# Differential forms and currents on Berkovich spaces

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- Asymptotic aspects of archimedean geometry
  - Conjectures of Kontsevich-Soibelman;
  - Degenerations of archimedean dynamics towards non-archimedean dynamics;
  - Asymptotic expansions of integrals.

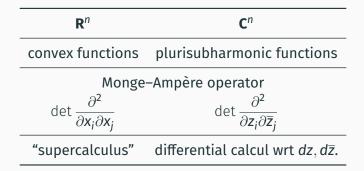
We build a theory of **real valued differential forms** and **currents** on analytic spaces in the sense of Berkovich giving rise to:

- (p,q) forms for  $p,q \leq n$  (dimension of space);
- integration of (*n*, *n*)-forms;
- by duality, currents;
- classical formulas, such as the Poincaré–Lelong formula;
- theory of metrized line bundles, their curvature forms,...

- Study of the tropical Dolbeauly cohomology of Berkovich spaces, relation with Chow groups (Jell, Wanner; Liu, Mikami);
- Tropical intersection theory (Gubler, Künnemann);
- Asymptotic expansions of archimedean integrals (Ducros, Hrushovski, Loeser);
- Non-archimedean degenerations of archimedean algebraic dynamics (Boucksom, Favre, Jonsson)...

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Let *U* be an open subset of  $\mathbf{R}^n$ .

A (super)form of type (p, q) on U is an element of

$$\mathscr{A}^{p,q}(U) = \mathscr{C}^{\infty}(U) \otimes \bigwedge^p (\mathbf{R}^n)^* \otimes \bigwedge^q (\mathbf{R}^n)^*.$$

In coordinates:

$$\alpha = \sum_{\substack{|I|=p\\|J|=q}} \alpha_{IJ}(x) \, \mathsf{d}' \, x_{i_1} \wedge \cdots \wedge \mathsf{d}' \, x_{i_p} \otimes \mathsf{d}'' \, x_{j_1} \wedge \cdots \wedge \mathsf{d}'' \, x_{j_q}.$$

**Bigraded algebra:**  $\mathscr{A}(U) = \bigoplus_{p,q} \mathscr{A}^{p,q}(U)$ , exterior product  $\land$ Involution J defined by J d' x = d'' x, J d'' x = -d' x. Notion of symmetric form: J $\alpha = \alpha$ .

### **Differential operators:**

$$\mathsf{d}'\colon \mathscr{A}^{p,q}(U)\to \mathscr{A}^{p+1,q}(U), \qquad \mathsf{d}''\colon \mathscr{A}^{p,q}(U)\to \mathscr{A}^{p,q+1}(U).$$

**Examples:** for  $f \in \mathscr{A}^{0,0}(U) = \mathscr{C}^\infty(U)$ ,

$$\mathsf{d}'\mathsf{d}''f = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \,\mathsf{d}' \,x_i \otimes \mathsf{d}'' \,x_j$$

$$(\mathsf{d}'\mathsf{d}''f)^n = n! \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \mathsf{d}' x_1 \wedge \mathsf{d}'' x_1 \wedge \cdots \wedge \mathsf{d}' x_n \wedge \mathsf{d}'' x_n.$$

**Integral:** For  $\alpha = f d' x_1 \wedge d'' x_1 \wedge \cdots \wedge d' x_n \wedge d'' x_n \in \mathscr{A}^{n,n}(U)$ , set

$$\int_U \alpha = \int_U f(x) \mathrm{d} x_1 \cdots \mathrm{d} x_n.$$

- *depends* on the choice of affine coordinates.

### **Currents**:

- currents = (continuous) linear forms on superforms;
- differential calculus defined by duality;
- (p,q)-forms define (n-p, n-q)-currents;
- positive forms, positive currents and their products (à la Bedford-Taylor), and formulas, such as:

$$(\mathsf{d}'\mathsf{d}'' \log \max(\mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_n))^n = \delta_0.$$

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## Analytic spaces in the sense of Berkovich

Let *k* be a field, complete for a non-archimedean absolute value.

With a *k*-algebra A (either of finite type, or affinoid), Berkovich associates an analytic **spectrum**  $\mathcal{M}(A)$ :

 $\mathscr{M}(\mathsf{A})$  is a set of multiplicative seminorms p on A, functions  $p\colon\mathsf{A}\to\mathbf{R}_+$  such that

- $p(f+g)\leqslant p(f)+p(g)$  for  $f,g\in$  A;
- p(fg) = p(f)p(g) for  $f, g \in A$ ;
- $p(\lambda) = |\lambda|$  for  $\lambda \in k$ .

**Topology:** the coarsest such that all maps  $f \mapsto p(f)$  are continuous

**Notation**: *p* is viewed as a point of  $\mathcal{M}(A)$ , hence p(f) is written |f(p)|.

#### Interests of Berkovich's theory:

- Good topology (locally contractible, locally compact);
- Interesting/fruitful interaction with real numbers;
- Possess both a topology and a Grothendieck topology.

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### **Other theories:**

- Naïve: has not enough local compactness for our work;
- Tate: has not enough points;
- Raynaud: lacks a "visualization framework"
- Huber: has too many points.

Let *k* be a field, complete for a non-archimedean absolute value.

**Torus:**  $G_m = \mathcal{M}(k[T, T^{-1}]);$ 

**tropicalization:** continuous map  $\mathbf{G}_{m} \rightarrow \mathbf{R}, x \mapsto \log |T(x)|$ 

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### Definition

Let X be a k-analytic space. A **moment** on X is a morphism  $f: X \to \mathbf{G}^d_m$ .

## **Tropicalization:** $f_{\text{trop}} : X \to \mathbb{R}^d$ .

If X is compact, then  $f_{trop}(X)$  is a compact polyhedral subspace of  $\mathbf{R}^{d}$ .

Let X be a k-analytic space.

## Definition

A function  $\varphi$  on X is **(G-)smooth** if (G-)locally, there exist

– a moment  $f \colon X \to \mathbf{G}^d_{\mathsf{m}}$ ,

– a smooth function u on an open neighborhood of  $f_{\rm trop}(X)$  in  ${\bf R}^d$ 

such that  $\varphi = \mathbf{u} \circ f_{\text{trop}}$ .

If *X* is locally holomorphically separated, topologically separated, then there are plenty of smooth functions:

- 1. Stone–Weierstrass:  $\mathscr{C}^{\infty}_{(c)}(X)$  is dense in  $\mathscr{C}_{(c)}(X)$  for the compact open topology;
- 2. If *X* is paracompact, then every open covering admits a smooth partition of unity.

## **Differential forms**

Formal construction of a sheaf  $\mathscr{A}_X^{p,q}$  and a G-sheaf  $\mathscr{A}_{X_G}^{p,q}$  from **tropical charts**: from

$$X \supset U \xrightarrow{f} \mathbf{G}^d_{\mathsf{m}}, \quad P \supset f_{\mathsf{trop}}(U), \quad \alpha \in \mathscr{A}^{p,q}(P)$$

get  $f^*\alpha$ .

#### Lemma

$$f_{\text{trop}} = g_{\text{trop}}$$
 implies  $f^* \alpha = g^* \alpha$ .

More or less formally, one obtains a  $d^\prime, d^\prime\prime$  differential calculus, a notion of currents...

### Theorem (Jell)

 $(\mathscr{A}_X^{0,*}, \mathsf{d}'')$  is a resolution of  $\mathbf{R}_X$ .

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**Gauss point**  $\eta_r \in \mathbf{G}_m^d$ , associated with  $r = (r_1, \dots, r_n) \in \mathbf{R}^d$ : it is the (semi)norm given by  $\sum a_m T^m \mapsto \sup_m |a_m| e^{mr}$ 

**Skeleton** of  $\mathbf{G}_{m}^{n}$ :  $S(\mathbf{G}_{m}^{d}) = \{\eta_{r}; r \in \mathbf{R}^{d}\} \simeq \mathbf{R}^{d}$  (as a top. space)

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Let  $f: X \to \mathbf{G}_{m}^{d}$  be a moment,  $n = \dim(X)$  (X equidimensional) Characteristic polyedron of f,  $\Sigma_{f} = \bigcup_{p: \mathbf{G}_{m}^{d} \to \mathbf{G}_{m}^{n}} (p \circ f)^{-1}(S(\mathbf{G}_{m}^{n}))$ 

It has a canonical polyhedral structure (Ducros).

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 (tropical dim. of  $f$  at  $x$ )  
 $\underset{\text{if } x \notin \partial(X)}{\longrightarrow}$ 

Let  $\alpha \in \mathscr{A}^{n,n}(X)$ .

Locally, the support of  $\alpha$  is contained in a polyhedral subspace of X, built from local skeletons.

One wants to integrate  $\alpha$  on this polyhedral subspace, cell by cell.

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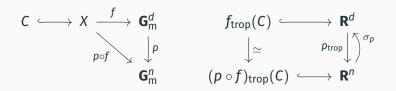
Solution: define "calibrations" of the cells.

$$f\colon X o {f G}^d_{\sf m}$$
 ,  $\Sigma_f$ 

 $\mathscr{C}$ , convenient cellular decomposition of  $\Sigma_f$ :

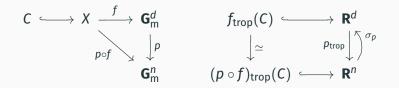
- the intersection of two cells, the boundary of a cell, are unions of cells;
- for every cell C,  $f_{\text{trop}} \colon C \xrightarrow{\sim} f_{\text{trop}}(C)$ , convex polyhedron of  $\mathbf{R}^d$ ;
- $\partial(X)$  does not meet open *n*-cells;
- each open *n*-cell is open in  $\Sigma_f$ .

Basic diagram for an *n*-cell C:



where  $\sigma_p$  is the unique affine section of  $p \circ f$  with image  $\langle f_{trop}(C) \rangle$ .

## Calibration



#### Theorem

- p ∘ f is finite and flat of some degree d<sub>C</sub>(p ∘ f) at each point of C.
- Up to sign, the n-vector  $d_{C}(p \circ f) \cdot \sigma_{p,*}(e_{1} \wedge \cdots \wedge e_{n}) \in \bigwedge^{n} \langle f_{trop}(C) \rangle$  does not depend on the choice of p.

### $\rightarrow$ canonical **calibration** of the cell C.

The canonical calibration of a cell C allows to integrate:

- any (n, n)-form on C
- any (n − 1, n)-form on ∂(C) (sign convention: outer normal)

By summing the contributions of *n*-cells of a convenient cellular decomposition, one gets:

• an integration map, 
$$\mathscr{A}^{n,n}_{c}(X) \to \mathbf{R}, \omega \mapsto \int_{X} \omega = \sum_{C} \int_{C} \omega$$

• a boundary integration map,  $\mathscr{A}_{c}^{n-1,n}(X) \to \mathbf{R}$ ,  $\omega \mapsto \int_{\partial(X)} \omega = \sum_{C} \int_{\partial(C)} \omega$ 

#### Theorem

The support of boundary integration is contained in  $\partial(X)$ .

More or less equivalent to the balancing condition in **tropical** geometry.

#### Theorem (Stokes formula)

For 
$$\omega \in \mathscr{A}^{n-1,n}_{\mathsf{c}}(X)$$
, one has  $\int_X \mathsf{d}' \, \omega = \int_{\partial(X)} \omega.$ 

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Let X be equidimensional,  $\dim(X) = n$ , without boundary As in differential geometry, **currents** are defined as (continuous) linear forms on differential forms:

$$\mathscr{D}^{p,q}(X) = (\mathscr{A}^{p,q}_{\mathsf{c}}(X))^* = \mathscr{D}_{n-p,n-q}(X).$$

 $d^\prime, d^{\prime\prime}\text{-differential calculus, involution J }-$  by duality (with a sign)

 $\alpha \in \mathscr{A}(X), T \in \mathscr{D}(X), T \wedge \alpha \colon \omega \mapsto \langle T, \alpha \wedge \omega \rangle.$ Sheaf property **Integration currents:**  $\varphi$ :  $Y \rightarrow X$  topologically proper map, dim(Y) = m,

$$\varphi_* \delta_{\mathsf{Y}} \left( : \alpha \mapsto \int_{\mathsf{Y}} \varphi^* \alpha \right) \in \mathscr{D}^{m,m}(\mathsf{X})$$
$$\varphi_* \delta_{\partial(\mathsf{Y})} \left( : \alpha \mapsto \int_{\partial(\mathsf{Y})} \varphi^* \alpha \right) \in \mathscr{D}^{m-1,m}(\mathsf{X})$$

Functions  $u: X \to \mathbf{R} \cup \{-\infty\}$  which are integrable on every compact skeleton define currents, e.g., continuous outside of a Zariski closed subset of empty interior.

#### Theorem

For  $f \in \mathscr{M}(X)^{ imes}$ , regular meromorphic function on X,

 $\mathsf{d}'\mathsf{d}''[\log|f|] = \delta_{\operatorname{div}(f)}.$ 

In complex analysis,  $d'd'' \log \sup(|z|, r)$  is the integration current on the circle of radius r.

#### Theorem

For  $f: X \rightarrow \mathbf{A}^1$ ,  $r \in \mathbf{R}$ ,

$$\mathsf{d}'\mathsf{d}''\left[\sup\left(\log\left|f\right|,r\right)\right]=\delta_{f^{-1}(\eta_r)},$$

integration current over the  $\mathscr{H}(\eta_r)$ -analytic space  $f^{-1}(\eta_r)$ .

Let  $\overline{L} = (L, \|\cdot\|)$  be a line bundle on X with a smooth metric. **Curvature form,** locally defined by  $c_1(\overline{L}) = d'd'' \log \|s\|^{-1}$ , for any local non-vanishing section s of L.

#### Proposition

Let  $\mathscr{X}$  be a proper k-scheme,  $\mathscr{L}$  a line bundle on  $\mathscr{X}$ ; take  $(X, L) = (\mathscr{X}^{an}, \mathscr{L}^{an})$ . Then  $\int_{X} c_{1}(\overline{L})^{n} = (c_{1}(\mathscr{L})^{n} | \mathscr{X}).$ 

**Remark** (Y. Liu): There are cycle classes in d"-cohomology, defined using the Gersten complex.

- Let  $\mathscr{X}$  be a proper formal  $k^{\circ}$ -scheme (normal, say), let  $\mathscr{L}$  be a line bundle on  $\mathscr{X}$
- take  $(X, L) = (\mathscr{X}_{\eta}, \mathscr{L}_{\eta}).$

Then *L* has a natural "formal" metric which is continuous, but not smooth in general, so that  $c_1(\overline{L})$  is a (1, 1)-current. Translating Bedford-Taylor theory from complex analysis to the current framework, we can consider **products** of these currents  $c_1(\overline{L})$ , using smooth approximations.

#### Theorem

Assume that k is endowed with a nontrivial discrete absolute value. Then

$$c_1(\overline{L})^n = \sum_{\xi \in X} m_{\xi}(c_1(\mathscr{L})^n | V_{\xi}) \delta_{\xi}.$$

Here,  $\xi$  runs over points of X which reduce to the generic point of a component  $V_{\xi}$  of the special fiber  $\widetilde{\mathscr{X}}$ , and  $m_{\xi}$  is its multiplicity.

The currents  $c_1(\overline{L})^p$  can be described in two different ways:

- By integration on suitable polyhedral subspaces of X theory of PL currents, which enjoys similar properties to tropical intersection theory;
- On a large open subset (which carries all of its mass), as a sum of integration currents on fibers;

## Perspectives

- Non-archimedean Arakelov geometry (Gubler-Künnemann);
- Local heights via analytic geometry rather than formal models;
- Relation with Chow groups and K-theory (Liu, Mikami);
- Study of psh functions (Thuillier, Maculan, Jell, Wanner, Boucksom-Favre-Jonsson,...);
- Monge-Ampère problem (Kontsevich-Tschinkel, B-F-J, Jell-Martin-G-K...);
- Non-archimedean limits of archimedean integrals (Ducros, Hrushovski, Loeser).