Université de Paris

Master Mathématiques fondamentales, 2e année, 2e semestre
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COHOMOLOGY OF COHERENT SHEAVES ON SCHEMES

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The exercises are independent one of another. You may solve them in any order.

EXERCICE 1

Let $\mathcal{C}$ be an abelian category, let $A = (A^p, d_A^p)_{p\in\mathbf{Z}}$ and $B = (B^p, d_B^p)_{p\in\mathbf{Z}}$ be complexes in $\mathcal{C}$.

1 Let $\varphi = (\varphi^p \colon A^p \to B^p)_{p\in\mathbf{Z}}$ be a family of morphisms in $\mathcal{C}$. For every $p \in \mathbf{Z}$, we set $C = C(\varphi)^p = A^p \oplus B^{p-1}$ and we define $d_C^p \colon C^p \to C^{p+1}$ by

$$d_C^p(a,b) = (d_A^p(a), \varphi^p(a) - d_B^{p-1}(b)),$$

for $a \in A^p$ and $b \in B^{p-1}$.
Prove that $(C^p, d_C^p)_{p\in\mathbf{Z}}$ is a complex if and only if φ defines a morphism of complexes from A to B.

2 We now assume that φ is a morphism of complexes. Define an exact sequence of complexes

$$0 \to B(-1) \to C \to A.$$

Prove that the complex C is exact if and only if φ is a homology (that is, if for every p, the morphism φ^p induces an isomorphism between cohomology objects $H^p(A) \to H^p(B)$).

EXERCICE 2

Let X be a noetherian scheme. In this exercise, we define the cohomological dimension of X to be the smallest integer d such that $H^p(X, \mathscr{F}) = 0$ for every quasicohérent sheaf $\mathscr{F}$ on X and any integer $p > d$.

1 Justify the existence of such an integer d What does it mean for X that $d = 0$?

2 For every open subset U of X, we write $\mathscr{N}(U)$ for the nilradical of $\mathscr{O}_X(U)$. Prove that $\mathscr{N}$ is a quasi-coherent sheaf of ideals. We denote by X_{red} the closed subscheme $V(\mathscr{N})$ it defines.

3 Prove that there exists an integer k such that $\mathscr{N}^k = 0$.

4 Prove that X and X_{red} have the same cohomological dimension.

5 Prove that X_{red} is affine if and only if X is affine.

6 Prove that the cohomological dimension of X is the upper bound of the cohomological dimensions of its irreducible components (viewed as reduced subschemes).

EXERCICE 3

Let A be a noetherian ring, let n be an integer $\geqslant 0$, let S be the graded ring $A[T_0, \dots, T_n]$ and let $\mathscr{F}$ be a coherent sheaf on $\mathbf{P}^n_A = \mathrm{Proj}(S)$.

1 Explain the graded S-module structure on $\bigoplus_{d\in\mathbf{Z}} \Gamma(\mathbf{P}^n_A, \mathscr{F}(d))$.

2 Give an example where this graded module is not of finite type.

3 For any integer $k \in \mathbf{Z}$, let

$$\Gamma_{\geqslant k}(\mathscr{F}) = \bigoplus_{d\geqslant k} \Gamma(\mathbf{P}^n_A, \mathscr{F}(d)).$$

Explain its graded S-module structure.

4 We suppose in this question that $\mathscr{F} = \mathscr{O}(e)$. Determine $\Gamma_{\geqslant k}(\mathscr{O}(e))$ for any integer k. Prove that it is an S-module of finite type.

5 Explain why there exists a finite sequence $(d_1, \dots, d_m)$ of integers and an epimorphism

$$\varphi\colon \bigoplus_{i=1}^{m} \mathscr{O}_{\mathbf{P}^n_A}(d_i) \to \mathscr{F}.$$

6 Prove that for any integer k large enough, the morphism φ induces a *surjective* homomorphism of S-modules, $\bigoplus_{i=1}^m \Gamma_{\geqslant k}(\mathscr{O}(d_i)) \to \Gamma_{\geqslant k}(\mathscr{F})$.

7 Deduce that for any integer k, $\Gamma_{\geqslant k}(\mathscr{F})$ is an S-module of finite type.

EXERCICE 4

Let K be a field, we set $S = K[T_0, \dots, T_n]$, $\mathbf{P}^n = \mathrm{Proj}(S)$, and we consider a coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_K$. For $m \in \mathbf{Z}$, we say that $\mathscr{F}$ is m-regular if $H^p(\mathbf{P}^n_K, \mathscr{F}(m-p)) = 0$ for any integer $p > 0$. The regularity of $\mathscr{F}$ is the greatest lower bound of the integers m such that $\mathscr{F}$ is m-regular.

1 Compute the regularity of $\mathscr{O}_{\mathbf{P}^n_K}$.

2 Let X be a closed subscheme of $\mathbf{P}^n_K$. Give a relation between the regularity of the ideal sheaf $\mathscr{I}_X$ and that of the sheaf $\mathscr{O}_X$ (viewed as a coherent sheaf on $\mathbf{P}^n_K$).

3 In the rest of this exercise, we assume that the dimension of X is zero. Prove that X is an affine scheme. Recall why $H^0(X, \mathscr{O}_X)$ is a finitely generated K-algebra; we will denote its dimension by d.

4 In this question, we assume that $n = 1$. Prove that there is an isomorphism $\mathscr{O}_{\mathbf{P}^1_K}(-d) \simeq \mathscr{I}_X$. Deduce from this the regularity of $\mathscr{I}_X$.

5 We now assume that $n \geqslant 2$. Prove that $\mathscr{I}_X$ is m-regular if and only if the morphism $\Gamma(\mathbf{P}^n_K, \mathscr{O}_{\mathbf{P}^n_K}(m-1)) \to \Gamma(X, \mathscr{O}_X(m-1))$ is surjective. Assuming that $\mathscr{I}_X$ is m-regular, conclude that $d \leqslant \binom{m-1+n}{n}$.

6 We assume that $n = 2$ and that X is a reduced subscheme consisting of three points a, b, c with residue field K. Compute the regularity of $\mathscr{I}_X$ according to whether the three points a, b, c are aligned or not.