to B$ such that $g_v = \varphi \circ f_v$ for every $v \in V$. Dually, a product A is endowed with maps $f_v \colon A \to A_v$ and satisfies the following universal property: for every object B of $\mathbf{C}$ and every family $(g_v \colon B \to A_v)$ of morphisms, there exists a unique morphism $\varphi \colon B \to A$ such that $g_v = f_v \circ \varphi$ for every $v \in V$.

c) Let Q be the quiver $\bullet \rightrightarrows \bullet$. A Q-diagram $\mathscr{A}$ consists in two objects M, N of $\mathbf{C}$ and two morphisms $f, g \colon M \to N$ in $\mathbf{C}$, hence can be represented as $\mathscr{A} = (\ M \overset{f}{\underset{g}{\rightrightarrows}} N \ )$.

A limit of this diagram $\mathscr{A}$ is called an *equalizer* of the pair (f, g). If $\mathbf{C}$ is the category of sets, or the category of groups, the subset E of M consisting of $m \in M$ such that $f(m) = g(m)$ is an equalizer of the diagram $\mathscr{A}$.

A colimit of $\mathscr{A}$ is called a *coequalizer* of the pair (f, g). If $\mathbf{C}$ is the category of sets, then the quotient of N by the smallest equivalence relation such that $f(m) \sim g(m)$ for every $m \in M$ is a coequalizer of the diagram $\mathscr{A}$. If $\mathbf{C}$ is the category of groups, then the quotient of N by the smallest normal subgroup containing the elements $f(m)g(m)^{-1}$, for $m \in M$, is a coequalizer of this diagram. If $\mathbf{C}$ is the category of abelian groups, or the category of modules over a ring, then the cokernel of $f - g$ is a coequalizer of this diagram.

Exercise **(2.3.6)**. — Let A be a ring, let S be a multiplicative subset of A. Let Q be the quiver whose vertex set is S and whose set of arrows is $S \times S$, an arrow (s, t) having source s and target st. Let M be an A-module and let $\mathscr{M} = ((M_s), f_{s,t})$ be the Q-diagram such that $M_s = M$ for every s, and $f_{s,t}$ is the multiplication by t. Show that the module $S^{-1}M$, endowed with the morphism $f_s \colon M_s \to S^{-1}M$ given by $m \mapsto m/s$, is a colimit of the diagram $\mathscr{M}$.

Proposition **(2.3.7)**. — *In the category of sets, every diagram has a limit and a colimit.*

Proof. — Let $Q = (V, E, s, t)$ be a quiver and let $\mathscr{A} = ((A_v), (f_e))$ be a Q-diagram of sets.

a) *Construction of a limit.* Let $A^* = \prod_{v \in V} A_v$ and let A be the subset of A^* consisting of families $(a_v)_{v \in V}$ such that $f_e(a_{s(e)}) = a_{t(e)}$ for every $e \in E$. For every $v \in V$, let $f_v \colon A \to A_v$ be the map deduced by restriction of the canonical projection from A^* to A_v. By construction, one has $f_e \circ f_{s(e)} = f_{t(e)}$ for every $e \in E$.

Let now B be a set and let $(g_v)_{v \in V}$ be a family such that $f_e \circ g_{s(e)} = g_{t(e)}$ for every $e \in E$. Let $\varphi^* \colon B \to A^*$ be the map given by $\varphi^*(b) = (g_v(b))_{v \in V}$. By the definition of A, one has $\varphi^*(b) \in A$ for every $b \in B$; the map $\varphi \colon B \to A$ deduced from φ^* satisfies $g_v = f_v \circ \varphi$ for every $v \in V$. Moreover, if $\psi \colon B \to A$ is a map such that $g_v = f_v \circ \psi$ for all $v \in V$, then $f_v(\psi(b)) = g_v(b)$, hence $\psi(b) = (g_v(b))_{v \in V} = \varphi(b)$. Consequently, $(A, (f_v))$ is a limit of the diagram $\mathscr{A}$, as was to be shown.

b) *Construction of a colimit.* Let A_* be the set of pairs (v, a), where $v \in V$ and $a \in A_v$. Let $\sim$ be the smallest equivalence relation on A_* such that $(s(e), a) \sim (t(e), f_e(a))$ for every $e \in E$ and every $a \in A_{s(e)}$; let $A = A_*/\sim$ be the quotient set; one writes $[v, a]$ for the class in A of an element $(v, a) \in A_*$. For every $v \in V$, let $f_v \colon A_v \to A$ be the map given by $a \mapsto [v, a]$. For every $e \in E$ and every $a \in A_{s(e)}$, one has

$$f_{t(e)}(f_e(a)) = [t(e), f_e(a)] = [s(e), a] = f_{s(e)}(a),$$

so that $f_{t(e)} \circ f_e = f_{s(e)}$; this shows that $(A, (f_v))$ is a cocone of the Q-diagram $\mathscr{A}$.

Let $(B, (g_v))$ be a cocone of this diagram. Let $\varphi \colon A \to B$ be a map such that $\varphi \circ f_v = g_v$ for every $v \in V$. For $v \in V$ and $a \in A_v$, one thus has $\varphi([v, a]) = \varphi(f_v(a)) = g_v(a)$. Since the map from A_* to A is surjective, thus shows that there exists at most one map $\varphi \colon A \to B$ such that $\varphi \circ f_v = g_v$ for every $v \in V$. Let us prove its existence. Let $\varphi_* \colon A_* \to B$ be the map given by $\varphi_*((v, a)) = g_v(a)$, whenever $v \in V$ and $a \in A_v$. For every $e \in E$ and every $a \in A_{s(e)}$, one has $\varphi_*((t(e), f_e(a))) = g_{t(e)}(f_e(a)) = g_{s(e)}(a) = \varphi_*((s(e), a))$. Consequently, the map φ_* is compatible with the equivalence relation $\sim$ and there exists a map $\varphi \colon A \to B$ such that $\varphi([v, a]) = \varphi_*((v, a)) = g_v(a)$ for every $v \in V$ and every $a \in A_v$. For every $v \in V$ and every $a \in A_v$, one has $\varphi(f_v(a)) = \varphi_*((v, a)) = g_v(a)$, hence $\varphi \circ f_v = g_v$.

This concludes the proof of the proposition. □

Corollary **(2.3.8)**. — *Let $\boldsymbol{C}$ be a category among the following: groups, abelian groups, rings, modules over a given ring, algebras. Then every diagram in $\boldsymbol{C}$ has a limit.*

Proof. — Let $Q = (V, E, s, t)$ be a quiver and let $((A_v), (f_e))$ be a Q-diagram in $\boldsymbol{C}$. The objects A_v are sets endowed with additional laws. The proof of the corollary consists in first considering the limit of the corresponding Q-diagram in the category of sets, and in observing that it is naturally an object of the category $\boldsymbol{C}$ which is a limit of the diagram in that category. We keep the notation introduced in the proof of proposition 2.3.7.

An arbitrary product of groups, rings, etc., has a canonical structure of a group, a ring, etc., so that the set $A^* = \prod_{v\in V} A_v$ is really an object of the category $\boldsymbol{C}$, and the projections $A^* \to A_v$ are morphisms in that category. Moreover, since the maps f_e are morphisms in the category $\boldsymbol{C}$, one checks readily that its subset A consisting of families $(a_v) \in A^*$ such that $f_e(a_{s(e)}) = a_{t(e)}$ is a subobject, hence an object of $\boldsymbol{C}$, and the maps $f_v \colon A \to A_v$ are morphisms of $\boldsymbol{C}$. By inspection of the proof, one checks that the map $\varphi \colon A \to B$ constructed there is a morphism in the category $\boldsymbol{C}$, so that $(A, (f_v))$ is a limit of the diagram $\mathscr{A}$ in the category $\boldsymbol{C}$. ☐

Remark **(2.3.9)**. — Let $\boldsymbol{C}$ be a category of algebraic structures, such as sets, groups, rings, modules, algebras,... It holds true that every diagram in $\boldsymbol{C}$ has a colimit. However, the colimit of this diagram in $\boldsymbol{C}$, which is a set with an algebraic structure, does in general not coincide with the colimit of the corresponding diagram of sets.

For example, the trivial group $\{e\}$ with one element is an initial object of the category of groups, while the initial object of the category of sets is the empty set.

Similarly, the coproduct of a family of sets is its "disjoint union", while the coproduct of a family of groups is its free product, and the coproduct of a family of abelian groups is its direct sum.

Coequalizers give another examples of this phenomenon: the coequalizer of a diagram of groups $H \underset{g}{\overset{f}{\rightrightarrows}} G$ is the quotient of G by the smallest normal subgroup containing the elements of the form $f(x)g(x)^{-1}$, for $x \in H$. For example, if G is simple and $f \neq g$, then $\operatorname{Coequal}(f, g)$ is the trivial group, while the coequalizer of this diagram in the category of sets is generally larger.

We now describe a particular type of quivers (associated to so called filtrant partially ordered sets), for which the colimit of a diagram in a category of a given algebraic structure is an algebraic structure on the set which is the colimit of the same diagram, viewed as a diagram of sets.

2.3.10. — Let I be a partially ordered set. An I-diagram consists in a family $(A_i)_{i\in I}$ of objects of $\mathbf{C}$, and of morphisms $f_{ij}\colon A_i \to A_j$ whenever i, j are elements of I such that $i \leqslant j$, subject to the conditions:

- One has $f_{ii} = \mathrm{id}_{A_i}$ for every $i \in I$;
- One has $f_{jk} \circ f_{ij} = f_{ik}$ for every triple (i, j, k) of elements of I such that $i \leqslant j \leqslant k$.

In other words, this is a functor from the category $\boldsymbol{I}$ associated with the partially ordered set I (see example 2.1.8) to the category $\boldsymbol{C}$. The morphisms f_{ij} are often omitted from the notation.

Let E be the set of pairs (i, j) of elements of I such that $i \leqslant j$ and let $\boldsymbol{I}$ be the quiver (I, E, s, t), where s and t are given by $s((i, j)) = i$ and $t((i, j)) = j$. An I-diagram naturally gives rise to an $\boldsymbol{I}$-diagram, whose eventual colimit (resp. limit) is called its colimit (resp. its limit).

Explicitly, a colimit of an I-diagram $((A_i), (f_{ij}))$ is an object A of the category **C** endowed with morphisms $f_i\colon A_i \to A$ satisfying $f_j \circ f_{ij} = f_i$ for all i, j such that $i \leqslant j$, and such that object B of $\boldsymbol{C}$, and every family $(g_i\colon A_i \to B)$ of morphisms such that $g_j \circ f_{ij} = g_i$, there exists a unique morphism $\varphi\colon A \to B$ such that $\varphi \circ f_i = g_i$ for every $i \in I$.

Similarly, a limit of an I-diagram $((A_i), (f_{ij}))$ is an object A of the category **C** endowed with morphisms $f_i\colon A \to A_i$ satisfying $f_{ij} \circ f_i = f_j$ for all i, j such that $i \leqslant j$, and such that object B of $\boldsymbol{C}$, and every family $(g_i\colon B \to A_i)$ of morphisms such that $f_{ij} \circ g_i = g_j$, there exists a unique morphism $\varphi\colon A \to B$ such that $g_i = f_i \circ \varphi$ for every $i \in I$.

2.3.11. — Let I be a partially ordered set. One says that I is *filtrant* if every finite subset has an upper bound in I. This means that I is non-empty and for every two elements $i, j \in I$, there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

If I is a filtrant partially ordered set, an I-diagram is also called a *direct system*, or an *inductive system*. In this case, colimits are also called *direct limits* or *inductive limits*, and limits are also called *inverse limits* or *projective limits*.

Proposition (**2.3.12**). — *Let $\mathbf{C}$ be a category among the following: groups, abelian groups, rings, modules, algebras. Every direct system in $\mathbf{C}$ has a colimit.*

Proof. — Let I be a filtrant partially ordered set and let $\mathscr{A} = ((\mathscr{A}_i), (f_{ij}))$ be a direct system indexed by I. Let $\boldsymbol{I}$ be the quiver associated with I, so that $\mathscr{A}$ is an $\boldsymbol{I}$-diagram. The objects of $\mathbf{C}$ are sets endowed with additional maps (binary laws, operations,...) subject to algebraic conditions. The proof consists in first considering a colimit $(\mathrm{A}, (f_i \colon \mathrm{A}_i \to \mathrm{A})$ of the diagram $\mathscr{A}$ in the category of sets, as given by proposition 2.3.7, and in observing that it is naturally a colimit in the category $\mathbf{C}$. For this, the hypothesis that the partially ordered set I is filtrant is essential. Let us keep the notation of the proof of proposition 2.3.7.

Let $\top$ be one of the binary laws of objects of the category $\mathbf{C}$, for example the group law if $\mathbf{C}$ is the category of groups. While A_* has not particular structure, let us prove that there is a unique law $\top$ on A such that $f_i(a\top b) = f_i(a)\top f_i(b)$ if $i \in \mathrm{I}$ and $a, b \in \mathrm{A}_i$. Indeed, we first define a map from $\mathrm{A}_* \times \mathrm{A}_*$ to A by $((i,a),(j,b)) \mapsto [k, f_{ik}(a)\top f_{jk}(b)]$, whenever k is an element of I such that $i \leqslant k$ and $j \leqslant k$. It is well defined; indeed, if $i \leqslant k'$ and $j \leqslant k'$, let $k'' \in \mathrm{I}$ be such that $k \leqslant k''$ and $k' \leqslant k''$; since $f_{kk''}$ is compatible with the law $\top$, one has

$$[k'', f_{ik''}(a)\top f_{jk''}(b)] = [k'', f_{kk''}(f_{ik}(a)\top f_{jk}(b))] = [k, f_{ik}(a)\top f_{jk}(b)],$$

and $[k'', f_{ik''}(a)\top f_{jk''}(b)] = [k', f_{ik'}(a)\top f_{jk'}(b)]$ by symmetry.

We then observe that this map passes to the quotient by the equivalence relation $\sim$ and defines a desired law $\top$ on A.

If the laws $\top$ on the A_i are commutative (resp. associative), one proves that the obtained law $\top$ on A is commutative (resp. associative) as well. Assume that for every i, the law $\top$ has a neutral element e_i in A_i; then, the classes $[i, e_i]$ (for $i \in \mathrm{I}$) are equal to a single element e of A which is a neutral element. Similarly, if every element of A_i has an inverse for the law $\top$, then every element of A has an inverse: the inverse of a class $[i, a]$ is the class $[i, b]$, where b is an inverse of a in A_i.

This treats the cases of groups and abelian groups. The case of rings is analogous: by what precedes, the colimit A is endowed with a natural addition and a natural multiplication compatibly with the maps $f_i \colon \mathrm{A}_i \to \mathrm{A}$; one then checks that the multiplication distributes over the addition.

Similarly, when $\boldsymbol{C}$ is the category of R-modules (for some ring R), one checks that A has a unique structure of R-module such that $x\cdot[i,a]=[i,x\cdot a]$ for every $x\in R$, every $i\in I$ and every $a\in A_i$.

To conclude the proof of the corollary, it remains first to observe that the maps f_i are morphisms in the category $\boldsymbol{C}$, and second to check that the map γ constructed in the proof of proposition 2.3.7 is a morphism in the category $\boldsymbol{C}$. □

2.3.13. — Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories, let $F\colon \boldsymbol{C}\to\boldsymbol{D}$ be a functor.

Let Q be a quiver and let $\mathscr{A}=((A_v),(\varphi_e))$ be a Q-diagram in $\boldsymbol{C}$. Assume that this diagram has a colimit $(A,(\varphi_v))$. Then $F(\mathscr{A})=((F(A_v)),(F(\varphi_e)))$ is a Q-diagram in $\boldsymbol{D}$ and the object $F(A)$, equiped with the family of morphisms $(F(\varphi_v))$, is a cocone on that diagram.

One says that the functor F commutes with colimits if for every such situation, the cocone $(F(A),((\varphi_v)))$ is a colimit of the diagram $F(\mathscr{A})$.

The definition, for the functor F, of commuting with limits is analogous: this means that for every diagram $\mathscr{A}$ as above which has a limit $(A,(\varphi_v))$, the cone $(F(A),(F(\varphi_v)))$ on the diagram $F(\mathscr{A})$ is a limit.

Definition **(2.3.14)**. — *One says that a functor is* right exact *if it commutes with every finite colimit, and that it is* left exact *if it commutes with every finite limit. One says that a functor is* exact *if it is both left exact and right exact.*

If $F\colon\boldsymbol{C}\to\boldsymbol{D}$ is a contravariant functor, one considers it as a functor from $\boldsymbol{C}^{o}$ to $\boldsymbol{D}$, so that we also have a definition of right or left exact contravariant functors.

Example **(2.3.15)**. — Let A be a ring, let S be a multiplicative subset of A. Let us consider the functor from the category of A-modules to that of $S^{-1}A$-modules which is given by $(M\mapsto S^{-1}M, f\mapsto S^{-1}f)$. Let us show that it commute with every colimit and with every finite limit.

We begin with the case of colimits. Let $Q=(V,E)$ be a quiver, let $\mathscr{M}=((M_v),(\varphi_e))$ be a Q-diagram of A-modules and let $(M,(\varphi_v)$ be its colimit. Let us then show that the cocone $(S^{-1}M,(S^{-1}\varphi_v))$ on the diagram $S^{-1}\mathscr{M}=((S^{-1}M_v),(S^{-1}\varphi_e))$ satisfies the universal property of a colimit. Let $(N,(\psi_v))$ be a cocone on this diagram, where N is an $S^{-1}A$-module. For every $v\in V$, ψ_v is a morphism from $S^{-1}M_v$ to N such that $\psi_{t(e)}\circ(S^{-1}\varphi_e)=\psi_{s(e)}$ for every $e\in E$. For every $v\in V$, let $\psi'_v\colon M_v\to N$ be the morphism given by $m\mapsto\psi_v(m/1)$; then

$(N, (\psi'_v))$ is a cocone on the initial diagram $\mathscr{M}$, so that there exists a unique morphism $\psi' \colon M \to N$ such that $\psi' \circ \varphi_v = \psi'_v$ for every $v \in V$. Since every element of S acts by automorphism on N, there exists a unique morphism $\psi \colon S^{-1}M \to N$ such that $\psi(m/s) = (1/s)\psi'(m)$ for every $m \in M$ and every $s \in S$. For every $v \in V$, every $m \in M$ and every $s \in S$, one has $\psi_v(m/1) = \psi'_v(m) = \psi'(\varphi_v(m)) = \psi(S^{-1}\varphi_v(m/1))$, hence $\psi_v = \psi \circ S^{-1}\varphi_v$. Conversely, every morphism $\widetilde{\psi} \colon S^{-1}M \to N$ such that $\widetilde{\psi} \circ S^{-1}\varphi_v = \psi_v$ for every v must satisfy $\widetilde{\psi}(\varphi_v(m)/1) = \psi(\varphi_v(m/1))$ for every v. Since M is a colimit of the diagram $\mathscr{M}$, the compositions of ψ and $\widetilde{\psi}$ with the canonical morphism from M to $S^{-1}M$ coincide with ψ'. This implies that $\psi = \widetilde{\psi}$.

Let us now prove that the functor $M \mapsto S^{-1}M$ commutes with every finite limit. Let thus $Q = (V, E)$ be a finite quiver and $\mathscr{M} = ((M_v), (\varphi_e))$ be a Q-diagram of A-modules; let $(M, (\varphi_v))$ be a limit of this diagram. Then $(S^{-1}M, (S^{-1}\varphi_v))$ is a cone on the diagram $S^{-1}\mathscr{M}$, and we need show that it satisfies its universal property. Let thus N be an $S^{-1}A$-module and let $(\psi_v)_{v\in V}$ be a family, where $\psi_v \colon N \to S^{-1}M_v$ is a morphism of A-mdules such that $S^{-1}\varphi_e \circ \psi_{s(e)} = \psi_{t(e)}$ for every $e \in E$.

Let $n \in N$. For every $v \in V$, let $m_v \in M$ and $s_v \in S$ be such that $\psi_v(n) = m_v/s_v$; since V is finite, we may replace s_v by $\prod_{v\in V} s_v$ and assume that all elements s_v are equal to a single element $s \in S$. For every $e \in E$, one then has $(S^{-1}\varphi_e)(m_{s(e)}/s) = m_{t(e)}/s$, hence there exists $s'_e \in S$ such that $s_e\varphi_e(m_{s(e)}) = s'_e m_{t(e)}$. Since E is finite, there exists an element $s' \in S$ such that $s'\varphi_e(m_{s(e)}) = s'm_{t'(e)}$ for every $e \in E$. It then follows from the universal property of a limit, applied to the morphisms $A \to M_v$, $a \mapsto as'm_v$, that there exists a unique element $m \in M$ such that $s'm_v = \varphi_v(m)$ for every $v \in V$. One then has $ss'\psi_v(n) = s'm_v = \varphi_v(m)$.

Define $\psi(n) = m/ss'$; this is an element of $S^{-1}M$ which does not depend on the choices of the elements s and s' such that $\psi_v(n) = m_v/s$ for every $v \in V$ and $s'\varphi_e(m_{s(e)}) = s'm_{t'(e)}$ for every $e \in E$. The map $\psi \colon N \to S^{-1}M$ is a morphism of $S^{-1}A$-modules and one has $(S^{-1}\varphi_v) \circ \psi = \psi_v$ for every $v \in V$.

It is moreover the unique such morphism. Let indeed $\widetilde{\psi}$ be a morphism of A-modules from N to $S^{-1}M$ such that $(S^{-1}\varphi_v) \circ \widetilde{\psi} = \psi_v$ for every $v \in V$. Let $n \in N$, let $m \in M$ and $s \in S$ be such that $\widetilde{\psi}(n) = m/s$. One then has $\psi_v(n) = \varphi_v(m)/s$ for every $v \in V$, so that, in the definition of ψ, one can take for $s, s', (m_v), m$ the elements $s, 1, (\varphi_v(m)), m$, which shows that $\psi(n) = m/s = \widetilde{\psi}(n)$.

2.4. Representable functors. Adjunction

2.4.1. — Let $\mathcal{C}$ be a locally small category and let P be an object of $\mathcal{C}$.

One defines a contravariant functor h_P from the category $\mathcal{C}$ to the category $\mathcal{S}et$ of sets, sometimes denoted $\mathrm{Hom}_{\mathcal{C}}(\bullet, P)$, as follows:

– For every object M of $\mathcal{C}$, one sets $h_P(M) = \mathcal{C}(M, P)$;

– For every morphism $f\colon M \to N$ in $\mathcal{C}$, $h_P(f)$ is the map $u \mapsto u \circ f$ from $\mathcal{C}(N, P)$ to $\mathcal{C}(M, P)$.

One says that a contravariant functor $G\colon \mathcal{C}^o \to \mathcal{S}et$ is *representable* if it is isomorphic to a functor of the form h_P; one then says that P represents the functor G.

Moreover, the assignment $P \mapsto h_P$ is a functor from the category $\mathcal{C}$ to the category $(\mathcal{C}^o, \mathcal{S}et)$ of contravariant functos from $\mathcal{C}$ to $\mathcal{S}et$.

2.4.2. — One can also define a functor k_P from the category $\mathcal{C}$ to the category of sets as follows:

– For every object M of $\mathcal{C}$, one sets $k_P(M) = \mathcal{C}(P, M)$;

– For every morphism $f\colon M \to N$ in $\mathcal{C}$, $k_P(f)$ is the map $u \mapsto f \circ u$ from $\mathcal{C}(P, M)$ to $\mathcal{C}(P, N)$.

This is functor is also denoted by $\mathrm{Hom}_{\mathcal{C}}(P, \bullet)$. It is also the functor h_P represented by the object P of the opposite category $\mathcal{C}^o$. Every functor which is isomorphic to a functor of this form is called a *corepresentable functor.* If F is isomorphic to k_P, one also says that P corepresents the functor F.

In fact, one often writes ''representable'' instead of ''corepresentable'', for the covariance of the functor immediately resolves the ambiguity.

2.4.3. — Algebra is full of *universal properties*: the free module on a given basis, quotient ring, quotient module, direct sum and product of modules, localization, algebra of polynomials on a given set of indeterminates. They are all of the following form: "in such algebraic situation, there exists an object and a morphism satisfying such property and such that every other morphism which satisfies this property factors through it".

The property for an object I to be an initial object can be rephrased as a property of the corepresentable functor $\mathrm{Hom}_{\mathcal{C}}(I, \bullet)$, namely that this functor coincides with (or, rather, is isomorphic to) the functor F that sends every object of $\mathcal{C}$ to a fixed set with one element.

This allows to rephrase the definition of an initial object as follows: an object I is an initial object if it *corepresents* the functor F defined above.

Objects that represent a given contravariant functor (resp. corepresent a given functor) are unique up to a unique isomorphism:

Proposition **(2.4.4)** (Yoneda's lemma). — *Let* $\mathbf{C}$ *be a category, let* A *and* B *be two objects of* $\mathbf{C}$.

a) *For any morphism of functors* φ *from* h_A *to* h_B*, there is a unique morphism* $f\colon A \to B$ *such that* $\varphi_M(u) = f \circ u$ *for every object* M *of* $\mathbf{C}$ *and any morphism* $u \in \mathbf{C}(M, A)$. *Moreover,* φ *is an isomorphism if and only if* f *is an isomorphism.*

b) *For any morphism of functors* φ *from* k_A *to* k_B*, there is a unique morphism* $f\colon B \to A$ *such that* $\varphi_M(u) = u \circ f$ *for every object* M *of* $\mathbf{C}$ *and any morphism* $u \in \mathbf{C}(A, M)$. *Moreover,* φ *is an isomorphism if and only if* f *is an isomorphism.*

Proof. — a) If there exists a morphism f such that $\varphi_M(u) = f \circ u$ for every $u \in \mathbf{C}(M, A)$, then one has $f = f \circ \mathrm{id}_A = \varphi_A(\mathrm{id}_A)$, hence the uniqueness of a morphism f as required. Conversely, let us show that the morphism $f = \varphi_A(\mathrm{id}_A) \in \mathbf{C}(A, B)$ satisfies the given requirement. To that aim, let us first recall the definition of a morphism of contravariant functors: for every object M of $\mathbf{C}$, one has a map $\varphi_M\colon h_A(M) \to h_B(M)$ such that $h_B(u) \circ \varphi_N = \varphi_M \circ h_A(u)$ for every two objects M and N of $\mathbf{C}$ and every morphism $u\colon M \to N$. In the present case, this means that for every object M of $\mathbf{C}$, φ_M is a map from $\mathbf{C}(M, A)$ to $\mathbf{C}(M, B)$ and that

$$\varphi_N(v) \circ u = h_B(u) \circ \varphi_N(v) = \varphi_M \circ (h_A(u))(v) = \varphi_M(v \circ u),$$

for every $v \in \mathbf{C}(N, A)$ and every $u \in (M, N)$. Consequently, taking $N = A$ and $v = \mathrm{id}_A$ in the above formula, one obtains

$$f \circ u = \varphi_A(\mathrm{id}_A) \circ u = \varphi_M(\mathrm{id}_A \circ u) = \varphi_M(u),$$

for every object M and every morphism $u \in \mathbf{C}(M, A)$.

Let us assume that f is an isomorphism and that g is its inverse. Then the assignement $\gamma_M(u) = g \circ u$ defines a morphism of functors γ from h_B to h_A which is an inverse of φ. Consequently, φ is an isomorphism. Conversely, assume that φ is an isomorphism and let ψ be its inverse. By what precedes, there is a unique morphism $g\colon B \to A$ such that $\psi_M(u) = g \circ u$ for every object $M \in \mathbf{C}$ and every $u \in \mathbf{C}(M, B)$. The morphism of functors $\psi \circ \varphi$ is the identity of h_A, and is given by $\psi_M \circ \varphi_M(u) = (g \circ f) \circ u$ for every $M \in \mathbf{C}$ and every $u \in \mathbf{C}(M, A)$. By the uniqueness property, one has $g \circ f = \mathrm{id}_A$. Similarly, $f \circ g = \mathrm{id}_B$. This shows that f is an isomorphism.

b) This follows from *a*), applied in the opposite category $\boldsymbol{C}^{\mathrm{o}}$. □

2.4.5. Adjunction. — Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories, let F be a functor from $\boldsymbol{C}$ to $\boldsymbol{D}$ and G be a functor from $\boldsymbol{D}$ to $\boldsymbol{C}$.

An *adjunction* for the pair (F, G) is the datum, for every object M of $\boldsymbol{C}$ and every object N of $\boldsymbol{D}$, of a *bijection*

$$\Phi_{M,N} \colon \boldsymbol{C}(M, G(N)) \xrightarrow{\sim} \boldsymbol{D}(F(M), N),$$

subject to the following relations: for every objects M, M' of $\boldsymbol{C}$, every morphism $f \in \boldsymbol{C}(M', M)$, every objects N, N' of $\boldsymbol{D}$, every morphism $g \in \boldsymbol{D}(N, N')$, and every morphism $u \in \boldsymbol{C}(M, G(N))$, one has the relation

$$\Phi_{M',N'}(G(g) \circ u \circ f) = g \circ \Phi_{M,N}(u) \circ F(f)$$

in $\boldsymbol{D}(F(M'), N')$.

If there exists an adjunction for the pair (F, G), one says that it is an *adjoint pair* of functors, or a pair of adjoint functors. One also says that F is a *left adjoint* of G, and that G is a *right adjoint* of F.

Proposition (**2.4.6**). — *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories, let* G *be a functor from $\boldsymbol{D}$ to $\boldsymbol{C}$. The following properties are equivalent:*

(i) *The functor* G *has a left adjoint;*

(ii) *For every object* M *of $\boldsymbol{C}$, the functor* $\mathrm{Hom}_{\boldsymbol{C}}(M, G(\bullet))$ *from $\boldsymbol{D}$ to $\boldsymbol{Set}$ is representable.*

Proof. — (i)⇒(ii). Let F be a functor from $\boldsymbol{C}$ to $\boldsymbol{D}$ which is a left adjoint of G and let $(\Phi_{M,N})$ be an adjunction for the pair (F, G).

Let M be an object of $\boldsymbol{C}$. Then, the bijections $\Phi_{M,N}$, for every object N of $\boldsymbol{D}$, define an isomorphism of functors from the functor $\boldsymbol{C}(M, G(\bullet))$ to the functor $\boldsymbol{D}(F(M), \bullet)$. Consequently, the object $F(M)$ of $\boldsymbol{D}$ represents the functor $\mathrm{Hom}_{\boldsymbol{C}}(M, G(\bullet))$ from $\boldsymbol{D}$ to $\boldsymbol{Set}$.

(ii)⇒(i). Assume conversely that for every object M of $\boldsymbol{C}$, the functor $\mathrm{Hom}_{\boldsymbol{C}}(M, G(\bullet))$ from $\boldsymbol{D}$ to $\boldsymbol{Set}$ is representable. For every such object M, let us *choose* an object $F(M)$ of $\boldsymbol{D}$ as well as an isomorphism of functors $\Phi_{M,\bullet}$ from $\boldsymbol{C}(M, G(\bullet))$ to $\boldsymbol{D}(F(M), \bullet)$. Let $f \colon M' \to M$ be a morphism in $\boldsymbol{C}$, let $F(f)$ be the unique morphism $f' \colon F(M) \to F(N)$ in $\boldsymbol{D}$ such that for every $u \in \mathcal{C}(\mathrm{M}, \mathrm{G}(\mathrm{N}))$, one has $\Phi_{\mathrm{M}',\mathrm{N}}(u \circ f) = \Phi_{\mathrm{M},\mathrm{N}}(u) \circ f'$. Since $\Phi_{\mathrm{M},\mathrm{N}}(u \circ \mathrm{id}_{\mathrm{M}}) = \Phi_{\mathrm{M},\mathrm{N}}(u) = \Phi_{\mathrm{M},\mathrm{N}}(u) \circ \mathrm{id}_{\mathrm{F}(\mathrm{M})}$, one has $\mathrm{F}(\mathrm{id}_{\mathrm{M}}) = \mathrm{id}_{\mathrm{F}(\mathrm{M})}$. Moreover, if $f \colon \mathrm{M}' \to \mathrm{M}$ and $g \colon \mathrm{M}'' \to \mathrm{M}'$ are morphisms in $\mathcal{C}$, then

$$\Phi_{\mathrm{M}'',\mathrm{N}}(u \circ g \circ f) = \Phi_{\mathrm{M},\mathrm{N}}(u \circ g) \circ \mathrm{F}(f) = \Phi_{\mathrm{M},\mathrm{N}}(u) \circ \mathrm{F}(g) \circ \mathrm{F}(f),$$

so that $\mathrm{F}(g \circ f) = \mathrm{F}(g) \circ \mathrm{F}(f)$. Consequently, the assignement $\mathrm{M} \mapsto \mathrm{F}(\mathrm{M})$ and $f \mapsto \mathrm{F}(f)$ is a functor, and the morphisms $\Phi_{\mathrm{M},\mathrm{N}}$ form an adjunction for the pair (F, G). In particular, G has a left adjoint. □

Example **(2.4.7)**. — Many universal constructions of algebra are particular instances of adjunctions when one of the functors is obvious.

a) Let G be the forgetful functor from the category of A-modules to the category of sets. Let F be the functor that associates to every set S the free A-module $\mathrm{A}^{(\mathrm{S})}$ on S, with basis $(\varepsilon_s)_{s \in \mathrm{S}}$. For every A-module M, every set S and and every function $f \colon \mathrm{S} \to \mathrm{M}$, there exists a unique morphism of A-modules $\varphi \colon \mathrm{A}^{(\mathrm{S})} \to \mathrm{M}$ which maps ε_s to $f(s)$ for every $s \in \mathrm{S}$. More precisely, the maps

$$\Phi_{\mathrm{S},\mathrm{M}} \colon \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}^{(\mathrm{S})}, \mathrm{M}) \to \mathrm{Fun}(\mathrm{S}, \mathrm{M}), \qquad \varphi \mapsto (s \mapsto \varphi(s))$$

define an adjunction, so that (F, G) is an adjoint pair.

b) The forgetful functor from the category of groups to the category of sets has a left adjoint which associates to every set S the free group on S.

c) Let A be a ring. The forgetful functor from the category of A-algebras to the category of sets has a left adjoint. It associates with every set S the ring of polynomials $\mathrm{A}[(\mathrm{X}_s)_{s \in \mathrm{S}}]$ with coefficients in A in the indeterminates $(\mathrm{X}_s)_{s \in \mathrm{S}}$.

Example **(2.4.8)**. — Let A and B be rings and let $f \colon \mathrm{A} \to \mathrm{B}$ be a morphism of rings. Let $\mathrm{G} \colon \boldsymbol{Mod}_{\mathrm{B}} \to \boldsymbol{Mod}_{\mathrm{A}}$ be the forgetful functor, that associates with a B-module M the associated A-module (the same underlying abelian group, with the structure of an A-module given by $a \cdot m = f(a)m$, for $a \in \mathrm{A}$ and $m \in \mathrm{M}$). In the other direction, the tensor product induces a functor $\mathrm{F} \colon \boldsymbol{Mod}_{\mathrm{A}} \to \boldsymbol{Mod}_{\mathrm{B}}$: one sets $\mathrm{F}(\mathrm{M}) = \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$ for every A-module M, and $\mathrm{F}(f) = f \otimes \mathrm{id}_{\mathrm{B}}$ for every morphism $f \colon \mathrm{M} \to \mathrm{N}$ of A-modules. For every A-module M and every B-module N, and every A-linear morphism $u \colon \mathrm{M} \to \mathrm{N}$, there exists a unique B-linear morphism $v \colon \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{N}$ such that $v(m \otimes b) = bu(m)$ for every $m \in \mathrm{M}$ and every $b \in \mathrm{B}$. (Indeed, the map $(m, b) \mapsto bu(m)$ from $\mathrm{M} \times \mathrm{B}$ to N is A-bilinear.) Set $\Phi_{\mathrm{M},\mathrm{N}}(u) = v$. The maps

$$\Phi_{\mathrm{M},\mathrm{N}} \colon \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}_{\mathrm{B}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}, \mathrm{N})$$

define an adjunction for the pair (F, G).

Exercise **(2.4.9)**. — Let $F: \mathbf{C} \to \mathbf{D}$ and $G: \mathbf{D} \to \mathbf{C}$ be functors such that the pair (F, G) is adjoint.

Let $Q = (V, E, s, t))$ be a quiver, let $\mathscr{A} = ((A_v), (f_e))$ be a Q-diagram in $\mathbf{C}$. Let $A = \varinjlim \mathscr{A}$ be a colimit of $\mathscr{A}$ and let $(f_v: A_v \to A)$ be the family of canonical maps. Prove that the family $F(\mathscr{A}) = ((F(A_v)), (F(f_e)))$ is a Q-diagram and that $(F(A), (F(f_v)))$ is a colimit of the Q-diagram $F(\mathscr{A})$. One says that F respects all colimits.

Similarly, prove that G respects all limits.

2.5. Exact sequences and complexes of modules

2.5.1. — An *exact sequence* of A-modules is a sequence $(f_n: M_n \to M_{n-1})$, indexed by $n \in \mathbf{Z}$, of morphisms of A-modules such that $\operatorname{Im}(f_{n+1}) = \operatorname{Ker}(f_n)$ for every integer n. One sometimes represents such an exact sequence by the diagram

$$\cdots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to \dots$$

If it is an eact sequence, one has in particular $f_n \circ f_{n+1} = 0$ for every integer n.

An exact sequence is said to be short if $M_n = 0$ except for (at most) three consecutive integers. One thus writes a short exact sequence as

$$0 \to N \xrightarrow{i} M \xrightarrow{p} P \to 0,$$

omitting the other null terms. The conditions for this diagram to be a short exact sequence are the following:

- The morphism i is injective;
- The image of i coincides with the kernel of p;
- The morphism p is surjective.

Consequently, the morphism i identifies N with a submodule of M, the kernel $\operatorname{Ker}(p)$ of p, and the morphism p identifies P with the Cokernel $\operatorname{Coker}(i) = M/\operatorname{Im}(i)$ of i.

It could be said that homological algebra is the science of creation and management of exact sequences. A first example is given by the following proposition.

Proposition **(2.5.2)** (Snake lemma). — *Let us consider a diagram of morphisms of* A*-modules:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & M & \xrightarrow{p} & P & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & N' & \xrightarrow{i'} & M' & \xrightarrow{p'} & P' & \longrightarrow & 0 \end{array}$$

in which the two rows are exact sequences, and the two squares are commutative, meaning that $i' \circ f = g \circ i$ and $p' \circ g = h \circ p$.

By restriction, the morphisms i and p induce morphisms $i_*\colon \mathrm{Ker}(f) \to \mathrm{Ker}(g)$ and $p_*\colon \mathrm{Ker}(g) \to \mathrm{Ker}(h)$; by passing to the quotients, the morphisms i' and p' induce morphisms $i'_*\colon \mathrm{Coker}(f) \to \mathrm{Coker}(g)$ and $p'_*\colon \mathrm{Coker}(g) \to \mathrm{Coker}(h)$. There exists a unique morphism $\partial\colon \mathrm{Ker}(h) \to \mathrm{Coker}(f)$ of A*-modules such that $\partial(p(x)) = \mathrm{cl}(y)$ for every $(x, y) \in \mathrm{M} \times \mathrm{N}'$ such that $g(x) = i'(y)$. Moreover, the diagram*

$$0 \to \mathrm{Ker}(f) \xrightarrow{i_*} \mathrm{Ker}(g) \xrightarrow{p_*} \mathrm{Ker}(h) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \mathrm{Coker}(f) \xrightarrow{i'_*} \mathrm{Coker}(g) \xrightarrow{p'_*} \mathrm{Coker}(h) \to 0$$

is an exact sequence.

Proof. — Let $x \in \mathrm{Ker}(f)$; then $g(i(x)) = i'(f(x)) = i'(0) = 0$, so that $i(x) \in \mathrm{Ker}(g)$. Similarly, let $y \in \mathrm{Ker}(g)$; one has $h(p(x)) = p'(g(x)) = p'(0)) = 0$, so that $p(x) \in \mathrm{Ker}(h)$. This shows the existence of the morphisms i_* and p_*.

Let $x' \in \mathrm{Im}(f)$ and let $x \in \mathrm{N}$ be such that $x' = f(x)$; then $i'(x') = i' \circ f(x) = g(i(x))$, $i'(x') \in \mathrm{Im}(g)$. Consequently, the kernel of the composition $\mathrm{N}' \xrightarrow{i'} \mathrm{M}' \to \mathrm{M}'/\mathrm{Im}(g) = \mathrm{Coker}(g)$ contains $\mathrm{Im}(f)$. Passing to the quotient, one obtains a morphism i'_* from $\mathrm{M}'/im(f) = \mathrm{Coker}(f)$ to $\mathrm{Coker}(g)$.

One constructs the morphism $p'_*\colon \mathrm{Coker}(g) \to \mathrm{Coker}(h)$ in the same way.

The morphism i_* is injective: let $x \in \mathrm{N}$ be such that $i_*(x) = 0$; Then $i(x) = 0$, hence $x = 0$.

Moreover, for every $x \in \mathrm{Ker}(f)$, one has $p_*(i_*(x)) = p(i(x)) = 0$, hence $i_*(x) \in \mathrm{Ker}(p_*)$. On the other hand, let $y \in \mathrm{Ker}(p_*)$; then $y \in \mathrm{Ker}(g)$ and $p(y) = 0$; since $\mathrm{Ker}(p) = \mathrm{Im}(i)$, there exists $x \in \mathrm{N}$ such that $y = i(x)$; one has $i(f(x)) = g(i(x)) = g(y) = 0$, hence $f(x)$ since i is injective; consequently, $y \in \mathrm{Im}(i_*)$. This shows that $\mathrm{Im}(i_*) = \ker(p_*)$.

We write $\mathrm{cl}(x)$ to denote the class in $\mathrm{Coker}(f)$ of an element $x \in \mathrm{N}'$, and similarly for the other two cokernels. Let $x \in \mathrm{N}'$; then $p'_*(i'_*(\mathrm{cl}(x)) = \mathrm{cl}(p(i(x))) = 0$; consequently, $\mathrm{Im}(i'_*) \subseteq \ker(p'_*)$. Let $y \in \mathrm{M}'$ be such that $\mathrm{cl}(y) \in \ker(p'_*)$; one thus has $\mathrm{cl}(p'(y)) = p'_*(\mathrm{cl}(y)) = 0$ in $\mathrm{Coker}(h)$, so that $p'(y) \in \mathrm{Im}(h)$; let then $x_1 \in \mathrm{P}$ be such that $p'(y) = h(x)$; since p is surjective, there exists $x \in \mathrm{M}$ such that $x_1 = p(x)$; one has $p'(g(x)) = h(p(x)) = h(x_1) = p'(y)$, hence $y - g(x) \in \ker(p')$; therefore, there exists $z \in \mathrm{N}'$ such that $y = g(x) + i'(z)$; this implies that $\mathrm{cl}(y) = \mathrm{cl}(i'(z)) = i'_*(\mathrm{cl}(z)) \in \mathrm{Im}(i'_*)$. We thus have shown that $\ker(p'_*) = \mathrm{Im}(i'_*)$.

Moreover, let $y \in \mathrm{Coker}(p')$, let $y' \in \mathrm{P}'$ be such that $y = \mathrm{cl}(y')$; since p' is surjective, there exists $x' \in \mathrm{M}'$ such that $y' = p'(x')$; one then has $y = \mathrm{cl}(y') = \mathrm{cl}(p'(y')) = p'_*(\mathrm{cl}(y'))$, which shows that p'_* is surjective.

It remains to construct the homomorphism ∂ and to show that $\mathrm{Im}(p_*) = \ker(\partial)$ and $\mathrm{Im}(\partial) = \ker(i'_*)$. Let Q be the submodule of $\mathrm{M} \times \mathrm{N}'$ consisting of pairs (x, y) such that $g(x) = i'(y)$. If $(x, y) \in \mathrm{Q}$, then $h(p(x)) = p'(g(x)) = p'(i'(y)) = 0$, hence $p(x) \in \ker(h)$. Let $q \colon \mathrm{Q} \to \ker(h)$ be the morphism of A-modules given by $q(x, y) = p(x)$; it is surjective. Let indeed $z \in \ker(h)$; since p is surjective, there exists $x \in \mathrm{M}$ such that $z = p(x)$; then $p'(g(x)) = h(p(x)) = h(z) = 0$, hence there exists $y \in \mathrm{N}'$ such that $g(x) = i'(y)$, and $z = q(x, y)$, as was to be shown. Consequently, there exists at more one morphism $\partial \colon \ker(h) \to \mathrm{Coker}(f)$ such that $\partial(q(x, y)) = \mathrm{cl}(y)$ for every $(x, y) \in \mathrm{Q}$. To prove the existence of the morphism ∂, it suffices to show that if $(x, y) \in \mathrm{Q}$ satisfies $q(x, y) = 0$, then $\mathrm{cl}(y) = 0$; but then, $p(x) = 0$, hence $x \in \mathrm{Im}(i)$, so that there exists $z \in \mathrm{N}$ such that $x = i(z)$; it follows that $i'(f(z)) = g(i(z)) = g(x) = i'(y)$, hence $y = f(z)$ since i' is injective; consequently, $y \in \mathrm{Im}(f)$ and $\mathrm{cl}(y) = 0$.

Let $x \in \mathrm{Ker}(g)$; then $(x, 0) \in \mathrm{Q}$, so that $\partial(p_*(x)) = \partial(q(x, y)) = \mathrm{cl}(0) = 0$; this shows that $\partial \circ p_* = 0$. Conversely, let $z \in \ker(\partial)$; let $(x, y) \in \mathrm{Q}$ be such that $q(x, y) = z$; one has $\partial(z) = \mathrm{cl}(y)$, hence $y \in \mathrm{Im}(f)$; consequently, there exists $t \in \mathrm{N}$ such that $y = f(t)$ and $g(x) = i'(y) = i'(f(t)) = g(i(t))$, so that $u = x - i(t) \in \ker(g)$;; then $z = p(x) = p(u + i(t)) = p(u) \in \mathrm{Im}(p_*)$. We have shown that $\ker(\partial) = \mathrm{Im}(p_*)$.

For every $(x, y) \in \mathrm{Q}$, one has $i'_*(\mathrm{cl}(y)) = \mathrm{cl}(i'(y)) = \mathrm{cl}(g(x)) = 0$ in $\mathrm{Coker}(g)$, so that $i'_* \circ \partial = 0$. Conversely, let $y' \in \ker(i'_*)$; let $y \in \mathrm{N}'$ be such that $y' = \mathrm{cl}(y)$; by definition, $0 = i'_*(y') = \mathrm{cl}(i'(y))$ in $\mathrm{Coker}(g)$, so that there exists $x_1 \in \mathrm{M}$ such that $i'(y) = g(x)$; one then has $(x, y) \in \mathrm{Q}$ and

$y' = \mathrm{cl}(y) = \partial(p(x))$; we thus have shown that $\mathrm{Im}(\partial) = \ker(i'_*)$ and this concludes the proof of the snake lemma. □

Corollary **(2.5.3)**. — a) *If f and h are injective, then g is injective. If f and h are surjective, then g is surjective.*

b) *If f is surjective and g is injective, then h is injective.*

c) *If g is surjective and h is injective, then f is surjective.*

Proof. — a) Assume that f and h are injective. The exact sequence given by the snake lemma begins with $0 \to 0 \xrightarrow{i_*} \ker(g) \xrightarrow{p_*} 0$. Necessarily, $\ker(g) = 0$.

If f and h are surjective, the exact sequence ends with $0 \xrightarrow{i'_*} \mathrm{Coker}(g) \xrightarrow{p'_*} 0$, so that $\mathrm{Coker}(g) = 0$ and f is surjective.

b) If f is surjective and g is injective, one has $\ker(g) = 0$ and $\mathrm{Coker}(f) = 0$. The middle of the exact sequence can thus be rewritten as $0 \xrightarrow{p_*} \ker(h) \xrightarrow{\partial} 0$, so that h est injective.

c) Finally, if g is surjective and h is injective, we have $\ker(h) = 0$, $\mathrm{Coker}(g) = 0$, hence an exact sequence $0 \xrightarrow{\partial} \mathrm{Coker}(f) \xrightarrow{i'_*} 0$, which implies that $\mathrm{Coker}(f) = 0$ and f is surjective.

□

2.6. Differential modules and their homology

2.6.1. — To construct exact sequences, it appears important to consider diagrams as in the definition but where one relaxes the conditions $\mathrm{Im}(f_{n+1}) = \mathrm{Ker}(f_n)$ of an exact sequence and only assumes the inclusions $\mathrm{Im}(f_{n+1}) \subseteq \mathrm{Ker}(f_n)$. Such diagrams are called *complexes*, but it will be technically convenient to define them as *graded differential modules.*

Definition **(2.6.2)**. — *Let* A *be a ring.*

A differential A*-module is an* A*-module* M *endowed with an endomorphism d_M such that $d_M \circ d_M = 0$.*

Let (M, d_M) *and* (N, d_N) *be differential* A*-modules. A morphism $f\colon M \to N$ is a morphism of differential modules if $d_N \circ f = f \circ d_M$.*

Let $\tilde{A} = A[T]/(T^2)$ and let ε be the class of T in $\tilde{A}$. With any differential module (M, d_M) one associates a $\tilde{A}$-module $\tilde{M}$ by setting $\tilde{M} = M$, endowed with the structure of module given by $(a + \varepsilon b) \cdot m = am + bd_M(m)$. Conversely, any

Ã-module defines a differential A-module with the same underlying A-module, and the differential being induced by the multiplication by ε.

A morphism of differential modules $f\colon \mathrm{M} \to \mathrm{N}$ is nothing but a morphism of the associated Ã-modules.

Let (M, d) and (N, d) be differential A-modules and let $f\colon \mathrm{M} \to \mathrm{N}$ be a morphism of differential modules. Then $\ker(f)$ is a differential submodule of M, and $\mathrm{Im}(f)$ is a differential submodule of N. Moreover, $\mathrm{Coker}(f)$ has a unique structure of a differential module such that the canonical surjection $\mathrm{N} \to \mathrm{Coker}(f)$ is a morphism of differential modules.

2.6.3. — Let (M, d) be a differential A-module. One associates with M the following A-modules:

- The module of *cycles*, $\mathrm{Z}(\mathrm{M}) = \ker(d)$;
- The module of boundaries, $\mathrm{B}(\mathrm{M}) = \mathrm{Im}(d)$;
- The module $\mathrm{H}(\mathrm{M}) = \mathrm{Z}(\mathrm{M})/\mathrm{B}(\mathrm{M})$ of *homologies*.

Observe that $f(\mathrm{Z}(\mathrm{M})) \subseteq \mathrm{Z}(\mathrm{N})$, and $f(\mathrm{B}(\mathrm{M})) \subseteq \mathrm{B}(\mathrm{N})$. Consequently, f induces a morphism $\mathrm{H}(f)\colon \mathrm{H}(\mathrm{M}) \to \mathrm{H}(\mathrm{N})$.

Let M, N, P be differential A-modules, let let $f\colon \mathrm{M} \to \mathrm{N}$ and $g\colon \mathrm{N} \to \mathrm{P}$ be morphisms of differential modules. Then $g \circ f$ is a morphism of differential modules and $\mathrm{H}(g \circ f) = \mathrm{H}(g) \circ \mathrm{H}(f)$.

2.6.4. — Let A be a ring. A *graded* A-*module* is an A-module M together with a family (M_n) of submodules, indexed by $\mathbf{Z}$, of which M is the direct sum. Elements of M_n are called *homogeneous* of degree n, the module M_n is called the homogeneous component of degree n of M.

The graduation is said to be bounded from below (resp. from above) if there exists an integer $m \in \mathbf{Z}$ such that $\mathrm{M}_n = 0$ for $n \leqslant m$ (resp. for $n \geqslant m$); it is bounded if it is bounded both from above and from below.

A submodule N of M is said to be *graded* if N is the direct sum of the submodules $\mathrm{N}_n = \mathrm{N} \cap \mathrm{M}_n$. If this is the case, the quotient module $\mathrm{P} = \mathrm{M}/\mathrm{N}$ admits a natural graduation such that $\mathrm{P}_n = \mathrm{M}_{\mathrm{N}}/\mathrm{N}_n$ for every integer n.

Let M, N be graded A-modules and let $f\colon \mathrm{M} \to \mathrm{N}$ be a morphism of A-modules. One says that f is *graded of degre r* if $f(\mathrm{M}_n) \subseteq \mathrm{N}_{n+r}$ for every $n \in \mathbf{Z}$. One also calls the induced morphism $f_n\colon \mathrm{M}_n \to \mathrm{N}_{n+r}$ the homogeneous component of degree n of f. If f is a graded morphism of graded A-modules, then $\mathrm{Ker}(f)$ is a graded submodule of M and $\mathrm{Im}(f)$ is a graded submodule of N, and $\mathrm{Coker}(f)$

has a natural structure of a graded module such that the canonical projection $N \to \mathrm{Coker}(f)$ is graded of degree 0.

Definition (**2.6.5**). — *Let* A *be a ring. A* graded differential A-module *is a differential* A*-module* (M, d_M) *such that* d_M *is homogeneous of some degree* r.

Let (M, d) be a graded differential A-module, let $r \in \mathbf{Z}$ be such that d has degree r. For every integer n, let M_n and $d_n \colon M_n \to M_{n+r}$ be the homogeneous components of degree n of M and d. Then $d_{n+r} \circ d_n = 0$.

Conversely, let (M_n) is a family of A-modules, let $r \in \mathbf{Z}$, and, for every n, let $d_n \colon M_n \to M_{n+r}$ be a morphism of A-modules. If $d_{n+r} \circ d_n$ for each n, then one defines a complex (M, d) of A-modules by setting $M = \bigoplus M_n$ and letting d be the unique endomorphism of M such that $d|_{M_n} = d_n$.

When $r = -1$, a graded differential A-module amounts to a diagram

$$\cdots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \dots$$

of morphisms of A-modules such that $d_n \circ d_{n+1} = 0$ for all n. One speaks of a *homological complex*, or simply a *complex*.

When $r = 1$, a graded differential A-module amounts to a diagram

$$\cdots \to M_{n-1} \xrightarrow{d_{n-1}} M_n \xrightarrow{d_n} M_{n+1} \to \dots$$

of morphisms of A-modules such that $d_n \circ d_{n-1} = 0$ for all n. One speaks of a *cohomological complex*. In this case, the custom is to indicate the grading as an upper index, as in the diagram

$$\cdots \to M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \to \dots$$

A *morphism of graded differential modules* is a morphism of differential modules which is a graded morphism of degree 0 of the underlying graded modules.

Lemma (**2.6.6**). — *Let* (M, d) *be a graded differential* A*-module of degree* r.

a) *The modules* $Z(M)$ *and* $B(M)$ *are graded submodules of* (M, d)*, and* $B_n(M) = B(M) \cap M_n = d(M_{n-r})$.

b) *The module* $H(M)$ *is a graded* A*-module in a natural way, whose homogeneous component of degree* n *is given by* $H_n(M) = Z_n(M)/B_n(M)$.

c) *Let* (N, d) *be a graded differential* A*-module of degree* r *and let* $f \colon M \to N$ *be a morphism of graded differential* A*-modules. Then the induced morphism* $H(f) \colon H(M) \to H(N)$ *is graded of degree* 0.

Proof. — For every n, let $Z_n(M) = Z(M) \cap M_n$ and $B_n(M) = B(M) \cap M_n$.

Let $x \in Z(M)$ and let (x_n) be the homogenous components of x; one has $d(x) = \sum d(x_n)$; for every n, $d(x_n) \in M_{n+r}$, hence $d(x_n) = 0$ for all n. Consequently, $x_n \in Z_n(M)$ for each n. This shows that $Z(M) = \bigoplus Z_n(M)$.

The inclusion $d(M_{n-r}) \subseteq B_n$ is obvious since d has degree r. Conversely, let $x \in B_n(M)$ and let $y \in M$ be such that $x = d(y)$. Let (y_m) be the homogeneous components of y; one has $d(y) = \sum d(y_m) = x$. Since $d(y_m) \in M_{m+r}$ and $x \in M_n$, this implies that $d(y_m) = 0$ for $m \neq n - r$ and $d(y_{n-r}) = x$. This shows that $x \in d(M_{n-r})$, so that $d(M_{n-r}) = B_n$.

Consequently, $\bigoplus B_n = \bigoplus d(M_{n-r}) = d(M) = B(M)$, so that $B(M)$ is a graded submodule of M.

Let $f: M \to N$ be a morphism of graded differential modules. Since $f(M_n) \subseteq N_n$, one has $f(Z_n(M)) \subseteq Z_n(N)$, hence $H(f)(H_n(M)) \subseteq H_n(N)$, showing that $H(f)$ is a graded morphism of degree 0. □

2.6.7. — One says that f is injective (resp. surjective) if it is injective (resp. surjective) as a morphism of A-modules.

Similarly, an exact sequence of complexes is a sequence of morphisms $(f_n: M_n \to M_{n-1})$ of complexes such that the associated sequence of A-modules is an exact sequence.

Theorem **(2.6.8)**. — *Let* (M, d_M), (N, d_N), (P, d_P) *be differential* A*-modules lying in an exact sequence*

$$0 \to M \xrightarrow{i} N \xrightarrow{p} P \to 0$$

of differential modules. Then there is a unique morphism $\partial: H(P) \to H(M)$ *of* A*-modules such that* $\partial(\mathrm{cl}(p(y))) = \mathrm{cl}(x)$ *for every pair* $(x, y) \in Z(M) \times p^{-1}(Z(P))$ *such that* $d_N(y) = i(x)$.

One has $\ker(H(i)) = \operatorname{Im}(\partial)$, $\ker(H(p)) = \operatorname{Im}(H(i))$ *and* $\ker(\partial) = \operatorname{Im}(H(p))$.

Moreover, if M, N, P *are graded differential* A*-modules whose differential have degree* r*, and if the morphisms* i *and* p *are graded of degree* 0*, then* ∂ *has degree* r.

In other words, one has an "exact triangle"

$$\begin{array}{ccc} & \mathrm{H}(\mathrm{P}) & \\ {\scriptstyle \partial}\swarrow & & \nwarrow{\scriptstyle \mathrm{H}(p)} \\ \mathrm{H}(\mathrm{M}) & \xrightarrow{\mathrm{H}(i)} & \mathrm{H}(\mathrm{N}) \end{array}$$

When M, N, P are homological complexes, one has $r = -1$ and this triangle can be rewritten as the long exact sequence

$$\cdots \to \mathrm{H}_{n+1}(\mathrm{P}) \xrightarrow{\partial} \mathrm{H}_n(\mathrm{M}) \xrightarrow{\mathrm{H}_n(i)} \mathrm{H}_n(\mathrm{N}) \xrightarrow{\mathrm{H}_n(p)} \mathrm{H}_n(\mathrm{P}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathrm{M}) \to \dots$$

When M, N, P are cohomological complexes, one has $r = 1$ and this triangle can be rewritten as the long exact sequence

$$\cdots \to \mathrm{H}_{n-1}(\mathrm{P}) \xrightarrow{\partial} \mathrm{H}_n(\mathrm{M}) \xrightarrow{\mathrm{H}_n(i)} \mathrm{H}_n(\mathrm{N}) \xrightarrow{\mathrm{H}_n(p)} \mathrm{H}_n(\mathrm{P}) \xrightarrow{\partial} \mathrm{H}_{n+1}(\mathrm{M}) \to \dots$$

Proof. — Let Q be the submodule of M × N consisting of pairs (x, y) such that $x \in \mathrm{Z}(\mathrm{M})$, $i(x) = dy$ and $p(y) \in \mathrm{Z}(\mathrm{P})$. Let $\zeta \in \mathrm{H}(\mathrm{P})$ and let $z \in \mathrm{Z}(\mathrm{P})$ be such that $\zeta = \mathrm{cl}(z)$. Since p is surjective, there exists $y \in \mathrm{N}$ such that $p(y) = z$; then $p(d(y)) = d(p(y)) = d(z) = 0$, so that $d(y) \in \ker(p)$. Consequently, there exists $x \in \mathrm{M}$ such that $d(y) = i(x)$; since $i(d(x)) = d(i(x)) = d^2(y) = 0$ and i is injective, one has $d(x) = 0$, that is, $x \in \mathrm{Z}(\mathrm{M})$. This shows that map from $\mathrm{Q} \to \mathrm{Z}(\mathrm{P})$ given by $(x, y) \mapsto p(y)$ is surjective.

As a consequence, there exists at most one morphism $\partial\colon \mathrm{H}(\mathrm{P}) \to \mathrm{H}(\mathrm{M})$ such that $\partial(\mathrm{cl}(p(y))) = \mathrm{cl}(x)$ for every $(x, y) \in \mathrm{Q}$. Noreover, to prove that such a morphism exists, it suffices to show that for every $(x, y) \in \mathrm{Q}$ such that $p(y) \in \mathrm{B}(\mathrm{P})$, one has $x \in \mathrm{B}(\mathrm{M})$. So let (x, y) be such a pair; let $z' \in \mathrm{P}$ be such that $p(y) = d(z')$ and let $y' \in \mathrm{N}$ be such that $z' = p(y')$; one has $p(y) = d(z') = d(p(y')) = p(d(y'))$, hence there exists $x' \in \mathrm{M}$ such that $y = d(y') + i(x')$; then $i(x) = d(y) = d(i(x')) = i(d(x'))$, hence $x = d(x'))$ since i is injective; consequently, $x \in \mathrm{B}(\mathrm{M})$ as was to be shown.

Let us show that ∂ is homogeneous of degree r. Let $\zeta \in \mathrm{H}_n(\mathrm{P})$. Let us revisit the argument showing that the map $(x, y) \mapsto \mathrm{cl}(p(y))$ from Q to H(P) is surjective. Let $z \in \mathrm{Z}_n(\mathrm{P})$ be such that $\zeta = \mathrm{cl}(z)$. Since p is surjective, there exists $y \in \mathrm{N}$ such that $z = p(y)$; let (y_m) be the homogeneous components of y; one has $y = \sum y_m$, hence $z = p(y) = \sum p(y_m)$; since z is homogeneous of degree n, and $p(y_m)$ is homogeneous of degree m, one has $p(y_m) = 0$ for $m \neq n$; consequently, $z = p(y_n)$. Since $p(d(y_n)) = d(p(y_n)) = 0$, there exists

$x \in M$ such that $d(y_n) = i(x)$; let (x_m) be the homogeneous components of M; one has $d(y_n) = \sum i(x_m)$. Since $d(y_n)$ is homogeneous of degree $n + r$ and $i(x_m)$ is homogeneous of degree m, one has $d(y_n) = i(x_{n+r})$. Moreover, $i(d(x_{n+r})) = d(i(x_{n+r})) = d^2(y_n) = 0$, so that $d(x_{n+r}) = 0$ because i is injective. Then $(x_{n+r}, y_n) \in Q$, $\zeta = \mathrm{cl}(p(y_n))$, hence $\partial(\zeta) = \mathrm{cl}(x_{n+r}) \in H_{n+r}(P)$, which shows that ∂ is homogeneous of degree r, as claimed.

To shorten notation, we write $i_* = H(i)$ and $p_* = H(p)$; let us show that $\mathrm{Im}(i_*) = \ker(p_*)$. Since $p \circ i = 0$, one has $p_* \circ i_* = 0$. Conversely, let $\eta \in \ker(p_*)$ and let $y \in H(N)$ be such that $\eta = \mathrm{cl}(y)$; one has $p_*(\eta) = \mathrm{cl}(p(y))$, hence $p(y) \in B(P)$. Let $z' \in P$ be such that $p(y) = d(z')$ and let $y' \in N$ be such that $z' = p(y')$; one has $p(y) = d(z') = d(p(y')) = p(d(y'))$, so that there exists $x \in M$ such that $y - d(y') = i(x)$. Then $\eta = \mathrm{cl}(y) = \mathrm{cl}(i(x)) = i_*(\mathrm{cl}(x))$, hence $\eta \in \mathrm{Im}(i_*)$.

Let us now show that $\mathrm{Im}(p_*) = \ker(\partial)$. Let $\zeta \in \mathrm{Im}(p_*)$; let $\eta \in H(N)$ be such that $\zeta = p_*(\eta)$ and let $y \in Z(N)$ be such that $\zeta = \mathrm{cl}(y)$. Then $d(y) = i(0)$, so that $(0, x) \in Q$. One thus has $\partial(\zeta) = \mathrm{cl}(0) = 0$. Conversely, let $\zeta \in \ker(\partial)$. Let $(x, y) \in Q$ be such that $\mathrm{cl}(p(y)) = \zeta$. One has $\mathrm{cl}(x) = \partial(\zeta) = 0$ in $H(M)$ so that $x \in B(M)$. Consequently, there exists $x' \in M$ such that $x = d(x')$; let $y' = y - i(x')$; one has $p(y') = p(y)$ and $d(y') = d(y) - d(i(x')) = d(y) - i(d(x')) = 0$, so that $y' \in Z(N)$. This implies that $\zeta = \mathrm{cl}(p(y')) = p_*(\mathrm{cl}(y')) \in \mathrm{Im}(p_*)$.

Let us finally show that $\mathrm{Im}(\partial) = \ker(i_*)$. Let $(x, y) \in Q$; one has $\partial(\mathrm{cl}(p(y)) = \mathrm{cl}(x)$, hence $i_*(\partial(\mathrm{cl}(p(y)))) = \mathrm{cl}(i(x)) = \mathrm{cl}(d(y)) = d(\mathrm{cl}(y)) = 0$. Conversely, let $\xi \in \ker(i_*)$ and let $x \in Z(N)$ be such that $\xi = \mathrm{cl}(x)$. Since $i_*(\xi) = \mathrm{cl}(i(x))$, there exists $y \in N$ such that $d(y) = i(x)$. Then $d(p(y)) = p(d(y)) = p(i(x)) = 0$, so that $p(y) \in N$. This implies that $(x, y) \in Q$ and that $\partial(\mathrm{cl}(p(y)) = \xi$, so that $\xi \in \mathrm{Im}(\partial)$. □

Remark **(2.6.9)**. — Let M, M', N, N', P, P' be differential modules and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & N & \xrightarrow{p} & P & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & M' & \xrightarrow{i'} & N' & \xrightarrow{p'} & P' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of differential modules whose two rows are exact sequences. Then the morphism $\partial\colon H(P) \to H(M)$ and $\partial'\colon H(P') \to H(M')$ satisfy

$$\partial' \circ H(h) = H(f) \circ \partial.$$

Let indeed $(x, y) \in Z(M) \times p^{-1}(Z(P))$ such that $d_N(y) = i(x)$. The definition of ∂ thus implies that

$$H(f) \circ \partial(\mathrm{cl}(p(y)) = H(f)(\mathrm{cl}(x)) = \mathrm{cl}(f(x)).$$

On the other hand, one has $f(x) \in Z(M')$, since $d_{M'}(f(x)) = f(d_M(x)) = 0$, $(p')(g(y)) \in Z(P')$, since $d_{P'}(p'(g(y)) = d_{P'}(h(p(y)) = h(d_P(p(y)) = h(0) = 0$. Moreover, $d_{N'}(g(y)) = g(d_N(y)) = g(i(x)) = i'(f(x))$, and it follows from the definition of ∂' that

$$\partial' \circ H(h)(\mathrm{cl}(p(y)) = \partial'(\mathrm{cl}(h(p(y)))) = \partial'(\mathrm{cl}(p'(g(y)))) = \mathrm{cl}(f(x)).$$

This shows that $\partial' \circ H(h) = H(f) \circ \partial$, as claimed.

Definition **(2.6.10)**. — *Let* (M, d_M) *and* (N, d_N) *be differential modules. Let* f, g *be morphisms of differential modules from* M *to* N. *An* homotopy *from* f *to* g *is an* A*-linear morphism* $u\colon M \to N$ *such that* $g - f = d_N \circ u + u \circ d_M$. *One says that* f *and* g *are homotopic if there exists a homotopy from* f *to* g.

Lemma **(2.6.11)**. — *Assume that* f *and* g *are homotopic. Then* $H(f) = H(g)$.

Proof. — Let $\xi \in H(M)$ and let $x \in Z(M)$ be such that $\xi = \mathrm{cl}(x)$. Then $f_*(\xi) = \mathrm{cl}(f(x))$ and $g_*(\xi) = \mathrm{cl}(g(x))$, hence

$$g_*(\xi) - f_*(\xi) = \mathrm{cl}(g(x) - f(x)) = \mathrm{cl}(d(u(x))) + \mathrm{cl}(u(d(x)) = 0$$

since $d(x) = 0$ and $d(u(x)) \in B(N)$. □

2.7. Projective modules and projective resolutions

Definition **(2.7.1)**. — *Let* A *be a ring and let* P *be an* A*-module. One says that* M *is* projective *if every surjective homomorphism* $p\colon M \to P$ *has a section, that is, there exists a morphism* $s\colon P \to M$ *such that* $p \circ s = \mathrm{id}_P$.

Proposition **(2.7.2)**. — *Let* A *be a ring and let* P *be an* A*-module. The following properties are equivalent.*

(i) *The* A*-module* P *is projective;*

(ii) *For every surjective morphism of* A*-modules* $p\colon M \to N$ *and every morphism* $f\colon P \to N$, *there exists a morphism* $\varphi\colon P \to M$ *such that* $f = p \circ \varphi$;

(iii) *The module* P *is a direct summand of a free* A*-module: there exists an* A*-module* Q *such that* $P \oplus Q$ *is a free* A*-module.*

Moreover, if M *is projective and finitely generated, then* M *is a direct summand of a free finitely generated* A*-module.*

In particular, a free A*-module is projective.*

Proof. — (i)⇒(ii). Let us assume that P is projective. Let $p\colon M \to N$ be a surjective morphism of A-modules and let $f\colon P \to N$ be a morphism. Let Q be the submodule of $P \times M$ consisting of pairs (x, y) such that $f(x) = p(y)$ and let $q\colon Q \to P$ be the morphism induced by the first projection. For every $x \in P$, there exists $y \in M$ such that $p(y) = f(x)$, because p is surjective; consequently, $(x, y) \in Q$, $q(x, y) = x$ and q is surjective. Since P is a projective A-module, there exists an A-morphism $s\colon P \to Q$ such that $q \circ s = \mathrm{id}_P$; for $x \in P$, write $s(x) = (x, \varphi(x))$. Then φ is a morphism from P to M; for every $x \in P$, one has $(x, \varphi(x)) \in Q$, hence $p(\varphi(x)) = f(x)$.

(ii)⇒(i). Indeed, the property of the definition of a projective module is the particular case of (ii) where $N = P$ and $f = \mathrm{id}_P$.

(i)⇒(iii). Let F be a free A-module and let $p\colon F \to P$ be a surjective homomorphism; if P is finitely generated, let us choose F to be finitely generated too. Let $r\colon P \to F$ be a section of p and let $F_1 = r(P)$. Since r is injective, P is isomorphic to F_2. Let $F_2 = \ker(p)$; this is a submodule of F. Let us check that $F = F_1 \oplus F_2$. For every $x \in F$, one has $x = r(p(x)) + (x - r(p(x)))$; by definition, $r(p(x)) \in F_1$, while $p(x - r(p(x))) = p(x) - (p \circ r)(p(x)) = p(x) - p(x) = 0$, so that $x - r(p(x)) \in F_2$; consequently, $F = F_1 + F_2$. Let moreover $x \in F_1 \cap F_2$. Then there exists $y \in P$ such that $x = r(y)$ and $p(x) = 0$; one thus has $y = p(r(y)) = p(x) = 0$, hence $x = 0$. This shows that (i)⇒(iii), as well as the two additional assertions.

(iii)⇒(i). Let $p\colon M \to P$ be a surjective morphism of A-modules. Let Q be a A-module such that $P \oplus Q$ is a free A-module. Let $(e_i)_{i \in I}$ be a basis of $P \oplus Q$; for every i, write $e_i = (x_i, y_i)$. For every $i \in I$, let us choose an element $z_i \in M$ such that $p(z_i) = x_i$. Since (e_i) is a basis of $P \oplus Q$, there exists a unique morphism $f\colon P \oplus Q \to M$ such that $f(e_i) = z_i$ for every i. Let $x \in P$, let (a_i) be the coordinates of $(x, 0) \in P \oplus Q$ in the basis (e_i). One has $x = \sum a_i x_i$ and $0 = \sum a_i y_i$, hence $f(x, 0) = \sum a_i z_i$ and $p(f(x, 0)) = \sum a_i x_i = x$. This shows that the map $r\colon x \mapsto f(x, 0)$ is a morphism from P to M such that $p \circ r = \mathrm{id}_P$. Consequently, P is projective. □

Corollary **(2.7.3)**. — *A direct sum of projective* A*-modules is projective.*

Theorem **(2.7.4)** (Kaplansky). — *Let* A *be a* local *ring. Every projective* A*-module is free.*

Proof. — Let M be a projective A-module. We only prove the proposition under the additional assumption that M is finitely generated. Let $\mathfrak{m}$ be the maximal ideal of A and let $k = A/\mathfrak{m}$ be its residue field. Then $M/\mathfrak{m}M$ is a finitely generated vector space over the field k; let $(x_1, \dots, x_n)$ be a family of elements of M whose classes modulo $\mathfrak{m}M$ form a basis of that vector space. Let us show that $(x_1, \dots, x_n)$ is a basis of M. Letting $p\colon A^n \to M$ be the morphism given by $p(a_1, \dots, a_n) = a_1x_1 + \dots + a_nx_n$, we need to prove that p is an isomorphism.

Let N be the image of p, that is, the submodule of M generated by $(x_1, \dots, x_n)$. By construction, one has $M = N + \mathfrak{m}M$, hence the quotient A-module M/N satisfies $M/N = \mathfrak{m}(M/N)$. By Nakayama's lemma, one thus has $M/N = 0$, hence $N = M$: the morphism p is surjective and the family $(x_1, \dots, x_n)$ generates M.

Since M is projective, there exists a morphism $r\colon M \to A^n$ such that $p \circ r = \mathrm{id}_M$. Let $M' = r(M)$ and $N = \ker(p)$; as shown in the proof of proposition 2.7.2, one has $A^n = M' \oplus N$; in particular, N is isomorphic to a quotient of A^n, hence is finitely generated. One has $k^n \simeq A^n/\mathfrak{m}A^n \simeq (M'/\mathfrak{m}M') \oplus (N/\mathfrak{m}N)$. By construction, M′ is isomorphic to M, hence $M'/\mathfrak{m}M'$ is an n-dimensional vector space over k. This implies that $N/\mathfrak{m}N = 0$, hence $N = \mathfrak{m}N$; by Nakayama's lemma, one has $N = 0$ hence p is injective. This concludes the proof. □

Definition **(2.7.5)**. — *Let* A *be a ring and let* M *be an* A*-module. A projective (resp. free) resolution of* M *is a homological complex* (P, d) *such that* P_n *is projective (resp. free) for every* n*,* $P_n = 0$ *for* $n < 0$*, together with a morphism* $p\colon P_0 \to M$*, such that the diagram*

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{p} M \to 0$$

is exact.

Theorem **(2.7.6)**. — a) *Every module has a free resolution;*

b) *If* A *is a noetherian ring and* M *is finitely generated* A*-module, then there exists a free resolution* (P, d) *of* M *such that* P_n *is finitely generated for every* n*;*

c) *Let* (P, d, p) *and* (P', d', p') *be projective resolutions of modules* M *and* M′*, and let* $f\colon M \to M'$ *be an* A*-morphism. There exists a morphism of graded differential modules* $\varphi\colon P \to P'$ *such that* $p' \circ \varphi_0 = f \circ p$*;*

d) *Two morphisms* φ *and* ψ *of graded differential modules from* P *to* P′ *such that* $p' \circ \varphi_0 = p' \circ \psi_0 = f \circ p$ *are homotopic.*

Proof. — We prove *a*) and *b*) at the same time. Let P_0 be a free A-module together with a surjective homomorphism $p\colon P_0 \to M$; if M is finitely generated, we choose P_0 to be finitely generated too. Let $P'_0 = \ker(p)$; if M is finitely generated and A is noetherian, then P_1 s finitely generated. We then choose a free A-module P_1 together with a morphism $d_1\colon P_1 \to P_0$ whose image is P'_0; in the "finitely generated case", we choose P_1 to be finitely generated. By induction, we construct the desired homological complex.

c) Since P_0 is projective and $p'\colon P'_0 \to M'$ is surjective; applying property (ii) of proposition 2.7.2 to the morphism $f \circ p\colon P_0 \to M'$, we conclude that there exists a morphism $\varphi_0\colon P_0 \to P'_0$ such that $p' \circ \varphi_0 = f \circ p$.

In particular, one has $p' \circ \varphi_0 \circ d_1 = f \circ p \circ d_1 = 0$ and $\operatorname{Im}(\varphi_0 \circ d_1) \subseteq \ker(p') = \operatorname{Im}(d'_1)$. Applying property (ii) of proposition 2.7.2 to the projective module P_1, to the surjective morphism from P'_1 to $\operatorname{Im}(d'_1)$ deduced from d'_1, and to the morphism $\varphi_0 \circ d_1\colon P_1 \to \operatorname{Im}(d'_1)$, there exists a morphism $\varphi_1\colon P_1 \to P'_1$ such that $d'_1 \circ \varphi_1 = \varphi_0 \circ d_1$.

By induction on n, we construct $\varphi_n\colon P_n \to P'_n$ such that $d'_n \circ \varphi_n = \varphi_{n-1} \circ d_{n-1}$ if $n \geqslant 1$. Then the morphism $\varphi\colon P \to P'$ which restricts to φ_n on P_n is a graded morphism of differential modules, and one has $f \circ p = p' \circ \varphi_0$.

d) Let φ and ψ be morphisms of graded differential modules such that $f \circ p = p' \circ \varphi_0 = p' \circ \psi_0$. For every n, let $\alpha_n = \psi_n - \varphi_n$; this is a morphism of A-modules from P_n to P'_n. Set $u_n = 0$ for $n < 0$.

One has $p' \circ \alpha_0 = p' \circ \psi_0 - p' \circ \varphi_0 = 0$, hence $\operatorname{Im}(\alpha_0) \subseteq \ker(p') = \operatorname{Im}(d'_1)$. Applying property (ii) of proposition 2.7.2, there exists a morphism $u_0\colon P_0 \to P'_1$ of A-modules such that $\alpha_0 = d'_1 \circ u_0$.

One has

$$d'_1 \circ u_0 \circ d_1 = \alpha_0 \circ d_1 = d'_1 \circ \alpha_1,$$

hence the image of the morphism $\alpha_1 - u_0 \circ d_1\colon P_1 \to P'_1$ is contained in $\ker(d'_1) = \operatorname{Im}(d'_2)$. Since P_1 is projective, there exists a morphism $u_1\colon P_1 \to P'_2$ such that $\alpha_1 - u_0 \circ d_1 = d'_2 \circ u_1$.

Assume that there exists, for each integer $m < n$, a morphism $u_m\colon P_m \to P'_{m+1}$ such that $\alpha_m = d'_{m+1} \circ u_m + u_{m-1} \circ d_m$, In particular, $\alpha_{n-1} = d'_n \circ u_{n-1} + u_{n-2} \circ d_{n-1}$, so that

$$d'_n \circ u_{n-1} \circ d_n = (\alpha_{n-1} - u_{n-2} \circ d_{n-1}) \circ d_n = \alpha_{n-1} \circ d_n = d'_n \circ \alpha_n.$$

As a consequence, the image of the morphism $\alpha_n - u_{n-1} \circ d_n \colon P_n \to P'_n$ is contained in $\ker(d'_n) = \operatorname{Im}(d'_{n+1})$. Since P_n is a projective module, there exists a morphism $u_n \colon P_n \to P'_{n+1}$ such that $\alpha_n = u_{n-1} \circ d_n + d'_{n+1} \circ u_n$.

By induction, this shows the existence of a graded morphism $u \colon P \to P'$ of graded degree 1 such that $\alpha = u \circ d + d' \circ u$. This is the required homotopy. □

2.8. Injective modules and injective resolutions

Definition **(2.8.1)**. — *Let* A *be a ring and let* I *be an* A*-module. One says that* I *is* injective *if every injective homomorphism* $i \colon I \to M$ *has a retraction, that is, there exists a morphism* $r \colon M \to I$ *such that* $r \circ i = \mathrm{id}_I$.

Proposition **(2.8.2)**. — *Let* A *be a ring and let* I *be an* A*-module. The following properties are equivalent.*

(i) *The* A*-module* I *is injective;*

(ii) *For every injective morphism of* A*-modules* $i \colon M \to N$ *and every morphism* $f \colon M \to I$, *there exists a morphism* $\varphi \colon N \to I$ *such that* $f = \varphi \circ i$;

(iii) *For every ideal* J *of* A *and every morphism* $f \colon J \to I$, *there exists an element* $x \in I$ *such that* $f(a) = ax$ *for every* $a \in J$.

Proof. — (i)⇒(ii). Let us assume that I is injective. Let $i \colon M \to N$ be an injective morphism of A-modules and let $f \colon M \to I$ be a morphism. Let Q be the submodule of $I \times N$ consisting of pairs of the form $(f(z), -i(z))$, for $z \in M$; let us write $[x, y]$ for the class in $(I \times N)/Q$ of an element $(x, y) \in I \times N$. The morphism $x \mapsto [x, 0]$ from I to $(I \times N)/Q$ is injective; indeed, if $(x, 0) \in Q$, then there exists $z \in M$ such that $x = f(z)$ and $i(z) = 0$; since i is injective, one has $z = 0$, hence $x = 0$. Since I is an injective module, there exists a morphism $g \colon (I \times N)/Q \to I$ such that $g([x, 0]) = x$ for every $x \in I$. Let $\varphi \colon N \to I$ be the morphism given by $\varphi(y) = g([0, y])$. For every $z \in M$, one has

$$\varphi(i(z)) = g([0, i(z)]) = g([f(z), 0]) - g([f(z), -i(z)]) = f(z),$$

hence $f = \varphi \circ i$.

(iii) is a particular case of (ii), where $i \colon M \to N$ is the injection of the ideal J into the ring A.

(iii)⇒(i). Let $f \colon I \to M$ be an injective morphism and let us show that there exists a morphism $r \colon M \to I$ such that $r \circ f = \mathrm{id}_I$. Let $\mathscr{F}$ be the set of all pairs (N, g), where N is a submodule of M containing $f(I)$ and $g \colon N \to I$ is a morphism of A-modules such that $g \circ f = \mathrm{id}_{\mathrm{I}}$. We order $\mathscr{F}$ by decreeing that $(\mathrm{N}, g) \prec (\mathrm{N}', g')$ if $\mathrm{N} \subseteq \mathrm{N}'$ and $g'|_{\mathrm{N}} = g$. Since f is injective, it induces an isomorphism from I onto its image $f(\mathrm{I})$; if $g_0 \colon f(\mathrm{I}) \to \mathrm{I}$ denotes the inverse isomorphism, then $(f(\mathrm{I}), g_0)$ is the unique minimal element of $\mathscr{F}$.

Let us show that the partially ordered set $\mathscr{F}$ is inductive. Let indeed $(\mathrm{N}_\alpha, g_\alpha)$ be a totally ordered family of elements of $\mathscr{F}$. Let $\mathrm{N}' = f(\mathrm{I}) \cup \bigcup_\alpha \mathrm{N}_\alpha$; this is a submodule of M. Moreover there exists a unique morphism $g \colon \mathrm{N}' \to \mathrm{I}$ such that $g|_{\mathrm{N}_\alpha} = g_\alpha$ for every α and $g|_{f(\mathrm{I})} = g_0$. The pair (N, g) belongs to $\mathscr{F}$ and is an upper bound of the family $(\mathrm{N}_\alpha, g_\alpha)$.

By Zorn's lemma, the set $\mathscr{F}$ has a maximal element (N, g). Let us prove by contradiction that $\mathrm{N} = \mathrm{M}$. Otherwise, let $m \in \mathrm{M} - \mathrm{N}$, let $\mathrm{N}' = \mathrm{N} + \mathrm{A}m$ and let $\mathrm{J} = \{a \in \mathrm{A}\,;\, am \in \mathrm{N}\}$. Let $i \colon \mathrm{J} \to \mathrm{I}$ be the morphism given by $i(a) = g(am)$ for $a \in \mathrm{J}$; by assumption, there exists an element $z \in \mathrm{I}$ such that $i(a) = az$ for every $a \in \mathrm{J}$. Let $x \in \mathrm{N}$ and $a \in \mathrm{J}$ be such that $x + am = 0$; one then has $g(x) = -g(am) = -i(a) = -az$, so that $g(x) + az = 0$. It follows that there exists a unique morphism $g' \colon \mathrm{N}' \to \mathrm{I}$ such that $g'(x + am) = g(x) + az$ for every $x \in \mathrm{N}$ and every $a \in \mathrm{J}$. The pair (N', g') is an element of $\mathscr{F}$ which contradicts the hypothesis that (N, g) is a maximal element. Consequently, $\mathrm{N} = \mathrm{M}$ and $g \colon \mathrm{M} \to \mathrm{I}$ is a morphism of A-modules such that $g \circ f = \mathrm{id}_{\mathrm{I}}$. This concludes the proof of the proposition. □

Corollary (**2.8.3**). — *Products of injective* A*-modules are injective.*

Proof. — Let $(\mathrm{M}_i)_{i \in \mathrm{I}}$ be a family of injective A-modules; let $\mathrm{M} = \prod_i \mathrm{M}_i$; for every i, let $p_i \colon \mathrm{M} \to \mathrm{M}_i$ be the projection of index i. Let J be an ideal of A, let $f \colon \mathrm{J} \to \mathrm{M}$ be a morphism. Then $p_i \circ f$ is a morphism from J to M_i, hence there exists an element $x_i \in \mathrm{M}_i$ such that $p_i(f(a)) = ax_i$ for every $a \in \mathrm{J}$. Let $x = (x_i)$; one has $f(a) = (p_i(f(a)) = (ax_i) = ax$ for every $a \in \mathrm{J}$. This proves that M is an injective module. □

Corollary (**2.8.4**). — *If* A *is a principal ideal domain, then an* A*-module is injective if and only if it is divisible. In particular,* $\mathbf{Q}/\mathbf{Z}$ *is an injective* $\mathbf{Z}$*-module.*

Proof. — Let M be an injective A-module, let $m \in \mathrm{M}$, let $a \in \mathrm{A}$ be any non-zero element. Let $f \colon (a) \to \mathrm{M}$ be the morphism given by $f(ab) = bm$, for $b \in \mathrm{B}$. Since M is injective, there exists an element $x \in \mathrm{M}$ such that $f(ab) = abx$ for every $b \in \mathrm{B}$; in particular, $f(a) = m = ax$. This shows that M is divisible.

Conversely, let M be a divisible A-module and let us prove, assuming that A is a principal ideal domain, that M is an injective module. Let J be an ideal of A, let $f\colon J \to M$ be a morphism of A-modules. Since A is a PID, there exists $a \in A$ such that $J = (a)$. If $a = 0$, then one can set $m = 0$. Let $m = f(a)$ and let $x \in M$ be such that $m = ax$; for every $b \in A$, one has $f(ab) = bf(a) = bm = abm$. By proposition 2.8.2, this shows that M is an injective A-module.

Since the $\mathbf{Z}$-module $\mathbf{Q}$ is divisible, so is its quotient $\mathbf{Q}/\mathbf{Z}$. The ring $\mathbf{Z}$ being a PID, this implies that $\mathbf{Q}/\mathbf{Z}$ is an injective $\mathbf{Z}$-module. □

2.8.5. — For every A-module M, one writes $M^* = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$, with its structure of A-module given by $a \cdot \varphi = (x \mapsto \varphi(ax))$ for every $a \in A$, $\varphi \in M^*$ and $x \in M$.

Lemma **(2.8.6)**. — *Let M be an A-module.*

a) *For every non-zero $x \in M$, there exists $\varphi \in M^*$ such that $\varphi(x) \neq 0$.*
b) *If M is a free A-module, then M^* is an injective A-module.*

Proof. — a) Let $J = \{a \in \mathbf{Z}\,;\, ax = 0\}$ and let n be the positive generator of this ideal, so that $\mathbf{Z}x \simeq \mathbf{Z}/n\mathbf{Z}$; since $x \neq 0$, one has $n = 0$ or $n \geqslant 2$. Let then $f\colon \mathbf{Z}x \to \mathbf{Q}/\mathbf{Z}$ given by $f(ax) = \frac{1}{2}a \pmod{\mathbf{Z}}$ if $n = 0$, and by $f(ax) = \frac{1}{n}a \pmod{\mathbf{Z}}$ if $n \geqslant 2$; one has $f(x) \neq 0$ by construction. Since $\mathbf{Q}/\mathbf{Z}$ is an injective $\mathbf{Z}$-module, there exists a morphism of abelian groups $\varphi\colon M \to \mathbf{Q}/\mathbf{Z}$ such that $\varphi|_{\mathbf{Z}x} = f$. One has $\varphi \in M^*$ and $\varphi(x) = f(x) \neq 0$.

b) We first show that A^* is an injective A-module. Let J be an ideal of A and let $f\colon J \to A^*$ be a morphism. For every $x \in J$, $f(x)$ is an additive map from A to $\mathbf{Q}/\mathbf{Z}$; let $\tilde f(x) = f(x)(1)$. This defines a morphism $\tilde f\colon J \to \mathbf{Q}/\mathbf{Z}$ of abelian groups. Since $\mathbf{Q}/\mathbf{Z}$ is an injective $\mathbf{Z}$-module, there exists a morphism $\tilde g\colon A \to \mathbf{Q}/\mathbf{Z}$ such that $\tilde g|_J = \tilde f$. For every $x \in A$, let $g(x)$ be the element $y \mapsto \tilde g(xy)$ of A^*; the map $g\colon A \to A^*$ is additive. It is in fact A-linear since for every $a, x, y \in A$, one has $g(ax)(y) = \tilde g(axy)$ and $(a \cdot g)(x)(y) = g(x)(ay) = \tilde g(axy)$. Let us show that $g|_J = f$: let $x \in J$ and $y \in A$; one has $g(x)(y) = \tilde g(xy) = \tilde f(xy)$ since $xy \in J$; consequently, $g(x)(y) = f(xy)(1) = f(x)(y)$ because f is A-linear; this shows that $g(x) = f(x)$. By proposition 2.8.2, we thus have proved that A^* is an injective A-module.

Let now M be a free A-module. It is isomorphic to a direct sum $A^{(I)}$ of copies of A, hence $M^* \simeq (A^*)^I$ is a product of copies of A^*. By corollary 2.8.3, M^* is an injective module.

□

Proposition (**2.8.7**). — *Let* M *be an* A*-module. There exists an injective* A*-module* I *and an injective morphism* $i\colon \mathrm{M} \to \mathrm{I}$.

Proof. — Let F be a free A-module and let $p\colon \mathrm{F} \to \mathrm{M}^*$ be a surjective morphism of A-modules. Then F^* is an injective A-module. Let $p^*\colon (\mathrm{M}^*)^* \to \mathrm{F}^*$ be the map given by $\varphi \mapsto \varphi \circ p$; it is an injective morphism of A-modules, because p is surjective.

For every $x \in \mathrm{M}$, let $j(x) \in (\mathrm{M}^*)^*$ be the map from M^* to $\mathbf{Q}/\mathbf{Z}$ given by $\varphi \mapsto \varphi(x)$; this defines a morphism of A-modules $j\colon \mathrm{M} \to (\mathrm{M}^*)^*$. Let $x \in \ker(j)$; this means that $\varphi(x) = 0$ for every $\varphi \in \mathrm{M}^*$. It thus follows from lemma 2.8.6 that $x = 0$. Consequently, j is injective.

The composition $p \circ j\colon \mathrm{M} \to \mathrm{F}^*$ is an injective morphism from M into an injective A-module, hence the proposition. □

Definition (**2.8.8**). — *Let* A *be a ring and let* M *be an* A*-module. An injective resolution of* M *is a cohomological complex* (I, d) *such that* I_n *is injective for every* n*,* $\mathrm{I}_n = 0$ *for* $n < 0$*, together with an injective morphism* $i\colon \mathrm{M} \to \mathrm{I}_0$*, such that the diagram*

$$0 \to \mathrm{M} \xrightarrow{i} \mathrm{I}_0 \xrightarrow{d_0} \mathrm{I}_1 \xrightarrow{d_1} \mathrm{I}_2 \to \dots$$

is exact.

Theorem (**2.8.9**). — a) *Every module has an injective resolution;*

b) *Let* (I, d, i) *and* (I', d', i') *be injective resolutions of modules* M *and* M′*, and let* $f\colon \mathrm{M} \to \mathrm{M}'$ *be an* A*-morphism. There exists a morphism of graded differential modules* $\varphi\colon \mathrm{I} \to \mathrm{I}'$ *such that* $\varphi_0 \circ i' = i \circ f$.

c) *Two morphisms* φ *and* ψ *of graded differential modules from* P *to* P′ *such that* $\varphi_0 \circ i' = \psi_0 \circ i' = i \circ f$ *are homotopic.*

Proof. — The proof is absolutely analogous to the proof of properties *a*), *c*), and *d*) of theorem 2.7.6. □

2.9. Abelian categories

The theory of *abelian categories* abstracts the main properties of modules over a ring within the framework of category theory.

2.9.1. Preadditive categories. — Let $\mathcal{C}$ be a category. One says that $\mathcal{C}$ is a preadditive category if for every objects M, N of $\mathcal{C}$, the set $\mathcal{C}(\mathrm{M},\mathrm{N})$ is endowed with the structure of an abelian group such that for every three objects M, N, P of $\mathcal{C}$, the composition map $\mathcal{C}(\mathrm{M},\mathrm{N}) \times \mathcal{C}(\mathrm{N},\mathrm{P}) \to \mathcal{C}(\mathrm{M},\mathrm{P})$ is bilinear.

Lemma (**2.9.2**). — *If* $(\mathrm{M},(p_i))$ *and* $(\mathrm{M}',(p'_i))$ *are products of the family* (M_i), *there exists a unique isomorphism* $f\colon \mathrm{M}' \to \mathrm{M}$ *such that* $p'_i = p_i \circ f$ *for every* i.

Proof. — Since M is a product, there exists a unique morphism $f\colon \mathrm{M}' \to \mathrm{M}$ such that $p'_i = p_i \circ f$ for every i. Since M′ is a product, there exists a morphism $g\colon \mathrm{M} \to \mathrm{M}'$ such that $p_i = p'_i \circ g$ for every i. Then $f \circ g \in \mathcal{C}(\mathrm{M},\mathrm{M})$ and $p_i \circ f \circ g = p'_i \circ g = p_i = p_i \circ \mathrm{id}_\mathrm{M}$ for every i; since M is a product, one thus has $f \circ g = \mathrm{id}_\mathrm{M}$. Reversing the rôles of M and M′, one proves that $g \circ f = \mathrm{id}_{\mathrm{M}'}$. This shows that f is an isomorphism. □

2.9.3. — Let $(\mathrm{M}_i)_{i\in\mathrm{I}}$ be a family of objects of the category $\mathcal{C}$. If it exists, one denotes by $\prod_{i\in\mathrm{I}} \mathrm{M}_i$ (resp. by $\bigoplus_{i\in\mathrm{I}} \mathrm{M}_i$) the product (resp. the coproduct) of the family (M_i).

One says that the category $\mathcal{C}$ admits products (resp. finite products) if every family (resp. every finite family) of objects of $\mathcal{C}$ has a product.

One says that the category $\mathcal{C}$ admits coproducts (resp. finite coproducts) if every family (resp. every finite family) of objects of $\mathcal{C}$ has a coproduct.

Lemma (**2.9.4**). — *Let* $\mathcal{C}$ *be a preadditive category. Let* $(\mathrm{M}_i)_{i\in\mathrm{I}}$ *be a* finite *family of objects of* $\mathcal{C}$ *and let* $(\mathrm{M},(p_i))$ *be a product of this family. There exists a family* (q_i), *where* $q_i \in \mathcal{C}(\mathrm{M}_i,\mathrm{M})$, *such that* $(\mathrm{M},(q_i))$ *is a coproduct of the family* (M_i).

Proof. — Let $j \in \mathrm{I}$. Since M is a product, there exists a unique morphism $q_j \in \mathcal{C}(\mathrm{M}_j,\mathrm{M})$ such that $p_i \circ q_j = 0$ if $i \neq j$ and $p_j \circ q_j = \mathrm{id}_{\mathrm{M}_j}$.

For $i \in \mathrm{I}$, let $u_i = q_i \circ p_i$; one has $u_i \in \mathcal{C}(\mathrm{M},\mathrm{M})$; let $u = \sum_{i\in\mathrm{I}} u_i$. For $j \in \mathrm{I}$, one has

$$p_j \circ u = p_j \circ \Big(\sum_{i\in\mathrm{I}} q_i \circ p_i\Big) = \sum_{i\in\mathrm{I}} p_j \circ q_i \circ p_i = p_j = p_j \circ \mathrm{id}_\mathrm{M}\,.$$

Since M is a product, this implies that $u = \mathrm{id}_\mathrm{M}$.

Let now Q be an object of $\mathcal{C}$ and let $(f_i)_{i\in\mathrm{I}}$ be a family, where $f_i \in \mathcal{C}(\mathrm{M}_i,\mathrm{Q})$; let us show that there exists a unique morphism $f \in \mathcal{C}(\mathrm{M},\mathrm{Q})$ such that $f \circ q_i = f_i$ for every i. Let $f = \sum_{i\in\mathrm{I}} f_i \circ p_i$; this is an element of $\mathcal{C}(\mathrm{M},\mathrm{Q})$. For every $j \in \mathrm{I}$, one has

$$f \circ q_j = \Big(\sum_{i \in \mathrm{I}} f_i \circ p_i \Big) \circ q_j = \sum_{i \in \mathrm{I}} f_i \circ p_i \circ q_j = f_i \circ \mathrm{id}_{\mathrm{M}_i} = f_i.$$

Conversely, let $g \in \boldsymbol{C}(\mathrm{M}, \mathrm{Q})$ be such that $g \circ q_i = f_i$ for every i. One has

$$g = g \circ \Big(\sum_{i \in \mathrm{I}} q_i \circ p_i \Big) = \sum_{i \in \mathrm{I}} q \circ q_i \circ p_i = \sum_{i \in \mathrm{I}} f_i \circ p_i = f.$$

This concludes the proof. □

Corollary (**2.9.5**). — *Let $\boldsymbol{C}$ be a preadditive category. Let $(\mathrm{M}_i)_{i \in \mathrm{I}}$ be a* finite *family of objects of $\boldsymbol{C}$ and let $(\mathrm{M}, (q_i))$ be a coproduct of this family. There exists a family (p_i), where $p_i \in \boldsymbol{C}(\mathrm{M}, \mathrm{M}_i)$, such that $(\mathrm{M}, (p_i))$ is a product of the family (M_i).*

Proof. — This follows from lemma 2.9.4 by passing the opposite category, which is also a preadditive category. □

2.9.6. Additive categories. — Let $\boldsymbol{C}$ be a preadditive category. One says that it is an *additive category* if every finite family of objects of $\boldsymbol{C}$ has a product and a coproduct.

A functor $\mathrm{F} \colon \boldsymbol{C} \to \boldsymbol{D}$ between additive categories is said to be additive if for all objects M, N of $\boldsymbol{C}$, the map $\boldsymbol{C}(\mathrm{M}, \mathrm{N}) \to \boldsymbol{D}(\mathrm{M}, \mathrm{N})$ induced by F is additive.

Exercise (**2.9.7**). — Let $\boldsymbol{C}$ be an additive category.

a) Show that the product of an empty family in $\boldsymbol{C}$ is both a terminal object and an initial object. It is denoted by 0.

b) Let M, N be objects of $\boldsymbol{C}$ and let $f, g \in \boldsymbol{C}(\mathrm{M}, \mathrm{N})$. Construct a canonical commutative square

$$\begin{array}{ccc} \mathrm{M} & \xrightarrow{\;f+g\;} & \mathrm{N} \\ \Big\downarrow{\scriptstyle \Delta_{\mathrm{M}}} & & \Big\uparrow{\scriptstyle \Delta'_{\mathrm{N}}} \\ \mathrm{M} \oplus \mathrm{M} & \xrightarrow[\;f \oplus g\;]{} & \mathrm{N} \oplus \mathrm{N} \end{array}$$

in $\boldsymbol{C}$, where Δ_{M}, Δ'_{N} and $f \oplus g$ are defined solely in terms of the structure of category of C. Conclude that the group laws on the morphism sets of an additive category is intrinsic.

2.9.8. Kernels. — Let $\mathcal{C}$ be an additive category and let $u\colon \mathrm{M} \to \mathrm{N}$ be a morphism in $\mathcal{C}$.

For u to be a monomorphism, it is necessary and sufficient that for every object P of $\mathcal{C}$, $v = 0$ is the only element of $\mathcal{C}(\mathrm{P}, \mathrm{M})$ such that $u \circ v = 0$. The necessity of this condition is obvious; conversely, if it holds and if $f, g \in \mathcal{C}(\mathrm{P}, \mathrm{M})$ satisfy $u \circ f = u \circ g$, then $u \circ (f - g) = 0$, so that $f - g = 0$ and $f = g$.

A *kernel* of u an equalizer of the pair $(u, 0)$; this is an object P together with a morphism $i\colon \mathrm{P} \to \mathrm{M}$ such that $u \circ i = 0$ and such that for every object Q of $\mathcal{C}$ and every morphism $f\colon \mathrm{Q} \to \mathrm{M}$ such that $u \circ f = 0$, there exists a unique morphism $\varphi\colon \mathrm{Q} \to \mathrm{P}$ such that $i \circ \varphi = f$. One sometimes says that i is a kernel of u. If (P, i) and (P', i') are kernels of u, there exists a unique morphism $\varphi\colon \mathrm{P} \to \mathrm{P}'$ such that $i \circ \varphi = i'$, and φ is an isomorphism.

Let (P, i) be a kernel of u. Then i is a monomorphism. Let indeed $f\colon \mathrm{Q} \to \mathrm{P}$ be a morphism such that $i \circ f = 0$. Applying the definition of a kernel to the morphism $0 = i \circ f\colon \mathrm{Q} \to \mathrm{M}$, we observe $\varphi = 0$ is the only element of $\mathcal{C}(\mathrm{Q}, \mathrm{P})$ such that $i \circ \varphi = 0$; consequently, $f = 0$. Let us also observe that P represents the functor $\mathrm{Q} \mapsto \operatorname{Ker}(u_*\colon \mathcal{C}(\mathrm{Q}, \mathrm{M}) \to \mathcal{C}(\mathrm{Q}, \mathrm{N}))$.

2.9.9. Cokernels. — Let $\mathcal{C}$ be an additive category and let $u\colon \mathrm{M} \to \mathrm{N}$ be a morphism in $\mathcal{C}$. The definition and the basic properties of a cokernel are obtained by passing to the opposite category.

For u to be an epimorphism, it is necessary and sufficient that for every object P of $\mathcal{C}$ and every $v \in \mathcal{C}(\mathrm{N}, \mathrm{P})$ such that $v \circ u = 0$, one has $v = 0$.

A *cokernel* of u is a coequalizer of the pair $(u, 0)$; this is an object P together with a morphism $p\colon \mathrm{N} \to \mathrm{P}$ such that $p \circ u = 0$ and such that for every object Q of $\mathcal{C}$ and every morphism $f\colon \mathrm{N} \to \mathrm{Q}$ such that $f \circ u = 0$, there exists a unique morphism $\varphi\colon \mathrm{P} \to \mathrm{Q}$ such that $\varphi \circ p = f$. It is a kernel of u in the opposite category $\mathcal{C}^{\mathrm{o}}$.

If (P, p) and (P', p') are cokernels of u, then there exists a unique morphism $\varphi\colon \mathrm{P} \to \mathrm{P}'$ such that $\varphi \circ p = p'$, and φ is an isomorphism.

If (P, p) is a cokernel of u, then p is an epimorphism.

2.9.10. — Let $\mathcal{C}$ be an additive category. One says that $\mathcal{C}$ is an *abelian category* if the following properties hold:

– Every morphism in $\mathcal{C}$ has a kernel and a cokernel;
– Every monomorphism is the kernel of some morphism;

– Every epimorphism is the cokernel of some morphism.

Exercise (**2.9.11**). — Let $\mathbf{C}$ be an abelian category and let $u\colon M \to N$ be a morphism in $\mathbf{C}$. Show the following properties:

a) The morphism u is a monomorphism if and only if its kernel is 0;

b) The morphism u is an epimorphism if and only if its cokernel is 0;

c) The morphism u is an isomorphism if and only if it is both an epimorphism and a monomorphism.

Example (**2.9.12**). — Let A be a ring, possibly noncommutative. The category $\mathbf{Mod}_A$ of (left) A-modules is an abelian category.

This is indeed a preadditive category. Moreover, in this category, all products exist and are given by the usual products of A-modules, all coproducts exist and are given by the direct sums of A-modules. Monomorphisms are the injective morphisms, epimorphisms are the surjective morphisms. Kernels and cokernels exist, and correspond to the usual notions. Moreover, an injective morphism $i\colon M \to N$ is the kernel of its cokernel, the morphism $p\colon N \to \operatorname{Coker}(i)$. Similarly, a surjective morphism $p\colon N \to P$ is the cokernel of its kernel $i\colon \ker(p) \to N$, hence the assertion.

One proves in a similar way that if A is (left) noetherian, then the category of finitely generated (left) A-modules is an abelian category.

Exercise (**2.9.13**). — Let $\mathbf{C}$ be an abelian category. Show that every monomorphism is the kernel of its cokernel, and that every epimorphism is the cokernel of its kernel.

2.10. Exact sequences in abelian categories

Let $\mathbf{C}$ be an abelian category.

Lemma (**2.10.1**). — *Let* M, N *be objects of* $\mathbf{C}$ *and let* $f\colon M \to N$ *be a morphism of* $\mathbf{C}$. *Let* $p\colon N \to \operatorname{Coker}(f)$ *be a cokernel of* f *and let* $j\colon \operatorname{Ker}(p) \to N$ *be a kernel of* p.

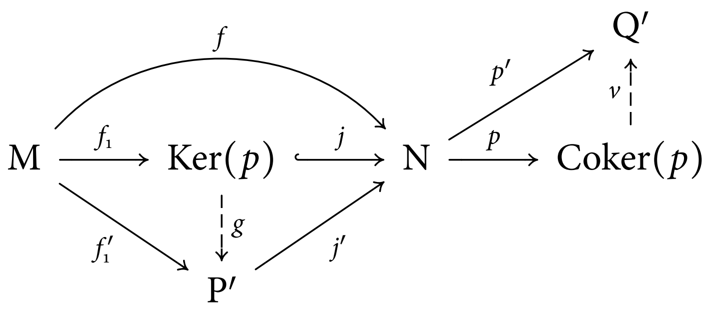
a) *There exists a unique morphism* $f_1\colon M \to \operatorname{Ker}(p)$ *such that* $f = j \circ f_1$.

b) *For every object* P' *of* $\mathbf{C}$, *every monomorphism* $j'\colon P' \to N$ *and every morphism* $f_1'\colon M \to P'$ *such that* $f = j' \circ f_1'$, *there exists a unique morphism* $g\colon \operatorname{Ker}(p) \to P'$ *such that* $f_1' = g \circ f_1$ *and* $j = j' \circ g$; *moreover,* g *is a monomorphism.*

c) *The morphism f_1 is an epimorphism.*

Proof. — a) One has $p \circ f = 0$ by the definition of a cokernel; by the definition of the kernel $\mathrm{Ker}(p)$, there exists a unique morphism f_1 such that $f = j \circ f_1$.

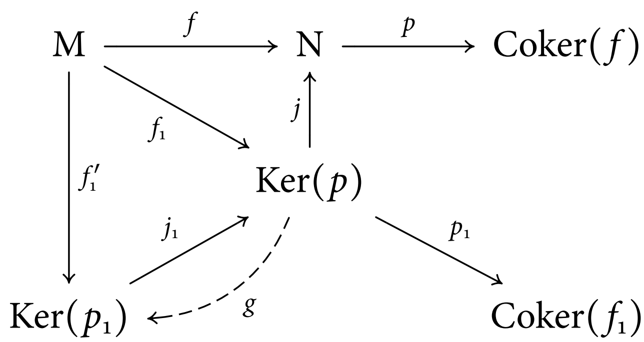
b) Since j' is a monomorphism, there exists an object N_1 of $\mathcal{C}$ and a morphism $p' \colon N \to Q'$ such that (P', j') is a kernel of p'. Then $p' \circ f = p' \circ j' \circ f_1' = 0$; consequently, there exists a unique morphism $v \colon \mathrm{Coker}(f) \to Q'$ such that $p' = v \circ p$. Then one has $p' \circ j = v \circ p \circ j = 0$; since (P', j') is a kernel of p', there exists a unique morphism $u \colon \mathrm{Ker}(p) \to P'$ such that $j = j' \circ u$. One then has $j' \circ f_1' = f = j \circ f_1 = j' \circ u \circ f_1$. Since j' is a monomorphism, this implies that $f_1' = u \circ f_1$.



Conversely, let $u' \colon \mathrm{Ker}(p) \to P'$ be a morphism such that $j = j' \circ u'$. One thus has $j' \circ u' = j = j' \circ u$, hence $u = u'$ because j' is a monomorphism.

Finally, since $j = j' \circ u$ and j is a monomorphism, the morphism u is a monomorphism as well.

c) Let (Q_1, p_1) be a cokernel of f_1 and let $(\mathrm{Ker}(p_1), j_1)$ be a kernel of p_1. Since $p_1 \circ f_1 = 0$, there exists a unique morphism $f_1' \colon M \to \mathrm{Ker}(p_1)$ such that $f_1 = j_1 \circ f_1'$. One then has $f = (j \circ j_1) \circ f_1'$, and $j \circ j_1$ is a monomorphism. By part *b*), there exists a monomorphism $g \colon \mathrm{Ker}(p) \to \mathrm{Ker}(p_1)$ such that $f_1' = g \circ f_1$ and $j \circ j_1 \circ g = j$.



Since j is a monomorphism, this implies that $j_1 \circ g = \mathrm{id}_{\mathrm{Ker}(p)}$. Consequently, $p_1 = p_1 \circ j_1 \circ g = 0$, hence f_1 is an epimorphism. □

Lemma (**2.10.2**). — *Let* M, N *be objects of* $\mathbf{C}$ *and let* $f\colon M \to N$ *be a morphism of* $\boldsymbol{C}$. *Let* $i\colon \mathrm{Ker}(f) \to M$ *be a kernel of* f *and let* $q\colon M \to \mathrm{Coker}(i)$ *be a cokernel of* i.

a) *There exists a unique morphism* $f_1\colon \mathrm{Coker}(i) \to N$ *such that* $f = f_1 \circ q$.

b) *For every object* Q' *of* $\boldsymbol{C}$, *every epimorphism* $q'\colon M \to Q'$ *and every morphism* $f_1'\colon Q' \to N$ *such that* $f = f_1' \circ q'$, *there exists a unique morphism* $g\colon Q' \to \mathrm{Coker}(i)$ *such that* $f_1' = f_1 \circ g$ *and* $q = g \circ q'$; *moreover,* g *is an epimorphism.*

c) *The morphism* f_1 *is a monomorphism.*

Proof. — It follows from lemma 2.10.1 by passing to the opposite category $\boldsymbol{C}^{\mathrm{o}}$. □

Proposition (**2.10.3**). — *Let* M, N *be objects of* $\mathbf{C}$ *and let* $f\colon M \to N$ *be a morphism. Let* $(\mathrm{Ker}(f), i)$ *be a kernel of* f, *let* $(\mathrm{Coker}(f), q)$ *be a cokernel of* f, *let* $(\mathrm{Ker}(q), j)$ *be a kernel of* q *and let* $(\mathrm{Coker}(i), p)$ *be a cokernel of* i. *There exists a unique morphism* $\varphi\colon \mathrm{Coker}(i) \to \mathrm{Ker}(q)$ *such that* $f = j \circ \varphi \circ p$, *and it is an isomorphism.*

Proof. — First observe that there exists at most one such morphism φ. Indeed, if $f = j \circ \varphi' \circ p$, then $j \circ \varphi \circ p = j \circ \varphi' \circ p$; since j is a monomorphism, this implies $\varphi \circ p = \varphi' \circ p$; since p is an epimorphism, this implies $\varphi = \varphi'$.

$$\begin{array}{ccccccc} \mathrm{Ker}(f) & \xrightarrow{\ i\ } & M & \xrightarrow{\ f\ } & N & \xrightarrow{\ q\ } & \mathrm{Coker}(f) \\ & & \downarrow{\scriptstyle p} & & \uparrow{\scriptstyle j} & & \\ & & \mathrm{Coker}(i) & \overset{\varphi}{\dashrightarrow} & \mathrm{Ker}(q) & & \end{array}$$

To construct such a morphism φ, let $f_1\colon \mathrm{Coker}(i) \to N$ be the unique morphism such that $f = f_1 \circ p$; it is a monomorphism (lemma 2.10.2). Then $q \circ f_1 \circ p = q \circ f = 0$. Since p is an epimorphism, $q \circ f_1 = 0$ and there exists a unique morphism $\varphi\colon \mathrm{Coker}(i) \to \mathrm{Ker}(q)$ such that $f_1 = j \circ \varphi$. Since f_1 is a monomorphism, φ is a monomorphism as well. One has $f = f_1 \circ p = j \circ \varphi \circ p$.

Let $f_2\colon M \to \mathrm{Ker}(q)$ be the unique morphism such that $f = j \circ f_2$; it is an epimorphism (lemma 2.10.1). Then $j \circ f_2 \circ i = f \circ i = 0$. Since j is a monomorphism, one has $f_2 \circ i = 0$, hence there exists a unique morphism $\psi\colon \mathrm{Coker}(i) \to \mathrm{Ker}(q)$ such that $f_2 = \psi \circ p$. Since f_2 is an epimorphism, ψ is an epimorphism as well. One has $f = j \circ f_2 = j \circ \psi \circ p$.

Consequently, $\varphi = \psi$. It is both a monomorphism and an epimorphism, hence it is an isomorphism. □

Remark **(2.10.4)**. — The objects $\mathrm{Coker}(i)$ and $\mathrm{Ker}(q)$ of the proposition are called the *image* of f and the *coimage* of f respectively, are are denoted $\mathrm{Im}(f)$ and $\mathrm{Coim}(f)$. To justify this terminology, observe that when $\boldsymbol{C}$ is the abelian category $\boldsymbol{Mod}_{\mathrm{A}}$ of A-modules over some ring A, one has $\mathrm{Coker}(i) = \mathrm{M}/\mathrm{Ker}(i) \simeq \mathrm{Im}(f)$.

In some books, the statement of the proposition is taken as a *definition* of an abelian category.

2.10.5. — Let $\boldsymbol{C}$ be an abelian category, let M, N, P be objects of **C** and let $f\colon \mathrm{M} \to \mathrm{N}$ and $g\colon \mathrm{N} \to \mathrm{P}$ be morphisms of $\boldsymbol{C}$ such that $g \circ f = 0$.

Let $i\colon \mathrm{Ker}(g) \to \mathrm{N}$ be a kernel of g; since $g \circ f = 0$, there exists a unique morphism $f'\colon \mathrm{M} \to \mathrm{Ker}(g)$ such that $f = i \circ f'$.

Let $p\colon \mathrm{N} \to \mathrm{Coker}(f)$ be a cokernel of f; since $g \circ f = 0$, there exists a unique morphism $g'\colon \mathrm{Coker}(f) \to \mathrm{P}$ such that $g = g' \circ p$.

Let (H_1, q) be a cokernel of f' and let (H_2, j) be a kernel of g'. The morphism $u = p \circ i\colon \mathrm{Ker}(g) \to \mathrm{Coker}(f)$ satisfies

$$g' \circ u \circ f' = (g' \circ p) \circ (i \circ f') = g \circ f = 0,$$

hence there exists a unique morphism $v\colon \mathrm{H}_1 \to \mathrm{H}_2$ such that $u = p \circ i = j \circ v \circ q$. We shall prove that *the morphism v is an isomorphism* by identifying it with the canonical isomorphism from $\mathrm{Coim}(u)$ to $\mathrm{Im}(u)$. In the case of a category of modules, observe that $\mathrm{H}_1 = \mathrm{Ker}(g)/\mathrm{Im}(f)$, while $\mathrm{H}_2 = \ker(g'\colon \mathrm{N}/\mathrm{Im}(f) \to \mathrm{P})$, and the morphism v is the obvious isomorphism between this modules. The proof for abelian categories is unfortunately more involved.

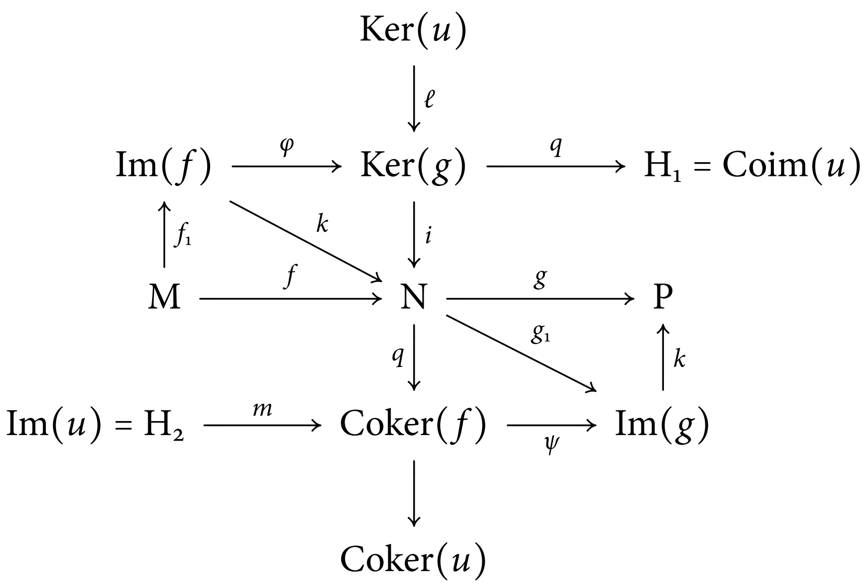
Let $k\colon \mathrm{Im}(f) \to \mathrm{N}$ and $f = k \circ f_1$ be the factorization of f given by lemma 2.10.1; the morphism f_1 is an epimorphism and there exists a unique morphism $\varphi\colon \mathrm{Im}(f) \to \mathrm{Ker}(g)$ such that $f' = \varphi \circ f_1$ and $k = i \circ \varphi$; one then has $f = i \circ f' = i \circ \varphi \circ f_1$.

Let $\ell\colon \mathrm{Ker}(u) \to \mathrm{Ker}(g)$ be a kernel of u. One has $u \circ \varphi \circ f_1 = p \circ i \circ \varphi \circ f_1 = p \circ f = 0$, hence $u \circ \varphi = 0$ since f_1 is an epimorphism. Consequently, there exists a unique morphism $\varphi'\colon \mathrm{Im}(f) \to \mathrm{Ker}(u)$ such that $\varphi = \ell \circ \varphi'$; since φ is a monomorphism, φ' is a monomorphism as well. Then $p \circ i \circ \ell = u \circ \ell = 0$, hence $i \circ \ell$ factors through the kernel of p, which, by definition, is the coimage of f. By proposition 2.10.3, $(\mathrm{Im}(f), k)$ represents the kernel of p, hence there exists a unique morphism $i'\colon \mathrm{Ker}(u) \to \mathrm{Im}(f)$ such that $k \circ i' = i \circ \ell$; since $i \circ \ell$ is a monomorphism, i' is a monomorphism too. Now, $k = i \circ \varphi = k \circ i' \circ \varphi'$; since k is a monomorphism, one has $i' \circ \varphi' = \mathrm{id}_{\mathrm{Im}(f)}$. This implies that i' is an epimorphism; moreover, $i' \circ \varphi' \circ i' = i'$, hence $\varphi' \circ i' = \mathrm{id}_{\mathrm{Ker}(u)}$. We have shown that φ' is an isomorphism from $\mathrm{Im}(f)$ to $\mathrm{Ker}(u)$.

One has $f' = \varphi \circ f_1 = \ell \circ \varphi' \circ f_1$. By definition, the pair (H_1, q) is a cokernel of f', but since $\varphi' \circ f_1$ is an epimorphism, (H_1, q) is also a cokernel of $\ell\colon \mathrm{Ker}(u) \to \mathrm{Ker}(g)$. In other words, we have identified H_1 with the coimage of u.

We can now apply the previous argument in the opposite category, or rework it patiently by reversing all arrows, and exchanging kernels and cokernels, images and coimages. This identifies (H_2, j) with the image of u.

The morphism v is then the unique morphism $\mathrm{Coim}(u) \to \mathrm{Im}(u)$ such that $u = q \circ v \circ j$. By proposition 2.10.3, it is an isomorphism.



Definition (**2.10.6**). — *The homology of a sequence* $\mathrm{M} \xrightarrow{f} \mathrm{N} \xrightarrow{g} \mathrm{P}$ *such that* $g \circ f = 0$ *is the object* H_1 *defined above.*

One says that this sequence is *exact* at N if $\mathrm{H}_1 = 0$. With the above notation, the following properties are equivalent:

(i) The sequence $\mathrm{M} \xrightarrow{f} \mathrm{N} \xrightarrow{g} \mathrm{P}$ is exact at N;
(ii) One has $\mathrm{H}_1 = 0$;
(iii) The morphism $f'\colon \mathrm{M} \to \mathrm{Ker}(g)$ deduced from f is an epimorphism;
(iv) One has $\mathrm{H}_2 = 0$;

(v) The morphism $g'\colon \mathrm{Coker}(f) \to \mathrm{P}$ deduced from g is a monomorphism.

2.10.7. — The notion of complex in an abelian category $\boldsymbol{C}$ can be developed in analogy with the corresponding concept for modules over a ring, However, some abelian categories do not always admit infinite coproducts, it is better to have a naïve definition of a graded differential object in $\boldsymbol{C}$ which avoids to considering a coproduct. So we shall just consider families $(\mathrm{M}_n)_{n\in\mathbf{Z}}$ of objects of $\boldsymbol{C}$ related by morphisms $d_n\colon \mathrm{M}_n \to \mathrm{M}_{n+r}$ such that $d_{n+r}\circ d_n = 0$ for all n. One speaks of a cohomological complex if $r = 1$, and of a homological complex if $r = -1$.

The definition of a morphism of complexes can be copied verbatim, as can that of a homotopy between two morphisms of complexes.

2.10.8. — Let $\mathrm{M} = (\mathrm{M}_n)$ be a homological complex in $\boldsymbol{C}$. Its homologies are defined by $\mathrm{H}_n(\mathrm{M}) = \mathrm{H}(\mathrm{M}_{n-1} \to \mathrm{M}_n \to \mathrm{M}_{n+1})$. A complex is an exact sequence if and only if is exact at each term, that is if and only if its homologies are 0.

A morphism of homological complexes $f = (f_n\colon \mathrm{M}_n \to \mathrm{N}_n)$ induces morphisms $\mathrm{H}_n(f)\colon \mathrm{H}_n(\mathrm{M}) \to \mathrm{H}_n(\mathrm{N})$. Two homotopic morphisms induce the same morphism.

Example (**2.10.9**). — Let $0 \to \mathrm{M} \xrightarrow{f} \mathrm{N} \xrightarrow{g} \mathrm{P} \to 0$ be a complex. The following properties are equivalent:

(i) This complex is an exact sequence;
(ii) The morphism f is a monomorphism and p is a cokernel of f;
(iii) The morphism g is an epimorphism and f is a kernel of g.

Definition (**2.10.10**). — *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be abelian categories and let* $\mathrm{F}\colon \boldsymbol{C} \to \boldsymbol{D}$ *be an additive functor. One says that* F *is* left exact *if for every exact sequence* $0 \to \mathrm{M} \xrightarrow{f} \mathrm{N} \xrightarrow{g} \mathrm{P} \to 0$, *the complex* $0 \to \mathrm{F}(\mathrm{M}) \xrightarrow{\mathrm{F}(f)} \mathrm{F}(\mathrm{N}) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(\mathrm{P})$ *is exact. One says that the functor* F *is* right exact *if for every such short exact sequence, the complex* $\mathrm{F}(\mathrm{M}) \xrightarrow{\mathrm{F}(f)} \mathrm{F}(\mathrm{N}) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(\mathrm{P}) \to 0$ *is exact. One says that the functor* F *is* exact *if it is both left and right exact.*[1]

If F is a contravariant additive functor from $\boldsymbol{C}$ to $\boldsymbol{D}$, it is viewed as a functor from the opposite category $\boldsymbol{C}^{\mathrm{o}}$ to the category $\boldsymbol{D}$, and this leads to analogous

(1) Compare with definition 2.3.14!

definitions. For example, such a contravariant functor F is left exact if for every short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ as above, the complex $0 \to F(P) \xrightarrow{F(g)} F(N) \xrightarrow{F(f)} F(M)$ is exact.

Example **(2.10.11)**. — Let $\mathbf{C}$ be an abelian category and let M be an object of $\mathbf{C}$.

a) *The functor* $\mathbf{C}(M, \bullet)$ *given by* $N \mapsto \mathbf{C}(M, N)$ *is a covariant left exact functor from* $\mathbf{C}$ *to the category* $\mathbf{Ab}$ *of abelian groups.*

Let $0 \to N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \to 0$ be a short exact sequence in $\mathbf{C}$ and let us consider the complex

$$0 \to \mathbf{C}(M, N_1) \xrightarrow{f'} \mathbf{C}(M, N_2) \xrightarrow{g'} \mathbf{C}(M, N_3)$$

deduced by application of the functor $\mathbf{C}(M, \bullet)$. Let $u \in \mathbf{C}(M, N_2)$ be a morphism such that $g'(u) = 0$, that is, $g \circ u = 0$. Since f is a kernel of g, there exists a unique morphism $v \in \mathbf{C}(M, N_1)$ such that $u = f \circ v = f'(v)$. This shows that f' is injective and that $\operatorname{Im}(f') = \operatorname{Ker}(g')$, as required.

b) *The functor* $\mathbf{C}(\bullet, M) \colon N \mapsto \mathbf{C}(N, M)$ *is a contravariant left exact functor from* $\mathbf{C}$ *to* $\mathbf{Ab}$.

This is deduced from the preceding case by considering the opposite category $\mathbf{C}^{o}$.

Remark **(2.10.12)**. — Let F be an additive functor between abelian categories.

Assume that F has a right adjoint. Then F respects all colimits. Since a cokernel of a morphism $f \colon M \to N$ is a colimit of the diagram $M \overset{f}{\underset{0}{\rightrightarrows}} N$, the functor F respects cokernels. Consequently, it is right exact.

By symmetry, if F has a left adjoint, it respects all limits, hence it respects kernels, so that it is left exact.

Exercise **(2.10.13)**. — Let F be an additive functor between abelian categories. If F is left (resp. right) exact, prove that F respects all finite limits (resp. colimits). That is, for every finite quiver Q and every Q-diagram $\mathscr{A}$ which has a limit (resp. a colimit) A, then F(A) is a limit (resp. a colimit) of $F(\mathscr{A})$.

Remark **(2.10.14)**. — As the first propositions of this section have shown, the manipulation of diagrams in abelian categories leads to much more involved arguments than what we are used to in the abelian category of modules over a ring. Indeed, in a category such as $\mathbf{Mod}_A$, to prove that a kernel is 0, it suffices to prove that each element of this kernel is 0. To that aim, we may do some "diagram chasing", pushing the element along morphisms, and lifting it along epimorphisms. This kind of argument is forbidden in general abelian categories whose objects have no elements to work with.

However, a theorem of Freyd–Mitchell shows that every (small) abelian category $\mathcal{C}$ possesses a fully faithful and exact functor F to a category of modules over some ring R. The properties of many canonical diagrams can then be established after applying the functor F, where classical proofs are possible.

2.11. Injective and projective objects in abelian categories

Let $\mathcal{C}$ be an abelian category. We have seen in example 2.10.11 that the functors $\mathcal{C}(\mathrm{M},\bullet)$ and $\mathcal{C}(\bullet,\mathrm{M})$ are left exact. They are not right exact in general and the definition of injective or projective objects in an abelian category essentially copies the one given for modules (definitions 2.7.1 and 2.8.1).

Definition **(2.11.1)**. — *Let $\mathcal{C}$ be an abelian category.*

a) *An object* I *of* $\mathcal{C}$ *is said to be* injective *if the left-exact functor* $\mathcal{C}(\bullet,\mathrm{I})$ *is exact.*

b) *An object* P *of* $\mathcal{C}$ *is said to be* projective *if the left-exact functor* $\mathcal{C}(\mathrm{P},\bullet)$ *is exact.*

In other words, an object I is injective if and only if, for every monomorphism $j\colon \mathrm{X}\to\mathrm{Y}$ and every morphism $f\colon\mathrm{X}\to\mathrm{I}$, there exists a morphism $g\colon\mathrm{Y}\to\mathrm{I}$ such that $f=g\circ j$. Taking $\mathrm{X}=\mathrm{I}$ and $f=\mathrm{id}_{\mathrm{I}}$, we see in particular that any monomorphism $j\colon\mathrm{X}\to\mathrm{I}$ admits a retraction, and one proves in the same way as for modules that this condition suffices.

Similarly, an object P is projective if and only if, for every epimorphism $p\colon\mathrm{X}\to\mathrm{Y}$ and every morphism $f\colon\mathrm{P}\to\mathrm{Y}$, there exists a morphism $g\colon\mathrm{P}\to\mathrm{X}$ such that $f=p\circ g$. Taking $\mathrm{Y}=\mathrm{P}$ and $f=\mathrm{id}_{\mathrm{P}}$, we see in particular that any epimorphism $p\colon\mathrm{X}\to\mathrm{P}$ admits a section, and one proves in the same way as for modules that this condition suffices.

Definition **(2.11.2)**. — *Let $\mathcal{C}$ be an abelian category.*

a) *One says that $\mathcal{C}$ has enough injectives if, for every object* X *of* $\mathcal{C}$, *there exists an injective object* I *of* $\mathcal{C}$ *and a monomorphism* $j\colon\mathrm{X}\to\mathrm{I}$.

b) *One says that* $\mathbf{C}$ *has enough projectives if, for every object* X *of* $\mathbf{C}$, *there exists a projective object* P *of* $\mathbf{C}$ *and an epimorphism* $p\colon \mathrm{P} \to \mathrm{X}$.

Example (**2.11.3**). — Let k be a ring. The injective (resp. projective) objects of the category $\mathbf{Mod}_k$ are exactly the injective (resp. projective) k-modules. Given proposition 2.8.7, this category has enough injectives. Similarly, any k-module M admits an epimorphism $p\colon \mathrm{P} \to \mathrm{M}$ with source a free k-module, for example the canonical morphism from $k^{(\mathrm{M})}$ to M which maps the basis element e_m to m, for every $m \in \mathrm{M}$. Since free modules are projective (proposition 2.7.2), this implies that the category $\mathbf{Mod}_k$ has enough projectives.

2.11.4. — The description of injective/projective objects in concrete abelian categories is generally a difficult question, as is the existence of such (nonzero) objects. Grothendieck introduced additional axioms on abelian categories which imply that a category has enough injectives (resp. projectives).

Definition (**2.11.5**). — *Let* $\mathbf{C}$ *be an abelian category.*

a) *A right resolution of an object* A *is a complex* $(\mathrm{X}_n, d_n^{\mathrm{X}})$ *in* $\mathbf{C}$ *such that* $\mathrm{X}_n = 0$ *for* $n < 0$, *together with a morphism* $\varepsilon\colon \mathrm{A} \to \mathrm{X}_0$ *such that the complex*

$$0 \to \mathrm{A} \to \mathrm{X}_0 \to \mathrm{X}_1 \to \dots$$

is exact.

b) *A right resolution is said to be injective of the objects* X_n *are injective, for all* $n \in \mathbf{N}$.

c) *A morphism of right resolutions is a morphism of complexes.*

Consider a complex $\mathrm{X} = (\mathrm{X}_n)$ such that $\mathrm{X}_n = 0$ for $n < 0$ together with a morphism $\varepsilon\colon \mathrm{A} \to \mathrm{X}_0$ in $\mathbf{C}$. Viewing the object A as a complex, whose only nonzero term is in degree 0, the morphism ε defines a morphism of complexes $\mathrm{A} \to \mathrm{X}$, which we still denote by ε. Then $(\mathrm{X}, \varepsilon)$ is a resolution if and only if the morphism of complexes $\varepsilon\colon \mathrm{A} \to \mathrm{X}$ is a homologism.

Proposition (**2.11.6**). — *Let* $\mathbf{C}$ *be an abelian category with enough injectives.*

a) *Every object* A *in* $\mathbf{C}$ *has an injective resolution.*

b) *Let* $\varepsilon\colon \mathrm{A} \to \mathrm{X}$ *and* $\eta\colon \mathrm{B} \to \mathrm{Y}$ *be resolutions, let* $f\colon \mathrm{A} \to \mathrm{B}$ *be a morphism in* $\mathbf{C}$. *If* Y_n *is injective for every* n, *then there exists a morphism of resolutions* $\varphi\colon \mathrm{X} \to \mathrm{Y}$ *that extends* f.

c) *Let* $\varphi, \psi\colon X \to Y$ *two morphisms of resolutions that extend a morphism* $f\colon A \to B$. *If* Y_n *is injective for every* n, *then* φ *and* ψ *are homotopic: there exist morphisms* $h_n\colon X_n \to Y_{n-1}$, *for all* $n \in \mathbf{N}$, *such that* $\varphi_n - \psi_n = d^Y_{n-1}h_n + h_{n+1}d^X_n$ *for every* n.

Proof. — The proof is analogous to the one made for projective resolutions of modules.

a) Since the abelian category $\boldsymbol{C}$ has enough injective, there exists a monomorphism $\varepsilon\colon A \to X_0$ from A to an injective object X_0. Denote by $p_0\colon X_0 \to X'_1$ its cokernel, so that we have an exact sequence $0 \to A \to X_0 \to X'_1 \to 0$. Let $\varepsilon_1\colon X'_1 \to X_1$ be a monomorphism from X'_1 to an injective object X_1, let $d_0 = \varepsilon_1 \circ p_0\colon X_0 \to X_1$ and let $p_1\colon X_1 \to X'_2$ be a cokernel of d_0. One has $d_0 \circ \varepsilon = \varepsilon_1 \circ p_0 \circ \varepsilon = 0$; moreover, since ε_1 is a monomorphism, the kernel of $d_0 = \varepsilon_1 \circ p_0$ is that of p_0, that is, the image of ε; in particular, the complex $A \to X_0 \to X_1$ is exact at X_0. Having constructed an exact sequence $0 \to A \to X_0 \xrightarrow{d_0} \dots \xrightarrow{d_{n-1}} X_n \xrightarrow{p_n} X'_{n+1} \to 0$, where $X_0, \dots, X_{n-1}$ are injective objects of $\boldsymbol{C}$, we can then introduce a monomorphism ε_n from X'_n to an injective object X_n, define the morphism $d_n\colon X_n \to X_{n+1}$ as the composition $\varepsilon_n \circ p_n$, and a morphism $p_{n+1}\colon X_{n+1} \to X'_{n+2}$ as the cokernel of d_n. By induction, this furnishes the desired injective resolution.

b) The morphism $\varepsilon\colon A \to X_0$ is a monomorphism; applying the definition of an injective object to the morphism $\eta \circ f\colon A \to Y_0$, we see that there exists a morphism $f_0\colon X_0 \to Y_0$ such that $f_0 \circ \varepsilon = \eta \circ f$. Assume that morphisms $f_m\colon X_m \to Y_m$ have been defined, for $m \leqslant n$ satisfying $f_m \circ d^X_{m-1} = d^Y_{m-1} \circ f_{m-1}$, for all integers m such that $1 \leqslant m \leqslant n$, and let us define a morphism $f_{n+1}\colon X_{n+1} \to Y_{n+1}$ such that $f_{n+1} \circ d^X_n = d^Y_n \circ f_n$. Since one has $d^Y_n \circ f_n \circ d^X_{n-1} = d^Y_n \circ d^Y_{n-1} \circ f_{n-1} = 0$, the morphism $d^Y_n \circ f_n\colon X_n \to Y_n$ vanishes on $\operatorname{Im}(d^X_{n-1})$, thus factors through a morphism $f'_n\colon X_n/\operatorname{Im}(d^X_{n-1}) \to Y_{n+1}$. By definition of an exact sequence, the morphism d_n factors through a monomorphism $d'_n\colon X_n/\operatorname{Im}(d_{n-1}) \to X_{n+1}$. Since Y_{n+1} is injective, there exists a morphism $f_{n+1}\colon X_{n+1} \to Y_{n+1}$ such that $f_{n+1} \circ d'_n = f'_n$. Since $X_n \to X_n/\operatorname{Im}(d_{n-1})$ is an epimorphism, this imples that $f_{n+1} \circ d^X_n = d^Y_n \circ f_n$. By induction, we obtain a sequence (f_n) of morphisms in $\boldsymbol{C}$ which is a morphism of resolutions and extends f.

c) We define h_0 as the zero morphism. By definition of a morphism of resolutions, one has $\varphi_0 \circ \varepsilon = \eta \circ f = \psi_0 \circ \varepsilon$, so that $(\varphi_0 - \psi_0) \circ \varepsilon = 0$. Consequently, the morphism $\varphi_0 - \psi_0$ induces a morphism $h'_0\colon X_0/\operatorname{Im}(\varepsilon) \to Y_0$. By definition of an exact sequence, the morphism $d^X_0\colon X_0 \to X_1$ induces a monomorphism $d'_0\colon X_0/\operatorname{Im}(\varepsilon) \to X_1$. Since Y_0 is injective, there exists a morphism $h_1\colon X_1 \to Y_0$

such that $h_1 \circ d'_0 = h'_0$; since $X_0 \to X_0/\operatorname{Im}(\varepsilon)$ is an epimorphism, one then has $h_1 \circ d_0^X = \varphi_0 - \psi_0$.

Assume that we have constructed morphisms $h_0, \dots, h_n$ such that $\varphi_m - \psi_m = d_{m-1}^Y h_m + h_{m+1} d_m^X$ for every integer m such that $0 \leqslant m < n$. Then one has

$$(\varphi_n - \psi_n - d_{n-1}^Y \circ h_n) \circ d_{n-1}^X = d_{n-1}^Y \circ (\varphi_{n-1} - \psi_{n-1} - h_n \circ d_{n-1}^X) = d_{n-1}^Y \circ d_{n-2}^Y h_{n-1} = 0,$$

so that $\varphi_n - \psi_n - d_{n-1}^Y \circ h_n$ factors through a morphism $h'_{n+1} \colon X_n/\operatorname{Im}(d_{n-1}^X) \to Y_n$. By definition of an exact sequence, the morphism $d_n^X \colon X_n \to X_{n+1}$ induces a monomorphism $d'_n \colon X_n/\operatorname{Im}(d_{n-1}^X) \to X_{n+1}$. Since Y_n is injective, there exists a morphism $h_{n+1} \colon X_{n+1} \to Y_n$ such that $h_{n+1} \circ d'_n = h'_{n+1}$; since $X_n \to X_n/\operatorname{Im}(d_{n-1}^X)$ is an epimorphism, one then has $h_{n+1} \circ d_n^X = \varphi_n - \psi_n - d_{n-1}^Y \circ h_n$.

By induction, we have constructed a family (h_n) of morphisms satisfying the desired relations. □

2.12. Derived functors

2.12.1. — Let $\boldsymbol{C}$ be an abelian category admitting enough injectives, let $\boldsymbol{D}$ be an abelian category and let $F \colon \boldsymbol{C} \to \boldsymbol{D}$ be an additive functor.

a) For every object A of $\boldsymbol{C}$, we choose an injective resolution $\varepsilon_A \colon A \to X$ and set

$$\boldsymbol{R}^n F(A) = H^n(F(X)),$$

where $F(X)$ is the complex $0 \to F(X_0) \to F(X_1) \to \dots.$

b) Let $f \colon A \to B$ be a morphism in the category $\boldsymbol{C}$. Consider a morphism $\varphi \colon X \to Y$ on the chosen injective resolutions of A and B that extends the morphism f. It gives rise to a morphism of complexes $F(\varphi) \colon F(X) \to F(Y)$; set

$$\boldsymbol{R}^n F(f) = H^n(F(\varphi)) \colon \boldsymbol{R}^n F(A) = H^n(F(X)) \to H^n(F(Y)) = \boldsymbol{R}^n F(B).$$

If $\psi \colon X \to Y$ is another morphism that extends the morphism f, we know that the morphisms φ and ψ are homotopic: there exists morphisms $h_n \colon X_{n+1} \to Y_n$ such that $\varphi_n - \psi_n = d_{n-1}^Y \circ h_n + h_{n+1} \circ d_n^X$ for all $n \in \mathbf{N}$. Since the functor F is additive, one has

$$F(\varphi)_n - F(\psi)_n = d_{n-1}^{F(Y)} \circ F(h_n) + F(h_{n+1}) \circ d_n^{F(X)}$$

for all $n \in \mathbf{N}$, so that the morphisms of complexes $F(\varphi)$ and $F(\psi)$ are homotopic. In particular, the maps they induce on the cohomology objects coincide:

$H^n(F(\varphi)) = H^n(F(\psi))$ for all $n \in \mathbf{N}$. In other words, the morphisms $\boldsymbol{R}^n F(f)$ do not depend on the choice of the morphisms of resolutions φ.

c) Let us show that these data define *functors* $\boldsymbol{R}^n F\colon \boldsymbol{C} \to \boldsymbol{D}$. We need to prove that $\boldsymbol{R}^n F(\mathrm{id}_A) = \mathrm{id}_{\boldsymbol{R}^n F(A)}$ and that $\boldsymbol{R}^n F(g \circ f) = \boldsymbol{R}^n F(g) \circ \boldsymbol{R}^n F(f)$ for morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\boldsymbol{C}$.

For the first relations, observe that if $A \to X$ is an injective resolution of A, then id_X defines a morphism of resolutions which extends id_A. By definition, one thus has $\boldsymbol{R}^n F(\mathrm{id}_A) = H^n(F(\mathrm{id}_X)) = \mathrm{id}_{H^n(F(X))} = \mathrm{id}_{\boldsymbol{R}^n F(A)}$.

Let now $A \to X$, $B \to Y$ and $C \to Z$ be injective resolutions of objects A, B and C. Fix a morphism of complexes $\varphi\colon X \to Y$ that extends f, as well as a morphism of complexes $\psi\colon Y \to Z$ that extends g. Then $\psi \circ \varphi\colon X \to Z$ is a morphism of complexes in $\boldsymbol{C}$ that extends $g \circ f$. Consequently,

$$\begin{aligned}\boldsymbol{R}^n F(g \circ f) &= \boldsymbol{H}^n(F(\psi \circ \varphi)) = \boldsymbol{H}^n(F(\psi) \circ F(\varphi)) \\ &= \boldsymbol{H}^n(F(\psi)) \circ \boldsymbol{H}^n(F(\varphi)) = \boldsymbol{R}^n F(g) \circ \boldsymbol{R}^n F(f),\end{aligned}$$

as was to be proved.

d) Let us show that these functors $\boldsymbol{R}^n F$ are additive. Let $f, g\colon A \to B$ be two morphisms in $\boldsymbol{C}$ and let $\varphi, \psi\colon X \to Y$ be two of complexes on the chosen injective resolutions of A and B which extend f and g. Then $\varphi + \psi$ extends $f + g$. Consequently, $\boldsymbol{R}^n(f + g) = H^n(F(\varphi + \psi)) = H^n(F(\varphi) + F(\psi))$, because F is additive, hence $\boldsymbol{R}^n(f + g) = H^n(F(\varphi)) + H^n(F(\psi)) = \boldsymbol{R}^n F(f) + \boldsymbol{R}^n F(g)$.

These additive functors $\boldsymbol{R}^n F\colon \boldsymbol{C} \to \boldsymbol{D}$ are called the (right) *derived functors* $\boldsymbol{R}^n F$ of F.

2.12.2. — Let A be an object of $\boldsymbol{C}$ and let $\varepsilon\colon A \to X$ be the chosen injective resolution of A. By definition, the kernel of $F(X_0) \to F(X_1)$ is the object $\boldsymbol{R}^0 F(A)$. Applying the functor $\boldsymbol{F}$ to the exact sequence $0 \to A \to X_0 \to \dots$ gives rise to a complex $0 \to F(A) \to F(X_0) \to F(X_1) \to \dots$, so that the morphism $F(A) \to F(X_0)$ factors through $\boldsymbol{R}^0 F(A)$.

This furnishes a morphism of functors (a “natural transformation”) $F \to \boldsymbol{R}^0 F$.

Assume that F is left exact. Applying F to the short exact sequence $0 \to A \to X_0 \to X_1$, we obtain a short exact sequence $0 \to F(A) \to F(X_0) \to F(X_1)$. That means that $F(\varepsilon)\colon F(A) \to F(X_0)$ induces an isomorphism $F(A) \equiv \boldsymbol{R}^0 F(A)$. In other words, the canonical morphism $F \to \boldsymbol{R}^0 F$ is an isomorphism.

2.12.3. — Let us consider an arbitrary right resolution $\eta\colon A \to Y$ of an object A. Because the chosen resolution $\varepsilon\colon A \to X$ is injective, there exists a morphism of resolutions $\varphi\colon Y \to X$ that extends id_A. Applying the additive functor F to the morphism of complexes φ gives a morphism $F(\varphi)\colon F(Y) \to F(X)$. Passing to the cohomology objects, we then obtain morphisms $H^n(F(Y)) \to H^n(F(X)) = \boldsymbol{R}^n F(A)$.

As above, these morphisms do not depend on the choice of the morphism φ. We shall call them canonical.

If, moreover, Y is an injective resolution, we can obtain in the same way a morphism of resolutions $\psi\colon X \to Y$, inducing morphisms $H^n(F(X)) = \boldsymbol{R}^n F(A) \to H^n(F(Y))$. The morphism $\psi \circ \varphi\colon Y \to Y$ is a morphism of injective resolutions that extends id_A, so that it is homotopic to id_Y. This imples that $F(\psi \circ \varphi)$ is homotopic to $F(\mathrm{id}_Y) = \mathrm{id}_{F(Y)}$ and one has $H^n(F(\psi)) \circ H^n(F(\varphi)) = \mathrm{id}_{H^n(F(Y))}$. Similarly, the morphism $\varphi \circ \psi\colon X \to X$ is a morphism of injective resolutions that extends id_A, so that it is homotopic to id_X. This imples that $F(\varphi \circ \psi)$ is homotopic to $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and one has $H^n(F(\varphi)) \circ H^n(F(\psi)) = \mathrm{id}_{\boldsymbol{R}^n F(A)}$. Consequently, the two morphisms $H^n F(\varphi)\colon H^n(F(Y)) \to \boldsymbol{R}^n F(A)$ and $H^n F(\psi)\colon \boldsymbol{R}^n F(A) \to H^n(F(Y))$ are inverse the one of the other. They are thus isomorphisms.

Informally, this shows that the derived functors $\boldsymbol{R}^n F$ do not depend on the choices of the injective resolutions of the objects of $\boldsymbol{C}$.

Lemma **(2.12.4)**. — *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence in the category* $\boldsymbol{C}$. *Let* $\varepsilon\colon A \to X$ *be an injective resolution of* A*; let* $\eta\colon C \to Z$ *be a resolution of* C.

There exists an resolution $\zeta\colon B \to Y$ *of* B *satisfying the following properties:*

(i) *For every* $n \in \mathbf{N}$, *one has* $Y_n = X_n \oplus Z_n$*;*

(ii) *The canonical morphisms* $i_n\colon X_n \to Y_n$ *and* $p_n\colon Y_n \to Z_n$ *induce morphisms of complexes* $i\colon X \to Y$ *and* $p\colon Y \to Z$ *fitting in an exact sequence* $0 \to X \to Y \to Z \to 0$*;*

(iii) *One has* $\zeta \circ f = i \circ \varepsilon$ *and* $p \circ \zeta = \eta \circ g$*: the morphisms i and p are morphisms of resolutions.*

Proof. — Denote by $f\colon A \to B$ and $g\colon B \to C$ the morphisms that appear in the given exact sequence. Since X_0 is injective, there exists a morphism $u_0\colon B \to X_0$ such that $u_0 \circ f = \varepsilon$. Let $\zeta\colon B \to Y_0$ be the morphism $(u_0, \eta \circ g)$.

Let us prove that it is a monomorphism. Pretending $\boldsymbol{C}$ is a category of modules, if an element $b \in \mathrm{B}$ belongs to $\ker(\zeta)$, then $\eta(g(b)) = 0$, hence $g(b) = 0$ because η is a monomorphism; then there exists $a \in \mathrm{A}$ such that $b = f(a)$ and $0 = u_0(b) = u_0(f(a)) = \varepsilon(a)$, so that $a = 0$; consequently, $b = 0$.

The canonical morphisms $\mathrm{X}_0 \to \mathrm{Y}_0$ and $\mathrm{Y}_0 \to \mathrm{Z}_0$ induce morphisms $f_0 \colon \mathrm{X}_0/\operatorname{Im}(\varepsilon) \to \mathrm{Y}_0/\operatorname{Im}(\zeta)$ and $g_0 \colon \mathrm{Y}_0/\operatorname{Im}(\zeta) \to \mathrm{Z}_0/\operatorname{Im}(\eta)$, and the snake lemma implies that the complex

$$0 \to \mathrm{X}_0/\operatorname{Im}(\varepsilon) \xrightarrow{f_0} \mathrm{Y}_0/\operatorname{Im}(\zeta) \xrightarrow{g_0} \mathrm{Z}_0/\operatorname{Im}(\eta) \to 0$$

is an exact sequence.

We apply the same argument to this exact sequence and the monomorphisms $d_0^{\mathrm{X}\prime} \colon \mathrm{X}_0/\operatorname{Im}(\varepsilon) \to \mathrm{X}_1$ and $d_0^{\mathrm{Z}\prime} \colon \mathrm{Z}_0/\operatorname{Im}(\eta) \to \mathrm{Z}_1$ deduced from the resolutions $\varepsilon \colon \mathrm{A} \to \mathrm{X}$ and $\eta \colon \mathrm{C} \to \mathrm{Z}$. This furnishes a monomorphism $d_0^{\mathrm{Y}\prime} \colon \mathrm{Y}_0/\operatorname{Im}(\zeta) \to \mathrm{Y}_1$ such that $d_0^{\mathrm{Y}\prime} \circ f_0 = (d_0^{\mathrm{X}\prime}, 0)$ and $d_0^{\mathrm{Z}\prime} \circ g_0 = p_2 \circ d_0^{\mathrm{Y}\prime}$. Define the morphism d_0^{Y} as the composition of $d_0^{\mathrm{Y}\prime}$ with the canonical epimorphism $\mathrm{Y}_0 \to \mathrm{Y}_0/\operatorname{Im}(\zeta)$.

This defines the first two levels of the complex Y, and we go on by induction. □

2.12.5. — Let $0 \to \mathrm{A} \xrightarrow{f} \mathrm{B} \xrightarrow{g} \mathrm{C} \to 0$ be an exact sequence in the category $\boldsymbol{C}$. Let $\varepsilon \colon \mathrm{A} \to \mathrm{X}$ and $\eta \colon \mathrm{C} \to \mathrm{Z}$ be the chosen injective resolution of A and C. The preceding lemma furnishes a resolution $\zeta \colon \mathrm{B} \to \mathrm{Y}$ of B which fits into an exact sequence of complexes $0 \to \mathrm{X} \to \mathrm{Y} \to \mathrm{Z} \to 0$, with $\mathrm{Y}_n = \mathrm{X}_n \oplus \mathrm{Z}_n$ for all n. Applying the functor F, it follows that $\mathrm{F}(\mathrm{Y}_n) = \mathrm{F}(\mathrm{X}_n) \oplus \mathrm{F}(\mathrm{Z}_n)$, and we obtain an exact sequence of complexes $0 \to \mathrm{F}(\mathrm{X}) \to \mathrm{F}(\mathrm{Y}) \to \mathrm{F}(\mathrm{Z}) \to 0$. The associated long exact sequence of cohomology objects writes

$$0 \to \boldsymbol{R}^0\mathrm{F}(\mathrm{A}) \to \mathrm{H}^0(\mathrm{F}(\mathrm{Y})) \to \boldsymbol{R}^0\mathrm{F}(\mathrm{C}) \to \boldsymbol{R}^1\mathrm{F}(\mathrm{A}) \to \dots$$

Since X_n and Z_n are injective objects, so is their sum Y_n (any product of injective objects is injective, and a Y_n is also the product $\mathrm{X}_n \times \mathrm{Z}_n$). This shows that the resolution $\zeta \colon \mathrm{B} \to \mathrm{Y}$ of B is injective, so that the canonical morphisms $\mathrm{H}^n(\mathrm{F}(\mathrm{Y})) \to \boldsymbol{R}^n\mathrm{F}(\mathrm{B})$ are isomorphisms. Replacing the objects $\mathrm{H}^n(\mathrm{F}(\mathrm{Y}))$ by them, we obtain the canonical long exact sequence of cohomology of derived functors:

$$0 \to \boldsymbol{R}^0\mathrm{F}(\mathrm{A}) \to \boldsymbol{R}^0\mathrm{F}(\mathrm{B}) \to \boldsymbol{R}^0\mathrm{F}(\mathrm{C}) \to \boldsymbol{R}^1\mathrm{F}(\mathrm{A}) \to \dots$$

If, moreover, F is left exact, then we can replace $\boldsymbol{R}^0\mathrm{F}$ by F in this exact sequence, and we have

$$0 \to \mathrm{F(A)} \to \mathrm{F(B)} \to \mathrm{F(C)} \to \boldsymbol{R}^1\mathrm{F(A)} \to \boldsymbol{R}^1\mathrm{F(B)} \to \boldsymbol{R}^1\mathrm{F(C)} \to \boldsymbol{R}^2\mathrm{F(A)} \to \dots$$

2.12.6. — These long exact sequences are functorial.

Definition (**2.12.7**). — *One says that a resolution* $\varepsilon\colon \mathrm{A} \to \mathrm{X}$ *is* F*-acyclic of an object* A *if* $\boldsymbol{R}^p\mathrm{F}(\mathrm{X}_n) = 0$ *for every n and every p* > 0.

For any integer d, we say that such a resolution is acyclic in degrees $\leqslant d$ if $\boldsymbol{R}^{p+1}\mathrm{F}(\mathrm{X}_n) = 0$ for all integers p, n such that $p + n \leqslant d$.

Proposition (**2.12.8**). — *Let* F *be an additive left exact functor. Let* $\varepsilon\colon \mathrm{A} \to \mathrm{X}$ *be a resolution of an object* A. *If it is* F*-acyclic in degrees* $\leqslant d$*, then the canonical morphisms* $\mathrm{H}^n(\mathrm{F(X)}) \to \boldsymbol{R}^n\mathrm{F(A)}$ *are isomorphisms, for all integers n such that* $n \leqslant d$.

Proof. — The case $n = 0$ is banal. The resolution $\varepsilon\colon \mathrm{A} \to \mathrm{X}$ induces a short exact sequence $0 \to \mathrm{A} \to \mathrm{X}_0 \to \mathrm{X}_1$; since F is left exact, applying F to that sequence furnishes a short exact sequence $0 \to \mathrm{F(A)} \to \mathrm{F(X_0)} \to \mathrm{F(X_1)}$, hence an isomorphism $\mathrm{F(A)} \xrightarrow{\sim} \mathrm{H}^0(\mathrm{F(X)})$.

For any integer $n \geqslant 0$, let $i_n\colon \mathrm{Z}_n \to \mathrm{X}_n$ be a kernel of $d_n^{\mathrm{X}}\colon \mathrm{X}_n \to \mathrm{X}_{n+1}$. Sice $d_{n+1}^{\mathrm{X}} \circ d_n^{\mathrm{X}} = 0$, the morphism d_n^{X} factors through Z_{n+1}; this furnishes complexes $0 \to \mathrm{Z}_n \xrightarrow{i_n} \mathrm{X}_n \xrightarrow{p_n} \mathrm{Z}_{n+1} \to 0$, for all $n \in \mathbf{N}$. Since $\varepsilon\colon \mathrm{A} \to \mathrm{X}$ is a resolution, these complexes are exact sequences; moreover, ε induces an isomorphism $\mathrm{A} \xrightarrow{\sim} \mathrm{Z}_0$.

Let n be an integer such that $1 \leqslant n \leqslant d$. Consider the short exact sequence $0 \to \mathrm{Z}_{n-1} \to \mathrm{X}_{n-1} \to \mathrm{Z}_n \to 0$. By the F-acyclicity condition on X_{n-1}, we have $\boldsymbol{R}^1\mathrm{F}(\mathrm{X}_{n-1}) = 0$, so that the long exact sequence associated with that short exact sequence starts as

$$0 \to \mathrm{F}(\mathrm{Z}_{n-1}) \xrightarrow{\mathrm{F}(i_{n-1})} \mathrm{F}(\mathrm{X}_{n-1}) \xrightarrow{\mathrm{F}(p_{n-1})} \mathrm{F}(\mathrm{Z}_n) \to \boldsymbol{R}^1\mathrm{F}(\mathrm{Z}_{n-1}) \to 0.$$

We now observe that this exact sequence identifies $\boldsymbol{R}^1\mathrm{F}(\mathrm{Z}_{n-1})$ with the cohomology object $\mathrm{H}^n(\mathrm{F(X)})$. Indeed, $i_n\colon \mathrm{Z}_n \to \mathrm{X}_n$ is a kernel of d_n^{X}, and since F is left exact, $\mathrm{F}(i_n)\colon \mathrm{F}(\mathrm{Z}_n) \to \mathrm{F}(\mathrm{X}_n)$ is a kernel of $\mathrm{F}(d_n^{\mathrm{X}})$. On the other hand, $d_{n-1}^{\mathrm{X}} = i_n \circ p_{n-1}$, so that $\mathrm{F}(d_{n-1}^{\mathrm{X}}) = \mathrm{F}(i_n) \circ \mathrm{F}(p_{n-1})$, which shows that the image of $\mathrm{F}(p_{n-1})$ coincides with the image of $\mathrm{F}(\mathrm{X}_{n-1})$ inside $\mathrm{F}(\mathrm{Z}_n)$.

When $n = 1$, this furnishes the desired isomorphism $\boldsymbol{R}^1\mathrm{F}(\mathrm{A}) \simeq \boldsymbol{R}^1\mathrm{F}(\mathrm{Z}_0) \simeq \mathrm{H}^1(\mathrm{F}(\mathrm{X}))$.

Let us assume that $n \geqslant 2$. Let m be an integer such that $0 \leqslant m \leqslant n-2$ and consider the short exact sequence $0 \to \mathrm{Z}_m \to \mathrm{X}_m \to \mathrm{Z}_{m+1} \to 0$; taking the corresponding long exact sequence associated with the functor F and its derived functors, we obtain isomorphisms $\boldsymbol{R}^p\mathrm{F}(\mathrm{Z}_{m+1}) \to \mathrm{R}^{p+1}\mathrm{F}(\mathrm{Z}_m)$, for all integers p, m such that $\boldsymbol{R}^p\mathrm{F}(\mathrm{X}_m) = \boldsymbol{R}^{p+1}\mathrm{F}(\mathrm{X}_m) = 0$. By the F-acyclicity condition on the resolution $\varepsilon\colon \mathrm{A} \to \mathrm{X}$, we may take $(p, m) = (1, n-2), (2, n-3), \dots, (n-1, 0)$, and the composition of these isomorphisms is an isomorphism $\boldsymbol{R}^1\mathrm{F}(\mathrm{Z}_{n-1}) \simeq \boldsymbol{R}^n\mathrm{F}(\mathrm{Z}_0) \simeq \boldsymbol{R}^n\mathrm{F}(\mathrm{A})$. Combining this isomorphism with the above isomorphism $\boldsymbol{R}^1\mathrm{F}(\mathrm{Z}_{n_1}) \simeq \mathrm{H}^n(\mathrm{F}(\mathrm{X}))$, we obtained an isomorphism $\mathrm{H}^n(\mathrm{F}(\mathrm{X})) \xrightarrow{\sim} \boldsymbol{R}^n\mathrm{F}(\mathrm{A})$.

However, this does not exactly prove that the canonical morphism $\mathrm{H}^n(\mathrm{F}(\mathrm{X})) \to \boldsymbol{R}^n\mathrm{F}(\mathrm{A})$ is an isomorphism. To that aim, we follow the same reasoning on the chosen injective resolution $\eta\colon \mathrm{A} \to \mathrm{I}$. Since the resolution $\eta\colon \mathrm{A} \to \mathrm{I}$ is injective, there exists a morphism of complexes $\varphi\colon \mathrm{X} \to \mathrm{I}$ such that $\varphi \circ \varepsilon = \eta$. Let $\mathrm{T}_n = \ker(d^{\mathrm{I}}_n)$, we obtain in the same way exact sequences $0 \to \mathrm{T}_n \to \mathrm{I}_n \to \mathrm{T}_{n+1} \to 0$; moreover, the morphism φ induces morphisms $\varphi'_n\colon \mathrm{Z}_n \to \mathrm{T}_n$ which induce, for each n, a morphism from the exact sequence $0 \to \mathrm{Z}_n \to \mathrm{X}_n \to \mathrm{Z}_{n+1} \to 0$ to the exact sequence $0 \to \mathrm{T}_n \to \mathrm{I}_n \to \mathrm{T}_{n+1} \to 0$. By fonctoriality of the cohomology long exact sequence, and keeping track of all morphisms introduced, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^n(\mathrm{F}(\mathrm{X})) & \xrightarrow{\sim} & \boldsymbol{R}^n(\mathrm{F}(\mathrm{A})) \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow \\ \mathrm{H}^n(\mathrm{F}(\mathrm{I})) & \xrightarrow{\sim} & \boldsymbol{R}^n(\mathrm{F}(\mathrm{A})) \end{array}$$

on the two horizontal arrows are the isomorphisms constructed in the proof, and the right vertical one is the identity. It follows that the left vertical arrow, which is the canonical morphism, is an isomorphism. □

CHAPTER 3

SHEAVES AND THEIR COHOMOLOGY

3.1. Presheaves and sheaves

Definition (**3.1.1**). — *Let* X *be a topological space. A* presheaf $\mathscr{F}$ *on* X *is the datum of a set* $\mathscr{F}(\mathrm{U})$ *for every open subset* U *of* X*, and of maps* $\rho_{\mathrm{UV}} \colon \mathscr{F}(\mathrm{U}) \to \mathscr{F}(\mathrm{V})$ *for every pair* (U, V) *of open subsets of* X *such that* $\mathrm{V} \subseteq \mathrm{U}$ *subject to the following conditions:*

– *If* U, V, W *are open subsets of* X *such that* $\mathrm{W} \subseteq \mathrm{V} \subseteq \mathrm{U}$*, one has* $\rho_{\mathrm{UW}} = \rho_{\mathrm{VW}} \circ \rho_{\mathrm{UV}}$*;*
– *For every open subset* U *of* X*, one has* $\rho_{\mathrm{UU}} = \mathrm{id}_{\mathscr{F}(\mathrm{U})}$.

Let U be an open subset of X. The set $\mathscr{F}(\mathrm{U})$ is also denoted by $\Gamma(\mathrm{U}, \mathscr{F})$; its elements are called the *sections* of $\mathscr{F}$ on U. The maps ρ_{UV} are called the *restriction maps.*; when $s \in \mathscr{F}(\mathrm{U})$, one also writes $s|_{\mathrm{V}}$ for $\rho_{\mathrm{UV}}(s)$.

Indeed, the basic intuition for presheaves is that of ‘‘generalized functions’’. Namely elements of $\mathscr{F}(\mathrm{U})$ have to be thought as of functions on U, and for $s \in \mathscr{F}(\mathrm{U})$, the element $\rho_{\mathrm{UV}}(s)$ of $\mathscr{F}(\mathrm{V})$ is a kind of restriction of s to V.

To avoid some possible confusions, the restriction maps of the presheaf $\mathscr{F}$ are sometimes denoted by $\rho^{\mathscr{F}}_{\mathrm{UV}}$.

Definition (**3.1.2**). — *Let* $\mathscr{F}, \mathscr{G}$ *be presheaves on the topological space* X*. A morphism of presheaves* $f \colon \mathscr{F} \to \mathscr{G}$ *is the datum, for every open subset* U *of* X*, of a map* $f(\mathrm{U}) \colon \mathscr{F}(\mathrm{U}) \to \mathscr{G}(\mathrm{U})$ *such that* $f(\mathrm{V}) \circ \rho^{\mathscr{F}}_{\mathrm{UV}} = \rho^{\mathscr{G}}_{\mathrm{UV}} \circ f(\mathrm{U})$ *for every pair* (U, V) *of open subsets of* X *such that* $\mathrm{V} \subseteq \mathrm{U}$.

When the maps $\mathscr{F}(\mathrm{U})$ is a subset of $\mathscr{G}(\mathrm{U})$ for every U, and the maps $f(\mathrm{U})$ are the inclusion maps, one says that $\mathscr{F}$ is a sub-presheaf of $\mathscr{G}$.

Morphisms of presheaves can be composed, etc., so that presheaves on the topological space X form a category $\boldsymbol{PreSh}_{\mathrm{X}}$.

3.1.3. — Let X be a topological space and let $\mathscr{F}$ be a presheaf on X. One says that $\mathscr{F}$ is a presheaf in abelian groups if the sets $\mathscr{F}(\mathrm{U})$ are endowed with structures of abelian groups and if the maps ρ_{UV} are morphism of abelian groups. A morphism of presheaves in abelian groups is a morphism of presheaves f such that the maps $f(\mathrm{U})$ are morphisms of abelian groups, for all open subsets U of X.

Similar definitions can be given for more general algebraic structures, such as modules, or rings, and even for general categories. A presheaf $\mathscr{F}$ with values in a category $\boldsymbol{C}$ is the datum of objects $\mathscr{F}(\mathrm{U})$ of $\boldsymbol{C}$ and of morphisms $\rho_{\mathrm{UV}} \in \boldsymbol{C}(\mathscr{F}(\mathrm{U}), \mathscr{F}(\mathrm{V}))$ satisfying the previous relations. A morphism of presheaves $f\colon \mathscr{F} \to \mathscr{G}$ with values in the category $\boldsymbol{C}$ is the datum of morphisms $f(\mathrm{U})$ in $\boldsymbol{C}$ satisfying the previous composition relations.

One can in fact give a compact definition of a presheaf with value in an arbitrary category $\boldsymbol{C}$. To this aim, define the category $\boldsymbol{Open}_{\mathrm{X}}$ of open subsets of X as follows: its objects are the open subsets of X, and its maps are the inclusions between open subsets. Explicitly, $\boldsymbol{Open}_{\mathrm{X}}(\mathrm{V}, \mathrm{U})$ is empty if $\mathrm{V} \not\subseteq \mathrm{U}$, and has exactly one element if $\mathrm{V} \subseteq \mathrm{U}$. A presheaf with values in a category $\boldsymbol{C}$ is a contravariant functor from the category $\boldsymbol{Open}_{\mathrm{X}}$ to the category $\boldsymbol{C}$; a morphism of presheaves is a natural transformation of functors.

Definition (**3.1.4**). — *Let $\mathscr{F}$ be a presheaf on the topological space* X. *Let* A *be a subspace of* X *and let $\mathscr{U}_{\mathrm{A}}$ be the set of open neighborhoods of* A *in* X*, endowed with the partial ordered opposite to inclusion. The colimit of the direct sytem of sets $(\mathscr{F}(\mathrm{U}))_{\mathrm{U}\in\mathscr{U}_{\mathrm{A}}}$ is called the set of germs of sections of the sheaf $\mathscr{F}$ on* A*; it is denoted by $\mathscr{F}_{\mathrm{A}}$. If* U *is an open neighborhood of* A *in* X *and $s \in \mathscr{F}(\mathrm{U})$, the canonical image s_{A} of s in $\mathscr{F}_{\mathrm{A}}$ is called the* germ *of s on* A.

When A is reduced to a single point $\{x\}$, the set $\mathscr{F}_{\mathrm{A}}$ is called the *stalk* of $\mathscr{F}$ at the point x, and is denoted by $\mathscr{F}_x$.

If $\mathscr{F}$ is a presheaf in abelian groups then $\mathscr{F}_{\mathrm{A}}$ is an abelian group, and the maps $s \mapsto s_{\mathrm{A}}$ from $\mathscr{F}(\mathrm{U})$ to $\mathscr{F}_{\mathrm{A}}$ are morphisms of abelian groups.

Let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on the topological space X. By the universal property of the limit, there exists a unique map $f_{\mathrm{A}}\colon \mathscr{F}_{\mathrm{A}} \to \mathscr{G}_{\mathrm{A}}$ between the sets of germs of sections at A such that $f_{\mathrm{A}}(s_{\mathrm{A}}) = f(s)_{\mathrm{A}}$ for every open neighborhood U of A and every section $s \in \mathscr{F}(\mathrm{U})$. If f is a morphism of presheaves of abelian groups, then f_{A} is a morphism of abelian groups.

Similar results hold for presheaves in algebraic structures, such as rings or modules, for which the notion of (filtrant) colimit makes sense.

Definition (**3.1.5**). — *Let* X *be a topological space and let* $\mathscr{F}$ *be a presheaf on* X. *One says that* $\mathscr{F}$ *is a* sheaf *if the following property holds: For every open subset* U *of* X, *every family* $(U_i)_{i\in I}$ *of open subsets of* X *such that* $U = \bigcup_{i\in I} U_i$, *every family* $(s_i)_{i\in I}$, *where* $s_i \in \mathscr{F}(U_i)$ *for every* i, *such that* $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, *there exists a unique element* $s \in \mathscr{F}(U)$ *such that* $s|_{U_i} = s_i$ *for every* $i \in I$.

In words, a presheaf $\mathscr{F}$ is a sheaf provided every family (s_i) of sections of $\mathscr{F}$ on open subsets U_i of X which coincide on the intersections $U_i \cap U_j$ can be "glued" uniquely to a section s of $\mathscr{F}$ on the union of the sets U_i.

This definition needs to be adapted for presheaves with values in a general category. Thus let $\mathscr{F}$ be a presheaf on X with values in a category $\boldsymbol{C}$. One says that $\mathscr{F}$ is a sheaf if for every open subset U of X and every family $(U_i)_{i\in I}$ of open subsets of X such that $U = \bigcup_{i\in I} U_i$, every object M of $\boldsymbol{C}$ and every family $(f_i\colon M \to \mathscr{F}(U_i))_{i\in I}$ of morphisms in $\boldsymbol{C}$ such that $\rho_{U_i,U_i\cap U_j} \circ f_i = \rho_{U_j,U_i\cap U_j} \circ f_j$ for every $i, j \in I$, there exists a unique morphism $f\colon M \to \mathscr{F}(U)$ in $\boldsymbol{C}$ such that $\rho_{U,U_i} \circ f = f_i$ for every $i \in I$.

Let Q be the quiver whose vertex set V is $I \times I$, whose set of arrows E is $I \times I \times \{0, 1\}$, and whose source and target maps are given by

$$s((i,j,0)) = (i,i), \quad s((i,j,1)) = (j,j), \quad t((i,j,0)) = t((i,j,1)) = (i,j).$$

The presheaf $\mathscr{F}$ gives rise to a Q-diagram in the category $\boldsymbol{C}$ whose value at the vertex (i, j) is $\mathscr{F}(U_i \cap U_j)$, whose value at an arrow $(i, j, 0)$ is the restriction morphism $\rho_{U_i,U_i\cap U_j}$. and whose value at an arrow $(i, j, 1)$ is the restriction morphism $\rho_{U_j,U_i\cap U_j}$. The above sheaf condition means that the object $\mathscr{F}(U)$ of $\boldsymbol{C}$, endowed with the morphisms ρ_{UU_i} for $i \in I$, represents the colimit of this Q-diagram.

3.1.6. — A morphism of sheaves is just a morphism of the underlying presheaves. In other words, sheaves of X form a full subcategory $\boldsymbol{Sh}_X$ of the category of presheaves on X.

Example (**3.1.7**). — Let X be an open subset of $\mathbf{R}^n$, or a differentiable manifold of class $\mathscr{C}^p$ for some $p \geqslant 1$.

a) Functions of class $\mathscr{C}^p$ on X give rise to a sheaf $\mathscr{C}^p_X$ on X. Precisely, for every open subset U of X, let $\mathscr{C}^p_X(U)$ be the set of all functions $\varphi\colon U \to \mathbf{R}$ of class $\mathscr{C}^p$; for $V \subseteq U$, let $\rho_{UV}\colon \mathscr{C}^p_X(U) \to \mathscr{C}^p_X(V)$ be the restriction map. This defines a presheaf $\mathscr{C}^p_X$ in **R**-algebras on X; this presheaf is a sheaf.

Indeed, let U be an open subset of X, let $(U_i)_{i\in I}$ be a family of open subsets of X whose union is U, and let (s_i) be a family of sections of $\mathscr{C}^p_X$, where $s_i \in \mathscr{C}^p_X(U_i)$, such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. There exist a unique function $s\colon U \to \mathbf{R}$ such that $s(x) = s_i(x)$ for $x \in U_i$. This function s satisfies $s|_{U_i} = s_i$ for every i, and is of class $\mathscr{C}^p$. Indeed, if $x \in U$ and $i \in I$ is such that $x \in U_i$, then U_i is a neighborhood of x in U on which the restriction of s is of class $\mathscr{C}^p$.

b) Bounded functions on X give rise to a presheaf $\mathscr{B}_X$ on X, where $\mathscr{B}_X(U)$ is the set of all functions $s\colon U \to \mathbf{R}$ which are bounded. However, this presheaf is generally not a sheaf, unless X is finite (hence 0-dimensional). Let indeed $a \in X$ be a point which is not isolated, let $U = X - \{a\}$, and let $s\colon U \to \mathbf{R}$ be the function $x \mapsto 1/d(a,x)$, where d is a distance on X compatible with its topology. For every integer $m \geqslant 1$, let U_m be the set of points $x \in U$ such that $d(a,x) \geqslant 1/m$. The union of the open sets U_m is equal to U, the restriction of s to U_m is bounded for every m, but s is not bounded.

c) Vector fields, distributions, etc. furnish other natural examples of sheaves on X. Tempered distributions form a sub-presheaf of the sheaf of distributions, but do not form a sheaf themselves.

Example **(3.1.8)**. — Let X be a topological space and let $\mathscr{F}$ be a presheaf on X. Let Y be an open subspace of X. One defines a presheaf $\mathscr{F}|_Y$ on Y by setting $\mathscr{F}|_Y(U) = \mathscr{F}(U)$ for every open subset U of Y, and by keeping the same restriction maps. If $f\colon \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, the maps $f(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$, for every open subset U of Y, define a morphism of presheaves $f|_Y\colon \mathscr{F}|_Y \to \mathscr{G}|_Y$.

If $\mathscr{F}$ is a presheaf of abelian groups, then so is $\mathscr{F}|_Y$.

If $\mathscr{F}$ is a sheaf, then so is $\mathscr{F}|_Y$.

Example **(3.1.9)**. — Let X be a topological space and let $\mathscr{F}, \mathscr{G}$ be presheaves of abelian groups on X. One defines a presheaf of abelian groups $\mathscr{H}$ on X by setting, for every open subset U of X, $\mathscr{H}(U) = \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$. (Note that $\mathscr{H}(U)$ is a set of morphism of sheaves from $\mathscr{F}|_U$ to $\mathscr{G}|_U$, and should not be confused with the morphisms of abelian groups from $\mathscr{F}(U)$ to $\mathscr{G}(U)$.) The restriction maps ρ_{UV} are defined as follows: let U and V be open subsets of X such that $V \subseteq U$ and let $f \in \mathscr{H}(U)$; one sets $\rho_{UV}(f) = f_V$. This is the *presheaf of morphisms* from $\mathscr{F}$ to $\mathscr{G}$; it is denoted by $\mathscr{H}om(\mathscr{F},\mathscr{G})$.

Assume that $\mathscr{G}$ is a sheaf; then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is a *sheaf*. Let indeed U be an open subset of X, let $(U_i)_{i\in I}$ be a family of open subsets of X such that $U = \bigcup_{i\in I} U_i$, let (φ_i) be a family, where $\varphi_i \in \mathscr{H}(U_i)$, such that $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$. Let us show that there exists a unique morphism of presheaves $\varphi\colon \mathscr{F}|_U \to \mathscr{G}|_U$ such that $\varphi|_{U_i} = \varphi_i$ for every i.

Let V be an open subset of U and let $s \in \mathscr{F}(U)$; for every i, set $V_i = V \cap U_i$ and $t_i = \varphi_i(s|_{V_i}) \in \mathscr{G}(V_i)$. For $i, j \in I$, one has

$$t_i|_{V_i\cap V_j} = \varphi_i(s|_{V_i})|_{V_i\cap V_j}\varphi_i(s|_{V_i\cap V_j}) = \varphi_j(s|_{V_i\cap V_j}) = t_j|_{V_i\cap V_j};$$

since $\mathscr{G}$ is a sheaf, there exists a unique section $t \in \mathscr{G}(V)$ such that $t|_{V_i} = t_i$ for every $i \in I$. Set $\varphi(V)(s) = t$. This defines a map $\varphi(V)\colon \mathscr{F}(V) \to \mathscr{G}(V)$. For $s, s' \in \mathscr{F}(V)$ and $i \in I$, one has

$$\begin{aligned}\varphi(V)(s+s')|_{V_i} &= \varphi_i(s|_{V_i} + s'|_{V_i}) \\ &= \varphi_i(s|_{V_i}) + \varphi_i(s|_{V_i}) \\ &= (\varphi(V)(s) + \varphi(V)(s'))|_{V_i};\end{aligned}$$

consequently, $\varphi(V)(s+s') = \varphi(V)(s) + \varphi(V)(s')$, hence $\varphi(V)$ is a morphism of abelian groups. Moreover, if V and W are open subsets of U such that $W \subseteq V$, then

$$\begin{aligned}\varphi(W)(s|_W)|_{W\cap U_i} &= \varphi_i(s|_{W\cap U_i}) \\ &= \varphi_i(s|_{V\cap U_i})|_{W\cap U_i} \\ &= \varphi(s)|_{W\cap U_i} \\ &= (\varphi(s)|_W)|_{W\cap U_i}\end{aligned}$$

for every i. Consequently, $\varphi(W)(s|_W) = \varphi(W)(s)|_W$. This shows that the family $\varphi = (\varphi(V))$ is a morphism of presheaves of abelian groups from $\mathscr{F}|_U$ to $\mathscr{G}|_U$.

Conversely, a morphism $\varphi'\colon \mathscr{F}|_U \to \mathscr{G}|_U$ such that $\varphi'|_{U_i} = \varphi$ is equal to φ. Indeed, for $V \subseteq U$ and $s \in \mathscr{F}(V)$, one has

$$\varphi'(V)(s)|_{V\cap U_i} = \varphi'(V)(s|_{V\cap U_i}) = \varphi_i(s|_{V\cap U_i} = \varphi(v)(s)|_{V\cap U_i},$$

so that $\varphi'(V)(s) = \varphi(V)(s)$; this shows that $\varphi' = \varphi$, as claimed.

Lemma **(3.1.10)**. — *Let* X *be a topological space and let* U *be an open subset of* X.

a) *Let* $\mathscr{F}$ *be a sheaf on* X *and let* $s, t \in \mathscr{F}(U)$ *be two sections such that* $s_x = t_x$ *for every* $x \in U$*. Then* $s = t$*.*

b) *Let* $\mathscr{F}, \mathscr{G}$ *be presheaves on* X *and let* $f, g\colon \mathscr{F} \to \mathscr{G}$ *be morphisms of presheaves on* X *such that* $f_x = g_x$ *for every* $x \in X$*. If* $\mathscr{G}$ *is a sheaf, then* $f = g$*.*

Proof. — a) Let $x \in U$; since $s_x = t_x$, there exists an open subset U_x of U containing x such that $s|_{U_x} = t|_{U_x}$. By the definition of a sheaf, applied to the family $(U_x)_{x\in U}$ of open subsets of X and to the family $(s|_{U_x})$, the section t is the unique element of $\mathscr{F}(U)$ whose restriction to U_x is equal to $s|_{U_x}$. One thus has $s = t$.

b) Let U be an open subset of X and let $s \in \mathscr{F}(U)$. We need to prove that s has the same image under the maps $f(U)$ and $g(U)$; let $t = f(U)(s)$ and $t' = g(U)(s)$. For every $x \in U$, one has $t_x = f_x(s_x) = g_x(s_x) = t'_x$. Since $\mathscr{G}$ is a sheaf, it follows from *a*) that $t = t'$, as claimed.

□

Lemma **(3.1.11)** (Glueing sheaves and morphisms of sheaves)

Let X *be a topological space, let* $(U_i)_{i\in I}$ *be a family of open subsets of* X *such that* $X = \bigcup_{i\in I} U_i$*.*

a) *Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be presheaves on* X*; assume that* $\mathscr{G}$ *is a sheaf. For every* $i \in I$*, let* $\varphi_i\colon \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ *be a morphism of presheaves. Assume that for every* $i, j \in I$*, the morphisms* φ_i *and* φ_j *coincide on* $U_i \cap U_j$*. Then there exists a unique morphism of presheaves* $\varphi\colon \mathscr{F} \to \mathscr{G}$ *such that* $\varphi_i = \varphi|_{U_i}$ *for every* $i \in I$*.*

If both $\mathscr{F}$ *and* $\mathscr{G}$ *are sheaves and* φ_i *is an isomorphism for every* $i \in I$*, then* φ *is an isomorphism.*

b) *For every* $i \in I$*, let* $\mathscr{F}_i$ *be a sheaf on* U_i*; for every pair* (i, j) *of elements of* I*, let* $\varphi_{ij}\colon \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ *be an isomorphism of sheaves. Assume that following properties hold:*

(i) *For* $i \in I$*, one has* $\varphi_{ii} = \mathrm{Id}_{\mathscr{F}_i}$*;*

(ii) *For* $i, j \in I$*, one has* $\varphi_{ij} = \varphi_{ji}^{-1}$*;*

(iii) *For* $i, j, k \in I$*, the morphisms* $\varphi_{jk} \circ \varphi_{ij}|_{U_i\cap U_j\cap U_k}$ *and* $\varphi_{ik}|_{U_i\cap U_j\cap U_k}$ *coincide.*

Then there exists a sheaf $\mathscr{F}$ *on* X*, and for every* $i \in I$*, an isomorphism* $\varphi_i\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$*, such that* $\varphi_{ij}\circ\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$ *for every pair* (i, j) *of elements of* I*. Moreover, if* $\mathscr{G}$ *is a sheaf on* x *and* (ψ_i) *a family of isomorphisms from* $\mathscr{G}_{U_i}$ *to* $\mathscr{F}_i$ *satisfying* *these requirements, there exists a unique morphism of sheaves* ψ *from* $\mathscr{F}$ *to* $\mathscr{G}$ *such that* $\psi|_{U_i} = \psi_i^{-1} \circ \varphi_i$*, and it is an isomorphism.*

Analogous results are valid for presheaves of abelian groups, rings, modules, etc.

Proof. — a) Let U be an open subset of X, let $s \in \mathscr{F}(U)$. For every $i \in I$, one has $\varphi_i(U \cap U_i)(s) \in \mathscr{F}(U \cap U_i)$ and for $i, j \in I$, one has

$$\begin{aligned}\varphi_i(U \cap U_i)(s)|_{U\cap U_i \cap U_j} &= \varphi_i(U \cap U_i \cap U_j)(s|_{U\cap U_i\cap U_j}) \\ &= \varphi_j(U \cap U_i \cap U_j)(s|_{U\cap U_i\cap U_j}) \\ &= \varphi_j(U \cap U_j)(s)|_{U\cap U_i\cap U_j}.\end{aligned}$$

Since $\mathscr{G}$ is a sheaf, there exists a unique section $\varphi(U)(s) \in \mathscr{G}(U)$ such that $\varphi(U)(s)|_{U_i} = \varphi_i(U \cap U_i)(s)$ for every $i \in I$. This defines a map $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$. The family $\varphi = (\varphi(U))$ is a morphism of presheaves such that $\varphi|_{U_i} = \varphi_i$. It is the unique such morphism.

Assume that both $\mathscr{F}$ and $\mathscr{G}$ are sheaves, and that φ_i is an isomorphism for every i. Then there exists a unique morphism of sheaves $\varphi'\colon \mathscr{G} \to \mathscr{F}$ such that $\varphi'|_{U_i} = \varphi_i^{-1}$ for every i. One has $\varphi' \circ \varphi = \mathrm{Id}_{\mathscr{F}}$, because it is the unique morphism of sheaves from $\mathscr{F}$ to itself whose restriction to U_i is the identity, for every $i \in I$. Similarly, $\varphi \circ \varphi' = \mathrm{Id}_{\mathscr{F}}$. This shows that φ is an isomorphism and concludes the proof of *a*).

b) Let U be an open subset of X; let $\mathscr{F}(U)$ be the set of all families $(s_i) \in \prod_{i\in I} \mathscr{F}_i(U \cap U_i)$ such that

$$s_j|_{U\cap U_i\cap U_j} = \varphi_{ij}(U \cap U_i \cap U_j)(s_i|_{U\cap U_i\cap U_j})$$

for every $i, j \in I$. If U and V are open subsets of X such that $V \subseteq U$, let $\rho_{UV}\colon \mathscr{F}(U) \to \mathscr{F}(V)$ be the map defined by $\rho_{UV}((s_i)) = (s_i|_{V\cap U_i})$. Then $\mathscr{F}$ is a presheaf on X.

Let us show that $\mathscr{F}$ is a sheaf. Let $(V_\lambda)_{\lambda\in L}$ be a family of open subsets of X, let $V = \bigcup_{\lambda\in L} V_\lambda$; for every $\lambda \in L$, let $s_\lambda \in \mathscr{F}(U_\lambda)$; assume that $s_\lambda|_{U_\lambda\cap U_\mu} = s_\mu|_{U_\lambda\cap U_\mu}$ for every $\lambda, \mu \in L$. One has $s_\lambda = (s_{\lambda,i})_{i\in I}$. Fix $i \in I$; for every $\lambda \in L$, one has $s_{\lambda,i} \in \mathscr{F}_i(V_\lambda \cap U_i)$; for every $\lambda, \mu \in L$, one has

$$s_{\lambda,i}|_{V_\lambda\cap V_\mu\cap U_i} = s_{\mu,i}|_{V_\lambda\cap V_\mu\cap U_i}.$$

Consequently, there exists a unique element $s_i \in \mathscr{F}_i(V \cap U_i)$ such that $s_i|_{V_\lambda\cap U_i} = s_{\lambda,i}$ for every $\lambda \in L$. Then

$$s_j|_{V\cap U_i\cap U_j} = \varphi_{ij}(V \cap U_i \cap U_j)(s_i|_{V\cap U_i\cap U_j}),$$

since for every $\lambda \in \mathrm{L}$, one has

$$s_j|_{\mathrm{V}_\lambda \cap \mathrm{U}_i \cap \mathrm{U}_j} = \varphi_{ij}(\mathrm{V}_\lambda \cap \mathrm{U}_i \cap \mathrm{U}_j)(s_i|_{\mathrm{V}_\lambda \cap \mathrm{U}_i \cap \mathrm{U}_j}).$$

This implies that $s = (s_i)_{i \in \mathrm{I}}$ is an element of $\mathscr{F}(\mathrm{V})$. Moreover, for every $\lambda \in \mathrm{L}$, one has

$$s|_{\mathrm{V}_\lambda} = (s_i|_{\mathrm{V}_\lambda \cap \mathrm{U}_i}) = (s_{\lambda,i}) = s_\lambda.$$

The section s is the only section of $\mathscr{F}$ on V such that $s|_{\mathrm{V}_\lambda} = s_\lambda$ for every $\lambda \in \mathrm{L}$. This concludes the proof that $\mathscr{F}$ is a sheaf on X.

Let $k \in \mathrm{I}$. For every open subset of X such that $\mathrm{U} \subseteq \mathrm{U}_k$, let $\varphi_k(\mathrm{U}) \colon \mathscr{F}(\mathrm{U}) \to \mathscr{F}_k(\mathrm{U})$ be the map given by $(s_i)_{i \in \mathrm{I}} \mapsto s_k$. The family $(\varphi_k(\mathrm{U}))$ is a morphism of sheaves from $\mathscr{F}|_{\mathrm{U}_k}$ to $\mathscr{F}_k$. By definition, for $s = (s_i) \in \mathscr{F}(\mathrm{U})$, and $i \in \mathrm{I}$, the section $s_i \in \mathscr{F}_i(\mathrm{U} \cap \mathrm{U}_i)$ satisfies (recall that $\mathrm{U} \subseteq \mathrm{U}_k$)

$$s_i = \varphi_{ki}(\mathrm{U} \cap \mathrm{U}_i)(s_k|_{\mathrm{U} \cap \mathrm{U}_i}).$$

Conversely, let $s \in \mathscr{F}(\mathrm{U}_k)$; for every $i \in \mathrm{I}$ define $s_i = \varphi_{ki}(\mathrm{U} \cap \mathrm{U}_i)(s|_{\mathrm{U} \cap \mathrm{U}_i})$. Since $\varphi_{kk} = \mathrm{Id}$, one has $s_k = s$. Moreover, for every $i, j \in \mathrm{I}$, one has

$$\begin{aligned}
\varphi_{ij}(\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j)&(s_i|_{\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j}) \\
&= \varphi_{ij}(\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j)(\varphi_{ki}(\mathrm{U} \cap \mathrm{U}_i)(s_k|_{\mathrm{U} \cap \mathrm{U}_i})|_{\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j}) \\
&= \varphi_{kj}(\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j)(s_k|_{\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j}) \\
&= s_j|_{\mathrm{U} \cap \mathrm{U}_i \cap \mathrm{U}_j}.
\end{aligned}$$

Consequently, the family (s_i) belongs to $\mathscr{F}(\mathrm{U})$ and is the unique element preimage of $\mathscr{F}(\mathrm{U})$ such that $\varphi_k(\mathrm{U})((s_i)) = s$. This implies that φ_k is an isomorphism of sheaves.

For $j, k \in \mathrm{I}$, every open subset U of $\mathrm{U}_j \cap \mathrm{U}_k$, and every family $(s_i) \in \mathscr{F}(\mathrm{U})$ one also has

$$\varphi_{jk}(\mathrm{U}) \circ \varphi_k(\mathrm{U})((s_i)) = \varphi_{jk}(\mathrm{U})(s_k) = s_j = \varphi_j(\mathrm{U})((s_i)),$$

hence $\varphi_{jk} \circ \varphi_k|_{\mathrm{U}_j \cap \mathrm{U}_k} = \varphi_j|_{\mathrm{U}_j \cap \mathrm{U}_k}$.

This concludes the proof of the first part of assertion b). The rest of the assertion follows from a). Let indeed $\mathscr{G}$ be a sheaf on X and let (ψ_i) be a family, where $\psi_i \colon \mathscr{G}|_{\mathrm{U}_i} \to \mathscr{F}_i$ such that $\varphi_{ij} \circ \psi_i|_{\mathrm{U}_i \cap \mathrm{U}_j} = \psi_j|_{\mathrm{U}_i \cap \mathrm{U}_j}$ for every $i, j \in \mathrm{I}$. For every $i \in \mathrm{I}$, $\theta_i = \varphi_i^{-1} \circ \psi_i$ is a morphism of sheaves from $\mathscr{G}|_{\mathrm{U}_i}$ to $\mathscr{F}|_{\mathrm{U}_i}$; For $i, j \in \mathrm{I}$, one has

$$\begin{aligned}
\theta_i|_{\mathrm{U}_i \cap \mathrm{U}_j} = \varphi_i^{-1} \circ \psi_i|_{\mathrm{U}_i \cap \mathrm{U}_j} &= \varphi_j^{-1}|_{\mathrm{U}_i \cap \mathrm{U}_j} \circ \varphi_{ij} \circ \varphi_{ji} \circ \psi_j|_{\mathrm{U}_i \cap \mathrm{U}_j} \\
&= \theta_j|_{\mathrm{U}_i \cap \mathrm{U}_j}.
\end{aligned}$$

By *a*), there exists a unique morphism of sheaves $\theta\colon \mathscr{G} \to \mathscr{F}$ such that $\theta|_{U_i} = \theta_i$ for every i, and it is an isomorphism. □

3.2. Some constructions of sheaves

3.2.1. Limits. — Let $Q = (V, E, s, t)$ be a quiver and let $\mathscr{F}_Q = ((\mathscr{F}_v), (\varphi_e))$ be a diagram of presheaves on a topological space X.

For every open subset U of X, this diagram induces a diagram $\mathscr{F}_Q(U) = ((\mathscr{F}_v(U)), (\varphi_e(U)))$ of sets. We denote its limit by $\mathscr{F}(U)$; for $v \in V$, let $\varphi_v(U)\colon \mathscr{F}(U) \to \mathscr{F}_v(U)$ be the canonical map.

Let U and V be open subsets of X such that $V \subseteq U$. The family of maps $(\rho^{\mathscr{F}_v}_{UV} \circ \varphi_v(U))$ is a cone on the diagram $\mathscr{F}_Q(V)$. Consequently, there exists a unique map $\rho^{\mathscr{F}}_{UV}\colon \mathscr{F}(U) \to \mathscr{F}(V)$ such that $\varphi_v(V) \circ \rho^{\mathscr{F}}_{UV} = \rho^{\mathscr{F}_v}_{UV} \circ \varphi_v(U)$ for every v.

The family of sets $(\mathscr{F}(U))$ and the family of maps $(\rho^{\mathscr{F}}_{UV})$ form a presheaf $\mathscr{F}$ on X.

Proposition (**3.2.2**). — a) *The presheaf* $\mathscr{F}$ *is a limit of the diagram* $\mathscr{F}_Q$ *in the category of presheaves on* X.

b) *If the* $\mathscr{F}_v$ *are sheaves, then* $\mathscr{F}$ *is a sheaf, and is a limit of the diagram* $\mathscr{F}_Q$ *in the category of sheaves on* X.

c) *The analogous results hold when* $\mathscr{F}_Q$ *is a diagram of presheaves in abelian groups, in rings, in* A*-modules, etc.*

Proof. — a) Let $(\mathscr{G}, (\psi_v))$ be a cone on the diagram $\mathscr{F}_Q$ of presheaves. For every open subset U of X, the set $\mathscr{G}(U)$, with the maps $\psi_v(U)$, is a cone on the diagram $\mathscr{F}_Q(U)$ of sets. Consequently, there exists a unique map $\theta(U)\colon \mathscr{G}(U) \to \mathscr{F}(U)$ such that $\varphi_v(U) \circ \theta(U) = \psi_v(U)$ for every v.

Let U and V be open subsets of X such that $V \subseteq U$. Since ψ_v is a morphism of presheaves, one has

$$\varphi_v(V) \circ \theta(V) \circ \rho^{\mathscr{G}}_{UV} = \psi_v(V) \circ \rho^{\mathscr{G}}_{UV} = \rho^{\mathscr{F}_v}_{UV} \circ \psi_v(U) = \rho^{\mathscr{F}_v}_{UV} \circ \varphi_v(U) \circ \theta(U)$$

for every v. Consequently, $\theta(V) \circ \rho^{\mathscr{G}}_{UV} = \rho^{\mathscr{F}}_{UV} \circ \theta(U)$.

This shows that the family $(\theta(U))$ is the unique morphism of presheaves from $\mathscr{G}$ to $\mathscr{F}$ such that $\psi_v = \varphi_v \circ \theta$ for every v.

b) Let $(U_i)_{i \in I}$ be a family of open subsets of X and let U be its union. For every $i \in I$, let $s_i \in \mathscr{F}(U_i)$; assume that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every i, j and let us show that there exists a unique element $s \in \mathscr{F}(\mathrm{U})$ such that $s|_{\mathrm{U}_i} = s_i$ for every $i \in \mathrm{I}$.

For every v, one has $\varphi_v(s_i) \in \mathscr{F}_v(\mathrm{U}_i)$ and $\varphi_v(s_i)|_{\mathrm{U}_i\cap\mathrm{U}_j} = \varphi_v(s_j)|_{\mathrm{U}_i\cap\mathrm{U}_j}$. Since $\mathscr{F}_v$ is a sheaf, there exists a unique element $s_v \in \mathscr{F}_v(\mathrm{U})$ such that $s_v|_{\mathrm{U}_i} = \varphi_v(s_i)$ for every $i \in \mathrm{I}$.

For every arrow e of Q, with source v and target v' one has $\varphi_e(s_v) = s_{v'}$, because these two sections of $\mathscr{F}_{v'}$ have the same restriction on U_i, for every $i \in \mathrm{I}$. Consequently, there exists a unique element $s \in \mathscr{F}(\mathrm{U})$ such that $\varphi_v(s) = s_v$ for every v.

Let $i \in \mathrm{I}$. One has $s|_{\mathrm{U}_i} = s_i$, because both sections of $\mathscr{F}(\mathrm{U}_i)$ map to $\varphi_v(s_i)$, for every v. Conversely, if s' is a section of $\mathscr{F}$ over U such that $s'|_{\mathrm{U}_i} = s_i$ for every i. One then has $\varphi_v(s')|_{\mathrm{U}_i} = \varphi_v(s'|_{\mathrm{U}_i}) = \varphi_v(s_i) = \varphi_v(s)|_{\mathrm{U}_i}$, hence $\varphi_v(s') = \varphi_v(s)$, because $\mathscr{F}_v$ is a sheaf. By definition of the presheaf $\mathscr{F}$, one then has $s' = s$.

c) Assume that the (pre)sheaves $\mathscr{F}_v$ are (pre)sheaves in abelian groups and the morphisms φ_e are morphisms of presheaves in abelian groups. For every open subset U of X, the set $\mathscr{F}(\mathrm{U})$ has a natural structure of an abelian group such that the morphisms $\varphi_v(\mathrm{U})\colon \mathscr{F}(\mathrm{U}) \to \mathscr{F}_v(\mathrm{U})$ are morphisms of abelian groups. Moreover, the maps $\rho^{\mathscr{F}}_{\mathrm{UV}}$ are morphisms of abelian groups, so that $\mathscr{F}$ is really a presheaf in abelian groups. In the proof of *a*), one checks that if morphisms ψ_v are morphisms of presheaves of abelian groups, then so is the morphism θ that we constructed.

The cases of (pre)sheaves of rings, etc. are analogous. □

Example **(3.2.3)**. — Let $\mathscr{G}$ be a presheaf on X and let $(\mathscr{F}_i)_{i\in\mathrm{I}}$ be a family of sub-presheaves of $\mathscr{G}$. Their intersection $\mathscr{F} = \bigcap_i \mathscr{F}_i$ is defined by $\mathscr{F}(\mathrm{U}) = \bigcap_i \mathscr{F}_i(\mathrm{U})$ for every open subset U of X; it is a sub-presheaf of $\mathscr{G}$. This presheaf is the colimit of the diagram of presheaves whose arrows are the inclusion morphisms $\mathscr{F}_i \hookrightarrow \mathscr{G}$.

If the $\mathscr{F}_i$ are sheaves, then so is $\mathscr{F}$.

3.2.4. — If $\mathscr{G}$ is a sheaf and $\mathscr{F}$ is a sub-presheaf of $\mathscr{G}$, there exists a smallest subsheaf $\mathscr{F}'$ of $\mathscr{G}$ which contains $\mathscr{F}$, called the subsheaf of $\mathscr{G}$ generated by $\mathscr{F}$. It is the intersection of all sub-sheaves of $\mathscr{G}$ which contain $\mathscr{F}$. For every open subset U of X, $\mathscr{F}'(\mathrm{U})$ is the set of sections $s \in \mathscr{G}(\mathrm{U})$ such that every point $x \in \mathrm{U}$ has an open neighborhood V contained in x such that $s|_{\mathrm{V}} \in \mathscr{F}(\mathrm{V})$.

Let indeed $\mathscr{F}'_0(U)$ be this subset. The family $\mathscr{F}'_0 = (\mathscr{F}'_0(U))$ is a sub-presheaf of $\mathscr{G}$ which contains $\mathscr{F}$. It also contains every sheaf that contains $\mathscr{F}$, hence contains $\mathscr{F}'$. It thus suffices to show that $\mathscr{F}'_0$ is a subsheaf of $\mathscr{G}$. Let then U be an open subset of X, let $(U_i)_{i\in I}$ be a family of open subsets of X such that $U = \bigcup_{i\in I} U_i$, let (s_i) be a family, where $s_i \in \mathscr{F}'_0(U_i)$ for every i, such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, for every $i, j \in I$. Since $\mathscr{G}$ is a sheaf, there exists a unique section $s \in \mathscr{G}(U)$ such that $s|_{U_i} = s_i$, for every $i \in I$. Moreover, $s \in \mathscr{F}'_0(U)$; let indeed $x \in U$, let $i \in I$ be such that $x \in U_i$, and let V be an open neighborhood of x contained in U_i such that $s_i|_V \in \mathscr{F}(V)$; then $s|_V = s_i|_V \in \mathscr{F}(V)$; consequently, $s \in \mathscr{F}'_0(U)$, as claimed.

3.2.5. Image of a morphism of sheaves. — Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on X. For every open subset U of X, let $\mathscr{I}_{\mathrm{pre}}(U) = \varphi(U)(\mathscr{F}(U))$. Then $\mathscr{I}_{\mathrm{pre}}$ is a sub-presheaf of $\mathscr{G}$.

Assume that $\mathscr{G}$ is a sheaf; one defines the subsheaf *image of* φ as the smallest subsheaf of $\mathscr{G}$ which contains the presheaf $\mathscr{I}_{\mathrm{pre}}$. It is denoted by $\mathrm{Im}(\varphi)$.

If $\mathscr{F}$ and $\mathscr{G}$ are (pre)sheaves in abelian groups and φ is a morphism of presheaves in abelian groups, then $\mathrm{Im}(\varphi)$ is subsheaf in abelian groups. Similar results hold for (pre)sheaves in rings, modules, etc.

Remark (**3.2.6**). — When $\mathscr{G}$ is a sheaf, the presheaf $\mathscr{I}_{\mathrm{pre}}$ is generally not a subsheaf of $\mathscr{G}$, even if $\mathscr{F}$ itself is a sheaf. For example, let $\mathscr{F}$ and $\mathscr{G}$ be both equal to the sheaf $\mathscr{C}_X$ of complex valued continuous functions on X, and let $\varphi\colon \mathscr{C}_X \to \mathscr{C}_X$ be given by $\varphi(U)(f) = \exp(f)$, for $f \in \mathscr{C}(U;\mathbf{C})$. If $X = \mathbf{C}^*$, there does not exist a continuous function $f\colon X \to \mathbf{C}$ such that $x = \exp(f(x))$, for every $x \in \mathbf{C}^*$. However, for every open subset U of X, small enough to be contained in a contractible subset of $\mathbf{C}^*$, there exists a function $f_U\colon U \to \mathbf{C}$ such that $x = \exp(f_U(x))$ for every $x \in U$. In other words, the identity function g (given by $g(x) = x$) does not belong to $\mathscr{I}_{\mathrm{pre}}(X)$, although every point of $\mathbf{C}^*$ has a neighborhood U such that $g|_U$ belongs to $\mathscr{I}_{\mathrm{pre}}(U)$.

Theorem (**3.2.7**). — *Let* X *be a topological space. Let* $\mathscr{F}$ *be a presheaf on* X. *There exists a* sheaf $\mathscr{F}^+$ *on* X *and a morphism of presheaves* $j\colon \mathscr{F} \to \mathscr{F}^+$ *which satisfies the following universal property: for every sheaf* $\mathscr{G}$ *on* X *and every morphism* $f\colon \mathscr{F} \to \mathscr{G}$ *of presheaves, there exists a unique morphism of sheaves* $\varphi\colon \mathscr{F}^+ \to \mathscr{G}$ *such that* $f = \varphi \circ j$.

Moreover, for every $x \in X$, *the map* $j_x\colon \mathscr{F}_x \to \mathscr{F}^+_x$ *is a bijection.*

If $(\mathscr{F}^+, j)$ and $(\mathscr{F}^\circ, j')$ are two morphisms satisfying this universal property, there exists a unique morphism $\varphi\colon \mathscr{F}^+ \to \mathscr{F}^\circ$ such that $j' = \varphi \circ j$, and this morphism is an isomorphism. This can be proved by the usual kind of arguments. One can also observe that the above universal property says that the sheaf $\mathscr{F}^+$ represents the functor $\mathscr{G} \mapsto \mathrm{Hom}(\mathscr{F}, \mathscr{G})$ on the category of sheaves on X.

The sheaf $\mathscr{F}^+$ is called the *sheaf associated* with $\mathscr{F}$.

Proof. — One first defines a presheaf $\mathscr{E}$ on X such that $\mathscr{E}(\mathrm{U}) = \prod_{x\in \mathrm{U}} \mathscr{F}_x$, for every open subset U on X, the restriction morphisms being the obvious ones: if $\mathrm{V} \subseteq \mathrm{U}$ and $s = (s_x)_{x\in\mathrm{U}} \in \mathscr{E}(\mathrm{U})$, then $s|_{\mathrm{V}} = (s_x)_{x\in\mathrm{V}}$. There is a morphism of presheaves $j\colon \mathscr{F} \to \mathscr{E}$, given by $s \mapsto (s_x)_{x\in\mathrm{U}}$, whenever U is an open subset of X and $s \in \mathscr{F}(\mathrm{U})$.

This presheaf $\mathscr{E}$ is in fact a sheaf. Let indeed U be an open subset of X, $(\mathrm{U}_i)_{i\in\mathrm{I}}$ a family of open subsets of X such that $\mathrm{U} = \bigcup_{i\in\mathrm{I}} \mathrm{U}_i$, and $(s_i)_{i\in\mathrm{I}}$ a family, where $s_i \in \mathscr{E}(\mathrm{U}_i)$, such that $s_i|_{\mathrm{U}_i\cap\mathrm{U}_j} = s_j|_{\mathrm{U}_i\cap\mathrm{U}_j}$ for every $i, j \in \mathrm{I}$. For $i \in \mathrm{I}$, write $s_i = (s_{i,x})_{x\in\mathrm{U}_i}$. Let $x \in \mathrm{U}$; if $i, j \in \mathrm{I}$ are such that $x \in \mathrm{U}_i \cap \mathrm{U}_j$, then $s_{i,x} = s_{j,x}$; let s_x be this common value and let $s = (s_x)_{x\in\mathrm{U}}$. Then s is an element of $\mathscr{E}(\mathrm{U})$ such that $s|_{\mathrm{U}_i} = (s_x)_{x\in\mathrm{U}_i} = (s_{i,x})_{x\in\mathrm{U}_i} = s_i$, and it is the unique such element.

Let now $\mathscr{F}^+$ be the image of the morphism j; it is the smallest subsheaf of $\mathscr{E}$ such that $\mathscr{F}^+(\mathrm{U})$ contains $j(\mathscr{F}(\mathrm{U}))$ for every open subset U of X. Moreover, a section $s \in \mathscr{E}(\mathrm{U})$ belongs to $\mathscr{F}^+(\mathrm{U})$ if and only if U can be covered by open subsets V for which there exists $t \in \mathscr{F}(\mathrm{V})$ such that $s|_{\mathrm{V}} = j(t)$.

By construction the morphism of presheaves $j\colon \mathscr{F} \to \mathscr{E}$ factors through a morphism from $\mathscr{F}$ to $\mathscr{F}^+$, which we still denote by j. It remains to show that for every $x \in \mathrm{X}$, the map $j_x\colon \mathscr{F}_x \to \mathscr{F}^+_x$ is a bijection and that the pair $(\mathscr{F}^+, j)$ satisfies the desired universal property.

Let $x \in \mathrm{X}$. For every open subset U that contains X, let $p_{\mathrm{U}}\colon \mathscr{F}^+(\mathrm{U}) \to \mathscr{F}_x$ be the canonical projection, given by $s \mapsto s_x$. If U and V are open neighborhoods of x such that $\mathrm{V} \subseteq \mathrm{U}$, one has $p_{\mathrm{V}}(s|_{\mathrm{V}}) = p_{\mathrm{U}}(s)$. By definition of the limit $\varinjlim \mathscr{F}^+(\mathrm{U})$, there exists a unique map $p\colon \mathscr{F}^+_x \to \mathscr{F}_x$ which maps the germ at x of a section $s = (s_y)_{y\in\mathrm{U}} \in \mathscr{F}^+(\mathrm{U})$ to s_x for every open neighborhood U of x. By construction, $p \circ j_x$ is the identity. In particular, j_x is injective. Let us show that j_x is surjective. Let $s \in \mathscr{F}^+_x$ be the germ of a section $t \in \mathscr{F}^+(\mathrm{U})$, for some open neighborhood U of x. By definition of $\mathscr{F}^+$, there exists an open neighborhood V of x such that $\mathrm{V} \subseteq \mathrm{U}$ and a section $t' \in \mathscr{F}(\mathrm{V})$ such that $t_y = j(t')_y$ for every $y \in \mathrm{V}$. By definition of the sheaf $\mathscr{E}$, one thus has $t|_{\mathrm{V}} = j(t')$. Moreover, t and $t|_{\mathrm{V}}$

have the same germ at x, so that $s = j_x(t')$. This concludes the proof that the map $j_x \colon \mathscr{F}_x \to \mathscr{F}^+_x$ is an isomorphism.

Let us now prove that the pair $(\mathscr{F}^+, j)$ satisfies the universal property of the theorem. Let $\mathscr{G}$ be a sheaf on X and let $f \colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Let U be an open subset of X and let $t \in \mathscr{F}^+(\mathrm{U})$. Let I be the set of pairs (V, s), where V is an open subset of U and $s \in \mathscr{F}(\mathrm{V})$ is such that $s_x = t_x$ for every $x \in \mathrm{V}$. Let $i = (\mathrm{V}, s)$ and $j = (\mathrm{V}', s') \in \mathrm{I}$ and let $\mathrm{W} = \mathrm{V} \cap \mathrm{V}'$. For every $x \in \mathrm{W}$, one has

$$f(s)_x = f_x(s_x) = f_x(t_x) = f_x(s'_x) = f(s')_x.$$

Since $\mathscr{G}$ is a sheaf, lemma 3.1.10 implies that $f(s)|_{\mathrm{W}} = f(s')|_{\mathrm{W}}$.

For $i = (\mathrm{V}, s) \in \mathrm{I}$, let $\mathrm{U}_i = \mathrm{V}$ and $u_i = f(s) \in \mathscr{G}(\mathrm{V})$. By the definition of a sheaf, applied to the family $(\mathrm{U}_i)_{i \in \mathrm{I}}$ of open subsets of X and to the family (u_i) of sections $\mathscr{G}$, there exists a unique section $u \in \mathscr{G}(\mathrm{U})$ such that $u|_{\mathrm{V}} = f(\mathrm{V})(s)$ for every pair $(\mathrm{V}, s) \in \mathrm{I}$. Set $\varphi(\mathrm{U})(t) = u$. This defines a map $\varphi(\mathrm{U}) \colon \mathscr{F}^+(\mathrm{U}) \to \mathscr{G}(\mathrm{U})$.

The morphism j maps a section $s \in \mathscr{F}(\mathrm{U})$ to the section $t = j(\mathrm{U})(s) = (s_x)_{x \in \mathrm{U}}$ of $\mathscr{F}^+$. By construction, one thus has $\varphi(\mathrm{U})(j(\mathrm{U})(s)) = f(\mathrm{U})(s)$ for every $s \in \mathscr{F}(\mathrm{U})$.

If U′ is an open subset of U, the definitions of $\varphi(\mathrm{U}')$ and $\varphi(\mathrm{U})$ imply at once that $\varphi(\mathrm{U}')(s|_{\mathrm{U}'}) = \varphi(\mathrm{U})(s)|_{\mathrm{U}'}$. We thus have defined a morphism of sheaves φ from $\mathscr{F}^+$ to $\mathscr{G}$, and $\varphi \circ j = f$.

Finally, let φ, ψ be two morphisms of sheaves from $\mathscr{F}^+$ to $\mathscr{G}$ such that $f = \varphi \circ j = \psi \circ j$, and let us show that $\varphi = \psi$. For every point $x \in \mathrm{X}$, one has $f_x = \varphi_x \circ j_x = \psi_x \circ j_x$, hence $\varphi_x = \psi_x$ since j_x is bijective. It follows from lemma 3.1.10 that $\varphi = \psi$. □

3.2.8. Colimits. — Let Q be a quiver and let $\mathscr{F}_{\mathrm{Q}} = ((\mathscr{F}_v), (\varphi_e))$ be a Q-diagram of presheaves on X. For every open subset U of X, let $\mathscr{F}_{\mathrm{pre}}(\mathrm{U}) = \varinjlim(\mathscr{F}_{\mathrm{Q}}(\mathrm{U}))$ be the colimit of the diagram of sets $((\mathscr{F}_v(\mathrm{U})), (\varphi_e(\mathrm{U})))$; let $\varphi_v(\mathrm{U}) \colon \mathscr{F}_v(\mathrm{U}) \to \mathscr{F}(\mathrm{U})$ be the canonical map. For every open subsets U, V of X such that $\mathrm{V} \subseteq \mathrm{U}$, there exists a unique map $\rho^{\mathscr{F}_{\mathrm{pre}}}_{\mathrm{UV}} \colon \mathscr{F}_{\mathrm{pre}}(\mathrm{U}) \to \mathscr{F}_{\mathrm{pre}}(\mathrm{V})$ such that $\rho^{\mathscr{F}_{\mathrm{pre}}}_{\mathrm{UV}} \circ \varphi_v(\mathrm{U}) = \varphi_v(\mathrm{V}) \circ \rho^{\mathscr{F}_{\mathrm{pre}}}_{\mathrm{UV}}$. The family $\mathscr{F}_{\mathrm{pre}} = ((\mathscr{F}_{\mathrm{pre}}(\mathrm{U})), (\rho^{\mathscr{F}}_{\mathrm{UV}}))$ is a presheaf; by construction, the maps $\varphi_v(\mathrm{U}) \colon \mathscr{F}_v(\mathrm{U}) \to \mathscr{F}_{\mathrm{pre}}(\mathrm{U})$ form a morphism of presheaves $\varphi_v \colon \mathscr{F}_v \to \mathscr{F}_{\mathrm{pre}}$.

Endowed with the family of morphisms (φ_v), *the presheaf $\mathscr{F}_{\mathrm{pre}}$ is a colimit of the diagram $\mathscr{F}_{\mathrm{Q}}$.* Let indeed $(\mathscr{G}, (\psi_v))$ be a cocone on this diagram. For every open subset U of X, the set $\mathscr{G}(U)$ is a cocone on the diagram $\mathscr{F}_Q(U)$, hence there exists a unique map $\theta(U)\colon \mathscr{F}_{pre}(U) \to \mathscr{G}(U)$ such that $\theta(U) \circ \psi_v(U) = \varphi_v(U)$ for every v. Let V and U be open subsets of X such that $V \subseteq U$. For every v, one has

$$\begin{aligned}\rho^{\mathscr{G}}_{UV} \circ \theta(U) \circ \varphi_v(U) &= \rho^{\mathscr{G}}_{UV} \circ \psi_v(U) \\ &= \psi_v(V) \circ \rho^{\mathscr{F}_v}_{UV} \\ &= \theta(V) \circ \varphi_v(V) \circ \rho^{\mathscr{F}_v}_{UV} \\ &= \theta(V) \circ \rho^{\mathscr{F}_{pre}}_{UV} \circ \varphi_v(U).\end{aligned}$$

It follows that $\rho^{\mathscr{G}}_{UV} \circ \theta(U) = \theta(V) \circ \rho^{\mathscr{F}_{pre}}_{UV}$, which proves that the family of maps $\theta = (\theta(U))$ is a morphism of presheaves. It satisfies $\psi_v = \theta \circ \varphi_v$ for every v, and it is the unique such morphism of presheaves.

3.2.9. — Let $\mathscr{F}_Q$ be a Q-diagram of sheaves. Generally, the presheaf $\mathscr{F}_{pre}$ which is the colimit of this diagram in the category of presheaves is not a sheaf. One thus defines the sheaf $\varinjlim(\mathscr{F}_Q)$ to be the sheaf associated to this presheaf $\mathscr{F}_{pre}$.

It is indeed a colimit of the diagram $\mathscr{F}_Q$ in the category of sheaves on X.

3.2.10. — Similarly, every diagram $\mathscr{F}_Q$ of sheaves of abelian groups has a colimit which is computed as follows. One begins by defining a presheaf $\mathscr{F}_{pre}$ on X such that $\mathscr{F}_{pre}(U)$ is the limit of the diagram $\mathscr{F}_Q(U)$ of abelian groups deduced from $\mathscr{F}_Q$. Then one shows that the sheaf $\mathscr{F}$ associated with this presheaf $\mathscr{F}_{pre}$ is a colimit of the initial diagram.

An analogous result for sheaves of rings, of modules, etc.

3.2.11. — Let Q be a quiver, let $\mathscr{F}_Q = ((\mathscr{F}_v), (\varphi_e))$ be a diagram of sheaves. Let $x \in X$. Taking the stalks at x, one obtains a natural diagram $\mathscr{F}_{Q,x} = ((\mathscr{F}_{v,x}), (\varphi_{e,x}))$ of sets.

If $(\mathscr{F}, (\varphi_v))$ is a colimit of the diagram $\mathscr{F}_Q$, then $(\mathscr{F}_x, (\varphi_{v,x}))$ is a colimit of the diagram $\mathscr{F}_{Q,x}$.

Let indeed $j_x\colon \{x\} \to X$ be the inclusion of the point x. Sheaves of sets (resp. modules,...) on a topological space reduced to one point x can be identify with the set (resp. module,...) of its global sections. Consequently, the stalk $\mathscr{F}_x$ of a sheaf $\mathscr{F}$ at x identifies with the sheaf $j^{-1}\mathscr{F}_x$. Since the functor j_x^{-1} associated with the continuous map j_x has a right adjoint (namely, the functor $j_{x,*}$), it commutes with arbitrary colimits.

If $(\mathscr{F}, (\varphi_v))$ *is a limit of the diagram* $\mathscr{F}_Q$ *and if the quiver* Q *is finite, then* $(\mathscr{F}_x, (\varphi_{v,x}))$ *is a limit of the diagram* $\mathscr{F}_{Q,x}$.

Let $(G, (\psi_v))$ be a cone on the diagram $\mathscr{F}_{Q,x}$; let us show that there exists a unique map $\psi\colon G \to \mathscr{F}_x$ such that $\psi_v = \varphi_{v,x} \circ \psi$ for every vertex v of Q, Let $g \in G$. For every vertex v, let U_v be an open neighborhood of x and $s_v \in \mathscr{F}_v(U_v)$ be such that $\psi_v(g)$ is the germ of s_v at x; since Q has finitely many vertices, we may replace each U_v be the intersection U of the family $(U_v)_v$ and s_v by $s_v|_U$; we thus assume that $s_v \in \mathscr{F}_v(U)$ for every v. For every arrow e of Q, one has the equality $\varphi_{e,x}(s_{o(e),x}) = s_{t(e),x}$ of germs at x; consequently, there exists an open neighborhood U_e of x contained in U such that $\varphi_e(U_e)(s_{o(e)}|_{U_e}) = s_{t(e)}|_{U_e}$. Since Q has finitely many arrows, we may replace U by the intersection $\bigcap_e U_e$. Since $\mathscr{F}$ is a limit of the diagram $\mathscr{F}_Q$, there exists a unique section $\psi_0(g) \in \mathscr{F}(U)$ such that $\varphi_v(U)(\psi_0(g)) = s$ for every vertex v. Let $\psi(g)$ be the germ of $\psi_0(g)$ at x. It does not depend of the choice of the open neighborhood U of x and of the sections $s_v \in \mathscr{F}_v(U)$ such that $\psi_v(g) = s_{v,x}$ for every vertex v and $\varphi_e(U)(s_{o(e)}) = s_{t(e)}$ for every arrow e of Q. By construction, the map $\psi\colon G \to \mathscr{F}_x$ satisfies $\varphi_{v,x} \circ \psi = \psi_v$, and it is the unique such map.

3.3. Direct and inverse images of sheaves

Let $f\colon X \to Y$ be a continuous map of topological spaces.

3.3.1. — Let $\mathscr{F}$ be a presheaf on X. For every open subset V of Y, the set $f^{-1}(V)$ is open in X, because f is continuous. One thus defines a presheaf $f_*\mathscr{F}$ on Y be setting $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for every open subset V of Y. If U and V are open subsets of Y such that $V \subseteq U$, the restriction map $\rho^{f_*\mathscr{F}}_{UV}$ from $f_*\mathscr{F}(U)$ to $f_*\mathscr{F}(V)$ is the map $\rho^{\mathscr{F}}_{f^{-1}(U),f^{-1}(V)}$ from $\mathscr{F}(f^{-1}(U))$ to $\mathscr{F}(f^{-1}(V))$.

Lemma (**3.3.2**). — *If* $\mathscr{F}$ *is a sheaf, then* $f_*\mathscr{F}$ *is also a sheaf.*

Proof. — Let indeed V be an open subset of Y, let $(V_i)_{i\in I}$ be a family of open subsets of Y such that $V = \bigcup_{i\in I} V_i$ and let (s_i) be a family of sections of $f_*\mathscr{F}$, where $s_i \in (f_*\mathscr{F})(V_i)$, such that the restrictions of s_i and s_j to $V_i \cap V_j$ coincide. Set $U = f^{-1}(V)$ and $U_i = f^{-1}(V_i)$. By definition of the presheaf $f_*\mathscr{F}$, s_i is an element of $\mathscr{F}(U_i)$ which we denote by t_i. One has $f^{-1}(V_i \cap V_j) = U_i \cap V_j$, and the restriction of t_i to $U_i \cap U_j$ corresponds with the restriction of s_i to $V_i \cap V_j$. Consequently, one has $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$ for every pair (i, j) of elements of I. Since $\mathscr{F}$ is a sheaf, there is a unique element $t \in \mathscr{F}(U)$ such that $t|_{U_i} = t_i$ for every i. This element t corresponds to a section $s \in (f_*\mathscr{F})(V)$ and one has $s|_{U_i} = s_i$ for every i; moreover, s is the only section possessing that property. This concludes the proof that the presheaf $\mathscr{F}$ is a sheaf. □

3.3.3. — Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X and let $u: \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For every open subset V of Y, denote by $(f_*u)(V)$ the map $u(f^{-1}(V))$ from $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ to $(f_*\mathscr{G})(V) = \mathscr{G}(f^{-1}(V))$. This is a morphism of presheaves.

One has $f_* \operatorname{id}_{\mathscr{F}} = \operatorname{id}_{f_*\mathscr{F}}$. If $v: \mathscr{G} \to \mathscr{H}$ is another morphism of presheaves, then $f_*(v \circ u) = (f_*v) \circ (f_*u)$.

Consequently, the assignments $\mathscr{F} \to f_*\mathscr{F}$ and $u \mapsto f_*u$ define a *functor* from the category $\boldsymbol{PreSh}_X$ of presheaves on X to the category $\boldsymbol{PreSh}_Y$ of presheaves on Y, and a functor from the category $\boldsymbol{Sh}_X$ of sheaves on X to the category $\boldsymbol{Sh}_Y$ of sheaves on Y.

3.3.4. — If $\mathscr{F}$ is a (pre)sheaf in abelian groups on X, then $f_*\mathscr{F}$ has a natural structure of a (pre)sheaf in abelian groups. If $u: \mathscr{F} \to \mathscr{G}$ is a morphism of (pre)sheaves in abelian groups on X, then f_*u is a morphism of (pre)sheaves in abelian groups on Y. In other words, one also has a functor (still denoted by f_*) from the category $\boldsymbol{Ab}_X$ of sheaves of abelian groups on X to the category $\boldsymbol{Ab}_Y$ of sheaves of abelian groups on Y.

A similar result holds more generally for (pre)sheaves with values in a category.

3.3.5. — Let $\mathscr{G}$ be a presheaf on Y. Let U and V be open subsets of X such that $U \subseteq V$. Then $f(U) \subseteq f(V)$; consequently, there exists a unique map ρ_{UV} from the set $\mathscr{G}_{f(V)}$ of germs of sections of $\mathscr{G}$ at $f(V)$ to the set $\mathscr{G}_{f(U)}$ of sets of germs of sections of $\mathscr{G}$ at $f(U)$ which associates with the germ at $f(V)$ of a section s of $\mathscr{G}$ on a neighborhood of $f(V)$ the germ of this section at $f(U)$.

The family $(\mathscr{G}_{f(U)})$ together with the maps ρ_{UV} is a presheaf on X, which we denote (temporarily) by $f^{-1}_{\text{pre}}(\mathscr{G})$.

Definition (**3.3.6**). — *If $\mathscr{G}$ is a sheaf on* Y, *one defines the sheaf $f^{-1}\mathscr{G}$ on* X *as the sheaf associated with this presheaf $f^{-1}_{\text{pre}}\mathscr{G}$.*

In other words, for every sheaf $\mathscr{F}$ on X and every morphism $v\colon f^{-1}_{\text{pre}}\mathscr{G} \to \mathscr{F}$ of presheaves of X, there exists a unique morphism $v'\colon f^{-1}\mathscr{G} \to \mathscr{F}$ such that $v = v' \circ j$, where $j\colon f^{-1}_{\text{pre}}\mathscr{G} \to f^{-1}\mathscr{G}$ is the canonical morphism of presheaves.

3.3.7. — Let $u\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on Y. There exists a unique morphism of presheaves from $f^{-1}_{\text{pre}}(\mathscr{F})$ to $f^{-1}_{\text{pre}}(\mathscr{G})$ which, for every open subset U of X, every open subset V of Y containing $f(\mathrm{U})$ and every section $s \in \mathscr{F}(\mathrm{U})$, associates with the germ of the section s at $f(\mathrm{U})$ the germ of the section $u(\mathrm{U})(s)$. We (temporarily) denote this morphism by $f^{-1}_{\text{pre}}(u)$.

Denote by j and k the canonical morphisms from $f^{-1}_{\text{pre}}\mathscr{F}$ to $f^{-1}\mathscr{F}$ and $f^{-1}_{\text{pre}}\mathscr{G}$ to $f^{-1}\mathscr{G}$ respectively. By the universal property of the associated sheaf, there exists a unique morphism of sheaves $f^{-1}u\colon f^{-1}\mathscr{F} \to f^{-1}\mathscr{G}$ such that $(f^{-1}u \circ j) = k \circ (f^{-1}_{\text{pre}}u)$.

One has $f^{-1}\,\mathrm{id}_{\mathscr{F}} = \mathrm{id}_{f^{-1}\mathscr{F}}$. If $v\colon \mathscr{G} \to \mathscr{H}$ is a morphism of sheaves, one has $f^{-1}(v \circ u) = (f^{-1}v) \circ (f^{-1}u)$.

In other words, the assignments $\mathscr{F} \mapsto f^{-1}\mathscr{F}$ and $u \mapsto f^{-1}u$ define a functor from the category of sheaves on Y to the category of sheaves on X.

3.3.8. — If $\mathscr{G}$ is a sheaf in abelian groups on Y, then $f^{-1}\mathscr{G}$ has a natural structure of a sheaf in abelian groups. If $u\colon \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves in abelian groups, then $f^{-1}u$ is also a morphism of sheaves in abelian groups. This gives a functor, still denoted by f^{-1}, from the category $\boldsymbol{Ab}_{\mathrm{Y}}$ of sheaves of abelian groups on Y to the category $\boldsymbol{Ab}_{\mathrm{X}}$.

An analogous result holds for sheaves in rings, modules, etc., for which colimits of direct systems is compatible with the colimit of the underlying direct system of sets.

A similar construction can also be made for sheaves with coefficients in a category $\boldsymbol{C}$, provided that colimits of direct systems exist in the category $\boldsymbol{C}$.

3.3.9. — Let $\mathscr{F}$ be sheaf on X. Let U be an open subset of X and let $s \in \mathscr{F}(\mathrm{U})$. Let V be an open subset of Y which contains $f(\mathrm{U})$; then $f^{-1}(\mathrm{V})$ contains U, so that there the restriction morphism $\rho^{\mathscr{F}}_{f^{-1}(\mathrm{V}),\mathrm{U}}$ defines a map from $(f_*\mathscr{F})(\mathrm{V}) = \mathscr{F}(f^{-1}(\mathrm{V}))$ to $\mathscr{F}(\mathrm{U})$. When V runs along the family of open neighborhoods of $f(\mathrm{U})$ in Y, these maps give rise to a map from the set $(f_*\mathscr{F})_{f(\mathrm{U})}$ of germs of sections of $f_*\mathscr{F}$ at $f(\mathrm{U})$ to $\mathscr{F}(\mathrm{U})$, hence to a map $\alpha_{\text{pre}}(\mathrm{U})$ from $f^{-1}_{\text{pre}}(f_*\mathscr{F})(\mathrm{U})$ to $\mathscr{F}(\mathrm{U})$. The family of maps $(\alpha_{\text{pre}}(\mathrm{U}))$ is a morphism of presheaves from the presheaf from $f^{-1}_{\mathrm{pre}}(f_*\mathscr{F})$ to the sheaf $\mathscr{F}$. Consequently, there exists a unique morphism of sheaves $\alpha_{\mathscr{F}}\colon f^{-1}(f_*\mathscr{F}) \to \mathscr{F}$ such that $\alpha_{\mathrm{pre}} = \alpha_{\mathscr{F}} \circ j$, where $j\colon f^{-1}_{\mathrm{pre}}(f_*\mathscr{F}) \to f^{-1}(f_*\mathscr{F})$ is the canonical morphism.

3.3.10. — Let $\mathscr{G}$ be a sheaf on Y. Let V be an open subset of Y, let $s \in \mathscr{G}(\mathrm{V})$ and let $\mathrm{U} = f^{-1}(\mathrm{V})$. Since $f(\mathrm{U}) \subseteq \mathrm{V}$, one may consider the germ of s at $f(\mathrm{U})$ which is an element of $f^{-1}_{\mathrm{pre}}\mathscr{G}(\mathrm{U})$; let $\beta(\mathrm{V})(s)$ be its image in $f^{-1}\mathscr{G}(\mathrm{U}) = f_*(f^{-1}\mathscr{G})(\mathrm{V})$. The maps $\beta(\mathrm{V})$ define a morphism of sheaves $\beta_{\mathscr{G}}$ from $\mathscr{G}$ to $f_*(f^{-1}\mathscr{G})$.

Theorem **(3.3.11)**. — *Let $\mathscr{F}$ be a sheaf on* X*, let $\mathscr{G}$ be sheaf on* Y*, and let $u\colon\mathscr{G} \to f_*\mathscr{F}$ be a morphism of sheaves. There exists a unique morphism of sheaves $v\colon f^{-1}(\mathscr{G}) \to \mathscr{F}$ such that $u = f_*(v) \circ \beta_{\mathscr{G}}$.*

If u is a morphism of sheaves of abelian groups (resp. of rings, etc.), then so is v.

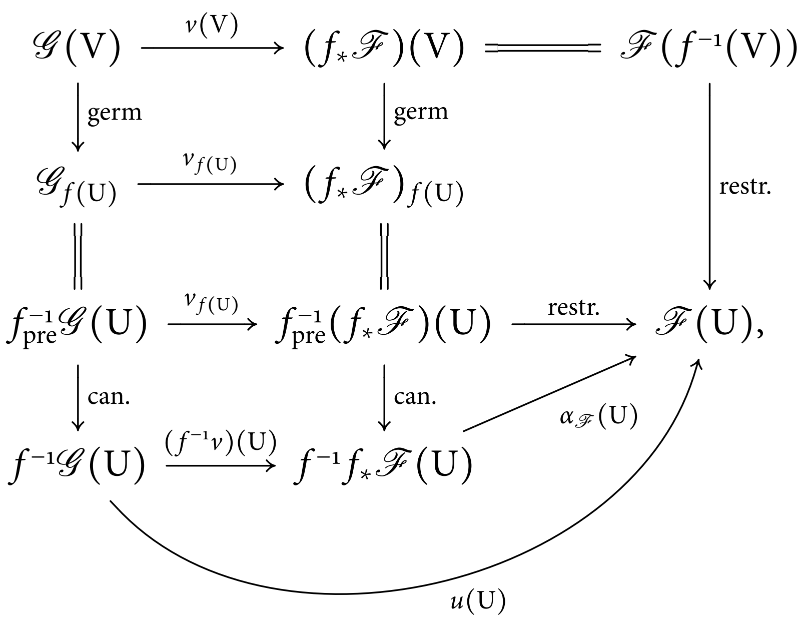
In other words, the map

$$\mathrm{Hom}(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F}), \qquad u \mapsto f_*(u) \circ \beta_{\mathscr{G}}$$

is a bijection, so that the pair (f^{-1}, f_*) of functors between the categories of sheaves on X and on Y is adjoint.

Proof. — Let $v\colon\mathscr{G} \to f_*\mathscr{F}$ be a morphism of sheaves on Y and let $u\colon f^{-1}\mathscr{G} \to \mathscr{F}$ be the morphism given by $u = \alpha_{\mathscr{F}} \circ (f^{-1}v)$.

Let U be an open subset of X, let V be an open subset of Y such that $f(\mathrm{U}) \subseteq \mathrm{V}$; one has $\mathrm{U} \subseteq f^{-1}(f(\mathrm{U})) \subseteq f^{-1}(\mathrm{V})$. Let us consider the commutative diagram of maps



where the arrows indicated "germ" map a section on V to its germ at $f(\mathrm{U})$, the vertical arrow indicated "restr." maps a section of $\mathscr{F}$ on $f^{-1}(\mathrm{V})$ to its restriction on U, the horizontal arrow indicated "restr." is the morphism induced by the natural restriction maps from the members of the directed system $((f_*\mathscr{F})(\mathrm{W}))_{\mathrm{W}\supseteq f(\mathrm{U})} = (\mathscr{F}(f^{-1}(\mathrm{W})))_{\mathrm{W}\supseteq f(\mathrm{U})}$ to $\mathscr{F}(\mathrm{U})$, and the arrows indicated "can." are the canonical morphisms from a presheaf to the associated sheaf.

Let us assume that $\mathrm{U} = f^{-1}(\mathrm{V})$ and let $s \in \mathscr{G}(\mathrm{V})$. Since $f(\mathrm{U}) = \mathrm{V}$ is open in Y, the maps "germ" in the diagram are bijections, as well as the vertical restriction map. The definition of $\beta_{\mathscr{G}}$ shows that $\beta_{\mathscr{G}}(s)$ is the image of s in $f^{-1}\mathscr{G}(\mathrm{U})$ under the composition of arrows of the left hand column of the above diagram. Consequently, $u(\mathrm{U})(\beta_{\mathscr{G}}(s))$ is the section $\nu(s) \in \mathscr{F}(\mathrm{U}) = f_*\mathscr{F}(\mathrm{V})$. Since $f_*u(\mathrm{V}) = u(\mathrm{U})$, it follows that $\nu(s) = (f_*u)(\mathrm{V})(\beta_{\mathscr{G}}(s))$. This shows that $\nu = (f_*u) \circ \beta_{\mathscr{G}}$.

Conversely, let u_1, u_2 be morphisms from $f^{-1}\mathscr{G}$ to $\mathscr{F}$ such that $\nu = (f_*u_1)\circ\beta_{\mathscr{G}} = (f_*u_2) \circ \beta_{\mathscr{G}}$. Let W be an open subset of X, let t be a section of $f^{-1}\mathscr{G}$ on W; let us prove that $u_1(t) = u_2(t)$ in $\mathscr{F}(\mathrm{W})$. Since $\mathscr{F}$ is a sheaf, it suffices to prove that every point x of W has a neighborhood U such that $u_1(t)|_{\mathrm{U}} = u_2(t)|_{\mathrm{U}}$. By definition of the sheaf $f^{-1}\mathscr{G}$, there exists an open neighborhood U of x in X, an open subset V of Y such that $f(\mathrm{U}) \subseteq \mathrm{V}$ and a section $s \in \mathscr{G}(\mathrm{V})$ such that $t|_{\mathrm{U}}$ is the image of s under the composition:

$$\mathscr{G}(V) \xrightarrow{\text{germ}} \mathscr{G}_{f(U)} = f_{\text{pre}}^{-1}\mathscr{G}(U) \xrightarrow{\text{can}} f^{-1}\mathscr{G}(U).$$

Observe that $\beta_{\mathscr{G}}(s)$ is an element of $f_*f^{-1}\mathscr{G}(V) = f^{-1}\mathscr{G}(f^{-1}(V))$; by the definition of $\beta_{\mathscr{G}}$, its restriction to U is thus equal to $t|_U$. Consequently,

$$u_1(t)|_U = u_1(t|_U) = u_1(\beta_{\mathscr{G}}(s)) = v(s) = u_2(\beta_{\mathscr{G}}(s)) = u_2(t|_U) = u_2(t)|_U.$$

This implies that $u_1(t) = u_2(t)$ and concludes the proof that $u_1 = u_2$. □

3.4. The abelian category of abelian sheaves

In this section, we show that the category of abelian sheaves is an abelian category. In fact, we treat a more general case.

3.4.1. — Let X be a topological space and let $\mathscr{A}$ be a sheaf of rings on X. An $\mathscr{A}$-(pre)module is a (pre)sheaf $\mathscr{F}$ in abelian groups such that for every open subset U of X, $\mathscr{F}(U)$ is endowed with a structure of an $\mathscr{A}(U)$-module, compatibly with the restriction maps: for every pair (U, V) of open subsets of X such that $V \subseteq U$, every $a \in \mathscr{A}(U)$ and every $s \in \mathscr{F}(U)$, one has $a|_V \cdot s|_V = (a \cdot s)|_V$.

Equivalently, $\mathscr{F}$ is an abelian (pre)sheaf endowed with the datum of a morphism of (pre)sheaves in (possibly non-commutative) rings: $\mathscr{A} \to \mathscr{E}nd(\mathscr{F})$.

3.4.2. — Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{A}$-modules. A morphism of $\mathscr{A}$-(pre)modules $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a morphism of (pre)sheaves in abelian groups such that $\varphi(U)(a \cdot s) = a \cdot \varphi(U)(s)$ for every open subset U of X, every $a \in \mathscr{A}(U)$ and every $s \in \mathscr{F}(U)$.

The identity is a morphism; the composition of two morphisms of $\mathscr{A}$-(pre)modules is a morphism of $\mathscr{A}$-(pre)modules. Consequently, $\mathscr{A}$-premodules and $\mathscr{A}$-modules form categories which we denote by $\boldsymbol{PreMod}_{\mathscr{A}}$ and $\boldsymbol{Mod}_{\mathscr{A}}$. They are additive category.

If $\mathscr{F}$ is an $\mathscr{A}$-premodule, then the associated sheaf $\mathscr{F}^+$ has a unique structure of an $\mathscr{A}$-module such that the canonical morphism $j\colon \mathscr{F} \to \mathscr{F}^+$ is $\mathscr{A}$-linear.

3.4.3. — Let $\mathscr{F}$ be an $\mathscr{A}$-module on X. Let $x \in X$. The stalk $\mathscr{F}_x$ has a unique structure of $\mathscr{A}_x$-module for which $a_x \cdot s_x = (a \cdot s)_x$, for every open neighborhood U of x, every $a \in \mathscr{A}(U)$ and every $s \in \mathscr{F}(U)$.

Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{A}$-modules. For every $x \in X$, the map $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ is a morphism of $\mathscr{A}_x$-modules.

3.4.4. — Let A be a ring and let A_X be the constant sheaf with value A. Every A_X-module is naturally a sheaf in A-modules. This gives rise to an equivalence of categories from the category of A_X-modules to the category of sheaves in A-modules.

3.4.5. — Every diagram of sheaves of $\mathscr{A}$-modules has a limit and a colimit. In particular, the category of $\mathscr{A}$-modules admits finite products and coproducts.

The limit is computed on each open set.

To compute the colimit, one first computes a presheaf of abelian groups and then takes the associated sheaf, which has a natural structure of an $\mathscr{A}$-module.

Let $x \in X$. The functor "stalk at x" from $\boldsymbol{Mod}_{\mathscr{A}}$ to $\boldsymbol{Mod}_{\mathscr{A}_x}$ commutes with all colimits, and with all finite limits.

3.4.6. Images, kernels and cokernels. — Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{A}$-modules.

Its *image* $\operatorname{Im}(\varphi)$ is the subsheaf of $\mathscr{G}$ generated by the sub-presheaf given by $U \mapsto \varphi(U)(\mathscr{F}(U))$. It is a sub-$\mathscr{A}$-module of $\mathscr{G}$.

The *kernel* of φ is the $\mathscr{A}$-submodule $\operatorname{Ker}(\varphi)$ of $\mathscr{F}$ whose sections over an open subset U of X are the elements of $\operatorname{Ker}(\varphi(U))$.

To justify the terminology, let j be the inclusion of $\operatorname{Ker}(\varphi)$ in $\mathscr{F}$, and let us show that $(\operatorname{Ker}(\varphi), j)$ is an equalizer of the pair $(\varphi, 0)$ of morphisms from $\mathscr{F}$ to $\mathscr{G}$. The morphism j is a monomorphism and one has $\varphi \circ j = 0 = 0 \circ j$. Let moreover $k\colon \mathscr{H} \to \mathscr{F}$ be a morphism of $\mathscr{A}$-modules such that $\varphi \circ k = 0$; for every open subset U of X and every section $s \in \mathscr{H}(U)$, one has $\varphi(U)(k(U)(s)) = 0$, hence $k(U)(s) = 0$; this shows that $k(U)(s) \in \operatorname{Ker}(\varphi)(U)$, so that the morphism k factors, necessarily uniquely, through $\operatorname{Ker}(\varphi)$.

A coequalizer $\operatorname{Coker}(\varphi)$ of the pair $(\varphi, 0))$ is called a *cokernel* of φ. The canonical morphism from $\mathscr{G}$ to $\operatorname{Coker}(\varphi)$ is an epimorphism.

Proposition (**3.4.7**). — *Let* X *be a topological space, let* $\mathscr{A}$ *be a sheaf of rings on* X, *let* $\mathscr{F}$ *and* $\mathscr{G}$ *be* $\mathscr{A}$*-modules and let* $\varphi\colon \mathscr{F} \to \mathscr{G}$ *be a morphism of* $\mathscr{A}$*-modules. Let* $j\colon \operatorname{Ker}(\varphi) \to \mathscr{F}$ *and* $p\colon \mathscr{G} \to \operatorname{Coker}(\varphi)$ *be the canonical morphisms.*

a) *The following properties are equivalent:*

(a) *The morphism* φ *is a monomorphism;*

(b) *One has* $\operatorname{Ker}(\varphi) = 0$*;*

(c) *For every* $x \in X$*, the morphism* φ_x *is injective;*

(d) *The pair* $(\mathscr{F}, \varphi)$ *is a kernel of* p.

b) *The following properties are equivalent:*

(a) *The morphism* φ *is an epimorphism;*

(b) *One has* $\operatorname{Coker}(\varphi) = 0$*;*

(c) *For every* $x \in X$*, the morphism* φ_x *is surjective;*

(d) *One has* $\operatorname{Im}(\varphi) = \mathscr{G}$*;*

(e) *The pair* $(\mathscr{G}, \varphi)$ *is a cokernel of the morphism* j.

c) *The morphism* φ *is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

Proof. — Recall that j is a monomorphism and p is an epimorphism.

a) (i)⇔(ii). One has $\varphi \circ j = 0$; consequently, if φ is a monomorphism, then $j = 0$ and $\operatorname{Ker}(\varphi) = 0$. Conversely, assume that $\operatorname{Ker}(\varphi) = 0$ and let $\psi\colon \mathscr{H} \to \mathscr{F}$ be a morphism of $\mathscr{A}$-modules such that $\varphi \circ \psi = 0$; then ψ factor through $\operatorname{Ker}(\varphi)$, so that $\psi = 0$.

(ii)⇔(iii). Since passing to stalks commute with finite limits, one has isomorphism $\operatorname{Ker}(\varphi)_x \simeq \operatorname{Ker}(\varphi_x)$ for every $x \in X$. If $\operatorname{Ker}(\varphi) = 0$, this implies that $\operatorname{Ker}(\varphi_x) = 0$, hence φ_x is injective; conversely, if φ_x is injective for every $x \in X$, then all stalks of the sheaf $\operatorname{Ker}(\varphi)$ are 0, hence $\operatorname{Ker}(\varphi) = 0$.

The implication (iv)⇒(i) is obvious, because kernels are monomorphisms.

b) (i)⇒(ii). One has $p \circ \varphi = 0$; if φ is an epimorphism, then $p = 0$ and $\operatorname{Coker}(\varphi) = 0$.

(ii)⇔(iii)⇔(iv). Let $x \in X$. Passing to stalks commute with colimits, hence $\operatorname{Coker}(\varphi)_x \simeq \operatorname{Coker}(\varphi_x)$; moreover, the stalk of the subsheaf $\operatorname{Im}(\varphi)$ of $\mathscr{G}$ at x is equal to $\operatorname{Im}(\varphi_x)$. If $\operatorname{Coker}(\varphi) = 0$, then for every $x \in X$, one has $\operatorname{Coker}(\varphi_x) = 0$, so that φ_x is surjective. If φ_x is surjective for every x, then the subsheaf $\operatorname{Im}(\varphi)$ then has the same stalks as $\mathscr{G}$, so that one has $\operatorname{Im}(\varphi) = \mathscr{G}$. Finally, if $\operatorname{Im}(\varphi) = \mathscr{G}$, then their stalks coincide, so that φ_x is surjective for every x; this implies that every stalk ot the sheaf $\operatorname{Coker}(\varphi)$ is zero, hence $\operatorname{Coker}(\varphi) = 0$.

(iii)⇒(i). Let us assume that φ_x is surjective for every x and let $\psi\colon \mathscr{G} \to \mathscr{H}$ be a morphism of $\mathscr{A}$-modules such that $\psi \circ \varphi = 0$; let us prove that $\psi = 0$. For every $x \in X$, one has $\psi_x \circ \varphi_x = 0$, hence $\psi_x = 0$ because φ_x is surjective. This implies that $\psi = 0$, as claimed.

The implication (v)⇒(i) is obvious, because cokernels are epimorphisms.

c) In any category, every epimorphism is a monomorphism and an epimorphism. Conversely, if φ is both a monomorphism and an epimorphism, then φ_x is bijective for every $x \in X$, so that φ is an isomorphism.

It remains to prove the implications (i)⇒(iv) in *a*) and (i)⇒(v) in *b*).

Since $p \circ \varphi = 0$, one has $\operatorname{Im}(\varphi) \subseteq \operatorname{Ker}(p)$. Moreover, one has $\operatorname{Ker}(p)_x = \operatorname{Im}(\varphi)_x =$ for every x. This implies that $\operatorname{Im}(\varphi) = \operatorname{Ker}(p)$. The morphism $\varphi' : \mathscr{F} \to \operatorname{Ker}(p)$ induced by φ is thus an epimorphism. If φ is a monomorphism, then φ' is a monomorphism as well, hence an isomorphism.

Let $k : \mathscr{F} \to \operatorname{Coker}(j)$ be a cokernel of j. Since one has $\varphi \circ j = 0$, there exists a unique morphism $\varphi' : \operatorname{Coker}(j) \to \mathscr{G}$ such that $\varphi = \varphi' \circ k$. Moreover, one has $\operatorname{Ker}(k_x) = \operatorname{Im}(j_x)$, so that $\operatorname{Ker}(\varphi'_x) = 0$ for every $x \in X$; this implies that φ' is a monomorphism. If φ is an epimorphism, then φ' is an epimorphism as well, hence it is an isomorphism. □

Theorem **(3.4.8)**. — *Let* X *be a topological space and let* $\mathscr{A}$ *be a sheaf of rings on* X. *The category of* $\mathscr{A}$*-modules is an abelian category.*

Proof. — The category of $\mathscr{A}$-modules is additive. We constructed kernels and cokernels, and proved that every monomorphism is a kernel, and that every epimorphism is a cokernel. The axioms defining an abelian category are satisfied, hence the theorem. □

3.4.9. — Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{A}$-modules.

Recall that the presheaf $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ of homomorphisms is defined by

$$\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})(U) = \operatorname{Hom}_{\mathscr{A}|_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

It is in fact an abelian sheaf. If, moreover, $\mathscr{A}$ is commutative, then it is a sheaf of $\mathscr{A}$-modules.

Observe that for every U, there is a canonical morphism

$$\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})(U) \to \operatorname{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U), \mathscr{G}(U)).$$

This morphism is neither surjective, nor injective in general.

3.4.10. — Let us assume that $\mathscr{A}$ is commutative. The *tensor product* sheaf $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ is an $\mathscr{A}$-module endowed with an universal bilinear morphism from $\mathscr{F} \times \mathscr{G}$.

To prove its existence, we first define a presheaf $\mathscr{T}_{\text{pre}}$ of $\mathscr{A}(U)$-modules by the formula

$$\mathscr{T}_{\text{pre}}(U) = \mathscr{F}(U) \otimes_{\mathscr{A}(U)} \mathscr{G}(U),$$

for every open subset U of X, and the restriction morphism $\rho_{UV}^{\mathscr{T}_{pre}}$ is defined as $\rho_{UV}^{\mathscr{F}} \otimes \rho_{UV}^{\mathscr{G}}$ whenever U and V are open subsets of X such that $V \subseteq U$. Let $\mathscr{T}$ be the sheaf associated with this presheaf and let $j \colon \mathscr{T}_{pre} \to \mathscr{T}$ be the canonical morphism. Then $\mathscr{T}$ is an $\mathscr{A}$-module. Moreover, the family $(b(U))$ of maps given by $b(U)(f,g) = j(U)(f \otimes g)$, for every open subset U of X, every $f \in \mathscr{F}(U)$ and every $g \in \mathscr{G}(U)$ is a morphism of sheaves $b \colon \mathscr{F} \times \mathscr{G} \to \mathscr{T}$. This morphism is $\mathscr{A}$-bilinear.

Let us prove that the pair $(\mathscr{T}, b)$ satisfies the following universal property: for every $\mathscr{A}$-module, every $\mathscr{A}$-bilinear morphism $c \colon \mathscr{F} \times \mathscr{G} \to \mathscr{P}$, there exists a unique $\mathscr{A}$-linear morphism $\gamma \colon \mathscr{T} \to \mathscr{G}$ such that $c = \gamma \circ b$. Let U be an open subset of X. The morphism $c(U) \colon \mathscr{F}(U) \times \mathscr{G}(U) \to \mathscr{P}(U)$ is $\mathscr{A}(U)$-bilinear; consequently, there exists a unique morphism $\gamma_{pre}(U) \colon \mathscr{F}(U) \otimes_{\mathscr{A}(U)} \mathscr{G}(U) \to \mathscr{P}(U)$ such that $\gamma_{pre}(U)(f \otimes g) = c(U)(f,g)$ for every $f \in \mathscr{F}(U)$ and every $g \in \mathscr{G}(U)$. The family $(\gamma_{pre}(U))$ is a morphism of presheaves in $\mathscr{A}$-modules from $\mathscr{T}_{pre}$ to $\mathscr{P}$. Consequently, there exists unique morphism $\gamma \colon \mathscr{T} \to \mathscr{P}$ of $\mathscr{A}$-modules such that $\gamma_{pre} = \gamma \circ j$. One has

$$\gamma(U) \circ b(U)(f,g) = \gamma(U) \circ j(U)(f \otimes g) = \gamma_{pre}(U)(f \otimes g) = c(U)(f,g)$$

for every open subset U of X, every $f \in \mathscr{F}(U)$ and every $g \in \mathscr{G}(U)$; this shows that $\gamma \circ b = c$. Conversely, this property implies that $\gamma \circ j = \gamma_{pre}$, so that γ is the unique morphisms of $\mathscr{A}$-modules which enjoys it.

3.4.11. — Let $\varphi \colon Y \to X$ be a continuous map of topological spaces, let $\mathscr{A}$ be a sheaf of rings on X and let $\mathscr{B}$ be sheaf of rings on Y.

Observe that $\varphi_*\mathscr{B}$ is a sheaf of rings on X, and that $\varphi^{-1}\mathscr{A}$ is a sheaf of rings on Y. Let moreover $\varphi^\sharp \colon \mathscr{A} \to \varphi_*\mathscr{B}$ be a morphism of sheaves of rings; it would be equivalent to give oneself the morphism $\varphi^\flat \colon \varphi^{-1}(\mathscr{A}) \to \mathscr{B}$ associated with $\varphi^\sharp$ by adjunction.

Let $\mathscr{F}$ be an $\mathscr{A}$-module and let $\mathscr{G}$ be an $\mathscr{B}$-module.

The sheaf $\varphi_*\mathscr{G}$ has a canonical structure of a $\varphi_*\mathscr{B}$-module. Using the morphism φ, we view it as an $\mathscr{A}$-module.

Similarly, the sheaf $\varphi^{-1}\mathscr{F}$ on Y has a canonical structure of a $\varphi^{-1}(\mathscr{A})$-module. Define a $\mathscr{B}$-module by the formula

$$\varphi^*\mathscr{F} = \mathscr{B} \otimes_{\varphi^{-1}(\mathscr{A}} \varphi^{-1}\mathscr{F}.$$

The assignments $\mathscr{G} \mapsto \varphi_*\mathscr{G}$ and $\varphi^*\colon \mathscr{F} \to \varphi^*\mathscr{F}$ give rise to functors between the category of $\mathscr{A}$-modules and that of $\mathscr{B}$-modules.

Let $u\colon \mathscr{F} \to \varphi_*\mathscr{G}$ be a morphism of $\mathscr{A}$-modules; let $u^\flat\colon \varphi^{-1}\mathscr{F} \to \mathscr{G}$ be the morphism of sheaves which is deduced from u by the adjunction property of the pair $(\varphi^{-1}, \varphi_*)$. Then $u^\flat$ is a morphism of $\varphi^{-1}\mathscr{A}$-modules. Consequently, there exists a unique morphism $\varphi^*u\colon \varphi^*\mathscr{F} \to \mathscr{G}$ of $\mathscr{B}$-modules such that $\varphi^*u(\mathrm{V})(b \otimes f) = b \cdot u^\flat(f)$ for every open subset V of Y, every $b \in \mathscr{B}(\mathrm{V})$ and every $f \in \varphi^{-1}\mathscr{F}(\mathrm{V})$.

The map $u \mapsto \varphi^*u$ is a bijection from $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \varphi_*\mathscr{G})$ to $\mathrm{Hom}_{\mathscr{B}}(\varphi^*\mathscr{F}, \mathscr{G})$. When $\mathscr{F}$ and $\mathscr{G}$ vary, these maps define an adjunction for the pair of functors (φ^*, φ_*).

3.5. Support, extension by zero

Definition (3.5.1). — *Let $\mathscr{F}$ be a sheaf of abelian groups on* X *and let $s \in \Gamma(\mathrm{X}, \mathscr{F})$. The* support *of s is the intersection of all closed subsets* A *of* X *such that the restriction of s to* X − A *is zero. It is denoted by* Supp(s).

Proposition (3.5.2). — *Let $\mathscr{F}$ be a sheaf of abelian groups on* X *and let $s \in \mathscr{F}(\mathrm{X})$.*

a) *The support of s is the smallest closed subset* A *such that the restriction of s to* X − A *is zero.*

b) *For every $x \in \mathrm{X}$, one has $x \in \mathrm{Supp}(s)$ if and only if $s_x = 0$.*

c) *Let $u\colon \mathscr{F} \to \mathscr{G}$ be a morphism of abelian sheaves on* X. *The support of $u(s)$ is contained in the support of s.*

Proof. — a) Since Supp(s) is defined as the intersection of a family of closed subsets, it is a closed subset of X. Taking complements, we see that X − Supp(s) is the union of all open subsets U of X such that $s|_{\mathrm{U}} = 0$. In particular, if U is an open subset of X such that $s|_{\mathrm{U}} = 0$, then $\mathrm{U} \subseteq \mathrm{X} - \mathrm{Supp}(s)$. Conversely, the family of open sets U such that $s|_{\mathrm{U}} = 0$ is an open covering of X − Supp(s); by the sheaf property of $\mathscr{F}$, the restriction of s to X − Supp(s) vanishes.

b) By construction of the fibers of a sheaf, $s_x = 0$ means that there exists an open neighborhood U of x such that $s|_{\mathrm{U}} = 0$, so that $\mathrm{U} \subseteq \mathrm{X} - \mathrm{Supp}(s)$ and $x \notin \mathrm{Supp}(s)$. Conversely, if $x \notin \mathrm{Supp}(s)$, then U − Supp($s$) is an open neighborhood of x to which the restriction of s vanishes; in particular, $s_x = 0$.

c) Let $U = X - \mathrm{Supp}(s)$. By definition of a morphism of abelian sheaves, one has $u(s)|_U = u(s|_U) = u(0) = 0$. Consequently, U is disjoint from the support of $u(s)$. This means exactly that $\mathrm{Supp}(u(s)) \subseteq \mathrm{Supp}(s)$, as claimed. □

3.5.3. — Let X be a topological space, let A be a subset of X; let us write $j\colon A \to X$ for the inclusion.

Let $\mathscr{F}$ be a sheaf of abelian groups on A. For every open subset U of X, let $j_!\mathscr{F}(U)$ be the set of all sections $s \in \mathscr{F}(A \cap U)$ whose support is closed in U.

By proposition 3.5.2, the support of any $s \in \mathscr{F}(A\cap U)$ is a closed subset of $A\cap U$. If A is closed in X, it is therefore closed in U, hence $j_!\mathscr{F}(U) = \mathscr{F}(U) = j_*\mathscr{F}(U)$.

Proposition **(3.5.4)**. — *Let* A *be a locally closed subset of* X *and let* $j\colon A \to X$ *be the inclusion. Let* $\mathscr{F}$ *be an abelian sheaf on* A.

a) *The assignment* $U \mapsto j_!\mathscr{F}(U)$ *is a subsheaf of* $j_*\mathscr{F}$.

b) *For every* $x \in X - A$, *one has* $(j_!\mathscr{F})_x = 0$.

c) *For every* $x \in A$, *the inclusion* $j_!\mathscr{F} \subseteq j_*\mathscr{F}$ *induces an isomorphism on fibers:* $(j_!\mathscr{F})_x = (j_*\mathscr{F})_x = \mathscr{F}_x$.

Proof. — a) Since $j_*\mathscr{F}(U) = \mathscr{F}(j^{-1}U) = \mathscr{F}(A\cap U)$ for every open subset U of X, we observe that $j_{\mathscr{F}}$ is a subassignment of $j_*\mathscr{F}$.

Let us show that it is a sub-presheaf. Let U and V be open subsets of V such that $U \subseteq V$ and let us consider a section of $j_*\mathscr{F}(V)$. Considered as an element s of $\mathscr{F}(A \cap V)$, its support S is closed in V. The restriction of that section to U is the restriction $s|_{A\cap U}$ whose support is $S \cap U$. It is therefore closed in U.

Let us show that it is a sub-sheaf of $j_*\mathscr{F}$. Let U be an open subset of X, let $s \in j_*\mathscr{F}(U) = \mathscr{F}(A \cap U)$ and let us assume that for every $x \in U$, there exists an open neighborhood U_x of x in U such that the support of $s|_{A\cap U_x}$ is closed in U_x; let us proove that the support S of s is closed in U. By assumption, $S \cap U_x$ is closed in U_x, because it is the support of $s|_{A\cap U_x}$. If $x \notin S$, then $x \notin S \cap U_x$, hence $U_x - (S \cap U_x)$ is a neighborhood of x, so that $U - S$ is a neighborhood of x. This implies that $s \in j_!\mathscr{F}(U)$.

b) Let $x \in X$ be such that $x \notin A$, let U be an open neighborhood of x and let $s \in j_!\mathscr{F}(U)$. Consider s as an element of $\mathscr{F}(A \cap U)$ and let S be its support; by assumption, this is a closed subset of $A \cap U$ which is also closed as a subset of U. Since $x \notin A$, one has $x \notin S$. Consequently, there exists an open neighborhood V of x which is contained in U such that the restriction of s to $A \cap V$ vanishes; then $s|_V = 0$ in $j_!\mathscr{F}(V)$. Consequently, $(j_!\mathscr{F})_x = 0$.

c) Let $x \in A$, let U be an open neighborhood of x and let $s_x \in (j_*\mathscr{F})_x$. By assumption, there exists an open neighborhood U of x and a section $s \in j_*\mathscr{F}(U) = \mathscr{F}(A \cap U)$ with germ s_x at x. Let S be the support of s; this is a closed subset of $A \cap U$.

Since A is locally closed, there exists an closed subset F of X and an open subset O of X such that $A = F \cap O$; since $x \in A$, one has $x \in F$ and $x \in O$. The intersection $U \cap O$ is an open neighborhood of x; the restriction of s to that open set corresponds to $s|_{A \cap U \cap O}$, and its support is $S \cap (U \cap O)$. Since S is closed in $A \cap U$, $S \cap (U \cap O)$ is closed in $A \cap (U \cap O)$. On the other hand, $A \cap (U \cap O) = F \cap (U \cap O)$ is closed in $U \cap O$, so that $S \cap (U \cap O)$ is closed in $U \cap O$. This implies that $s|_{U \cap O}$ belongs to $j_!\mathscr{F}(U \cap O)$, and this section has the prescribed germ at x. □

3.5.5. — Let A be a locally closed subset of X and let $u : \mathscr{F} \to \mathscr{G}$ be a morphism of abelian sheaves on A. The morphism $j_*u : j_*\mathscr{F} \to j_*\mathscr{G}$ maps $j_!\mathscr{F}$ into $j_!\mathscr{G}$. Indeed, if $s \in j_!\mathscr{F}(U) \subseteq \mathscr{F}(A \cap U)$, then $\operatorname{Supp}(u(s)) \subseteq \operatorname{Supp}(s)$, so that its closure in U, being contained in the closed subset $\operatorname{Supp}(s)$, is contained in $A \cap U$; since $\operatorname{Supp}(u(s))$ is closed in $A \cap U$, it is therefore closed in U. This defines an additive functor $j_! : \boldsymbol{Ab}_A \to \boldsymbol{Ab}_X$.

Proposition **(3.5.6)**. — *Let* A *be a locally closed subset of* X *and let* $j : A \to X$ *be the inclusion. The additive functor* $j_! : \boldsymbol{Ab}_A \to \boldsymbol{Ab}_X$ *is exact and fully faithful. It induces an equivalence of categories from* $\boldsymbol{Ab}_A$ *to the subcategory of abelian sheaves on* X *whose fibers outside of* A *are zero. The functor* j^{-1} *furnishes a quasi-inverse.*

Proof. — At the level of fibers, the functor $j_!$ induces the identity functor, or the zero functor; it is in particular exact.

Let $\mathscr{G}$ be an abelian sheaf on X such that $\mathscr{G}_x = 0$ for every $x \in X - A$. Let us show that the canonical morphism $\beta_{\mathscr{G}} : \mathscr{G} \to j_*j^{-1}\mathscr{G}$ factors through $j_!j^{-1}\mathscr{G}$. Let indeed U be an open subset of X. Then $j_*j^{-1}\mathscr{G}(U) = j^{-1}\mathscr{G}(A \cap U)$ and the morphism $\beta_{\mathscr{G}}(U) : \mathscr{G}(U) \to j^{-1}\mathscr{G}(A \cap U)$ factors through the morphism $s \mapsto (s|_V)_V$ from $\mathscr{G}(U)$ to $j^{-1}_{\text{pre}}\mathscr{G}(A \cap U) = \operatorname{colim}_{V \supseteq A \cap U} \mathscr{G}(V) \simeq \operatorname{colim}_{U \supseteq V \supseteq A \cap U} \mathscr{G}(V)$. Moreover, for every $s \in \mathscr{G}(U)$, the support of s is closed in U, hence the support of $s|_V$ is closed in V, for every open subset V of X such that $U \supseteq V \supseteq A \cap U$. This implies that the image of s belongs to $j_!j^{-1}\mathscr{G}(U)$. The resulting morphism of sheaves, $\mathscr{G} \to j_!j^{-1}\mathscr{G}$, induces an isomorphism on fibers: this is tautological for

$x \in A$, and follows from the fact that $\mathscr{G}_x = 0$ otherwise. This morphism is thus an isomorphism.

Let then $\mathscr{F}$ be an abelian sheaf on A. Since $j_!\mathscr{F} \subseteq j_*\mathscr{F}$, the canonical morphism $\alpha_{\mathscr{F}}\colon j^{-1}j_*\mathscr{F} \to \mathscr{F}$ induces a morphism $\varepsilon_{\mathscr{F}}\colon j^{-1}j_!\mathscr{F} \to \mathscr{F}$. Let O be an open subset of X containing A such that A is closed in O. Let V be an open subset of A. The canonical morphisms

$$\begin{aligned} j^{-1}_{\mathrm{pre}} j_!\mathscr{F}(V) = \operatorname*{colim}_{U \supseteq V} j_!\mathscr{F}(U) &\simeq \operatorname*{colim}_{O \supseteq U \supseteq V} j_!\mathscr{F}(U) \\ &\to \operatorname*{colim}_{O \supseteq U \supseteq V} j_*\mathscr{F}(U) = \operatorname*{colim}_{O \supseteq U \supseteq V} \mathscr{F}(A \cap U) \to \mathscr{F}(V) \end{aligned}$$

are isomorphisms. They induce an isomorphism from the presheaf $j^{-1}_{\mathrm{pre}} j_!\mathscr{F}$ to the sheaf $\mathscr{F}$, so that the corresponding morphism from $j^{-1}j_!\mathscr{F}$ to $\mathscr{F}$ is an isomorphism as well.

Let $\mathscr{F}$ and $\mathscr{G}$ be abelian sheaves on A and let $v\colon j_!\mathscr{F} \to j_!\mathscr{G}$ be a morphism of abelian sheaves. There exists a a unique morphism of sheaves $u\colon \mathscr{F} \to \mathscr{G}$ the diagram

$$\begin{array}{ccc} j^{-1}j_!\mathscr{F} & \xrightarrow{j^{-1}v} & j^{-1}j_!\mathscr{G} \\ {\scriptstyle \varepsilon_{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\mathscr{G}}} \\ \mathscr{F} & \xrightarrow{u} & \mathscr{G}. \end{array}$$

Since $j_!u$ induces the morphism v_x on the fibers at x, one then has $j_!u = v$.

This concludes the proof of the proposition. □

3.5.7. — Let A be a locally closed subset of X and let $j\colon A \to X$ be the inclusion. Let us construct a right adjoint $j^!$ to the functor $j_!$.

Let $\mathscr{G}$ be an abelian sheaf on X. For every open subset U of X, let $\Gamma_A(U;\mathscr{G})$ be the subset of all $s \in \mathscr{G}(U)$ whose support is contained in $A \cap U$. This is a subgroup of $\Gamma(U;\mathscr{G})$. The assignment $U \mapsto \Gamma_A(U;\mathscr{G})$ defines a subsheaf $\Gamma_A(\mathscr{G})$ of $\mathscr{G}$. This construction is functorial in $\mathscr{G}$.

If $x \in X - A$ and $s \in \Gamma_A(U;\mathscr{G})$, then $x \notin \mathrm{Supp}(s)$, so that there exists an open neighborhood V of x which is contained in U such that $s|_V = 0$. In particular, $s_x = 0$. This proves that the fibers of $\Gamma_A(\mathscr{G})$ vanish at all points x of $X - A$.

We set $j^!\mathscr{G} = j^{-1}\Gamma_A(\mathscr{G})$; this defines a functor from $\boldsymbol{Ab}_X$ to $\boldsymbol{Ab}_A$. By the preceding proposition, one has functorial isomorphisms $j_!j^!\mathscr{G} \simeq \Gamma_A(\mathscr{G})$, hence a monomorphism $\varepsilon_{\mathscr{G}}\colon j_!j^!\mathscr{G} \to \mathscr{G}$.

Let $\mathscr{F}$ be an abelian sheaf on A and let $u\colon j_!\mathscr{F} \to \mathscr{G}$ be morphism of abelian sheaves. Since the fibers of $j_!\mathscr{F}$ outside A are zero, the morphism u factors uniquely through a morphism $u'\colon j_!\mathscr{F} \to \Gamma_A(\mathscr{G}) = j_!j^!\mathscr{G}$, and there exists a unique morphism $v\colon \mathscr{F} \to j^{\mathscr{G}}$ such that $u' = j_!v$. This furnishes the desired adjunction for $(j_!, j^!)$.

Its unit and counit are the morphisms

$$\varepsilon_{\mathscr{G}}\colon j_!j^!\mathscr{G} \to \mathscr{G}$$

and

$$\eta_{\mathscr{F}}\colon \mathscr{F} \to j^!j_!\mathscr{F}$$

respectively. The counit is an isomorphism of functors, the unit is a monomorphism; consequently, $j_!$ is fully faithful, and $j^!$ is full.

Example **(3.5.8)**. — There are two important particular cases of this pair $(j_!, j^!)$ of functors.

First of all, if A is closed, then $j_!$ coincides with j_*. Since j_* has a right adjoint, it is right exact, which allows to recover that this functor is exact.

On the other hand, if A is open, then $j^!$ coincides with j^{-1}. Note that j^{-1} is exact and has both a left adjoint and a right adjoint, namely $j_!$ and j_* respectively.

3.5.9. — Let A be a closed subset of X and let $U = X - A$ be its complementary subset; let $i\colon A \to X$ and $j\colon U \to X$ be the inclusion maps. Let $\mathscr{F}$ be an abelian sheaf on X. The unit of the $(j_!, j^!)$ adjunction and the counit of the (i^{-1}, i_*) furnish an exact sequence

$$0 \to j_!j^!\mathscr{F} \to \mathscr{F} \to i_*i^{-1}\mathscr{F} \to 0.$$

Indeed, exactness can be checked on fibers; for $x \in A$, this sequence reduces to $0 \to 0 \to \mathscr{F}_x \to \mathscr{F}_x \to 0$, and for $x \in U$, it reduces to $0 \to \mathscr{F}_x \to \mathscr{F}_x \to 0 \to 0$.

The abelian sheaves $j_!j^!\mathscr{F}$ and $i_*i^{-1}\mathscr{F}$ on X are generally denoted by $\mathscr{F}_U$ and $\mathscr{F}_A$.

3.6. Cohomology of abelian sheaves

3.6.1. — Let X, Y be topological spaces, let $\mathscr{A}$ be a sheaf of rings on X and let $\mathscr{B}$ be sheaf of rings on Y. Let $\varphi\colon Y \to X$ be a continuous map together with a morphism of sheaves of rings $\varphi^\sharp\colon \mathscr{A} \to \varphi_*(\mathscr{B})$ or, equivalently, a morphism of sheaves of rings $\varphi^\flat\colon \varphi^{-1}(\mathscr{A}) \to \mathscr{B}$. In the terminology of definitions 4.2.1

and 4.2.2, we say that $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are ringed spaces, and that the pair $(\varphi, \varphi^\sharp)$ is a morphism of ringed spaces.

This datum induces a functor $\varphi_*\colon \boldsymbol{Mod}_{\mathscr{B}} \to \boldsymbol{Mod}_{\mathscr{A}}$ which, as we have seen, admits a left adjoint φ^*. In particular, the functor φ_* is left exact.

However, it is not right exact in general; applying the theory of derived functors to the functor φ_*, we will be able to quantify this defect of right exactness. To apply that machinery, we need to prove that the abelian category $\boldsymbol{Mod}_{\mathscr{A}}$ has enough injectives.

Theorem **(3.6.2)** (Grothendieck). — *Let X be a topological space and let $\mathscr{A}$ be a sheaf of rings on X. The abelian category $\boldsymbol{Mod}_{\mathscr{A}}$ of $\mathscr{A}$-modules has enough injectives.*

The proof proposed by Grothendieck (1957) consists in applying the general existence theorem, by showing that the category $\boldsymbol{Mod}_{\mathscr{A}}$ has a generator (namely, the direct sum $\bigoplus_U (j_U)_* \mathscr{O}_U$, indexed by all open subsets U of X) and satisfies the axiom (AB_5). The following proof, due to Godement (1958), is more direct.

Proof. — Let $\mathscr{M}$ be an $\mathscr{A}$-module. For every point $x \in X$, its fiber $\mathscr{M}_x$ is an $\mathscr{A}_x$-module. By the existence of injective modules, there exists an embedding of $\mathscr{M}_x$ into an injective $\mathscr{A}_x$-module I_x. For every point $x \in X$, let $i_x\colon \{x\}$ to X be the inclusion and let $\mathscr{I}$ be the sheaf on X given by $\prod_{x\in X} j_{x,*}(I_x)$, where we view I_x as a sheaf on $\{x\}$; Its sections on an open subset U of X identifies with the product set $\prod_{x\in U} I_x$, the restriction maps being given by the canonical projections. It is in fact a sheaf in abelian groups. Moreover, the $\mathscr{A}_x$-module structures on the module I_x, for every $x \in X$, endow it with the structure of an $\mathscr{A}$-module.

Moreover, the injections $\mathscr{M}_x \hookrightarrow I_x$ furnish a morphism of $\mathscr{A}$-modules $\mathscr{M} \to \mathscr{I}$ which is a monomorphism.

Let us prove that $\mathscr{I}$ is an injective object in the abelian category of $\mathscr{A}$-modules. We have to prove that the functor $\operatorname{Hom}_{\mathscr{A}}(\cdot, \mathscr{I})$ is exact. Using the definition of $\mathscr{I}$ as a product and the adjunctions $(i_x^*, i_{x,*})$, this functor is isomorphic to the functor $\mathscr{F} \mapsto \prod_{x\in X} \operatorname{Hom}_{\mathscr{A}_x}(\mathscr{F}_x, I_x)$. Since a product of exact functor is exact, it suffices to prove that for every $x \in X$, the functor $\mathscr{F} \mapsto \operatorname{Hom}_{\mathscr{A}_x}(\mathscr{F}_x, I_x)$ is exact. On the other hand, this functor is the composition of the fiber functor $\mathscr{F} \mapsto \mathscr{F}_x$, which is exact, and the functor $F \mapsto \operatorname{Hom}_{\mathscr{A}_x}(F, I_x)$, which is as well exact, because I_x is an injective $\mathscr{A}_x$-module. This concludes the proof. □

3.6.3. — Let X be a topological space and let $\mathscr{A}$ be a sheaf of rings on X. By the preceding theorem, the category $\boldsymbol{Mod}_{\mathscr{A}}$ is known to have enough injectives; consequently, one can apply the theory of derived functors to any additive left exact functor on this category. In particular, we have the following ones:

– The functor of global sections $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$, from $\boldsymbol{Mod}_{\mathscr{A}}$ to $\boldsymbol{Mod}_{A}$, where $A = \Gamma(X, \mathscr{A})$. Its derived functors are called the cohomology modules and are denoted by $H^n(X, \mathscr{F})$.

– But we can also forget the A-module structure and consider this functor from $\boldsymbol{Mod}_{\mathscr{A}}$ to $\boldsymbol{Ab}$.

– More generally, one can consider global sections with support in a closed subspace W of X; this gives rise to cohomology with support $H^n_W(X, \mathscr{F})$.

– More generally, functors of the form $\mathscr{F} \mapsto \mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{F})$, where $\mathscr{E}$ is an $\mathscr{A}$-module; the derived functors are called the Ext-modules of $\mathscr{F}$, and are denoted $\mathrm{Ext}^n_{\mathscr{A}}(\mathscr{E}, \mathscr{F})$.

– The direct image functor $\varphi_*\colon \boldsymbol{Mod}_{\mathscr{A}} \to \boldsymbol{Mod}_{\mathscr{B}}$, whenever $\varphi\colon X \to Y$ be a continuous map of topological spaces, $\mathscr{B}$ is a sheaf of rings and $\varphi^\sharp\colon \mathscr{B} \to \varphi_*\mathscr{A}$ be a morphism of sheaves of rings on Y. Its derived functors are called *higher direct images*, and are denoted by $\mathscr{F} \mapsto \boldsymbol{R}^n\varphi_*\mathscr{F}$.

The definition of these functors depends of the choice of $\mathscr{A}$ and $\mathscr{B}$, so that their derived functors a priori depend on these sheaves of rings. The definition of the derived functors implies that the dependence on $\mathscr{B}$ is mostly irrelevant: the abelian sheaves underlying the cohomology objects of a complex of $\mathscr{B}$-modules are the cohomology objects of the same complex, viewed as a complex of abelian sheaves.

These functors do not really depend on $\mathscr{A}$: for example, the global sections of an $\mathscr{A}$-module coincide with its global sections as an abelian sheaf. However, the categories $\boldsymbol{Mod}_{\mathscr{A}}$ and $\boldsymbol{Ab}_X$ differ, and they don't have the same injective objects, so that the cohomology of an $\mathscr{A}$-module $\mathscr{F}$ and that of the underlying abelian sheaf could differ in principle. The theory of flasque sheaves will allow us to see that this discrepancy does not happen.

3.6.4. — Cohomology is not only functorial with respect to morphism of sheaves, but also with respect to morphism of schemes. Let $f\colon Y \to X$ be continuous map and let $\mathscr{F}$ be an abelian sheaf on X. Let us construct a morphism of abelian groups $f^*\colon H^n(X, \mathscr{F}) \to H^n(Y, f^{-1}\mathscr{F})$.

Let $\varepsilon\colon \mathscr{F} \to \mathscr{I}_0 \to \mathscr{I}_1 \to \dots$ be the injective resolution chosen to compute $H^n(X, \mathscr{F})$.

By definition, one has $H^n(X, \mathscr{F}) = H^n(\Gamma(X, \mathscr{Y}_\bullet))$.

The definition of the sheaves $f^{-1}\mathscr{I}_m$ furnishes morphisms $\Gamma(X, \mathscr{I}_m) \to \Gamma(Y, f^{-1}\mathscr{I}_m)$ from which we deduce a morphism of complexes $\Gamma(X, \mathscr{I}_\bullet) \to \Gamma(Y, f^{-1}\mathscr{I}_\bullet)$ and morphisms of cohomology groups

$$H^n(\Gamma(X, \mathscr{I}_\bullet)) \to H^n(\Gamma(Y, f^{-1}\mathscr{I}_\bullet)).$$

Since the functor $f^{-1}\colon \boldsymbol{Ab}_X \to \boldsymbol{Ab}_Y$ is exact, it induces a resolution $f^{-1}\varepsilon\colon f^{-1}\mathscr{F} \to f^{-1}\mathscr{I}_0 \to \dots$ of $f^{-1}\mathscr{F}$. We thus have a canonical morphism

$$H^n(\Gamma(Y, f^{-1}\mathscr{I}_\bullet)) \to H^n(Y, f^{-1}\mathscr{G}).$$

Combining these two morphisms furnishes the desired morphisms $f^*\colon H^n(X, \mathscr{F}) \to H^n(Y, f^{-1}\mathscr{F})$, for $n \in \mathbf{N}$.

If $f = \mathrm{id}_X$, then f^* is the identity. If $g\colon Z \to Y$ is another continuous map, then one has $(f \circ g)^* = g^* \circ f^*$.

3.6.5. —

Proposition (**3.6.6**). — *Let $\mathscr{F}$ be an abelian sheaf on* X *and let n be an integer such that $n \geqslant 1$ and let $\xi \in H^n(X, \mathscr{F})$. For every point $x \in X$, there exists an open neighborhood* U *of x such that $\xi|_U = 0$.*

In other words, cohomology vanishes locally.

Proof. — Let $\varepsilon\colon \mathscr{F} \to \mathscr{I}_\bullet$ be an injective resolution of $\mathscr{F}$. By definition, $H^n(X, \mathscr{F})$ is the nth cohomology group of the complex $\Gamma(X, \mathscr{I}_\bullet)$. The class ξ is represented by an element $c \in \Gamma(X, \mathscr{I}_n)$ such that $d_n(c) = 0$. Since $n \geqslant 1$, the complex $\mathscr{I}_\bullet$ is exact at $\mathscr{I}_n$ and one has $\ker(d_n) = \operatorname{Im}(d_{n-1})$. The class c belongs to the image sheaf $d_{n-1}(\mathscr{I}_{n-1})$. Since the fiber of this sheaf at x is the image $d_{n-1}(\mathscr{I}_{n-1,x})$, there exists an element $c'_x \in \mathscr{I}_{n-1,x}$ such that $c_x = d_{n-1}(c'_x)$. There exists an open neighborhood U of x and $c' \in \mathscr{I}_{n-1}(U)$ with germ c'_x at x; then one has $c_x = d_{n-1}(c')_x$, so that $c|_U$ and $d_{n-1}(c')$ have the same germ at x. Consequently, there exists an open neighborhood V of x such that $V \subseteq U$ and such that $c|_V = d_{n-1}(c')|_V = d_{n-1}(c'|_V)$. The cohomology class $\xi|_V$ is represented by $c|_V = d_{n-1}(c'|_V)$, hence it vanishes. □

3.7. Flasque sheaves

Definition (**3.7.1**). — *One says that a sheaf* $\mathscr{F}$ *on* X *is* flasque[(1)] *if for every open subset* U *of* X*, the restriction map* $\mathscr{F}(X) \to \mathscr{F}(U)$ *is surjective.*

Example (**3.7.2**). — *On an* irreducible *topological space, every constant sheaf (with non-empty fiber) is flasque.*

Let S be a non-empty set and let S_X be the constant sheaf on X with value S; by definition, its sections over an open subset U of X are the locally constant functions from U to S.

Let U be an open subset of X and let $f \in S_X(U)$. If $U = \varnothing$, any constant function from X to S restricts to f; since S is non-empty, this proves that the restriction map from $S_X(X)$ to $S_X(U)$ is surjective. Otherwise, U is irreducible (proposition 1.10.3), hence is connected; consequently, the locally constant function f is constant, and it extends (uniquely, U being non-empty) to a constant function from X to S.

Lemma (**3.7.3**). — *Let* X *be a topological space.*

a) *Let* $\mathscr{F}$ *be a flasque sheaf on* X*. The sheaf* $\mathscr{F}|_U$ *is flasque, for every open subset* U *of* X*.*

b) *Let* $f\colon X \to Y$ *be a continuous map and let* $\mathscr{F}$ *be a sheaf on* X*. The sheaf* $f_*\mathscr{F}$ *on* Y *is flasque.*

c) *Let* $(\mathscr{F}_i)_{i\in I}$ *be a family of flasque sheaves on* X*. Their product* $\prod_{i\in I}\mathscr{F}_i$ *is flasque.*

d) *Let* $\mathscr{F}$ *be a flasque sheaf on* X *and let* $\mathscr{F}'$ *be retract of* $\mathscr{F}$*. Then* $\mathscr{F}'$ *is flasque.*

Proof. — a) Let V be an open subset of U and let $s \in \mathscr{F}(V)$. Since $\mathscr{F}$ is flasque, there exists $s' \in \mathscr{F}(X)$ such that $s'|_V$. Then $t = s'|_U$ is an element of $\mathscr{F}(U)$ such that $t|_V = s$. This proves that $\mathscr{F}|_U$ is flasque.

b) Let U be an open subset of Y and let $s \in f_*\mathscr{F}(U)$. By definition, s is a section t of $\mathscr{F}(f^{-1}(U))$. Since $\mathscr{F}$ is flasque, there exists a section $t' \in \mathscr{F}(X)$ such that $t'|_{f^{-1}(U)} = t$. Then t' can be viewed as a section s' of $\mathscr{F}(Y)$ and $t'|_U = t$. This proves that $f_*\mathscr{F}$ is flasque.

c) Let U be an open subset of Y and let s be a section of $\prod_{i\in I}\mathscr{F}_i(U)$, corresponding to a family (s_i), where $s_i \in \mathscr{F}_i(U)$ for all i. For every $i \in I$, there exists

(1) This is the French word used by Godement (1958); some authors say *flabby*.

a section $t_i \in \mathscr{F}_i(X)$ such that $t_i|_U = s_i$, because $\mathscr{F}_i$ is flasque. Then the family $t = (t_i)$ is a section of $\prod \mathscr{F}_i$ on X such that $t|_U = s$.

d) By assumption, $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$ and there exists a morphism of sheaves $r\colon \mathscr{F} \to \mathscr{F}'$ which is the identity on $\mathscr{F}'$. Let U be an open subset of X and let $s' \in \mathscr{F}'(U)$. Let $t \in \mathscr{F}(X)$ be a global section extending s', as a section of $\mathscr{F}$. Then $r(t) \in \mathscr{F}'(X)$ and $r(t)|_U = r(t|_U) = r(s') = s'$. This proves that $\mathscr{F}'$ is flasque. □

Example **(3.7.4)**. — If X is discrete, every section of an étale space is continuous, so that a sheaf $\mathscr{F}$ on X is flasque if and only if its fibers $\mathscr{F}_x$ are non-empty, for all $x \in X$. Indeed, In particulier, every sheaf on X is flasque in this case.

Let X^δ be the set X endowed with the discrete topology. The identity $p\colon X^\delta \to X$ is continuous. For every abelian sheaf $\mathscr{F}$ on X, one lets $G(\mathscr{F}) = p_*p^*\mathscr{F}$. This is a flasque sheaf on X, and the unit $\eta_{\mathscr{F}}\colon \mathscr{F} \to G(\mathscr{F})$ is a monomorphism. Explicitly, one has $G(\mathscr{F})(U) = \prod_{x\in U} \mathscr{F}_x$, for every open subset U of X, the restriction morphisms are the morphisms $\prod_{x\in U} \mathscr{F}_x \to \prod_{x\in V} \mathscr{F}_x$, for $V \subseteq U$ which are surjective. In fact, they even have a section.

Example **(3.7.5)**. — *An injective sheaf $\mathscr{F}$ on* X *is flasque.*

Let indeed U be an open subset of X and let $s \in \mathscr{F}(U)$. Let $j\colon U \to X$ be the canonical inclusion; let $f\colon j_!\mathbf{Z}_U \to \mathscr{F}$ be the unique morphism corresponding to the morphism $\mathbf{Z}_U \to j^*\mathscr{F}$ which maps 1 to s. The canonical morphism $u\colon j_!\mathbf{Z}_U \to \mathbf{Z}_X$ is injective; since $\mathscr{F}$ is injective, there exists a unique morphism $g\colon \mathbf{Z}_X \to \mathscr{F}$ such that $g \circ u = f$; it corresponds to a section $t \in \mathscr{F}(X)$ such that $t|_U = s$. Consequently, the restriction morphism $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective, as as to be shown.

Proposition **(3.7.6)**. — *Let $\mathscr{F}$ be a sheaf on* X*. Assume that for every open subset* U *of* X *and every $s \in \mathscr{F}(U)$, there exists an open covering $\mathscr{V}$ of* X *such that for every* $V \in \mathscr{V}$*, there exists $t_V \in \mathscr{F}(V)$ such that $t_V|_{U\cap V} = s|_{U\cap V}$. Then $\mathscr{F}$ is flasque.*

Proof. — Let U be an open subset of X and let $s \in \mathscr{F}(U)$. Let us show that there exists $t \in \mathscr{F}(X)$ such that $t|_U = s$. Let $\mathscr{R}$ be the set of pairs (V, t), where V is an open subset of X and $t \in \mathscr{F}(V)$. The relation $\leq$ defined by $(V, t) \leq (V', t')$ if and only if $V \subseteq V'$ and $t'|_V = t$ is an ordering relation on $\mathscr{R}$; moreover, the ordered set $\mathscr{R}$ is inductive. By Zorn's lemma, we may consider a maximal element (W, t) of $\mathscr{R}$ such that $(U, s) \leq (W, t)$; let us show that $W = X$. By hypothesis, there exists an open covering $\mathscr{V}$ of X and, for every $V \in \mathscr{V}$, an element $t_V \in \mathscr{F}(V)$ such that $t_V|_{U\cap W} = t|_{W\cap V}$. If $W \neq X$, there exists an open subset $V \in \mathscr{V}$ such that $V \not\subseteq W$; then, there exists a unique section $t' \in (W \cup V)$ which restricts to t on W and to t_V on V. In particular, $(W \cup V, t')$ is an element of $\mathscr{R}$ such that $(W, t) \leq (W \cup V, t')$, contradicting the hypothesis that (W, t) were maximal. □

Corollary **(3.7.7)**. — *Let $\mathscr{F}$ be a sheaf on* X. *Assume that there exists an open covering $\mathscr{V}$ of* X *such that $\mathscr{F}|_V$ is flasque, for every* $V \in \mathscr{F}$. *Then $\mathscr{F}$ is flasque.*

Proof. — Indeed, the condition of the proposition is satisfied: since $\mathscr{F}|_V$ is flasque, there exists $t \in \mathscr{F}(V)$ which restricts to $s|_{U\cap V}$. □

Proposition **(3.7.8)**. — *Let* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *be an exact sequence of abelian sheaves. Assume that $\mathscr{F}'$ is flasque.*

a) *For every open subset* U *of* X, *the sequence* $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ *is exact.*

b) *If $\mathscr{F}$ is flasque, then $\mathscr{F}''$ is flasque as well.*

Proof. — a) It suffices to treat the case where $U = X$, the general case follows from it by applying the result to the sheaves deduced by restriction to U.

It is a general fact that the sequence $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is exact, except possibly at $\mathscr{F}''(X)$.

Let $s \in \mathscr{F}''(X)$. Let us show that there exists $t \in \mathscr{F}(X)$ with image s. Let $\mathscr{F}_s$ be the subsheaf of $\mathscr{F}$ consisting of sections t with image s in $\mathscr{F}''$. The exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ implies that $\mathscr{F}_s$ is locally isomorphic to $\mathscr{F}'$: whenever U is an open subset of X such that $s|_U$ has a lift $t \in \mathscr{F}(U)$, the morphism $\mathscr{F}'|_U \to \mathscr{F}_s|_U$ is an isomorphism of sheaves, hence is flasque, because $\mathscr{F}'|_U$ is flasque. Consequently, $\mathscr{F}_s$ is flasque and admits a global section t. By construction, $t \in \mathscr{F}(X)$ maps to s in $\mathscr{F}''(X)$.

b) Assume now that $\mathscr{F}$ is flasque as well; let us show that $\mathscr{F}''$ is flasque. Let U be an open subset of X and let $s \in \mathscr{F}''(U)$. By what precedes, there exists $t \in \mathscr{F}(U)$ which maps to s. Since $\mathscr{F}$ is flasque, there exists $t' \in \mathscr{F}(X)$ such that $t'|_U = t$. Then the image s' of t' in $\mathscr{F}''(X)$ satisfies $s'|_U = s$. Consequently, $\mathscr{F}''$ is flasque. □

3.7.9. — To be added :

a) Existence of flasque resolutions
b) Injective sheaves are flasque

c) Acyclicity of flasque sheaves for Γ or f_*

d) Cohomology of an $\mathscr{A}$-module coincides with its cohomology as an abelian sheaf.

e) Mayer–Vietoris exact sequence

f) Cohomology of a sheaf on a closed subset and cohomology of its extension by zero.

g) Higer direct images and cohomology of preimages.

h) Cohomology and directed colimits

3.8. Cohomological dimension

The main objective of this section is to prove a theorem of Grothendieck (1957) about the vanishing of the cohomology groups of abelian sheaves.

Definition **(3.8.1)**. — *Let* X *be a topological space. The* cohomological dimension *of* X *is the least upper bound of all integers n such that there exists an abelian sheaf* $\mathscr{F}$ *on* X $H^n(X, \mathscr{F}) \neq 0$. *We denote it as* $\operatorname{coh dim}(X)$.

This is an element of $\mathbf{N} \cup \{+\infty\}$. It follows from the definition that for any integer $n > \operatorname{coh dim}(X)$ and any abelian sheaf $\mathscr{F}$ on X, one has $H^n(X, \mathscr{F}) = 0$.

Proposition **(3.8.2)**. — *Let* X *be a topological space and let* $(\mathscr{F}_i)_{i \in I}$ *be a direct system of sheaves on* X.

a) *If* X *is quasi-compact, then the canonical map* $\operatorname{colim}_i \mathscr{F}_i(X) \to (\operatorname{colim}_i \mathscr{F}_i)(X)$ *is injective.*

b) *If* X *is noetherian, then this map is bijective.*

Proof. — Let us denote by $\mathscr{F}$ the colimit sheaf $\operatorname{colim}_i \mathscr{F}_i$.

a) Let $s, s' \in \operatorname{colim}_i \mathscr{F}_i(X)$ be two elements having the same image in $\mathscr{F}(X)$. By definition, s and s' are represented by some elements $s_i \in \mathscr{F}_i(X)$ and $s'_j \in \mathscr{F}_j(X)$, for some $i, j \in I$. Since I is a directed set, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Up to replacing s_i and s'_j by their images in $\mathscr{F}_k$, we may assume that $i = j = k$.

There exists an open covering $(U_\alpha)_{\alpha \in A}$ of X such that $s|_{U_\alpha} = s'|_{U_\alpha}$ in $\operatorname{colim}_i \mathscr{F}_i(U_\alpha)$, for any $\alpha \in A$. Let $\alpha \in A$. The equality $s|_{U_\alpha} = s'|_{U_\alpha}$ implies that there exists an element $k_\alpha \in I$ such that $i \leq k_\alpha$ and such that the images of $s_i|_{U_\alpha}$ and $s'_i|_{U_\alpha}$ in $\mathscr{F}_{k_\alpha}(U_\alpha)$ coincide.

Since X is quasi-compact, there exists a finite subset A' of A such that $X = \bigcup_{\alpha\in A'} U_\alpha$. Let $k \in I$ be an element such that $k_\alpha \leq k$ for all $\alpha \in A'$. The images of $s_i|_{U_\alpha}$ and $s'_i|_{U_\alpha}$ in $\mathscr{F}_k(U_\alpha)$ coincide for all $\alpha \in A'$. Since the U_α, for $\alpha \in A'$, cover X, it follows that the images of s_i and s'_i in $\mathscr{F}_k(X)$ coincide. Consequently, $s = s'$ in $\operatorname{colim}_i \mathscr{F}_i(X)$

b) Let $s \in \mathscr{F}(X)$. By definition, the sheaf $\mathscr{F}$, is associated with the presheaf $U \mapsto \operatorname{colim}_i \mathscr{F}_i(U)$; consequently, there exists an open covering $(U_\alpha)_{\alpha\in A}$ of X and, for any $\alpha \in A$, an element $i_\alpha \in I$ and an element $s_\alpha \in \mathscr{F}_{i_\alpha}(U_\alpha)$ that represents $s|U_\alpha$. Since X is quasicompact, there exists a finite subset A' of A such that the U_α, for $\alpha \in A'$, cover X. Changing notation, we may thus assume that $A = \{1, \dots, n\}$. For any integer p such that $0 \leqslant p \leqslant n$, set $V_p = U_1 \cup \dots \cup U_p$.

By induction, we will construct an element $j_p \in I$ and a section $s_p \in \mathscr{F}_{j_p}(V_p)$ whose class in $\operatorname{colim}_i \mathscr{F}_i(V_p)$ maps to $s|_{V_p}$.

For $p = 0$, one has $V_0 = \varnothing$, and we take for j_0 an arbitrary element of I (which is non-empty) and for s_0 the only section of $\mathscr{F}_{i_0}(\varnothing)$.

Assume that $1 \leqslant p \leqslant n$ and that j_{p-1}, s_{p-1} have been constructed. Let also $t \in \mathscr{F}_{j_p}(U_p)$ whose class in $\operatorname{colim}_i \mathscr{F}_i(U_p)$ represents $s|_{U_p}$. The restrictions to $V_{p-1} \cap U_p$ of s_{p-1} and t define clases in $\operatorname{colim}_i \mathscr{F}_i(V_{p-1} \cap U_p)$ which both map to $s|_{V_{p-1}\cap U_p}$ in $\mathscr{F}(V_{p-1} \cap U_p)$. Since X is a noetherian topological space, the open subset $V_{p-1} \cap U_p$ is quasi-compact, and assertion *a*) implies that there exists an element $j_p \in I$ such that $j_{p-1} \leq j_p$, $i_p \leq j_p$ and such the images of $s_{p-1}|_{V_{p-1}\cap U_p}$ and $t|_{V_{p-1}\cap U_p}$ in $\mathscr{F}_{j_p}(V_{p-1} \cap U_p)$ coincide. Consequently, the images of s_{p-1} in $\mathscr{F}_{j_p}(V_{p-1})$ and of t in $\mathscr{F}_{j_p}(U_p)$ coincide on $V_{p-1} \cap U_p$, hence there exists a unique section $s_p \in \mathscr{F}_{j_p}(V_p)$ which coincides with them on V_{p-1} and U_p. The image of s_p in $\mathscr{F}(V_p)$ is equal to $s|_{V_p}$ since it restricts to $s|_{V_{p-1}}$ on V_{p-1} and to $s|_{U_p}$ on U_p. This concludes the induction step.

When $p = n$, we have $V_n = X$ and an element $s_n \in \mathscr{F}_{j_n}(X)$ which incudes s in $\mathscr{F}(X)$. This concludes the proof of the proposition. □

Corollary **(3.8.3)**. — *Let* X *be a noetherian topological space and let* $(\mathscr{F}_i)_{i\in I}$ *be a direct system of sheaves on* X. *The presheaf* $\operatorname{colim}_i \mathscr{F}_i$ *is a sheaf.*

Proof. — Let $\mathscr{F}_{\mathrm{pre}}$ be the presheaf $\operatorname{colim}_i \mathscr{F}_i$ and let $\mathscr{F}$ be the associated sheaf. By definition, $\mathscr{F}_{\mathrm{pre}}(U) = \operatorname{colim}_i \mathscr{F}_i(U)$ for every open subset U of X. Since X is noetherian, any open subset of X is noetherian, and proposition 3.8.2 asserts that the canonical morphism of presheaves $\mathscr{F}_{\mathrm{pre}} \to \mathscr{F}$ induces a bijection

$\mathscr{F}_{\mathrm{pre}}(\mathrm{U}) \to \mathscr{F}(\mathrm{U})$, for any open subset U of X. Consequently, this morphism is an isomorphism and $\mathscr{F}_{\mathrm{pre}}$ is a sheaf. □

Corollary **(3.8.4)**. — *Let* X *be a noetherian topological space. The colimit* $\operatorname{colim}_i \mathscr{F}_i$ *of a direct system* $(\mathscr{F}_i)_{i\in \mathrm{I}}$ *of flasque sheaves on* X *is flasque.*

Proof. — Let U be an open subset of X and let $s \in (\operatorname{colim}_i \mathscr{F}_i)(\mathrm{U})$. By the proposition, applied to the noetherian topological space U, there exists an element $i \in \mathrm{I}$ and a section $s_i \in \mathscr{F}_i(\mathrm{U})$ with image s. Since $\mathscr{F}_i$ is flasque, there exists a section $t_i \in \mathscr{F}_i(\mathrm{X})$ such that $t_i|_{\mathrm{U}} = s_i$, and tts image t in $\mathscr{F}(\mathrm{X})$ satisfies $t|_{\mathrm{U}} = s$. This proves that the sheaf $\operatorname{colim}_i \mathscr{F}_i$ is flasque. □

Corollary **(3.8.5)**. — *Let* X *be a noetherian topological space, let* $(\mathscr{F}_i)$ *be a directed system of abelian sheaves on* X*. For any integer* n*, the canonical morphism* $\operatorname{colim}_i \mathrm{H}^n(\mathrm{X}, \mathscr{F}_i) \to \mathrm{H}^n(\mathrm{X}, \operatorname{colim}_i \mathscr{F}_i)$ *is an isomorphism of abelian groups.*

Proof. — Let $\mathscr{F} = \operatorname{colim}_i \mathscr{F}_i$ be the colimit sheaf of this directed system. For every $i \in \mathrm{I}$, consider the Godement resolution $\mathscr{F}_i \to \mathrm{G}_\bullet(\mathscr{F}_i)$ by flasque sheaves. By functoriality, we obtain a morphism $\mathscr{F} \to \operatorname{colim} \mathrm{G}(\mathscr{F}_i)$ which, by right-exactness of direct colimits, is a resolution of $\mathscr{F}$. By the preceding corollary, the sheaves $\operatorname{colim}_i \mathrm{G}_n(\mathscr{F}_i)$ are flasque, for all $n \in \mathbf{N}$. By the proposition, the complex with terms $(\operatorname{colim}_i \mathrm{G}_n(\mathscr{F}_i))(\mathrm{X})$ coincides with the complex with terms $\operatorname{colim}_i(\mathrm{G}_n(\mathscr{F}_i)(\mathrm{X}))$. By exactness of direct colimits, its cohomology is $\operatorname{colim}_i \mathrm{H}^n(\mathrm{X}, \mathscr{F}_i)$, as claimed. □

Theorem **(3.8.6)**. — *Let* X *be a* noetherian *topological space and let* $n \in \mathbf{N}$*. The following assertions are equivalent:*

(i) *One has* $n \leqslant \operatorname{coh\,dim}(\mathrm{X})$*;*

(ii) *For any integer* m *such that* $m > n$ *and any abelian sheaf* $\mathscr{F}$ *on* X*, one has* $\mathrm{H}^m(\mathrm{X}, \mathscr{F}) = 0$*;*

(iii) *For any open subset* U *of* X *and any integer* m *such that* $m > n$*, one has* $\mathrm{H}^m(\mathrm{X}, \mathbf{Z}_{\mathrm{U}}) = 0$*;*

Proof. — The equivalence (i)⇔(ii) is exactly the definition 3.8.1 of the cohomological dimension.

Assertion (iii) is the particular case of (ii), applied with $\mathscr{F} = \mathbf{Z}_{\mathrm{U}} = j_!\mathbf{Z}$, where $j \colon \mathrm{U} \to \mathrm{X}$ is the inclusion map.

Let us now prove the implication (iii)⇒(ii). Let us assume that $H^m(X, \mathbf{Z}_U) = 0$ for any open subset U of X and any integer $m > n$. Let $\mathscr{F}$ be an abelian sheaf on X and let us prove that $H^m(X, \mathscr{F}) = 0$ for any integer $m > n$.

We split the proof in four steps.

a) Let us prove the result under the assumption that there exists a locally closed subset A of X such that $\mathscr{F} = \mathbf{Z}_A$.

By definition of a locally closed subset, there exists an open subset U of X such that A is closed in U. Consequently, there exists an open subset V of U such that $A = U \setminus V$. One then has an exact sequence

$$0 \to \mathbf{Z}_V \to \mathbf{Z}_U \to \mathbf{Z}_A \to 0$$

of abelian sheaves on X. Let m be an integer such that $m > n$; we have a short exact sequence

$$H^m(X, \mathbf{Z}_U) \to H^m(X, \mathbf{Z}_A) \to H^{m+1}(X, \mathbf{Z}_V),$$

as a part of the long exact sequence of cohomology associated with the initial short exact sequence. By assumption, $H^m(X, \mathbf{Z}_U) = H^{m+1}(X, \mathbf{Z}_V) = 0$; consequently, $H^m(X, \mathbf{Z}_A) = 0$.

b) Let us prove the result under the assumption that there exists an open subset U of X such that $\mathscr{F}$ is an abelian subsheaf of $\mathbf{Z}_U$.

By lemma 3.8.7 below, there exists a finite sequence $(\mathscr{L}_0, \dots, \mathscr{L}_p)$ of subsheaves of $\mathscr{L}$ such that $0 = \mathscr{L}_0 \subseteq \dots \subseteq \mathscr{L}_p = \mathscr{L}$, and for every integer q such that $1 \leqslant p \leqslant p$, a locally closed subset A_q of U such that $\mathscr{L}_q / \mathscr{L}_{q-1} \simeq \mathbf{Z}_{A_q}$.

Let us prove by induction on q that $H^m(X, \mathscr{L}_q) = 0$ for all integers $m > n$. This is trivial if $q = 0$, since $\mathscr{L}_0 = 0$. Let us assume that $q \geqslant 1$ and that the result holds for $q - 1$.

Let m be an integer such that $m > n$. From the long exact sequence in cohomology associated with the short exact sequence $0 \to \mathscr{L}_{q-1} \to \mathscr{L}_q \to \mathbf{Z}_{A_q} \to 0$, we get a short exact sequence

$$H^m(X, \mathscr{L}_{q-1}) \to H^m(X, \mathscr{L}_q) \to H^m(X, \mathbf{Z}_{A_q}).$$

By induction, the first term is zero, and step *a*) asserts that the third term vanishes as well. Consequently, $H^m(X, \mathscr{L}_q) = 0$, as was to be shown.

For $q = p$, we obtain $H^m(X, \mathscr{L})$ for all integers $m > n$, as claimed.

c) Let us now prove that the result holds under the assumption that there exists an open subset U of X and an epimorphism $\mathbf{Z}_U \to \mathscr{F}$.

Let $\mathscr{L}$ be its kernel. For any integer m such that $m > n$, one has the short exact sequence

$$\mathrm{H}^m(\mathrm{X}, \mathbf{Z}_\mathrm{U}) \to \mathrm{H}^m(\mathrm{X}, \mathscr{F}) \to \mathrm{H}^{m+1}(\mathrm{X}, \mathscr{L}),$$

as a part of the long exact sequence of cohomology associated with the short exact sequence $0 \to \mathscr{L} \to \mathscr{Z}_\mathrm{U} \to \mathscr{F} \to 0$. By assumption (iii), one has $\mathrm{H}^m(\mathrm{X}, \mathbf{Z}_\mathrm{U}) = 0$, and by step *b*), we have $\mathrm{H}^{m+1}(\mathrm{X}, \mathscr{L}) = 0$. Consequently, $\mathrm{H}^m(\mathrm{X}, \mathscr{F}) = 0$.

d) We now treat the general case. Let I be the set of all pairs (U, s), where U is an open subset of X and $s \in \mathscr{F}(\mathrm{U})$; for $i = (\mathrm{U}, s) \in \mathrm{I}$, we write $\mathrm{U}_i = \mathrm{U}$ and $s_i = s$ and let $j_i \colon \mathrm{U}_i \to \mathrm{X}$ be the inclusion map. For any $i \in \mathrm{I}$, the section $s_i \in \mathscr{F}(\mathrm{U}_i)$ defines a morphism $\mathscr{Z} \to j_i^{-1}\mathscr{F}$, hence, using the adjunction for $(j_{i!}, j_i^!)$, a morphism $\mathbf{Z}_{\mathrm{U}_i} = j_{i!}\mathscr{Z} \to \mathscr{F}$. The direct sum of these morphisms

$$\bigoplus_{i \in \mathrm{I}} \mathbf{Z}_{\mathrm{U}_i} \to \mathscr{F}$$

is an epimorphism. For any finite subset L of I, let $\mathscr{F}_\mathrm{L}$ be the image of $\bigoplus_{i\in\mathrm{L}} \mathbf{Z}_{\mathrm{U}_i}$ in $\mathscr{F}$. The set of all finite subsets of I is a directed set, and the family $(\mathscr{F}_\mathrm{L})_\mathrm{L}$ of subsheaves of $\mathscr{F}$ is a direct system, with colimit $\mathscr{F}$. By corollary 3.8.5, it suffices to prove that $\mathrm{H}^m(\mathrm{X}, \mathscr{F}_\mathrm{L}) = 0$ for all integers m such that $m > n$.

We may thus assume that there exists an integer p, open subsets $\mathrm{U}_1, \dots, \mathrm{U}_p$ of X, and sections $s_i \in \mathscr{F}(\mathrm{U}_i)$ inducing an epimorphism $\bigoplus_{i=1}^p \mathbf{Z}_{\mathrm{U}_i} \to \mathscr{F}$. For any integer q such that $0 \leqslant q \leqslant p$, let $\mathscr{F}_q$ be the image of the direct sum $\bigoplus_{i=0}^q \mathbf{Z}_{\mathrm{U}_i}$. This gives rise to a filtration

$$0 = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \dots \subseteq \mathrm{F}_p = \mathscr{F},$$

and for every integer $q \in \{1, \dots, p\}$, the quotient sheaf $\mathscr{F}_q / \mathscr{F}_{q-1}$ is a quotient of $\mathbf{Z}_{\mathrm{U}_q}$.

We now argue by induction on q that $\mathrm{H}^m(\mathrm{X}, \mathscr{F}_q) = 0$ for all integers q such that $0 \leqslant q \leqslant p$ and all integers $m > n$. The case $q = 0$ is trivial, since $\mathscr{F}_0 = 0$. Now assume that $q \geqslant 1$ and that the result holds for $q - 1$. Let us consider the exact sequence $0 \to \mathscr{F}_{q-1} \to \mathscr{F}_q \to \mathscr{F}_q / \mathscr{F}_{q-1} \to 0$ of abelian sheaves on X. Let m be an integer such that $m > n$. From the associated long exact sequence of cohomology, we get an exact sequence

$$\mathrm{H}^m(\mathrm{X}, \mathscr{F}_{q-1}) \to \mathrm{H}^m(\mathrm{X}, \mathscr{F}_q) \to \mathrm{H}^m(\mathrm{X}, \mathscr{F}_q / \mathscr{F}_{q-1}).$$

By induction, one has $\mathrm{H}^m(\mathrm{X}, \mathscr{F}_{q-1}) = 0$, and step *c*) asserts that $\mathrm{H}^m(\mathrm{X}, \mathscr{F}_q / \mathscr{F}_{q-1}) = 0$. Consequently, $\mathrm{H}^m(\mathrm{X}, \mathscr{F}_q) = 0$. This concludes the proof by induction.

For $q = p$, we have $\mathscr{F}_p = \mathscr{F}$, so that $\mathrm{H}^m(\mathrm{X}, \mathscr{F}) = 0$ for all integers m such that $m > n$. This concludes the proof of the implication (iii)⇒(ii). □

Lemma **(3.8.7)**. — *Let* X *be a noetherian topological space, let* U *be an open subset of* X *and let* $\mathscr{L}$ *be a subsheaf of* $\mathbf{Z}_\mathrm{U}$. *There exists a finite sequence* $(\mathscr{L}_0, \dots, \mathscr{L}_n)$ *of subsheaves of* $\mathscr{L}$ *such that* $0 = \mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \cdots \subseteq \mathscr{L}_n = \mathscr{L}$ *and, for every* $m \in \{1, \dots, n\}$, *a locally closed subset* A_m *of* U *such that* $\mathscr{L}_m / \mathscr{L}_{m-1} \simeq \mathbf{Z}_{\mathrm{A}_m}$.

Proof. — It suffices to treat the case U = X.

For every $x \in \mathrm{X}$, the fiber $\mathscr{L}_x$ is a subgroup of $\mathbf{Z}_x = \mathbf{Z}$, so that there exists a unique integer $n(x) \in \mathbf{N}$ such that $\mathscr{L}_x = n(x)\mathbf{Z}$.

Let $x \in \mathrm{X}$; there exists an open neighborhood U of x and a section $s \in \mathscr{L}(\mathrm{U})$ with fiber $n(x)$ at x. Then, for every $y \in \mathrm{U}$, one has $s_y = n(x) \in \mathscr{L}_y = n(y)\mathbf{Z}$, so that $n(y)$ divides $n(x)$. In particular, if $n(x) \neq 0$, this implies that $1 \leqslant n(y) \leqslant n(x)$.

Let U be the set of $x \in \mathrm{X}$ such that $n(x) \neq 0$; by what precedes, this is an open subset of X. For every $n \in \mathbf{N}$, let U_n be the set of $x \in \mathrm{U}$ such that $n(x) \leqslant n$; by what precedes, it is an open subset of U and the sequence $(\mathrm{U}_n)_n$ is increasing. For $m \geqslant 1$, set $\mathrm{A}_m = \mathrm{U}_m - \mathrm{U}_{m_1}$; this is a locally closed subset of X and one has an isomorphism $\mathscr{L}_{\mathrm{U}_m} / \mathscr{L}_{\mathrm{U}_{m_1}} \simeq \mathbf{Z}_{\mathrm{A}_m}$.

The union of the sets U_n is equal to U. Since X is noetherian, U is quasi-compact, so that there exists $n \in \mathbf{N}$ such that $\mathrm{U}_n = \mathrm{U}$. Then $\mathscr{L} = \mathscr{L}_\mathrm{U} = \mathscr{L}_{\mathrm{U}_n}$ and the sequence $(\mathscr{L}_{\mathrm{U}_0}, \dots, \mathscr{L}_{\mathrm{U}_n})$ satisfies the conditions of the lemma. □

Theorem **(3.8.8)**. — *Let* X *be a* noetherian *topological space. One has* $\operatorname{coh\,dim}(\mathrm{X}) \leqslant \dim(\mathrm{X})$.

Proof. — There is nothing to prove if $\dim(\mathrm{X}) = \infty$. We may thus assume that the dimension of X is finite and prove the result by induction on this dimension, assuming that it holds for all noetherian topological spaces of dimension $< \dim(\mathrm{X})$.

a) We first prove the theorem under the assumption that X is irreducible. Let us prove that $\mathrm{H}^n(\mathrm{X}, \mathbf{Z}_\mathrm{U}) = 0$ for all open subsets U of X and all integers n such that $n > \dim(\mathrm{X})$. If $\mathrm{U} = \varnothing$, then $\mathbf{Z}_\mathrm{U} = 0$ and the result is obvious. Otherwise, set $\mathrm{A} = \mathrm{X} - \mathrm{U}$; this is a closed subset of X, distinct from X. Since X is irreducible, any strictly increasing sequence of irreducible subsets of A can be extended by adding X, so that $\dim(\mathrm{A}) < \dim(\mathrm{X})$. We now consider the long exact sequence in cohomology associated with the canonical exact sequence

$$0 \to \mathbf{Z}_U \to \mathbf{Z} \to \mathbf{Z}_A \to 0;$$

this furnish in particular a short exact sequence

$$H^{m-1}(X, \mathbf{Z}_A) \to H^m(X, \mathbf{Z}_U) \to H^m(X, \mathbf{Z}).$$

Since A is closed, one has $H^{m-1}(X, \mathbf{Z}_A) = H^{m-1}(A, \mathbf{Z})$, which is zero by the induction hypothesis for A. Since X is irreducible, the constant sheaf $\mathbf{Z}$ is flasque (example 3.7.2) and one has $H^m(X, \mathbf{Z}) = 0$. Consequently, $H^m(X, \mathbf{Z}_U) = 0$.

b) Let us now prove the general case. Since X is noetherian, it has only finitely many irreducible components (proposition 1.10.9, say $X_1, \dots, X_m$. Let $\mathscr{F}$ be an abelian sheaf on X. The inclusions $X_p \to X$ induce, by adjunction, morphisms $\mathscr{F} \to \mathscr{F}_{X_p}$ whose fiber at a point x is an isomorphism if $x \in X_p$, and is zero otherwise. Consider the induced morphism $\varphi \colon \mathscr{F} \to \bigoplus_{p=1}^m \mathscr{F}_{X_p}$; by the preceding description, its fiber at a point x is injective, and it is an isomorphism if x belongs to exactly one irreducible component. Denoting its cokernel by $\mathscr{G}$, we obtain a short exact sequence

$$0 \to \mathscr{F} \to \bigoplus_{p=1}^{m} \mathscr{F}_{X_p} \to \mathscr{G} \to 0$$

is an exact sequence. Let $Y = \bigcup_{1 \leqslant p < q \leqslant m} (X_p \cap X_q)$; this is a closed subset of X and we have $\mathscr{G}_x = 0$ for $x \notin Y$, because φ_x is then an isomorphism, so that $\mathscr{G} = \mathscr{G}_Y$. Moreover, one has $\dim(Y) < \dim(X)$, because any strictly increasing sequence of irreducible closed subsets of Y is contained in some $X_p \cap X_p$, and can be extended by adding X_p.

Let now $n \in \mathbf{N}$ be such that $n > \dim(X)$. The long exact sequence of cohomology associated with the short exact sequence $0 \to \mathscr{F} \to \bigoplus \mathscr{F}_{X_p} \to \mathscr{G} \to 0$ furnishes a short exact sequence

$$H^{n-1}(X, \mathscr{G}) \to H^n(X, \mathscr{F}) \to \bigoplus_{p=1}^{m} H^n(X, \mathscr{F}_{X_p}).$$

Since Y is closed, we then have $H^n(X, \mathscr{G}) = H^n(X, \mathscr{G}_Y) = H^n(Y, i^{-1}\mathscr{G})$, where $i \colon Y \to X$ is the inclusion map. Since $n > \dim(X)$ and $\dim(X) > \dim(Y)$, one has $n - 1 > \dim(Y)$, so that the induction hypothesis implies $H^{n-1}(Y, i^{-1}\mathscr{G}) = 0$. This proves that $H^{n-1}(X, \mathscr{G}) = 0$.

Let $p \in \{1, \dots, m\}$. Since X_p is closed, one has $H^n(X, \mathscr{F}_{X_p}) = H^n(X_p, i_p^{-1}\mathscr{F})$, where $i_p \colon X_p \to X$ is the inclusion map. By hypothesis, one has $n > \dim(X) \geqslant$

$\dim(X_p)$, so that step *a*) implies $H^n(X_p, i_p^{-1}\mathscr{F}) = 0$, so that $H^n(X, \mathscr{F}_{X_p}) = 0$. Consequently, $H^n(X, \bigoplus_p \mathscr{F}_{X_p}) = 0$.

This implies that $H^n(X, \mathscr{F}) = 0$, as was to be shown. □

CHAPTER 4

SCHEMES

4.1. Sheaves associated to modules on spectra of rings

4.1.1. — Let A be a ring and let X = Spec(A) be its spectrum. Recall that it is the set of prime ideals of A, endowed with the spectral (or Zariski) topology whose closed subsets are those of the form

$$V(E) = \{\mathfrak{p} \in \operatorname{Spec}(A)\,;\, E \subseteq \mathfrak{p}\},$$

for some subset E of A. For every subset Z of Spec(A), we also defined

$$\mathfrak{j}(Z) = \bigcap_{\mathfrak{p}\in Z} \mathfrak{p} = \{a \in A\,;\, a \in \mathfrak{p},\ \forall \mathfrak{p} \in Z\},$$

and that the operations V and $\mathfrak{j}$ define bijections, inverse one from the other, from the set of radical ideals of A to the set of closed subsets of Spec(A).

The *algebraic geometry of schemes* considers these topological spaces Spec(A) as its building blocks. In some sense, the prime spectrum of a ring is seen as a more fundamental object than the ring itself. This suggests an adjustment of the notation.

As in any topological space, elements of X are called *points*; a point of X is thus denoted by a letter, such as x, and the corresponding prime ideal of A will be denoted $\mathfrak{p}_x$. With this notation, one thus has

$$\mathfrak{j}(Z) = \bigcap_{x\in Z} \mathfrak{p}_x.$$

Then, the quotient ring $A/\mathfrak{p}_x$ is an integral domain, and its field of fractions will be denoted $\kappa(x)$; it is called the *residue field* of X at x. One has morphisms of rings:

$$A \to A/\mathfrak{p}_x \to \kappa(x).$$

For $f \in A$ and $x \in \operatorname{Spec}(A)$, one writes $f(x)$ for the image of f in the residue field $\kappa(x)$; with this notation, the condition $f \in \mathfrak{p}_x$ is then equivalent to the condition $f(x) = 0$. For $E \subseteq A$ and $Z \subseteq \operatorname{Spec}(A)$, one thus has

$$V(E) = \{x \in X\,;\, f(x) = 0\ \forall f \in E\} \quad \text{and} \quad \mathfrak{j}(Z) = \{f \in A\,;\, f(x) = 0\ \forall x \in Z\}.$$

For $f \in A$, one also has

$$V(f) = \{x \in X\,;\, f(x) = 0\} \quad \text{and} \quad D(f) = \{x \in X\,;\, f(x) \neq 0\}.$$

The subsets $D(f)$, for $f \in A$, form a basis of open subsets of $\operatorname{Spec}(A)$. For $f, g \in A$, the conditions (i) $g \in \sqrt{(f)}$, (ii) $V(g) \supseteq V(f)$, and (iii) $D(g) \subseteq D(f)$, are equivalent.

4.1.2. — Let A be a ring and let M be an A-module. Let us define a presheaf of A-modules $\widetilde{M}_{\text{pre}}$ on X.

Let U be an open subset of $\operatorname{Spec}(A)$ and let $S(U)$ be the set of all $f \in A$ such that $f(x) \neq 0$ for every $x \in U$. The set $S(U)$ is a multiplicative subset of A. It contains 1. Moreover, if $f, g \in S(U)$ and $x \in U$, then $(fg)(x) = f(x)g(x)$ in the residue field $\kappa(x)$, hence $(fg)(x) \neq 0$. Let $j_U \colon M \to S(U)^{-1}M$ be the canonical morphism of A-modules, given by $m \mapsto m/1$.

Let U and V be open subsets of $\operatorname{Spec}(A)$ such that $V \subseteq U$. By definition, one has $S(U) \subseteq S(V)$. Let $\rho^M_{UV} \colon S(U)^{-1}M \to S(V)^{-1}M$ be the unique morphism of A-modules such that $j_V = \rho^M_{UV} \circ j_U$.

Consequently, the modules $\widetilde{M}_{\text{pre}}(U) = S(U)^{-1}M$ and the morphisms ρ^M_{UV} define define a *presheaf* of A-modules on X.

Let $u \colon M \to N$ be a morphism of A-modules. The morphisms $S(U)^{-1}M \to S(U)^{-1}N$ deduced from u form a morphism of presheaves $u_*^{\text{pre}} \colon \widetilde{M}_{\text{pre}} \to \widetilde{N}_{\text{pre}}$. One has $(\mathrm{Id}_M)_*^{\text{pre}} = \mathrm{Id}$ and $(v \circ u)_*^{\text{pre}} = v_*^{\text{pre}} \circ u_*^{\text{pre}}$.

4.1.3. — If B is an A-algebra, then $\widetilde{B}_{\text{pre}}$ is even a presheaf of A-algebras. Indeed, the A-modules of fractions $S(U)^{-1}B$ are A-algebras, and the morphisms ρ^B_{UV} are morphisms of A-algebras.

If $u \colon B \to C$ is a morphism of A-algebras, then the associated morphism $u_*^{\text{pre}} \colon \widetilde{B}_{\text{pre}} \to \widetilde{C}_{\text{pre}}$ of presheaves of A-modules is a morphism of presheaves of A-algebras.

Remark **(4.1.4)**. — Let A be a ring and let M be an A-module. Let $f \in A$ and let $U = D(f)$. By assumption, an element g belongs to $S(U)$ if and only if

$V(g) \subseteq V(f)$, that is if and only if $f \in \sqrt{(g)}$. In particular, the multiplicative subset $S_f = \{1, f, f^2, \dots\}$ is contained in $S(U)$. Let us observe that the canonical morphism φ from $S_f^{-1}M$ to $S(U)^{-1}M$ is an isomorphism.

Let $m \in M$ and $n \geqslant 0$ be such that $\varphi(m/f^n) = 0$ in $S(U)^{-1}M$. Then there exists $g \in S(U)$ such that $gm = 0$. Since $f \in \sqrt{(g)}$, there exists $p \geqslant 0$ and $h \in A$ such that $f^p = gh$; then $f^p m = 0$, hence $m/f^n = 0$ in $S_f^{-1}M$.

Conversely, let $m \in M$ and let $g \in S(U)$. By the same argument, there exists $p \geqslant 0$ and $h \in A$ such that $f^p = gh$. One has $f, g, h \in S(U)$ and $m/g = mh/gh = mh/f^p$ in $S(U)^{-1}M$. Consequently, $m/g = \varphi(mh/f^p)$ belongs to the image of φ.

Definition **(4.1.5)**. — *Let* A *be a ring and let* M *be an* A*-module. One defines the sheaf* $\widetilde{M}$ *to be the sheaf of* A*-modules associated with this presheaf* $\widetilde{M}_{\mathrm{pre}}$.

If $u\colon M \to N$ is a morphism of A-modules, the morphism of sheaves $\widetilde{M} \to \widetilde{N}$ associated with the morphism u_*^{pre} of presheaves is denoted u_*, or $\tilde{u}$.

If B is an A-algebra, then the sheaf $\widetilde{B}$ is a sheaf of A-algebras. If $u\colon B \to C$ is a morphism of A-algebras, then the associated morphism u_* is a morphism of sheaves of A-algebras.

If B is an A-algebra and M is a B-module, then $\widetilde{M}$ is a $\widetilde{B}$-module.

Lemma **(4.1.6)**. — *Let* $x \in X$ *and let* S_x *be the multiplicative subset* $A - \mathfrak{p}_x$ *of* A*. Let* M *be an* A*-module. The canonical morphism from* M *to* $\widetilde{M}_x$ *induces an isomorphism of* $A_{\mathfrak{p}_x}$*-modules from the stalk* $\widetilde{M}_x$ *of the sheaf* $\widetilde{M}$ *with the module of fractions* $M_{\mathfrak{p}_x} = S_x^{-1}M$ *deduced from* M *and the multiplicative subset* S_x*. If* M *is an* A*-algebra, then this isomorphism is an isomorphism of* $A_{\mathfrak{p}_x}$*-algebras.*

Proof. — Since the canonical morphism from $\widetilde{M}_{\mathrm{pre},x}$ to $\widetilde{M}_x$ is an isomorphism, it suffices to prove that the canonical morphism from M to $\widetilde{M}_{\mathrm{pre},x}$ is itself an isomorphism. By definition, $\widetilde{M}_{\mathrm{pre},x}$ is the colimit $\varinjlim S(U)^{-1}M$, where U ranges over all open subsets of X which contain x. For every such U, one has $S(U) \subseteq A - \mathfrak{p}_x$; let $\varphi\colon \widetilde{M}_{\mathrm{pre},x} \to M_{\mathfrak{p}_x}$ be the canonical morphism. It is surjective: for $f \in A - \mathfrak{p}_x$ and $m \in M$, the element m/f of $M_{\mathfrak{p}_x}$ is the image by φ of the class of the element m/f of $S(D(f))^{-1}M$. It is also injective: if, for an open neighborhood U of x, $f \in S(U)$, and $m \in M$, one has $\varphi([m/f]) = 0$, there exists $g \in A - \mathfrak{p}_x$ such that $gm = 0$; this implies that $m/f = 0$ in $S(D(g))^{-1}M$, hence $[m/f] = 0$ in $\widetilde{M}_{\mathrm{pre},x}$. □

Remark **(4.1.7)**. — Let A be a ring, let $X = \mathrm{Spec}(A)$ be its spectrum; let $f \in A$; let M be an A-module. Recall (proposition 1.5.10) that the canonical morphism of rings $A \to A_f$ induces a homeomorphism from $\mathrm{Spec}(A_f)$ to the open subset $D(f)$ of $\mathrm{Spec}(A)$. Under this homeomorphism, the sheaf $\widetilde{M}_f$ on $\mathrm{Spec}(A_f)$ identifies with the restriction $\widetilde{M}|_{D(f)}$ to $D(f)$ of the sheaf $\widetilde{M}$ on X.

Indeed, for every $g \in A$, one has $D(fg) \subseteq D(f)$, $\widetilde{M}_{\mathrm{pre}}(D(fg)) = M_{fg}$, while $\widetilde{M_f}_{\mathrm{pre}}(D(g)) = (M_f)_g$, so that both presheaves $\widetilde{M}_{\mathrm{pre}}|_{D(f)}$ and $\widetilde{M_f}_{\mathrm{pre}}$ on $D(f)$ are canonically identified.

Theorem **(4.1.8)**. — *Let* A *be a ring, let* $X = \mathrm{Spec}(A)$ *be its spectrum; let* M *be an* A*-module and let* $\widetilde{M}$ *be the associated sheaf of* $\mathscr{O}_X$*-modules. For every open subset* U *of* X, *let* $\theta_U \colon S(U)^{-1}M \to \widetilde{M}(U)$ *be the canonical morphism.*

For every $f \in A$, *the morphism* $\theta_{D(f)}$ *is an isomorphism. In particular, the canonical morphism from* M *to* $\widetilde{M}(X)$ *is an isomorphism.*

Proof. — Let $f \in A$ and let $U = D(f)$.

We first show that θ_U is injective. Let $m \in M$ and let $g \in S(U)$ be such that $\theta_U(m/g) = 0$. In particular, for every $x \in U$, its germ $\theta_U(m/g)_x$ at x vanishes, hence $m/g = 0$ in $M_{\mathfrak{p}_x}$. Let I be the set of elements $a \in A$ such that $am = 0$; it is an ideal of A. By assumption, for every $x \in U$, there exists $a \in A - \mathfrak{p}_x$ such that $am = 0$, that is, $V(I) \cap U = \varnothing$. In other words, one has $V(I) \subseteq V(f)$, hence $f \in \sqrt{I}$. Consequently, there exists an integer $n \geqslant 0$ such that $f^n \in I$. One has $f^n m = 0$, hence $m/g = 0$ in M_f, and $m/g = 0$ in $S(U)^{-1}M$ since $f \in S(U)$.

Let us now show that θ_X is surjective. Let $\mu \in \widetilde{M}(X)$ and let us show that there exists $m \in M$ such that $\mu = \theta_X(m)$. Let $x \in X$; by the construction of the sheaf associated to a presheaf, there exists an open neighborhood U_x of x, elements $f_x \in S(U_x)$ and $m_x \in M$ such that $\mu|_{U_x} = \theta_{U_x}(m_x/f_x)$.

Since the open sets of the form $D(h)$ form a basis of open subsets of X, there exists $h_x \in A$ such that $D(h_x) \subseteq U_x \cap D(f_x)$ and $x \in D(h_x)$. Then $h_x \notin \mathfrak{p}_x$ and one has $\mu|_{D(h_x)} = \theta_{D(h_x)}(m_x/f_x)$. Moreover, since $D(h_x) \subseteq D(f_x)$, there exists $g_x \in A$ such that $f_x g_x = h_x^{n_x}$. Then $m_x/f_x = g_x m_x / h_x^{n_x}$. We may then replace f_x and h_x by $h_x^{n_x}$, and replace m_x by $g_x m_x$; this simplifies the notation in so that $U_x = D(f_x)$ and $\mu|_{U_x} = \theta_{U_x}(m_x/f_x)$.

Let $x, y \in X$. One has $\mu|_{U_x \cap U_y} = \theta_{U_x \cap U_y}(m_x/f_x) = \theta_{U_x \cap U_y}(m_y/f_y)$. Consequently, the morphism $\theta_{U_x \cap U_y}$ maps the element $m_x/f_x - m_y/f_y$ of $S(U_x \cap U_y)^{-1}M$ to 0. Since $U_x \cap U_y = D(f_x f_y)$, the injectivity part implies that $m_x/f_x = m_y/f_y$

in $S(U_x \cap U_y)^{-1}M$; by the remark 4.1.4, one even has $m_x/f_x = m_y/f_y$ in the module $M_{f_x f_y}$. By the definition of modules of fractions, this means that there exists an integer $n_{xy} \geqslant 0$ such that $(f_x f_y)^{n_{xy}} f_y m_x = (f_x f_y)^{n_{xy}} f_x m_y$.

Since $x \in D(f_x)$, the open sets $D(f_x)$ cover Spec(A), hence the intersection of the sets $V(f_x)$ is empty. This implies that the family $(f_x)_{x\in X}$ generates the unit ideal of A; as a consequence, there exist a finite subset Σ of X which generates the unit ideal. Let $n = \sup_{x,y\in\Sigma}(n_{xy})$; for every $x, y \in \Sigma$, one has $f_x^{n+1} f_y^n m_y = f_x^n f_y^{n+1} m_x$. Since the family $(f_x^{n+1})_{x\in\Sigma}$ generates the unit ideal, there exists a family $(h_x)_{x\in\Sigma}$ such that $\sum_{x\in\Sigma} f_x^{n+1} h_x = 1$. Let then

$$m = \sum_{x\in\Sigma} h_x f_x^n m_x.$$

For every $x \in \Sigma$, one has

$$f_x^{n+1} m = \sum_{y\in\Sigma} f_x^{n+1} f_y^n h_y m_y = \sum_{y\in\Sigma} f_x^n f_y^{n+1} h_y m_x = f_x^n m_x \sum_{y\in\Sigma} f_y^{n+1} h_y = f_x^n m_x.$$

Consequently, $m/1 = m_x/f_x$ in M_{f_x} and $\theta_{U_x}(m/1) = \theta_{U_x}(m_x/f_x) = \theta_{U_x}(\mu)$ in $\widetilde{M}(U_x)$. Since the open sets $(U_x)_{x\in\Sigma}$ cover X and $\widetilde{M}$ is a sheaf, this shows that $\mu = \theta_X(m)$ and concludes the proof that the map θ_X is surjective.

It remains to show that the map $\theta_{D(f)}$ is surjective for every element $f \in A$. Given remark 4.1.7, this can be deduced from the preceding part by replacing A by the ring of fractions A_f and M with the module of fractions M_f. One can also redo explicitly the proof. In both cases, details are left to the reader. □

Corollary **(4.1.9)**. — *Let* A *be a ring and let* X = Spec(A). *Let* M *be an* A*-module and let* $\mathscr{N}$ *be a* $\widetilde{A}$*-module. For every morphism* $\varphi\colon M \to \mathscr{N}(X)$ *of* A*-modules, there exists a unique morphism* $\tilde\varphi\colon \widetilde{M} \to \mathscr{N}$ *of* $\widetilde{A}$*-modules such that* $\tilde\varphi(X) = \varphi$.

This corollary has two important consequences.

Firstly, it can be reformulated as saying that the pair of functors ($M \mapsto \widetilde{M}, \mathscr{N} \mapsto \mathscr{N}(X)$) from the category $\boldsymbol{Mod}_A$ of A-modules to the category $\boldsymbol{Mod}_{\widetilde{A}}$ of $\widetilde{A}$-modules on X is adjoint. In particular, the functor $M \mapsto \widetilde{M}$ respects all colimits, and the functor $\mathscr{N} \mapsto \mathscr{N}(X)$ respects all limits (exercise 2.4.9).

Secondly, implied to $\widetilde{A}$-modules of the form $\mathscr{N} = \widetilde{N}$, it implies that the functor given by $\mathscr{F} \mapsto \mathscr{F}(\mathrm{Spec}(A))$ from the full subcategory of the category of $\widetilde{A}$-modules on X whose objects are of the form $\widetilde{M}$, to the category of A-modules is an equivalence of categories. Indeed, the functor $M \mapsto \widetilde{M}$ is a quasi-inverse.

Proof. — Let $\varphi\colon \mathrm{M} \to \mathscr{N}(\mathrm{X})$ be a morphism of A-modules. For every open subset U of X, let $\widetilde{\varphi}_{\mathrm{pre}}(\mathrm{U})\colon \mathrm{S}(\mathrm{U})^{-1}\mathrm{M} \to \mathscr{N}(\mathrm{U})$ be the morphism of A-modules given by $\widetilde{\varphi}_{\mathrm{pre}}(m/s) = (1/s)\varphi(m)$, where, for $s \in \mathrm{S}(\mathrm{U})$, $1/s$ is considered as an element of $\widetilde{\mathrm{A}}(\mathrm{U})$. The family $(\widetilde{\varphi}_{\mathrm{pre}}(\mathrm{U}))$ is a morphism of presheaves on X. Let $j\colon \widetilde{\mathrm{M}}_{\mathrm{pre}} \to \widetilde{\mathrm{M}}$ be the canonical morphism from the presheaf $\widetilde{\mathrm{M}}_{\mathrm{pre}}$ to the associated sheaf. There exists a unique morphism of sheaves $\widetilde{\varphi}\colon \widetilde{\mathrm{M}} \to \mathscr{N}$ such that $\widetilde{\varphi}(\mathrm{U})(j(\mathrm{U})(m/s)) = \widetilde{\varphi}_{\mathrm{pre}}(\mathrm{U})(m/s) = (1/s)\varphi(m)$ for every $m \in \mathrm{M}$, every open subset U of X and every $s \in \mathrm{S}(\mathrm{U})$. This is a morphism of $\widetilde{\mathrm{A}}$-modules, and one has $\widetilde{\varphi}(\mathrm{X}) = \varphi$. Conversely, let $\psi\colon \widetilde{\mathrm{M}} \to \mathscr{N}$ be any morphism of $\widetilde{\mathrm{A}}$-modules such that $\psi(\mathrm{X}) = \varphi$. For every open subset U of X, every $m \in \mathrm{M}$ and every $s \in \mathrm{S}(\mathrm{U})$, one necessarily has

$$\begin{aligned}\psi(\mathrm{U})(j(\mathrm{U})(m/s)) &= (1/s)\cdot\psi(\mathrm{U})\circ j(\mathrm{U})(m/1)\\ &= (1/s)\cdot\varphi(\mathrm{X})(m)|_{\mathrm{U}} = \widetilde{\varphi}(\mathrm{U})(j(\mathrm{U})(m/s)),\end{aligned}$$

hence $\psi\circ j = \widetilde{\varphi}\circ j$. Consequently, $\psi = \widetilde{\varphi}$, as claimed. □

Corollary **(4.1.10)**. — *Let* A *be a ring. The assignment* $\mathrm{M} \mapsto \widetilde{\mathrm{M}}$ *and* $\varphi \mapsto \widetilde{\varphi}$ *is a functor from the category of* A*-module to the category of* $\widetilde{\mathrm{A}}$*-modules. This functor commutes with all colimits, with all finite limits, and is fully faithful.*

Proof. — We have already noted that this functor is fully faithful. Since it has a right adjoint, it commutes with every colimit, finite or not (see exercise 2.4.9). Let us now show that it commutes with every finite limit.

Let $\mathrm{Q} = (\mathrm{V}, \mathrm{E})$ be a *finite* quiver and let $\mathscr{M} = (\mathrm{M}_v)$ be a Q-diagram of A-modules, let $(\mathrm{M}, (\varphi_v))$ be its limit. Let $(\mathscr{N}, (\psi_v))$ be a cone on the diagram $\widetilde{\mathscr{M}}$ of $\widetilde{\mathrm{A}}$-modules which is associated with $\mathscr{M}$.

By definition, for every v, $\psi_v\colon \widetilde{\mathrm{N}} \to \widetilde{\mathrm{M}}_v$ is a morphism of $\widetilde{\mathrm{A}}$-modules such that $\psi_{t(e)} \circ \widetilde{\psi}_e = \varphi_{s(e)}$ for every $e \in \mathrm{E}$. Then $(\mathscr{N}(\mathrm{X}), (\psi_v(\mathrm{X})))$ is a cone on the diagram $\mathscr{M}$ of A-modules, hence there exists a unique morphism of A-modules $\theta\colon \mathscr{N}(\mathrm{X}) \to \mathrm{M}$ such that $\psi_v(\mathrm{X}) = \varphi_v \circ \theta$ for every $v \in \mathrm{V}$.

Let $a \in \mathrm{A}$ and let S_a be the multiplicative subset $\mathrm{S}_a = \{1, a, a^2, \dots\}$ of A. Since the functor $\mathrm{M} \mapsto \mathrm{S}_a^{-1}\mathrm{M}$ commutes with finite limits (it is exact, see example 2.3.15), the cone $(\mathrm{S}_a^{-1}\mathrm{M}, (\mathrm{S}_a^{-1}\varphi_v))$ is a limit of the diagram $(\mathrm{S}_a^{-1}\mathrm{M}_v)$. Since the canonical morphism from $\mathrm{S}_a^{-1}\mathrm{M}_v$ to $\widetilde{\mathrm{M}}(\mathrm{D}(a))$ is an isomorphism, the cone $(\mathrm{S}_a^{-1}\mathrm{M}, (\mathrm{S}_a^{-1}\varphi_v))$ is a limit of the diagram $(\widetilde{\mathrm{M}}(\mathrm{D}(a))$. Since $(\mathscr{N}(\mathrm{D}(a)), (\psi_v(\mathrm{D}(a))))$ is also a cone on this diagram, there exists a unique morphism of $S_a^{-1}A$-modules $\theta_a\colon \mathscr{N}(D(a)) \to S_a^{-1}M$ such that $\psi_v(D(a)) = \varphi_v(D(a)) \circ \theta_a$ for every $v \in V$.

Let now U be an open subset of X. There exists a unique morphism of $\mathscr{O}_X(U)$-modules $\theta(U)\colon \mathscr{N}(U) \to \widetilde{M}(U)$ such that $\theta(U)(s)|_{D(a)} = \theta_a(s|_{D(a)})$ for every $a \in A$ such that $D(a) \subseteq U$. Moreover, the family $(\theta(U))$ is a morphism of $\mathscr{O}_X$-modules from $\mathscr{N}$ to $\widetilde{M}$ such that $\psi_v = \varphi_v \circ \theta$, and it is the unique such morphism. □

Example (**4.1.11**). — Here are two particularly important examples:

a) Let $\varphi\colon M \to N$ be a morphism of A-modules, and let $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$ be the associated morphism between the corresponding $\widetilde{A}$-modules on Spec(A). Then the $\widetilde{A}$-modules associated with $\operatorname{Ker}(\varphi)$ and $\operatorname{Coker}(\varphi)$ are respectively a kernel and a cokernel of $\widetilde{\varphi}$.

b) Let (M_i) be a family of A-modules, and let $M = \bigoplus_{i \in I} M_i$ be its direct sum (coproduct). Then $\widetilde{M}$ is a direct sum of the family $(\widetilde{M}_i)$ of $\widetilde{A}$-modules.

4.2. Locally ringed spaces

Definition (**4.2.1**). — *A* ringed space *is a topological space* X *endowed with a sheaf of rings* $\mathscr{O}_X$*, which is called its structure sheaf.*

When we talk of a ringed space, we often omit the sheaf of rings from the notation.

Definition (**4.2.2**). — *Let* $(X, \mathscr{O}_X)$ *and* $(Y, \mathscr{O}_Y)$ *be ringed spaces. A* morphism of ringed spaces *from* X *to* Y *is a pair* $(\varphi, \varphi^\sharp)$ *consisting of a continuous map* $\varphi\colon X \to Y$ *and morphism of sheaves of rings* $\varphi^\sharp\colon \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$.

Concretely, given a continuous map φ of topological space, the morphism $\varphi^\sharp$ amounts to the datum, for every open subset U of Y, of a morphism of rings $\varphi^\sharp(U)\colon \mathscr{O}_Y(U) \to \mathscr{O}_X(\varphi^{-1}(U))$, subject to the following compatibility with restrictions: if U and V are open subsets of Y such that $V \subseteq U$, then $\varphi^\sharp(V)(s|_V) = \varphi^\sharp(U)(s)|_V$ for every $s \in \mathscr{O}_Y(U)$.

Instead of $\varphi^\sharp\colon \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$, it is equivalent to give oneself the morphism $\varphi^\flat\colon \varphi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ deduced by the adjunction property of the pair of functors $(\varphi^{-1}, \varphi_*)$ (theorem 3.3.11).

4.2.3. — Let $(\varphi, \varphi^\sharp)$ be a morphism of ringed spaces from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$. Let $x \in X$ and let $y = \varphi(x)$. There is a unique morphism of rings $\varphi^\sharp_x \colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ such that $\varphi^\sharp_x(f_y) = \varphi^\sharp(U)(f)_x$ for every open neighborhood U of y and every section $f \in \mathscr{O}_Y(U)$.

Definition **(4.2.4)**. — *A* locally ringed space *is a ringed space such that the stalks of its structure sheaf are local rings.*

A morphism from a locally ringed space $(X, \mathscr{O}_X)$ *to a locally ringed space* $(Y, \mathscr{O}_Y)$ *is a morphism* $(\varphi, \varphi^\sharp)$ *of ringed spaces such that for every* $x \in X$*, the associated morphism* $\varphi^\sharp_x \colon \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$ *is a local morphism of local rings.*

Recall from §1.1.7 that a morphism of local rings is said to be local if the image of every non-invertible element is not invertible.

We keep the notation of the previous definition. Let $(X, \mathscr{O}_X)$ be a locally ringed space. For every point $x \in X$, the *residue field* of the local ring $\mathscr{O}_{X,x}$ is usually denoted by $\kappa(x)$. The image in $\kappa(x)$ of a germ $f \in \mathscr{O}_{X,x}$ is denoted by $f(x)$; for every open neighborhood U of x and every section $f \in \mathscr{O}_X(U)$, the image of the germ f_x in $\kappa(x)$ is denoted by $f(x)$.

Let $(\varphi, \varphi^\sharp)$ be a morphism of locally ringed spaces. Let $x \in X$. Since the morphism $\varphi^\sharp_x$ is local, it induces, by passing to the residue fields, a morphism of fields from $\kappa(\varphi(x))$ to $\kappa(x)$. If U is an open neighborhood of $\varphi(x)$ and $f \in \mathscr{O}_Y(U)$, then the element $\varphi^\sharp(U)(f)(x)$ of $\kappa(x)$ is the image of the element $f(\varphi(x))$ of $\kappa(\varphi(x))$.

4.2.5. — Let $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$ be morphisms of locally ringed spaces. Their composition $\psi \circ \varphi$ is defined as follows: the underlying continuous map is the usual composition, and the morphism of sheaves $(\psi \circ \varphi)^\sharp \colon \mathscr{O}_Z \to (\psi \circ \varphi)_* \mathscr{O}_X$ is given by $\psi_*(\varphi^\sharp) \circ \psi^\sharp$. For every $x \in X$, the morphism

$$(\psi \circ \varphi)^\sharp_x \colon \mathscr{O}_{X,x} \to \mathscr{O}_{Z,\psi(\varphi(x))}$$

is the composition of $\varphi_x \colon \mathscr{O}_{X,x} \to \mathscr{O}_{Y,\varphi(x)}$ and of $\psi_{\varphi(x)} \colon \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{Z,\psi(\varphi(x))}$; it is thus a morphism of local rings.

Locally ringed spaces form a category.

Example **(4.2.6)**. — a) Let X be an open subset of $\mathbf{R}^n$ or, more generally, a $\mathscr{C}^\infty$-manifold. Let $\mathscr{C}^\infty_X$ be the sheaf of $\mathscr{C}^\infty$-functions on X. For every point $x \in X$, the ring $\mathscr{C}^\infty_{X,x}$ is the ring of germs of $\mathscr{C}^\infty$-functions in a neighborhood of x; this is a local ring whose maximal ideal $\mathfrak{m}_x$ is the ideal of germs of functions which vanish at x. In particular, the residue field $\kappa(x)$ is equal to $\mathbf{R}$, and for every open neighborhood U of x, the "value" $\varphi(x) \in \kappa(x)$ of a section $\varphi \in \mathscr{C}^\infty_X(U)$ is the actual value of φ at x.

Let X and Y be $\mathscr{C}^\infty$-manifolds. The definition of a morphism $f: X \to Y$ says that f is a continuous map such that for every open subset V of Y and every $\mathscr{C}^\infty$-function φ on V, the composition $\varphi \circ f$ is $\mathscr{C}^\infty$ on $f^{-1}(V)$. Then the assignment $\varphi \mapsto \varphi \circ f$ induces a morphism of sheaves $f^\sharp: \mathscr{C}^\infty_Y \to f_*\mathscr{C}^\infty_X$, so that the pair $(f, f^\sharp)$ is a morphism of locally ringed spaces.

Conversely, let $(f, f^\sharp): (X, \mathscr{C}^\infty_X) \to (Y, \mathscr{C}^\infty_Y)$ be a morphism of locally ringed spaces. This first implies that f is continuous. Moreover, we have explained that for every open subset V of Y and every function $f \in \mathscr{C}^\infty_Y(V)$, one has $f^\sharp(V)(\varphi)(x) = \varphi(f(x))$. Consequently, the morphism of sheaves $f^\sharp$ is given by composition of functions.

In conclusion, morphisms of $\mathscr{C}^\infty$-manifolds coincide with the morphisms of the associated locally ringed spaces.

b) Let $(X, \mathscr{O}_X)$ be a locally ringed space and let U be an open subset of X. The pair $(U, \mathscr{O}_X|_U)$ is a locally ringed space.

Let $j: U \to X$ be the inclusion. For every open subset V of X, one has $(j_*(\mathscr{O}_X|_U))(V) = \mathscr{O}_X(U \cap V)$; let $j^\sharp(V)$ be the restriction morphism. This defines a morphism of sheaves $j^\sharp: \mathscr{O}_X \to j_*\mathscr{O}_X|_U$. For every $x \in U$, the morphism $j^\sharp_x: \mathscr{O}_{X,x} \to (\mathscr{O}_X|_U)_x$ induced by $j^\sharp$ is an isomorphism. Consequently, $(j, j^\sharp)$ is a morphism of locally ringed spaces.

Let moreover $f: Y \to X$ be a morphism of locally ringed spaces. If $f(Y) \subseteq U$, there exists a unique morphism of locally ringed spaces $g: Y \to U$ such that $f = j \circ g$.

c) Let A be a ring. Endowed with the sheaf of rings $\widetilde{A}$, the topological space $\operatorname{Spec}(A)$ is a locally ringed space. (Such locally ringed spaces are the fundamental bricks of algebraic geometry, and are called *affine schemes*.) Recall indeed from lemma 4.1.6 that the stalk of the sheaf $\widetilde{A}$ at a point $x \in \operatorname{Spec}(A)$ identifies with the local ring $A_{\mathfrak{p}_x}$.

d) Let A be a ring. For every $f \in A$, the canonical homeomorphism of $D(f)$ to $\operatorname{Spec}(A_f)$ identifies the restriction to $D(f)$ of the structure sheaf $\widetilde{A}$ with the structure sheaf $\widetilde{A}_f$ of $\operatorname{Spec}(A_f)$. As a consequence, $D(f)$ is an affine scheme.

Lemma (**4.2.7**). — *Let* X *be locally ringed space, let* $\mathscr{O}_X$ *be its structure sheaf. Let* U *be an open subset of* X, *let* $f \in \mathscr{O}_X(U)$ *and let* $D(f) = \{x \in U\,;\, f(x) \neq 0\}$. *Then* $D(f)$ *is the largest open subset of* U *the restriction to which* f *is invertible.*

Proof. — Let V be an open subset of U such that $f|_V$ is invertible and let $g \in \mathscr{O}_X(V)$ be such that $f|_V g = 1$. Then, for every $x \in V$, one has $f(x)g(x) = 1$, hence $x \in D(f)$; consequently, $V \subseteq D(f)$.

Let $x \in D(f)$. Since $f(x) \neq 0$, the germ f_x of f at x is invertible, because it does not belong to the maximal ideal of the local ring $\mathscr{O}_{X,x}$. Consequently, f_x is invertible, hence there exists an open neighborhood V of x contained in U and an element $g \in \mathscr{O}_X(V)$ such $f_x g_x = 1$. This implies that there exists an open neighborhood W of x contained in V such that $f|_W g|_W = 1$: this shows that $f|_W$ is invertible. In particular, $W \subseteq D(f)$, so that $D(f)$ is open in U.

For every $x \in D(f)$, let W_x be an open neighborhood of x contained in $D(f)$ and let $g_x \in \mathscr{O}_X(W_x)$ be an inverse of $f|_{W_x}$. For every pair (x, y) of elements of $D(f)$, the restrictions of g_x and g_y to $W_x \cap W_y$ are both equal to the inverse of $f|_{W_x \cap W_y}$. By the sheaf condition, there exists a unique element $g \in \mathscr{O}_X(D(f))$ such that $g|_{W_x} = g_x$ for every $x \in D(f)$. One then has $(fg)|_{W_x} = f|_{W_x} g_x = 1$ for every x, hence $f|_{D(f)} g = 1$ since the union of the open subsets W_x is equal to $D(f)$. □

Theorem (**4.2.8**). — *Let* $(X, \mathscr{O}_X)$ *be a locally ringed space; let* A *be a ring. For every morphism of rings* $u\colon A \to \mathscr{O}_X(X)$, *there exists a unique morphism* $\varphi = (\varphi, \varphi^\sharp)$ *of locally ringed spaces from* X *to* Spec(A) *such that* $u = \varphi^\sharp(\mathrm{Spec}(A))$.

Proof. — We first establish the uniqueness of such a morphism $(\varphi, \varphi^\sharp)$ by analysing properties which follow from the condition $\varphi^\sharp(\mathrm{Spec}(A)) = u$.

For every point $x \in X$, let $\mathfrak{p}_x$ be the kernel of the canonical morphism $f \mapsto f(x)$ from $\mathscr{O}_X(X)$ to $\kappa(x)$; it is a prime ideal of $\mathscr{O}_X(X)$, because $\kappa(x)$ is a field, hence an integral domain. For $f \in A$, one has $f(\varphi(x)) = \varphi^\sharp(\mathrm{Spec}(A))(f)(x) = u(f)(x)$, so that the conditions $f \in \mathfrak{p}_{\varphi(x)}$ and $u(f) \in \mathfrak{p}_x$ are equivalent. In other words, one has the equality $\mathfrak{p}_{\varphi(x)} = u^{-1}(\mathfrak{p}_x)$. This shows that the point $\varphi(x)$ of Spec(Spec(A)) is the prime ideal $u^{-1}(\mathfrak{p}_x)$ of A. This also shows that $\varphi^{-1}(D(f)) = D(u(f))$. Since $u(f)$ is invertible on $D(u(f))$, there exists a unique morphism of rings from $u_f\colon A_f \to \mathscr{O}_X(D(u(f)))$ such that $u_f(a/1) = u(a)|_{D(u(f))}$ for every $a \in A$. Since $\mathscr{O}_{\mathrm{Spec}(A)}(D(f)) = A_f$, this also implies the equality $\varphi^\sharp(D(f)) = u_f$. Since the open subsets of Spec(A) of the form $D(f)$ constitute a basis of open subsets, we conclude from this analysis that there exists at most one morphism $(\varphi, \varphi^\sharp)$ of locally ringed spaces such that $\varphi^\sharp(\mathrm{Spec}(A)) = u$.

Let us now show its existence.

For $x \in X$, define $\mathfrak{p}_x$ as above. We first construct a map $\varphi\colon X \to \mathrm{Spec}(A)$ by defining $\varphi(x) \in \mathrm{Spec}(A)$ as the prime ideal $u^{-1}(\mathfrak{p}_x)$ of A.

By construction, a point $x \in X$ belongs to $\varphi^{-1}(D(f))$ if and only if $f \notin u^{-1}(\mathfrak{p}_x)$, that is, if and only if $u(f)(x) \neq 0$; in other words, we have $\varphi^{-1}(D(f)) = D(u(f))$; it is thus open in X. Since the open subsets of Spec(A) of the form $D(f)$ constitute a basis of open subsets of Spec(A), this implies that the map φ is continuous.

Let us now show that there exists a morphism of sheaves $\varphi^\sharp\colon \mathscr{O}_{\mathrm{Spec}(A)} \to \varphi_*\mathscr{O}_X$ such that $\varphi^\sharp(\mathrm{Spec}(A)) = u$. For every $f \in A$, the restriction to $D(u(f))$ of the element $u(f) \in \mathscr{O}_X(X)$ is invertible and we define $\varphi^\sharp(D(f))$ to be the unique morphism of rings from $\mathscr{O}_{\mathrm{Spec}(A)}(D(f)) = A_f$ which maps $a/1$ to $u(f)|_{D(u(f))}$. If f and g are elements of A such that $D(g) \subseteq D(f)$, one has $\varphi^\sharp(D(f))(a)|_{D(g)} = \varphi^\sharp(D(g))(a|_{D(g)})$, for every $a \in A_f$, because both sides coincide on the image of A in A_f. Let U be an open subset of Spec(A); let $(f_i)_{i\in I}$ be a family of elements of A such that $U = \bigcup_{i\in I} D(f_i)$; one then has $\varphi^{-1}(U) = \bigcup_{i\in I} D(u(f_i))$. Let $a \in \mathscr{O}_{\mathrm{Spec}(A)}(U)$; for $i \in I$, let $a_i = a|_{D(f_i)}$. For $i, j \in I$, one has $D(f_i) \cap D(f_j) = D(f_i f_j)$, and $D(u(f_i)) \cap D(u(f_j)) = D(u(f_i f_j))$; moreover, $\varphi^\sharp(D(f_i))(a_i) \in \mathscr{O}_X(D(u(f_i)))$ and $\varphi^\sharp(D(f_j))(a_j) \in \mathscr{O}_X(D(u(f_j)))$ coincide with $\varphi^\sharp(D(f_i f_j))(a|_{D(f_i f_j)})$ on $D(u(f_i f_j))$. Consequently, there exists a unique element $\varphi^\sharp(U)(a) \in \mathscr{O}_X(\varphi^{-1}(U))$ whose restriction to $D(u(f_i))$ is equal to $\varphi^\sharp(D(f_i))(a_i)$. The map $\varphi^\sharp(U)$ is a morphism of rings. The family $(\varphi^\sharp(U))$ of morphisms is a morphism of rings of sheaves from $\mathscr{O}_{\mathrm{Spec}(A)}$ to $\varphi_*\mathscr{O}_X$. By construction, one has $\varphi^\sharp(\mathrm{Spec}(A)) = u$. This concludes the proof. □

Lemma **(4.2.9)** (Glueing locally ringed spaces). — *Let* $(X_i)_{i\in I}$ *be a family of locally ringed spaces. For every pair* (i, j) *of elements of* I, *let* X_{ij} *be an open subset of* X_i *and let* $\varphi_{ij}\colon X_{ij} \to X_{ji}$ *be an isomorphism of locally ringed space. Assume that the following properties hold:*

(i) *For every i, one has* $X_{ii} = X_i$ *and* $\varphi_{ii} = \mathrm{Id}$*;*

(ii) *For every i and j, one has* $\varphi_{ij} = \varphi_{ji}^{-1}$*;*

(iii) *For every i, j, k, one has* $\varphi_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$ *and the restriction of* φ_{ik} *to the open subset* $X_{ij} \cap X_{ik}$ *of* X_i *coincides with the restriction of* $\varphi_{jk} \circ \varphi_{ij}$.

Then there exists a locally ringed space X, *a family* $(U_i)_{i\in I}$ *of open subsets of* X, *and a family* $(\varphi_i)_{i\in I}$ *such that for every* $i, j \in I$, *the following properties hold:*

(i) *The morphism* φ_i *is an isomorphism of locally ringed space from* U_i *to* X_i;
(ii) *One has* $X_{ij} = \varphi_i(U_i \cap U_j)$;
(iii) *The morphisms* $\varphi_{ij} \circ \varphi_i$ *and* φ_j *coincide on* $U_i \cap U_j$.

Proof. — Let us first define the topological space X_* to be the union of the family $(X_i)_{i\in I}$: a point of X_* is a pair (i, x) such that $x \in X_i$. One then defines a relation on X_* by setting $(i, x) \sim (j, y)$ if $x \in X_{ij}$ and $y = \varphi_{ij}(x)$. This is an equivalence relation. Let X be the quotient topological space $X_*/\sim$: this is the set of equivalence classes of points of X_* endowed with the quotient topology, for which a subset Ω of X is open if and only if its preimage in X_* by the canonical map $\pi\colon X_* \to X$ is itself open. The map π is continuous.

Let $i \in I$ and let U be an open subset of X_i; one has

$$\pi^{-1}(\pi(\{i\} \times U)) = \bigcup_{j\in I} \{j\} \times \varphi_{ij}(X_{ij} \cap U),$$

so that $\pi^{-1}(\pi(U))$ is open in X_*. By definition of the quotient topology, $\pi(U)$ is open in X. Since every open subset of X_* is a union of open subsets of the form $\{i\} \times U_i$, where U_i is an open subset of X_i, this shows that π is an open map.

For every $i \in I$, let $U_i = \pi(\{i\} \times X_i)$; it is an open subset of X and the family $(U_i)_{i\in I}$ is an open covering of X. Moreover, the map π induces a continuous and open bijection π_i from X_i to U_i; as a consequence, π_i is a homeomorphism.

For $i \in I$, let $\mathscr{O}_{U_i}$ be the sheaf of rings $\pi_{i,*}\mathscr{O}_{X_i}$ on U_i; equivalently, one has $\mathscr{O}_{X_i} = \pi_i^{-1}\mathscr{O}_{U_i}$. For $i, j \in I$, the isomorphism φ_{ij} of locally ringed spaces induces an isomorphism of sheaves of rings

$$\theta_{ij}\colon \mathscr{O}_{U_i}|_{U_i\cap U_j} = (\pi_{i,*}\mathscr{O}_{X_i})|_{U_i\cap U_j} = (\pi_i|_{X_{ij}})_*(\mathscr{O}_{X_i}|_{X_{ij}}) \xrightarrow{\varphi_{ij}^\sharp} \varphi_{ij}(\pi_j|_{X_{ji}})_*(\mathscr{O}_{X_j}|_{X_{ji}}) = \mathscr{O}_{U_j}|_{U_i\cap U_j}.$$

Assumptions (i), (ii), (iii) imply that these isomorphisms satisfy the relations of lemma 3.1.11. Consequently, there exists a sheaf of rings $\mathscr{O}_X$ on X and isomorphisms $\theta_i\colon \mathscr{O}_X|_{U_i} \simeq \mathscr{O}_{U_i}$ such that $\theta_{ij} \circ \theta_i|_{U_i\cap U_j} = \theta_j|_{U_i\cap U_j}$.

Let $x \in X$, let $i \in I$ be such that $x \in U_i$; let $y \in X_i$ be such that $\pi_i(y) = x$. The isomorphism θ_i induces an isomorphism of the stalk $\mathscr{O}_{X,x}$ with the stalk $\mathscr{O}_{U_i,x}$ which is itself isomorphic to $\mathscr{O}_{X_i,y}$; in particular, it is a local ring. This shows that $(X, \mathscr{O}_X)$ is a locally ringed space. □

Remark **(4.2.10)**. — The locally ringed space X defined by the lemma is called the locally ringed space defined by glueing the family $(X_i)_{i\in I}$ along the open subspaces X_{ij} by means of the isomorphisms φ_{ij}. It satisfies the following universal property: For every locally ringed space Y, every family of morphisms $(\psi_i)_{i\in I}$, where $\psi_i\colon X_i \to Y$ is a morphism of locally ringed spaces such that $\psi_j \circ \varphi_{ij} = \psi_i|_{X_{ij}}$, there exists a unique morphism $\psi\colon X \to Y$ such that $\psi \circ \varphi_i = \psi_i$.

4.3. Schemes

Definition **(4.3.1)**. — *Let* $(X, \mathscr{O}_X)$ *be a locally ringed space.*

a) *One says that* X *is an* affine scheme *if it is isomorphic to* $(\operatorname{Spec}(A), \widetilde{A})$.

b) *One says that* X *is a* scheme *if every point of* X *has an open neighborhood* U *such that the locally ringed space* $(U, \mathscr{O}_X|_U)$ *is an affine scheme.*

c) *A morphism of schemes is a morphism of the underlying locally ringed spaces.*

Example **(4.3.2)**. — a) Every affine scheme is a scheme. If a scheme X is affine, then it is isomorphic to $\operatorname{Spec}(\mathscr{O}_X(X))$.

b) The locally ringed space induced on every open subset of an affine scheme is a scheme. Indeed, if $X = \operatorname{Spec}(A)$ and U is an open subset of X, then every point of x has a neighborhood in U of the form $D(f)$, for some $f \in A$. By remark 4.1.7, the locally ringed spaced induced on $D(f)$ is an affine scheme, isomorphic to $\operatorname{Spec}(A_f)$.

In particular, the set of open subsets U of X such that $(U, \mathscr{O}_X|_U)$ is an affine scheme is a basis of the topology of X.

c) The coproduct (disjoint union) of a a family of schemes is a scheme.

d) Let $(X, \mathscr{O}_X)$ be a scheme and let U be an open subset of X. Then $(U, \mathscr{O}_X|_U)$ is a scheme; one says that it is an *open subscheme* of X. If, moreover, U is affine, then one says that it is an *affine open subscheme* of X.

Example **(4.3.3)**. — Let X and Y be schemes; assume that Y is an affine scheme, say $Y = \operatorname{Spec}(A)$. By theorem 4.2.8, for every morphism of rings $u\colon A \to \mathscr{O}_X(X)$, there exists a unique morphism of schemes $f\colon X \to Y$ such that $f^\sharp(Y) = u$.

In particular, there exists a unique morphism of schemes $f\colon X \to \operatorname{Spec}(\mathscr{O}_X(X))$ such that $f^\sharp = \mathrm{Id}$. Moreover, X is an affine scheme if and only if f is an isomorphism.

Exercise **(4.3.4)**. — Let k be a field, let $A = k[x, y]$ and let $X = \operatorname{Spec}(A)$. Let $U = X - V(x, y)$. Then, $(U, \mathscr{O}_X|_U)$ is a locally ringed space which is *not* an affine scheme.

Definition **(4.3.5)**. — *Let* S *be a scheme. An* S-scheme *is a scheme* X *equipped with a morphism of schemes* $f: X \to S$. *If* (X, f) *and* (Y, g) *are* S*-schemes, a morphism of* S*-schemes* $\varphi: X \to Y$ *is a morphism of schemes such that* $g \circ \varphi = f$.

If (X, f) is an S-scheme, the morphism f is called the structural morphism of X. In practice, the morphism f is omitted from the notation; for example, one thus may write: "Let X be an S-scheme; let f be its structural morphism."

Assume that k is a ring and that $S = \operatorname{Spec}(k)$. An S-scheme is also called a k-scheme, and a morphism of S-schemes is also called a k-morphism. By definition, a k-scheme is just a scheme X equipped with a morphism of rings from k to $\mathscr{O}_X(X)$, so that the structure sheaf of X is a sheaf in k-algebras. In particular, an affine k-scheme is the spectrum of a k-algebra. Moreover, a morphism of schemes $\varphi: X \to Y$ is a morphism of k-schemes if the morphism of sheaves $\varphi^\sharp: \mathscr{O}_Y \to \varphi_* \mathscr{O}_X$ is a morphism of sheaves in k-algebras.

Example **(4.3.6)**. — The category of locally ringed spaces admits coproducts (disjoint unions), and the coproduct of any family of schemes is a scheme.

Let us moreover remark that the coproduct of a finite family of affine schemes is affine. So let $(A_i)_{i \in I}$ be a finite family of rings; for every i, let $X_i = \operatorname{Spec}(A_i)$. Let $A = \prod_{i \in I} A_i$ and let $X = \operatorname{Spec}(A)$; for every i, the projection of index i, $p_i: A \to A_i$, induces a morphism j_i from X_i to X.

For every i, let ε_i be the element of A all of whose components are 0, except for the component of index i which is equal to 1. Let $m \in I$. One has $j_m^{-1}(D(\varepsilon_m)) = D(p_m(\varepsilon_m)) = D(1) = X_m$. Moreover, the morphism p_m extends to a surjective morphism from A_{ε_m} to A_m; this morphism is in fact an isomorphism, so that j_m induces an isomorphism from X_m to $D(\varepsilon_m)$.

Finally, $\varepsilon_m \varepsilon_n = 0$ for every pair (m, n) of distinct elements of I, so that $D(\varepsilon_m) \cap D(\varepsilon_n) = \varnothing$.

This proves that the affine scheme $X = \operatorname{Spec}(\prod_{i \in I} A_i)$ is the coproduct of the (finite) family $(\operatorname{Spec}(A_i))$ in the category of locally ringed spaces.

Proposition **(4.3.7)** (Glueing schemes). — *Let* $(X_i)_{i \in I}$ *be a family of schemes. For every* $i \in I$, *let* X_{ij} *be an open subschemes of* X_i*; for every pair* (i, j) *of elements of* I, *let* $\varphi_{ij}\colon X_{ij} \to X_{ji}$ *be an isomorphism of schemes. Assume that these isomorphisms satisfy the conditions of lemma 4.2.9. Then the locally ringed space* X *obtained by glueing the schemes* X_i *along the open subschemes* X_{ij} *by means of the isomorphisms* φ_{ij} *is a scheme.*

Proof. — Indeed, X is the union of open subsets which are isomorphic, as locally ringed spaces, to the schemes X_i. Consequently, every point of X has an open neighborhood which is an affine scheme, hence X is a scheme. □

Example **(4.3.8)** (Affine spaces). — Let k be a ring. The affine space of dimension n over k is defined by $\mathbf{A}^n_k = \operatorname{Spec}(k[T_1, \dots, T_n])$. Since $k[T_1, \dots, T_n]$ is a k-algebra, this is k-scheme.

For every k-scheme X, one has $\operatorname{Hom}_k(X, \mathbf{A}^n_k) = \mathscr{O}_X(X)^n$. In particular, for every k-algebra A, one has $\operatorname{Hom}_k(\operatorname{Spec}(A), \mathbf{A}^n_k) = A^n$.

Example **(4.3.9)** (Projective spaces). — Let k be a ring. The projective space of dimension n over k is defined by glueing $n+1$ affine schemes $U_0, \dots, U_n$ isomorphic to $\mathbf{A}^n_k$. Precisely, let $U = \mathbf{A}^{n+1}_k = \operatorname{Spec}(k[T_0, \dots, T_n])$ and, for every $i \in \{0, \dots, n\}$, let $U_i = \operatorname{Spec}(k[T_0, \dots, T_n]/(T_i - 1)) = V(T_i - 1)$.

For every pair (i, j), let U_{ij} be the open subscheme $D(T_j)$ of U_i; it is affine, isomorphic to $\operatorname{Spec}(k[T_0, \dots, T_n]/(T_i - 1)[1/T_j])$. There exists a unique morphism of schemes $\varphi_{ij}\colon U_{ij} \to U_{ji}$ such that

$$\varphi_{ij}^\sharp\colon k[T_0, \dots, T_n]/(T_j - 1)[1/T_i] \to k[T_0, \dots, T_n]/(T_i - 1)[1/T_j]$$

maps T_m to $T_i T_m / T_j$ for every m. Indeed, the morphism from $k[T_0, \dots, T_n]$ to $k[T_0, \dots, T_n]/(T_i - 1)[1/T_j]$ which maps T_m to $T_i T_m/T_j$ for every m maps T_j to $T_i = 1$, hence it passes to the quotient by $(T_j - 1)$, and it maps T_i to $1/T_j$ which is invertible, hence it extends to $k[T_0, \dots, T_n]/(T_j - 1)[1/T_i]$.

One can check that the glueing conditions of proposition 4.3.7 are satisfied. The scheme obtained is called the projective space of dimension n over k; it is denoted by $\mathbf{P}^n_k$. Since the schemes U_i are k-schemes, and are glued via morphisms of k-schemes, this is a k-scheme.

We shall prove later that $\mathbf{P}^n_k$ is not an affine scheme when $n \geqslant 1$.

Example **(4.3.10)**. — Let X be a scheme. Let x be a point of X and let $\kappa(x)$ be its residue field. Let us define a canonical morphism φ from $\operatorname{Spec}(\kappa(x))$ to X.

The space $\operatorname{Spec}(\kappa(x))$ has exactly one point, and the underlying continuous map of topological spaces is just the one with image x. Let us now describe the morphism $\varphi^\sharp \colon \mathscr{O}_X \to \varphi_* \mathscr{O}_{\operatorname{Spec}(\kappa(x))}$. For every open subset U of X which contains x, one has $\varphi_*(\mathscr{O}_{\operatorname{Spec}(\kappa(x))})(U) = \kappa(x)$, and $\varphi^\sharp(U)$ is the canonical "evaluation morphism" $\mathscr{O}_X(U) \to \kappa(x)$. On the other hand, if U is an open subset of X such that $x \notin U$, then $\varphi_*(\mathscr{O}_{\operatorname{Spec}(\kappa(x))})(U) = 0$, and $\varphi^\sharp(U)$ is the zero morphism.

Let us give an alternate description. The morphism φ factors through every open subscheme of X which contains x. Let thus U be an affine open subscheme of X such that $x \in U$ and let A be a ring such that $U = \operatorname{Spec}(A)$. The point x corresponds to a prime ideal $\mathfrak{p}_x$ of A, and the morphism $\varphi \colon \operatorname{Spec}(\kappa(x)) \to \operatorname{Spec}(A)$ is nothing but the morphism deduced from the ring morphism $A \to A/\mathfrak{p}_x \to \kappa(x)$.

4.4. Some properties of schemes

Definition **(4.4.1)**. — *One says that a scheme* X *is* reduced *if for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ is reduced. One says that it is* integral *if it is irreducible and reduced.*

Recall that a ring is said to be reduced if no-nonzero element is nilpotent. Since the fraction rings of a reduced ring are reduced, the spectrum of a ring A is a reduced ring if and only if the affine scheme $\operatorname{Spec}(A)$ is reduced. Moreover, the affine scheme $\operatorname{Spec}(A)$ is integral if and only if the ideal (0) is its (necessarily unique) minimal prime ideal, that is, if and only if A is an integral domain.

An open subscheme of a reduced scheme is reduced.

Since a non-empty open subset of an irreducible topological space is irreducible (prop. 1.10.3), a non-empty open subscheme of an integral scheme is integral.

Proposition **(4.4.2)**. — *Let* X *be a scheme.*

a) *Let $f \in \mathscr{O}_X(X)$ be such that $V(f) = X$. If* X *is reduced, then $f = 0$.*

b) *If* X *is reduced, then the ring $\mathscr{O}_X(U)$ is reduced for every open subscheme* U *of* X.

c) *Conversely, if every point of* X *has an affine open neighborhood* U *such that $\mathscr{O}_X(U)$ is reduced, then* X *is reduced.*

Proof. — a) Let $U = \operatorname{Spec}(A)$ be an affine open subscheme of X and let $a = f|_U$. One has $V(a) = \operatorname{Spec}(A)$, hence a is nilpotent in A. This implies that

f_x is nilpotent in $\mathscr{O}_{X,x}$ for every $x \in U$, hence $f_x = 0$. Consequently, the germ of f at every point of X vanishes, hence $f = 0$.

b) Let us assume that X is reduced and let us prove that $\mathscr{O}_X(U)$ is reduced for every open subset U of X. Let $f \in \mathscr{O}_X(U)$ and let n be a positive integer such that $f^n = 0$. One then has $V(f) = V(f^n) = X$, hence $f = 0$ by *a*).

c) Let U be an affine open subscheme of X and let $A = \mathscr{O}_X(U)$. Under the canonical isomorphism from U with Spec(A), a point $x \in U$ corresponds to a prime ideal $\mathfrak{p}$ of A, and the local ring $\mathscr{O}_{X,x}$ corresponds to the ring of fractions $\mathscr{A}_\mathfrak{p}$. Let $f \in A_\mathfrak{p}$ be a nilpotent element; let $a \in A$ and $s \in A - \mathfrak{p}$ be such that $f = a/s$ and let $n \in \mathbf{N}$ be such that $f^n = 0$. Then $a^n/s^n = 0$, hence $a^n/1 = 0$ in $A_\mathfrak{p}$, so that there exists $t \in A - \mathfrak{p}$ such that $ta^n = 0$; one then has $(ta)^n = 0$. If A is reduced, then $ta = 0$, hence $a/s = 0$; this proves that $A_\mathfrak{p}$ is reduced. □

Proposition **(4.4.3)**. — *Let* X *be a non-empty scheme. The following conditions are equivalent:*

(i) *The scheme* X *is integral;*

(ii) *For every non-empty open subset* U *of* X*, the ring* $\mathscr{O}_X(U)$ *is an integral domain;*

(iii) *For every non-empty affine open subscheme* U *of* X*, the ring* $\mathscr{O}_X(U)$ *is an integral domain.*

(iv) *The scheme* X *is connected, and every point of* X *has an affine open neighborhood* U *such that* $\mathscr{O}_X(U)$ *is an integral domain.*

Proof. — (i)⇒(ii). Let us assume that X is an integral scheme and let us prove that the ring $\mathscr{O}_X(U)$ is an integral domain for every non-empty open subset of X; we may assume that $U = X$. Since 1 is invertible in $\mathscr{O}_X(X)$, one has $D(1) = X$ (see lemma 4.2.7), hence $1 \neq 0$; this shows that $\mathscr{O}_X(X) \neq 0$. Let then f and g be elements of $\mathscr{O}_X(X)$ such that $fg = 0$. Then $X = V(fg) = V(f) \cup V(g)$. Since X is irreducible, this implies that $X = V(f)$ or $X = V(g)$. Since X is reduced, one has $f = 0$ or $g = 0$.

(ii)⇒(iii) is obvious.

(iii)⇒(iv). Let us assume that $\mathscr{O}_X(U)$ is an integral domain for every non-empty affine open subset U of X, and let us prove that X is irreducible; this will imply that X is connected and non-empty. First of all, it is non-empty: indeed, the empty scheme is, and its ring of functions, being equal to 0, is not an integral domain. By contradiction, let us consider two distinct irreducible components Y and Z of X; by definition of an irreducible component, one has $Y \cap Z \neq Y$ and $Y \cap Z \neq Z$, for these equalities mean that one of Y or Z is contained in the other. Let then $y \in Y$ and $z \in Z$ be points such that $y \notin Z$ and $z \notin Y$. Let U be an affine open neighborhood of y which is contained in $Y - (Y \cap Z)$ and let V be an affine open neighborhood of z which is contained in $Z - (Y \cap Z)$. Then U and V are disjoint open subsets of X and $U \cup V$ is isomorphic to $\operatorname{Spec}(\mathscr{O}_X(U) \times \mathscr{O}_X(V))$ (see example 4.3.6), hence is affine. Since the ring $\mathscr{O}_X(U) \times \mathscr{O}_X(V)$ is not an integral domain (one has $(1, 0) \cdot (0, 1) = (0, 0) = 0$), we obtain a contradiction. This proves that X is irreducible.

(iv)⇒(i). The conditions imply that X is reduced, so that we need to prove that it is irreducible. It is non-empty by hypothesis.

To prove that it irreducible, we prove that every non-empty open subset U of X is dense. Let $x \in \overline{U}$ and let V be an affine open neighborhood of x such that $\mathscr{O}_X(V)$ is an integral domain. Then V is irreducible, hence its open subset $U \cap V$ is dense. Since $\overline{U} \cap V$ is a closed subset of V which contains $U \cap V$, we deduce that $V \subseteq \overline{U}$. We have proved that $\overline{U}$ is open in X. Since $\overline{U}$ is non-empty and X is connected, this implies that $\overline{U} = X$, hence U is dense in X. Consequently, X is irreducible. □

Example **(4.4.4)**. — Let k be an integral domain.

a) The affine space $\mathbf{A}^n_k$ is the spectrum of the integral domain $k[T_1, \dots, T_n]$, hence it is an integral scheme.

b) The projective space $\mathbf{P}^n_k$ is an integral scheme.

Indeed, by its very construction, $\mathbf{P}^n_k$ is the union of $(n+1)$ open affine subschemes $U_0, \dots, U_n$, and each of them is isomorphic to the affine space $\mathbf{A}^n_k$, hence is integral. Moreover, for every pair (i, j) of integers such that $0 \leqslant i < j \leqslant n$, $U_i \cap U_j$ is isomorphic to $\operatorname{Spec}(k[T_1, \dots, T_n, 1/T_n])$, hence is non-empty. This implies that $\mathbf{P}^n_k$ is connected. It thus follows from the previous proposition that $\mathbf{P}^n_k$ is an integral scheme.

Proposition **(4.4.5)**. — *Let* X *be a scheme. For every closed irreducible subset* Z *of* X*, there exists a unique point* $z \in X$ *such that* $Z = \overline{\{z\}}$.

This point is called the *generic point* of Z.

Proof. — Let x be a point of Z and let U be an affine open subscheme of X such that $x \in U$. Let A be a ring such that $U = \operatorname{Spec}(A)$. By proposition 1.10.3, $Z \cap U$ is an irreducible closed subset of U, and one has $Z = \overline{Z \cap U}$. It then follows from proposition 1.10.2 that there exists a prime ideal $\mathfrak p$ of A such that $Z\cap U = V(\mathfrak p)$. Let z be the point of Z corresponding to the prime ideal $\mathfrak p \in \operatorname{Spec}(A) = U$. One has $V(\mathfrak p) = \overline{\{\mathfrak p\}}$ in $\operatorname{Spec}(A)$, so that $\overline{\{z\}}$ contains $Z\cap U$; since it is closed in X, it also contains $\overline{Z\cap U} = Z$. Conversely, $z\in Z$ and Z is closed, hence $\overline{\{z\}}\subseteq Z$.

Conversely, let z' be a point of Z such that $\overline{\{z'\}} = Z$. Since $X - U$ is a closed subset of X which does not contain Z, it does not contain z', hence $z'\in U$. Consequently, z' corresponds to a prime ideal $\mathfrak p'$ of A, and

$$Z\cap U = \overline{\{z'\}}\cap U = V(\mathfrak p') = V(\mathfrak p)$$

in $\operatorname{Spec}(A)$. This implies that $\mathfrak p' = \mathfrak p$, hence $z' = z$. □

Proposition **(4.4.6)**. — *An affine scheme is quasi-compact. More generally, a scheme is quasi-compact if and only if it is the union of finitely many affine open subschemes.*

Recall that a topological space X is said to be *quasi-compact* if every open cover of X admits a finite sub-cover, namely if for every family $(U_i)_{i\in I}$ of open subsets of X such that $X = \bigcup_{i\in I} U_i$, there exists a finite subset J of I such that $X = \bigcup_{i\in J} U_i$. This is the French terminology, where "compact" means "quasi-compact and Hausdorff", hence "compact" in the American terminology for which compact spaces are called "compact Hausdorff".

A subset of a topological space is said to be quasi-compact if it is so with the induced topology. It follows readily from the definition that a finite union of quasi-compact subsets of a topological space is quasi-compact.

Proof. — Let A be a ring and let $X = \operatorname{Spec}(A)$. Let $(U_i)_{i\in I}$ be a family of open subsets of X such that $X = \bigcup_{i\in I} U_i$. For every $i\in I$, let $(f_{i,j})_{J_i}$ be a family of elements of A such that $U_i = \bigcup_{j\in J_i} D(f_{i,j})$. Let J be the union of the family (J_i); an element of J is just a pair (i,j) where $i\in I$ and $j\in J_i$. One thus has $X = \bigcup_{(i,j)\in J} D(f_{i,j})$, hence $\varnothing = \bigcap_{(i,j)\in J} V(f_{i,j})$. Consequently, the ideal of A generated by the $f_{i,j}$ contains 1, and there exists a finite subset J_0 of J and a family $(a_{i,j})_{(i,j)\in J_0}$ of elements of A such that $1 = \sum_{(i,j)\in J_0} a_{i,j}f_{i,j}$. This implies $\operatorname{Spec}(A) = \bigcup_{(i,j)\in J_0} D(f_{i,j})$. If I_0 is the image of J_0 by the projection $(i,j)\mapsto i$, one then has $\operatorname{Spec}(A) = \bigcup_{i\in I_0} U_i$. This shows that affine schemes are quasi-compact.

Conversely, let X be a scheme and let $(U_i)_{i\in I}$ be a covering of X by open affine subschemes. If X is quasi-compact, there exists a finite subfamily of (U_i) which covers X; if I is finite, then X is quasi-compact since U_i is quasi-compact for every i, and every finite union of quasi-compact subsets of a topological space is quasi-compact. □

Proposition **(4.4.7)**. — *Any non-empty quasi-compact scheme possesses a closed point.*

Proof. — Let X be a non-empty quasi-compact scheme. It can be written as a finite union of affine open subschemes, say n, and we argue by induction on n.

We may write $X = U \cup V$, where U is a non-empty affine scheme and V is a union of strictly less than n open subschemes.

Consequently, there exists a ring A such that $U = \operatorname{Spec}(A)$. Since $U \neq \varnothing$, the ring A is nonzero, hence admits a maximal ideal which, in turn, corresponds to a closed point x of U. Let Z be the closure of $\{x\}$ in X. If $Z = \{x\}$, then x is a closed point of X. Let us assume that $Z \neq \{x\}$. Since x is a closed point of U, one has $Z \cap U = \{x\}$ and $Z - \{x\}$ is a non-empty closed subset of Z which is contained in V. By induction, there exists a point $y \in Z - \{x\}$ which is closed in Z. In particular, y is a closed point of X. Otherwise, by induction, we may write $X = U \cup V$, □

Lemma **(4.4.8)**. — *Let $f\colon X \to S$ be a morphism of schemes. The following properties are equivalent:*

(i) *For every quasi-compact open subset* U *of* S, $f^{-1}(U)$ *is quasi-compact;*

(ii) *For every affine open subset* U *of* S, $f^{-1}(U)$ *is quasi-compact;*

(iii) *Every point of* S *has an affine open neighborhood* U *such that* $f^{-1}(U)$ *is quasi-compact.*

If these properties hold, one says that *the morphism f is quasi-compact.*

Observe that a morphism of affine schemes is quasi-compact. Let indeed $\varphi\colon A \to B$ be a morphism of rings. For every $a \in A$, the equality ${}^a\varphi^{-1}(D(a)) = D(\varphi(a))$ proves that ${}^a\varphi^{-1}(D(a))$ is affine. Since every quasi-compact open subset U of $\operatorname{Spec}(A)$ is the union of a finite family of open subsets of the form $D(a)$, this implies that ${}^a\varphi^{-1}(U)$ is quasi-compact.

Moreover, if a morphism $f\colon Y \to X$ is quasi-compact, then for every open subset U of X, the induced morphism $f_U\colon f^{-1}(U) \to U$ is quasi-compact as well.

Proof. — The implication (i)⇒(ii) follows from the fact that affine schemes are quasi-compact, and the implication (ii)⇒(iii) holds true because every point of S has an affine open neighborhood.

Let us now assume that (iii) holds true.

Let U be an affine open subset of S such that $f^{-1}(U)$ is quasi-compact. Let A be a ring such that $U = \mathrm{Spec}(A)$. Since $f^{-1}(U)$ is quasi-compact, it can be written as a finite union of affine open subsets $V_1, \dots, V_n$ of $f^{-1}(U)$. For every i, let B_i be a ring such that $V_i = \mathrm{Spec}(B_i)$; the morphism $f|_{V_i}$ corresponds to a ring morphism $u_i : A \to B_i$. For every $a \in A$, one has $(f|_{V_i})^{-1}(D(a)) = D(u_i(a)) = \mathrm{Spec}(A_{u_i(a)})$, so that $(f|_{V_i})^{-1}(D(a))$ is affine; consequently, $f^{-1}(D(a)) = \bigcup_{i=1}^{n}(f|_{V_i})^{-1}(D(a))$ is quasi-compact.

Let now W be a quasi-compact open subset of S. Let $s \in S$; let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of s such that $f^{-1}(U)$ is quasi-compact and let W_s be an open subset of U of the form $D(a)$, for $a \in A$, such that $W_s \subseteq U \cap W$. By what precedes, $f^{-1}(W_s)$ is quasi-compact. Since W is the union of the family $(W_s)_{s \in W}$ of open sets and is quasi-compact, there exists a finite subset Σ of S such that $W = \bigcup_{s \in \Sigma} W_s$. Then $f^{-1}(W) = \bigcup_{s \in \Sigma} f^{-1}(W_s)$ is quasi-compact, as was to be shown. □

Definition (**4.4.9**). — *Let* $f : X \to S$ *be a morphism of schemes. One says that* f *is* quasi-separated *if for every affine open subscheme* U *of* S *and every pair* (V, V') *of affine open subsets of* X *contained in* $f^{-1}(U)$, *the intersection* $V \cap V'$ *is quasi-compact.*

One says that a scheme X *is quasi-separated if the canonical morphism from* X *to* $\mathrm{Spec}(\mathbf{Z})$ *is quasi-separated.*

In other words, a scheme X is quasi-separated if and only if the intersection of any two quasi-compact open subsets of X is quasi-compact.

Definition (**4.4.10**). — *One says that a scheme is* locally noetherian *if every point has a neighborhood isomorphic to the spectrum of a noetherian ring. One says that it is* noetherian *if it is locally noetherian and quasi-compact.*

Proposition (**4.4.11**). — a) *The underlying topological space of a noetherian scheme is noetherian.*

b) *Every open subscheme of a locally noetherian scheme is locally noetherian.*

c) *Every open subscheme of a noetherian scheme is noetherian.*

d) *Let* X *be an affine scheme. If* X *is noetherian, then* $\mathscr{O}_X(X)$ *is a noetherian ring.*

Proof. — a) If X is a noetherian scheme, it is the union of finitely many open subschemes which are spectra of noetherian rings. Each of them being a noetherian topological space, X is a noetherian topological space.

b) Let X be a locally noetherian scheme and let U be an open subscheme of X. Let $x \in \mathrm{U}$ and let $\mathrm{W} = \mathrm{Spec}(\mathrm{A})$ be an affine open neighborhood of x, where A is a noetherian ring. Let $a \in \mathrm{A}$ be such that $x \in \mathrm{D}(a)$ and $\mathrm{D}(a) \subseteq \mathrm{U} \cap \mathrm{W}$. Then $\mathrm{D}(a) \simeq \mathrm{Spec}(\mathrm{A}_a)$ is an affine open neighborhood of x contained in U; moreover, the ring A_a is generated by $1/a$ over A, hence is a noetherian ring. This shows that U is locally noetherian.

c) With the same notation, U is both quasi-compact (because it is a noetherian topological space) and locally noetherian, hence is noetherian.

d) Let A be a ring, let $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$. Let (I_n) be an increasing sequence of ideals of A. Every point $x \in \mathrm{X}$ has an affine open neighborhood U_x in X such that $\mathscr{O}(\mathrm{U}_x)$ is a noetherian ring. Let then $a_x \in \mathrm{A}$ be such that $x \in \mathrm{D}_\mathrm{X}(a_x) \subseteq \mathrm{U}_x$; one thus has $\mathrm{D}_\mathrm{X}(a_x) = \mathrm{D}_{\mathrm{U}_x}(a_x)$, hence $\mathscr{O}(\mathrm{D}_\mathrm{X}(a_x)) = \mathscr{O}(\mathrm{U}_x)_{a_x} = \mathrm{A}_{a_x}$. Since $\mathscr{O}(\mathrm{U}_x)_{a_x}$ is generated by $1/a_x$ over $\mathscr{O}(\mathrm{U}_x)$, it is a noetherian ring. Consequently, A_{a_x} is a noetherian ring. Since X is quasi-compact, there exists a finite family (a_i) of elements of A such that $\mathrm{X} = \bigcup_i \mathrm{D}(a_i)$ and A_{a_i} is a noetherian ring for every i.

Let us now show that A is noetherian. Let (I_n) be a strictly increasing sequence of ideals of A, and let I be its union. For every i, there exists an integer n_i such that $\mathrm{I}_n = \mathrm{I}$ for $n \geqslant n_i$. Let $n \geqslant \sup(n_i)$ and let us show that $\mathrm{I}_n = \mathrm{I}$. Let thus $u \in \mathrm{I}$ and let J be the set of elements $v \in \mathrm{A}$ such that $uv \subseteq \mathrm{I}_n$; this is an ideal of A. Moreover, for every i, one has $v/1 \in \mathrm{I}_n \cdot \mathrm{A}_{a_k}$, hence there exists an integer k_i such that $a_i^{k_i} \in \mathrm{J}$. Let $k = \sup(k_i)$. Since $\mathrm{X} = \bigcup_i \mathrm{D}(a_i)$, the ideal of A generated by the family (a_i) contains 1, as does the ideal generated by the family (a_i^k). Consequently, $1 \in \mathrm{J}$ and $u \in \mathrm{I}_n$. □

4.4.12. — Let X be a scheme and let Z be a subset of X; let $x \in \mathrm{Z}$. We introduced in definition 1.11.2 the dimension of Z and its dimension at x, respectively denoted by $\dim(\mathrm{Z})$ and $\dim_x(\mathrm{Z})$, as well as its codimension, denoted by $\mathrm{codim}(\mathrm{Z})$. Recall that $\dim(\mathrm{Z})$ is the supremum of the lengths of chains of closed irreducible subsets of Z, while $\dim_x(\mathrm{Z})$ is the supremum of the lengths of chains of closed irreducible subsets of Z containing x. On the other hand, if Z is a closed irreducible subset of X, then $\mathrm{codim}(\mathrm{Z})$ is the supremum of the lengths of chains of closed irreducible subsets of X containing Z. in particular, if x is the generic point of Z, then $\operatorname{codim}(Z) = \dim_x(X)$. In general, one defines $\operatorname{codim}(Z)$ as the infimum of the codimensions of the closed irreducible subsets of X contained in Z.

Recall also the following properties, for an arbitrary closed subset Z of X:

a) The dimension of Z is the supremum of the dimensions of its irreducible components;

b) Each irreducible component of X has codimension 0 in X;

c) For every closed irreducible subset Z of X, one has $\operatorname{codim}(Z) + \dim(Z) \leqslant \dim(X)$;

d) If Y and Z are irreducible closed subsets of X such that $Y \subseteq Z$, then $\dim(Y) \leqslant \dim(Z)$ and $\operatorname{codim}(Z) \leqslant \operatorname{codim}(Y)$.

e) If $X = \operatorname{Spec}(A)$ is affine and $Z = V(\mathfrak{p})$, then $\operatorname{codim}(V(\mathfrak{p})) = \dim(A_{\mathfrak{p}})$.

f) For every open subset U of X such that $Z \cap U \neq \varnothing$, one has $\operatorname{codim}(Z) = \operatorname{codim}_U(Z \cap U)$ and $\dim_x(Z) = \dim_x(Z \cap U)$ for every $x \in Z \cap U$. In particular, for every point $x \in U$, one has $\dim_x(U) = \dim_x(X)$. This follows from the fact that the map $Z \mapsto Z \cap U$ induces a bijection from the set of closed irreducible subsets of X which meet U to the set of closed irreducible subsets of U.

Example **(4.4.13)**. — Let k be a field and let X be an integral k-scheme of finite type. By this, we mean that X is irreducible and that every point of X has an affine open neighborhood U such that $\mathscr{O}_X(U)$ is an integral domain which is finitely generated as a k-algebra.

Let x be the generic point of X. Let U be an affine open neighborhood of x such that $A = \mathscr{O}_X(U)$ is a finitely generated k-algebra and an integral domain. The point x of X corresponds to the prime ideal (0) of A, hence the local ring $\mathscr{O}_{X,x}$, isomorphic to the ring of fractions $A_{(0)} = \operatorname{Frac}(A)$, is a finitely generated field extension of k. It is called the *field of rational functions on* X and is denoted by $R(X)$. By theorem 1.11.6, one has $\dim(U) = \operatorname{tr.deg}_k(R(X))$. It then follows from the definition of the dimension that $\dim(X) = \operatorname{tr.deg}_k(R(X))$.

Let Z be an irreducible closed subset of X, let z be its generic point and let $U = \operatorname{Spec}(A)$ be an affine open neighborhood as above such that $z \in U$. Let $\mathfrak{p}$ be the prime ideal of A corresponding to Z. One thus has

$$\dim_z(X) = \dim_z(U) = \operatorname{codim}(Z) = \dim(A_{\mathfrak{p}})$$

and

$$\dim(Z) = \dim(Z \cap U) = \dim(A/\mathfrak{p}).$$

It then follows from theorem 1.13.6 that

$$\dim(Z) + \operatorname{codim}(Z) = \dim(X).$$

Moreover, all maximal chains of closed irreducible subsets of X have lengths $\dim(X)$. One says that X is *catenary*.

Example **(4.4.14).** — Let K be a field. It follows from corollary 1.11.7 that for every integer $n \geqslant 0$, one has $\dim(\mathbf{A}^n_K) = n$. By the preceding example, one also has $\dim(\mathbf{P}^n_K) = n$.

4.5. Products of schemes

4.5.1. — Let $\mathcal{C}$ be a category. Let S be an object of $\mathcal{C}$, and let $(X_i)_{i\in I}$ be objects of $\mathcal{C}$ endowed with morphisms $f_i\colon X_i \to S$ in $\mathcal{C}$. Let Q be the quiver whose set of vertices is the disjoint union of I and a point s, and with exactly one arrow from every point $i \in I$ to s, and none other. The morphisms f_i give rise to a Q-diagram in $\mathcal{C}$. By definition, a limit of this diagram is called a *fiber product* of the family (X_i, f_i). Explicitly, a fiber product is an object P of $\mathcal{C}$, equipped with morphisms $p_i\colon P \to X_i$ for every i, and $p\colon P \to S$, such that $p = f_i \circ p_i$ for every i, and such that for every object T of $\mathcal{C}$, and every family $(g_i)_{i\in I}$, where $g_i\colon T \to X_i$ is a morphism in $\mathcal{C}$, every morphism $g\colon T \to S$ such that $f_i \circ g_i = g$ for every $i \in I$, there exists a unique morphism $\psi\colon T \to P$ such that $g_i = p_i \circ \psi$ for every i and $g = p \circ \psi$.

When I has two elements, the above diagram takes the form

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

and a fiber product is usually denoted $X \times_S Y$. One also says that the (commutative) square

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

is *cartesian*. Then, for every object T of $\mathcal{C}$, the maps p and q induce maps $p_T\colon \mathcal{C}(T, X \times_S Y) \to \mathcal{C}(T, X)$ and $q_T\colon \mathcal{C}(T, X \times_S Y) \to \mathcal{C}(T, Y)$. The resulting map

$$(p_T, q_T): \mathbf{C}(T, X \times_S Y) \to \mathbf{C}(T, X) \times \mathbf{C}(T, Y)$$

is a bijection from $\mathbf{C}(T, X \times_S Y)$ to the subset $\mathbf{C}(T, X) \times_{\mathbf{C}(T,S)} \mathbf{C}(T, Y)$ of pairs (φ, ψ) in $\mathbf{C}(T, X) \times \mathbf{C}(T, Y)$ such that $f \circ \psi = g \circ \varphi$.

This can also be rephrased by introducing the category $\mathbf{C}_S$ of objects of $\mathbf{C}$ "over S", whose objects are pairs (X, f), where $f: X \to S$ is a morphism, and whose morphisms from (X, f) to (Y, g) is a morphism $\varphi: X \to Y$ in $\mathbf{C}$ such that $g \circ \varphi = f$. Rephrasing the previous definition, a fiber product of a family (X_i) of objects over S is nothing but a product of this family in the category $\mathbf{C}_S$.

Lemma **(4.5.2)**. — *Let k be a ring, let $S = \operatorname{Spec}(k)$. Let I be a finite set; for every $i \in I$, let A_i be a k-algebra and let $X_i = \operatorname{Spec}(A_i)$. Let $A = \bigotimes_i A_i$ be the tensor product of these k-algebras; for every $i \in I$, it is an A_i-algebra. Then the affine scheme $\operatorname{Spec}(A)$ is a product of the family (X_i) of S-schemes.*

Proof. — For every $i \in I$, let $f_i: X_i \to S$ be the morphism induced by the morphism $k \to A_i$ (i.e., by the structure of k-algebra of A_i). Let T be a scheme, let $g: T \to S$ be a morphism of schemes, and let $(g_i)_{i \in I}$ be a family, where $g_i: T \to X_i$ is a morphism of S-schemes; we thus have $f_i \circ g_i = g$ for every $i \in I$.

Let $\gamma: k \to \mathscr{O}_T(T)$ be the morphism $g^\sharp(\operatorname{Spec}(k))$. For every i, let $\gamma_i: A_i \to \mathscr{O}_T(T)$ be the morphism $g_i^\sharp(\operatorname{Spec}(A_i))$; this is a morphism of k-algebras, because $f_i \circ g_i = g$. The map

$$(a_i)_{i \in I} \mapsto \prod_{i \in I} \gamma_i(a_i)$$

is k-multilinear. Consequently, there exists a unique morphism of k-algebras

$$u: \bigotimes_{i \in I} A_i \to \mathscr{O}_T(T)$$

such that $u(\otimes_{i \in I} a_i) = \prod_{i \in I} \gamma_i(a_i)$. By theorem 4.2.8, there exists a unique morphism $\varphi: T \to \operatorname{Spec}(A)$ such that $u = \varphi^\sharp(\operatorname{Spec}(A))$. It is a morphism of S-schemes, because u is a morphism of k-algebras.

Let $i \in I$. The morphism $g_i^\sharp(\operatorname{Spec}(A_i)): A_i \to \mathscr{O}_T(T)$ is the composition of $f_i^\sharp(\operatorname{Spec}(A_i)): A_i \to A$ and of $u = \varphi^\sharp(\operatorname{Spec}(A))$; by theorem 4.2.8, one has $f_i \circ \varphi = g_i$.

Conversely, every morphism $\psi: T \to \operatorname{Spec}(A)$ of k-schemes such that $f_i \circ \psi = g_i$ for every i induces a morphism $\psi^\sharp(\operatorname{Spec}(A)): A \to \mathscr{O}_T(T)$ such that $\psi^\sharp(\operatorname{Spec}(A)) \circ f_i^\sharp(\operatorname{Spec}(k)) = g_i^\sharp(\operatorname{Spec}(k))$. Since A is generated by the images of the algebras A_i, one has $\psi^\sharp(\operatorname{Spec}(A)) = u$. By theorem 4.2.8, this implies that $\psi = \varphi$. □

Lemma (**4.5.3**). — *Let* S *be a scheme, let* S_1 *be an open subscheme of* S*, let* $(X_i)_{i\in I}$ *be a finite family of* S_1*-schemes; for every* i*, let* $f_i\colon X_i \to S_1$ *be the structural morphism. Assume that this family of* S_1*-schemes admits a product* P*; for every* i*, let* $p_i\colon P \to X_i$ *be the canonical morphism.*

Let V *be an open subscheme of* S*; for every* $i \in I$*, let* U_i *be an open subscheme of* X_i *such that* $f_i(U_i) \subseteq V$*. Let* $Q = \bigcap_{i\in I} p_i^{-1}(U_i)$*. It is an open subset of* P *and the induced scheme* $(Q, \mathscr{O}_P|_Q)$ *is a fiber product of the family* $(U_i, f_i|_{U_i})$ *of* V*-schemes.*

Proof. — Since p_i is continuous and U_i is open, Q is an open subset of P. Let (T, g) be a V-scheme; for every i, let $g_i\colon T \to U_i$ be a morphism of V-schemes. Composing g with the inclusion from V to S, and, for every i, the morphism g_i with the inclusion from U_i to X_i, we can view T as an S-scheme endowed for every i with an S-morphism to X_i.

Since $g = f_i \circ g$, one has in fact $g(T) \subseteq S_1$, which allows us to view T as an S_1-schemes, and the morphisms g_i as morphisms of S_1-schemes. Consequently, there exists a unique morphism $\psi_1\colon T \to P$ of S-schemes such that $p_i \circ \psi_1 = g_i$ for every i. Since the image of g_i is contained in U_i, one has $\psi(T) \subseteq p_i^{-1}(U_i)$ for every i, hence $\psi_1(T) \subseteq Q$. Consequently, the morphism ψ_1 induces a morphism $\psi\colon T \to Q$ of S_1-schemes such that $p_i|_Q \circ \psi = g_i$ for every i. It is also a morphism of S-schemes.

This is in fact the unique such morphism. Let indeed $\tilde\psi\colon T \to Q$ be a morphism of S-schemes such that $p_i|_Q \circ \tilde\psi = g_i$ for every i. Then $\tilde\psi$ can be considered as a morphism of S_1-schemes from T to P and one has $p_i \circ \tilde\psi = g_i = p_i \circ \psi$ for every i. Since P is a product of the family (X_i) of S_1-schemes, this implies that $\tilde\psi = \psi$. □

Theorem (**4.5.4**). — *The category of schemes admits all finite fiber products.*

Proof. — Let I be a finite set, let S be scheme, let $(X_i)_{i\in I}$ be a family of S-schemes; for every i, let $f_i\colon X_i \to S$ be the structural morphism. We need to show that the family (X_i) of S-schemes has a product. By lemma 4.5.2, this family has a product if S and all the schemes X_i are affine. In general, the construction of the desired product will consist in glueing the fiber products of families $(U_i \to V)_{i\in I}$, where V is an open affine subscheme of S and, for every i, U_i is an open affine subscheme of X_i such that $f_i(U_i) \subseteq V$.

Let $(S_\lambda)_{\lambda\in L}$ be an covering of S by open affine subschemes. For every $\lambda \in L$ and every $i \in I$, let $(U_{i,m})_{m\in M_{i,\lambda}}$ be a covering of $f_i^{-1}(S_\lambda)$ by open affine subschemes. Let M be the union of the family $M_{i,\lambda}$: an element of M is a pair $(\lambda, (m_i))$ where $\lambda \in L$ and $m_i \in M_{i,\lambda}$ for every $i \in I$.

For every $m = (\lambda, (m_i)) \in M$, let P_m be the product of the family $(U_{i,m_i})_{i\in I}$ of affine S_λ-schemes; by lemma 4.5.3, it is also a product of this family in the category of S-schemes. For every i, let $p_{m,i}\colon P_m \to X_i$ be the canonical morphism (its image is contained in U_{i,m_i}). Let also $g_m\colon P_m \to S$ be the morphism $f_i \circ p_{m,i}$, for every $i \in I$; one has $g_m(P_m) \subseteq S_\lambda$.

Let $m = (\lambda, (m_i))$ and $m' = (\lambda', (m'_i))$ be elements of M. For every $i \in I$, let $V_i = U_{i,m_i} \cap U'_{i,m_i}$, and let $P_{mm'} = \bigcap_{i\in I}(p_{m,i})^{-1}(V_i)$; by lemma 4.5.3, the open subscheme $P_{mm'}$ of P_m is a product of the family (V_i). By symmetry, $P_{m'm}$ is also a product of this family. Consequently, there exists a unique morphism of S-schemes $\varphi_{mm'}\colon P_{mm'} \to P_{m'm}$ such that $p_{m',i} \circ \varphi_{mm'} = p_{m,i}$ for every $i \in I$, and it is an isomorphism.

Let P be the scheme obtained by glueing the family of schemes $(P_m)_{m\in M}$ along their open subschemes $P_{mm'}$ via the isomorphisms $\varphi_{mm'}$. For every $m \in M$, let $\varphi_m\colon P_m \to P$ be the canonical morphism; by definition, it induces an isomorphism of P_m onto an open subscheme W_m of P, and one has $\varphi_{m'} \circ \varphi_{mm'} = \varphi_m$ for every pair (m, m') of elements of M. For every $i \in I$, there exists a unique morphism of schemes $p_i\colon P \to X_i$ such that $p_i \circ \varphi_m = p_{m,i}$ for every $m \in P$. Similarly, there exists a unique morphism of schemes $g\colon P \to S$ such that $g \circ \varphi_m = g_m$ for every $m \in M$; one has $g = f_i \circ p_i$ for every $i \in I$. Consequently, P is an S-scheme (via g) and the morphisms p_i are morphisms of S-schemes.

Let $m \in M$ and let $U_m = \bigcap_{i\in I} p_i^{-1}(U_{i,m_i})$. Let us show that $U_m = W_m$. The inclusion $W_m \subseteq U_m$ follows from the equality $p_i \circ \varphi_m = p_{i,m_i}$, for every $i \in I$. Conversely, let $m' \in M$. By lemma 4.5.3 the isomorphism $\varphi_{m'}\colon P_{m'} \to W_{m'}$ induces an isomorphism from $P_{m'm}$ with $U_m \cap W_{m'}$. Since $\varphi_{m'} = \varphi_m \circ \varphi_{mm'}$, it follows that $U_m \cap W_{m'} = \varphi_m(P_{mm'}) \subseteq W_m$. Consequently,

$$U_m = U_m \cap \left(\bigcup_{m'\in M} W_{m'}\right) = \bigcup_{m'\in M} (U_m \cap W_{m'}) \subseteq W_m.$$

This shows that $U_m = W_m$, as claimed.

Let us now show that the S-scheme P, equipped with the family of morphisms $(p_i)_{i\in I}$ is a product of the family $(X_i)_{i\in I}$ of S-schemes. We need to check the universal property: Let T be an S-scheme; for every $i \in I$, let $h_i\colon T \to X_i$ be an

S-morphism; let us show that there exists a unique morphism of S-schemes $\psi\colon T \to P$ such that $p_i \circ \psi = h_i$ for every i.

For every $m \in M$, let $T_m = \bigcap_{i\in I} h_i^{-1}(U_{i,m_i})$. Since P_m is a product of the family $(U_{i,m_i})_i$ of S-schemes, there exists a unique morphism of S-schemes $\psi'_m\colon T_m \to P_m$ such that $p_{m,i} \circ \psi'_m = h_i|_{T_m}$ for every $i \in I$. Let $\psi_m = \varphi_m \circ \varphi'_m$.

Let $m, m' \in M$ and let $V = T_m \cap T_{m'}$, so that the morphism $\psi'_m|_V$ factors through $P_{mm'}$, Then, the morphism $\varphi_{mm'} \circ \psi'_m|_V$ from V to $P_{m'}$ satisfies

$$p_{m',i} \circ \varphi_{mm'} \circ \psi'_m|_V = p_{m,i} \circ \psi'_m|_V = h_i|_V = p_{m',i} \circ \psi'_{m'}|_V.$$

Since $P_{m'}$ is a product of the family $(U_{i,m'_i})_i$ of S-schemes, one thus has $\psi'_{m'}|_V = \varphi_{mm'} \circ \psi'_m|_V$. In particular, the morphisms $\psi_m = \varphi_m \circ \psi_m$ and $\psi_{m'} = \varphi_{m'} \circ \psi'_{m'}$ coincide on V. As a consequence, there exists a unique morphism of S-schemes $\psi\colon T \to P$ such that $\psi|_{T_m} = \psi_m$ for every $m \in M$. Moreover, for every such m and every $i \in I$, one has

$$p_i \circ \psi|_{T_m} = p_i \circ \varphi_m \circ \psi'_m = p_{m,i} \circ \psi'_m = h_i|_{T_m}.$$

This implies that $p_i \circ \psi = h_i$ for every $i \in I$.

Conversely, let $\tilde\psi\colon T \to P$ be a morphism such that $p_i \circ \tilde\psi = h_i$ for every $i \in I$. Let us show that $\tilde\psi = \psi$. Let $m \in M$. Observe that one has $T_m = \psi^{-1}(U_m) = \tilde\psi^{-1}(U_m) = \bigcap_{i\in I}(p_i \circ \tilde\psi)^{-1}(U_{i,m_i}) = \bigcap_{i\in I} h_i^1(U_{i,m_i}) = \psi^{-1}(U_m) = T_m$. Moreover, $p_i \circ \tilde\psi|_{T_m} = h_i|_{T_m} = p_i \circ \psi|_{T_m}$. Since U_m is a product of the family $(U_{i,m_i})_i$ of S-schemes, the two morphisms from T_m to U_m induced by ψ and $\tilde\psi$ are equal. In other words, $\psi|_{T_m} = \tilde\psi|_{T_m}$. Since the family $(T_m)_{m\in M}$ of open subschemes cover P, this implies that $\tilde\psi = \psi$. □

Corollary **(4.5.5)**. — *Let* S *be a scheme, let* X *and* Y *be* S*-schemes. Every non-empty finite family* $(f_i)_{i\in I}$ *of* S*-morphisms from* X *to* Y *has an equalizer in the category of* S*-schemes.*

Recall that an equalizer (Z, g) of the family (f_i) is a scheme Z endowed with a morphism $g\colon Z \to X$ such that all morphisms $f_i \circ g$ are equal, and such that for every scheme T and every morphism $h\colon T \to X$ such that all morphisms $f_i \circ h$ are equal, there exists a unique morphism $k\colon T \to Z$ such that $h = g \circ k$.

Proof. — Let Y^I be the product of I copies of X over S; for every $i \in I$, let $p_i\colon Y^I \to Y$ be the projection of index i. Let $f\colon X \to Y^I$ be the unique S-morphism such that $p_i \circ f = f_i$ for every $i \in I$. Let also $\delta\colon Y \to Y^I$ be the diagonal morphism, namely, the unique morphism such that $p_i \circ \delta = \mathrm{id}_Y$ for every $i \in I$; it is an

S-morphism. Let Z be the fiber product of the morphisms f and g; let $p\colon Z \to X$ and $q\colon Z \to Y$ be the canonical projections. Let us show that (Z, p) is an equalizer of the family (f_i).

By definition, for every $i \in I$, one has $f_i \circ p = p_i \circ f \circ p = p_i \circ g \circ q = q$, so that all morphisms $f_i \circ p$ are equal. Let T be a scheme, let $h\colon T \to X$ be a morphism of schemes such that all morphisms $f_i \circ h$ are equal to a common morphism $j\colon T \to Y$. Then $p_i \circ f \circ h = f_i \circ h = j = p_i \circ g \circ j$, so that there exists a unique morphism $k\colon T \to Z$ such that $h = p \circ k$ and $j = q \circ k$. Conversely, if $k'\colon T \to Z$ is a morphism such that $h = p \circ k'$, then $g \circ q \circ k' = f \circ p \circ k' = f \circ h$. For every element i of I, one then has $q \circ k' = p_i \circ g \circ q \circ k' = p_i \circ f \circ h = f_i \circ h = j$; Since I is non-empty, this proves that $q \circ k' = j$. By the definition of the fiber product Z, one then has $k' = k$. □

Remark **(4.5.6)**. — If the morphisms $f_i\colon X_i \to S$ are quasi-compact, then the morphism $g\colon P \to S$ from the fiber product of the family (X_i) to S is quasi-compact as well.

Indeed, the construction of P shows that for every $\lambda \in L$, the open subset $f_i^{-1}(S_\lambda)$ of X_i is the union of a finite family of affine open subschemes, so that one can assume that the sets $M_{i,\lambda}$ are finite, for every $i \in I$ and every λ. For every λ, $g^{-1}(S_\lambda)$ is the union of the affine schemes $P_{(\lambda,(m_i))}$, for $(m_i) \in \prod_i M_{i,\lambda}$. Consequently, $g^{-1}(S_\lambda)$ is quasi-compact. This concludes the proof.

4.5.7. — Let $f\colon X \to S$ be a morphism of schemes. Let T be a scheme and let $u\colon T \to S$ be a morphism of schemes. Let X_T be the fiber product $X \times_S T$, and let $f_T\colon X_T \to T$ be the second projection. The T-scheme (X_T, f_T) is called the T-*scheme deduced from* X *by base change to* T.

Let Y be an S-scheme and let $g\colon Y \to S$ be its structural morphism. and let $\varphi\colon X \to Y$ be a morphism of S-schemes. There exists a unique morphism $\varphi_T\colon X_T \to Y_T$ of T-schemes such that $q \circ \varphi_T = \varphi \circ p$, where $p\colon X_T \to X$ and $q\colon Y_T \to Y$ are the first projections. This morphism φ_T is called the *morphism deduced from* φ *by base change to* T.

The assignments $X \mapsto X_T$ and $\varphi \mapsto \varphi_T$ define a functor u^* from the category $\boldsymbol{Sch}_S$ of S-schemes to the category $\boldsymbol{Sch}_T$ of T-schemes.

Let s be a point of S and let $j_s\colon \operatorname{Spec}(\kappa(s)) \to S$ be the associated morphism. The $\operatorname{Spec}(\kappa(s))$-scheme $X \times_S \operatorname{Spec}(\kappa(s)) \to \operatorname{Spec}(\kappa(s))$ is called the *fiber* of f at s; it is denoted by X_s. This terminology is justified by the fact that the underlying continous map to the first projection $X_s \to X$ induces a homeomorphism from X_s to the closed subset $f^{-1}(s)$ of X with the induced topology.

4.6. Group schemes

4.6.1. — Let $\mathbf{C}$ be a category which admits finite products and a terminal object p.

By Yoneda's lemma (proposition 2.4.4), the datum of a morphism $m\colon G \times G \to G$ is equivalent to the data of functorial maps $m_A\colon \mathbf{C}(A, G) \times \mathbf{C}(A, G) \to \mathbf{C}(A, G)$, that is, such that $m_B(f, g) \circ \varphi = m_A(f \circ \varphi, g \circ \varphi)$ for every pair (A, B) of objects of $\mathbf{C}$, every morphism $\varphi \in \mathbf{C}(A, B)$ and every pair (f, g) in $\mathbf{C}(B, G)$.

A *group object* in the category $\mathbf{C}$ is an object G of $\mathbf{C}$ endowed with a morphism $m\colon G \times G \to G$ such that for every object A of $\mathbf{C}$, the map m_A is a group law on the set $\mathbf{C}(A, G)$. Let m be such a morphism.

The associativity of the group laws m_A means that $m_A \circ (m_A \times \mathrm{id}_{\mathbf{C}(A,G)}) = m_A \circ (\mathrm{id}_{\mathbf{C}(A,G)} \times m_A)$ for every object A of $\mathbf{C}$. Applying again Yoneda's lemma, it thus translates into the equality

$$m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m) \tag{4.6.1.1}$$

of morphisms from $G \times G \times G$ to G.

Let $e_A \in \mathbf{C}(A, G)$ be the unit element of the group law m_A and let $i_A\colon \mathbf{C}(A, G) \to \mathbf{C}(A, G)$ be its inversion. For every morphism $\varphi\colon A \to B$, the map $\mathbf{C}(B, G) \to \mathbf{C}(A, G)$ deduced from φ is a morphism of groups. In particular, it maps e_B to e_A; in other words, $e_B \circ \varphi = e_A$. Similarly, for every $f \in \mathbf{C}(B, G)$, one has $i_B(f) \circ \varphi = i_A(f \circ \varphi)$. Consequently, the family of maps (i_A) is a morphism of functors from h_G to itself. By Yoneda's lemma, there exists a unique morphism $i\colon G \to G$ such that $i_A(f) = i \circ f$ for every object A of $\mathbf{C}$ and every $f \in \mathbf{C}(A, G)$. Concretely, one has $i = i_G(\mathrm{id}_G)$.

The fact that for every object A, the map i_A is the inversion of $\mathbf{C}(A, G)$ is equivalent to the relation

$$m \circ (i \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times i) = e_G = e \circ t_G \tag{4.6.1.2}$$

in $\mathbf{C}(G, G)$.

Similarly, the formula $e_B \circ f = e_A$ means that the assignment $A \mapsto e_A$ is a morphism of functors from the functor h_p (such that $h_p(A)$ is a set with one element, for every object A) to the functor h_G. Consequently, there exists a unique morphism $e: p \to G$ such that $e_A = e \circ t_A$ for every object A of $\mathcal{C}$, where $t_A: A \to p$ is the unique morphism to the terminal object p. Similarly, the fact that e_A is the neutral element of $\mathcal{C}(A, G)$, for every object A, translates into the formula

(4.6.1.3)
$$m \circ (\mathrm{id}_G \times e) = m \circ (e \times \mathrm{id}_G) = \mathrm{id}_G .$$

Conversely, if G is an object of $\mathcal{C}$, endowed with three morphisms $m: G \times G \to G$, $e: p \to G$ and $i: G \to G$ satisfying the relations (4.6.1.1), (4.6.1.3) and (4.6.1.2), then it is a group object in $\mathcal{C}$.

Furthermore, the group laws m_A are commutative if and only if one has

(4.6.1.4)
$$m \circ s = m,$$

where $s: G \times G \to G \times G$ is the unique morphism such that $p_1 \circ s = p_2$ and $p_2 \circ s = p_1$. One then says that this group object is commutative.

Definition **(4.6.2)**. — *Let* S *be a scheme. A (commutative)* S*-group scheme is a (commutative) group object in the category* $\mathbf{Sch}_S$ *of* S*-schemes.*

4.6.3. The additive group. — Let $\mathbf{G}_a = \mathrm{Spec}(\mathbf{Z}[T])$. For every scheme A, one has

$$\mathrm{Hom}(A, \mathbf{G}_a) = \mathrm{Hom}(\mathbf{Z}[T], \mathscr{O}_A(A)) = \mathscr{O}_A(A),$$

and this set is naturally an additive group, functorially in A. It thus defines a commutative group scheme $(\mathbf{G}_a, m)$ The morphism $m: \mathbf{G}_a \times_S \mathbf{G}_a \to \mathbf{G}_a$ corresponds to the morphism of rings $\mathbf{Z}[T] \to \mathbf{Z}[T] \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ given by $T \mapsto 1 \otimes T + T \otimes 1$.

4.6.4. The multiplicative group. — Let $\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[T, 1/T])$ be the open subscheme $D(T)$ of $\mathbf{G}_a$. For every scheme A, one has

$$\mathrm{Hom}(A, \mathbf{G}_m) = \mathrm{Hom}(\mathbf{Z}[T, 1/T], \mathscr{O}_A(A)) = \mathscr{O}_A(A)^\times.$$

Again, this set is naturally a group for multiplication, functorially in A, so that $\mathbf{G}_m$ is a commutative group scheme. Its multiplication $m: \mathbf{G}_m \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{G}_m \to \mathbf{G}_m$ corresponds to the unique morphism of rings $\mathbf{Z}[T, 1/T] \to \mathbf{Z}[T, 1/T] \otimes_{\mathbf{Z}} \mathbf{Z}[T, 1/T]$ given by $T \mapsto T \otimes T$.

4.6.5. The general linear group. — Let n be an integer and let $\Delta \in \mathbf{Z}[(\mathrm{T}_{i,j})_{1\leqslant i,j\leqslant n}]$ be the determinant polynomial; let G be the open subset $\mathrm{D}(\Delta)$ in $\mathrm{Spec}(\mathbf{Z}[(\mathrm{T}_{i,j})_{1\leqslant i,j\leqslant n}])$. In particular, G is affine, and one has

$$\mathscr{O}_{\mathrm{G}}(\mathrm{G}) = \mathbf{Z}[(\mathrm{T}_{i,j}), 1/\Delta].$$

For every scheme A, $\mathrm{Hom}(\mathrm{A}, \mathrm{G})$ is the set of matrices M with coefficients in the ring $\mathscr{O}_{\mathrm{A}}(\mathrm{A})$ such that $\det(\mathrm{M})$ is invertible. It thus identifies with the group $\mathrm{GL}(n, \mathscr{O}_{\mathrm{A}}(\mathrm{A}))$. When A varies, the group laws on these groups endows the scheme G with a structure of an S-group scheme (non-commutative if $n \geqslant 2$). The morphism $m\colon \mathrm{G}\times\mathrm{G} \to \mathrm{G}$ corresponds to the morphism

$$\mathbf{Z}[(\mathrm{T}_{i,j}), 1/\Delta] \to \mathbf{Z}[(\mathrm{T}_{i,j}), 1/\Delta] \otimes \mathbf{Z}[(\mathrm{T}_{i,j}), 1/\Delta]$$

given by

$$\mathrm{T}_{i,j} \mapsto \sum_{k=1}^{n} \mathrm{T}_{i,k} \otimes \mathrm{T}_{k,j}.$$

4.6.6. — Let G and H be two group schemes over S. A morphism of group schemes $\varphi\colon \mathrm{G} \to \mathrm{H}$ is a morphism of S-schemes such that for every S-scheme T, the map $\varphi_{\mathrm{T}}\colon \mathrm{G}(\mathrm{T}) \to \mathrm{H}(\mathrm{T})$ is a morphism of groups. Note that by the Yoneda lemma, a functorial family (φ_{T}) of morphism of groups comes from a unique morphism of S-schemes, hence from a unique morphism of group schemes.

Equivalently, a morphism of schemes $\varphi\colon \mathrm{G} \to \mathrm{H}$ is a morphism of group schemes if one has $m_{\mathrm{H}}\circ(\varphi,\varphi) = \varphi\circ m_{\mathrm{G}}$, where $m_{\mathrm{G}}\colon \mathrm{G}\times_{\mathrm{S}}\mathrm{G} \to \mathrm{G}$ and $m_{\mathrm{H}}\colon \mathrm{H}\times_{\mathrm{S}}\mathrm{H} \to \mathrm{H}$ are the group laws.

For example, there is a unique morphism of group schemes $\det\colon \mathrm{GL}(n) \to \mathbf{G}_{\mathrm{m}}$ such that $\det_{\mathrm{T}}$ is the determinant morphism from $\mathrm{GL}(n, \mathscr{O}_{\mathrm{T}}(\mathrm{T}))$ to $\mathbf{G}_{\mathrm{m}}(\mathrm{T}) = \mathscr{O}_{\mathrm{T}}(\mathrm{T})^{\times}$.

4.6.7. — Let G and H be two group S-schemes. Then the product $\mathrm{G}\times_{\mathrm{S}}\mathrm{H}$ has a unique structure of group scheme such that the canonical projections from $\mathrm{G}\times_{\mathrm{S}}\mathrm{H}$ to G and H are morphisms of group schemes.

4.7. Coherent and quasi-coherent modules on schemes

Definition **(4.7.1)**. — *Let* $(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ *be a locally ringed space and let* $\mathscr{M}$ *be an* $\mathscr{O}_{\mathrm{X}}$-*module. One says that* $\mathscr{M}$ *is* quasi-coherent *if every point* $x \in \mathrm{X}$ *has an open* *neighborhood* U *such that* $\mathscr{M}|_U$ *is isomorphic to the cokernel of a morphism of* $\mathscr{O}_X|_U$*-modules of the form*

$$\mathscr{O}_U^{(J)} \to \mathscr{O}_U^{(I)}.$$

Theorem **(4.7.2)**. — *Let* A *be a ring, let* X *be the affine scheme* Spec(A) *and let* $\mathscr{M}$ *be an* $\mathscr{O}_X$*-module on* X*. The following properties are equivalent:*

(i) *The* $\mathscr{O}_X$*-module* $\mathscr{M}$ *is quasi-coherent;*

(ii) *For every* $f \in A$*, the canonical morphism* $\mathscr{M}(X)_f \to \mathscr{M}(D(f))$ *is an isomorphism of* A_f*-modules.*

(iii) *There exists an* A*-module* M *such that* $\mathscr{M}$ *is isomorphic to* $\widetilde{M}$*;*

Observe that property (ii) is the conjonction of two properties:

(ii′) For every $f \in A$ and every section $s \in \mathscr{M}(D(f))$, there exists a section $s' \in \mathscr{M}(X)$ and an integer $n \geqslant 0$ such that $s^n f = s'|_{D(f)}$;

(ii″) For every $f \in A$ and every section $s \in \mathscr{M}(X)$ such that $s|_{D(f)} = 0$, there exists an integer $n \geqslant 0$ such that $s^n f = 0$.

Proof. — Let us assume that (ii) holds and let $M = \mathscr{M}(X)$. Let us consider the canonical morphism of sheaves $\theta\colon \widetilde{M} \to \mathscr{M}$. By the definition of the $\mathscr{O}_X$-module $\widetilde{M}$, $\theta(U)$ is an isomorphism whenever U is an open subset of X of the form D(f). Since these subsets form a basis of open subsets of X, this implies that θ is an isomorphism. We thus have shown the implication (ii)⇒(iii).

Assume that $\mathscr{M} = \widetilde{M}$ and let us show that it is a quasi-coherent $\widetilde{A}$-module. Let $p\colon A^{(I)} \to M$ be a surjective morphism of A-modules, and let $\varphi\colon A^{(J)} \to A^{(I)}$ be a morphism of A-modules such that $\operatorname{Im}(\varphi) = \operatorname{Ker}(p)$, so that $M \simeq \operatorname{Coker}(p)$. Since the functor $M \mapsto \widetilde{M}$ commutes with all colimits and with all finite limits (corollary 4.1.10, see also example 4.1.11), the $\widetilde{A}$-module $\widetilde{M}$ is a cokernel of the morphism $\widetilde{\varphi}\colon \mathscr{O}_X^{(J)} \to \mathscr{O}_X^{(I)}$ of $\mathscr{O}_X$-modules. This proves that (iii)⇒(i).

Finally, let $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_X$-module on X. Let (U_λ) be a family of open subsets of X such that $X = \bigcup_{\lambda\in L} U_\lambda$ and such $\mathscr{M}|_{U_\lambda}$ is isomorphic to a cokernel of a morphism of $\mathscr{O}_{U_\lambda}$-modules, say $\varphi_\lambda\colon \mathscr{O}_{U_\lambda}^{(I_\lambda)} \to \mathscr{O}_{U_\lambda}^{(J_\lambda)}$. We may assume that U_λ is a distinguished open subset of the form $D(f_\lambda)$, for some $f_\lambda \in A$. By corollary 4.1.10, there exists an A_{f_λ}-module M_λ and an isomorphism $\widetilde{M_\lambda} \simeq \mathscr{M}|_{D(f_\lambda)}$ of $\mathscr{O}_{D(f_\lambda)}$-modules.

Since Spec(A) is quasi-compact, there exists a finite subset L′ of L such that $\bigcup_{\lambda\in L'} D(f_\lambda) = X$; we may thus assume that the set L is finite.

Let $f \in A$ and let $s \in \mathscr{M}(X)$ be such that $s|_{D(f)} = 0$. For every $\lambda \in L$, consider the section $s_\lambda = s|_{D(f_\lambda)}$. Since $s_\lambda|_{D(f)} = 0$, there exists an integer $n_\lambda \geqslant 0$ such that $f^{n_\lambda} s|_{D(f_\lambda)} = 0$. Let $n = \sup_{\lambda \in L}(n_\lambda)$. One has $f^n s|_{D(f_\lambda)} = 0$ for every $\lambda \in L$, hence $f^n s = 0$.

Let $f \in A$ and let $s \in \mathscr{M}(D(f))$. For every $\lambda \in L$, consider the section $s|_{D(ff_\lambda)}$ of the sheaf $\mathscr{M}|_{D(f_\lambda)}$ on its distinguished open subset $D(f_\lambda f) = D(f) \cap D(f_\lambda)$. There exists a section $s'_\lambda \in \mathscr{M}(D(f_\lambda))$ and an integer $n_\lambda \geqslant 0$ such that $f^{n_\lambda} s|_{D(ff_\lambda)} = s'_\lambda|_{D(ff_\lambda)}$. Let $n = \sup_{\lambda \in L}(n_\lambda)$; let us replace n_λ by n and s'_λ by $f^{n-n_\lambda} s'_\lambda$, we assume that $f^n s$ and s_λ coincide on $D(ff_\lambda)$. As a consequence, for $\lambda, \mu \in L$, the sections s_λ and s_μ coincide on $D(ff_\lambda f_\mu)$. This implies that there exists an integer $m(\lambda, \mu)$ such that $f^{m(\lambda,\mu)}(s_\lambda - s_\mu) = 0$. Let $m = \sup(m(\lambda, \mu))$; replace n by $n + m$ and s_λ by $f^m s_\lambda$. Then has $f^n s$ and s_λ coincide on $D(ff_\lambda)$; s_λ and s_μ coincide on $D(f_\lambda f_\mu)$. Consequently, there exists a unique section $s' \in \mathscr{M}(X)$ such that $s'|_{D(f_\lambda)} = s_\lambda$ for every $\lambda \in L$. Since $s'|_{D(ff_\lambda)} = f^n s|_{D(f_\lambda)}$ for every λ, this implies that $s'|_{D(f)} = f^n s$. We thus have proved the implication (i)⇒(ii). □

Corollary **(4.7.3)**. — [1] *Let* X *be scheme, let* $\mathscr{M}$ *be a quasi-coherent sheaf on* X *and let* $f \in \Gamma(X, \mathscr{O}_X)$.

The restriction morphism $\Gamma(X, \mathscr{M}) \to \Gamma(X_f, \mathscr{M})$ *of* $\Gamma(X, \mathscr{O}_X)$*-modules extends uniquely to a morphism* $\varphi \colon \Gamma(X, \mathscr{M})_f \to \Gamma(X_f, \mathscr{M})$.

a) *If* X *is quasi-compact, then* φ *is injective.*

b) *If* X *is quasi-compact and quasi-separated, then* φ *is bijective.*

Corollary **(4.7.4)**. — *Let* X *be a scheme. A colimit of a diagram of quasi-coherent* $\mathscr{O}_X$*-modules is quasi-coherent, a limit of a finite diagram of quasi-coherent* $\mathscr{O}_X$*-modules is quasi-coherent. In particular, for every morphism* $\varphi \colon \mathscr{M} \to \mathscr{N}$ *of quasi-coherent* $\mathscr{O}_X$*-modules, the* $\mathscr{O}_X$*-modules* Ker(φ), Im(φ) *and* Coker(φ) *are quasi-coherent.*

Proof. — Let U be an affine open subscheme of X, say U = Spec(A) for some ring A. Let $\mathscr{D}$ be a diagram of quasi-coherent $\mathscr{O}_X$-modules and let $\mathscr{M}$ be its colimit. By passing to the A-modules of sections on U, the diagram $\mathscr{D}$ furnishes a diagram $\mathscr{D}(U)$ of A-modules. Moreover, if M is the colimit of this diagram, then

[1] To be proved. In fact, extract this corollary from the proof of the theorem and reorganize the beginning of this section…

$\widetilde{M}$ is the colimit of the diagram $\mathscr{D}|_U$ of $\mathscr{O}_U$-modules. This implies that $\mathscr{M}|_U$ is isomorphic to $\widetilde{M}$, hence is quasi-coherent. Consequently, $\mathscr{M}$ is quasi-coherent.

In particular, the cokernel of a morphism of quasi-coherent $\mathscr{O}_X$-modules is quasi-coherent.

If $\mathscr{D}$ is finite, the same argument shows that a limit of $\mathscr{D}$ is quasi-coherent. This implies that the kernel of a morphism $\varphi\colon \mathscr{M} \to \mathscr{N}$ of quasi-coherent $\mathscr{O}_X$-modules is quasi-coherent. Since the image of φ is isomorphic to the kernel of the canonical morphism from $\mathscr{N}$ to $\operatorname{Coker}(\varphi)$, it is a quasi-coherent $\mathscr{O}_X$-module as well. □

Corollary **(4.7.5)**. — a) *Let* A *be a ring and let* $X = \operatorname{Spec}(A)$*; let* M *and* N *be* A-*modules. There exists a unique isomorphism of* $\mathscr{O}_X$-*modules* $\varphi\colon \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \to \widetilde{M \otimes_A N}$ *such that* $\varphi(X)$ *induces the canonical homomorphism* $\widetilde{M}(X) \otimes_{\widetilde{A}(X)} \widetilde{N}(X) \to M \otimes_A N$.

b) *The tensor product of two quasi-coherent* $\mathscr{O}_X$-*modules on a scheme is quasi-coherent.*

Proof. — a) Recall that the sheaf $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is the sheaf associated with the presheaf on $\operatorname{Spec}(A)$ given by

$$U \mapsto \widetilde{M}(U) \otimes_{\widetilde{A}(U)} \widetilde{N}(U).$$

The canonical morphism $S(U)^{-1}M \to \widetilde{M}(U)$ In particular, For every open subset U of X, let $\varphi_U\colon M \otimes_A N \to \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}(U)$ be the morphism of A-modules induced by the bilinear map $(m, n) \mapsto m|_U \otimes n|_U$. It induces a morphism

b) Let X be a scheme, let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent $\mathscr{O}_X$-modules. Let U be an open subset of X such that $(U, \mathscr{O}_X|_U)$ is an affine scheme, isomorphic to the spectrum of a ring A. Then the restriction to U of the sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the sheaf $\mathscr{F}|_U \otimes_{\mathscr{O}_X|_U} \mathscr{G}|_U$. Let M and N be A-modules such that $\mathscr{F}|_U$ and $\mathscr{G}|_U$ are equal with $\widetilde{M}$ and $\widetilde{N}$ respectively. By part *a*), the sheaf $\mathscr{F}|_U \otimes_{\mathscr{O}_X|_U} \mathscr{G}|_U$ is associated with the A-module $M \otimes_A N$. It is thus quasi-coherent. By definition of a scheme, every point of X has a neighborhood U which is an affine scheme. This proves that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module.

□

Corollary **(4.7.6)**. — *Let* A *and* B *be rings, let* $X = \operatorname{Spec}(A)$, *let* $Y = \operatorname{Spec}(B)$*; let* $\varphi\colon A \to B$ *be a morphism of rings and let* $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *be the associated* *morphism of schemes. For every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$, $f_*\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module.*

Note that if a priori, $f_*\mathscr{M}$ is a $f_*\mathscr{O}_Y$-module, we can use the canonical morphism $f^\sharp\colon \mathscr{O}_X \to f_*\mathscr{O}_Y$ to view it as an $\mathscr{O}_X$-module.

Proof. — Let M be the B-module $\mathscr{M}(Y)$; since $\mathscr{M}$ is quasi-coherent, we may assume that $\mathscr{M} = \widetilde{M}$. One has $f_*\mathscr{M}(X) = \mathscr{M}(f^{-1}(X)) = M$, where we view M as an A-module via the morphism φ.

Let then $a \in A$. By definition, one has

$$f_*\mathscr{M}(D(a)) = \mathscr{M}(f^{-1}(D(a))) = \mathscr{M}(D(\varphi(a))) = M_{\varphi(a)},$$

so that the canonical morphism $(f_*\mathscr{M})(X)_a \to (f_*\mathscr{M})(D(\varphi(a)))$ identifies with the tautological isomorphism from M_a to $M_{\varphi(a)}$. This implies that $f_*\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module. □

Corollary **(4.7.7)**. — *Let $f\colon Y \to X$ be a morphism of schemes and let $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the $\mathscr{O}_Y$-module $f^*\mathscr{M}$ is quasi-coherent.*

Moreover, for every affine open subscheme V of Y and every affine open subscheme U of X such that $f(V) \subseteq U$, the canonical homomorphism $\mathscr{O}_Y(V) \otimes_{\mathscr{O}_X(U)} \mathscr{M}(U) \to f^*\mathscr{M}(V)$ is an isomorphism.

Proof. — Let $y \in Y$ and let $x = f(y)$, let U be an open neighborhood of x such that $\mathscr{M}|_U$ is isomorphic to the cokernel of a morphism $\varphi\colon \mathscr{O}_X|_U^{(J)} \to \mathscr{O}_X|_U^{(I)}$. Let $V = f^{-1}(U)$. Since the functor f^* is right exact and commutes with direct sums, the $\mathscr{O}_Y|_V$-module $f^*\mathscr{M}|_V$ is isomorphic to the cokernel of the morphism $f^*\varphi\colon \mathscr{O}_Y|_V^{(J)} \to \mathscr{O}_Y|_V^{(I)}$ deduced from φ. This proves that $f^*\mathscr{M}|_V$ is quasi-coherent.

Let now U and V be affine open subschemes of X and Y respectively such that $f(V) \subseteq U$. Let $A = \mathscr{O}_X(U)$ and $B = \mathscr{O}_Y(V)$; let $M = \mathscr{M}(U)$, so that one can identify $\mathscr{M}|_U$ with $\widetilde{M}$. Moreover, one has $(f^*\mathscr{M})|_V = g^*(\mathscr{M}|_U)$, where $g\colon V \to U$ is the morphism of schemes deduced from f by restriction. Let $\varphi\colon A^{(J)} \to A^{(I)}$ be a morphism of A-modules such that $M = \operatorname{Coker}(\varphi)$. By what precedes, the $\mathscr{O}_V$-module $\mathscr{N}|_V = g^*\mathscr{M}|_U$ is the cokernel of the morphism $g^*\varphi\colon \mathscr{O}_V^{(J)} \to \mathscr{O}_V^{(I)}$. Since V is affine, $\mathscr{N}(V)$ is the cokernel of the morphism $g^*\varphi(V)\colon B^{(J)} \to B^{(I)}$ deduced from φ by base-change to B. It is thus isomorphic to $B \otimes_A M$, as claimed. □

4.7.8. — Let X be a ringed space. Let $\mathscr{M}$ be an $\mathscr{O}_X$-module. Let $(s_i)_{i\in I}$ be a family of global sections of $\mathscr{M}$.

Let $\mathscr{O}_X^{(I)}$ be the direct sum of copies of $\mathscr{O}_X$, indexed by I; for $i \in I$, denote by $j_i\colon \mathscr{O}_X \to \mathscr{O}_X^{(I)}$ the canonical injection with index i. For every $i \in I$, there exists a unique morphism of $\mathscr{O}_X$-modules, $\varphi_i\colon \mathscr{O}_X \to \mathscr{M}$, such that $\varphi_i(X)(1) = s_i$. Consequently, there exists a unique morphism of $\mathscr{O}_X$-modules $\varphi\colon \mathscr{O}_X^{(I)} \to \mathscr{M}$ such that $\varphi \circ j_i = \varphi_i$ for every $i \in I$. It is in fact the unique morphism of $\mathscr{O}_X$-modules such that $\varphi(X) \circ j_i(X)(1) = s_i$.

By construction, the $\mathscr{O}_X$-module $\mathscr{O}_X^{(I)}$ can be identified with the submodule of $\mathscr{O}_X^I$ whose sections over an open subset U consist of families $(f_i)_{i\in I}$ of elements of $\mathscr{O}_X(U)$ such that for every point $x \in U$, there exists an open neighborhood V of x in U such that $f_i|_V = 0$ for all but finitely many $i \in I$. Consequently, the morphism φ is given by $\varphi(U)((f_i)) = \sum_{i\in I} f_i s_i|_U$ for every open subset U of X and every section $(f_i)_{i\in I} \in \mathscr{O}_X^{(I)}(U)$; the sum looks infinite but is locally finite.

One says that the family $(s_i)_{i\in I}$ generates $\mathscr{M}$ (resp. is a *frame* of $\mathscr{M}$) if this morphism φ is an epimorphism (resp. an isomorphism). If such a family exists, then one says that $\mathscr{M}$ is *globally generated* (resp. is *free*).

Definition (**4.7.9**). — *Let* X *be a ringed space and let* $\mathscr{M}$ *be an* $\mathscr{O}_X$*-module.*

a) *One says that* $\mathscr{M}$ *is* locally free *(resp.* invertible*) if every point* $x \in X$ *has a neighborhood* U *such that* $\mathscr{M}|_U$ *is a free* $\mathscr{O}_U$*-module (resp. is isomorphic to* $\mathscr{O}_U$*).*

b) *One says that* $\mathscr{M}$ *is of* finitely generated *(or of* finite type*) if every point of* X *has a neighborhood* U *such that* $\mathscr{M}|_U$ *is generated by a finite family of global sections.*

c) *One says that* $\mathscr{M}$ *is of* finitely presented *(or of* finite presentation*) if every point of* X *has a neighborhood* U *such that* $\mathscr{M}|_U$ *is isomorphic to the cokernel of a morphism* $p\colon \mathscr{O}_U^J \to \mathscr{O}_U^I$*, where* I *and* J *are finite sets.*

We notice that if U is an open neighborhood of x satisfying each of the given conditions, then any open subset contained in V satisfies them as well.

The definition of a finitely presented $\mathscr{O}_X$-module is the same as that of a quasi-coherent $\mathscr{O}_X$-module, the only difference being on the requirement that I and J be finite sets. In particular, *a finitely presented* $\mathscr{O}_X$*-module is quasi-coherent.* Moreover, condition *c*) can also be rephrased by saying that $\mathscr{M}|_U$ is globally generated by some finite family (s_i) of sections of $\mathscr{M}(U)$, and that the kernel of the associated morphism $\varphi\colon \mathscr{O}_X^I \to \mathscr{M}$ is itself generated by a finite family of global sections.

Proposition (**4.7.10**). — *Let* X *be a ringed space and let* $0 \to \mathscr{M} \xrightarrow{k} \mathscr{N} \xrightarrow{p} \mathscr{P} \to 0$ *be an exact sequence of* $\mathscr{O}_X$*-modules.*

a) *If* $\mathscr{N}$ *is finitely generated, then* $\mathscr{P}$ *is finitely generated.*

b) *If* $\mathscr{M}$ *and* $\mathscr{P}$ *are finitely generated, then* $\mathscr{N}$ *is finitely generated.*

c) *If* $\mathscr{P}$ *is finitely presented and* $\mathscr{N}$ *is finitely generated, then* $\mathscr{M}$ *is finitely generated.*

Proof. — a) Let $x \in X$. Let U be an open neighborhood of x and let $(s_i)_{i\in I}$ be a finite family of sections of $\mathscr{N}(U)$ such that the morphism $\varphi\colon \mathscr{O}_X|_U^I \to \mathscr{N}|_U$ given by $(f_i) \mapsto \sum f_i s_i$ is an epimorphism. Then the morphism $p \circ \varphi\colon \mathscr{O}_X|_U^I \to \mathscr{P}_U$ is an epimorphism as well, which implies that $\mathscr{P}$ is finitely generated.

b) Let $x \in X$, let U be an open neighborhood of x, small enough so that there exists a finite family $(s_i)_{i\in I}$ of sections of $\mathscr{P}(U)$ which generates $\mathscr{P}|_U$, and a finite family $(t_j)_{j\in J}$ of sections of $\mathscr{M}(U)$ which generates $\mathscr{M}|_U$. Since the morphism p is surjective, there exists for each $i \in I$ an open neighborhood U_i of x in U and a section $s'_i \in \mathscr{N}(U_i)$ such that $p(U_i)(s'_i) = s_i|_{U_i}$. Replacing U by the open neighborhood $\bigcap_{i\in I} U_i$ of x, the sections s'_i and t_j by their restrictions, we assume that $s_i = p(U)(s'_i)$ for every i. Let us then prove that $\mathscr{N}|_U$ is generated by the union of the families $(k(t_j))_{j\in J}$ and $(s'_i)_{i\in I}$. Let indeed V be an open subset of U and let $s \in \mathscr{N}(V)$. Let $y \in V$. By assumption, there exists an open neighborhood V' of y in V and elements $(f_i)_{i\in I}$ of $\mathscr{O}_X(V')$ such that $p(V)(s)|_{V'} = \sum f_i s_i|_{V'}$. Let $t = s|_{V'} - \sum f_i s_i|_{V'}$; by construction, $p(V')(t) = 0$, so that t belongs to $\ker(p)(V')$. Consequently, there exists an open neighborhood V'' of y in V' and elements $(g_j)_{j\in J}$ of $\mathscr{O}_X(V'')$ such that $t|_{V''} = \sum g_j k(t_j)|_{V''}$. Then $s|_{V''} = \sum g_j k(t_j)|_{V''} + \sum f_i|_{V''} s'_i|_{V''}$, which concludes the proof that the the union of the families $(k(t_j))_{j\in J}$ and $(s'_i)_{i\in I}$ generates $\mathscr{N}|_U$.

c) Let $x \in X$. Let us choose an open neighborhood U of x and a presentation $\mathscr{O}_U^m \xrightarrow{\psi} \mathscr{O}_U^n \xrightarrow{\varphi} \mathscr{P}|_U$. There exists an open neighborhood U of x and a morphism $u\colon \mathscr{O}_U^n \to \mathscr{N}|_U$ such that $p \circ u = \varphi$. Then $p \circ u \circ \psi = \varphi \circ \psi = 0$; it follows that there exists a unique morphism $v\colon \mathscr{O}_U^m \to \mathscr{M}|_U$ such that $u \circ \psi = k \circ v$.

Let us now observe that the canonical morphism $\overline{k}\colon \operatorname{Coker}(v) \to \operatorname{Coker}(u)$ deduced from k is an isomorphism. While this can be proved by a variant of the snake lemma in the category of abelian sheaves, let us do it by hand. Since the stalk of the cokernel is the cokernel of the morphism induced between stalks, ($\mathrm{Coker}(v)_y = \mathrm{Coker}(v_y)$, and similarly for u), it suffices to prove that for every point $y \in \mathrm{U}$, the induced morphism $\overline{k}_y \colon \mathrm{Coker}(v_y) \to \mathrm{Coker}(u_y)$ is an isomorphism. Let thus $m \in \mathscr{M}_y$ be such that $\overline{k}_y(\overline{m}) = 0$, where $\overline{m}$ is the image of m in $\mathrm{Coker}(v_y)$; let then $s \in \mathscr{O}_{\mathrm{X},y}^n$ by such that $k_y(m) = u_y(s)$; one has $\varphi_y(s) = p_y \circ u_y(s) = p_y \circ k_y(m) = 0$, hence there exists $t \in \mathscr{O}_{\mathrm{X},y}^m$ such that $s = \psi(t)$; this implies that $k_y(m) = u_y \circ \psi_y(t) = k_y \circ v_y(t)$; since k_y is injective, one thus has $m = v_y(t)$, hence $\overline{m} = 0$; this shows that $\overline{k}_y$ is injective. Let then $n \in \mathscr{N}_y$; since φ_y is surjective, there exists $s \in \mathscr{O}_{\mathrm{X},y}^n$ such that $p_y(n) = \varphi_y(s)$; then $p_y(n - u_y(s)) = 0$, so that there exists $m \in \mathscr{M}_y$ such that $n = u_y(s) + k_y(m)$; one then has $\overline{n} = \overline{k}_y(\overline{m})$ in $\mathrm{Coker}(u_y)$, which shows that $\overline{k}_y$ is surjective.

Since $\mathscr{N}|_{\mathrm{U}}$ is finitely generated, so is $\mathrm{Coker}(u)$, which proves that $\mathrm{Coker}(v)$ is finitely generated. Applying assertion *a*) to the exact sequence $0 \to \mathscr{O}_{\mathrm{U}}^m \xrightarrow{v} \mathscr{M}|_{\mathrm{U}} \to \mathrm{Coker}(v) \to 0$, we conclude that $\mathscr{M}|_{\mathrm{U}}$ is finitely generated, as claimed. □

Proposition **(4.7.11)**. — *Let* A *be a ring, let* X = Spec(A), *let* M *be an* A*-module. The* $\mathscr{O}_{\mathrm{X}}$*-module* $\widetilde{\mathrm{M}}$ *is finitely generated (resp. finitely presented) if and only if the* A*-module* M *is finitely generated (resp. finitely presented).*

Proof. — Let us assume that the A-module M is finitely generated, let $(s_i)_{i\in\mathrm{I}}$ be a finite generating family of elements of M, and let $\varphi \colon \mathscr{O}_{\mathrm{X}}^{\mathrm{I}} \to \widetilde{\mathrm{M}}$ be the associated morphism. For every $f \in \mathrm{A}$, the morphism $\varphi(\mathrm{D}(f))$ identifies with the morphism from $\mathrm{A}_f^{\mathrm{I}}$ to M_f deduced from the morphism $\varphi(\mathrm{X})$ by passing to the modules of fractions; it is thus surjective. Since the open subsets of X of the form $\mathrm{D}(f)$ constitute a basis of open subsets of X, this implies that $\mathrm{Im}(\varphi) = \widetilde{\mathrm{M}}$, hence φ is an epimorphism. Consequently, $\widetilde{\mathrm{M}}$ is a finitely generated $\mathscr{O}_{\mathrm{X}}$-module.

Assume now that M is a finitely presented A-module. If (s_i) is as above, then the kernel K of the canonical morphism from A^{I} to M is finitely generated. By what precedes, $\widetilde{\mathrm{K}}$ is a finitely generated $\mathscr{O}_{\mathrm{X}}$-module. Since $\widetilde{\mathrm{M}}$ is the cokernel of the morphism $\widetilde{\mathrm{K}} \to \mathscr{O}_{\mathrm{X}}^{\mathrm{I}}$, we conclude that $\widetilde{\mathrm{M}}$ is finitely presented.

Conversely, let us assume that $\widetilde{\mathrm{M}}$ is a finitely generated $\mathscr{O}_{\mathrm{X}}$-module. Let $(\mathrm{U}_i)_{i\in\mathrm{I}}$ be a family of open subsets of X such that $\mathscr{M}|_{\mathrm{U}_i}$ is generated by finitely many sections and such that $\mathrm{X} = \bigcup_{i\in\mathrm{I}} \mathrm{U}_i$. We may assume that there exists $f_i \in \mathrm{A}$ such that $\mathrm{U}_i = \mathrm{D}(f_i)$. Since Spec(A) is quasi-compact, there exists a finite subset I′ of I such that $\mathrm{X} = \bigcup_{i\in\mathrm{I}'} \mathrm{U}_i$. We thus may assume that I is a finite set.

For every i, let $(s_{i,j})_{j\in J_i}$ be a finite family of elements of $\mathscr{M}(U_i) = M_{f_i}$ which generates $\mathscr{M}|_{U_i}$. For every $i \in I$ and every $j \in J$, there exists an integer $n_{i,j}$ such that $f_i^{n_{i,j}} s_{i,j}$ belongs to the image of M in M_{f_i}, say $f_i^{n_{i,j}} s_{i,j} = m_{i,j}/1$, for some $m_{i,j} \in M$. Let M′ be the submodule of M generated by the family $(m_{i,j})_{\substack{i\in I\\ j\in J_i}}$. For every $i \in I$, one has $M'_{f_i} = M_{f_i}$, so that $(M/M')_{f_i} = 0$. Every global section of the quasi-coherent $\mathscr{O}_X$-module associated with the A-module M/M′ is locally 0, hence is 0; consequently, $M/M' = 0$ and $M' = M$. This shows that M is finitely generated.

Assume now that $\widetilde{M}$ is finitely presented. It is thus finitely generated, so that the A-module M is finitely generated. Let $(s_i)_{i\in I}$ be finite generating family of elements of M and let $\varphi\colon A^I \to M$ be the associated surjective morphism of A-modules and let $K = \operatorname{Ker}(\varphi)$. Let then $\widetilde{\varphi}\colon \mathscr{O}_X^I \to \widetilde{M}$ be the corresponding morphism of $\mathscr{O}_X$-modules; it is surjective and its kernel is $\widetilde{K}$. By prop. 4.7.10, its kernel $\widetilde{K}$ is a finitely generated $\mathscr{O}_X$-module. By what precedes, K is a finitely generated A-module; this shows that M is finitely presented, as claimed. □

Proposition **(4.7.12)**. — *Let* X *be a scheme and let* $\mathscr{M}$ *and* $\mathscr{N}$ *be quasi-coherent* $\mathscr{O}_X$*-modules. If* $\mathscr{M}$ *is finitely presented, then the* $\mathscr{O}_X$*-module* $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ *is quasi-coherent.*

Proof. — It suffices to treat the case where X is affine; let then $A = \mathscr{O}_X(X)$, $M = \mathscr{M}(X)$ and $N = \mathscr{N}(X)$. Let us define a morphism of A-modules $\varphi\colon \operatorname{Hom}_A(M, N) \to \mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$. Let thus $f \in \operatorname{Hom}_A(M, N)$; since $\widetilde{N}(X) = N$, there exists a unique morphism of sheaves $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that $\widetilde{f}(X) = f$. The map $f \mapsto \widetilde{f}$ is a morphism of A-modules from $\operatorname{Hom}_A(M, N)$ to $\operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$. The latter module being the set of global section of the sheaf $\mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$, there exists a unique morphism of sheaves of A-modules

$$\Phi\colon \widetilde{\operatorname{Hom}_A(M, N)} \to \mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$$

such that $\Phi(X)(f) = \widetilde{f}$ for every $f \in \operatorname{Hom}_A(M, N)$.

For every $a \in A$, the morphism $\Phi(D(a))$ is the canonical morphism from $\operatorname{Hom}_A(M, N)_a$ to $\operatorname{Hom}_{A_a}(M_a, N_a)$; by lemma 4.7.13 below, it is an isomorphism.

Since the subsets of Spec(A) of the form $D(a)$ are a basis of open subsets, this implies that the morphism Φ is an isomorphism of sheaves. □

Lemma **(4.7.13)**. — *Let* A *be a ring, let* S *be a multiplicative subset of* A*, let* M *and* N *be* A*-modules. There exists a unique morphism of* A*-modules,*

$$\theta\colon S^{-1}\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N),$$

which, for $u\in\operatorname{Hom}_A(M,N)$*, maps* $u/1$ *to the morphism given by* $m/s\mapsto u(m)/s$*. If* M *is finitely generated, then* θ *is injective; if* M *is finitely presented, then* θ *is an isomorphism.*

Proof. — Let $\theta_1\colon\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N)$ being the map underlying the functor $S\mapsto S^{-1}M$; by definition, $\theta_1(u)(m/s)=u(m)/s$, for $u\in\operatorname{Hom}_A(M,N)$, $m\in M$, and $s\in S$. Since the target of θ_1 consists of an $S^{-1}A$-module, there exists a unique morphism $\theta\colon S^{-1}\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N)$ such that $\theta(u/s)=s^{-1}\theta_1(u)$.

Let us now assume that M is finitely generated and let us show that the morphism θ is injective. Let $(m_1,\dots,m_r)$ be a finite generating family; let $\psi\colon A^r\to M$ be the morphism given by $(a_1,\dots,a_r)\mapsto\sum a_im_i$. Consider an element of $\operatorname{Ker}(\theta)$; let us write it as u/s, where $s\in S$ and $u\in\operatorname{Hom}_A(M,N)$. By assumption, for every $i\in\{1,\dots,r\}$, one has $u(m_i/1)=0$ hence there exists an element $s_i\in S$ such that $s_iu(m_i)=0$. Let $t=s_1\dots s_r$; one has $tu(m_i)=0$ for every i, hence $tu(m)=0$ for every $m\in M$. In other words, $tu=0$; this implies that $u/s=0$.

Let us now assume that M is finitely presented. Let $P=\operatorname{Ker}(\psi)$; it is a finitely generated A-module. Let $v\colon S^{-1}M\to S^{-1}N$ be a morphism of $S^{-1}A$-modules. There exists an element $s\in S$ and a family $(n_1,\dots,n_r)$ of elements of N such that $v(m_i/1)=n_i/s$, for every i. Let $u_1\colon A^r\to N$ be the morphism given by $u_1(a_1,\dots,a_r)=\sum a_in_i$. For every $p=(a_1,\dots,a_r)\in P$, one has $u_1(p)=v(\psi(p)=v(0)=0$ in $S^{-1}N$; since P is finitely generated, there exists an element $t\in S$ such that $tu_1(p)$ for every $p\in P$. Passing to the quotient by P, there exists a morphism $u\colon M\to N$ such that $u\circ\psi=tu_1$. It follows that $u(m)/1=(t/s)v(m/1)$, for every $m\in M$, hence $\theta(u)=(t/s)v$. Finally, $v=\theta(ts^{-1}u)$, which shows that θ is surjective. □

Exercise **(4.7.14)**. — Give examples of a ring A, of a multiplicative subset S of A and of A-modules M and N such that the canonical morphism θ of lemma 4.7.13 is not injective (resp. is injective but not bijective).

Definition (**4.7.15**). — *Let* X *be a ringed space and let* $\mathscr{M}$ *be an* $\mathscr{O}_X$*-module. One says that* $\mathscr{M}$ *is* coherent *if it is of finite type and if, for every open subset* U *of* X*, and every finite family* $(s_i)_{i\in I}$ *of elements of* $\mathscr{M}(U)$*, the kernel of the associated morphism* $\varphi\colon \mathscr{O}_X^I \to \mathscr{M}$ *is of finite type.*

It follows from the definition that a coherent $\mathscr{O}_X$-module is finitely presented; in particular, it is quasi-coherent. Similarly, any finitely generated submodule of a coherent $\mathscr{O}_X$-module is coherent.

Exercise (**4.7.16**). — Let X be a ringed space. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-modules.

a) Assume that $\mathscr{F}$ is finitely generated and $\mathscr{G}$ is coherent. Then $\operatorname{Im}(\varphi)$ is coherent and $\operatorname{Ker}(\varphi)$ is finitely generated.

b) If $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\operatorname{Ker}(\varphi)$ and $\operatorname{Coker}(\varphi)$ are coherent.

c) If, out of $\operatorname{Ker}(\varphi)$, $\mathscr{F}$, and $\operatorname{Im}(\varphi)$, two $\mathscr{O}_X$-modules are coherent, then so is the third one.

Lemma (**4.7.17**). — *Let* A *be a ring and let* $X = \operatorname{Spec}(A)$. *Assume that the scheme* X *is noetherian. Then the following properties hold:*

a) *The ring* A *is noetherian;*

b) *The sheaf of rings* $\mathscr{O}_X$ *is coherent.*

c) *For every* A*-module* M*, the quasi-coherent module* $\widetilde{M}$ *is coherent if and only if* M *is finitely generated.*

Proof. — a) Let J be an ideal of A; let us prove that J is finitely generated. let us prove that the associated sheaf of ideals $\widetilde{J} \subseteq \mathscr{O}_X$ is a finitely generated $\mathscr{O}_X$-module. Let thus $x \in X$ and let U be an affine open neighborhood of x of the form $\operatorname{Spec}(B)$, where B is a noetherian ring. Then $\widetilde{J}(U)$ is an ideal of $\mathscr{O}_X(B) = B$; since B is noetherian, it is finitely generated, so that $\widetilde{J}|_U$ is a finitely generated $\mathscr{O}_U$-module. This implies that the $\mathscr{O}_X$-module $\widetilde{J}$ is finitely generated and it follows from proposition 4.7.11 that the ideal J is finitely generated.

b) The sheaf $\mathscr{O}_X$ is generated by its global section 1, hence it is finitely generated. Let U be an open subset of X, let $(s_i)_{i\in I}$ be a finite family of elements of $\mathscr{O}_X(U)$ and let $\varphi\colon \mathscr{O}_U^I \to \mathscr{O}_X$ be the associated morphism. We need to show that $\operatorname{Ker}(\varphi)$ is finitely generated.

Let $x \in U$ and let $f \in A$ be an element such that $x \in D(f) \subseteq U$. Then $\operatorname{Ker}(\varphi)|_{D(f)}$ is a quasi-coherent $\mathscr{O}_X$-module associated to the kernel K_f of the morphism $\varphi(D(f))$ of A_f-modules given by $(a_i) \mapsto \sum_i a_i s_i|_{D(f)}$. Observe that for every $f \in A$, the ring of fractions A_f is noetherian, because it is generated by $1/f$ as an A-algebra. Consequently, the A_f-module K_f, being a submodule of the finitely generated A_f-module A_f^I, is finitely generated as well. This implies that $\operatorname{Ker}(\varphi)|_{D(f)}$ is finitely generated, and concludes the proof that $\operatorname{Ker}(\varphi)$ is finitely generated.

c) If $\widetilde{M}$ is coherent, it is finitely generated; by prop. 4.7.11, the A-module M is finitely generated. Conversely, let us assume that M is finitely generated, so that there exists an integer $n \geqslant 0$, a submodule P of A^n such that M is isomorphic to the quotient A^n/P. This implies that the $\mathscr{O}_X$-module $\widetilde{M}$ is isomorphic to the quotient of $\mathscr{O}_X^n$ by the finitely generated submodule $\widetilde{P}$. Since $\mathscr{O}_X$ is coherent, assertion *c*) of exercise 4.7.16 implies that $\mathscr{O}_X^n$ is coherent; it then follows from assertion *b*) of that exercise that $\widetilde{M}$ is coherent. □

Theorem **(4.7.18)**. — *Let* X *be a locally noetherian scheme and let* $\mathscr{M}$ *is a quasi-coherent* $\mathscr{O}_X$*-module. The following properties are equivalent:*

(i) *The* $\mathscr{O}_X$*-module* $\mathscr{M}$ *is coherent;*
(ii) *The* $\mathscr{O}_X$*-module* $\mathscr{M}$ *is finitely presented;*
(iii) *The* $\mathscr{O}_X$*-module* $\mathscr{M}$ *is finitely generated.*

Proof. — The implications (i)⇒(ii) and (ii)⇒(iii) have already been discussed. Assuming that $\mathscr{M}$ is finitely generated, it remains to prove that it is coherent. Let $x \in X$ and let U be an affine open neighborhood of X. Since X is locally noetherian, U is isomorphic to the spectrum of a noetherian ring A (one has $A = \mathscr{O}_X(U)$). Since $\mathscr{M}$ is finitely generated, proposition 4.7.11 implies that the A-module $\mathscr{M}(U)$ is finitely generated. Consequently, $\mathscr{M}|_U$ is a coherent $\mathscr{O}_U$-module. This concludes the proof that $\mathscr{M}$ is coherent. □

4.8. Schemes associated with graded algebras

4.8.1. — Let A be a *graded ring.* By definition, there exists a family $(A_n)_{n\in\mathbf{N}}$ of additive subgroups of A such that $A = \bigoplus A_n$ and such that $A_n \cdot A_m \subseteq A_{n+m}$ for every $n, m \in \mathbf{N}$. Elements of A_n are said to be homogeneous of degree n. For $a \in A$, one can write $a = \sum_{n\in\mathbf{N}} a_n$, where $a_n \in A_n$ for every integer n. The element a_n is called the homogeneous component of degree n of a.

If A is a k-algebra and all subgroups A_n are k-submodules, then we say that A is a *graded k-algebra*.

An A-module M is said to be graded if there exists a family $(M_n)_{n\in\mathbf{Z}}$ of additive subgroups of M such that $M = \bigoplus M_n$ and $A_n \cdot M_m \subseteq M_{n+m}$ for every $n \in \mathbf{N}$ and every $m \in \mathbf{Z}$. The homogeneous components of an element of M are defined similarly as those of an element of A.

A submodule N of a graded module M is called to be graded if is equal to the direct sum $\bigoplus_{n\in\mathbf{Z}}(N \cap M_n)$.

4.8.2. — An ideal I of A is said to be homogeneous if it satisfies the equivalent conditions:

(i) The ideal I is generated by homogeneous elements;
(ii) The homogeneous components of every element of I belong to I;
(iii) The ideal I is a graded submodule of A.

The subgroup $A_+ = \bigoplus_{n>0} A_n$ of A is a homogeneous ideal of A, called the *irrelevant ideal.*

Lemma **(4.8.3)**. — *The radical of a homogeneous ideal of* A *is a homogeneous ideal.*

Proof. — Let I be a homogeneous ideal of A. Let $f \in \sqrt{I}$ and let (f_n) be the family of its homogenous components; we need to show that $f_n \in \sqrt{I}$ for every integer $n \geqslant 0$. Otherwise, there exists a largest integer d such that $f_d \notin \sqrt{I}$. Let $f' = \sum_{n\leqslant d} f_n$; by assumption, one has $f - f' \in \sqrt{I}$, hence $f' \in \sqrt{I}$. Let $e \geqslant 0$ be an integer such that $(f')^e \in I$. The homogeneous component of degree de of f' is equal to $(f_d)^e$; since I is a homogeneous ideal, one has $(f_d)^e \in I$, hence $f_d \in \sqrt{I}$. This contradicts the definition of d. □

Lemma **(4.8.4)**. — *Let* A *be a graded ring and let* I *be a homogeneous ideal of* A *which does not contain* A_+. *Assume that for every pair* (a, b) *of homogeneous elements of* A *such that* $a \notin I$ *and* $b \notin I$, *one has* $ab \notin I$. *Then* I *is a prime ideal.*

Proof. — One has $I \neq A$. Let a, b be elements of $A - I$ and let us show that $ab \notin I$. Let (a_n) and (b_n) be their homogeneous components; there exists a largest integer d such that $a_d \notin I$ and a largest integer e such that $a_e \notin I$. Let $a' = \sum_{n\leqslant d} a_n$ and $b' = \sum_{n\leqslant e} b_n$; one has $a - a' \in I$ and $b - b' \in I$, so that

$$c = a'b' = a'(b' - b) + a'b = a'(b' - b) + (a' - a)b + ab \in I$$

On the other hand, the homogeneous components (c_n) of c are given by

$$c_n = \sum_{\substack{p\leqslant d \\ n-p\leqslant e}} a_p b_{n-p};$$

in particular, $c_{d+e} = a_d b_e \notin \mathrm{I}$. Since I is a homogenous ideal, this implies that $c \notin \mathrm{I}$, as was to be shown. □

4.8.5. — Let Proj(A) be the set of homogeneous prime ideals of A which do not contain the irrelevant ideal A_+.

For every subset E of A consisting of homogeneous elements, one defines $\mathrm{V}_+(\mathrm{E})$ as the set of $\mathfrak{p} \in \mathrm{Proj}(\mathrm{A})$ such that $\mathrm{E} \subseteq \mathfrak{p}$, and $\mathrm{D}_+(\mathrm{E}) = \mathrm{Proj}(\mathrm{A}) - \mathrm{V}_+(\mathrm{E})$.

The subsets of Proj(A) of the form $\mathrm{V}_+(\mathrm{E})$ are the closed subsets of a topology on Proj(A), called the Zariski topology. In fact, one has $\mathrm{Proj}(\mathrm{A}) \subseteq \mathrm{Spec}(\mathrm{A})$, and it is the topology induced by the Zariski topology of Spec(A).

The topological space Proj(A) is called the *homogeneous spectrum* of A.

For every subset Z of Proj(A), let $\mathrm{j}_+(\mathrm{Z})$ be the set of all $f \in \mathrm{A}_+$ such that $\mathrm{Z} \subseteq \mathrm{V}_+(f)$. This is a homogeneous ideal of A, contained in A_+, which is equal to its radical.

4.8.6. — Let A be a graded algebra and let M be a graded A-module. Let S be a multiplicative subset of A consisting of homogeneous elements. The A-module $\mathrm{S}^{-1}\mathrm{M}$ inherits of a graduation, such that, for any homgeneous element $m \in \mathrm{M}$ and any $s \in \mathrm{S}$, the degree of a fraction m/s is equal to $\deg(m) - \deg(s)$. Let $\mathrm{M}_{(\mathrm{S})}$ be the submodule of $\mathrm{S}^{-1}\mathrm{M}$ consisting of elements of degree 0, that is to say, of the form m/s, where $m \in \mathrm{M}$ and $s \in \mathrm{S}$ are homogeneous of the *same* degree.

This construction applies in particular when M = A. Then $\mathrm{A}_{(\mathrm{S})}$ is a subring of $\mathrm{S}^{-1}\mathrm{A}$ that contains A_0.

In general, $\mathrm{M}_{(\mathrm{S})}$ is an $\mathrm{A}_{(\mathrm{S})}$-module.

This construction $\mathrm{M} \mapsto \mathrm{M}_{(\mathrm{S})}$ gives rise to a functor on the category of graded A-modules, with respect to the homomorphisms that preserve the degree. This functor is exact. Indeed, if $0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0$ is an exact sequence of graded A-modules, it induces an exact sequence $0 \to \mathrm{M}'_{\mathrm{S}} \to \mathrm{M}_{\mathrm{S}} \to \mathrm{M}''_{\mathrm{S}} \to 0$, by exactness of classical localization. Taking the elements of degree 0, we obtain the exact sequence $0 \to \mathrm{M}'_{(\mathrm{S})} \to \mathrm{M}_{(\mathrm{S})} \to \mathrm{M}''_{(\mathrm{S})} \to 0$.

Example (**4.8.7**). — Let A be a graded ring, let M be a graded A-module and let $f \in \mathrm{A}$ be a homogeneous element of degree 1.

In the quotient module $M/(f-1)M$, the element f acts by identity, so that the A-module $M/(f-1)M$ inherits from a structure of an A_f-module hence. Moreover, there exists a unique morphism of A_f-modules $M_f \to M/(f-1)$ that extends the canonical morphism $M \to M/(f-1)$. By restriction, it induces a morphism of $A_{(f)}$-modules, $\varphi\colon M_{(f)} \to M/(f-1)$.

In the other direction, there exists a unique morphism of abelian groups from M to $M_{(f)}$ that maps an homogeneous element $m \in M_d$ to m/f^d, for every $d \in \mathbf{Z}$. For any homogeneous element $m \in M_d$, it maps $(f-1)m = fm - m$ to $fm/f^{d+1} - m/f^d = 0$ (recall that $fm \in M_{d+1}$). Consequently, there exists a unique morphism of abelian groups $\psi\colon M/(f-1) \to M_{(f)}$ that maps the class of an element $m \in M_d$ to m/f^d. This morphism ψ is $A_{(f)}$-linear.

In the particular case where $M = A$, these are morphism of A_0-algebras.

Let $d \in \mathbf{Z}$ and let $m \in M_d$; then

$$\psi \circ \varphi(m/f^d) = \psi([m]) = m/f^d,$$

so that $\psi \circ \varphi = \mathrm{id}$. Similarly, for $m \in M_d$, one has

$$\varphi \circ \psi([m]) = \varphi(m/f^d) = [m],$$

so that $\varphi \circ \psi = \mathrm{id}$. In particular, the morphisms φ and ψ are isomorphisms.

4.8.8. — Let A be a graded ring, let f be a homogeneous element of A of strictly positive degree, say d. The natural diagram of rings

$$A \longrightarrow A_f \hookleftarrow A_{(f)}$$

gives rise to a commutative diagram of topological spaces

$$\begin{array}{ccccccc}
\mathrm{Proj}(A) & \hookleftarrow & D_+(f) & \dashrightarrow^{\psi_f} & & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Spec}(A) & \hookleftarrow & D(f) & \xrightarrow{\sim} & \mathrm{Spec}(A_f) & \xrightarrow[{}^a j]{} & \mathrm{Spec}(A_{(f)}),
\end{array}$$

in which the dashed arrow represents a continuous map $\psi_f\colon D_+(f) \to \mathrm{Spec}(A_{(f)})$; we will prove that *the map ψ_f is a homeomorphism.*

Concretely, the map ψ_f is defined as follows. Let $\mathfrak{p}$ be a homogeneous prime ideal of A which does not contain f. Then the ideal $\psi_f(\mathfrak{p})$ is the set of elements of $A_{(f)}$ of the form a/f^n, where $a \in \mathfrak{p} \cap A_{nd}$ and $n \in \mathbf{N}$.

It follows from its definition that map ψ_f is continuous. More precisely, let $n \in \mathbf{N}$ and let $g \in A_{nd}$, let us show the two relations

$$\psi_f^{-1}(D(g/f^n)) = D_+(fg) \quad \text{and} \quad \psi_f(D_+(fg)) = \psi_f(D_+(f)) \cap D(g/f^n).$$

If $g \in \mathfrak{p}$, then $g/f^n \in \psi_f(\mathfrak{p})$ by definition of this prime ideal. Conversely, if $g/f^n \in \psi_f(\mathfrak{p})$, there exists an integer $m \geqslant 0$ and $a \in \mathfrak{p} \cap A_{dm}$ such that $g/f^n = a/f^m$; this implies that there exists $p \geqslant 0$ such that $f^{m+p}g = f^{n+p}a$; in particular, $f^{m+p}g \in \mathfrak{p}$, hence $g \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal which does not contain f. Since $D_+(fg) = D_+(f) \cap D_+(g)$, this concludes the proof of the two indicated relations. The first one implies that ψ_f is continuous, and the second one that it induces an open map onto its image.

Let us now show that ψ_f is injective. Let $\mathfrak{q}, \mathfrak{q}' \in D_+(f)$ be such that $\psi_f(\mathfrak{q}) = \psi_f(\mathfrak{q}')$; let us show that $\mathfrak{q} = \mathfrak{q}'$. Let a be a homogeneous element of $\mathfrak{q}$ and let n be its degree; then a^d is an element of degree nd of $\mathfrak{q}$, so that $a^d/f^n \in A_{(f)}$; the definition of $\psi_f(\mathfrak{p})$ shows that $a^d/f^n \in \psi_f(\mathfrak{p})$, hence $a^d/f^n \in \psi_f(\mathfrak{q}')$, hence $a^d \in \mathfrak{q}'$. Since $\mathfrak{q}'$ is a prime ideal of A, one then has $a \in \mathfrak{q}'$. This implies the inclusion $\mathfrak{q} \subseteq \mathfrak{q}'$, and the other follows by symmetry. Consequently, ψ_f is injective.

Let $\mathfrak{q}$ be a prime ideal of $A_{(f)}$. For every integer $n \geqslant 0$, let $\mathfrak{p}_n$ be the set of elements $x \in A_n$ such that $x^d/f^n \in \mathfrak{q}$. Observe that $\mathfrak{p}_n$ is an additive subgroup of A_n. Let indeed $x, y \in A_n$; it follows from Newton's binomial formula that $(x-y)^{2d}/f^{2n} \in \mathfrak{q}$; since $\mathfrak{q}$ is a prime ideal, we thus have $(x-y)^d/f^n \in \mathfrak{q}$, hence $x - y \in \mathfrak{p}_n$. Let then $\mathfrak{p} = \bigoplus_{n \geqslant 0} \mathfrak{p}_n$. If $a \in A_m$ and $x \in \mathfrak{p}_n$, then $(ax)^d/f^{n+m} = (a^d/f^m)(x^d/f^n) \in \mathfrak{q}$, hence $ax \in \mathfrak{p}$; this implies that $\mathfrak{p}$ is a homogeneous ideal of A. Since $1 \notin \mathfrak{q}$, one has $f \notin \mathfrak{p}$. Let $a \in A_m$ and $b \in A_n$ be such that $ab \in \mathfrak{p}$; then $(ab)^d/f^{n+m} = (a^d/f^m)(b^d/f^n) \in \mathfrak{q}$; since $\mathfrak{q}$ is prime, at least one of a^d/f^m and b^d/f^n belongs to $\mathfrak{q}$, which means that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Consequently, $\mathfrak{p}$ is a prime ideal, hence a member of $D_+(f)$.

Let us show that $\psi_f(\mathfrak{p}) = \mathfrak{q}$. Let indeed n be an integer and $x \in \mathfrak{p}_{nd}$; by assumption, $x^d/f^{nd} \in \mathfrak{q}$, hence $x/f^n \in \mathfrak{q}$ because $\mathfrak{q}$ is a prime ideal; consequently, $\psi_f(\mathfrak{p}) \subseteq \mathfrak{q}$. Conversely, an element of $\mathfrak{q}$ has the form x/f^n, with $n \in \mathbf{N}$ and $x \in A_{nd}$; then $x^d/f^{nd} \in \mathfrak{q}$, hence $x \in \mathfrak{p}$ by definition of $\mathfrak{p}$; consequently, $x/f^n \in \psi_f(\mathfrak{p})$.

We have shown that ψ_f is a continuous, bijective, and open map from $D_+(f)$ to $\operatorname{Spec}(A_{(f)})$. Therefore, it is a homeomorphism.

Proposition **(4.8.9)**. — *Let* A *be a graded ring.*

a) *Let* E *be a set of homogeneous elements of* A_+, *let* $f \in A_+$. *The following propositions are equivalent:*

(i) *One has* $D_+(f) \subseteq D_+(E)$;

(ii) *One has* $V_+(f) \supseteq V_+(E)$;

(iii) *There exists an integer* $n \geqslant 0$ *such that* f^n *belongs to the homogeneous ideal of* A *generated by* E.

b) *The maps* $E \mapsto V_+(E)$ *and* $Z \mapsto j_+(Z)$ *induce bijections, inverse one of the other, between the set of closed subsets of* Proj(A) *and the set of homogeneous radical ideals of* A *which are contained in* A_+.

Proof. — a) The equivalence (i)⇔(ii) is obvious, and it follows from the definitions of $V_+(f)$ and $V_+(E)$ that (iii) implies (i). Let us now show that (i) implies (iii).

Let E be a set of homogeneous elements of A contained in A_+, let I be the homogeneous ideal of A generated by E, and let $f \in A_+$ be a homogeneous elements of A. Let d be the degree of f. For every homogeneous element $g \in A$, of degree n, set $g' = g^d/f^n \in A_{(f)}$. Let E′ be the set of elements g', for $g \in E$. We have proved that $\psi_f(D_+(f) \cap D_+(g)) = V(g')$; consequently, $\psi_f(D_+(f) \cap V_+(E)) = V(E')$.

Assume now that $D_+(f) \subseteq D_+(E)$ or, equivalently, such that $D_+(f) \cap V_+(E) = \varnothing$. It follows that $V(E') = \varnothing$, hence the ideal I′ of $A_{(f)}$ generated by E′ contains 1. There thus exists an almost null family $(b_g)_{g \in E}$ of elements of $A_{(f)}$ such that $1 = \sum b_g g'$. Each element b_g is of the form c/f^m, for some homogeneous element c of degree md; consequently, there exists an almost null family $(c_g)_{g \in E}$ of homogeneous elements of A and an integer $m \geqslant 0$ such that $f^m = \sum c_g g^d$. In particular, $f^m \in I$ and $f \in \sqrt{I}$.

b) Let I be a homogeneous ideal of A and let $I_+ = I \cap A_+$; let us show that $V_+(I) = V_+(I_+)$. The inclusion $V_+(I) \subseteq V_+(I_+)$ follows from the definition, since $I_+ \subseteq I$. Conversely, let $\mathfrak{p} \in V_+(I_+)$. One thus has $I_+ \subseteq \mathfrak{p}$ but $\mathfrak{p} \not\supseteq A_+$. Let $f \in A_+$ be such that $f \notin \mathfrak{p}$. For every $a \in I_0$, one has $af \in I_+$, hence $af \in \mathfrak{p}$; since $\mathfrak{p}$ is prime, this implies that $a \in \mathfrak{p}$; consequently, $I_0 \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a homogeneous ideal of A, one has $I \subseteq \mathfrak{p}$, hence $\mathfrak{p} \in V_+(I)$. We thus have shown that $V_+(I) = V_+(I_+)$, as claimed.

Let E be a family of homogeneous elements of A and let $Z = V_+(E)$. Let I be the ideal generated by E; one has $Z = V_+(I)$. Moreover, $Z = V_+(\sqrt{I})$ since, for a prime ideal $\mathfrak{p}$, the conditions $I \subseteq \mathfrak{p}$ and $\sqrt{I} \subseteq \mathfrak{p}$ are equivalent.

Moreover, $\sqrt{I}$ is a homogeneous ideal of A, and one has $Z = V_+(\sqrt{I})$. One has $\sqrt{I} \cap A_+ \subseteq j_+(Z)$. Moreover, it follows from *a*) that every element f of $j_+(Z) \cap A_+$ belongs to $\sqrt{I}$. This shows that $j_+(Z) = \sqrt{I} \cap A_+$.

Consequently, for a homogeneous radical ideal $I \subseteq A_+$, one has $j_+(V_+(I)) = I$.

Let Z be a closed subset of Proj(E). By what precedes, there exists a radical and homogeneous ideal I of A, contained in A_+ such that $V_+(I) = Z$. One then has $j_+(Z) = I$, hence $V_+(j_+(Z)) = V_+(I) = Z$. □

4.8.10. — Let U be an open subset of Proj(A) and let S(U) be the set of homogeneous elements $s \in A$ such that $s \notin \mathfrak{p}$, for every homogeneous prime ideal $\mathfrak{p} \in U$. If $V \subseteq U$, one has $S(U) \subseteq S(V)$. One defines the sheaf $\widetilde{M}$ on Proj(A) as the sheaf associated with the presheaf given by $U \mapsto M_{(S(U))}$.

For every integer $p \in \mathbf{Z}$, the *twist* of order p of M is the graded A-module $M(p)$ whose underlying A-module is M, but whose grading is shifted by p: $M(p)_n = M_{p+n}$ for every integer n.

Lemma **(4.8.11)**. — *Let* A *be a graded algebra, let* f *be a homogeneous element of* A *of degree* $d > 0$ *and let* $U = D_+(f)$. *The element* f *belongs to* S(U). *For every graded* A*-module* M, *the canonical morphism of graded modules from* M_f *to* $M_{S(U)}$ *is an isomorphism. In particular, it induces an isomorphism from* $M_{(f)}$ *to* $M_{(S(U))}$.

Proof. — One has $f \in S(U)$ by the very definition of $D_+(f)$. Let $\varphi \colon M_f \to M_{S(U)}$ be the canonical morphism.

Let $x \in M_f$; there exists $m \in M$ and an integer $n \geqslant 0$ such that $x = m/f^n$. If $x \in \operatorname{Ker}(\varphi)$, there exists $s \in S(U)$ such that $sm = 0$. Since $s \in S(U)$, $D_+(f)$ is contained in $D_+(s)$, so that there exists an integer $p \geqslant 0$ and $t \in A$ such that $f^p = st$. Consequently, $f^p m = 0$, hence $x = 0$. This shows that φ is injective.

Let x be an element of $M_{S(U)}$; let $m \in M$ and $s \in S(U)$ be such that $x = m/s$. By the preceding argument, there exists an integer $p \geqslant 0$ and $t \in A$ such that $f^p = st$. Then $x = (tm)/f^p$, so that $x \in \operatorname{Im}(\varphi)$.

Since φ is compatible with the natural gradings of M_f and $M_{S(U)}$, it induces an isomorphism from $M_{(f)}$ to $M_{(S(U))}$. □

Proposition **(4.8.12)**. — *Let* A *be a graded ring, let* f *be a homogeneous element of* A *of strictly positive degree and let* M *be a graded* A*-module.*

a) *Under the homeomorphism* ψ_f, *the sheaf* $\widetilde{\mathrm{M}}|_{\mathrm{D}_+(f)}$ *is transformed to the quasi-coherent sheaf on* $\mathrm{Spec}(\mathrm{A}_{(f)})$ *associated to the* $\mathrm{A}_{(f)}$*-module* $\mathrm{M}_{(f)}$.

b) *The ringed space* $(\mathrm{Proj}(\mathrm{A}), \widetilde{\mathrm{A}})$ *is a scheme.*

c) *For every graded* A*-module* M, *the* $\widetilde{\mathrm{A}}$*-module* $\widetilde{\mathrm{M}}$ *on* $\mathrm{Proj}(\mathrm{A})$ *is quasi-coherent.*

Proof. — a) Let d be the degree of f. For every homogeneous element $g \in \mathrm{A}$ of strictly positive degree n, denote by U_g the open subset $\mathrm{D}_+(f) \cap \mathrm{D}_+(g) = \mathrm{D}_+(fg)$ of $\mathrm{D}_+(f)$. By the previous lemma, the module of fractions $\mathrm{M}_{(\mathrm{S}(\mathrm{U}_g))}$ identifies with $\mathrm{M}_{(fg)}$. Observe also that the natural morphism from $\mathrm{M}_{(f)}$ to $\mathrm{M}_{(fg)}$ induces an isomorphism from $(\mathrm{M}_{(f)})_{g^d/f^n}$ to $\mathrm{M}_{(fg)}$. On the other hand, we have proved that $\psi_f(\mathrm{U}_g) = \mathrm{D}(g^d/f^n)$. Consequently, the presheaf given by $\mathrm{U} \mapsto \mathrm{M}_{(\mathrm{S}(\mathrm{U}))}$ on $\mathrm{D}_+(f)$ identifies, via ψ_f, with the sheaf $\widetilde{\mathrm{M}_{(f)}}$, at least on distinguished open subsets. This identifies the associated sheaf $\widetilde{\mathrm{M}}|_{\mathrm{D}_+(f)}$ with the sheaf $\widetilde{\mathrm{M}_{(f)}}$.

b) By *a*), the restriction of the ringed space $(\mathrm{Proj}(\mathrm{A}), \widetilde{\mathrm{A}})$ to the open subset $\mathrm{D}_+(f)$ is an affine scheme. By definition of the homogeneous spectrum, the open subsets of this form cover $\mathrm{Proj}(\mathrm{A})$, since for every $\mathfrak{p} \in \mathrm{Proj}(\mathrm{A})$, there exists $f \in \mathrm{A}_+$ such that $f \notin \mathfrak{p}$. Consequently, the ringed space $(\mathrm{Proj}(\mathrm{A}), \widetilde{\mathrm{A}})$ is a scheme.

c) Let M be a graded A-module. The restriction to $\mathrm{D}_+(f)$ of the $\widetilde{\mathrm{A}}$-module $\widetilde{\mathrm{M}}$ is quasi-coherent, since it identifies with the quasi-coherent $\mathscr{O}_{\mathrm{D}_+(f)}$-module associated to the $\mathrm{A}_{(f)}$-module $\mathrm{M}_{(f)}$. Consequently, it is quasi-coherent. □

Example **(4.8.13)**. — Let k be a ring and let $\mathrm{A} = k[\mathrm{T}_0, \dots, \mathrm{T}_n]$ be the ring of polynomials in $(n+1)$ indeterminates with coefficients in k. Let us endow the ring A with the graduation by degree.

For every $i \in \{0, \dots, n\}$, let U_i be the open subset $\mathrm{D}_+(\mathrm{T}_i)$ of $\mathrm{Proj}(\mathrm{A})$ and let ψ_i be the k-isomorphism from $\mathrm{D}_+(\mathrm{T}_i)$ to the affine scheme $\mathrm{X}_i = \mathrm{Spec}(k[\mathrm{U}_0, \dots, \mathrm{U}_n]/(\mathrm{U}_i - 1))$ such that $\psi_i^\sharp(\mathrm{U}_j) = \mathrm{T}_j/\mathrm{T}_i$ for every j. For every pair (i,j), let X_{ij} be the open subscheme $\mathrm{D}(\mathrm{U}_j)$ of X_i. One has $\psi_i(\mathrm{D}_+(\mathrm{T}_i\mathrm{T}_j)) = \mathrm{D}(\mathrm{U}_j) = \mathrm{X}_{ij} = \mathrm{Spec}(k[\mathrm{U}_0, \dots, \mathrm{U}_n]/(\mathrm{U}_i - 1)[1/\mathrm{U}_j])$ and the isomorphism $\psi_{ij} = \psi_j \circ \psi_i^{-1}$ from the open subscheme $\mathrm{X}_{ij} = \mathrm{D}(\mathrm{U}_j)$ of X_i to the open subscheme $X_{ji} = D(U_i)$ of X_j is given by

$$\psi_{ij}^\sharp(U_m) = (\psi_i^{-1})^\sharp \circ \psi_j^\sharp(U_m) = (\psi_i^{-1})^\sharp(T_m/T_j) = U_m(\psi_i^{-1})^\sharp(T_i/T_j) = U_m U_i/U_j.$$

By definition, the scheme $\mathbf{P}^n_k$ is defined by gluing the family (X_i) along the open subschemes (X_{ij}) by means of the isomorphisms ψ_{ij}. For every i, let $\varphi_i \colon X_i \to X$ be the canonical open immersion and let U_i be its image. By what precedes, there exists a unique morphism $\varphi \colon \mathrm{Proj}(A) \to \mathbf{P}^n_k$ such that $\varphi|_{D+(T_i)} = \varphi_i \circ \psi_i$ and it is an isomorphism.

Proposition **(4.8.14)**. — *Assume that* A_0 *is a noetherian ring and that* A *is a finitely generated* A_0*-algebra.*

a) *For every strictly positive integer* d*, the ring* $\bigoplus_{d|n} A_n$ *is noetherian;*
b) *The scheme* Proj(A) *is noetherian* (ie, *quasi-compact and locally noetherian);*
c) *For every finitely generated graded* A*-module* M*, the* $\widetilde{A}$*-module* $\widetilde{M}$ *on* Proj(A) *is coherent.*

Proof. — a) Let $f_1, \dots, f_m$ be homogeneous elements of A such that $A = A_0[f_1, \dots, f_m]$. By hypothesis, the morphism of A_0-algebras $\varphi \colon A_0[T_1, \dots, T_m] \to A$ such that $\varphi(T_i) = f_i$ for every i is surjective. Since the ring $A_0[T_1, \dots, T_m]$ is noetherian (theorem 1.9.3), so is A. This also implies that for every integer n, the A_0-module A_n is generated by the elements of the form $f_1^{n_1} \dots f_m^{n_m}$ such that $n_1 d_1 + \dots + n_m d_m = n$, hence is finitely generated.

Let d be a strictly positive integer and let us consider the graded ring $A' = \bigoplus_{d|n} A_n$. Writing $n_i = q_i d + r_i$, with $0 \leqslant r_i < d$, we have

$$f_1^{n_1} \dots f_m^{n_m} = (f_1^d)^{q_1} \dots (f_m^d)^{q_m} (f_1^{r_1} \dots f_m^{r_m}),$$

so that the A_0-algebra A' is generated by $f_1^d, \dots, f_m^d$ and by the finite set of elements of the form $f_1^{r_1} \dots f_m^{r_m}$ such that d divides $\sum d_i r_i$.

b) Let f be a homogeneous element of strictly positive degree and let us show that the ring $A_{(f)}$ is noetherian. The isomorphism $A[T]/(fT-1) \simeq A_f$ implies that A_f is noetherian. If f has degree 1, then every element of A_f can be written uniquely under the form af^n, where $a \in A_{(f)}$ and $n \in \mathbf{Z}$, so that the ring $A_{(f)}$ is isomorphic to the quotient of the ring A_f by its (non-homogeneous) ideal $(f-1)$. This implies that $A_{(f)}$ is a noetherian ring. In fact, note that

$$A_{(f)} \simeq A_f/(f-1) \simeq A[T]/(fT-1, f-1) \simeq A[T]/(T-1, f-1) \simeq A/(f-1).$$

Let us now treat the general case; let d be the degree of f. Similarly, every element of A_f of degree divisible by d can be written uniquely under the form af^n, where $a \in A_{(f)}$ and $n \in \mathbf{Z}$, so that the ring $A_{(f)}$ is isomorphic to the quotient of the graded ring $A'_f = \bigoplus_{d|n}(A_f)_n$ by the (non-homogeneous) ideal $(f-1)$. By *a*), A' is a noetherian ring, hence so are A'_f and $A_{(f)} \simeq A'_f/(f-1)$.

This shows that the affine open subscheme $D_+(f)$ of $\operatorname{Proj}(A)$ is the spectrum of a noetherian ring. It first follows that $\operatorname{Proj}(A)$ is a locally noetherian scheme. Since the ring A is noetherian, its ideal A_+ is finitely generated, say $A_+ = (f_1, \dots, f_m)$. Consequently, one has $\operatorname{Proj}(A) = \bigcup_{i=1}^m D_+(f_i)$, which shows that it is quasi-compact.

c) Let M be a finitely generated graded A-module. For every homogeneous element f of strictly positive degree, say d, the restriction $\widetilde{M}|_{D_+(f)}$ identifies with the quasi-coherent module on $\operatorname{Spec}(A_{(f)})$ associated with the $A_{(f)}$-module $M_{(f)}$.

First assume that $d = 1$. In that case, M_f is finitely generated as an A_f-module. Moreover, every element of M_f can be written uniquely under the form $f^n m$, where $m \in M_{(f)}$ and $n \in \mathbf{Z}$. Consequently, $M_f/(f-1)M_f$ is isomorphic to $M_{(f)}$. This implies that $M_{(f)}$ is a finitely generated $A_{(f)}$-module.

In the general case, one proves as above that $M' = \bigoplus_{d|n} M_d$ is finitely generated as an A'-module, where $A' = \bigoplus_{d|n} A_d$. Then M'_f is a finitely generated A'_f-module, and $M_{(f)} = M'_{(f)} \simeq M'_f/(f-1)$ is a finitely generated $A_{(f)}$-module.

This proves that $\widetilde{M}$ is a coherent $\widetilde{A}$-module on $\operatorname{Proj}(A)$.

□

4.8.15. — Let A be a graded ring. The assignment $M \mapsto \widetilde{M}$ is a functor from the category of graded A-module to the category of quasi-coherent $\widetilde{A}$-modules on the homogeneous spectrum $\operatorname{Proj}(A)$. This functor is exact, but has less good properties than the analogous functor on spectra (which is an equivalence of categories). In particular, it is neither fully faithful, nor essentially surjective in general.

Definition **(4.8.16)**. — *Let* A *be a graded ring and let* M *be a graded* A*-module.*

a) *We say that* M *is almost null if there exists* $m \in \mathbf{N}$ *such that* $A_n = 0$ *for* $n \geqslant m$.

b) *We say that* M *is almost finitely generated if there a finitely generated graded submodule* N *of* M *such that* M/N *is almost null.*

c) *A morphism* $u\colon M \to N$ *of graded* A*-modules is said to be almost injective (resp. almost surjective) if* $\ker(u)$ *is almost null (resp. if* $\operatorname{Coker}(u)$ *is almost null).*

The class of almost null graded A-modules is a thick[2] subcategory of the category $\boldsymbol{Mod}_A$, and we can consider the quotient category obtained by inverting all almost isomorphisms.

Proposition **(4.8.17)**. — *Let* A *be a graded ring and let* M *be a graded* A*-module.*

a) *If* M *is almost null, then the quasi-coherent sheaf* $\widetilde{M}$ *on* $\operatorname{Proj}(A)$ *is null.*

b) *If* M *is almost finitely generated, then the quasicoherent sheaf* $\widetilde{M}$ *is finitely generated.*

Proof. — a) Let us assume that M is almost null and let us prove that $\widetilde{M} = 0$. It suffices to prove that all of its fibers are 0. So let $x \in \operatorname{Proj}(A)$ and let P be the corresponding homogeneous prime ideal. By construction, elements of $\widetilde{M}_x$ are fractions of the form m/f, where $f \in A - P$ is homogeneous of some degree $n > 0$ and $m \in M_n$. For every integer q, one has $m/f = mf^q/f^{1+q}$; since $mf^q \in M_{(n+1)q}$, we have $mf^q = 0$ for q large enough, so that $m/f = 0$. We thus have $\widetilde{M}_x = 0$ for all $x \in \operatorname{Proj}(A)$, hence $\widetilde{M} = 0$.

b) Let M′ be a graded submodule of M which is finitely generated and such that M/M′ is almost null. By *a*), one then has $\widetilde{M/M'} = 0$, so that $\widetilde{M'} \simeq \widetilde{M}$. It thus suffices to prove that the quasi-coherent sheaf $\widetilde{M'}$ is finitely generated. Changing notation, we may thus assume that M itself is finitely generated.

Then, for every homogeneous element f whose degree is strictly positive, the $A_{(f)}$-module $M_{(f)}$ is finitely generated □

Proposition **(4.8.18)**. — *Let* A *be a graded ring whose irrelevant ideal* A_+ *is finitely generated and let* M *be a graded* A*-module which is almost finitely generated.*

If $\widetilde{M} = 0$, *then* M *is almost null.*

Proof. — Let N be a finitely generated graded submodule of M such that M/N is almost null. Then $\widetilde{M/N} = 0$, so that $\widetilde{M} = \widetilde{N} = 0$, by exactness of the functor $M \mapsto \widetilde{M}$. It suffices to prove that N is almost null, so that we may assume that M is finitely generated.

Let m be any homogeneous element of M, let n be its degree, and let $J = \operatorname{Ann}_A(m)$; this is a homogeneous ideal of A.

(2) Define?

Let us prove that $V_+(J) = \varnothing$. Let $x \in \operatorname{Proj}(A)$. We view m as a global section μ of $\widetilde{M}(n)$. Since A is generated by A_1, one has $\widetilde{M}(n) = \widetilde{M} \otimes \mathscr{O}(n)$, and the vanishing of $\widetilde{M}$ implies that $\widetilde{M}(n) = 0$. In particular, $\mu = 0$. In particular, $m/1 = 0$ in the localized module M_x, which means that there exists a homogeneous element $f \in A$ such that $fm = 0$ and $f \notin P_x$. In other words, $J \nsubseteq P_x$, hence $x \notin V_+(J)$.

As a consequence, $\sqrt{J} = A_+$.

Applying this property to a finite family of homogeneous generators of M, we conclude that $\sqrt{\operatorname{Ann}_A(M)} = A_+$. Since the ideal A_+ is finitely generated, there exists an integer $p \geqslant 0$ such that $(A_+)^p \subseteq \operatorname{Ann}_A(M)$.

Let T be a finite family of homogeneous generators of M and let m be an upper bound for their degrees. By what precedes, we have $M_n = 0$ for every integer $n \geqslant m + p$. Consequently, M is almost null. ☐

Corollary **(4.8.19)**. — *Let* A *be a graded ring.*

a) *If* A *is almost null, then* $\operatorname{Proj}(A) = \varnothing$.

b) *If* $\operatorname{Proj}(A) = \varnothing$ *and if the ideal* A_+ *is finitely generated, then* A *is almost null.*

Proof. — Indeed, the equality $\operatorname{Proj}(A) = \varnothing$ means that $\widetilde{A} = 0$. By the preceding proposition, this holds if and only if A is almost null. ☐

4.9. Locally free modules

4.9.1. — Let X be a scheme and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. For every $x \in X$, let

$$d_{\mathscr{M}}(x) = \dim_{\kappa(x)}(\mathscr{M}_x \otimes_{\mathscr{O}_{X,x}} \kappa(x)).$$

Proposition **(4.9.2)**. — *Let* X *be a scheme and let* $\mathscr{M}$ *be a finitely generated quasi-coherent* $\mathscr{O}_X$*-module. The function* $d_{\mathscr{M}}$ *is upper semi-continuous: for every* $n \in \mathbf{N}$, *the set of points* $x \in X$ *such that* $d_{\mathscr{M}}(x) \geqslant n$ *is closed in* X, *and the set of points* $x \in X$ *such that* $d_{\mathscr{M}}(x) \leqslant n$ *is open in* X.

The result does not hold without the hypothesis that $\mathscr{M}$ finitely generated, and quasi-coherent.

Proof. — We may assume that X is affine, say $X = \operatorname{Spec}(A)$; let M be the A-module $\mathscr{M}(X)$. Let $n \in \mathbf{N}$ and let $x \in X$ be such that $d_{\mathscr{M}}(x) \leqslant n$; let $\mathfrak{p}$ be the corresponding prime ideal of A. Let thus $m_1, \dots, m_n$ be elements of M which generate $M \otimes_A \kappa(\mathfrak{p})$; let N be the submodule of M generated by $m_1, \dots, m_n$. One has $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$; moreover, $M \otimes_A \kappa(\mathfrak{p}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})$, and similarly for N. Consequently, we have

$$M_\mathfrak{p} = N_\mathfrak{p} + \mathfrak{p}M_\mathfrak{p}.$$

By Nakayama's lemma (corollary 1.3.3), this implies the equality $M_\mathfrak{p} = N_\mathfrak{p}$.

Let (x_i) be a finite generating family for M; for every i, there exists $s_i \in A - \mathfrak{p}$ such that $s_i x_i \in N$. Let s be the product of the s_i; one has $sx_i \in N$ for every i, hence $sM \subseteq N$. Consequently, M_s is generated by the family $(m_1/1, \dots, m_n/1)$. Therefore, for every $y \in D(s)$, $M \otimes_A \kappa(\mathfrak{p}_y)$ is generated by the images of $m_1, \dots, m_n$, so that $d_{\mathscr{M}}(y) \leqslant n$. □

4.9.3. — If $\mathscr{M}$ is free, i.e., if there exists a set I such that $\mathscr{M} \simeq \mathscr{O}_X^{(I)}$, then $d_{\mathscr{M}}(x) = \mathrm{Card}(I)$ for every $x \in X$: the function $d_{\mathscr{M}}$ is constant on X.

Recall that one says that $\mathscr{M}$ is locally free if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{M}|_U$ is a free $\mathscr{O}_X|_U$-module. In that case, the function $d_{\mathscr{M}}$ is locally constant on X. If, moreover, $d_{\mathscr{M}}(x)$ is finite for every $x \in X$, then $\mathscr{M}$ is finitely generated and one says that $\mathscr{M}$ is locally free of finite rank. One says that $\mathscr{M}$ is locally free of rank n if it is locally free and if $d_{\mathscr{M}}(x) = n$ for every point $x \in X$.

When $X = \mathrm{Spec}(A)$ is affine and M is an A-module, one says that M is locally free (resp. locally free of rank n) if the $\mathscr{O}_X$-module $\mathscr{M}$ is locally free (resp. locally free of rank n).

Proposition (**4.9.4**). — *Let* A *be a ring, let* $X = \mathrm{Spec}(A)$, *and let* M *be an* A-*module. The following properties are equivalent:*

(i) *The* $\mathscr{O}_X$-*module* $\widetilde{M}$ *is locally free of finite rank;*

(ii) *The* A-*module* M *is finitely generated and projective.*

(iii) *There exists an integer* n *and an* A-*module* N *such that* $M \oplus N \simeq A^n$.

(iv) *The* A-*module* M *is finitely presented, and for every* $\mathfrak{p} \in \mathrm{Spec}(A)$, *the* $A_\mathfrak{p}$-*module* $M_\mathfrak{p}$ *is free;*

(v) *For every* $\mathfrak{p} \in \mathrm{Spec}(A)$, *there exists an element* $f \in A - \mathfrak{p}$ *such that* M_f *is a finitely generated free* A_f-*module;*

Proof. — (i)⇒(ii). Let $p\colon N \to N'$ be a surjective morphism of A-modules and let $f\colon M \to N'$ be a morphism of A-modules. We need to show that there exists a morphism $g\colon M \to N$ such that $f = p \circ g$. To that aim, let us set $P = \mathrm{Hom}_A(M, N)$, $P' = \mathrm{Hom}_A(M, N')$, and let $p_*\colon P \to P'$ be the morphism of A-modules induced by p. It suffices to prove that p_* is surjective and, to that aim, that the morphism of sheaves $\widetilde{p_*}\colon \widetilde{\mathrm{P}} \to \widetilde{\mathrm{P}'}$ is surjective. Since M is finitely generated, the canonical morphism from $\widetilde{\mathrm{P}}$ to $\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{\mathrm{M}}, \widetilde{\mathrm{N}})$ is an isomorphism, as is the canonical morphism from $\widetilde{\mathrm{P}'}$ to $\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{\mathrm{M}}, \widetilde{\mathrm{N}'})$ Let f is an element of A such that $\widetilde{\mathrm{M}}|_{\mathrm{D}(f)}$ is free; then $\widetilde{p_*}|_{\mathrm{D}(f)}$ is surjective. This implies that $\widetilde{p_*}\colon \widetilde{\mathrm{P}} \to \widetilde{\mathrm{P}'}$ is a surjective morphism of quasi-coherent $\mathscr{O}_X$-modules. In particular, the morphism $p_* = \widetilde{p_*}(X)\colon \mathrm{P} \to \mathrm{P}'$ is surjective, as was to be shown.

(ii)⇒(iii). This follows from proposition 2.7.2, (ii)⇒(iii).

(iii)⇒(iv). Since they are isomorphic to a quotient module of A^n, both A-modules M and N are finitely generated; consequently, M is finitely presented. For every $\mathfrak{p} \in \mathrm{Spec}(\mathrm{A})$, one has an isomorphism $\mathrm{M}_{\mathfrak{p}} \oplus \mathrm{N}_{\mathfrak{p}} \simeq \mathrm{A}_{\mathfrak{p}}^n$. In particular, $\mathrm{M}_{\mathfrak{p}}$ is a finitely generated projective $\mathrm{A}_{\mathfrak{p}}$-module, hence is free (theorem 2.7.4).

(iv)⇒(v). Let $\mathfrak{p} \in \mathrm{Spec}(\mathrm{A})$, let $n \in \mathbf{N}$ and let $m_1, \dots, m_n$ be elements of M such that $(m_1/1, \dots, m_n/1)$ is a basis of $\mathrm{M}_{\mathfrak{p}}$. Let $\varphi\colon \mathrm{A}^n \to \mathrm{M}$ be the morphism of A-modules defined by $\varphi(a_1, \dots, a_n) = \sum a_i m_i$. One has $\mathrm{Coker}(\varphi)_{\mathfrak{p}} = \mathrm{Coker}(\varphi_{\mathfrak{p}}) = 0$. Since M is finitely generated, $\mathrm{Coker}(\varphi)$ is finitely generated too, hence there exists an element $f \in \mathrm{A} - \mathfrak{p}$ such that $\mathrm{Coker}(\varphi)_f = 0$. This implies that φ_f is surjective. Then, M_f being finitely presented, the kernel of φ_f is finitely generated. Moreover, one has $\mathrm{Ker}(\varphi_f)_{\mathfrak{p}} = 0$. Consequently, there exists $g \in \mathrm{A} - \mathfrak{p}$ such that $\mathrm{Ker}(\varphi_{fg}) = \mathrm{Ker}(\varphi_f)_g = 0$. This implies that φ_{fg} is an isomorphism. Consequently, M_{fg} is free and finitely generated.

(v)⇔(i). The quasi-coherent sheaf $\widetilde{\mathrm{M}}$ on X associated to a free A-module $\mathrm{M} = \mathrm{A}^{(\mathrm{I})}$ is isomorphic to $\widetilde{\mathrm{A}}^{(\mathrm{I})}$, so that $\widetilde{\mathrm{M}}$ is free if and only if M is free. Moreover, $\widetilde{\mathrm{M}}$ is finitely generated if and only if M is finitely generated. Let now $f \in \mathrm{A}$. Applying this remark to the A_f-module M_f, we see that the $\mathscr{O}_X|_{\mathrm{D}(f)}$-module $\widetilde{\mathrm{M}}|_{\mathrm{D}(f)}$ is free of finite rank if and only if M_f is free and finitely generated. This shows that (i) and (v) are equivalent. □

Corollary **(4.9.5)**. — *Let* A *be a principal ideal domain and let* X = Spec(A). *Every locally free* $\mathscr{O}_X$*-module of rank n is trivial, i.e., is isomorphic to* $\mathscr{O}_X^n$.

Proof. — If A is a principal ideal domain and m is an integer, then every submodule of A^m is free. □

Corollary **(4.9.6)**. — *Let* X *be a scheme, let* $\mathscr{M}$ *and* $\mathscr{N}$ *be locally free finitely generated* $\mathscr{O}_X$*-modules and let* $\varphi\colon \mathscr{M} \to \mathscr{N}$ *be a* surjective *homomorphism. Then* Ker(φ) *is locally free; moreover, if* X *is affine, then* φ *has a section.*

Proof. — Let us first assume that X is affine. Let $A = \mathscr{O}_X(X)$, let $M = \mathscr{M}(X)$ and $N = \mathscr{N}(X)$, and let $f\colon M \to N$ be the morphism $\varphi(X)$. The A-modules M and N are finitely generated and projective, and the morphism f is surjective. In particular, there exists a morphism $g\colon N \to M$ such that $f \circ g = \mathrm{id}_N$, hence M is isomorphic to $N \oplus \mathrm{Ker}(f)$. Since M is projective and finitely generated, there exists an integer m and an A-module M′ such that $M \oplus M' \simeq A^m$; then $(M' \oplus N) \oplus \mathrm{Ker}(f) \simeq A^m$, which shows that $\mathrm{Ker}(f)$ is projective and finitely generated. In this case, this shows that $\mathrm{Ker}(\varphi)$ is a finitely generated locally free $\mathscr{O}_X$-module, and that φ has a section.

In general, this implies that for every affine open subscheme U of X, $\mathrm{Ker}(\varphi)|_U$ is a finitely generated locally free $\mathscr{O}_X|_U$-module. Consequently, $\mathrm{Ker}(\varphi)$ is a finitely generated locally free $\mathscr{O}_X$-module, as claimed. ☐

Proposition **(4.9.7)**. — *Let* X *be a scheme and let* $\mathscr{M}$ *be a quasi-coherent* $\mathscr{O}_X$-*module of finite presentation.*

a) *If* $\mathscr{M}$ *is locally free, then the function* $x \mapsto d_{\mathscr{M}}(x)$ *on* X *is locally constant.*

b) *Conversely, if* X *is reduced and the function* $d_{\mathscr{M}}$ *is locally constant, then* $\mathscr{M}$ *is locally free.*

Proof. — If $\mathscr{M}$ is free, then $d_{\mathscr{M}}$ is constant. It thus suffices to prove that $\mathscr{M}$ is locally free if $d_{\mathscr{M}}$ is constant and X is reduced. We may even assume that X is an affine scheme. Let then $A = \mathscr{O}_X(X)$ and $M = \mathscr{M}(X)$; the ring A is reduced, the A-module M is finitely presented and we need to prove that it is locally free of rang n, assuming that for every $\mathfrak{p} \in \mathrm{Spec}(A)$, one has $\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p}) = n$. By proposition 4.9.4, we need to prove that for every prime ideal $\mathfrak{p}$ of A, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free of rank n. Replacing A by $A_{\mathfrak{p}}$ and M by $M_{\mathfrak{p}}$, we may thus assume that A is a local ring; let $\mathfrak{m}$ be its maximal ideal.

Let $(m_1, \dots, m_n)$ be elements of M whose images in $M \otimes_A \kappa(\mathfrak{m})$ constitute a basis of that vector space. Let $f\colon A^n \to M$ be the morphism of A-modules given by $f(a_1, \dots, a_n) \sum a_i m_i$. One has $M = \mathrm{Im}(f) + \mathfrak{m}M$, by assumption; it thus follows from Nakayama's lemma (corollary 1.3.3) that f is surjective. Let N be its kernel. Let $\mathfrak{p}$ be a prime ideal of A. Let $f(\mathfrak{p})\colon \kappa(\mathfrak{p})^n \to M \otimes_A \kappa(\mathfrak{p})$ be the morphism deduced from f; it is surjective by right exactness of the tensor product; since, $M \otimes_A \kappa(\mathfrak{p})$ has dimension n, by assumption, this implies that $f(\mathfrak{p})$ is an isomorphism. Now, the injection j from N to A^n induces a morphism $\overline{j}(\mathfrak{p})\colon N \to \kappa(\mathfrak{p})^n$ whose image is zero, since it is contained in $\mathrm{Ker}(f(\mathfrak{p}))$.

Necessarily, $N \subseteq \mathfrak{p}^n$. This holds for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, and the intersection of them is $\{0\}$, because A is reduced. Consequently, $N = 0$ and f is an isomorphism. □

4.9.8. — All standard constructions from linear algebra (direct sums of modules over some ring, tensor products, symmetric and exterior powers, sheaves of homomorphisms, duals,...) associate free modules with free modules. Thanks to the above proposition, they translate from the context of free modules over a ring to that of locally free sheaves of finite rank over a scheme.

Assume that $\mathscr{M}$ and $\mathscr{N}$ are locally free sheaves of ranks m and n on X. Then $\mathscr{M} \oplus \mathscr{N}$ is locally free of rank $m+n$; $\mathrm{Hom}(\mathscr{M}, \mathscr{N})$ and $\mathscr{M} \otimes \mathscr{N}$ are locally free of rank mn. In particular, $\mathscr{M}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$ is locally free of rank m; moreover, the canonical morphism from $\mathscr{M}^\vee \otimes_{\mathscr{O}_X} \mathscr{N}$ to $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is an isomorphism. For every integer $p \geqslant 0$, the exterior power $\bigwedge^p \mathscr{M}$ is locally free of rank $\binom{m}{p}$, and the symmetric power $S^p\mathscr{M}$ is locally free of rank $\binom{m+p-1}{p}$.

In particular, the "maximal" exterior power of $\mathscr{M}$ is locally free of rank 1; it is called the determinant of $\mathscr{M}$ and is denoted by $\det(\mathscr{M})$. One has an isomorphism $\det(\mathscr{M} \oplus \mathscr{N}) \simeq \det(\mathscr{M}) \otimes \det(\mathscr{N})$.

Proposition **(4.9.9)**. — *Let* X *be a scheme and let* $\mathscr{M}$ *be a quasi-coherent* $\mathscr{O}_X$-*module. The following properties are equivalent:*

(i) *The* $\mathscr{O}_X$-*module* $\mathscr{M}$ *is locally free of rank* 1*;*

(ii) *The canonical morphism* $\mathscr{M}^\vee \otimes_{\mathscr{O}_X} \mathscr{M} \to \mathscr{O}_X$ *is an isomorphism;*

(iii) *There exists a quasi-coherent* $\mathscr{O}_X$-*module* $\mathscr{N}$ *such that* $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ *is isomorphic to* $\mathscr{O}_X$.

Proof. — (i)⇒(ii). We may assume that $\mathscr{M}$ is free, hence possesses a frame (ε); then $\mathscr{M}^\vee$ is free as well, and possesses a frame (φ), characterized by the relation $\varphi(\varepsilon) = 1$. The indicated canonical morphism maps $(a\varphi)\otimes(b\varepsilon)$ to ab; it identifies with the isomorphism of $\mathscr{O}_X \otimes_{\mathscr{O}_X}$ with $\mathscr{O}_X$.

(ii)⇒(iii). Indeed, one may take $\mathscr{N} = \mathscr{M}^\vee$.

(iii)⇒(i). We may assume that X is affine, say $X = \mathrm{Spec}(A)$; then $M = \mathscr{M}(X)$ and $N = \mathscr{N}(X)$ are two A-modules such that there exists an isomorphism $\varphi \colon M \otimes_A N \simeq A$.

Let us assume for the moment that there exists a split tensor $m \otimes n$ such that $\varphi(m \otimes n) = 1$. Let us consider the unique morphism ψ from $M \otimes_A N \otimes_A M$ to M such that $\psi(x, y, z) = \varphi(z \otimes y)x$ for every $x \in M$, $y \in N$ and $z \in M$. Now, if $x \in M$ is such that $x \otimes n = 0$, one has $x = \psi(x \otimes n \otimes m) = \psi(0) = 0$. Then, for every $x \in M$, the element $x' = x - \varphi(x \otimes n)m$ of M satisfies $\varphi(x' \otimes n) = \varphi(x \otimes n) - \varphi(x \otimes n)\varphi(m \otimes n) = 0$; since φ is an isomorphism, one has $x' \otimes n$, hence $x' = 0$. This shows that the map from M to A given by $x \mapsto \varphi(x \otimes n)$ is an isomorphism, with inverse given by $a \mapsto am$. Consequently, m is a basis of M.

In general, since φ is an isomorphism, there exist families (m_i) and (n_i) of elements of M and N respectively such that $\varphi(\sum m_i \otimes n_i) = 1$. For every i, let us set $a_i = \varphi(m_i \otimes n_i)$. By localization, φ induces an isomorphism $\varphi_{a_i}\colon M_{a_i} \otimes_{A_{a_i}} N_{a_i}$, which maps the split tensor $a_i^{-1} m_i \otimes n_i$ in $M_{a_i} \otimes_{A_{a_i}} N_{a_i}$ to 1. Consequently, M_{a_i} is a free A_{a_i}-module of rank 1.

Since $1 = \sum a_i$, the open subsets $D(a_i)$ cover Spec(A). Consequently, M is locally free of rank 1. □

4.9.10. — Let X be a scheme. In view of the preceding proposition, a locally free $\mathscr{O}_X$-module of rank 1 is also called an *invertible sheaf*. Let Pic(X) be the set of isomorphism classes of invertible sheaves. The tensor product of $\mathscr{O}_X$-modules endowes Pic(X) with the structure of a group. The neutral element is the class of the sheaf $\mathscr{O}_X$. If $\mathscr{M}$ is an invertible sheaf, the inverse of the class of $\mathscr{M}$ is the class of its dual $\mathscr{M}^\vee$.

4.9.11. Locally free sheaves and cohomology. — Let X be a scheme, let n be an integer and let $\mathscr{M}$ be a locally free $\mathscr{O}_X$-module of rank n on X.

Let $\mathscr{U}$ be an covering of X by open subschemes of X. We say that $\mathscr{M}$ is $\mathscr{U}$-free on $\mathscr{U}$ if for every open subscheme $U \in \mathscr{U}$, the restriction $\mathscr{M}|_U$ is free, i.e., is isomorphic to $\mathscr{O}_X|_U^n$.

Let us assume that this is the case. For every $U \in \mathscr{U}$, let us choose an isomorphism $s^U\colon \mathscr{O}_U^n \to \mathscr{M}|_U$.

Let $U, V \in \mathscr{U}$. Since $s^U|_{U\cap V}$ and $s^V|_{U\cap V}$ are two isomorphism from $\mathscr{O}_{U\cap V}^n$ to $\mathscr{M}|_{U\cap V}$, there exists a unique isomorphism $A_{UV} \in GL(n, \mathscr{O}(U \cap V))$ such that $s^U|_{U\cap V} \circ A_{UV} = s^V|_{U\cap V}$. Let $U, V, W \in \mathscr{U}$; on $U \cap V \cap W$, one has

$$s^U \circ A_{UW} = s^W = s^V \circ A_{VW} = s^U \circ A_{UV} \circ A_{VW};$$

consequently, the family $z^1(s) = (A_{UV})_{U,V\in\mathscr{U}}$ satisfies the following *cocycle relation:*

$$A_{UW} = A_{UV} A_{VW} \quad \text{in } GL(n, \mathscr{O}(U \cap V \cap W)).$$

In particular, one has $A_{UU} = I_n$ and $A_{UV} = A_{VU}^{-1}$. Let $Z^1(\mathscr{U}, GL(n))$ be the set of all families (A_{UV}) satisfying this cocycle relation. An element of $Z^1(\mathscr{U}, GL(n))$ is called a Čech 1-cocycle with values in $GL(n)$ on X, and the element $z^1(s)$ is called the Čech 1-cocycle associated to the family $s = (s^U)_{U\in\mathscr{U}}$ of trivializations.

Let $(t^U)_{U\in\mathscr{U}}$ be another family, where t^U is an isomorphism from $\mathscr{O}_U^n$ to $\mathscr{M}|_U$. For every $U \in \mathscr{U}$, there exists a unique matrix $B_U \in GL(n, \mathscr{O}(U))$ such that $t^U = s^U \circ B_U$. Let then (U, V) be a pair of elements of $\mathscr{U}$; on $U \cap V$, one has

$$t^V = s^V \circ B_V = s^U \circ A_{UV} \circ B_V = t^U \circ B_U^{-1} \circ A_{UV} \circ B_V.$$

Consequently, the Čech 1-cocycle $z^1(t)$ associated to the family $t = (t^U)_{U\in\mathscr{U}}$ is given by

$$z^1(t) = (B_U^{-1} A_{UV} B_V)_{U,V\in\mathscr{U}}.$$

Let $B^1(\mathscr{U}, GL(n))$ be the set of families $(B_U)_{U\in\mathscr{U}}$, where $B_U \in GL(n, \mathscr{O}(U))$. This is a group, and this group acts on $Z^1(\mathscr{U}, GL(n))$ by the above formula: $(B_U) \cdot (A_{UV}) = (B_U^{-1} A_{UV} B_V)$. The set of equivalence classes is denoted by $H^1(\mathscr{U}, GL(n))$ and is called the first set of Čech cohomology of $\mathscr{U}$ with values in $GL(n)$.

The set $Z^1(\mathscr{U}, GL(n))$ admits a privileged element, namely the 1-cocycle given by $A_{UV} = I_n$ for every pair (U, V). Its class in $H^1(\mathscr{U}, GL(n))$ is called the trivial class.

When $n = 1$, the set $Z^1(\mathscr{U}, GL(n))$ has a natural structure of an abelian groups, and the abelian group $B^1(\mathscr{U}, GL(n))$ acts on $Z^1(\mathscr{U}, GL(n))$ via a morphism of groups. Consequently, the set $H^1(\mathscr{U}, GL(n))$ has a natural of an abelian group; the trivial class is its neutral element.

Theorem **(4.9.12)**. — *Let* X *be a scheme and let* $\mathscr{U}$ *be an open covering of* X. *The previous construction furnishes a bijection* $c_{\mathscr{U}}$ *from the set of isomorphism classes of* $\mathscr{U}$*-free sheaves of rank n on* X *to the set* $H^1(\mathscr{U}, GL(n))$. *When* $n = 1$, *this bijection is an isomorphism of abelian groups.*

4.9.13. — Let us define a category whose objects are open coverings of X. Let $\mathscr{U}$ and $\mathscr{V}$ be open coverings of X; call any map $j\colon \mathscr{U} \to \mathscr{V}$ such that $U \subseteq j(U)$ for every $U \in \mathscr{U}$. a *morphism* from $\mathscr{U}$ to $\mathscr{V}$. Such a map j exists if and only the open covering $\mathscr{U}$ is finer than the open overing $\mathscr{V}$.

Moreover, the map j allows to define maps

(4.9.13.1) $$j^*\colon Z^1(\mathscr{V},\mathrm{GL}(n)) \to Z^1(\mathscr{U},\mathrm{GL}(n)),$$

(4.9.13.2) $$j^*\colon B^1(\mathscr{V},\mathrm{GL}(n)) \to B^1(\mathscr{U},\mathrm{GL}(n)),$$

and

(4.9.13.3) $$j^*\colon H^1(\mathscr{V},\mathrm{GL}(n)) \to H^1(\mathscr{U},\mathrm{GL}(n)).$$

In fact, associating to a given open covering $\mathscr{U}$ the set of Čech cocycles and the first Čech cohomology group is a contravariant functor from the category to open coverings to the category of (pointed) sets, and to the category of groups when $n = 1$.

Finally, we define the first Čech cohomology set of X with values in $\mathrm{GL}(n)$ by the colimit

$$H^1(X,\mathrm{GL}(n)) = \varinjlim_{j^*} H^1(\mathscr{U},\mathrm{GL}(n)),$$

indexed by the category of open coverings of $\mathscr{U}$.

Corollary **(4.9.14)**. — *There exists a unique map $\mathscr{M} \mapsto c(\mathscr{M})$ from the set of locally free sheaves of rank n on X to $H^1(X,\mathrm{GL}(n))$ such that $c(\mathscr{M})$ is the class of the Čech cohomology class $c(\mathscr{U},\mathscr{M})$, for every open covering $\mathscr{U}$ of X and every $\mathscr{U}$-trivial sheaf of rank n, $\mathscr{M}$. It is bijective, and an isomorphism of abelian groups if $n = 1$.*

Proof. — We observe that if $j\colon \mathscr{U} \to \mathscr{V}$ is a morphism of open coverings of X and $\mathscr{M}$ is a locally free sheaf of rank n on X which is $\mathscr{V}$-trivial, then $j^*(c_{\mathscr{V}}(\mathscr{M})) = c_{\mathscr{U}}(\mathscr{M})$. This implies the existence of the map $\mathscr{M} \mapsto c(\mathscr{M})$. Its bijective character follows from the fact that the maps $c_{\mathscr{U}}$ are bijective, and that for every locally free sheaf of rank n, $\mathscr{M}$, on X, there exists an open covering $\mathscr{U}$ such that $\mathscr{M}$ is $\mathscr{U}$-trivial. □

Remark **(4.9.15)**. — The constructions from linear algebra described in §4.9.8 associate free modules with free modules. on locally free sheaves of finite rank have a reflection on cohomology classes. For example, if $\mathscr{M}$ and $\mathscr{N}$ are locally free sheaves of ranks m and n on X, and $\mathscr{U}$ is an open covering of X such that $\mathscr{M}$ and $\mathscr{N}$ are $\mathscr{U}$-trivial, then $\mathscr{M} \oplus \mathscr{N}$, $\mathscr{H}om_{\mathscr{O}_X}$, $\mathscr{M} \otimes \mathscr{N}$,… are $\mathscr{U}$-trivial as well.

For example, assume that $\mathscr{M}$ and $\mathscr{N}$ are represented by cocycles $z_{\mathscr{U}}(\mathscr{N}) \in Z^1(\mathscr{U}, \mathrm{GL}(m))$ and $z_{\mathscr{U}}(\mathscr{N}) \in Z^1(\mathscr{U}, \mathrm{GL}(n))$, associated with given trivializations. The proof that $\mathscr{M} \oplus \mathscr{N}$, etc., are $\mathscr{U}$-trivial furnishes explicit $\mathscr{U}$-trivializations of these $\mathscr{O}_X$-modules, hence a particular cocycle. More precisely, the following formulas hold:

(4.9.15.1) $$z_{\mathscr{U}}(\mathscr{M} \oplus \mathscr{N}) = \begin{pmatrix} z_{\mathscr{U}}(\mathscr{M}) & \\ & z_{\mathscr{U}}(\mathscr{N}) \end{pmatrix},$$

(4.9.15.2) $$z_{\mathscr{U}}(\mathscr{M}^\vee) = \left(z_{\mathscr{U}}(\mathscr{M}^\vee)^{\mathrm{t}}\right)^{-1}$$

(4.9.15.3) $$z_{\mathscr{U}}(\mathscr{M} \otimes \mathscr{N}) = z_{\mathscr{U}}(\mathscr{M}) \otimes z_{\mathscr{U}}(\mathscr{N}) \qquad \text{(Kronecker product)},$$

(4.9.15.4) $$z_{\mathscr{U}}(\det(\mathscr{M})) = \det(z_{\mathscr{U}}(\mathscr{M})), \dots$$

Remark **(4.9.16)** (Comparison with differential geometry)

Let M be a $\mathscr{C}^k$-manifold; denote its sheaf of $\mathscr{C}^k$-functions by $\mathscr{C}^k_X$. Let n be an integer. A *vector bundle of rank n* on M is a manifold E endowed with a morphism $p\colon E \to M$, structures of real vector spaces on the fibers $E_x = p^{-1}(x)$, for $x \in M$, satisfying the following local triviality property: for every point x of M, there exists an open neighborhood U of x, an isomorphism of manifolds $\varphi_U\colon p^{-1}(U) \to \mathbf{R}^n \times U$ such that $\mathrm{pr}_2 \circ \varphi_U = p_U$ and such that for every $y \in U$, the map $\mathrm{pr}_1 \circ \varphi_U$ induces a linear bijection from $p^{-1}(y)$ to $\mathbf{R}^n$.

Given an open covering $\mathscr{U}$ of M and such a trivialization φ_U, for every open subset $U \in \mathscr{U}$, one defines a $\mathscr{C}^k$-map $f_{UV}\colon U \cap V \to \mathrm{GL}(n, \mathbf{R})$, for every pair (U, V) of elements of $\mathscr{U}$. Equivalently, one can view f_{UV} as an element of $\mathrm{GL}(n, \mathscr{C}^k(U \cap V))$. The family (f_{UV}) satisfies the cocycle relation: on $U \cap V \cap W$, one has $f_{UV} f_{VW} = f_{UW}$. The cohomology classes of this cocycle in $H^1(\mathscr{U}, \mathrm{GL}(n))$ and in $H^1(X, \mathrm{GL}(n))$ do not depend on the choice of the local trivializations φ_U and on the chosen open covering $\mathscr{U}$.

Let then $\mathscr{E}$ be the sheaf of $\mathscr{C}^k$-sections of E: for every open subset U of X, $\mathscr{E}(U)$ is the set of all $\mathscr{C}^k$-morphisms $s\colon U \to E$ such that $p \circ s = \mathrm{id}_U$. The vector space laws on the fibers $p^{-1}(m)$ endow this sheaf with the structure of a sheaf in $\mathbf{R}$-vector spaces. In fact, it is naturally a $\mathscr{C}^k_M$-module, and this module is locally free of rank n.

The sheaf of sections of a projection $\mathrm{pr}_2\colon \mathbf{R}^n \times U \to U$ identifies with the sheaf $(\mathscr{C}^k_U)^n$. Consequently, the trivialization φ_U of E on an open set $U \in \mathscr{U}$ gives rise to an isomorphism of $\mathscr{E}|_U$ with $\mathscr{C}^k_U$. In particular, $\mathscr{E}$ is $\mathscr{U}$-trivial; moreover its cohomology class coincides with that of E.

Conversely, given a locally free sheaf $\mathscr{F}$ of rank n on M, one can define a vector bundle of rank n on M whose sheaf of sections is equal to $\mathscr{F}$. For that, it suffices to choose an open covering $\mathscr{U}$ of M such that $\mathscr{F}$ is $\mathscr{U}$-trivial, trivialisations s^{U}, for $\mathrm{U} \in \mathscr{U}$, and to use the associated cocycle $z_{\mathscr{U}}(\mathscr{F})$ to glue trivial vector bundles $\mathbf{R}^n \times \mathrm{U}$.

This more geometric point of view on locally free sheaves given by the notion of vector bundle has an analogue in algebraic geometry.

Namely, if X is a scheme, a *vector bundle of rank n* on X is a scheme E endowed with an affine morphism $p\colon \mathrm{E} \to \mathrm{X}$, a locally free sheaf of rank n, $\mathscr{E}$, and an isomorphism $\mathrm{Sym}^* \mathscr{E} \xrightarrow{\sim} p_* \mathscr{O}_{\mathrm{E}}$ of quasi-coherent $\mathscr{O}_{\mathrm{X}}$-algebras. Then the sheaf of sections of p is isomorphic to the dual sheaf $\mathscr{E}^\vee$ of $\mathscr{E}$, and the X-scheme E is isomorphic to the spectrum $\mathrm{Spec}(\mathrm{Sym}^* \mathscr{E}^\vee)$ of the quasi-coherent $\mathscr{O}_{\mathrm{X}}$-algebra $\mathrm{Sym}^* \mathscr{E}^\vee$.

The reason for this duality can be explained as follows. Observe that if k is a ring, then $\mathbf{A}^n_k \simeq \mathrm{Spec}(k[\mathrm{T}_1, \dots, \mathrm{T}_n])$, and $k[\mathrm{T}_1, \dots, \mathrm{T}_n]$ is the symmetric algebra on a free k-module V of rank n, and then, $\mathrm{T}_1, \dots, \mathrm{T}_n$ are *linear forms* on V.

4.10. Invertible sheaves and divisors

Proposition **(4.10.1)**. — *Let* A *be a unique factorization domain and let* K *be its field of fractions.*

a) *If* $\dim(\mathrm{A}) = 1$, *then* A *is a principal ideal domain.*

b) *Let* $x \in \mathrm{K}$ *be such that* $x \in \mathrm{A}_{\mathfrak{p}}$ *for every prime ideal* $\mathfrak{p}$ *of height* 1 *in* A. *Then* $x \in \mathrm{A}$.

Proof. — a) Let a, b be non-zero elements of A; assume that a and b are coprime and let $\mathrm{I} = (a, b)$. Assume that $\mathrm{I} \neq \mathrm{A}$ and let $\mathfrak{p}$ be a maximal ideal of A containing I. Since $\dim(\mathrm{A}) = 1$, one has $\mathrm{ht}(\mathfrak{p}) = 1$. Since A is a unique factorization domain, there exists an irreducible element $\pi \in \mathrm{A}$ such that $\mathfrak{p} = (\pi)$; this implies that $\pi | a$ and $\pi | b$, and contradicts the hypothesis that a and b are coprime. Consequently, $\mathrm{I} = \mathrm{A}$. More generally for every pair (a, b) of non-zero elements of A, the ideal (a, b) they generate is the principal ideal generated by their gcd.

For any non-zero element $a \in \mathrm{A}$, let $v(a)$ denote the number of irreducible factors of a, counted with multiplicities. One thus has $v(a) = 0$ if a is a unit, $v(a) = 1$ if a is irreducible, and $v(ab) = v(a) + v(b)$. Let I be a non-zero ideal of A and let $a \in I - \{0\}$ be an element such that $v(a)$ is minimal. Let $b \in I - \{0\}$ and let $d = \gcd(a, b)$. One has $d \in (a, b) \subseteq I$, and $v(d) \leqslant v(a)$. By the choice of a, one has $v(d) = v(a)$. Since d divides a, this implies that there exists a unit $u \in A^\times$ such that $a = ud$; then $a|b$. We thus have shown that $I = (a)$.

b) Let us write $x = a/b$, where a, b are coprime elements of A. Let π be an irreducible factor of b. The ideal (π) is prime, and its height is equal to 1. By assumption, there exists $c \in A - (\pi)$ such that $cx \in A$. Set $d = cx$; one has $ac = bd$. Since π divides b, it is prime to a; since it divides ac it divides c. This contradiction imples that no irreducible element of A divides b, so that b is a unit. We thus have shown that $x \in A$. □

4.10.2. — Let X be an integral noetherian scheme and let $K = R(X)$ be the field of rational functions on X. Let us moreover assume that $\mathscr{O}_{X,x}$ is a unique factorization domain, for every $x \in X$.

Let $X^{(1)}$ be the set of points $x \in X$ such that $\dim(\mathscr{O}_{X,x}) = 1$. The closure $\overline{\{x\}}$ of a point $x \in X^{(1)}$ is an integral closed subscheme of codimension 1 in X; conversely, the generic point of an irreducible closed subset of X belongs to $X^{(1)}$.

Let $x \in X^{(1)}$. By proposition 4.10.1, the local ring $\mathscr{O}_{X,x}$ is a principal ideal ring with field of fractions R(X). Let p_x be any generating element of the maximal ideal of $\mathscr{O}_{X,x}$. For every non-zero element f of R(X), there exists a unique integer $n \in \mathbf{Z}$ such that f/p_x^n is a unit in $\mathscr{O}_{X,x}$; we denote this integer by $\operatorname{ord}_x(f)$ and we call it the order of vanishing of f along D, where $D = \overline{\{x\}}$. The map $\operatorname{ord}_x \colon R(X)^\times \to \mathbf{Z}$ given by $f \mapsto \operatorname{ord}_x(f)$ is a morphism of abelian groups.

Lemma **(4.10.3)**. — *Let $f \in R(X)^\times$.*

a) *The set of elements $x \in X^{(1)}$ such that $\operatorname{ord}_x(f) \neq 0$ is finite;*
b) *One has $f \in \mathscr{O}_X(X)$ if and only if $\operatorname{ord}_x(f) \geqslant 0$ for every $x \in X^{(1)}$;*
c) *One has $f \in \mathscr{O}_X(X)^\times$ if and only if $\operatorname{ord}_x(f) = 0$ for every $x \in X^{(1)}$;*

Proof. — a) There exists non-empty open subscheme U of X such that f belongs to the image of $\mathscr{O}(U)^\times$ in $R(X)^\times$. One has $\operatorname{ord}_x(f) = 0$ for every point $x \in X^{(1)} \cap U$. Since X is noetherian, the closed subset $X - U$ has finitely many irreducible components, all of codimension $\geqslant 1$; in particular $X^{(1)} \cap (X - U)$ is finite.

b) If $f \in \mathscr{O}_X(X)$, then $f \in \mathscr{O}_{X,x}$ for every $x \in X$; in particular, $\operatorname{ord}_x(f) \geqslant 0$ for every $x \in X^{(1)}$. Conversely, assume that $\operatorname{ord}_x(f) \geqslant 0$ for every $x \in X^{(1)}$. Let y

be a point of X; it follows from proposition 4.10.1 that $f \in \mathscr{O}_{X,y}$. Consequently, $f \in \mathscr{O}_X(X)$.

c) Let $g = f^{-1}$. One has $f \in \mathscr{O}_X(X)^\times$ if and only if $f \in \mathscr{O}_X(X)$ and $g \in \mathscr{O}_X(X)$. It thus follows from *b*) that $f \in \mathscr{O}_X(X)^\times$ if and only if $\operatorname{ord}_x(f) \geqslant 0$ and $\operatorname{ord}_x(g) \geqslant 0$ for every $x \in X^{(1)}$. Since $\operatorname{ord}_x(g) = -\operatorname{ord}_x(f)$ for every $x \in X^{(1)}$, this is equivalent to $\operatorname{ord}_x(f) = 0$ for every $x \in X^{(1)}$. ☐

4.10.4. — Let $\operatorname{Div}(X) = \mathbf{Z}^{(X^{(1)})}$ be the free abelian group on $X^{(1)}$; an element of Div(X) is called a cycle of codimension 1, or a *divisor*, on X. Formally, a divisor is a function with finite support from $X^{(1)}$ to **Z**, but it is customary to write a divisor under the form $\sum n_D D$, where D ranges over the set $X^{(1)}$ or, equivalently, over the set of irreducible closed subsets of codimension 1 in X.

A divisor $\sum n_D D$ is said to be effective if $n_D \geqslant 0$ for every D.

By the preceding lemma, there is a morphism of abelian groups

$$\operatorname{div}\colon R(X)^\times \to \operatorname{Div}(X), \qquad \operatorname{div}(f) = \sum_{x \in X^{(1)}} \operatorname{ord}_x(f)[x].$$

Moreover, $f \in \mathscr{O}_X(X)$ if and only if $\operatorname{div}(f)$ is effective; and $f \in \mathscr{O}_X(X)^\times$ if and only if $\operatorname{div}(f) = 0$.

Let U be a non-empty open subscheme of X. Then $U^{(1)} = U \cap X^{(1)}$, and the restriction of functions induces a morphism of abelian groups from Div(X) to Div(U). Similarly, the generic point of X belongs to U and the restriction map from R(X) to R(U) induces an isomorphism of fields. For every $f \in R(X)^\times$, one has

$$\operatorname{div}_X(f)|_U = \operatorname{div}_U(f|_U).$$

In particular, $f \in \mathscr{O}_X(U)$ if and only if $\operatorname{div}(f)|_U$ is effective, and $f \in \mathscr{O}_X(U)^\times$ if and only if $\operatorname{div}(f)|_U = 0$.

Lemma **(4.10.5).** — *Let* X *be a noetherian integral scheme, let* ξ *be its generic point, let* $\mathscr{L}$ *be an invertible* $\mathscr{O}_X$*-module. There exists a unique map*

$$\mathscr{L}_\xi - \{0\} \to \operatorname{Div}(X), \qquad s \mapsto \operatorname{div}(s)$$

such that, for every open subset U *of* X*, every basis* ε *of* $\mathscr{L}|_U$*, and every rational function* f *on* X, $\operatorname{div}(f\varepsilon)|_U = \operatorname{div}(f)|_U$.

Proof. — By definition of an invertible $\mathscr{O}_X$-module, the scheme X is covered by open subsets U such that $\mathscr{L}|_U$ has a basis ε; then the map $f \mapsto f\varepsilon_\xi$ is a bijection from R(X) to $\mathscr{L}_\xi$. The given formula thus defines $\operatorname{div}(s)|_U$, hence the uniqueness of such a map. To prove its existence, let ε' be a second basis of $\mathscr{L}|_U$. Then there exists an element $u \in \mathscr{O}_X(U)^\times$ such that $\varepsilon' = u\varepsilon$, hence $\operatorname{div}(u)|_U = 0$. For $f, f' \in R(X)^\times$, the relation $f\varepsilon = f'\varepsilon'$ is equivalent to the relation $f = uf'$; it implies that

$$\operatorname{div}(f)|_U = \operatorname{div}(u)|_U + \operatorname{div}(f')|_U.$$

□

Lemma **(4.10.6)**. — *Let* A *be a unique factorization domain and let* M *be an invertible* A*-module. Let f and g be coprime elements of* A*; such that* M_f *and* M_g *are free. Then* M *is free.*

Proof. — Let m, n be elements such that $m/1$ is a basis of M_f and $n/1$ is a basis of M_g. Then $m/1$ and $n/1$ are bases of M_{fg}, hence there exists an invertible element $a/b \in A_{fg}^\times$ such that $(a/b)n/1 = m/1$. Since M is torsion-free, this implies $an = bm$, and we may moreover assume that a and b are coprime. The irreducible factors of a and b are among those of fg; let us write $a = a_1a2$ and $b = b_1b_2$, where the irreducible factors of a_1 and b_1 divide f, and the irreducible factors of a_2 and b_2 divide g. Then $a_1a_2n = b_1b_2m\ b_2m/1 = (a_1a_2/b_1)n/1$ in M_g, so that b_1 divides a_1a_2 in A_g, hence b_1 divides a_1, hence we may assume that $b_1 = 1$. Similarly, we may assume that $a_2 = 1$. In other words, the irreducible factors of b divide g, and those of a divide f. Now, $an/1$ generates M_f, and $bm/1$ generates M_g. This implies that the restriction of the invertible sheaf $\widetilde{M}$ to $\operatorname{Spec}(A) - V(f, g)$ is free.

□

4.10.7. — Let X be a locally noetherian integral scheme. For simplicity, we assume that X is noetherian and integral. Let $\kappa(X)$ be its field of fractions; this is the local ring of X at its generic point. Let also $\mathscr{M}_X$ be the constant sheaf on X with value $\kappa(X)$: for every non-empty open subset U of X, one has $\mathscr{M}_X(U) = \kappa(X)$. If U is affine, say $U = \operatorname{Spec}(A)$, then $\kappa(X) = \operatorname{Frac}(A)$ is an A-algebra, and $\mathscr{M}_X|_U = \widetilde{\kappa(X)}$; in particular, $\mathscr{M}_X$ is a quasi-coherent $\mathscr{O}_X$-module.

Theorem **(4.10.8)**. — *Let* A *be a unique factorization domain and let* $X = \operatorname{Spec}(A)$. *Then* $\operatorname{Pic}(X) = 0$*: every invertible sheaf on* X *is free.*

Proof. — Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Let $\mathscr{U}$ be an open covering of X such that $\mathscr{L}$ is $\mathscr{U}$-trivial; since X is quasi-compact, we may also assume that $\mathscr{U}$ is finite and that every open subset $U \in \mathscr{U}$ is of the form $D(a)$, for some $a \in A$.

Let us show the following result: let $a_1, a_2 \in \mathrm{A}$ be non-zero elements and let a be their gcd; if $\mathscr{L}|_{\mathrm{D}(a_1)}$ and $\mathscr{L}|_{\mathrm{D}(a_2)}$, then $\mathscr{L}|_{\mathrm{D}(a)}$ is trivial. For $i \in \{1, 2\}$, let indeed s_i be an isomorphism from $\mathscr{O}_{\mathrm{X}}|_{\mathrm{D}(a_i)}$ to $\mathscr{L}|_{\mathrm{D}(a_i)}$; let f be the unique element of $\mathscr{O}_{\mathrm{X}}(\mathrm{D}(a_1) \cap \mathrm{D}(a_2))$ such that $s_1|_{\mathrm{D}(a_1)\cap\mathrm{D}(a_2)} = f s_2|_{\mathrm{D}(a_1)\cap\mathrm{D}(a_2)}$.

(Unfinished) □

4.11. Graded modules and quasi-coherent sheaves on homogeneous spectra

Lemma **(4.11.1)**. — *Let k be a ring and let* A *be a graded k-algebra; let* $\mathrm{X} = \mathrm{Proj}(\mathrm{A})$. *Let* M *be a graded* A*-module, let d be an integer such that $d \geqslant 0$.*

a) *There exists a unique morphism of quasi-coherent sheaves, $\theta \colon \widetilde{\mathrm{M}} \otimes_{\mathscr{O}_{\mathrm{X}}} \mathscr{O}_{\mathrm{X}}(d) \to \widetilde{\mathrm{M}(d)}$, such that $\theta((m/f^p) \otimes (g/f^q)) = gm/f^{p+q}$ for every homogeneous element $f \in \mathrm{A}_+$, every homogeneous element $g \in \mathrm{A}$ such that $\deg(g) = q \deg(f) + d$, and every homogeneous element $m \in \mathrm{M}$ such that $\deg(m) = p \deg(f)$.*

b) *For every $m \in \mathrm{M}_d$, there exists a unique section $s_m \in \Gamma(\mathrm{X}, \widetilde{\mathrm{M}(d)})$ such that $s_m|_{\mathrm{D}_+(f)} = m/1$, for every $f \in \mathrm{A}_+$. The map $m \mapsto s_m$ is a k-morphism from M_d to $\Gamma(\mathrm{X}, \widetilde{\mathrm{M}(d)})$.*

c) *Let $f \in \mathrm{A}_d$. The section $s_f|_{\mathrm{D}_+(f)}$ is a basis of $\mathscr{O}_{\mathrm{X}}(d)|_{\mathrm{D}_+(f)}$, and the restriction to $\mathrm{D}_+(f)$ of the morphism θ is an isomorphism.*

Proof. — a) The given formula describes the restriction of θ to an arbitrary affine open subset $\mathrm{D}_+(f)$. It thus suffices to check that these requirements are compatible, a verification left to the reader.(3)

b) This is straightforward.

c) Let us first prove that for every open subscheme U of $\mathrm{D}_+(f)$, and every section $s \in \Gamma(\mathrm{U}, \widetilde{\mathrm{M}(d)})$, there exists a unique element $t \in \Gamma(\mathrm{U}, \widetilde{\mathrm{M}})$ such that $s = \theta(t \otimes s_f)$. We may assume that there exists $g \in \mathrm{A}_+$ such that $\mathrm{U} = \mathrm{D}_+(fg)$; then there exists an homogeneous element $m \in \mathrm{M}$ such that $s = m/(fg)^p$, and $d = \deg(m) - p\deg(f) - p\deg(g)$. The formula $s = f(gm)/(fg)^{p+1}$ expresses s as $\theta(t \otimes s_f)$, where $t \in \Gamma(\mathrm{U}, \widetilde{\mathrm{M}})$ is represented by $(gm)/(fg)^{p+1}$, a homogeneous fraction of degree 0. Since $\mathrm{A}_{(f)}$ is a subring of A_f in which the element f is invertible, this is the unique such expression.

Applied to M = A, this shows that s_f is a basis of $\mathscr{O}_{\mathrm{X}}(d)|_{\mathrm{D}_+(f)}$. □

(3)...!

Proposition **(4.11.2)**. — *Let k be a ring and let* A *be a graded k-algebra which is generated by* A_1 *as an* A_0*-algebra; let* X = Proj(A).

a) *For every integer $d \in \mathbf{Z}$, the quasi-coherent sheaf $\mathscr{O}_X(d)$ is invertible.*

b) *For every graded* A*-module* M *and every integer d, the canonical morphism $\theta\colon \widetilde{M} \otimes \mathscr{O}_X(d) \to \widetilde{M(d)}$ is an isomorphism.*

c) *In particular, for every pair (d, e) of integers, one has an isomorphism $\mathscr{O}_X(d) \otimes \mathscr{O}_X(e) \simeq \mathscr{O}_X(d+e)$.*

Proof. — Let $d \in \mathbf{N}$. For every $f \in A_1$, the restriction of $\mathscr{O}_X(d)$ to the open subscheme $D_+(f) = D_+(f^d)$ of X is locally free of rank 1. Since A is generated by elements of A_1, these affine open subschemes consistute an open covering of X, so that $\mathscr{O}_X(d)$ is locally free of rank 1. For the same reason, the morphism $\theta_M\colon \widetilde{M}\otimes\mathscr{O}_X(d) \to \widetilde{M(d)}$ is an isomorphism. In particular, for every integer $e \in \mathbf{Z}$, the morphism $\theta_{A(e)}$ is an isomorphism from $\mathscr{O}_X(e)\otimes\mathscr{O}_X(d)$ to $\mathscr{O}_X(d+e)$. Taking $e = -d$, this implies that $\mathscr{O}_X(-d)$ is isomorphic to the dual of $\mathscr{O}_X$, hence is locally free of rank 1 as well.

This establishes the proposition, except for the isomorphism of part *c*) when $d < 0$. To prove this remaining case, we can start from the isomorphism $\mathscr{O}_X(e) \simeq \mathscr{O}_X(-d) \otimes \mathscr{O}_X(d+e)$; tensoring both sides by $\mathscr{O}_X(d)$, we obtain an isomorphism

$$\mathscr{O}_X(d) \otimes \mathscr{O}_X(e) \simeq \mathscr{O}_X(d) \otimes \mathscr{O}_X(-d) \otimes \mathscr{O}_X(d+e),$$

hence the required isomorphism if we use the fact that $\mathscr{O}_X(d) \otimes \mathscr{O}_X(-d)$ is isomorphic to $\mathscr{O}_X$. □

Example **(4.11.3)**. — Let k be a ring; the case of $X = \mathbf{P}^n_k = \operatorname{Proj}(k[T_0, \dots, T_n])$ is extremly important for algebraic geometry. The graded k-algebra $A = k[T_0, \dots, T_n]$ being generated by elements of degree 1, namely, $T_0, \dots, T_n$, the quasi-coherent sheaf $\mathscr{O}_X(1)$ is locally free of rank 1.

Moreover, let us show that for every integer d, the k-linear morphism $P \mapsto s_P$ from A_d to $\Gamma(\mathbf{P}^n_k, \mathscr{O}_X(d))$ is an isomorphism; in particular, $\Gamma(\mathbf{P}^n_k, \mathscr{O}_X(d)) = 0$ for $d < 0$. Let thus $\sigma \in \Gamma(\mathbf{P}^n_k, \mathscr{O}_X(d))$. For every $i \in \{0, \dots, d\}$, there is a unique polynomial $P_i \in k[T_0/T_i, \dots, T_n/T_i]$ such that $\sigma|_{D_+(T_i)} = P_i s_{T_i}^{\otimes d}$. On $D_+(T_iT_j)$, one thus has $s_{T_i} = (T_i/T_j)s_{T_j}$, leading to the equality $P_iT_i^d = P_jT_j^d$ of rational functions. Let P be this common rational function; looking at the formula $P = P_iT_i^d$, we see that its denominator is a power of T_i; but switching to $j \neq i$ shows that its denominator is a power of T_j. Consequently, P is a polynomial. Since P_i is homogeneous of degree 0, P is homogeneous of degree d; in particular, one has $P = 0$ if $d < 0$, and $\sigma = 0$. Finally, viewed as an element of $A(d)_{(T_i)}$, one has $s_{T_i}|_{D_+(T_i)} = T_i/1$, hence $\sigma|_{D_+(T_i)} = P_i T_i^d/1 = P/1$, so that $\sigma|_{D_+(T_i)} = s_P|_{D_+(T_i)}$; consequently, $\sigma = s_P$, and the polynomial P is the unique one such that this relation holds.

4.11.4. — Let us assume that A is generated by A_1 as an A_0-algebra. For every quasi-coherent sheaf $\mathscr{F}$ on Proj(A), we define a graded abelian group by

$$\Gamma(\mathscr{F}) = \bigoplus_{n\in\mathbf{Z}} \Gamma(\mathrm{Proj}(A), \mathscr{F}(n)).$$

Any homogeneous element $a \in A$ of degree m defines a global section of $\mathscr{O}(m)$; consequently, for $m \in \Gamma(\mathrm{Proj}(A), \mathscr{F}(n))$, we have $a \otimes m \in \Gamma(\mathrm{Proj}(A), \mathscr{F}(m+n))$. This endows the graded abelian group $\Gamma(\mathscr{F})$ with the structure of a graded A-module.

The association $\mathscr{F} \mapsto \Gamma(\mathscr{F})$ is functorial.

4.11.5. — Let M be a graded A-module. There is a canonical morphism of graded A-modules $\alpha_M \colon M \to \Gamma(\widetilde{M})$. It associates with a homogeneous element $m \in M$ of degree n the corresponding global section of $\widetilde{M}(n))$. These morphisms are functorial in M.

Let $\mathscr{F}$ be a quasi-coherent sheaf on Proj(A). Let $f \in A$ be a homogeneous element with degree $n > 0$. On $D_+(f)$, a section of $\widetilde{\Gamma(\mathscr{F})}$ takes the form s/f^p, where $s \in \Gamma(\mathrm{Proj}(A), \mathscr{F}(np))$. On the other hand, we may view f as a global section of $\mathscr{O}(n)$, and it does not vanish on $D_+(f)$, so that s/f^p may be interpreted as a section of $\mathscr{F}$ on $D+(f)$. This defines a morphism of sheaves $\beta_{\mathscr{F}} \colon \widetilde{\Gamma(\mathscr{F})} \to \mathscr{F}$. It is functorial in $\mathscr{F}$.

For any graded A-module M, one has

$$\beta_{\widetilde{M}} \circ \widetilde{\alpha_M} = \mathrm{id}_{\widetilde{M}} .$$

For any quasi-coherent sheaf $\mathscr{F}$ on Proj(A), one has

$$\Gamma(\beta_{\Gamma(\mathscr{F})}) \circ \alpha_{\Gamma(\mathscr{F})} = \mathrm{id}_{\Gamma(\mathscr{F})} .$$

Proposition (**4.11.6**). — *Assume that the ideal* A_+ *is generated by finitely many elements of degree* 1. *Then, for every quasi-coherent sheaf* $\mathscr{F}$ *on* Proj(A), *the morphism* $\beta_{\mathscr{F}} \colon \widetilde{\Gamma(\mathscr{F})} \to \mathscr{F}$ *is an isomorphism.*

Corollary (**4.11.7**). — *Assume that the ideal* A_+ *is generated by finitely many elements of degree* 1.

a) *Every quasi-coherent sheaf on* Proj(A) *is of the form* $\widetilde{M}$, *for some graded* A-*module* M.

b) *Every finitely generated quasi-coherent sheaf on* Proj(A) *is of the form* $\widetilde{M}$, *for some finitely generated graded* A-*module* M.

CHAPTER 5

MORPHISMS OF SCHEMES

5.1. Morphisms of finite type, morphisms of finite presentation

Definition **(5.1.1)**. — *Let* A *be a ring and let* B *be an* A*-algebra. One says that* B *is a* finitely presented A*-algebra if there exists a family* $(b_1, \dots, b_n)$ *of elements of* B *such that the unique morphism of* A*-algebras* $\varphi\colon A[T_1, \dots, T_n] \to B$ *such that* $\varphi(T_i) = b_i$ *for every* i *is surjective and its kernel is a finitely generated ideal.*

Recall that one says that B is a finitely generated A-algebra if there exists a finite family $(b_1, \dots, b_n)$ of elements of B such that the morphism $\varphi\colon A[T_1, \dots, T_n]$ of A-algebras such that $\varphi(T_i) = b_i$ for every i is surjective.

If the ring A is noetherian, then the ring $A[T_1, \dots, T_n]$ is noetherian as well, so that every finitely generated A-algebra is finitely presented.

If $f\colon A \to B$ is a ring morphism, it endowes B with the structure of an A-algebra and we also say that f is of finite type (resp. is of finite presentation) to mean that the A-algebra B is of finite type (resp. of finite presentation).

Example **(5.1.2)**. — We have seen in example a) of §1.2.5 that for every element a of A, the morphism $\varphi\colon A[T] \to A_a$ of A-algebras such that $\varphi(T) = 1/a$ is surjective and its kernel is generated by $(1 - aT)$. Consequently, the A-algebra A_a is finitely presented.

Lemma **(5.1.3)**. — *Let* A *be a ring and let* B *be a finitely presented* A*-algebra. For every integer* m*, the kernel of every surjective morphism* $\varphi\colon A[X_1, \dots, X_m] \to B$ *of* A*-algebras is finitely generated.*

Proof. — Let $\varphi\colon A[X_1, \dots, X_m] \to B$ be a surjective morphism of A-algebras. Let n be an integer and let $\psi\colon A[Y_1, \dots, Y_n] \to B$ be a surjective morphism whose kernel is finitely generated.

For every $i \in \{1, \dots, m\}$, let $P_i \in A[Y_1, \dots, Y_n]$ be a polynomial such that $\psi(P_i) = \varphi(X_i)$; let $\alpha: A[X] \to A[Y]$ be the unique morphism of A-algebras such that $\alpha(X_i) = P_i$, for every i; one has $\psi \circ \alpha = \varphi$. For every $j \in \{1, \dots, n\}$, let $Q_j \in A[X_1, \dots, X_m]$ be a polynomial such that $\varphi(Q_j) = \psi(Y_i)$; let $\beta: A[Y] \to A[X]$ be the unique morphism of A-algebras such that $\beta(Y_j) = Q_j$, for every j; one has $\varphi \circ \beta = \psi$.

Let (N_k) be a finite family of polynomials in A[Y] which generates $\operatorname{Ker}(\psi)$ and let I be the ideal of A[X] generated by the polynomials $X_i - \beta \circ \alpha(X_i)$ and the polynomials $\beta(N_k)$. It is is finitely generated, by construction; to conclude the proof of the lemma, it suffices to prove that it equals $\operatorname{Ker}(\varphi)$.

Observe that for every polynomial $P \in A[X]$, one has $P - \beta \circ \alpha(P)) \in \operatorname{Ker}(\varphi)$, since $\varphi \circ \beta \circ \alpha(P) = \psi \circ \alpha(P) = \varphi(P)$. Moreover, $\beta(N_k) \in \operatorname{Ker}(\varphi)$, for every k, since $\varphi \circ \beta(N_k) = \psi(N_k) = 0$. In particular, the ideal I is contained in $\operatorname{Ker}(\varphi)$.

Let us then observe that for every polynomial $P \in A[X]$, one has $P - \beta \circ \alpha(P) \in I$. Indeed, if $p: A[X] \to A[X]/I$ is the canonical surjection, then p and $p \circ \alpha \circ \beta$ are two morphisms of A-algebras from A[X] to A[X]/I which coincides on $X_1, \dots, X_m$; Their equalizer is a sub-algebra of A[X] which contains the indeterminates $X_1, \dots, X_n$, hence is equal to the whole of A[X]. Let finally $P \in \operatorname{Ker}(\varphi)$. Then $\alpha(P) \in \operatorname{Ker}(\psi)$, since $\psi \circ \alpha(P) = \varphi(P) = 0$. Since the ideal I contains the image by β of a generating family of $\operatorname{Ker}(\psi)$, one has $\beta(\alpha(P)) \in I$. Finally, the relation $P = (P - \beta \circ \alpha(P)) + \beta \circ \alpha(P)$ shows that $P \in I$. This concludes the proof. □

Lemma **(5.1.4)**. — *Let* A *be a ring, let* B *be an* A*-algebra and let* C *be a* B*-algebra.*

a) *If* B *is finitely generated over* A *and* C *is finitely generated over* B*, then* C *is finitely generated over* A.

b) *If* C *is finitely generated over* A*, then it is finitely generated over* B.

c) *If* B *is finitely presented over* A *and* C *is finitely presented over* B*, then* C *is finitely presented over* A.

Proof. — We write $f: A \to B$ and $g: B \to C$ for the canonical ring morphisms.

a) Let $b_1, \dots, b_m \in B$ such that $B = A[b_1, \dots, b_m]$; let $c_1, \dots, c_n \in C$ such that $C = B[c_1, \dots, c_n]$. Then, the subring $A[g(b_1), \dots, g(b_m), c_1, \dots, c_n]$ of C is a finitely generated A-algebra which contains the image of B under g, as well as $c_1, \dots, c_n$; it is thus equal to C, which shows that C is a finitely generated A-algebra.

b) Let $c_1, \dots, c_n$ be elements of C such that $C = A[c_1, \dots, c_n]$. Then one has $C = B[c_1, \dots, c_n]$, since this subring of C contains the image of A and the elements $c_1, \dots, c_n$. Consequently, C is a finitely generated B-algebra.

c) Let $b_1, \dots, b_m \in B$ such that $B = A[b_1, \dots, b_m]$; let $\varphi \colon A[X_1, \dots, X_m] \to B$ be the unique morphism of A-algebras such that $\varphi(X_i) = b_i$ for every i; by lemma 5.1.3, $\operatorname{Ker}(\varphi)$ is finitely generated. Similarly, let $c_1, \dots, c_n \in C$ such that $C = B[c_1, \dots, c_n]$ and let $\psi \colon B[Y_1, \dots, Y_n] \to C$ be the unique morphism of B-algebras such that $\psi(Y_j) = c_j$ for every j; then $\operatorname{Ker}(\psi)$ is finitely generated. Let $\theta \colon A[X_1, \dots, X_m, Y_1, \dots, Y_n] \to C$ be the unique morphism of A-algebras such that $\theta(X_i) = g(b_i)$ for every i and $\theta(Y_j) = c_j$ for every j.

To shorten the notation, we write $A[X]$ for $A[X_1, \dots, X_m]$, etc. Let $(P_1, \dots, P_r)$ be polynomials in $A[X]$ generating $\operatorname{Ker}(\varphi)$. Let $(Q_1, \dots, Q_s)$ be polynomials in $B[Y]$ generating $\operatorname{Ker}(\psi)$. Let us extend φ to a morphism φ' from $A[X_1, \dots, X_m, Y_1, \dots, Y_n]$ to $B[Y_1, \dots, Y_n]$ such that $\varphi(X_i) = \varphi'(X_i)$ for every i, and $\varphi'(Y_j) = Y_j$ for every j; it is surjective. Consequently, there exist polynomials $(Q'_1, \dots, Q'_s)$ in $A[X, Y]$ such that $\varphi'(Q'_j) = Q_j$ for every j.

One has $P_1, \dots, P_r, Q'_1, \dots, Q'_s \in \operatorname{Ker}(\theta)$. Conversely, let $R \in \operatorname{Ker}(\theta)$. Since $\theta = \psi \circ \varphi'$, one has $\psi(\varphi'(R)) = 0$. Consequently, there are polynomials $R_j \in B[Y]$ such that $\varphi'(R) = \sum R_j Q_j$. Since φ' is surjective, there are polynomials $R'_j \in A[X, Y]$ such that $R_j = \varphi(R'_j)$ for every j. Then $R - \sum R'_j Q'_j \in \operatorname{Ker}(\psi)$, so that there are polynomials S_i in $A[X]$ such that $R = \sum S_i P_i + \sum R'_j Q'_j$. This shows that $\operatorname{Ker}(\theta) \subseteq (P_1, \dots, P_r, Q'_1, \dots, Q'_s)$, hence the equality. This proves that C is a finitely presented A-algebra, as claimed. □

Lemma **(5.1.5)**. — *Let* A *be a ring, let* B *and* C *be* A*-algebras.*

a) *If* B *is finitely generated (resp. finitely presented), then* $B \otimes_A C$ *is a finitely generated (resp. finitely presented)* C*-algebra;*

b) *If* B *and* C *are finitely generated (resp. finitely presented), then so is* $B \otimes_A C$.

Proof. — a) Let n be an integer and let $\varphi \colon A[X_1, \dots, X_n] \to B$ be a surjective morphism of A-algebras. Then the morphism $\varphi \otimes_A \mathrm{id}_C \colon A[X_1, \dots, X_n] \otimes_A C \to B$ is surjective. Since the natural morphism from $A[X_1, \dots, X_n] \otimes_A C$ to $C[X_1, \dots, X_n]$ is an isomorphism, this implies that $B \otimes_A C$ is a finitely generated C-algebra.

Assume that B is finitely presented and let $N = \ker(\varphi)$; it is a finitely generated ideal of $A[X_1, \dots, X_n]$. Since the kernel of $\varphi \otimes_A \mathrm{id}_C$ is generated by N, it is finitely generated as well, and $B \otimes_A C$ is a finitely presented C-algebra.

b) Assertion *b*) then follows from *a*) and from lemma 5.1.4. □

Definition **(5.1.6)**. — *Let $f \colon Y \to X$ be a morphism of schemes.*

One says that f is locally of finite type *(resp. is* locally of finite presentation*) if for every point y of Y, there exists an affine open neighborhood V of y in Y and an affine open neighborhood U of $f(y)$ in X such that $\mathscr{O}_Y(V)$ is a finitely generated $\mathscr{O}_X(U)$-algebra (resp. a finitely presented $\mathscr{O}_X(U)$-algebra).*

One says that f is of finite type *(resp. is* of finite presentation*) if it is locally of finite type (resp. locally of finite presentation) and quasi-compact.*[1]

Remark **(5.1.7)**. — If f is locally of finite type and X is locally noetherian, then f is locally of finite presentation.

Let indeed $y \in Y$ and let $x = f(y)$. Let U be an affine open neighborhood of x and let V be an affine open neighborhood of y contained in $f^{-1}(U)$ such that $\mathscr{O}_Y(V)$ is a finitely generated $\mathscr{O}_X(U)$-algebra. Since U is locally noetherian, $\mathscr{O}_X(U)$ is a noetherian ring. Consequently, $\mathscr{O}_Y(V)$ is a finitely presented $\mathscr{O}_X(U)$-algebra.

Lemma **(5.1.8)**. — *Let $f \colon Y \to X$ be a morphism of schemes. Assume that f is locally of finite type (resp. locally of finite presentation). Let $y \in Y$, let $x = f(y)$, let U be an affine open neighborhood of x and let V be an open neighborhood of y. There exists an affine open neighborhood V′ of y which is contained in $f^{-1}(U) \cap V$ such that $\mathscr{O}_Y(V')$ is a finitely generated (resp. a finitely presented) $\mathscr{O}_X(U)$-algebra.*

Proof. — By assumption, there exists an affine open neighborhood V_1 of y in Y, and an affine open neighborhood U_1 of x in X such that $\mathscr{O}_Y(V_1)$ is a finitely generated $\mathscr{O}_X(U_1)$-algebra (resp. a finitely presented $\mathscr{O}_X(U_1)$-algebra).

Let $a \in \mathscr{O}_X(U_1)$ be such that $x \in D(a)$ and $D(a) \subseteq U \cap U_1$; let $U_2 = D(a)$ and let $V_2 = f^{-1}(U_2) \cap V_1$. Then U_2 and V_2 are affine open neighborhoods of x and y respectively such that $f(U_2) \subseteq V_2$. One has $\mathscr{O}_X(U_2) = \mathscr{O}_X(U_1)_a$, $\mathscr{O}_Y(V_2) = \mathscr{O}_Y(V_1)_a \simeq \mathscr{O}_Y(V_1) \otimes_{\mathscr{O}_X(U_1)} \mathscr{O}_X(U_2)$, so that the morphism $\mathscr{O}_X(U_2) \to \mathscr{O}_Y(V_2)$

[1] The standard definition of a morphism of finite presentation imposes that it be quasi-separated. I need to correct this at some point.

is deduced from the morphism $\mathscr{O}_X(U_1) \to \mathscr{O}_Y(V_1)$ by base change; it is thus finitely generated (resp. finitely presented).

Let then $a' \in \mathscr{O}_X(U)$ be such that $x \in D(a')$ and $D(a') \subseteq U_2$; let $U_3 = D(a')$ and let $V_3 = f^{-1}(U_3) \cap V_2$. By the same argument, U_3 and V_3 are affine open neighborhoods of x and y respectively, one has $f(U_3) \subseteq V_3$ and the corresponding morphism $\mathscr{O}_X(U_3) \to \mathscr{O}_Y(V_3)$ is finitely generated (resp. finitely presented).

Let now $b \in \mathscr{O}_Y(V_3)$ be such that $y \in D(b)$ and $D(b) \subseteq V \cap V_3$; let $V' = D(b)$. Then V' is an affine open neighborhood of y contained in V. By example 5.1.2, the morphism $\mathscr{O}_Y(V_3) \to \mathscr{O}_Y(V') = \mathscr{O}_Y(V_3)_b$ is finitely presented, as well as the morphism $\mathscr{O}_X(U) \to \mathscr{O}_X(U_3) = \mathscr{O}_X(U)_{a'}$. Consequently, the composition

$$\mathscr{O}_X(U) \to \mathscr{O}_X(U_3) \to \mathscr{O}_Y(V_3) \to \mathscr{O}_Y(V')$$

is finitely generated (resp. is finitely presented). This concludes the proof of the lemma. □

Corollary **(5.1.9)**. — *Let $f: Y \to X$ be a morphism of schemes. Let U be an open subscheme of X and let V be an open subscheme of $f^{-1}(U)$. If f is locally of finite type (resp. locally of finite presentation), then the morphism $f|_V: V \to U$ deduced from f by restriction is locally of finite type (resp. locally of finite presentation) as well.*

Corollary **(5.1.10)**. — *If f is of finite type (resp. of finite presentation), then for every open subscheme U of X, the morphism $f_U: f^{-1}(U) \to U$ deduced from f is of finite type (resp. of finite presentation).*

Proof. — By corollary 5.1.9, the morphism f_U is locally of finite type (resp. of finite presentation). Since it is also quasi-compact, this implies the corollary. □

Proposition **(5.1.11)**. — *Let A be a ring, let B be an A-algebra, let $X = \operatorname{Spec}(A)$, let $Y = \operatorname{Spec}(B)$ and let $f: Y \to X$ be the associated morphism of schemes. If f is of finite type (resp. of finite presentation), then B is a finitely generated (resp. a finitely presented) A-algebra.*

Proof. — By lemma 5.1.8, every point y of Y has an affine open neighborhood V'_y such that $\mathscr{O}_Y(V'_y)$ is a finitely generated (resp. a finitely presented) A-algebra. Let then $b_y \in B$ be an element such that $y \in D(b_y)$ and $D(b_y) \subseteq V'_y$; let $V_y = D(b_y)$. One has $\mathscr{O}_Y(V_y) = \mathscr{O}_Y(V'_y)_{b'_y}$, where $b'_y = b_y|_{V_y}$. Consequently, $\mathscr{O}_Y(V_y)$ is a finitely generated (resp. finitely presented) A-algebra; observe moreover that $\mathscr{O}_Y(V_y) = B_{b_y}$.

Since Y is affine, it is quasi-compact and there exists a finite subset Σ of Y such that $Y = \bigcup_{y\in\Sigma} V_y$. The ideal of B generated by the family $(b_y)_{y\in\Sigma}$ contains 1, hence there exists a family $(c_y)_{y\in\Sigma}$ of elements of B such that $1 = \sum_{y\in\Sigma} b_y c_y$.

Let us now prove that the A-algebra B is finitely generated. For every $y \in \Sigma$, let S_y be a finite subset of B such that the A-algebra B_{b_y} is generated by S_y and $1/b_y$. Let S be a finite subset of B containing the sets S_y, the elements b_y, as well as the elements c_y, for $y \in \Sigma$. Let then $\varphi\colon A[(X_s)_{s\in S}] \to B$ be the unique morphism of A-algebras such that $\varphi(X_s) = s$ for every $s \in S$.

Let $B' = \operatorname{Im}(\varphi)$ and let us show that $B' = B$. Let $M = B/B'$; this is a B'-module such that $M_{b_y} = 0$ for every $y \in \Sigma$. Since B' contains the elements c_y, the ideal of B' generated by the elements b_y contains 1; therefore, one has $M = 0$, hence $B' = B$.

Let us now assume that for every $y \in \Sigma$, the A-algebra B_{b_y} is finitely presented. Let us then prove that the kernel N of φ is a finitely generated $A[X]$-module; for this, it suffices to prove that the *quasi-coherent sheaf* $\widetilde{N}$ on $\operatorname{Spec}(A[X])$ is finitely generated.

Let $y \in \Sigma$ and let $P_y \in A[X]$ be such that $\varphi(P) = b_y$ (for example, one may take $P = X_{b_y}$). Then $D(P_y) = \operatorname{Spec}(A[X,T]/(1-TP_y))$; moreover, the morphism φ_y from $A[X,T]$ to B_{b_y} that coincides with φ on $A[X]$ and such that $\varphi(T) = 1/b_y$ is surjective, and its kernel N_y is finitely generated since B_{b_y} is a finitely presented A-module. Since $\widetilde{N}(D(P_y))$ is the image of N_y in $A[X,T]/(1-TP_y)$, it is finitely generated as well.

Let $V = \bigcup_{y\in\Sigma} D(P_y)$. Let us show that V is an open subset of $\operatorname{Spec}(A[X])$ which contains $V(N)$. Let indeed $\mathfrak{p}$ be a prime ideal of $A[X]$ which contains N. Its image $\varphi(\mathfrak{p})$ in B is a prime ideal of B, because φ is surjective. Consequently, there exists $y \in \Sigma$ such that $b_y \notin \varphi(\mathfrak{p})$, because these elements b_y generate the unit ideal of B, hence $\mathfrak{p} \in D(P_y)$.

Let then $U = \operatorname{Spec}(A[X]) - V(N)$ be the complementary open subset to $V(N)$. One has $\widetilde{N}|_U = \mathscr{O}_{\operatorname{Spec}(A[X])}|_U$, hence $\widetilde{N}|_U$ is finitely generated.

We thus have shown that the quasi-coherent sheaf $\widetilde{N}$ on $\operatorname{Spec}(A[X])$ is finitely generated. By proposition 4.7.11, the $A[X]$-module N is finitely generated. In other words, N is a finitely generated ideal, and B is a finitely presented A-algebra. □

Corollary (**5.1.12**). — *Let* $f\colon Y \to X$ *be a morphism of schemes. Assume that* f *is of finite type (resp. of finitely presentation). For every affine open subset* U *of* X, *there exists a finite family* (V_i) *of affine open subschemes of* Y *such that* $f^{-1}(U) = \bigcup_i V_i$ *and* $\mathscr{O}_Y(V_i)$ *is a finitely generated (resp. a finitely presented)* $\mathscr{O}_X(U)$*-algebra for every* i.

Proof. — Since the open subscheme $f^{-1}(U)$ is quasi-compact, it is the union of a finite family (V_i) of affine open subschemes. For each i, the morphism from V_i to U induced by f is locally finitely generated (resp. locally finitely presented); by the preceding proposition, the $\mathscr{O}_X(U)$-algebra $\mathscr{O}_Y(V_i)$ is then finitely generated (resp. finitely presented). This concludes the proof of the corollary. □

Proposition (**5.1.13**). — *Let* S *be a scheme, let* X, Y *be* S*-schemes, let* f, g *be their structural morphisms.*

a) *Let* $h\colon X \to Y$ *be a morphism of* S*-schemes. If* f *is locally finitely generated, then* h *is locally finitely generated.*

b) *If* h *and* g *are locally finitely generated (resp. locally finitely presented), then* f *is locally finitely generated (resp. locally finitely presented).*

c) *If* f *is locally finitely generated (resp. locally finitely presented), then so is* $f \times_S \mathrm{id}_Y\colon X \times_S Y \to Y$.

d) *If both* f *and* g *are locally finitely generated (resp. locally finitely presented), then so is* $f \times_S g\colon X \times_S Y \to S$.

5.2. Subschemes and immersions

Definition (**5.2.1**). — *Let* $\varphi\colon Y \to X$ *be a morphism of schemes.*

a) *One says that it is an* open immersion *if it is a homeomorphism from* Y *to an open subset of* X *and if for every* $y \in Y$, *the morphism of local rings* $\varphi^\sharp_y$ *is bijective.*

b) *One says that* φ *is an* immersion *if it induces a homeomorphism from* Y *to a locally closed subspace of* X *and if for every* $y \in Y$, *the morphism of local rings* $\varphi^\sharp_y\colon \mathscr{O}_{X,\varphi(y)} \to \mathscr{O}_{Y,y}$ *is surjective.*

c) *One says that it is an* closed immersion *if it is an immersion and if* $\varphi(Y)$ *is closed in* X.

Let X be a topological space. Recall that a subspace Z of X is said to be *locally closed* if it can be written as the intersection of an open and of a closed subspace. This means that for every point $x \in Z$, there exists an open neighborhood U of x

in X such that $Z \cap U$ is closed in U. The union of all such open sets is the largest open subset U of X such that $T \cap U$ is closed in U.

Consequently, if $\varphi\colon Z \to X$ is an immersion and if U is the largest open subset of X such that $\varphi(Z)$ is closed in U, then φ induces a closed immersion from Z to U.

If $\varphi\colon Z \to X$ is an immersion of schemes whose underlying map of topological spaces is an inclusion, we also say that Z is a *subscheme* of X.

Remark **(5.2.2)**. — An immersion is a monomorphism in the category of schemes.

Example **(5.2.3)**. — a) Let X be a scheme and let U be an open subset of X. Then $(U, \mathscr{O}_X|_U)$ is a scheme and the canonical morphism $\varphi\colon U \to X$ of locally ringed spaces is an open immersion.

b) Let A be a ring, let $X = \operatorname{Spec}(A)$; let I be an ideal of A, let $Y = \operatorname{Spec}(A/I)$ and let $\varphi\colon Y \to X$ be the morphism of schemes deduced from the canonical surjection from A to A/I. Let us prove that φ is a closed immersion.

By proposition 1.5.10, we already know that φ induces a homeomorphism from Y to the closed subset $V(I)$ of X. Let $y \in Y$ and let $x = \varphi(y)$; then $\mathfrak{p}_x$ is a prime ideal of A containing I and $\mathfrak{p}_y$ is the corresponding ideal of A/I. The morphism of local rings $\varphi^\sharp_y\colon \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ identifies with the canonical morphism from $A_{\mathfrak{p}_x}$ to $(A/I)_{\mathfrak{p}_y}$, which is indeed surjective.

By construction the ring morphism $\varphi^\sharp(X)\colon \mathscr{O}_X(X) \to \varphi_*\mathscr{O}_Y(X)$ identifies with the canonical surjection from A to A/I. Since X is affine and the $\mathscr{O}_X$-modules $\mathscr{O}_X$ and $\varphi_*\mathscr{O}_Y$ are quasi-coherent, the morphism of sheaves $\varphi^\sharp$ is surjective.

c) Let $\varphi\colon Y \to X$ be an immersion of schemes. For every open subscheme U of X, the morphism $\varphi_U\colon \varphi^{-1}(U) \to U$ deduced from φ by restriction is an immersion. If, moreover, $\varphi(Y) \cap U$ is closed in U, then it is a closed immersion.

Conversely, let $\varphi\colon Y \to X$ be a morphism of schemes. Let us assume that every point of Y has an open neighborhood U such that the morphism $\varphi_U\colon \varphi^{-1}(U) \to U$ is an immersion. Then φ is an immersion.

Indeed, φ is injective and induces an open map from Y to $\varphi(Y)$; consequently, it defines a homeomorphism from Y to its image, which is locally closed in X. Moreover, for every point $y \in Y$, the morphism $\varphi^\sharp_y\colon \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ induced by φ coincides with the morphism $\varphi^\sharp_{U,y}$ whenever U is an open subset of X such that

$\varphi(y) \in \mathrm{U}$. If φ_{U} is an immersion, then $\varphi^\sharp_{\mathrm{U},y}$ is surjective, hence $\varphi^\sharp_y$ is surjective as well.

We shall see that these examples are archetypal immersions.

Lemma (**5.2.4**). — *Let* $\varphi\colon \mathrm{Y} \to \mathrm{X}$ *be an open immersion. Then* $\varphi(\mathrm{Y})$ *is an open subset of* X *and* φ *induces an isomorphism from* Y *to the scheme* $(\varphi(\mathrm{Y}), \mathscr{O}_{\mathrm{X}}|_{\varphi(\mathrm{Y})})$.

Proof. — By definition of an open immersion, φ induces a homeomorphism from Y to an open subset V of X. Moreover, for every $y \in \mathrm{Y}$, the morphism $\varphi^\sharp_y\colon \mathscr{O}_{\mathrm{X},\varphi(y)} \to \mathscr{O}_{\mathrm{Y},y}$ is an isomorphism of local rings. Let $\psi\colon \mathrm{Y} \to \mathrm{V}$ be the induced morphism of locally ringed spaces; it is a homeomorphism. If we use ψ to identify Y and V, then $\varphi^\sharp$ is a morphism of sheaves on Y which induces an isomorphism on stalks; it is thus an isomorphism. □

Lemma (**5.2.5**). — *Let* $\varphi\colon \mathrm{Y} \to \mathrm{X}$ *be a morphism of schemes which induces a homeomorphism from* Y *to a locally closed subset of* X. *Let* $y \in \mathrm{Y}$ *and let* $x = \varphi(y)$, *let* V *be an open neighborhood of* y *in* Y. *There exists an affine open neighborhood* U *of* x *such that* $\varphi^{-1}(\mathrm{U})$ *is an affine open neighborhood of* y *contained in* V.

Proof. — By the definition of a locally closed subset, there exists an open subset Ω of X such that $\varphi(\mathrm{Y})$ is a closed subset of Ω, and the morphism from Y to Ω deduced from φ is closed.

Let U_1 be an affine open neighborhood of x which is contained in Ω and let V_1 be an affine open neighborhood of y contained in $\varphi^{-1}(\mathrm{U}_1) \cap \mathrm{V}$. Let $\varphi_1\colon \mathrm{V}_1 \to \mathrm{U}_1$ be the morphism of schemes deduced from φ by restriction; let $\mathrm{A}_1 = \mathscr{O}_{\mathrm{X}}(\mathrm{U}_1)$, $\mathrm{B}_1 = \mathscr{O}_{\mathrm{Y}}(\mathrm{V}_1)$ and let $u = \varphi_1^\sharp\colon \mathrm{A}_1 \to \mathrm{B}_1$ be the morphism of rings associated with φ_1.

Then $\mathrm{Z}_1 = \varphi(\mathrm{Y}) \cap \mathrm{U}$ is closed in U, and $\varphi(\mathrm{V}_1)$ is an open subset of Z_1; consequently, there exists an open subset U_2 of U_1 such that $\varphi(\mathrm{V}_1) = \varphi(\mathrm{Y}) \cap \mathrm{U}_2$. Let $a \in \mathrm{A}_1$ be any element such that $x \in \mathrm{D}(a)$ and $\mathrm{D}(a) \subseteq \mathrm{U}_2$. Then $\mathrm{U} = \mathrm{D}(a)$ is an affine open neighborhood of x in U_1, and $\varphi^{-1}(\mathrm{U})$ is an open neighborhood of y contained in V_1. Moreover, $\varphi^{-1}(\mathrm{U})$ is affine since it is equal to $\mathrm{D}(u_1(a))$; finally, the relation $\varphi(\varphi^{-1}(\mathrm{U})) = \varphi(\mathrm{Y}) \cap \mathrm{U}$ shows that it is closed in U. □

Proposition (**5.2.6**). — *Let* $\varphi\colon \mathrm{Y} \to \mathrm{X}$ *be a morphism of schemes. The following properties are equivalent:*

(i) *For every affine open subscheme* $\mathrm{U} = \operatorname{Spec}(\mathrm{A})$ *of* X, *there exists an ideal* I *of* A *and an isomorphism of* A*-schemes* $\psi_{\mathrm{U}}\colon \varphi^{-1}(\mathrm{U}) \to \operatorname{Spec}(\mathrm{A}/\mathrm{I})$;

(ii) *Every point of* X *has an affine open neighborhood* $U = \mathrm{Spec}(A)$ *such that there exists an ideal* I *of* A *and an isomorphism of* A*-schemes* $\psi_U \colon \varphi^{-1}(U) \to \mathrm{Spec}(A/I)$*;*

(iii) *The morphism* φ *induces a homeomorphism from* Y *to a closed subset of* X*, and the morphism of sheaves* $\varphi^\sharp \colon \mathscr{O}_X \to \varphi_* \mathscr{O}_Y$ *is surjective;*

(iv) *The morphism* φ *is a closed immersion.*

If they hold, then the $\mathscr{O}_X$*-algebra* $\varphi_*\mathscr{O}_Y$ *is quasi-coherent.*

Proof. — In each of these situations, every point of X has an affine open neighborhood U such that $\varphi^{-1}(U)$ is affine; this is obvious in cases (i) and (ii), and follows from lemma 5.2.5 in cases (iii) and (iv). Then the restriction to U of the $\mathscr{O}_X$-module $\varphi_*\mathscr{O}_Y$ is isomorphic to the sheaf $(\varphi_U)_*(\mathscr{O}_{\varphi^{-1}(U)})$. By corollary 4.7.6, the latter sheaf is a quasi-coherent $\mathscr{O}_U$-module. Consequently, $\varphi_*\mathscr{O}_Y$ is a quasi-coherent $\mathscr{O}_X$-module.

The implication (i)⇒(ii) follows from the fact that every point of a scheme has an affine open neighborhood.

Assume that (ii) holds. Let $U = \mathrm{Spec}(A)$ be an open affine subscheme of X and let I be an ideal of A such that there exists an A-isomorphism $\psi_U \colon \varphi^{-1}(U) \to \mathrm{Spec}(A/I)$. By example 5.2.3, *b*), we see that the morphism φ induces a homeomorphism from $\varphi^{-1}(U)$ to the closed subset $V(I)$ of U, and the morphism of local rings $\varphi^\sharp_y \colon \mathscr{O}_{X,\varphi(y)} \to \mathscr{O}_{Y,y}$ is surjective for every $y \in \varphi^{-1}(U)$. It also follows from that example that the morphism of local rings $\varphi^\sharp_y \colon \mathscr{O}_{X,\varphi(y)} \to \mathscr{O}_{Y,y}$ is surjective for every $y \in \varphi^{-1}(U)$. Since X is covered by such affine open subsets, this implies that φ induces a homeomorphism from Y to a closed subset of X, that $\varphi^\sharp$ is surjective, and that $\varphi^\sharp_y$ is surjective for every $y \in Y$. We thus have proved the implications (ii)⇒(iii) and (ii)⇒(iv).

We now assume (iii). To prove that φ is a closed immersion, it suffices to prove that the morphism $\varphi^\sharp_y \colon \mathscr{O}_{X,\varphi(y)} \to \mathscr{O}_{Y,y}$ is surjective for every $y \in Y$. Let U be an affine open subset of X such that $\varphi^{-1}(U)$ is an affine open subset of X. Let $A = \mathscr{O}_X(U)$, let $B = \mathscr{O}_Y(\varphi^{-1}(U))$ and let $u = \varphi^\sharp(U)$. Since $\varphi^\sharp$ is surjective and the sheaves $\mathscr{O}_X$ and $\varphi_*\mathscr{O}_Y$ are quasi-coherent, the ring morphism u is surjective. Let then $y \in \varphi^{-1}(U)$; it corresponds to a prime ideal $\mathfrak{q}$ of B, the point $\varphi(y)$ corresponds to the prime ideal $\mathfrak{p} = u^{-1}(\mathfrak{q})$, and the morphism $\varphi^\sharp_y$ identifies with the morphism $A_\mathfrak{p} \to B_\mathfrak{q}$; it is thus surjective.

Let us finally assume that φ is a closed immersion and let us prove (i). To simplify the notation, we may replace X by U and assume that $X = \mathrm{Spec}(A)$.

Let $B = \mathscr{O}_Y(Y)$ and let $u\colon A \to B$ be the morphism of rings corresponding to the morphism of schemes $\varphi\colon Y \to \operatorname{Spec}(A)$; let $I = \operatorname{Ker}(u)$, so that u factors through the quotient A/I. Let $\psi\colon Y \to \operatorname{Spec}(A/I)$ be the morphism of schemes associated with the ring morphism $A/I \to B$, and let $j\colon \operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$ be the closed immersion defined by the ideal I; one has $\varphi = j \circ \psi$. It follows from the definitions that ψ is a closed immersion as well. We may thus assume that $I = (0)$, in other words, that the morphism u is injective. We then need to prove that φ, or, equivalently, u, is an isomorphism.

Let us first show that φ is surjective. Let $x \in X - \varphi(Y)$. Since $\varphi(Y)$ is closed, there exists an affine open neighborhood U of x such that $\varphi(Y) \cap U = \varnothing$, or, equivalently, that $\varphi^{-1}(U) = \varnothing$. Let $a \in A$ be such that $x \in D(a)$ and $D(a) \subseteq U$. Then $\varphi(Y) \subseteq V(a)$; in other words, one has $\varphi(a) \in \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of B. Consequently, $\varphi(a)$ is nilpotent. Since φ is injective, a is nilpotent as well, which contradicts the hypothesis that $x \in D(a)$.

Since φ is a closed continuous bijection, it is a homeomorphism from Y to X. In particular, for every $y \in Y$, the canonical morphism from $(\varphi_*\mathscr{O}_Y)_{\varphi(y)}$ to $\mathscr{O}_{Y,y}$ is an isomorphism. The morphisms $\varphi^\sharp_y$ being surjective, for every $y \in Y$, the morphism of sheaves $\varphi^\sharp$ is surjective. Since X is affine and the sheaves $\mathscr{O}_X$ and $\varphi_*\mathscr{O}_Y$ are quasi-coherent $\mathscr{O}_X$-modules, this implies that u is surjective; therefore, u is an isomorphism. This concludes the proof of the proposition. □

Corollary **(5.2.7)**. — *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be immersions (resp. closed immersions, resp. open immersions) of schemes. Then $f \circ g\colon Z \to X$ is an immersion (resp. a closed immersion, resp. an open immersion).*

Proof. — Let $z \in Z$, let $y = g(z)$ and $x = f(y)$. Let V be an open neighbhorood of y in Y such that the map $g_V\colon g^{-1}(V) \to V$ is closed. Let then U be an open neighborhood of x in X such that the map $f_U\colon f^{-1}(U) \to U$ is closed and such that $f^{-1}(U) \subseteq V$. The map from $g^{-1}(f^{-1}(U))$ to $f^{-1}(U)$ deduced from g_V is then closed, hence the map from $g^{-1}(f^{-1}(U))$ to U deduced from $f \circ g$ by restriction is closed. Consequently, $f \circ g$ induces a homeomorphism from Z to a locally closed subset of X.

If f and g are closed immersions, then $f \circ g$ is closed, and $f \circ g$ induces a homeomorphism from Z to a closed subset of X.

Moreover, the morphism $(f \circ g)^\sharp_z\colon \mathscr{O}_{X,x} \to \mathscr{O}_{Z,z}$ is the composition of the morphisms $f^\sharp_y$ and $g^\sharp_z$; it is thus surjective.

This shows that $f \circ g$ is an immersion, and a closed immersion if f and g are closed immersions.

If f and g are open immersions, they induce isomorphisms from Y to an open subscheme of X, and from Z to an open subscheme of Y. Their composition induces an isomorphism from Z to an open subscheme of X, hence is an open immersion. □

Corollary **(5.2.8)**. — *Let $f\colon Y \to X$ and $g\colon Z \to X$ be morphisms of schemes. Let us prove that if f is an immersion (resp. a closed immersion, resp. an open immersion), then so is the morphism f_Z deduced from f by base change to Z.*

Proof. — a) We first assume that f is an open immersion. Then $U = f(Y)$ is an open subset of X and f induces an isomorphism from Y to U, so that the morphism f_Z identifies with the open immersion from $g^{-1}(U)$ to U.

b) Assume that f is closed immersion. Let $U = \operatorname{Spec}(A)$ be an affine open subset of X, let I be an ideal of A such that $V = f^{-1}(U)$ is X-isomorphic to $\operatorname{Spec}(A/I)$. Let $W = \operatorname{Spec}(B)$ be an affine open subset of Z such that $g(V) \subseteq U$, in particular, B is an A-algebra. Since the natural ring morphism from $(A/I)\otimes_A B$ to B/IB is an isomorphism, we see that $V \times_U W$ is an affine open subset of $Y \times_Z X$, isomorphic to $\operatorname{Spec}(B/IB)$; by restriction, the morphism f_Z induces a morphism from $V \times_U W$ to W which identifies with the closed immersion of $\operatorname{Spec}(B/IB)$ to $\operatorname{Spec}(B)$. Since every point of Z has an affine open neighborhood W whose image is contained in an affine open subset of X, this proves that f_Z is a closed immersion.

c) In the general case, let U be the largest open subset of X such that f induces a closed immersion from Y to U. Then f_Z is the composition of the closed immersion from $Y \times_U g^{-1}(U)$ to $g^{-1}(U)$, and of the open immersion from $g^{-1}(U)$ to Z. It is thus an immersion.

□

5.2.9. — Let X be a scheme and let Z be a closed subset; let $j\colon Z \to X$ be the inclusion.

Let $\mathscr{O}_Z$ be a sheaf of rings on Z such that $(Z, \mathscr{O}_Z)$ is a scheme and let $j^\sharp\colon \mathscr{O}_X \to j_*\mathscr{O}_Z$ be a morphism of sheaves such that $(j, j^\sharp)$ is an immersion. Then $j^\sharp$ is surjective, and its kernel $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_X$. Moreover, if U is an affine open subscheme $U = \operatorname{Spec}(A)$ of X, then $I = \mathscr{I}(U)$ is an ideal of A, and j induces a homeomorphism from $Z \cap U$ to the closed subset $V(\mathscr{I}(U))$ of $\operatorname{Spec}(A)$, and $(j, j^\sharp)$ induces an isomorphism of schemes from $(j^{-1}(U), \mathscr{O}_Z|_{j^{-1}(U)})$ to $\operatorname{Spec}(A/I)$.

Conversely, let $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$ such that for every affine open subscheme $U = \operatorname{Spec}(A)$ of X, denoting by I the ideal $\mathscr{I}(U)$ of A, one has $V(I) = Z \cap U$. Then $\mathscr{O}_Z = j^{-1}(\mathscr{O}_X/\mathscr{I})$ is a sheaf of rings on Z. Let $j^\sharp \colon \mathscr{O}_X \to j_*\mathscr{O}_Z$ be the morphism of sheaves deduced from the canonical surjection of $\mathscr{O}_X$ to $\mathscr{O}_X/\mathscr{I}$. Then $(j, j^\sharp)$ is a closed immersion, and $\mathscr{I} = \operatorname{Ker}(j^\sharp)$.

One says that Z is the *closed subscheme* of X defined by the quasi-coherent ideal $\mathscr{I}$, and one denotes it by $Z = V(\mathscr{I})$.

The inclusion of quasi-coherent ideals gives rise to a natural order relation on closed subschemes: the larger the ideal, the smaller the subscheme. We will say that $V(\mathscr{I})$ is supported by Z to mean that the closed subspace of X underlying the subscheme $V(\mathscr{I})$ is equal to Z.

Proposition **(5.2.10)**. — *Let X be a scheme and let Z be a closed subset of X. There is a unique structure of closed subscheme on Z such that for every $x \in Z$, the local ring $\mathscr{O}_{Z,z}$ has no non-zero nilpotent element. It is defined by the largest quasi-coherent ideal $\mathscr{I}$ such that $Z = V(\mathscr{I})$.*

Proof. — For every open subset U of X, let $\mathscr{I}(U)$ be the set of $f \in \mathscr{O}_X(U)$ such that $f(x) = 0$ for every $x \in Z$. This defines a a sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_X$.

To prove that $\mathscr{I}$ is quasi-coherent, it suffices to prove that its restriction to every affine open subscheme of X is quasi-coherent. Let thus $U = \operatorname{Spec}(A)$ be an affine open subscheme of X and let $I = \mathscr{I}(U) = \mathfrak{j}(Z \cap U)$. Then I is a radical ideal of A and is the largest ideal of A such that $V(I) = Z \cap U$. Let $a \in A$. One has $A_a = \mathscr{O}_U(D(a))$, and the inclusion $I_a \subseteq \mathscr{I}(D(a))$ follows from the definition. Conversely, let $f \in \mathscr{I}(D(a))$; let $g \in A$ and $n \in \mathbf{N}$ be such that $f = g/a^n$; by assumption, one has $g \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ containing I such that $a \notin \mathfrak{p}$; it follows that $ag \in I$, hence $f = ag/a^{n+1} \in I_a$. This proves that $\mathscr{I}$ is quasi-coherent.

The underlying topological space to the subscheme $V(\mathscr{I})$ is equal to Z. One has $Z \cap U \simeq \operatorname{Spec}(A/I)$. For every $x \in Z \cap U$, 0 is the only nilpotent element of $\mathscr{O}_{Z,z}$, because the ideal I is radical.

Moreover, if $\mathscr{J}$ is a quasi-coherent ideal such that $V(\mathscr{J})$ has support Z, then $\mathscr{J}(U) = \mathfrak{j}(Z \cap U) = \mathscr{I}(U)$ for every affine open subscheme U of X. $\square$

5.3. Affine morphisms, finite morphisms

Definition (**5.3.1**). — *Let* $f: Y \to X$ *be a morphism of schemes. One says that* f *is affine if for every open affine subscheme* U *of* X, $f^{-1}(U)$ *is an affine scheme.*

5.3.2. — Here is a general way to construct affine morphisms. Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_X$-algebra.

For every affine open subset U of X, let $Y_U = \operatorname{Spec}(\mathscr{A}(U))$; this is an affine scheme equiped with a morphism f_U to $\operatorname{Spec}(\mathscr{O}_X(U)) = U$.

For every pair (U, W) of affine open subschemes of X such that $W \subseteq U$, the restriction morphism $\mathscr{A}(U) \to \mathscr{A}(W)$ induces a morphism $\varphi'_{UW}: Y_W \to Y_U$ such that $f_W \circ \varphi'_{UW} = f_U$. Since $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra, the restriction morphism induces an isomorphism $\mathscr{A}(W) \simeq \mathscr{A}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(W)$. Consequently, the morphism φ'_{UW} induces an isomorphism φ_{UW} from Y_W to the open subscheme $f_U^{-1}(W)$ of Y_U.

Let U and V be affine open subschemes of X. There exists a unique isomorphism of schemes ψ_{UV} from the open subscheme $f_U^{-1}(U \cap V)$ of Y_U to the open subscheme $f_V^{-1}(U \cap V)$ of Y_V whose restriction to $f_U^{-1}(W)$ is equal to $\varphi_{VW} \circ \varphi_{UW}^{-1}$, for every affine open subscheme W of $U \cap V$.

We can now glue the schemes (Y_U) along the open subschemes Y_{UV} by means of these isomorphisms ψ_{UV}. This defines a scheme Y, as well as a morphism of schemes $\psi: Y \to X$, and isomorphisms $\psi_U: \psi^{-1}(U) \to Y_U$ for every affine open subscheme U of X, such that $\psi|_U = f_U \circ \psi_U$ and such that the morphisms $\psi_{UV} \circ \psi_U$ and ψ_V coincide on $|_{\psi^{-1}(U \cap V)}$. This X-scheme is called the *spectrum* of the quasi-coherent $\mathscr{O}_X$-algebra, and is denoted by $\operatorname{Spec}(\mathscr{A})$.

By construction, for every affine open subscheme U of X, U is isomorphic to $\operatorname{Spec}(\mathscr{O}_X(U))$, $\psi^{-1}(U)$ is isomorphic to $\operatorname{Spec}(\mathscr{A}(U))$, and the morphism $\psi_U: \psi^{-1}(U) \to U$ identifies with the morphism of affine schemes deduces with the ring morphism $\mathscr{O}_X(U) \to \mathscr{A}(U)$.

Example (**5.3.3**). — It follows from proposition 5.2.6 that a morphism $f: Y \to X$ is a closed immersion if and only if it is affine and the morphism $f^\sharp \mathscr{O}_X \to f_* \mathscr{O}_Y$ is surjective.

Let, moreover, $\mathscr{I}$ be the kernel of the morphism $f^\sharp: \mathscr{O}_X \to f_* \mathscr{O}_Y$. It is a quasi-coherent $\mathscr{O}_X$-module and the quotient sheaf $\mathscr{O}_X/\mathscr{I}$ is a quasi-coherent $\mathscr{O}_X$-algebra. Then f induces an isomorphism from Y to the closed subscheme $V(\mathscr{I}) = \operatorname{Spec}(\mathscr{O}_X/\mathscr{I})$.

5.3.4. — The spectrum of a quasi-coherent sheaf of algebras satisfies a universal property.

Let $f\colon Y \to X$ be a morphism of schemes, let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_X$-algebra and let $u\colon \mathscr{A} \to f_*\mathscr{O}_Y$ be a morphism of $\mathscr{O}_X$-algebras. Let $g\colon \mathrm{Spec}(\mathscr{A}) \to X$ be the canonical morphism.

Let U be an affine open subscheme of X and let $f_U\colon f^{-1}(U) \to U$ be the morphism deduced from f by restriction. The identification $(f_U)_*\mathscr{O}_{f^{-1}(U)} = f_*\mathscr{O}_Y|_U$ and the morphism $g(U)\colon \mathscr{A}(U) \to f_*\mathscr{O}_Y(U)$ give rise to a morphism of schemes $\varphi_U\colon f^{-1}(U) \to \mathrm{Spec}(\mathscr{A}(U)) = g^{-1}(U)$. These morphisms glue together and define a morphism of X-schemes $\varphi\colon Y \to \mathrm{Spec}(\mathscr{A})$.

Proposition **(5.3.5)**. — *Let $f\colon Y \to X$ be a morphism of schemes. Assume that every point of* X *has an affine open neighborhood* U *such that $f^{-1}(U)$ is affine. Then the $\mathscr{O}_X$-algebra $f_*\mathscr{O}_Y$ is quasi-coherent, and there exists an* X*-isomorphism from* Y *to* $\mathrm{Spec}(f_*\mathscr{O}_Y)$*. In particular, the morphism f is affine.*

Proof. — Let us first prove that $f_*\mathscr{O}_Y$ is quasi-coherent. Let $x \in X$ and let U be an affine open neighborhood of x such that $f^{-1}(U)$ is affine; let $f_U\colon f^{-1}(U) \to U$ be the morphism of schemes deduced by restriction. By definition, $f_*\mathscr{O}_Y|_U$ is isomorphic to $(f_U)_*\mathscr{O}_{f^{-1}(U)}$. It thus follows from corollary 4.7.6 that the sheaf $f_*\mathscr{O}_Y|_U$ is a quasi-coherent $\mathscr{O}_U$-algebra. Consequently, $f_*\mathscr{O}_Y$ is a quasi-coherent $\mathscr{O}_X$-algebra.

We consider the spectrum $Z = \mathrm{Spec}(f_*\mathscr{O}_Y)$ of this algebra, and its canonical morphism $g\colon Z \to X$ to X. Let $\varphi\colon Y \to Z$ be the canonical morphism of X-schemes associated with $f_*\mathscr{O}_Y$; let us prove that it is an isomorphism.

Let U be an affine open subscheme of X such that $f^{-1}(U)$ is affine, say Spec(B). Then one has $f_*\mathscr{O}_Y(U) = B$, and the morphism φ identifies with the identical morphism from $f^{-1}(U) = \mathrm{Spec}(B)$ to $g^{-1}(U) = \mathrm{Spec}(B)$. Consequently, φ is an isomorphism. ☐

Corollary **(5.3.6)**. — *Let $f\colon Y \to X$ be an affine morphism of schemes and let* Z *be an* X*-scheme. The morphism $f_Z\colon Y_Z \to Z$ deduced from f by base-change to* Z *is affine.*

Proof. — Let $g\colon Z \to X$ be the structural morphism. Every point of Z has an affine open neighborhood U such that $g(U)$ is contained in an affine open subset V of X. Then $f_Z^{-1}(U)$ identifies with to the fiber product $f^{-1}(V) \times_V U$ of affine schemes, hence is affine. ☐

Definition **(5.3.7)**. — *Let $f\colon Y \to X$ be a morphism of schemes. One says that f is* finite *if it is affine and if $f_*\mathscr{O}_Y$ is a finitely generated $\mathscr{O}_X$-module.*

Lemma **(5.3.8)**. — *Let $f\colon Y \to X$ be a morphism of schemes. Assume that every point $x \in X$ has an affine open neighborhood U such that $f^{-1}(U)$ is an affine open subscheme of Y and such that $\mathscr{O}_Y(f^{-1}(U))$ is a finitely generated $\mathscr{O}_X(U)$-module. Then f is a finite morphism.*

Proof. — By proposition 5.3.5, f is an affine morphism. Let $\mathscr{A} = f_*\mathscr{O}_Y$. It is a quasi-coherent $\mathscr{O}_X$-module; let us prove that it is finitely generated. By hypothesis, every point of X has an affine open neihborhood U such that $\mathscr{A}(U)$ is a finitely generated $\mathscr{O}_X(U)$-module. By proposition 4.7.11, $\mathscr{A}|_U$ is then a finitely generated $\mathscr{O}_X|_U$-module. Consequently, $\mathscr{A}$ is a finitely generated $\mathscr{O}_X$-module, as was to be shown. ☐

Remark **(5.3.9)**. — a) Let A be a ring and let B be an A-algebra. Let $X = \operatorname{Spec}(A)$, let $Y = \operatorname{Spec}(B)$ and let $f\colon Y \to X$ be the associated morphism. The following properties are equivalent:

(a) The morphism f is finite;
(b) The A-module B is finitely generated;
(c) The A-algebra B is finitely generated and integral.

Assume that they hold, and let $I = \ker(A \to B)$. The first theorem of Cohen-Seidenberg (theorem 1.11.4) then implies that $f(Y) = V(I)$.

b) Let k be a field and let A be a non-zero finitely generated k-algebra. Let n be a nonnegative integer and $f\colon k[T_1, \dots, T_n] \to A$ be an integral injective morphism of k-algebras. The associated morphism of schemes ${}^a f\colon \operatorname{Spec}(A) \to \mathbf{A}^n_k$ is then finite and surjective. This is the geometric formulation of Noether's normalization lemma (theorem 1.6.1).

c) Assume that k is infinite and let X be a non-empty closed subscheme of $\mathbf{A}^m_k$. It follows from exercise 1.6.5 that there exists an integer n such that $0 \leqslant n \leqslant m$ and a linear morphism $p\colon \mathbf{A}^m_k \to \mathbf{A}^n_k$ which induces a finite and surjective morphism $p_X\colon X \to \mathbf{A}^n_k$.

5.4. Separated and proper morphisms

Definition (**5.4.1**). — *Let* X *be an* S*-scheme and let* p_1 *and* p_2 *be the two projections from* $X \times_S X$ *to* X*. The* diagonal morphism δ *is the unique morphism of* S*-schemes from* X *to* $X \times_S X$ *such that* $p_1 \circ \delta = p_2 \circ \delta = \mathrm{id}_X$.

Lemma (**5.4.2**). — *Let* $f\colon X \to S$ *be a morphism of schemes.*

a) *The diagonal morphism* $\delta\colon X \to X \times_S X$ *is an immersion.*
b) *If* f *is affine, then* δ *is a closed immersion.*
c) *If* f *is a monomorphism, then* δ *is an isomorphism.*

Proof. — a) Let $x \in X$ and let $s = f(x)$. Let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of x in X whose image is contained in an affine open neighborhood $V = \mathrm{Spec}(R)$ of s in S. Then $W = p_1^{-1}(U) \cap p_2^{-1}(U)$ is an affine open subscheme of $X \times_S X$ which contains $\delta(x)$, isomorphic to $\mathrm{Spec}(A \times_R A)$. Moreover, $\delta^{-1}(W) = U$ and the induced morphism $\delta_W\colon U \to W$ corresponds to the morphism of R-algebras $\gamma\colon A \times_R A \to A$ such that $\gamma(a \otimes b) = ab$. Since it is surjective, the morphism δ_W is a closed immersion. Consequently, δ is an immersion.

b) If f is affine, then we may take $U = f^{-1}(V)$, and the open subschemes of $X \times_S X$ of the form $W = p_1^{-1}(U) \cap p_2^{-1}(U)$ cover $X \times_S X$. For each such W, the morphism $\delta_W\colon \delta^{-1}(W) \to W$ is a closed immersion, so that δ is a closed immersion.

c) Let T be an S-scheme and let u, v be two S-morphisms from T to X. This means that $f \circ u = f \circ v$. Since f is a monomorphism, one then has $u = v$. Consequently, for every S-scheme T, the morphism δ induces a bijection from $\mathrm{Hom}_S(T, X)$ to $\mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, X) = \mathrm{Hom}_S(X \times_S X)$. In other words, the morphism δ induces an isomorphism of functors from h_X to $h_{X \times_S X}$; by Yoneda's lemma, δ is an isomorphism.

□

Corollary (**5.4.3**). — *Let* S *be a scheme, let* X *and* Y *be schemes, and let* $f, g\colon Y \to X$ *be two* S*-morphisms. Let* (Z, j) *be an equalizer of the pair* (f, g)*. Then* $j\colon T \to Y$ *is an immersion of* S*-schemes; if* X *is separated over* S*, then* j *is a closed immersion.*

Proof. — Recall the construction of an equalizer done in corollary 4.5.5. Let p and q be the two projections from $X \times_S X$ to X; let $\delta\colon X \to X \times_S X$ be the diagonal immersion. Let $h\colon Y \to X \times_S X$ be the unique S-morphism such that

$p \circ h = f$ and $q \circ h = g$. Let $T = Y \times_{X \times_S X} X$ be the fiber product of the pair (h, δ) of morphisms to $X \times_S X$ and let $\varphi \colon T \to Y$ be the first projection. Then φ is an S-morphism and it is shown in the proof of corollary 4.5.5 that (T, φ) is an equalizer of the pair (f, g). We thus observe that φ is the morphism of schemes deduced from δ by base change to Y. This shows that φ is an immersion, and is a closed immersion if δ is itself a closed immersion, that is, if X is separated over S. □

Lemma **(5.4.4)**. — *A morphism of schemes $f \colon X \to S$ is quasi-separated if and only if its diagonal immersion is quasi-compact.*

Proof. — □

Definition **(5.4.5)**. — *One says that a morphism of schemes $f \colon X \to S$ is* separated *if the diagonal immersion is a closed immersion.*

One says that a scheme X *is* separated *if the canonical morphism from* X *to* $\operatorname{Spec}(\mathbf{Z})$ *is separated.*

Since a closed immersion is quasi-compact, a separated morphism is quasi-separated, and a separated scheme is quasi-separated.

Proposition **(5.4.6)**. — *Let $f \colon X \to S$ be a morphism of schemes. The following assertions are equivalent:*

(i) *The morphism f is separated;*

(ii) *The image of the diagonal immersion is a closed subset of $X \times_S X$;*

(iii) *For every* S*-scheme* T *and every pair (u, v) of* S*-morphisms from* T *to* X*, the equalizer of u and v is a closed subscheme of* T.

Proof. — (i)⇔(ii). If f is separated, then the diagonal immersion is a closed immersion by definition, so that its image is a closed subset of $X \times_S X$. Conversely, an immersion is a closed immersion if and only if its image is closed, hence the converse implication.

(iii)⇒(ii). Let us apply the hypothesis to $T = X \times_S X$ and to the two projections to X. Their equalizer being the diagonal subscheme, it follows that f is separated.

The implication (i)⇒(iii) follows from corollary 5.4.3. □

Proposition **(5.4.7)**. — a) *Let $f \colon X \to S$ be a morphism of schemes. Assume that every point of* S *has an open neighborhood* U *such that the induced morphism $f_U \colon f^{-1}(U) \to U$ is separated. Then f is separated.*

b) *An affine morphism, an immersion of schemes is a separated morphism.*

c) *Let* $f\colon X \to S$ *be a separated morphism of schemes and let* T *be an* S*-scheme. Then the morphism* f_T *deduced from* f *by base-change to* T *is separated.*

d) *Let* $f\colon Z \to Y$ *and* $g\colon Y \to X$ *be morphisms of schemes. If* f *and* g *are separated, then* $g \circ f$ *is separated; if* $g \circ f$ *is separated, then* f *is separated.*

e) *Let* S *be a scheme, let* $f\colon Y \to X$ *and* $f'\colon Y' \to X'$ *be morphisms of* S*-schemes. If* f *and* f' *are separated, then the morphism* $(f, f')\colon Y \times_S Y' \to X \times_S X'$ *is separated.*

Proof. — a) Let $g\colon X \to X \times_S X$ be the diagonal immersion. To prove that g is a closed immersion, it suffices to establish that every point of $X \times_S X$ has an open neighborhood V such that $g_V\colon g^{-1}(V) \to V$ is a closed immersion. Let z be point of $X \times_S X$ and let s be its image in S; let U be an open neighborhood of s such that f_U is separated. Then $V = f^{-1}(U) \times_U f^{-1}(U)$ is an open neighborhood of z, and the immersion g_V identifies with the diagonal immersion associated with the morphism $f_U\colon f^{-1}(U) \to U$. By hypothesis, g_V is a closed immersion. This proves that g is a closed immersion, as claimed.

b) If $f\colon X \to S$ is an immersion, then it is a monomorphism hence the diagonal morphism $g\colon X \to X \times_S X$ is an isomorphism. Consequently, f is separated.

We have already explained that affine morphisms are separated. In fact, by *a*), it would suffice to prove that a morphism of affine schemes is separated, which is at the heart of the proof that the diagonal morphism is an immersion.

c) The diagonal morphism $g_T\colon X_T \to X_T \times_T X_T$ associated with f_T is obtained from the diagonal morphism $g\colon X \to X \times_S X$ by base change to T. If the morphism f is separated, then the diagonal g is a closed immersion, hence so is g_T, so that the morphism f_T is separated.

d) Let us assume that f and g are separated and let us show that $g \circ f$ is separated. We make use of the criterion 5.4.6. Let T be a Z-scheme and let (u, v) be a pair of Z-morphisms from T to X. Since g is separated, the equalizer (T_1, h_1) of the pair $(f \circ u, f \circ v)$ is a closed subscheme of T. Since f is separated, the equalizer (T_2, h_2) of the pair $(u \circ h_1, v \circ h_2)$ is a closed subscheme of T_2. Let $h = h_1 \circ h_2\colon T_2 \to T$; it is the composition of two closed immersions, hence is a closed immersion. Let us observe that (T_2, h) is the equalizer of the pair (u, v). One has $u \circ h = u \circ h_1 \circ h_2 = v \circ h_1 \circ h_2 = v \circ h$. Let moreover $k\colon U \to T$ be a morphism such that $u \circ k = v \circ k$ and let us show that there exists a unique morphism $k'\colon U \to T_2$ such that $k = h \circ k'$. Since h is a monomorphism, there exists at most one such morphism, hence we just need to prove its existence.

One has $f \circ u \circ k = f \circ v \circ k$, so that there exists a morphism $k_1 \colon \mathrm{U} \to \mathrm{T}_1$ such that $k = h_1 \circ k_1$. Consequently, $u \circ h_1 \circ k_1 = v \circ h_1 \circ k_1$, so that there exists a morphism $k_2 \colon \mathrm{U} \to \mathrm{T}_2$ such that $k_1 = h_2 \circ k_2$. It follows that $k = h_1 \circ h_2 \circ k_2 = h \circ k_2$, and the morphism k_2 satisfies the given requirement.

Let us now assume that $g \circ f$ is separated. Let T be a Y-scheme and let $(u, v) \colon \mathrm{T} \to \mathrm{Z}$ be a pair of morphisms of Y-schemes. Composing its structural morphism with g, we may view T as an X-scheme; then u and v are morphisms of X-schemes. Since $g \circ f$ is separated, the equalizer E of the pair (u, v) is then a closed subscheme of T. This proves that f is separated.

e) Let $g \colon \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ and $g' \colon \mathrm{Y}' \to \mathrm{Y}' \times_{\mathrm{X}'} \mathrm{Y}'$ be the diagonal immersions. Since f and f' are assumed to be separated, they are closed immersions. Let p and p' be the projections from $\mathrm{X} \times_{\mathrm{S}} \mathrm{X}'$ to X and X′ respectively; let q and q' be the projections from $\mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}'$ to Y and Y′ respectively. Let $\varphi \colon \mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}' \to \mathrm{X} \times_{\mathrm{S}} \mathrm{X}'$ be the morphism (f, f'): it is characterized by the relations $p \circ \varphi = f \circ q$ and $p' \circ \varphi = f' \circ q'$. The fiber product $(\mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}')_{\mathrm{X} \times_{\mathrm{S}} \mathrm{X}'} (\mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}')$ identifies with $(\mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}) \times_{\mathrm{S}} (\mathrm{Y}' \times_{\mathrm{X}'} \mathrm{Y}')$ and the diagonal morphism $\gamma \colon (\mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}') \to (\mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}')_{\mathrm{X} \times_{\mathrm{S}} \mathrm{X}'} (\mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}')$ associated with the morphism (g, g'). It is thus a closed immersion. □

Corollary **(5.4.8)**. — *Let* X *and* S *be schemes and let* $f \colon \mathrm{X} \to \mathrm{S}$ *be a morphism of schemes. The following conditions are equivalent:*

(i) *The morphism* f *is separated;*

(ii) *The inverse image* $f^{-1}(\mathrm{U})$ *of every affine open subset* U *of* S *is a separated scheme;*

(iii) *Every point of* S *has an open neighborhood* U *such that* $f^{-1}(\mathrm{U})$ *is a separated scheme.*

Proof. — Let $g \colon \mathrm{S} \to \operatorname{Spec}(\mathbf{Z})$ be the canonical morphism.

(i)⇒(ii). Let U be an affine open subset of S, let $f_{\mathrm{U}} \colon f^{-1}(\mathrm{U}) \to \mathrm{U}$ be the morphism deduced from f by restriction, so that the unique morphism from $f^{-1}(\mathrm{U})$ to $\operatorname{Spec}(\mathbf{Z})$ is equal to $g|_{\mathrm{U}} \circ f_{\mathrm{U}}$. If f is separated, then f_{U} is separated; since U is affine, $g|_{\mathrm{U}}$ is separated; it follows from assertion *d*) of proposition 5.4.7 that $f^{-1}(\mathrm{U})$ is a separated scheme.

(ii)⇒(iii) because every point of U has an affine open neighborhood.

(iii)⇒(i). By proposition 5.4.7, *a*), it suffices to prove that every point of S has an open neighborhood U such that the morphism $f_{\mathrm{U}} \colon f^{-1}(\mathrm{U}) \to \mathrm{U}$ is separated.

Choose U so that $f^{-1}(U)$ is a separated scheme. Then $g|_U \circ f_U$ is separated by definition, and the above proposition, *d*), implies that f_U is separated. □

Proposition **(5.4.9)**. — *Let* $f\colon X \to S$ *be a morphism of schemes and let* $(U_i)_{i\in I}$ *be a family of open subschemes of* X *such that* $X = \bigcup_{i\in I} U_i$. *For every pair* (i, j) *of elements of* I, *let* p_i *and* p_j *be the two projections from* $U_i \times_S U_j$ *to* U_i *and* U_j *respectively, and let* $g_{ij}\colon U_i \cap U_j \to U_i \times_S U_j$ *be the unique morphism such that* $p_i \circ g_{ij}$ *and* $p_j \circ g_{ij}$ *are the canonical inclusions of* $U_i \cap U_j$ *into* U_i *and* U_j *respectively. Then* f *is separated if and only if the morphism* g_{ij} *is a closed immersion for every pair* (i, j).

Proof. — Let $g\colon X \to X \times_S X$ be the diagonal immersion. For every pair (i, j) of elements of I, one has $g^{-1}(U_i \times_S U_j) = U_i \cap U_j$, and the morphism g_{ij} is deduced from g by restriction to these open sets. Since the open subsets of $X \times_S X$ of the form $U_i \times_S U_j$ cover $X \times_S X$, the morphism g is a closed immersion if and only if g_{ij} is a closed immersion for every pair (i, j). This establishes the proposition. □

This statement is helpful to decide the separatedness of schemes which are constructed by glueing.

Corollary **(5.4.10)**. — *Let* X *be the* S*-scheme obtained by glueing a family* $(X_i)_{i\in I}$ *of* S*-schemes along open subschemes* X_{ij} *by means of isomorphisms* φ_{ij}. *For every pair* (i, j) *of elements of* I, *let* $\gamma_{ij}\colon X_{ij} \to X_i \times_S X_j$ *be the morphism whose first component if the injection of* X_{ij} *into* X_i, *and whose second component is the morphism* φ_{ij}. *Then* X *is separated if and only if the morphism* γ_{ij} *is a closed immersion for every pair* (i, j).

Proof. — For $i \in I$, let $\varphi_i\colon X_i \to X$ be the canonical inclusion, and let $U_i = \varphi_i(X_i)$, so that φ_i induces an isomorphism from X_i to U_i. Under these isomorphisms, the morphisms g_{ij} of the proposition identify with the morphisms γ_{ij} of the corollary. This concludes the proof. □

Corollary **(5.4.11)**. — *Let* A *be a ring and let* $S = \operatorname{Spec}(A)$. *Let* X *be an* A*-scheme. The following properties are equivalent:*

(i) *The scheme* X *is separated;*

(ii) *For every pair* (U, V) *of affine open subschemes of* X, *the intersection* $U \cap V$ *is affine, and* $\mathscr{O}_X(U \cap V)$ *is generated by the images of* $\mathscr{O}_X(U)$ *and* $\mathscr{O}_X(V)$ *by the restriction morphisms;*

(iii) *There exists an open cover* $(U_i)_{i\in I}$ *of* X *by affine open subschemes such that for every pair* (i,j) *of elements of* I, *the scheme* $U_i \cap U_j$ *is affine, and its ring* $\mathscr{O}_X(U_i\cap U_j)$ *is generated by the images of* $\mathscr{O}_X(U_i)$ *and* $\mathscr{O}_X(U_j)$ *by the restriction morphisms.*

Proof. — Let $\delta\colon X\to X\times_S X$ be the diagonal immersion.

(i)⇒(ii). Let us assume that X is separated and let U, V be affine open subschemes of X. Then $U\times_S V$ is an affine open subscheme of $X\times_S X$, and $\delta^{-1}(U\times V)$ is equal to $U\cap V$. Since δ is a closed immersion, by assumption, it follows that $U\cap V$ is affine. Moreover, $\mathscr{O}_X(U\cap V)$ is a quotient of $\mathscr{O}_X(U)\otimes_A \mathscr{O}_X(V)$; consequently, it is generated by the images of $\mathscr{O}_X(U)$ and $\mathscr{O}_X(V)$.

The implication (ii)⇒(iii) follows from the definition of a scheme, namely, that every point of X has an affine open neighborhood.

(iii)⇒(ii). By restriction, the diagonal immersion δ induces a morphism from $\delta^{-1}(U_i\times_S U_j)=U_i\cap U_j$ to $U_i\times_S U_j$. Under the conditions of (iii) imply, this is a morphism of affine schemes which is a closed immersion, since the associated morphism of rings is surjective. Since the family $(U_i\times_S U_j)_{i,j\in I}$ covers $X\times_S X$, this implies that δ is a closed immersion. Consequently, X is separated. □

Corollary **(5.4.12)**. — *For every ring k, the projective space of dimension n over k,* $\mathbf{P}^n_k$, *is separated.*

Proof. — Let $X=\mathbf{P}^n_k$; let us recall that it is the k-scheme obtained by glueing a family $(X_i)_{0\leqslant i\leqslant n}$ of affine schemes, each of them isomorphic to $\mathbf{A}^n_k$. Let (i,j) be a pair of elements of $\{0,\dots,n\}$. To check the criterion of the previous corollary, we may assume that $i\neq j$ and, up to a permutation of indices, that $i=0$ and $j=n$. Then $X_0=\operatorname{Spec}(k[S_1,\dots,S_n])$, $X_n=\operatorname{Spec}(k[T_0,\dots,T_{n-1}])$, one has $X_{0n}=D(S_n)=\operatorname{Spec}(k[S_1,\dots,S_n,1/S_n])$, $X_{n0}=D(T_0)=\operatorname{Spec}(k[T_0,\to,T_{n-1},1/T_0])$ and $\varphi_{0n}\colon X_{0n}\to X_{n0}$ is the unique morphism of k-schemes such that $\varphi^\sharp_{0n}(S_i)=T_i/T_0$, for every $i\in\{1,\dots,n-1\}$ and $\varphi^\sharp_{0n}(S_n)=1/T_0$. We observe that X_{n0} is affine and that $\mathscr{O}_X(X_{n0})=k[T_1,\dots,T_n,1/T_n]$ is generated by $\mathscr{O}_X(X_n)=k[T_1,\dots,T_n]$ and by $1/T_0$ which belongs to $\mathscr{O}_X(X_0)$ by $\varphi^\sharp_{0n}$. This concludes the proof that $\mathbf{P}^n_k$ is separated. □

Definition **(5.4.13)**. — *Let* $f\colon X\to S$ *be a morphism of schemes. One says that* f *is* proper *if it is of finite type, separated, and universally closed.*

Let us precise that f is *universally closed* if and only if for every S-scheme T, the morphism $f_T\colon X_T \to T$ deduced from f by base change to T is closed.

Proposition **(5.4.14).** — a) *Let $f\colon X \to S$ be a morphism of schemes. Assume that every point of* S *has an open neighborhood* U *such that the induced morphism $f_U\colon f^{-1}(U) \to U$ is proper. Then f is proper.*

b) *A closed immersion of schemes is a proper morphism.*

c) *Let $f\colon X \to S$ be a proper morphism of schemes and let* T *be an* S*-scheme. Then the morphism f_T deduced from f by base-change to* T *is proper.*

d) *Let $f\colon Z \to Y$ and $g\colon Y \to X$ be morphisms of schemes. If f and g are proper, then $g \circ f$ is proper.*

e) *Let* S *be a scheme, let $f\colon Y \to X$ and $f'\colon Y' \to X'$ be morphisms of* S*-schemes. If f and f' are proper, then the morphism $(f, f')\colon Y \times_S Y' \to X \times_S X'$ is proper.*

Proof. — a) Assume that every point of S has an open neighborhood U such that $f_U\colon f^{-1}(U) \to U$ is proper. Then f is of finite type and separated. For every closed subset Z of X, one has $f(Z) \cap U = f_U(Z \cap f^{-1}(U))$, so that $f(Z) \cap U$ is closed in U for every open subset U of S such that f_U is closed. This implies that $f(Z)$ is closed, so that f is a closed map. More generally, let (T, g) be an S-scheme and let $W = g^{-1}(U)$; one has $(f_T)^{-1}(W) = f^{-1}(U) \times_S W$, and the $(f_T)_W\colon (f_T)^{-1}(W) \to W$ deduced from f_T identifies with the morphism $(f_U)_W$ deduced from f_U by base change to W. If f_U is closed, then $(f_U)_W$ is closed. Since T is covered by such open subsets W, this implies that f_T is closed.

b) Let f be a closed immersion. It is of finite type and separated, and closed. For every S-scheme T, f_T is again a closed immersion, hence is closed. This proves that f is a proper morphism.

c) Let $f\colon X \to S$ be a proper morphism and let T be an S-scheme. Then f_T is of finite type, and is separated; it is also closed, and in fact universally closed since for every T-scheme U, the morphism $(f_T)_U$ identifies with the morphism f_U deduced from f by base change to U. Consequently, f_T is a proper morphism.

d) The morphism $g \circ f$ is of finite type, and is separated. For every S-scheme T, one has $(g \circ f)_T = g_T \circ f_T$; since the composition of closed maps is a closed map, this implies that $g \circ f$ is universally closed. Consequently, $g \circ f$ is a proper morphism.

e) The morphism (f, f') is the composition of the morphism $f_{Y'}\colon Y \times_S Y' \to X \times_S Y'$ deduced from f by base change to Y' and of the morphism $f'_X\colon X \times_S Y' \to X \times_S X'$ deduced from f' by base change to X. It is thus proper. □

Proposition **(5.4.15)**. — *A finite morphism is proper and has finite fibers.*

A difficult theorem of Chevalley asserts the converse: a proper morphism with finite fibers is finite.

Proof. — Let $f\colon Y \to X$ be a finite morphism. Then f is affine, hence it is separated. Let us prove that f is closed. Let Z be a closed subset of Y; to prove that $f(Z)$ is closed in X, it suffices to prove that for every affine open subscheme U of X, $f(Z) \cap U = f(f^{-1}(U) \cap Z)$ is closed in U. We may thus assume that X and Y are affine, say $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$, where B is an A-algebra which is finitely generated as a B-module. Let J be an ideal of B such that $Z = V(J)$. Let φ be the composition $A \to B \to B/J$ and let I be its kernel. The associated ring morphism $A/I \to B/J$ is injective and integral, since B/J is an finitely generated A/I-module. By the first theorem of Cohen-Seidenberg (theorem 1.11.4), the associated morphism from $\operatorname{Spec}(B/J)$ to $\operatorname{Spec}(A/I)$ is surjective. Since the canonical surjection from A to A/I induces a homeomorphism from $\operatorname{Spec}(A/I)$ to the closed subset $V(I)$ of $\operatorname{Spec}(A)$, this implies that $f(Z) = V(I)$. In particular, $f(Z)$ is closed in X.

For every X-scheme Z, the morphism of schemes $f_T\colon Y_Z \to Z$ deduced from f by base change is finite; by what precedes, it is closed as well. This proves that the morphism f is proper.

Let us now prove that its fibers are finite. As above, we may assume that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Let $x \in X$; then its fiber $f^{-1}(x)$ identifies with $\operatorname{Spec}(B \otimes_A \kappa(x))$, where $\kappa(x)$ is the residue field of X at x. The $\kappa(x)$-algebra $B \otimes_A \kappa(x)$ is finitely generated as a $\kappa(x)$-vector space, hence it has finite length. In particular, it is an artinian ring and it follows from lemma 1.12.7 that its spectrum is finite. □

Theorem **(5.4.16)**. — *The canonical morphism* $f\colon \mathbf{P}^n_{\mathbf{Z}} \to \operatorname{Spec}(\mathbf{Z})$ *is proper.*

Proof. — This morphism is separated and of finite type, so we just need to prove that it is universally closed. Let T be a scheme and let $f_T\colon \mathbf{P}^n_T \to T$ be the morphism deduced from f by base-change to T; let us prove that f_T is closed.

It is enough to treat the case where T is an affine scheme, which brings us to proving that the canonical morphism $f_k\colon \mathbf{P}^n_k \to \operatorname{Spec}(k)$ is closed for every ring k.

Recall that the scheme $\mathbf{P}^n_k$ is isomorphic to the projective spectrum of the graded ring $A = k[T_0, \dots, T_n]$; let A_+ be the ideal $(T_0, \dots, T_n)$. Let $Z \subseteq \mathbf{P}^n_k$ be a closed subset and let $J = j_+(Z)$ be its homogeneous ideal. For every integer $d \geqslant 0$, let $A_d \subseteq R[T_0, \dots, T_n]$ be the k-submodule of homogeneous polynomials of degree d and let $J_d = J \cap A_d$.

By assumption $y \notin f(Z)$, hence $V_+(\mathfrak{p}R[T_0, \dots, T_n]) \cap Z = \varnothing$. Consequently, the homogeneous ideal

$$V_+(j_+(Z) + \mathfrak{p}R[T_0, \dots, T_n]) = Z \cap V_+(\mathfrak{p}R[T_0, \dots, T_n]) = \varnothing.$$

Consequently, the smallest radical ideal of A which contains $j_+(Z) + \mathfrak{p}R[T_0, \dots, T_n]$ is equal to A_+. In particular, for every i, there exists an integer d_i such that $d_i \geqslant 0$, a homogeneous polynomial $P_i \in j_+(Z)$ and a homogeneous polynomial $Q_i \in \mathfrak{p}R[T_0, \dots, T_n]$ such that $T_i^{d_i} = P_i + Q_i$.

Let $d = \sum_{i=0}^n d_i$. By construction, every monomial of degree m belongs to $j_+(Z) + \mathfrak{p}R[T_0, \dots, T_n]$, hence the equality $A_d = J_d + \mathfrak{p}A_d$. By Nakayama's lemma (corollary 1.3.2) applied to the finitely generated k-module A_d/J_d and the ideal $\mathfrak{p}$ of k, there exists an element $a \in k$ such that $a - 1 \in \mathfrak{p}$ and such that $aA_d \subseteq J_d$. In particular, $aT_i^d \in J_d$ for every integer $i \in \{0, \dots, n\}$.

This implies that the ideal J contains the ideal $a(T_0^d, \dots, T_n^d)$, so that $Z = V_+(J) \subseteq V((a))$. Consequently, $f(Z) \subseteq V(a)$; moreover, $a \notin \mathfrak{p}$. In other words, the set $\operatorname{Spec}(k) - f(Z)$ contains the neighborhood $D(a)$ of $\mathfrak{p}$. This shows that $f(Z)$ is closed and concludes the proof that the morphism $f\colon \mathbf{P}^n_{\mathbf{Z}} \to \operatorname{Spec}(\mathbf{Z})$ is proper. □

Definition **(5.4.17)**. — *Let $f\colon X \to S$ be a morphism. One says that f is projective if there exists an integer $n \geqslant 0$ and a closed immersion of S-schemes, $g\colon X \to \mathbf{P}^n \times_{\operatorname{Spec}(\mathbf{Z})} S$.*

By theorem 5.4.16, the projection from $\mathbf{P}^n \times S$ to S is proper. It thus follows from proposition 5.4.14 that a projective morphism is proper.

5.5. Flat morphisms

Definition **(5.5.1)**. — *Let* A *be a ring and let* M *be an* A*-module. one says that* M *is* flat *(over* A*) if for every injective morphism* $u\colon N \to N'$ *of* A*-modules, the morphism* $\mathrm{id}_M \otimes u\colon M \otimes_A N \to M \otimes_A N'$ *is injective.*

One says that M *is* faithfully flat *(over* A*) if it is flat and if* $M \otimes_A N \neq 0$ *for every non-zero* A*-module* N.

This definition can be reformulated as follows.

Lemma **(5.5.2)**. — *Let* A *be a ring, and let* T_M *be the "tensorization by* M*" functor, from the category of* A*-modules to itself.*

a) *The* A*-module* M *is flat if and only if the functor* T_M *is exact.*

b) *The following assertions are equivalent: (i) The* A*-module* M *is faithfully flat; (ii) For any morphism* $u\colon N \to N'$ *of* A*-modules, then u is injective if and only if* $\mathrm{id}_M \otimes u$ *is injective; (iii) The functor* T_M *is exact and conservative.*

An functor T is called *conservative* if every morphism u such that $T(u)$ is an isomorphism is itself an isomorphism.

Proof. — a) By definition, the functor T_M is given by $T_M(N) = M \otimes_A N$ and $T_M(u) = \mathrm{id}_M \otimes u$ for every A-module N and every morphism u of A-modules. Recall that this functor is right exact; indeed, the universal property of the tensor product expresses the functor T_M as a left-adjoint of some functor. In particular, for every exact sequence $N'' \to N \to N' \to 0$ of A-modules, the associated sequence $M \otimes_A N'' \to M \otimes_A N \to M \otimes_A N' \to 0$ is exact. The definition of a flat module thus says that M is flat if and only if this functor T_M is exact: for every exact sequence $0 \to N'' \to N \to N' \to 0$ of A-modules, the associated sequence $0 \to M \otimes_A N'' \to M \otimes_A N \to M \otimes_A N' \to 0$ is exact.

b) Assume that M is flat. Let $u\colon N \to N'$ be morphism of A-modules. Then $\mathrm{Coker}(\mathrm{id}_M \otimes u) = M \otimes_A \mathrm{Coker}(u)$, and $\mathrm{Ker}(\mathrm{id}_M \otimes u) = M \otimes_A \mathrm{Ker}(u)$. Consequently, $\mathrm{id}_M \otimes u$ is surjective (resp. injective) if and only if u is surjective (resp. injective). It follows that $\mathrm{id}_M \otimes u$ is an isomorphism if and only if u is an isomorphism, that is, if the functor T_M is conservative.

Conversely, let us assume that M is flat and that the functor T_M is conservative. Let then N be an A-module such that $M \otimes_A N = 0$ and let $u\colon 0 \to N$ be the zero morphism; then $T_M(u) = 0$ is the isomorphism from 0 to $0 = M \otimes_A N$, so that u is an isomorphism: this shows that $N = 0$.

The same argument shows that M is faithfully flat over A if and only if, for every morphism u of A-modules such that $T_M(u)$ is injective, then u is injective. □

Proposition **(5.5.3)**. — *Let* A *be a ring.*

a) *The* A*-module* A *is faithfully flat.*

b) *A filtrant colimit of flat* A*-modules is flat.*

c) *A direct sum* $\bigoplus_i M_i$ *of a family* (M_i) *of* A*-modules is flat if and only if* M_i *is flat for every* i.

d) *Every projective* A*-module is flat.*

e) *For every multiplicative subset* S *of* A*, the* A*-module* $S^{-1}A$ *is flat.*

f) *Let* M *and* N *be flat (resp. faithfully flat)* A*-modules. Then* $M \otimes_A N$ *is flat (resp. faithfully flat).*

Proof. — a) Under the canonical isomorphism $A \otimes_A N \simeq N$ given by $a \otimes n \mapsto an$, a morphism $\mathrm{id}_A \otimes u$ identifies with u. In other words, the functor T_A is isomorphic with the identical functor. It is thus exact and conservative.

b) Let $((M_i)_{i\in I}, (\varphi_{ij})$ be a diagram of flat A-modules indexed by a filtrant partially ordered set, and $M = \varinjlim_i M_i$; for $i \in I$, let $\varphi_i \colon M_i \to M$ be the canonical morphism. Let then $u \colon N \to N'$ be an injective morphism of A-modules and let us show that $\mathrm{id}_M \otimes u$ is injective. Let x be any element of its kernel; there exists an element $i \in I$ and $x_i \in M_i \otimes N$ such that $x = (\varphi_i \otimes \mathrm{id}_N)(x_i)$. Consequently, one has

$$\begin{aligned}(\varphi_i \otimes \mathrm{id}_N)(T_{M_i}(x_i)) &= (\varphi_i \otimes \mathrm{id}_N) \circ (\mathrm{id}_{M_i} \otimes u)(x_i)\\ &= \varphi_i \otimes u(x_i)\\ &= (\mathrm{id}_M \otimes u) \circ (\varphi_i \otimes \mathrm{id}_N)(x_i)\\ &= T_M(u)(x) = 0.\end{aligned}$$

Since the tensor product is a right exact functor, the canonical morphism from $\varinjlim(M_i \otimes N)$ to $M \otimes N$ is an isomorphism. This implies that there exists $j \in I$ such that $j \geqslant i$ and such $(\varphi_{ij} \otimes \mathrm{id}_N)(T_{M_i}(x_i)) = 0$. Let then $x_j = \varphi_{ij}(x_i)$; one has $T_{M_j}(x_j) = 0$. Since M_j is a flat A-module, this implies that $x_j = 0$. Finally, $x = \varphi_i(x_i) = \varphi_j(\varphi_{ij}(x_i)) = \varphi_j(x_j) = 0$. This shows that the morphism $T_M(u)$ is injective and concludes the proof that M is a flat A-module.

c) Let M be the direct sum of the family (M_i); for every i, let $p_i \colon M \to M_i$ be the projection of index i. Under the isomorphism $M \otimes_A N \simeq \bigoplus_{i\in I} M_i \otimes_A N$ associated with the family $(p_i \otimes \mathrm{id}_N)$, a morphism $\mathrm{id}_M \otimes u$ identifies with the morphism $\bigoplus \mathrm{id}_{M_i} \otimes u$. Consequently, $\mathrm{id}_M \otimes u$ is injective if and only if $\mathrm{id}_{M_i} \otimes u$ is injective for every i.

d) If follows from *b*) that a free A-module is flat; it is moreover faithfully flat if it is non-zero. If M is a projective A-module, there exists an A-module N such that $M \oplus N$ is free; then $M \oplus N$ is flat; by *b*), M is flat as well.

e) For every A-module N, there exists a unique morphism from $S^{-1}A \otimes_A N$ to $S^{-1}N$ which maps a tensor $1 \otimes n$ to the fraction $n/1$. This morphism is an isomorphism. Indeed[(2)], there exists a unique morphism from $S^{-1}N$ to $S^{-1}A \otimes_A N$ which maps $n/1$ to $1 \otimes n$. Both compositions of these morphisms are the identity. Consequently, the functor $T_{S^{-1}A}$ is isomorphic to the localization functor. By example 2.3.15, this latter functor is exact, so that $S^{-1}A$ is a flat A-module, as claimed.

f) Given the associativity isomorphisms $(M \otimes_A N) \otimes_A P \simeq M \otimes_A (N \otimes_A P)$ of the tensor product, the functor $T_{M \otimes_A N}$ is the composition $T_M \circ T_N$ of the exact functors T_M and T_N, hence is exact. □

Proposition **(5.5.4)**. — *Let* A *be a ring and let* B *be an* A*-algebra.*

a) *For every flat (resp. faithfully flat)* A*-module* M*, the* B*-module* $M \otimes_A B$ *is flat (resp. faithfully flat).*

b) *Assume that* B *is flat over* A*. Then for every flat* B*-module* M*, the* A*-module* M *is flat.*

c) *Assume that* B *is faithfully flat over* A*. Then, for every* A*-module* M*, the* B*-module* $M \otimes_A B$ *is flat (resp. faithfully flat) if and only if* M *is flat (resp. faithfully flat) over* A*.*

Proof. — a) For every B-module N, there is an isomorphism from $(M \otimes_A B) \otimes_B N$ with $M \otimes_A N$, given by $(m \otimes b) \otimes n \mapsto m \otimes (bn)$, for $m \in M$, $b \in B$ and $n \in N$. Thanks to these isomorphisms, the functor $T_{M \otimes_A B}$ identifies with the composition of the functor T_M with the forgetful functor from the category of B-modules to the category of A-modules. Since the latter functor is exact, this implies that $T_{M \otimes_A B}$ is exact is T_M is.

Assume moreover that M is faithfully flat and let N be a B-module such that $(M \otimes_A B) \otimes_B N = 0$. Then $M \otimes_A N = 0$, hence $N = 0$. This shows that $M \otimes_A B$ is a faithfully flat B-module.

(2) This should be already in the notes, but I can't find it!

b) Let us denote by M_A the A-module associated with M. Since B is an A-algebra, every module of the form $B \otimes_A N$, for an A-module N is naturally a B-module, and the functor T_B can be viewed as a functor from the category of A-modules to the category of B-modules. Since B is flat over A, this functor is exact. Under the isomorphisms $M_A \otimes_A N \simeq M \otimes_B (B \otimes_A N)$, the functor T_{M_A} identifies with the composition of the functor T_M and the functor T_B. Consequently, it is exact as well, and M_A is a flat A-module.

c) Let us assume that $M \otimes_A B$ is a flat B-module. Let $u\colon N \to N'$ be an injective morphism of A-modules; since B is faithfully flat over A, the morphism $u_B = u \otimes \mathrm{id}_B\colon N \otimes_A B \to N' \otimes_A B$ is injective. Consequently, the morphism $\mathrm{id}_M \otimes u_B$ from $M \otimes_A N \otimes_A B$ to $M \otimes_A N' \otimes_A B$ is injective. Since B is faithfully flat over A, this implies that the morphism $\mathrm{id}_M \otimes u$ is injective as well. Consequently, M is a flat A-module.

□

Proposition (5.5.5). — *Let* A *be a ring and let* M *be an* A*-module. The following properties are equivalent:*

(i) *The* A*-module* M *is flat (resp. faithfully flat);*
(ii) *For every* $\mathfrak{p} \in \mathrm{Spec}(A)$*, the* $A_\mathfrak{p}$*-module* $M_\mathfrak{p}$ *is flat (resp. faithfully flat);*
(iii) *For every* $\mathfrak{m} \in \mathrm{Spm}(A)$*, the* $A_\mathfrak{m}$*-module* $M_\mathfrak{m}$ *is flat (resp. faithfully flat).*

Proof. — (i)⇒(ii) follows from the fact that flatness is preserved by base change, and (ii)⇒(iii) is obvious.

Let us assume that $M_\mathfrak{m}$ is flat over $A_\mathfrak{m}$ for every $\mathfrak{m}$, and let $u\colon N \to N'$ be an injective morphism of A-modules, let $v = \mathrm{id}_M \otimes u$ and let us prove that v is injective. Let $\mathfrak{m} \in \mathrm{Spm}(A)$; the morphism $u_\mathfrak{m}$ is injective, hence $M_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module. Since the morphism $v_\mathfrak{m}$ identifies with $\mathrm{id}_{M_\mathfrak{m}} \otimes u_\mathfrak{m}$, we conclude that $v_\mathfrak{m}$ is injective. By exactness of localization, the canonical morphism from $\mathrm{Ker}(v)_\mathfrak{m}$ $\mathrm{Ker}(v_\mathfrak{m})$ is an isomorphism, hence $\mathrm{Ker}(v)_\mathfrak{m} = 0$. This this holds for every maximal ideal $\mathfrak{m}$ of A, one has $\mathrm{Ker}(v) = 0$ (lemma 1.2.9), hence v is injective.

This shows that the three statements concerning flatness are equivalent. Let us check thee equivelence of their counterparts for faithfull flatness. Let N be a A-module such that $M \otimes_A N = 0$. For every prime ideal $\mathfrak{p}$ of A, one has an isomorphism of $A_\mathfrak{p}$-modules, $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$, so that the implication (i′)⇒(ii″) holds, and the implication (ii′)⇒(iii′) is again obvious. Finally, if (iii′) holds and if

$M \otimes_A N = 0$, then $M_{\mathfrak m} \otimes_{A_{\mathfrak m}} N_{\mathfrak m} = 0$ for every maximal ideal $\mathfrak m$ of A, hence $N_{\mathfrak m} = 0$; by lemma 1.2.9, one has $N = 0$. □

Exercise **(5.5.6)**. — Let A be a ring and let M be an A-module.

a) Prove that M is flat if and only if, for every ideal I of A, the canonical morphism from $I \otimes_A M$ to IM is an isomorphism.

b) Assume that A is a principal ideal domain. Prove that M flat if and only if it is torsion free. Prove that **Q** is a flat **Z**-module which is not projective.

Exercise **(5.5.7)**. — Let A be a ring and let M be an A-module. A relation in M is an expression of the form $\sum_{i=1}^n a_i x_i = 0$, where (a_i) is a family of elements of A, and (x_i) is a family of elements of M. A relation is said to be trivial if there exists a family (b_{ij}) of elements of A and a family (y_j) of elements of M such that $x_i = \sum_{j=1}^m b_{ij} y_j$ for all i, and $\sum_{i=1}^n a_i b_{ij} = 0$ for all j.

Prove that M is flat if and only if every relation in M is trivial.

Proposition **(5.5.8)**. — *Let* A *be a ring and let* M *be a flat* A*-module. The following properties are equivalent:*

(i) *The* A*-module* M *is faithfully flat;*
(ii) *For every prime ideal* $\mathfrak p$ *of* A*, one has* $M \otimes_A \kappa(\mathfrak p) \neq 0$*;*
(iii) *For every maximal ideal* $\mathfrak m$ *of* A*, one has* $M \otimes_A \kappa(\mathfrak m) \neq 0$.

Proof. — (i)⇒(ii) follows from the definition, since $\kappa(\mathfrak p) \neq 0$.

(ii)⇒(iii) is obvious.

(iii)⇒(i). Let N be an A-module such that $M \otimes_A N = 0$. Let $x \in N$ and let $I = \{a \in A\,;\, ax = 0\}$ be its annihilator. Let $g\colon A/I \to N$ be the unique morphism which maps the class of an element $a \in A$ to ax; it is injective. Since M is flat, the morphism $\mathrm{id}_M \otimes g$ is injective as well, hence $M \otimes_A (A/I) = 0$, that is, $M = IM$. If $I = A$, then $1 \in I$ and $x = 0$. Otherwise, there exists a maximal ideal $\mathfrak m$ of A such that $I \subseteq \mathfrak m$; one then has $M = IM = \mathfrak m M$, which contradicts the assumption that $M \otimes_A \kappa(\mathfrak m) \neq 0$. □

Corollary **(5.5.9)**. — *Let* $f\colon A \to B$ *be a flat ring morphism. Then* f *is faithfully flat if and only if the map* ${}^a f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective.*

Proof. — Let $\mathfrak p$ be a prime ideal of A. The prime ideals $\mathfrak q$ of B such that $f^{-1}(\mathfrak q) = \mathfrak p$ are in bijection with the prime ideals of $B_{\mathfrak p}$ which contain $\mathfrak p B_{\mathfrak p}$. Consequently, $\mathfrak p$ belongs to the image of ${}^a f$ if and only if $B_{\mathfrak p}/\mathfrak p B_{\mathfrak p} \neq 0$. Since the latter ring is isomorphic to $B \otimes_A \kappa(\mathfrak{p})$, this shows that ${}^a f$ is surjective if and only if f is faithfully flat. □

Corollary **(5.5.10)**. — *A local morphism of local rings which is flat is faithfully flat.*

Proof. — Let $f\colon A \to B$ be a flat local morphism of local rings. Let $\mathfrak{p}$ be the maximal ideal of A and let $\mathfrak{q}$ be the maximal ideal of B. By assumption, one has ${}^a f(\mathfrak{q}) = \mathfrak{p}$, hence $B/\mathfrak{p}B \neq 0$. Consequently, f satisfies the assumption (iii) of proposition 5.5.8, hence f is faithfully flat. □

Definition **(5.5.11)**. — *Let* X *be a scheme and let* $\mathscr{M}$ *be an* $\mathscr{O}_X$*-module. One says that* $\mathscr{M}$ *is flat if, for every* $x \in X$*, the* $\mathscr{O}_{X,x}$*-module* $\mathscr{M}_x$ *is flat.*

If $\mathscr{M}$ is flat, then $\mathscr{M}|_U$ is a flat $\mathscr{O}_U$-module. Conversely, if every point x of X has an open neighborhood U such that $\mathscr{M}|_U$ is a flat $\mathscr{O}_U$-module, then $\mathscr{M}$ is flat.

Together with proposition 5.5.5, these remarks imply the following proposition.

Proposition **(5.5.12)**. — *Let* X *be a scheme and let* $\mathscr{M}$ *be a quasi-coherent* $\mathscr{O}_X$*-module. The following properties are equivalent:*

(i) *The* $\mathscr{O}_X$*-module* $\mathscr{M}$ *is flat;*

(ii) *For every affine open subset* U *of* X*, the* $\mathscr{O}_X(U)$*-module* $\mathscr{M}(U)$ *is flat;*

(iii) *Every point of* X *has an affine open neighborhood* U *such that the* $\mathscr{O}_X(U)$*-module* $\mathscr{M}(U)$ *is flat.*

Example **(5.5.13)**. — Let X be a scheme. The following properties of flat $\mathscr{O}_X$-modules follow directly from the definition and from proposition 5.5.3.

a) The $\mathscr{O}_X$-module $\mathscr{O}_X$ is flat.

b) A direct sum $\bigoplus \mathscr{M}_i$ of a family $(\mathscr{M}_i)$ of $\mathscr{O}_X$-modules is flat if and only if $\mathscr{M}_i$ is flat for every i.

c) A finitely presented $\mathscr{O}_X$-module is flat if and only if it is locally free.

d) Let $\mathscr{M}$ and $\mathscr{N}$ be flat $\mathscr{O}_X$-modules; then $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ is flat.

Definition **(5.5.14)**. — *Let* $f\colon Y \to X$ *be a morphism of schemes. One says that* f *is flat if* $\mathscr{O}_{Y,y}$ *is a flat* $\mathscr{O}_{X,f(y)}$*-module, for every* $y \in Y$.

One says that f *is faithfully flat if it is flat and surjective.*

If Y is an X-scheme, then one also says that Y is flat (resp. faithfully flat) over X to mean that its structural morphism is flat.

There is a more general definition that is often useful in more advanced topics of algebraic geometry. Let $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_Y$-module. One says that $\mathscr{M}$ is f-flat at a point $y \in Y$ if $\mathscr{M}_y$ is flat over $\mathscr{O}_{X,f(y)}$. One says that it is f-flat if it is f-flat at every point of Y.

Given this definition, saying that f is flat is equivalent to saying that $\mathscr{O}_Y$ is f-flat.

Lemma **(5.5.15)**. — *Let $f\colon Y \to X$ be a morphism of schemes. The following properties are equivalent:*

(i) *The morphism f is flat;*

(ii) *For every open affine subscheme U of X and every affine subscheme V of $f^{-1}(U)$, the ring $\mathscr{O}_Y(V)$ is a flat $\mathscr{O}_X(U)$-module;*

(iii) *For every point $y \in Y$, there exists an affine open neighborhood V of y in Y, and an affine open neighborhood U of $f(y)$ in X such that $f(V) \subseteq U$ and such that the ring $\mathscr{O}_Y(V)$ is a flat $\mathscr{O}_X(U)$-module.*

In particular, a morphism of affine schemes $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is flat if and only if B is a flat A-module. By corollary 5.5.9, it is then faithfully flat if and only if B is a faithfully flat A-module.

Proposition **(5.5.16)**. — a) *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be flat morphisms, then $f \circ g$ is flat.*

b) *Let $f\colon Y \to X$ and $g\colon Z \to X$ be morphisms of schemes. If f is flat, then the morphism $f_Z\colon Y_Z \to Z$ deduced from f by base change to Z is flat. If f_Z is flat and g is faithfully flat, then f is flat.*

c) *Let $f\colon Y \to X$ and $g\colon Z \to X$ be morphisms of schemes. If f and g are flat, then the canonical morphism $h\colon Y \times_X Z \to X$ is flat.*

Proposition **(5.5.17)** (Going down for flat morphisms). — *Let $f\colon A \to B$ be a flat morphism of rings. Let $(\mathfrak{p}_0, \dots, \mathfrak{p}_n)$ be a chain of prime ideals of A and let $\mathfrak{q}_n$ be a prime ideal of B such that ${}^a f(\mathfrak{q}_n) = \mathfrak{p}_n$. There exists a chain $(\mathfrak{q}_0, \dots, \mathfrak{q}_n)$ of prime ideals of B such that ${}^a f(\mathfrak{q}_m) = \mathfrak{p}_m$ for every $m \in \{0, \dots, n\}$.*

Proof. — By induction, we may assume that $n = 1$. Let us then consider the flat morphism of local rings $g\colon A_{\mathfrak{p}_1} \to B_{\mathfrak{q}_1}$ deduced from f by localization. It is surjective, hence there exists a prime ideal $\mathfrak{q}_0$ of B contained in $\mathfrak{q}_1$ such that

$\mathfrak{p}_0 A_{\mathfrak{p}_1} = g^{-1}(\mathfrak{q}_0 B_{\mathfrak{q}_1})$. Necessarily, $\mathfrak{p}_0 = f^{-1}(\mathfrak{q}_0)$, and this concludes the proof of the proposition. □

Proposition **(5.5.18)**. — *Let* A *and* B *be noetherian local rings, let* $\mathfrak{m}_A$ *and* $\mathfrak{m}_B$ *denote their maximal ideals and let* $\varphi\colon A \to B$ *be a local morphism. Then*

$$\dim(B) \leqslant \dim(A) + \dim(B/\mathfrak{m}_A B).$$

If φ *is flat, then equality holds:*

$$\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B).$$

Proof. — Let $d = \dim(A)$ and let $(a_1, \dots, a_d)$ be a family of elements of $\mathfrak{m}_A A$ such that $\mathfrak{m}_A = \sqrt{(a_1, \dots, a_d)}$. Let $e = \dim(B/\mathfrak{m}_A B)$, and let $(b_1, \dots, b_e)$ be elements of $\mathfrak{m}_B$ such that $\mathfrak{m}_B = \sqrt{(b_1, \dots, b_e) + \mathfrak{m}_A B}$. Then $(\varphi(a_1), \dots, \varphi(a_d), b_1, \dots, b_e)$ is an ideal of B, contained in $\mathfrak{m}_A B$. Moreover, the radical of this ideal contains $\mathfrak{m}_A B$ and $(b_1, \dots, b_e)$, hence it is equal to $\mathfrak{m}_A B$. This implies that

$$\dim(B) \leqslant d + e = \dim(A) + \dim(B/\mathfrak{m}_A B).$$

Let us now assume that φ is flat. Let $(\mathfrak{p}_0, \dots, \mathfrak{p}_d)$ be a chain of prime ideals of A and let $(\mathfrak{q}_d, \dots, \mathfrak{q}_{d+e})$ be a chain of prime ideals of B containing $\mathfrak{m}_A B$. By the going-down proposition for flat morphisms (proposition 5.5.17), there exist prime ideals $\mathfrak{q}_0, \dots, \mathfrak{q}_{d-1}$ of B such that ${}^a\varphi(\mathfrak{q}_i) = \mathfrak{p}_i$ for every i, and such $\mathfrak{q}_0 \subseteq \dots \subseteq \mathfrak{q}_d$. Then $(\mathfrak{q}_0, \dots, \mathfrak{q}_d, \dots, \mathfrak{q}_{d+e})$ is a chain of prime ideals of B, hence $\dim(B) \geqslant d + e$. □

Theorem **(5.5.19)**. — *Let* K *be a field, let* X *and* Y *be* K*-schemes of finite type; Assume that* X *is irreducible and that* Y *is equidimensional. Let* $f\colon Y \to X$ *be a flat* K*-morphism. For every* $x \in X$*, the fiber* Y_x *is equidimensional and*

$$\dim(Y_x) = \dim(Y) - \dim(X).$$

Geometrically, this theorem says that given a flat morphism $f\colon Y \to X$ as in the statement of the theorem, all fibers of f have the same dimension which is the difference of the dimensions of Y and X. Flatness is thus seen as a property that the fibers of a morphism behave in a reasonable way.

Proof. — Let y be a closed point of Y_x. Let Z be an irreducible component of Y containing y. Since Y is equidimensional, one has $\dim(Z) = \dim(Y)$, hence

$$\dim(\mathscr{O}_{Z,y}) = \dim(Z) - \dim(\overline{\{y\}}) = \dim(Y) - \dim(\overline{\{y\}}).$$

Consequently,

$$\dim(\mathscr{O}_{X,y}) = \dim(Y) - \dim(\overline{\{y\}}).$$

On the other hand, y is the generic point of $\overline{\{y\}}$, which is a closed subscheme of Y, hence is a K-scheme of finite type; we thus have $\dim(\overline{\{y\}}) = \operatorname{tr.deg}_K(\kappa(y))$. Similarly, $\dim(\overline{\{x\}}) = \operatorname{tr.deg}_K(\kappa(x))$. Moreover, $\kappa(y)$ is a finite extension of $\kappa(x)$, because y is a closed point of Y_x. Consequently,

$$\dim(\overline{\{y\}}) = \operatorname{tr.deg}_K(\kappa(y)) = \operatorname{tr.deg}_K(\kappa(x)) = \dim(\overline{\{x\}}) = \dim(X) - \dim(\mathscr{O}_{X,x}).$$

This implies the relation

$$\dim(\mathscr{O}_{Y,y}) - \dim(\mathscr{O}_{X,x}) = \dim(Y) - \dim(X).$$

On the other hand, since y is a closed point of Y_x, one has

$$\dim_y(Y_x) = \dim(\mathscr{O}_{Y_x,y}) = \dim(\mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{Y,y}),$$

since

$$\mathscr{O}_{Y_x,y} = \mathscr{O}_{Y,y} \otimes \kappa(x) = \mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{Y,y}.$$

Proposition 5.5.18 then shows that $\dim_y(Y_x) \geqslant \dim(Y) - \dim(X)$, with equality if f is flat at y. In particular, $\dim(Y_x) \geqslant \dim(Y) - \dim(X)$. If f is flat, then $\dim_y(Y_x) = \dim(Y) - \dim(X)$ for every closed point $y \in Y_x$. It first follows that $\dim(Y_x) = \dim(Y) - \dim(X)$. If Y_x were not equidimensional, it would possess an irreducible component T of dimension $< \dim(Y) - \dim(X)$; let then y be a closed point of T which does not belong to the union of the other components; one has $\dim_y(Y_x) = \dim(T)$, a contradiction. □

Exercise **(5.5.20)**. — Let $f\colon \mathbf{A}^2_K \to \mathbf{A}^2_K$ be the morphism given by $f(x,y) = (xy, y)$. Let $U = \mathbf{A}^2_K - V(x,y)$. Prove that $f_U\colon f^{-1}(U) \to U$ is an isomorphism. Let $P = V(x,y)$. Prove that $f^{-1}(P) \simeq \mathbf{A}^1_K$. It thus follows from theorem 5.5.19 that f is not flat; prove this fact directly.

5.6. The module of relative differential forms

Definition **(5.6.1)**. — *Let k be a ring, let A be a k-algebra and let M be an A-module. A map $d\colon A \to M$ is called a k-*derivation *if it is k-linear and if one has*

$$d(ab) = ad(b) + bd(a)$$

for every pair (a,b) of elements of A.

For every integer n such that $n \geqslant 1$ and every $a \in A$, one proves by induction that

$$d(a^n) = na^{n-1}d(a)$$

Let $a, b \in A$; if b is invertible, then $d(b \cdot (a/b)) = (a/b)d(b) + bd(a/b)$, so that

$$d(a/b) = b^{-2}(bd(a) - ad(b)).$$

In particular, $d(1) = d(1/1) = 0$; consequently, $d(a) = ad(1) = 0$ for every element a in the image of k.

The set $\mathrm{Der}_k(A, M)$ of k-derivations from A to M is an A-submodule of the A-module M^A. When $k = \mathbf{Z}$, one simply says that d is a derivation; the module $\mathrm{Der}_{\mathbf{Z}}(A, M)$ is simply denoted by $\mathrm{Der}(A, M)$.

If $f: M \to N$ is a morphism of A-modules and $d: A \to M$ is a k-derivation, then $f \circ d$ is a k-derivation. This defines a map $f_*: \mathrm{Der}_k(A, M) \to \mathrm{Der}_k(A, N)$; it is a morphism of A-modules.

Example **(5.6.2)**. — Let k be a ring, let I be a set and let $A = k[(T_i)_{i \in I}]$ be the ring of polynomials with coefficients in k in the family of indeterminates $(T_i)_{i \in I}$.

a) For every $i \in I$, the map $P \mapsto \partial P/\partial T_i$ is a k-derivation from A to A.

b) Let M be an A-module. The map $\mathrm{Der}_k(A, M) \to M^I$ which associates, with every k-derivation $d: A \to M$, the family $(d(T_i))_{i \in I}$ is an isomorphism of A-modules.

Let us denote this map by φ. It is A-linear. Moreover, for every multi-index $(n_i) \in \mathbf{N}^{(I)}$ and every k-derivation $d: A \to M$, one has

$$d\Big(\prod_i T_i^{n_i}\Big) = \sum_{i \in I} n_i T_i^{n_i - 1} \prod_{\substack{j \in I \\ j \neq i}} T_j^{n_j} d(T_i);$$

this sum is finite since $n_i = 0$ for all but finitely many elements $i \in I$. Consequently,

$$d(P) = \sum_{i \in I} \frac{\partial P}{\partial T_i} d(T_i),$$

where, again, this sum is in fact finite because a polynomial P depends on finitely many indeterminates, hence $\partial P/\partial T_i = 0$ for all but finitely many $i \in I$. This shows that the morphism φ is injective. Moreover, if $(m_i)_{i \in I}$ is a family of elements of M, then the map

$$P \mapsto \sum_{i \in I} \frac{\partial P}{\partial T_i} m_i$$

is a k-derivation; consequently, φ is surjective.

Exercise **(5.6.3)**. — Let k be a ring, let A be a k-algebra and let M be an A-module. Let M_ε be the abelian group $M_\varepsilon = A \oplus M$, endowed with the multiplication law given by $(a, m)\cdot(a', m') = (aa', am' + a'm)$. Show that M_ε is a ring and that the map from M_ε to A given by $(a, m) \mapsto a$ is a morphism of rings. Let $d\colon A \to M$ be a map. prove that the map from A to M_ε given by $a \mapsto (a, d(a))$ is a ring morphism if and only if d is a derivation.

Proposition **(5.6.4)**. — *Let k be a ring and let* A *be a k-algebra.*

a) *There exists an* A*-module* $\Omega^1_{A/k}$ *and a k-derivation* $d_{A/k}\colon A \to \Omega^1_{A/k}$ *satisfying the following universal property: for every* A*-module* M *and every derivation* $d\colon A \to M$*, there exists a unique* A*-linear morphism* $\varphi\colon \Omega^1_{A/k} \to M$ *such that* $\varphi \circ d_{A/k} = d$.

b) *If* A *is a finitely generated k-algebra, then* $\Omega^1_{A/k}$ *is a finitely generated* A*-module.*

c) *If* A *is a finitely presented k-algebra, then* $\Omega^1_{A/k}$ *is a finitely presented* A*-module.*

Any A-module $\Omega^1_{A/k}$ such in the proposition is called a *module of differential forms* of A over k. Since it satisfies a universal property, the pair $(\Omega^1_{A/k}, d_{A/k})$ is well defined up to isomorphism.

In fact, the assignment $M \mapsto \mathrm{Der}_k(A, M)$ is a functor from the category of A-modules to itself; the functorial isomorphisms

$$\mathrm{Hom}_A(\Omega^1_{A/k}, M) \to \mathrm{Der}_k(A, M), \qquad f \mapsto f \circ d_{A/k}$$

show that this functor is corepresentable.

Lemma **(5.6.5)**. — *Let k be a ring, let* B *be a k-algebra; Assume that there exists a pair* $(\Omega^1_{B/k}, d_{B/k})$ *satisfying the universal property of a module of differentials of* B*. Let* I *be an ideal of* B *and let* $A = B/I$*; let* $\Omega^1_{A/k}$ *be the* A*-module* $\Omega^1_{B/k}/(I\Omega^1_{B/k} + Bd_{B/k}(I))$*; let* $p\colon B \to A$ *and* $q\colon \Omega^1_{B/k} \to \Omega^1_{A/k}$ *be the canonical surjections.*

a) *There exists a unique map* $d_{A/k}\colon A \to \Omega^1_{A/k}$ *such that* $d_{A/k}(p(a)) = q(d_{B/k}(a))$ *for every* $a \in B$*; it is a k-derivation.*

b) *The pair* $(\Omega^1_{A/k}, d_{A/k})$ *satisfies the universal property of a module of differentials of* A.

c) *If* $\Omega^1_{B/k}$ *is a finitely generated* B*-module, then* $\Omega^1_{A/k}$ *is a finitely generated* A*-module.*

d) *If* $\Omega^1_{B/k}$ *is a finitely presented* B*-module and* I *is a finitely generated ideal, then* $\Omega^1_{A/k}$ *is a finitely presented* A*-module.*

Proof. — a) A priori, $\Omega^1_{A/k}$ is defined as a B-module; since the elements of I act by 0 in $\Omega^1_{A/k}$, it is a A-module. Moreover, the map $q \circ d_{B/k}$ is k-linear and its kernel contains I; consequently, there exists a unique k-linear morphism $d_{A/k}\colon A \to \Omega^1_{A/k}$ such that $d_{A/k} \circ p = q \circ d_{B/k}$.

b) Let now M be a A-module and let $d\colon A \to M$ be a k-derivation. The surjective morphism $p\colon B \to A$ endowes M with the structure of a B-module, and the map $a \mapsto d(p(a))$ is a k-derivation from B to M; consequently, there exists a B-linear morphism $f\colon \Omega^1_{B/k} \to M$ such that $d \circ p = f \circ d_{B/k}$. For every $a \in I$, one has $f(d_{B/k}(a)) = d(p(a)) = 0$, hence $d_{B/k}(I) \subseteq \operatorname{Ker}(f)$. Moreover, for every $\omega \in \Omega^1_{B/k}$ and every $a \in I$, one has $f(a\omega) = af(\omega) = 0$, since M is an A-module; consequently, $I\Omega^1_{B/k} \subseteq \operatorname{Ker}(f)$. Consequently, there exists a B-linear morphism $g\colon \Omega^1_{A/k} \to M$ such that $f = g \circ q$; this is an A-linear morphism. Finally, one has

$$d \circ p = f \circ d_{B/k} = g \circ q \circ d_{B/k} = g \circ q \circ d_{B/k} = g \circ d_{A/k} \circ p.$$

Since p is surjective, this implies that $d = g \circ d_{A/k}$. Finally, if $g'\colon \Omega^1_{A/k} \to M$ is an A-linear morphism such that $d = g' \circ d_{A/k}$, one has $d \circ p = g' \circ q \circ d_{B/k} = g \circ q \circ d_{B/k}$, hence $g' \circ q = g \circ q$; by the universal property of $d_{B/k}$. Since q is surjective, this implies $g = g'$.

c) Let us assume that $\Omega^1_{B/k}$ is finitely generated as a B-module. Since $\Omega^1_{A/k}$ is a quotient of $\Omega^1_{B/k}$, it is finitely generated as a B-module, hence as an A-module since the morphism from B to A is surjective.

d) Let us finitely assume that $\Omega^1_{B/k}$ is finitely presented as a B-module and that I is a finitely generated ideal. Then $\Omega^1_{B/k}/I\Omega^1_{B/k}$ is finitely presented as well, and $\Omega^1_{A/k}$ is the quotient of that module by the A-submodule generated by the images elements of the form $d_{B/k}(a)$, for $a \in I$. Let $(b_1, \dots, b_n)$ be a finite family generating I. For every family $a \in I$ and every family $(a_1, \dots, a_n)$ of elements of B such that $a = a_1 b_1 + \dots + a_n b_n$, one has

$$d_{B/k}(a) = \sum_{i=1}^{n} a_i d_{B/k}(b_i) + \sum_{i=1}^{n} b_i d_{B/k}(a_i).$$

Consequently, $\Omega^1_{A/k}$ is isomorphic to the quotient of $\Omega^1_{B/k}/I\Omega^1_{B/k}$ by the finitely generated submodule generated by the images of the elements $d_{B/k}(b_i)$, for $1 \leqslant i \leqslant n$. It is thus finitely presented.

□

Proof of proposition 5.6.4. — As any k-algebra, A is isomorphic to the quotient of a polynomial algebra $B = k[(T_\lambda)_{\lambda\in L}]$ by an ideal J. For example, the unique morphism from $k[(T_a)_{a\in A}]$ to A such that $T_a \mapsto a$ is surjective. If A is finitely generated, we may even assume that the set L is finite; if, moreover, A is finitely presented, then the ideal J is finitely generated. By example 5.6.2, the k-algebra B admits a module of differentials, namely the module $\Omega^1_{B/k} = B^{(L)}$. By lemma 5.6.5, the k-algebra A admits the quotient $\Omega^1_{A/k} = B^{(L)}/(JB^{(L)} + Bd_{B/k}(J))$ as a module of differentials. It also follows from this lemma that $\Omega^1_{A/k}$ is a finitely generated (resp. finitely presented) A-module if A is a finitely generated (resp. finitely presented) k-algebra. □

Remark **(5.6.6)**. — We detail a few consequences of the above explicit construction of the A-module $\Omega^1_{A/k}$.

a) Let us assume that $A = k[(X_\lambda)]_{\lambda\in L}$ is a polynomial algebra. Then the family $(d_{A/k}(X_\lambda))_{\lambda\in L}$ is a basis of $\Omega_{A/k}$; in particular, this A-module is free.

b) Let $a = (a_\lambda)_{\lambda\in L}$ be a family of elements of A which generates A as a k-algebra. The family $(d_{A/k}(a_\lambda))$ generates $\Omega_{A/k}$ as an A-module. The kernel of the associated morphism from $A^{(L)}$ to $\Omega_{A/k}$ is generated by families $(\partial P/\partial X_\lambda(a))$, where $P \in k[(X_\lambda)]_{\lambda\in L}$ generates the kernel I of the morphism from $k[(X_\lambda)]$ to A given by $P \mapsto P(a)$.

c) Let us put ourselves in the context of lemma 5.6.5. Let $x, y \in B$; in the A-module $A \otimes_B \Omega_{B/k}$, one has

$$1 \otimes d_{B/k}(xy) = 1 \otimes (xd_{B/k}(y) + yd_{A/k}(x)) = p(x) \otimes d_{B/k}(y) + p(y) \otimes d_{B/k}(x).$$

In particular, if $x, y \in I$, then $p(x) = p(y) = 0$ and $1 \otimes d_{B/k}(xy) = 0$. It follows that there exists a unique morphism of A-modules, $\delta\colon I/I^2 \to A \otimes_B \Omega_{B/k}$ which maps the class modulo I^2 of an element $x \in I$ to $1 \otimes d_{B/k}(x)$. Its image is the quotient of $\Omega_{B/k}$ by the submodule generated by $I\Omega_{B/k}$ and $d_{B/k}(I)$.

There exists a unique morphism $q'\colon A \otimes_B \Omega_{B/k} \to \Omega_{A/k}$ such that $q'(a \otimes d_{B/k}(b)) = ad_{A/k}(p(b))$ for every $a \in A$ and every $b \in B$. The kernel of the surjection $q\colon \Omega_{B/k} \to \Omega_{A/k}$ is generated by $I\Omega_{B/k} + d_{B/k}(I)$. Consequently, the kernel of q' is generated by the image of δ.

This shows that the natural diagram of A-modules

$$I/I^2 \xrightarrow{\delta} A \otimes_B \Omega_{B/k} \xrightarrow{q'} \Omega_{A/k} \to 0$$

is an exact sequence (*conormal exact sequence*).

5.6.7. — Let k be a ring and let A be a k-algebra. We give an alternate construction of the module of k-differentials. It is more abstract, but important; as we will see, it embodies the algebraic nature of "first order variation" well known in calculus.

Let $m\colon A\otimes_k A \to A$ be the unique morphism of k-algebras such that $m(a\otimes b) = ab$ for every pair (a, b) of elements of A, and let I be its kernel.

Let j_1 and j_2 be the maps from A to $A \otimes_k A$ given by $j_1(a) = a \otimes 1$ and $j_2(a) = 1 \otimes a$; they are morphisms of k-algebras. Obviously, one has $j_2(a) - j_1(a) \in I$, for every $a \in A$. Consequently, if $u \in I$, then $j_1(a)u \equiv j_2(a)u \pmod{I^2}$, for every $a \in A$: this shows that the two morphisms j_1 and j_2 induce the *same* stucture of an A-module on I/I^2. Let then $d\colon A \to I/I^2$ be the map given by $d(a) = (j_2(a) - j_1(a)) \pmod{I^2}$.

Let us show that *the map d is a k-derivation on* A. This map is additive; moreover, for every $s \in k$ and every $a \in A$, one has

$$d(sa) \equiv 1 \otimes sa - sa \otimes 1 \equiv s(1 \otimes a - a \otimes 1) = sd(a) \pmod{I^2}.$$

This shows that d is a k-linear map. Let then $a, b \in A$. One has

$$\begin{aligned} j_2(ab) - j_1(ab) &= 1 \otimes ab - ab \otimes 1 \\ &= j_2(a)(1 \otimes b - b \otimes 1) + b \otimes a - ab \otimes 1 \\ &= j_2(a)(j_2(b) - j_1(b)) + j_1(b)(1 \otimes a - a \otimes 1) \\ &= j_2(a)(j_2(b) - j_1(b)) + j_1(b)(j_2(a) - j_1(a)), \end{aligned}$$

so that $d(ab) = ad(b) + bd(a)$.

Let us then prove that the pair $(I/I^2, d)$ satisfies the universal property of a module of differentials. Let M be an A-module and let $f\colon A \to M$ be an M-valued k-derivation on A. First of all, the relation

$$a \otimes b = j_1(a)(1 \otimes b - b \otimes 1) + ab \otimes 1$$

implies that I is generated, as an A-module (under j_1), by elements of the form $1 \otimes b - b \otimes 1$. As a consequence, the image of d generates I/I^2 as an A-module, so that there exists at most one morphism of A-modules $\varphi\colon I/I^2 \to M$ such that

$f = \varphi \circ d$. On the other hand, the map $(a, b) \mapsto af(b)$ from $A \times A$ to M is k-bilinear, so that there exists a unique k-linear morphism $g\colon A \otimes_k A \to M$ such that $g(a \otimes b) = af(b)$ for every pair $(a, b) \in A \times A$. When $A \otimes_k A$ is viewed as an A-module via j_1, this map g is A-linear. Let $u, v \in I$; write $u = \sum_i a_i \otimes b_i$ and $v = \sum_j a'_j \otimes b'_j$; by definition, one has $\sum a_i b_i = \sum a'_j b'_j = 0$, hence

$$\begin{aligned} g(uv) &= \sum_{i,j} g(a_i a'_j \otimes b_i b'_j) \\ &= \sum_{i,j} a_i a'_j f(b_i b'_j) \\ &= \sum_{i,j} a_i a'_j b_i f(b'_j) + \sum_{i,j} a_i a'_j b'_j f(b_i) \\ &= \Big(\sum_i a_i b_i\Big)\Big(\sum_j a'_j f(b'_j)\Big) + \Big(\sum_j a'_j b'_j\Big)\Big(\sum_i a_i f(b_i)\Big) \\ &= 0. \end{aligned}$$

Consequently, g vanishes on I^2. Let φ be the induced A-linear morphism from I/I^2 to M. For every $a \in A$, one has

$$\varphi(d(a)) = g(1 \otimes a - a \otimes 1) = f(a) - af(1) = f(a),$$

and $f = \varphi \circ d$, as claimed.

5.6.8. — Let k be a ring, let A and B be k-algebras and let $f\colon A \to B$ be a morphism of k-algebras. The map $d_{B/k} \circ f\colon A \to \Omega^1_{B/k}$ is a k-derivation on A; consequently, there exists a unique A-linear morphism $\varphi\colon \Omega^1_{A/k} \to \Omega^1_{B/k}$ such that $d_{B/k} \circ f = \varphi \circ d_{A/k}$. Let $\varphi\colon B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k}$ be the associated morphism of B-modules.

Lemma (**5.6.9**). — *Let* S *be a multiplicative subset of* A, *let* $B = S^{-1}A$ *and let* $f\colon A \to B$ *be the canonical morphism. Then the associated morphism* $\varphi\colon S^{-1}A \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k}$ *is an isomorphism.*

Proof. — Let $d'_1\colon S \times A \to S^{-1}A \otimes_A \Omega^1_{A/k}$ given by

$$d'_1(s, a) = s^{-1} \otimes d_{A/k}(a) - s^{-2} a d_{A/k}(s),$$

for $a \in A$ and $s \in S$. For $a \in A$, $s, t \in S$, one has

$$\begin{aligned} d'_1(st, at) &= (st)^{-1} \otimes d_{A/k}(at) - (st)^{-2} at d_{A/k}(st) \\ &= (st)^{-1} t d_{A/k}(a) + (st)^{-1} a d_{A/k}(t) \\ &\qquad - (st)^{-2} at^2 d_{A/k}(s) - (st)^{-2} ast d_{A/k}(t) \\ &= s^{-1} d_{A/k}(a) - s^{-2} a d_{A/k}(s) \\ &= d'_1(s, a). \end{aligned}$$

Consequently, if $a, b \in A$ and $s, t \in S$ are such that $a/s = b/t$, let $u \in S$ such that $uta = sub$; then

$$d'_1(s, a) = d'_1(stu, uta) = d'_1(stu, sub) = d'_1(t, b).$$

This shows that there exists a unique map $d' \colon S^{-1}A \to S^{-1}A \otimes_A \Omega^1_{A/k}$ such that $d'(a/s) = d'_1(s, a)$ for every $a \in A$ and every $s \in S$. This map d' is a k-derivation (*exercise*...). Consequently, there exists a unique $S^{-1}A$-linear morphism $\psi \colon \Omega^1_{B/k} \to B \otimes_A \Omega^1_{A/k}$ such that $d' = \psi \circ d_{B/k}$.

For $a \in A$ and $s \in S$, one has

$$\begin{aligned} \varphi \circ \psi(d_{B/k}(a/s)) &= \varphi(d'(a/s)) \\ &= \varphi(s^{-1} \otimes d_{A/k}(a) - s^{-2} a d_{A/k}(s)) \\ &= s^{-1} d_{B/k}(a/1) - s^{-2} a d_{B/k}(s/1) \\ &= d_{B/k}(a/s), \end{aligned}$$

so that $\varphi \circ \psi \circ d_{B/k} = d_{B/k}$; by the universal property of the module of differentials, one has $\varphi \circ \psi = \mathrm{id}$. Moreover, for every $a \in A$, one has

$$\psi \circ \varphi(1 \otimes d_{A/k}(a)) = \psi(d_{B/k}(a/1)) = 1 \otimes d_{A/k}(a).$$

Since the elements of $B \otimes_A \Omega^1_{A/k}$ of the form $1 \otimes d_{A/k}(a)$ generate this B-module, this implies that $\psi \circ \varphi = \mathrm{id}$.

We thus have proved that φ is an isomorphism. □

5.6.10. — Let k be a ring, let A and B be k-algebras and let $f \colon A \to B$ be a morphism of k-algebras. The map $d_{B/A} \colon B \to \Omega^1_{B/A}$ is a k-derivation on B; consequently, there exists a unique morphism $\psi \colon \Omega^1_{B/k} \to \Omega^1_{B/A}$ of B-modules such that $d_{B/A} = \psi \circ d_{B/k}$.

Proposition (**5.6.11**). — *Let k be a ring, let* A *and* B *be k-algebras and let $f\colon \mathrm{A}\to\mathrm{B}$ be a morphism of k-algebras. The diagram*

$$\mathrm{B}\otimes_{\mathrm{A}}\Omega^1_{\mathrm{A}/k}\xrightarrow{\varphi}\Omega^1_{\mathrm{B}/k}\xrightarrow{\psi}\Omega^1_{\mathrm{B}/\mathrm{A}}\to 0$$

is an exact sequence.

Proof. — For every $b\in\mathrm{B}$, one has $d_{\mathrm{B}/\mathrm{A}}(b)=\psi(d_{\mathrm{B}/k}(b))$. Since $\Omega^1_{\mathrm{B}/\mathrm{A}}$ is generated, as a B-module, by elements of the form $d_{\mathrm{B}/\mathrm{A}}(b)$, for $b\in\mathrm{B}$, the morphism ψ is surjective. Let M be the image of φ; it is the B-submodule of $\Omega^1_{\mathrm{B}/k}$ generated by elements of the form $d_{\mathrm{B}/k}(f(a))$, for $a\in\mathrm{A}$. Let us show that $\mathrm{M}=\mathrm{Ker}(\psi)$. For every $a\in\mathrm{A}$, one has

$$\psi(d_{\mathrm{B}/k}(f(a)))=d_{\mathrm{B}/\mathrm{A}}(f(a))=d_{\mathrm{B}/\mathrm{A}}(a\cdot 1)=0$$

since $d_{\mathrm{B}/\mathrm{A}}$ is an A-derivation. This shows that $\mathrm{M}\subseteq\mathrm{Ker}(\psi)$. Let $\psi_1\colon\Omega^1_{\mathrm{B}/k}/\mathrm{M}\to\Omega^1_{\mathrm{B}/\mathrm{A}}$ be the induced homomorphism. Let $d\colon\mathrm{B}\to\Omega^1_{\mathrm{B}/k}/\mathrm{M}$ be the map given by $b\mapsto[d_{\mathrm{B}/k}(b)]$. It is a k-linear derivation; in fact, one has $d(f(a))=0$ for every $a\in\mathrm{A}$, by definition of M, so that d is an A-derivation. Consequently, there exists a unique B-linear morphism $\theta_1\colon\Omega^1_{\mathrm{B}/\mathrm{A}}\to\Omega^1_{\mathrm{B}/k}/\mathrm{M}$ such that $\theta_1\circ d_{\mathrm{B}/\mathrm{A}}=d$. For every $b\in\mathrm{B}$, one has

$$\theta_1\circ\psi_1(d(b))=\theta_1\circ\psi_1([d_{\mathrm{B}/k}(b)])=\theta_1(d_{\mathrm{B}/\mathrm{A}}(b))=d(b);$$

since the elements of the form $d_{\mathrm{B}/k}(b)$ generate the B-module $\Omega^1_{\mathrm{B}/k}$, this implies that $\theta_1\circ\psi_1=\mathrm{id}$. In particular, ψ_1 is injective, hence $\mathrm{M}=\mathrm{Ker}(\psi)$. This concludes the proof of the proposition. □

5.6.12. — One can extend to schemes the definition of the module of differentials. Let $f\colon\mathrm{Y}\to\mathrm{X}$ be a morphism of schemes. Recall that the canonical morphism $f^\sharp\colon\mathscr{O}_\mathrm{X}\to f_*\mathscr{O}_\mathrm{Y}$ induces, by adjunction, a ring morphism $f^\flat\colon f^{-1}(\mathscr{O}_\mathrm{X})\to\mathscr{O}_\mathrm{Y}$. In particular, every $\mathscr{O}_\mathrm{Y}$-module can be considered, via $f^\flat$, as an $f^{-1}(\mathscr{O}_\mathrm{X})$-module. An $\mathscr{O}_\mathrm{Y}$-derivation from $\mathscr{O}_\mathrm{Y}$ to a quasi-coherent $\mathscr{O}_\mathrm{Y}$-module $\mathscr{M}$ is a $f^{-1}(\mathscr{O}_\mathrm{X})$-linear morphism $d\colon\mathscr{O}_\mathrm{Y}\to\mathscr{M}$ such that for every open subscheme U of Y, every element $a,b\in\mathscr{O}_\mathrm{Y}(\mathrm{U})$, one has $d(ab)=ad(b)+bd(a)$.

Proposition (**5.6.13**). — *Let $f\colon\mathrm{Y}\to\mathrm{X}$ be a morphism of schemes.*

There exists a quasi-coherent $\mathscr{O}_Y$*-module* $\Omega^1_{Y/X}$ *on* Y *and an* $f^{-1}(\mathscr{O}_X)$*-linear derivation* $d_{Y/X}\colon \mathscr{O}_Y \to \Omega^1_{Y/X}$ *which satisfies the universal property: for every* $\mathscr{O}_Y$*-module* $\mathscr{M}$ *and every* $\mathscr{O}_X$*-derivation* $d\colon \mathscr{O}_Y \to \mathscr{M}$*, there exists a unique* $\mathscr{O}_Y$*-linear morphism* $\varphi\colon \Omega^1_{Y/X} \to \mathscr{M}$ *such that* $d = \varphi \circ d_{Y/X}$.

If f *is locally finitely generated, then* $\Omega^1_{Y/X}$ *is a finitely generated* $\mathscr{O}_Y$*-module. If* f *is locally finitely presented, then* $\Omega^1_{Y/X}$ *is a finitely presented* $\mathscr{O}_Y$*-module.*

One can construct this sheaf by reduction to the case of affine schemes, where it reduces to the module $\Omega_{A/k}$. A more geometric construction is also possible, whose affine counterpart was described in §5.6.7. Translated in the language of calculus, this construction builds on the following remark: if f is a smooth function on an open subset U of $\mathbf{R}^n$, Taylor's formula writes

$$f(y) - f(x) = (d_x f)(y - x) + \text{terms of order} \geqslant 2.$$

The differential term $(d_x f)(y-x)$ appear as function $f(y) - f(x)$ on U^2 vanishing on the diagonal (namely, when $x = y$) modulo those vanishing at a higher order. This is the I/I^2 of §5.6.7. Let us know describe this construction in the context of schemes.

Proof. — Let $\delta_{Y/X}\colon Y \to Y \times_X Y$ be the diagonal immersion. Its image, $\Delta_{Y/X}$, is a locally closed subscheme of $Y \times_X Y$, and $\delta_{Y/X}$ induces an isomorphism from Y to $\Delta_{Y/X}$. Let thus W be the largest open subscheme of $Y \times_X Y$ in which $\Delta_{Y/X}$ is a closed subscheme, and let $\mathscr{I}$ be the ideal sheaf of $\Delta_{Y/X}$ in W. One then defines a quasi-coherent $\mathscr{O}_Y$-module by the formula

$$\Omega_{Y/X} = \delta^*_{Y/X}(\mathscr{I}/\mathscr{I}^2).$$

Let also $d_{Y/X}\colon \mathscr{O}_Y \to \Omega_{Y/X}$ be the map given by

$$d_{Y/X}(f) = \mathrm{pr}_2^*(f) - \mathrm{pr}_1^*(f) \pmod{\mathscr{I}^2},$$

for every open subscheme U of Y and every $f \in \mathscr{O}_Y(U)$.

If f is locally finitely generated (resp. locally finitely presented), then $\mathscr{I}$ is a finitely generated (resp. finitely presented) quasi-coherent $\mathscr{O}_W$-module, so that $\Omega_{Y/X}$ is a finitely generated (resp. a finitely presented) $\mathscr{O}_Y$-module. □

5.7. Smooth morphisms

5.7.1. — Let X be a locally noetherian scheme and let x be a point of X. One says that x is *regular point* of X if the local ring $\mathscr{O}_{X,x}$ is regular. The dual $(\mathfrak{m}_x/\mathfrak{m}_x)^\vee$

of the $\kappa(x)$-vector space $\mathfrak{m}_x/mathfrakm_x^2$ is denoted by $\mathrm{T}_x(\mathrm{X})$, and is called *Zariski's tangent space* to X at x. One has $\dim(\mathscr{O}_{\mathrm{X},x}) \leqslant \dim(\mathrm{T}_x(\mathrm{X}))$, with equality if and only if X is regular at x. If this holds, then $\mathfrak{m}_x$ is generated by $\dim(\mathscr{O}_{\mathrm{X},x})$ elements, and the ring $\mathscr{O}_{\mathrm{X},x}$ is an integral domain (proposition 1.14.6).

Proposition (**5.7.2**). — *Let k be a field and let* X *be a k-scheme; let* $x \in \mathrm{X}(k)$. *After the canonical derivation* $d_{\mathrm{X}/k,x}\colon \mathfrak{m}_x \to \Omega_{\mathrm{X}/k,x}$ *induces an isomorphism from* $\mathfrak{m}_x/\mathfrak{m}_x^2$ *to* $\kappa(x) \otimes \Omega_{\mathrm{X}/k,x}$.

Proof. — Let us apply the conormal exact sequence with $\mathrm{B} = \mathscr{O}_{\mathrm{X},x}$ and $\mathrm{I} = \mathfrak{m}_x$; it furnishes an exact sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\delta} k \otimes_{\mathscr{O}_{\mathrm{X},x}} \Omega_{\mathrm{X}/k,x} \to \Omega_{k/k} \to 0.$$

Since $\Omega_{k/k} = 0$, this shows that δ is surjective.

Let then $\theta \in \mathrm{T}_x(\mathrm{X}) = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$. There exists a unique k-linear map $d_\theta\colon \mathscr{O}_{\mathrm{X},x} \to k$ which maps 1 to 0 and which maps an element $a \in \mathfrak{m}_x$ to the image by θ of its class in $\mathfrak{m}_x/\mathfrak{m}_x^2$. This map d_θ is a k-derivation. Consequently, there exists a linear $\mathscr{O}_{\mathrm{X},x}$-morphism, $\varphi\colon \Omega_{\mathrm{X}/k,x} \to k$ such that $d_\theta = \varphi \circ d_{\mathrm{X}/k,x} = \varphi \circ \delta$. Moreover, for every $a \in \mathfrak{m}_x$, one has $\varphi \circ \delta(a) = d_\theta(a) = \theta(a)$. Assume that $a \in \ker(\delta)$; one then has $\theta(a) = 0$ for every $\theta \in \mathrm{T}_x(\mathrm{X})$. Consequently, the image of a in $\mathfrak{m}_x/\mathfrak{m}_x^2$ vanishes, as was to be shown. □

CHAPTER 6

COHOMOLOGY OF QUASI-COHERENT SHEAVES

6.1. Cohomology of affine schemes

The main theorems of this section and of the next one are variants for schemes of theorems proved by Serre (1955): for quasi-compact schemes, the vanishing of the cohomology of all quasi-coherent sheaves characterizes affine schemes.

Theorem **(6.1.1)** (Serre). — *Let* X *be an affine scheme and let* $\mathscr{F}$ *be an quasi-coherent sheaf on* X. *Then* $H^i(X, \mathscr{F}) = 0$ *for all* $i > 0$.

The proof is a simplification of the one given by Kempf (1980).

Proof. — We prove by induction on the integer n that $H^i(X, \mathscr{F}) = 0$ for all schemes X, all quasi-coherent schemes $\mathscr{F}$ on X and all integers i such that $0 < i \leqslant n$. Let n be an integer, X be a scheme and $\mathscr{F}$ be a quasi-coherent sheaf on X; let A be the ring such that $X = \operatorname{Spec}(A)$. The result is trivial for $n = 0$, so we assume $n \geqslant 1$. By induction we have $H^i(X, \mathscr{F}) = 0$ for all integers i such that $0 \leqslant i \leqslant n-1$, and it suffices to prove that $H^n(X, \mathscr{F}) = 0$. Let $\alpha \in H^n(X, \mathscr{F})$ be any class. By proposition 3.6.6, any point $x \in X$ has an open neighborhood U_x such that $\alpha|_{U_x} = 0$. We may assume that U_x is a basic oben set, of the form $D(f_x)$, with $f_x \in A$. Since X is quasi-compact (exercise 1.5.9), there exists a finite subset S of X such that $X = \bigcup_{x \in S} D(f_x)$. For $x \in S$, write $\mathscr{F}_x$ for the sheaf on X given by $(j_x)_* \mathscr{F}|_{U_x}$; if $\mathscr{F}$ is defined by an A-module M, then $\mathscr{F}_x$ is defined by the A-module M_{f_x}, hence is quasi-coherent.

Restricting a section $s \in \mathscr{F}(U)$ to the intersections $U \cap U_x$ furnishes a morphism $\mathscr{F} \to \bigoplus_{x \in S} \mathscr{F}_x$ of quasi-coherent sheaves; let $\mathscr{G}$ be its cokernel. The long exact sequence in cohomology furnishes a short exact sequence

$$\bigoplus_{x \in S} H^{n-1}(X, \mathscr{F}_x) \to H^{n-1}(X, \mathscr{G}) \to H^n(X, \mathscr{F}) \to \bigoplus_{x \in S} H^n(X, \mathscr{F}_x).$$

If $n \geqslant 2$, then by the induction hypothesis, one has $H^{n-1}(X, \mathscr{G}) = 0$, so that the last map is injective. If $n = 1$, the first map is surjective, because the functor Γ is exact on affine schemes; consequently, the map $H^0(X, \mathscr{G}) \to H^1(X, \mathscr{F})$ is zero and the last map is injective as well. On the other hand, for any $x \in S$, the cohomology group $H^n(X, \mathscr{F}_x)$ identifies with $H^n(U_x, \mathscr{F}_x)$, the image of α in that group is $\alpha|_{U_x}$. Consequently, that image vanishes, and this proves $\alpha = 0$. □

Corollary (**6.1.2**). — *Let $f: X \to Y$ be an affine morphism of schemes and let $\mathscr{F}$ be a quasi-coherent sheaf on* X. *For every integer i, the canonical map $H^i(Y, f_*\mathscr{F}) \to H^i(X, \mathscr{F})$ is an isomorphism.*

Proof. — Let $\mathscr{F} \to \mathscr{G}_0 \to \mathscr{G}_1 \to \dots$ be a flasque resolution of $\mathscr{F}$. Let V be an affine open subset of Y; then $U = f^{-1}(V)$ is an affine open subset of X, because f is affine. By theorem 6.1.1, the sequence

$$0 \to \mathscr{F}(U) \to \mathscr{G}_0(U) \to \mathscr{G}_1(U) \to \dots$$

is exact, which shows that

$$f_*\mathscr{F} \to f_*\mathscr{G}_0 \to f_*\mathscr{G}_1 \to \dots$$

is a flasque resolution of $f_*\mathscr{F}$. Taking global sections, the complex $f_*\mathscr{G}_\bullet(Y)$ thus computes the cohomology $H^\bullet(Y, f_*\mathscr{F})$, but this complex is nothing but the complex $\mathscr{G}_\bullet(X)$ which computes $H^\bullet(X, \mathscr{F})$. □

6.1.3. — Let $\mathscr{F}$ be a abelian sheaf on a topological space X. Let $\mathscr{U} = (U_i)_{i\in I}$ be an open covering of X indexed by a *totally ordered* set I.

For any finite subset J of I, let $U_J = \bigcup_{i\in J} U_i$; it is an open subset and we let $j_J: U_J \to X$ be the inclusion map. Let also $\mathscr{F}_J$ be the abelian sheaf $j_{J,*}j_J^{-1}\mathscr{F}$ on X.

Moreover, for $i \in I - J$, let $\varphi_{J,i}: \mathscr{F}_J \to \mathscr{F}_{J\cup\{i\}}$ be the morphism of abelian sheaves induced by $s \mapsto s|_{U_i}$, and let $\varepsilon_{J,i}$ be $(-1)^m$, where m is the number of elements $j \in J$ such that $j < i$.

For every integer $m \in \{-1\} \cup \mathbf{N}$, one defines an abelian sheaf $\mathscr{F}_m$ on X by

$$\mathscr{F}_m = \bigoplus_{|J|=m+1} j_{J,*}j_J^{-1}\mathscr{F}$$

and a morphism of abelian sheaves

$$d_m^{\mathscr{F}}: \mathscr{F}_m \to \mathscr{F}_{m+1}$$

which maps a section $s \in \mathscr{F}_J$ to $\sum_{j\in I-J} \varepsilon_{J,i}\varphi_{J,i}$.

For $J = \varnothing$, one has $U_J = X$ and $\mathscr{F}_J = \mathscr{F}$, so that $F_{-1} = \mathscr{F}$. We write $\varepsilon^{\mathscr{F}}$ instead of $d^{\mathscr{F}}_{-1}$.

For $m = 0$, one has $\mathscr{F}_0 = \bigoplus_{i\in I} \mathscr{F}_i$ and the morphism $\varepsilon^{\mathscr{F}}$ is given by

$$s \mapsto \bigoplus_{i\in I} s|_{U_i}.$$

For $m = 1$, one has $\mathscr{F}_1 = \bigoplus_{i<j} \mathscr{F}_{i,j}$ and the morphism $d^{\mathscr{F}}_0$ maps a family $(s_i)_{i\in I}$ to the family $(s_i|_{U_j} - s_j|U_i)_{i<j}$.

This defines a complex $\check{C}_{\mathscr{U}}(\mathscr{F})$ on X and it follows from the sheaf condition on X that this is *resolution* of $\mathscr{F}$.(1) It is called the Čech complex of $\mathscr{F}$ associated with the open covering $\mathscr{U}$..

Corollary **(6.1.4)** (Čech cohomology). — *Let* X *be a scheme and let* $\mathscr{F}$ *be a quasi-coherent sheaf on* X. *Let* $\mathscr{U} = (U_1, \dots, U_n)$ *be a finite covering of* X *by affine open subschemes all of whose intersections are affine. The Čech complex* $\check{C}_{\mathscr{U}}(\mathscr{F})$ *of* $\mathscr{F}$ *associated with* $\mathscr{U}$ *is acyclic and computes the cohomology of* $\mathscr{F}$

Note that the hypothesis about the intersections holds if X is separated.(2)

Proof. — All open subsets U_J, for non-empty finite subsets J of $\{1, \dots, n\}$, that appear in the Çech resolution are affine open subsets of X and the morphism $j_J \colon U_J \to X$ is affine, because it is affine above every U_i, for $i \in I$. Consequently, $\check{C}_{\mathscr{U},m}$ is a finite direct sum of quasi-coherent sheaves of the form $j_{J,*}j^{-1}\mathscr{F}$. They are thus acyclic. ☐

Corollary **(6.1.5)**. — *Let* $f \colon X \to Y$ *be a morphism of schemes which is quasi-compact and separated. Let* $\mathscr{F}$ *be a quasi-coherent sheaf on* X. *Then the higher direct images* $\boldsymbol{R}^p f_*\mathscr{F}$ *on* Y *are quasi-coherent sheaves. Moreover, for every affine open subset* V *of* Y*, the canonical morphism* $H^p(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)}) \to \boldsymbol{R}^p f_*\mathscr{F}(V)$ *is an isomorphism.*

6.2. Serre's characterization of affine schemes

Theorem **(6.2.1)** (Serre). — *Let* X *be a scheme which is quasi-compact and quasi-separated. The following properties are equivalent:*

(1) Move this paragraph to the chapter on sheaves, and give more complete arguments.

(2) This is not proved explicitly, but follows from the definition, because the product of affine schemes is affine, and their intersection is the preimage under the diagonal morphism, which is a closed immersion when X is separated.

(i) *The scheme* X *is affine;*

(ii) *One has* $H^i(X, \mathscr{F}) = 0$ *for every quasi-coherent sheaf* $\mathscr{F}$ *on* X *and any integer* $i > 0$*;*

(iii) *One has* $H^1(X, \mathscr{I}) = 0$ *for every quasi-coherent sheaf of ideals* $\mathscr{I}$ *on* X.

Remark **(6.2.2)**. — Let X be a scheme which is a disjoint union $\bigcup_{i\in I} U_i$ of affine subschemes; such a scheme is separated. If X is quasi-compact, there exists a finite subset S of I such that $X = \bigcup_{i\in S} U_i$, and the open sets U_i, for $i \notin S$, are empty. In this case, X is affine. If X is not quasi-compact, then X is not affine, because affine schemes are quasi-compact. However, one can prove that $H^i(X, \mathscr{F}) = 0$ for any quasi-coherent sheaf on X and any integer $i > 0$, so that X satisfies the hypothesis (ii). Affine schemes are quasi-compact and separated. This shows that one cannot remove the assumption that the scheme be quasi-compact and quasi-separated in theorem 6.2.1.

I do not know of an example of a quasi-compact scheme that satisfies (ii) but isn't affine.

Proof of theorem 6.2.1. — The implication (i)⇒(ii) is theorem 6.1.1 and the implication (ii)⇒(iii) is obvious. We now assume that X is a quasi-compact scheme such that $H^1(X, \mathscr{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathscr{I}$ on X and prove that X is an affine scheme. Set $A = \Gamma(X, \mathscr{O}_X)$. The proof requires two steps which we state as independent lemmas.

Lemma **(6.2.3)**. — *For every quasi-coherent subsheaf* $\mathscr{F}$ *of* $\mathscr{O}_X^n$*, one has* $H^1(X, \mathscr{F}) = 0$.

Proof. — Let us prove the lemma by induction on n, the case $n = 0$ being obvious since then $\mathscr{F} = 0$.

Let $u \colon \mathscr{O}_X^n \to \mathscr{O}_X$ be the morphism of quasi-coherent sheaves given by $(a_1, \dots, a_n) \mapsto a_n$; its kernel is isomorphic to $\mathscr{O}_X^{n-1}$. Let $\mathscr{F}' = \ker(u) \cap \mathscr{F}$ and $\mathscr{I} = u(\mathscr{F})$, so that one has an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{I} \to 0.$$

The associated cohomology long exact sequence furnishes an exact sequence

$$H^1(X, \mathscr{F}') \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{I}).$$

Now, $H^1(X, \mathscr{F}') = 0$ by induction, and $H^1(X, \mathscr{I}) = 0$ since $\mathscr{I}$ is a quasi-coherent submodule of $\mathscr{O}_X$, that is, a quasi-coherent ideal sheaf on X. This implies that $H^1(X, \mathscr{F}) = 0$, as was to be shown. □

Lemma **(6.2.4)**. — *There exists a finite family* $(f_1, \dots, f_n)$ *in* A *such that* X_{f_i} *is affine for every* $i \in \{1, \dots, n\}$, *and* $\bigcup_{i=1}^n X_{f_i} = X$.

Proof. — We first prove that *every closed point* $x \in X$ *admits an affine open neighborhood of the form* X_f. Let U be an open affine neighborhood of x in X, let Z be the reduced closed subscheme of X supported by the closed subset $(X - U) \cup \{x\}$, and let $\mathscr{I}_Z$ be its sheaf of ideals. Since $x \notin U$, the scheme Z is the disjoint union of $X - U$ and $\{x\}$, and there exists a section $g \in \Gamma(Z, \mathscr{O}_Z)$ such that $g|_{X-U} = 0$ and $g(x) = 1$. The long cohomology exact sequence associated with the short exact sequence $0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$ furnishes the short exact sequence

$$\Gamma(X, \mathscr{O}_X) \to \Gamma(Z, \mathscr{O}_Z) \to H^1(X, \mathscr{I}_Z).$$

Since $H^1(X, \mathscr{I}_Z) = 0$, by assumption, there exists $f \in \Gamma(X, \mathscr{O}_X)$ such that $f|_Z = g$, so that $f|_{X-U} = 0$ and $f(x) = 1$. The first equality implies that $X_f \subseteq U$, and the second one that $x \in X_f$. Since $X_f \subseteq U$, the open subscheme X_f identifies with the basic open subscheme $D(f|_U)$ of U; in particular, X_f is affine.

Let us now prove that X *is the union of all open subschemes of* X *of the form* X_f, *for* $f \in A$, *which are affine*. Let U be that union and let Z be its complementary subset; we have to prove that $Z = \varnothing$. A closed point of Z is also a closed point of X, because Z is closed, hence the first part of the proof implies that Z contains no closed point On the other hand, Z is quasi-compact, as a closed subset of the quasi-compact scheme X. By proposition 4.4.7, we conclude that $Z = \varnothing$.

Since X is quasi-compact, it is covered by a finite subfamily of this family (X_f), and this concludes the proof of the lemma. □

We can now conclude the proof of theorem 6.2.1. Let $\pi: X \to \operatorname{Spec}(A)$ be the morphism from theorem 4.2.8 associated to the identity morphism of A; we have to prove that π is an isomorphism (see example 4.3.3).

Let $(f_1, \dots, f_n)$ be a finite family in A such that, for every i, X_{f_i} is an open affine subscheme of X, and such that their union is equal to X.

Let us prove that *the ideal* $\langle f_1, \dots, f_n \rangle$ *it generates is equal to* A. Let $u: \mathscr{O}_X^n \to \mathscr{O}_X$ be the morphism of quasi-coherent sheaves given by $(a_1, \dots, a_n) \mapsto \sum a_i f_i$. Its image is a quasi-coherent ideal sheaf $\mathscr{I}$ on X which contains every f_i; since for every point $x \in X$, there exists x such that $x \in X_{f_i}$, we then have $(f_i)_x \in \mathscr{I}_x$, hence $1 \in \mathscr{I}_x$. Consequently, $\mathscr{I} = \mathscr{O}_X$. Considering the cohomology long exact sequence associated with the exact sequence $0 \to \ker(u) \to \mathscr{O}_X^n \to \mathscr{O}_X \to 0$, we obtain an exact sequence

$$H^0(X, \mathscr{O}_X^n) \xrightarrow{u} H^0(X, \mathscr{O}_X) \to H^1(X, \ker(u)).$$

By lemma 6.2.3, one has $H^1(X, \ker(u)) = 0$, so that the first map is surjective. In particular, there exists $(a_1, \dots, a_n) \in H^0(X, \mathscr{O}_X)^n = A^n$ such that $u(a_1, \dots, a_n) = 1$, which means $\sum_{i=1}^n a_i f_i = 1$, as claimed.

In other words, the union in $\operatorname{Spec}(A)$ of the basic open subschemes $D(f_i)$, for $i \in \{1, \dots, n\}$, is equal to $\operatorname{Spec}(A)$. To prove that $\pi \colon X \to \operatorname{Spec}(A)$ is an isomorphism, it thus suffices to prove that for every i, the induced morphism $\pi^{-1}(D(f_i)) \to D(f_i)$ is an isomorphism.

Fix $i \in \{1, \dots, n\}$. By construction, one has $\pi^{-1}(D(f_i)) = X_{f_i}$, and the morphism induced by π corresponds with the canonical morphism $X_{f_i} \to D(f_i) = \operatorname{Spec}(A_{f_i})$, by theorem 4.7.2[3]. Since X_{f_i} is affine, that morphism is an isomorphism. This concludes the proof. □

6.3. Cohomology of the projective space

6.3.1. — Let k be a ring, let n be a natural integer and let $A = k[T_0, \dots, T_n]$, graded by total degree. The projective space of dimension n over k is the scheme $\mathbf{P}_k^n = \operatorname{Proj}(A)$; it is a proper k-scheme, covered by affine schemes $D_+(f)$, for all homogeneous elements $f \in A$. The affine ring of $D_+(f)$ is denoted by $A_{(f)}$, and is the degree-0 part of the classical localization A_f.

Every graded A-module M gives rise to a quasi-coherent sheaf $\widetilde{M}$ on $\mathbf{P}_k^n$. Recall that for any homogeneous element $f \in A_d$, $H^0(D_+(f), \widetilde{M})$ identifies with the degree-0 part of the A-module M_f.

Shifting its degree by d (so that its degree m part becomes M_{m+d} defines the graded A-module $M(d)$, with associated quasi-coherent sheaf $\widetilde{M}(d)$. Taking $M = A$, we obtain the quasi-coherent sheaves $\mathscr{O}_{\mathbf{P}_k^n}(d)$, which are locally free of rank 1; moreover $\mathscr{M} \otimes \mathscr{O}_{\mathbf{P}_k^n}(d)$ is canonically isomorphic to $\mathscr{M}(d)$.

(3) more precisely, by the extension of its implication (i)⇒(ii), corollary 4.7.3, to arbitrary quasi-compact and quasi-separated schemes

Theorem **(6.3.2)**. — *Let k be a noetherian ring, let n be a natural integer such that $n \geqslant 1$ and let $A = k[T_0, \dots, T_n]$.*

a) *For any integer $d \in \mathbf{Z}$, the canonical morphism $A_d \to H^0(\mathbf{P}^n_k, \mathscr{O}(d))$ is an isomorphism.*

b) *For any integer i such that $0 < i < n$ and any integer $d \in \mathbf{Z}$, one has $H^i(\mathbf{P}^n_k, \mathscr{O}(d)) = 0$.*

c) *There exists an isomorphism $H^n(\mathbf{P}^n_k, \mathscr{O}(-n-1)) \simeq A$; for every integer $d \in \mathbf{Z}$, the bilinear map*

$$H^0(\mathbf{P}^n_k, \mathscr{O}(-n-1-d)) \times H^n(\mathbf{P}^n_k, \mathscr{O}(d))) \to H^n(\mathbf{P}^n_k, \mathscr{O}(-n-1)) \simeq A$$

is invertible.

Proof. — One ingredient of the proof consists in proving the theorem for all integers d at once by considering the graded k-modules

$$\bigoplus_{d \in \mathbf{Z}} H^i(\mathbf{P}^n_k, \mathscr{O}(d)).$$

Since $\mathbf{P}^n_k$ is noetherian, it follows from corollary 3.8.5 that this direct sum identifies with

$$H^i(\mathbf{P}^n_k, \bigoplus_{d \in \mathbf{Z}} \mathscr{O}(d)).$$

The quasi-coherent sheaf $\mathscr{O}(*) = \bigoplus_{d \in \mathbf{Z}} \mathscr{O}(d)$ is associated with the graded ring

$$A(*) = \bigoplus_{d \in \mathbf{Z}} A(d) = A[T, T^{-1}],$$

where T is a new homogeneous indeterminate of degree 1. Then, for any homogeneous element f, one has

$$A[T, T^{-1}]_{(f)} = \bigoplus_{p \in \mathbf{Z}} A(p)_{(f)} T^{-p} = A_f,$$

so that by considering all degrees at once, the homogeneous localization appears as a classical localization.

The affine open subsets $D_+(T_i)$, for $i \in \{0, \dots, n\}$, cover $\mathbf{P}^n_k$; for all non-empty subsets I of $\{0, \dots, n\}$, the intersections of the $D_+(T_i)$, for $i \in I$, is the open subset $D_+(\prod_{i \in I} T_i)$, hence is affine. We may thus compute the cohomology of the quasi-coherent sheaf $\mathscr{O}(*)$ using the Čech complex associated with this covering, namely

$$\bigoplus_{i=0}^{n} A_{T_i} \xrightarrow{d_0} \bigoplus_{i_0 < i_1} A_{T_{i_0} T_{i_1}} \to \xrightarrow{d_{n-1}} \to A_{T_0 \dots T_n}.$$

Let us now compute the cohomology of that complex of A-modules.

a) To compute $H^0(\mathbf{P}^n_k, \mathscr{O}(*))$, we have to compute the kernel of d_0. Let $a = (a_i/T_i^{m_i})$ be an element of $\bigoplus_i A_{T_i}$ such that $d_0(a) = 0$, where $a_0, \dots, a_n \in A$. Let us prove that for every i, a_i is divisible by $T_i^{m_i}$ in A.

Let $i \in \{0, \dots, n\}$. Since $n \geqslant 1$, there exists $j \in \{0, \dots, n\}$ such that $i \neq j$, and the relation $d_0(a) = 0$ implies that $a_i/T_i^{m_i} = a_j/T_j^{m_j}$ in $A_{T_iT_j}$. Since all T_i are regular in A, this implies that $T_j^{m_j} a_i = T_i^{m_i} a_j$; in particular, $T_i^{m_i}$ divides $T_j^{m_j} a_i$. By inspection on the nonzero monomials of a_i, we conclude that $T_i^{m_i}$ divides a_i.

We may thus assume that a is of the form $(a_0/1, \dots, a_n/1)$, and the relation $d_0(a) = 0$ implies that $a_i = a_j$ for all i, j. Consequently, there exists a unique element $f \in A$ such that $a = (f/1, \dots, f/1)$, and this concludes the proof that the canonical morphism of graded rings $A \to H^0(\mathbf{P}^n_k, \mathscr{O}(*))$ is an isomorphism.

b) We now treat the case of degree n, that is, we compute the graded k-module

$$\operatorname{Coker}(d_{n-1}) = A_{T_0 \dots T_n} \Big/ \left(\sum_{i=0}^{n} A_{T_0 \dots \widehat{T_i} \dots T_n} \right),$$

where $T_0 \dots \widehat{T_i} \dots T_n$ is the product of all T_j, with T_i excluded. The k-module $A_{T_0 \dots T_n}$ is free, with basis the family of monomials $T^p = T_0^{p_0} \dots T_n^{p_n}$, indexed by all $p \in \mathbf{Z}^{n+1}$, and T^p has degree $|p| = p_0 + \dots + p_n$. The submodule $A_{T_0 \dots \widehat{T_i} \dots T_n}$ is generated by these monomials T^p such that $p_i \geqslant 0$. Consequently, $\operatorname{Coker}(d_{n-1})$ is a free k-module with basis the classes $[T^p]$ of the monomials T^p, for all $p \in \mathbf{Z}^n$ such that $p_i \leqslant -1$ for all i.

In particular, we observe that $H^n(\mathbf{P}^n_k, \mathscr{O}(d)) = 0$ if $d > -n$.

The cohomology group $H^n(\mathbf{P}^n_k, \mathscr{O}(-n-1))$ is the degree-$(-n-1)$ part of this module; it is free of rank 1, with basis the monomial $T_0^{-1} \dots T_n^{-1}$.

Let us now study the bilinear map β,

$$H^0(\mathbf{P}^n_k, \mathscr{O}(-n-1-d)) \times H^n(\mathbf{P}^n_k, \mathscr{O}(d)) \to H^n(\mathbf{P}^n_k, \mathscr{O}(-n-1)) = k[T_0^{-1} \dots T_n^{-1}].$$

The k-module $H^0(\mathbf{P}^n_k, \mathscr{O}(-n-1-d))$ is free, with basis the family of monomials T^p, with $|p| = -n-1-d$. For any $q \in \mathbf{Z}^n$, $\beta(T^p, [T^q])$ is the image of $[T^{p+q}]$, hence is 1 for $q = -p - \mathbf{1}$ and is 0 otherwise. Under this bilinear map, the basis (T^p) of $H^0(\mathbf{P}^n_k, \mathscr{O}(-n-1-d))$ and the family $([T^{-\mathbf{1}-p}])$ (both indexed by $p \in \mathbf{N}^{n+1}$ such that $|p| = -n-1-d$) are dual bases one of the other.

c) Let us now prove that $H^p(\mathbf{P}^n_k, \mathscr{O}(d)) = 0$ for all integers d and all integers p such that $0 < p < n$. The proof runs by induction on n. The assertion being trivial for $n = 1$, we assume that $n \geqslant 2$.

We will prove that the multiplication by T_n is both nilpotent and injective on $H^p(\mathbf{P}^n_k, \mathscr{O}(*))$.

Lemma **(6.3.3)**. — *Let* A *be a graded ring and let* $M' \to M \to M''$ *be a complex of graded* A*-modules, where the morphisms are homogeneous of degree* 0. *For any homogeneous element* $f \in A$, *the canonical morphism*

$$H(M'_{(f)} \to M_{(f)} \to M''_{(f)}) \to H(M' \to M \to M'')_{(f)}$$

is an isomorphism.

Proof. — The analogous statement for localization holds, by exactness of localization: the canonical morphism

$$H(M'_f \to M_f \to M''_f) \to H(M' \to M \to M'')_f$$

is an isomorphism. Since f is homogeneous, the complex $M'_f \to M_f \to M''_f$ is a direct sum of complexes of various degrees, and the complex $M'_{(f)} \to M_{(f)} \to M''_{(f)}$ is its subcomplex corresponding to the degree 0 part. This identifies $H(M'_f \to M_f \to M''_f)$ with the degree 0-part of $H(M' \to M \to M'')_f$, that is, with $H(M' \to M \to M'')_{(f)}$, as claimed. ☐

By this lemma, the localization

$$H^\bullet(\mathbf{P}^n_k, \mathscr{O}(*))_{(T_n)}$$

can be computed as the cohomology of the complex

$$\bigoplus_{i=0}^{n} A_{T_i T_n} \to \bigoplus_{i_0 < i_1} A_{T_{i_0} T_{i_1} T_n} \to \dots$$

which identifies withe the Čech complex of the quasi-coherent sheaf $\mathscr{O}(*)$ for the open covering $(D_+(T_i T_n))_{0 \leqslant i \leqslant n}$ of $D_+(T_n)$. Since $D_+(T_n)$ is affine, we thus have

$$H^p(\mathbf{P}^n_k, \mathscr{O}(*))_{(T_n)} = 0$$

for all integers $p > 0$.

In other words, for every $p > 0$ and every class $\xi \in H^p(\mathbf{P}^n_k, \mathscr{O}(*))$, there exists an integer q such that $T_n^q \xi = 0$. To conclude the proof, it suffices to prove that multiplication by T_n is injective if, moreover, $p < n$.

Let Z be the closed subscheme $V_+(T_n)$, identified with $\mathbf{P}^{n-1}_k$, with homogeneous graded ring $A' = k[T_0, \dots, T_{n-1}]$. Let $i \colon Z \to \mathbf{P}^n_k$ be the corresponding closed immersion. Multiplication by T_n induces an exact sequence of quasi-coherent sheaves

$$0 \to \mathscr{O}(d-1) \xrightarrow{T_n} \mathscr{O}(d) \to i_*\mathscr{O}(d) \to 0.$$

us study the associated cohomology long exact sequence. It starts with

$$0 \to A_{d-1} \xrightarrow{T_n} A_d \to A'_d \to H^1(\mathbf{P}^n_k, \mathscr{O}(d-1)) \xrightarrow{T_n} H^1(\mathbf{P}^n_k, \mathscr{O}(d));$$

since the canonical morphism $A_d \to A'_d$ is surjective, the next morphism

$$H^1(\mathbf{P}^n_k, \mathscr{O}(d-1)) \xrightarrow{T_n} H^1(\mathbf{P}^n_k, \mathscr{O}(d))$$

is injective.

By the induction hypothesis, we also have $H^p(Z, \mathscr{O}(d)) = 0$ for all integers p such that $0 < p < n-1$. Consequently, if $2 \leqslant p < n$, we obtain a short exact sequence

$$H^{p-1}(Z, \mathscr{O}(d)) = 0 \to H^p(\mathbf{P}^n_k, \mathscr{O}(d-1)) \xrightarrow{T_n} H^p(\mathbf{P}^n_k, \mathscr{O}(d)).$$

Consequently, for any integer p such that $0 < p < n$, multiplication by T_n induces injective maps

$$H^p(\mathbf{P}^n_k, \mathscr{O}(d-1)) \to H^p(\mathbf{P}^n_k, \mathscr{O}(d)),$$

for all $d \in \mathbf{Z}$.

Given a class $\xi \in H^p(\mathbf{P}^n_k, \mathscr{O}(d))$ (where $0 < p < n$ and $d \in \mathbf{Z}$) and an integer q such that $T_n^q \xi = 0$, we thus have $\xi = 0$. This concludes the proof. ☐

Remark **(6.3.4)**. — Theorem 6.3.2 doesn't hold as such for $n = 0$, which is an exceptional case since $\mathbf{P}^0_k = \operatorname{Proj}(k[T_0]) = \operatorname{Spec}(k)$ is affine. In this case, assertion *a*) is obvious for $d \geqslant 0$ but false for $d < 0$, assertion *b*) is empty, and assertion *c*) holds obviously.

Theorem **(6.3.5)** (Serre). — *Let k be a noetherian ring, let n be a natural integer, let* X *be a closed subscheme of* $\mathbf{P}^n_k$ *and let* $\mathscr{F}$ *be a coherent sheaf on* X.

a) *For any integer p,* $H^p(X, \mathscr{F})$ *is a finitely generated k-module.*

b) *There exists an integer* $d(\mathscr{F})$ *suchat that for all integers* $d \geqslant d(\mathscr{F})$, $\mathscr{F}(d)$ *is generated by its global sections, and* $H^p(X, \mathscr{F}(d)) = 0$ *for all* $p > 0$.

Proof. — By replacing $\mathscr{F}$ with $i_*\mathscr{F}$, where $i\colon X \to \mathbf{P}^n_k$ is the canonical closed immersion, we reduce to the case where $X = \mathbf{P}^n_k$. We deduce from theorem 6.3.2 that the theorem holds for $\mathscr{F} = \mathscr{O}(m)$, at least if $n \geqslant 1$. For $n = 0$, it also holds.

Indeed, $\mathbf{P}^0_k = \operatorname{Spec}(k)$ is affine, $\mathscr{F}$ is of the form $\widetilde{M}$, for some finitely generated k-module M, because k is noetherian, and $H^0(\mathbf{P}^0_k, \mathscr{F}) = M$; moreover, any quasi-coherent sheaf on an affine scheme is generated by its global sections, and all higher cohomology groups vanish by Serre's theorem 6.1.1.

For any integer $i \in \{0, \dots, n\}$, the coherent sheaf on $D_+(T_i)$ obtained by restriction of $\mathscr{F}$ is generated by its global sections, because $D_+(T_i)$ is affine. Let S_i be a finite subset of $\Gamma(D_+(T_i), \mathscr{F})$ that generates $\mathscr{F}|_{D_+(T_i)}$. For any $s \in S_i$, there exists an integer $m_s \geqslant 0$ such that $T_i^{m_s} s$ is the restriction of a section $t \in \Gamma(\mathbf{P}^n_k, \mathscr{F}(m_s))$. Replacing all the integers m_s by their least upper bound, we have a finite set of sections $t \in \Gamma(\mathbf{P}^n_k, \mathscr{F}(m))$ such that for any $i \in \{0, \dots, n\}$, the sections t/T_i^m generate $\mathscr{F}$ on $D_+(T_i)$. These sections furnish a surjective morphism of coherent sheaves

$$\mathscr{O}^N_{\mathbf{P}^n_k} \to \mathscr{F}(m)$$

hence, by tensoring it with $\mathscr{O}(-m)$, a surjective morphism

$$\mathscr{O}_{\mathbf{P}^n_k}(-m)^N \to \mathscr{F}.$$

Denoting its kernel by $\mathscr{G}$ and taking the tensor prodct by $\mathscr{O}(d)$, we thus have an exact sequences of coherent sheaves

$$0 \to \mathscr{G}(d) \to \mathscr{O}_{\mathbf{P}^n_k}(d-m)^N \to \mathscr{F}(d) \to 0.$$

We now prove both statements by descending induction on p.

a) Using the Čech complex associated with the open covering $(D_+(T_i))$ of $\mathbf{P}^n_k$, we see that $H^p(\mathbf{P}^n_k, \mathscr{F}) = 0$ for any coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_k$ and any integer $p > n$; in particular, it is finitely generated.

Taking the cohomology long exact sequence associated to the short exact sequence

$$0 \to \mathscr{G} \to \mathscr{O}_{\mathbf{P}^n_k}(-m)^N \to \mathscr{F} \to 0,$$

we obtain an exact sequence

$$H^p(\mathbf{P}^n_k, \mathscr{O}(-m))^N \to H^p(\mathbf{P}^n_k, \mathscr{F}) \to H^{p+1}(\mathbf{P}^n_k, \mathscr{G}).$$

The explicit computation of theorem 6.3.2 shows that $H^p(\mathbf{P}^n_k, \mathscr{O}(-m))$ is finitely generated, and by induction, $H^{p+1}(\mathbf{P}^n_k, \mathscr{G})$ is finitely generated too. Since k is noetherian, we conclude that $H^p(\mathbf{P}^n_k, \mathscr{F})$ is finitely generated.

b) Similarly, the Čech complex associated with the open covering $(D_+(T_i))$ of $\mathbf{P}^n_k$ shows that $H^p(\mathbf{P}^n_k, \mathscr{F}(d)) = 0$ for $p > n$ and any integer d.

Assume that for any coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_k$, there exists an integer $d(\mathscr{F})$ such that $\mathrm{H}^q(\mathbf{P}^n_k, \mathscr{F}(d)) = 0$ for all integers $d \geqslant d(\mathscr{F}))$ and all $q > p$. Let d be any integer such that $d \geqslant \sup(m-n, d(\mathscr{G}))$. Considering the cohomology long exact sequence associated with the short exact sequence

$$0 \to \mathscr{G}(d) \to \mathscr{O}_{\mathbf{P}^n_k}(d-m)^{\mathrm{N}} \to \mathscr{F}(d) \to 0,$$

we get an exact sequence

$$\mathrm{H}^p(\mathbf{P}^n_k, \mathscr{O}(d-m))^{\mathrm{M}} \to \mathrm{H}^p(\mathbf{P}^n_k, \mathscr{F}(d)) \to \mathrm{H}^{p+1}(\mathbf{P}^n_k, \mathscr{G}(d)).$$

Since $d \geqslant m-n$, we have $\mathrm{H}^p(\mathbf{P}^n_k, \mathscr{O}(d-m)) = 0$ by theorem 6.3.2; since $d \geqslant d(\mathscr{G})$, the induction hypothesis implies that $\mathrm{H}^{p+1}(\mathbf{P}^n_k, \mathscr{G}(d)) = 0$. It follows that $\mathrm{H}^p(\mathbf{P}^n_k, \mathscr{F}(d)) = 0$ for $d \geqslant \sup(m, d(\mathscr{G}))$.

We thus obtain that for any coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_k$, there exists an integer $d(\mathscr{F})$ such that $\mathrm{H}^p(\mathbf{P}^n_k, \mathscr{F}(d)) = 0$ for all $d > d(\mathscr{F})$ and all $p > 0$.

Taking $p = 0$, we obtain that there exists an integer $d(\mathscr{F})$ such that morphism

$$\mathrm{H}^0(\mathbf{P}^n_k, \mathscr{O}(d-m))^{\mathrm{M}} \to \mathrm{H}^0(\mathbf{P}^n_k, \mathscr{F}(d))$$

is surjective if $d \geqslant d(\mathscr{F})$. If, moreover, $d \geqslant m$, we see from theorem 6.3.2 that the coherent sheaf $\mathscr{O}(d-m)$ is generated by its global sections. Consequently, the same holds for $\mathscr{F}(d)$. ☐

Corollary **(6.3.6)**. — *Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a morphism of noetherian schemes. Assume that f is locally projective: for every point $y \in \mathrm{Y}$, there exists an open neighborhood V of y such that $f^{-1}(\mathrm{V})$ is isomorphic, as a V-scheme, to a closed subscheme of $\mathbf{P}^n_{\mathrm{V}}$.*

Then, for every coherent sheaf $\mathscr{F}$ on X and any integer p, the higher direct image $\boldsymbol{R}^p f_* \mathscr{F}$ is a coherent sheaf on Y.

Proof. — Let $p \in \mathbf{N}$. The property for the $\mathscr{O}_{\mathrm{Y}}$-module $\boldsymbol{R}^p f_* \mathscr{F}$ to be coherent is local on Y; that allows us to assume that Y is affine, say $\mathrm{Y} = \mathrm{Spec}(k)$, and X is a closed subscheme of $\mathbf{P}^n_k$. Since $\mathbf{P}^n_k$ is proper over k, we see that f is proper, hence, in particular, quasi-compact and separated. By corollary 6.1.5 the $\mathscr{O}_{\mathrm{Y}}$-module $\boldsymbol{R}^p f_* \mathscr{F}$ is quasi-coherent, and corresponds with the k-module $\mathrm{H}^p(\mathrm{X}, \mathscr{F})$ which is finitely generated, by theorem 6.3.5. Since Y is a noetherian scheme, the ring k is noetherian (proposition 4.4.11). It then follows from lemma 4.7.17 that $\boldsymbol{R}^p f_* \mathscr{F}$ is coherent. ☐

Remark **(6.3.7)**. — Grothendieck has shown that theorem 6.3.5 holds for any proper k-scheme X, and the preceding corollary holds for any proper morphism of noetherian schemes. His proof relies on a *dévissage* from the projective case, together with Chow's lemma that if X is a proper k-scheme, there exists a proper morphism $h\colon X' \to X$ and a dense open affine subscheme U of X such that X′ is isomorphic to a closed subscheme of some projective space $\mathbf{P}^n_k$ and h induces an isomorphism from $h^{-1}(U)$ to U.

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