Contemporary Mathematics
Volume **605**, 2013
`http://dx.doi.org/10.1090/conm/605/12115`

Diophantine geometry and analytic spaces

Antoine Chambert-Loir

1. Introduction

Diophantine Geometry can be roughly defined as the *geometric* study of *Diophantine equations*. Historically, and for most mathematicians, those equations are polynomial equations with integer coefficients and one seeks for integer, or rational, solutions; generalizations to number fields come naturally. However, it has been discovered in the XIXth century that number fields share striking similarities with finite extensions of the field $k(t)$ of rational functions with coefficients in a field k, the analogy being the best when k is a finite field. From this point of view, rings of integers of number fields are analogues of rings of regular functions on a regular curve. Namely, both rings are Dedekind (i.e., integrally closed, one-dimensional, Noetherian) domains.

When one studies Diophantine Geometry over number fields, the geometric shape defined by the polynomial equations over the complex numbers plays an obvious important role. Be it sufficient to recall the statement of Mordell conjecture (proved by Faltings [**19**]): a Diophantine equation whose associated complex shape is a compact Riemann surface of genus at least 2 has only finitely many solutions. Over function fields, such a role can only be played by analytic geometry over non-Archimedean fields, a much more recent theory than its complex counterpart.

The lecture the author gave at the Bellairs Workshop in Number Theory was devoted to a survey of recent works in Diophantine Geometry over function fields, where analytic geometry over non-Archimedean fields in the sense of Berkovich [**5**] took a significant place. Since this topic was not the main one of the conference, the talk had been deliberately informal and the present notes aim at maintaining this character, in the hope that they will be useful for geometers of all obediences, be it Diophantine, tropical, complex, non-Archimedean. . .

2. The standard height function

In all the sequel, we fix a field F which can be, either the field $\mathbf{Q}$ of rational numbers (arithmetic case), or the field $k(T)$ of rational functions with coefficients in a given field k (geometric case). This terminology will be explained later. We let $\overline{F}$ be an algebraic closure of F.

2010 *Mathematics Subject Classification.* 14G40, 14G22, 14K.

The standard height function $h\colon \mathbf{P}^n(\overline{F}) \to \mathbf{R}$ is a function measuring the "complexity" of a point in projective space with homogeneous coordinates in $\overline{F}$.

We begin by describing it on the subset $\mathbf{P}^n(F)$ of F-rational points.

2.1. Arithmetic case. Let $\mathbf{x}$ be a point in $\mathbf{P}^n(\mathbf{Q})$. We may assume that its homogeneous coordinates $[x_0 : \cdots : x_n]$ are chosen so as to be coprime integers; they are then well defined up to a common sign and one defines

$$h(\mathbf{x}) = \log \max(|x_0|, \dots, |x_n|). \tag{2.1}$$

One first observes a *finiteness property:* for any real number B, there are only finitely many points $\mathbf{x} \in \mathbf{P}^n(\mathbf{Q})$ such that $h(\mathbf{x}) \leq B$. Indeed, this bound gives only finitely many possibilities for each coordinate.

The height function behaves well under morphisms. Let $f\colon \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ be a rational map given by homogeneous forms $f_0, \dots, f_m \in \mathbf{Q}[X_0, \dots, X_n]$ of degree d, without common factor. Its exceptional locus E is the closed subspace in $\mathbf{P}^n$ defined by the simultaneous vanishing of all f_is. Then, one proves easily that there exists a constant c such that $h\big(f(\mathbf{x})\big) \leq dh(\mathbf{x}) + c$, for any point $\mathbf{x} \in \mathbf{P}^n(\mathbf{Q})$ such that $\mathbf{x} \notin E$. The converse inequality is more subtle and relies on the *Nullstellensatz*: *Let X be a closed subscheme of $\mathbf{P}^n$ such that $X \cap E = \varnothing$; then, there exists a constant c_X such that $h\big(f(\mathbf{x})\big) \geq dh(\mathbf{x}) - c_X$ for any $\mathbf{x} \in X(\mathbf{Q})$.*

To add on the subtlety behind these apparently simple estimates, let me remark that it is easy, given an explicit map f, to write down an explicit acceptable constant c; however, giving an explicit constant c_X requires a quite nontrivial statement called the *effective arithmetic Nullstellensatz*; see, for example, D'Andrea, Krick, and Sombra [**14**] for a recent and sharp version.

2.2. Geometric case. Let now $\mathbf{x}$ be a point in $\mathbf{P}^n\big(k(T)\big)$. Again, we may choose a system of homogeneous coordinates $[x_0 : \cdots : x_n]$ of $\mathbf{x}$ consisting of polynomials in $k[T]$ without common factors. Such a system is unique up to multiplication by a common nonzero constant. Let us define the height of $\mathbf{x}$ by the formula

$$h(\mathbf{x}) = \max(\deg x_0, \dots, \deg x_n). \tag{2.2}$$

If the base field k is finite, then the height satisfies a similar finiteness property as in the arithmetic case: since there are only finitely many polynomials $f_i \in k[T]$ of given degree, the set of points $\mathbf{x} \in \mathbf{P}^n\big(k(T)\big)$ such that $h(\mathbf{x}) \leq B$ is finite, for any B.

The height function has exactly the same properties with respect to morphism as in the arithmetic case.

2.3. Geometric interpretation (geometric case). In the geometric case, the height can be given a *geometric interpretation*, free of homogeneous coordinates. Indeed, let C be the projective line over k, that is, the unique projective regular k-curve with function field $F = k(T)$. Any point $\mathbf{x} \in \mathbf{P}^n(F)$ can be interpreted as a morphism $\varphi_{\mathbf{x}}\colon C \to \mathbf{P}^n$ of k-schemes. When $\varphi_{\mathbf{x}}$ is generically one-to-one, then $h(\mathbf{x})$ can be computed as the degree of the the rational curve C, as embedded in $\mathbf{P}^n$ through $\varphi_{\mathbf{x}}$. In the general case, one has

$$h(\mathbf{x}) = \deg \varphi_{\mathbf{x}}^* \mathscr{O}(1), \tag{2.3}$$

that is, $h(\mathbf{x})$ is the degree of the pull-back to C of the tautological line bundle on $\mathbf{P}^n$.

2.4. Extension to the algebraic closure (geometric case). The previous geometric interpretation suggests a way to define the height on the whole of $\mathbf{P}^n(\overline{F})$. Namely, let E be a finite extension of F; it is the field of rational functions on a projective regular curve C_E defined over a finite extension of k. Any point $\mathbf{x} \in \mathbf{P}^n(E)$ can be interpreted as a morphism $\varphi_{\mathbf{x}} \colon C_E \to \mathbf{P}^n$ and one sets

$$h(\mathbf{x}) = \frac{1}{[E:F]} \deg \varphi_{\mathbf{x}}^* \mathscr{O}(1). \tag{2.4}$$

One checks that the right-hand side of this formula does not depend on the actual choice of a finite extension E such that $\mathbf{x} \in \mathbf{P}^n(E)$, thus defining a function $h \colon \mathbf{P}^n(\overline{F}) \to \mathbf{R}$.

2.5. Absolute values. Using absolute values, one can give a general definition of the standard height function, valid for any finite extension of F.

Recall that an absolute value on a field F is a map $|\cdot| \colon F \to \mathbf{R}_{\geq 0}$ subject to the following axioms: $|0| = 0$, $|1| = 1$, $|ab| = |a||b|$ and $|a+b| \leq |a| + |b|$ for any $a, b \in F$. Two absolute values $|\cdot|$ and $|\cdot|'$ are said to be equivalent if there exists a positive real number λ such that $|a|' = |a|^\lambda$ for any $a \in F$. The trivial absolute value on F is defined by $|a|_0 = 1$ for any $a \in F^*$.

Let M_F be the set of nontrivial absolute values of F, up to equivalence. Any class $v \in F$ possesses a preferred, normalized, representative, denoted $|\cdot|_v$, so that the *product formula* holds:

$$\prod_{v \in M_F} |a|_v = 1 \quad \text{for any } a \in F^\times. \tag{2.5}$$

It connects the nontrivial absolute values (on the left-hand side) and the trivial one (on the right-hand side).

The field $F = \mathbf{Q}$ possesses the usual Archimedean absolute value, denoted $|\cdot|_\infty$. Absolute values nonequivalent to that satisfy the *ultrametric inequality* $|a+b| \leq \max(|a|, |b|)$, and each of them is associated to a prime number p. The corresponding normalized p-adic absolute value is characterized by the equalities $|p|_p = 1/p$ and $|a|_p = 1$ for any integer a which is prime to p. Therefore, $M_{\mathbf{Q}} = \{\infty, 2, 3, 5, 7, \dots\}$. A similar description applies to number fields, the normalized ultrametric absolute values are in correspondence with the maximal ideals of the ring of integers, while the Archimedean absolute values correspond to real or pair of conjugate complex embeddings of the field.

All absolute values of the field $F = k(T)$ which are trivial on k are ultrametric. They correspond to the closed points of the projective line $\mathbf{P}^1_k$ (whose field of rational functions is precisely F). For example, there is a unique absolute value on $k(T)$ which maps a polynomial P to $\mathrm{e}^{\deg(P)}$, and it corresponds to the point at infinity of $\mathbf{P}^1_k$. Similarly, let $A \in k[T]$ be a monic, irreducible, polynomial; there is a unique absolute value on $k(T)$ which maps A to $\mathrm{e}^{-\deg(A)}$ and maps to 1 any polynomial which indivisible by A; this absolute value corresponds to the closed point (A) of the affine line.

More generally, if E is a finite extension of F, the set M_E is naturally in bijection with the set of closed points of the unique projective normal curve C_E with function field E.

The product formula is nothing but the formula that claims that the number of zeroes of a rational function on a curve is equal to the number of poles (in both cases, counted with multiplicity).

In this language, the height function on $\mathbf{P}^n(\overline{F})$ can be defined as

$$h(\mathbf{x}) = \frac{1}{[E:F]} \sum_{v \in M_E} \log\max(|x_0|_v, \dots, |x_n|_v), \tag{2.6}$$

where E is a finite extension of F and $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbf{P}^n(E)$.

2.6. Properties. In the arithmetic case, or, in the geometric case over a *finite* base field k, the height function satisfies an important *finiteness principle*, due to [**29**]: for any real number B and any positive integer d, the set of points $\mathbf{x} \in \mathbf{P}^n(\overline{F})$ such that $[F(\mathbf{x}) : F] \leq d$ and $h(\mathbf{x}) \leq B$ is finite. Obviously, this property does not hold in the geometric case, when the base field is infinite.

In all cases, the height function has a similar behavior with respect to morphisms. *Let $f \colon \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ be a rational map defined by homogeneous polynomials $(f_0, \dots, f_m)$ of degree d, without common factor; let $E \subset \mathbf{P}^n$ be the locus defined by $f_0, \dots, f_m$. Then, there exists a constant c_f such that $h\big(f(\mathbf{x})\big) \leq dh(\mathbf{x}) + c_f$ for any $\mathbf{x} \in \mathbf{P}^n(\overline{F})$ such that $\mathbf{x} \notin E$. Let X be a closed subscheme of $\mathbf{P}^n$ such that $X \cap E = \varnothing$; then, there exists a real number c_X such that $h\big(f(\mathbf{x})\big) \geq dh(\mathbf{x}) - c_X$ for any $\mathbf{x} \in X(\overline{F})$.*

3. Heights for line bundles, canonical heights

3.1. Heights for line bundles. For applications, it is important to understand precisely the behavior of heights under morphisms. This is embodied in the following fact, called the *height machine*. Let $\mathscr{F}(X(\overline{F}); \mathbf{R})$ be the vector space of real valued functions on $X(\overline{F})$, and let $\mathscr{F}_{\mathrm{b}}(X(\overline{F}); \mathbf{R})$ be its subspace of bounded functions. There exists a unique additive map

$$h \colon \operatorname{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R} \to \mathscr{F}(X(\overline{F}); \mathbf{R}) / \mathscr{F}_{\mathrm{b}}(X(\overline{F}); \mathbf{R}), \quad L \mapsto h_L$$

such that for any closed embedding $f \colon X \hookrightarrow \mathbf{P}^n_F$ of X into a projective space,

$$h_{f^*\mathscr{O}(1)} \equiv h_{\mathbf{P}^n} \circ f \pmod{\mathscr{F}_{\mathrm{b}}(X(\overline{F}); \mathbf{R})},$$

where we have denoted $h_{\mathbf{P}^n}$ the height on projective space that we had define in the previous section.

Uniqueness comes from the fact that $\operatorname{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ is generated by line bundles of the form $f^*\mathscr{O}(1)$, for some closed embedding f. The existence follows from basic properties of the height on projective spaces, namely its behavior under Segre and Veronese embeddings.

Moreover, the previous formula holds not only for embeddings f, but for any morphisms f. As a consequence, one get the desirable functoriality: *if $f \colon Y \to X$ is a morphism of projective algebraic varieties over F and $L \in \operatorname{Pic}(X) \otimes \mathbf{R}$, then $h_L \circ f \equiv h_{f^*L}$ (modulo bounded functions).*

Any function in the class h_L deserves to be called a *height function on X with respect to L*. However, it may be desirable to point out specific height functions with good properties. In the following paragraphs, we show some cases where this is indeed possible.

3.2. Algebraic dynamics (Tate, Silverman). Let X be a projective variety over F, and assume that it carries a dynamical system $\varphi\colon X\to X$, and a real line bundle $L\in\operatorname{Pic}(X)\otimes\mathbf{R}$ such that that $\varphi^*L\simeq L^q$, for some real number $q>1$. We shall say that (X,φ,L) is a *polarized dynamical system*, and that q is its weight. Let h^0_L be some arbitrary representative of h_L; then, the following formula

$$\hat h_L(x)=\lim_{n\to\infty} q^{-n}h^0_L\big(\varphi^n(x)\big)$$

defines a height function $\hat h_L$ on $X(\overline F)$ with respect to L, which is independent of the choice of h^0_L. Moreover, it satisfies the following functional equation

$$\hat h_L\big(\varphi(x)\big)=q\hat h_L(x),\quad \text{for any } x\in X(\overline F).$$

In fact, it is the unique height function with respect to L which satisfies this functional equation. We cal it the *canonical height function.*

Abelian varieties, that is, projective group varieties, furnish especially beautiful examples of this situation. If X is an Abelian variety over F, let $[n]$ be the multiplication by an integer n, an endomorphism of X; in particular, $[-1]$ is the inversion on X. Then, for any ample line bundle L on X which is symmetric (that is, $[-1]^*L\simeq L$), one has $[n]^*L\simeq L^{n^2}$ for any integer n. The various canonical height functions, for all integers $n\geq 2$, coincide and are called the *Néron-Tate height* on X relative to the line bundle L.

Similarly, projective spaces, for the maps $[x_0:\dots:x_n]\mapsto[x_0^q:\dots:x_n^q]$ (for some integer $q\geq 2$) and any line bundle, and more generally toric varieties are also interesting examples.

There are also nice examples for some K3-surfaces, first described by Silverman [**35**]. (There, it is useful to work with $\operatorname{Pic}(X)\otimes\mathbf{R}$, rather than $\operatorname{Pic}(X)$.)

3.3. Height functions for geometric ground fields. Assume that $F=k(C)$ is the field of rational functions on a regular curve C which is projective, geometrically irreducible over a field k. Let X be a projective variety over F and L be a real line bundle on X. A projective k-variety $\mathscr{X}$ together with a flat morphism $\pi\colon\mathscr{X}\to C$, the generic fiber of which is X, is called a *model* of X over C; any line bundle $\mathscr{L}\in\operatorname{Pic}(\mathscr{X})\otimes\mathbf{R}$ which gives back L on X is called a model of L.

Now, let $x\in X(\overline F)$; it is defined over a finite extension E of F which is the field of rational functions on a regular integral curve C', finite over C. By projectivity of X and regularity of C', the point x is the generic fiber of a morphism $\varepsilon_x\colon C'\to X$. Then, one can define

$$h_{\mathscr{L}}(x)=\frac{1}{[C':C]}\deg_{C'}\varepsilon_x^*\mathscr{L}.$$

The function $h_{\mathscr{L}}$ is a height function with respect to L.

3.4. Arakelov geometry. This point of view offers a sophisticated, and powerful, way to mimic the geometric case in order to obtain actual height functions in the arithmetic case. Let X be a projective variety over a number field F, let L be a real line bundle on X.

Let $\mathscr{X}$ be a model of X over the ring of integers $\mathfrak{o}_F$, let $\mathscr{L}\in\operatorname{Pic}(\mathscr{X})\otimes\mathbf{R}$ be a model of L. If we observe the analogy between function fields and number fields under the point of view offered by the product formula, we see that $\mathfrak{o}_F$ behaves as the ring of regular functions of an affine curve. Consequently, to get a height function, we need to compactify somehow the spectrum of $\mathfrak{o}_F$ taking into account the Archimedean places of F. This is where Arakelov's ideas come in.

For any Archimedean place v of F, we set $\mathbf{C}_v = \overline{F_v} \simeq \mathbf{C}$ and indicate by a subscript v a v-adic completion, or base change to v-adic completion. Let us endow the holomorphic line bundle L_v on the complex analytic space $X(\mathbf{C}_v)$ deduced from L with a continuous Hermitian metric. Such a metric is a way to define the *size* of sections of L_v. It can be defined as a continuous function on the total space of L_v inducing a Hermitian norm on each fiber $L_{v,x} \simeq \mathbf{C}_v$ above a point $x \in X(\mathbf{C}_v)$. In other words, any holomorphic section s of L_v over an open subset U of $X(\mathbf{C}_v)$ is given a norm $\|s\|$, which is a continuous function $U \to \mathbf{R}_+$, in such a way that $\|fs\|(x) = |f(x)|\|s\|(x)$ for any $x \in U$ and any holomorphic function f on U, and that $\|s\|(x) \neq 0$ if $s(x) \neq 0$.

Let $\overline{\mathscr{L}} = \big(\mathscr{L}, (\|\cdot\|_v)\big)$ be the datum of such a model $\mathscr{L}$ and of Hermitian metrics at all Archimedean places v of F. It is called an Hermitian line bundle over $\mathscr{X}$. (It is customary in Arakelov geometry to impose that these metrics on $\mathscr{L}$ be conjugation invariant, but this hypothesis is not necessary here.) Algebraic operations on line bundles such as taking duals, or tensor products, can be done at the level of Hermitian metrics, so that there is a group $\widehat{\operatorname{Pic}}(\mathscr{X})$ of isomorphism classes of Hermitian line bundles on $\mathscr{X}$.

Now, for $x \in X(\overline{F})$, there is a finite extension E of F such that $x \in X(E)$, and a morphism $\varepsilon_x \colon \operatorname{Spec} \mathfrak{o}_E \to \mathscr{X}$ which extends x. We can then define

$$h_{\overline{\mathscr{L}}}(x) = \frac{1}{[E:F]} \widehat{\deg}\, \varepsilon_x^* \overline{\mathscr{L}},$$

where $\widehat{\deg}$ means the Arakelov degree, an analogue for Hermitian line bundles over $\operatorname{Spec} \mathfrak{o}_E$ of the geometric degree of line bundles over complete curves.

Let us recall shortly the definition of this degree. Let $\overline{\mathscr{M}}$ be a Hermitian line bundle over $\operatorname{Spec} \mathfrak{o}_E$. The module of global sections is a projective $\mathfrak{o}_E$-module M of rank 1 and for any Archimedean place v of E, $M_v = M \otimes_E \mathbf{C}_v$ is endowed with a Hermitian scalar product. Then, one has

$$\widehat{\deg}(\overline{\mathscr{M}}) = \log \frac{\operatorname{Card}(M/\mathfrak{o}_E m)}{\prod_v \|m\|_v},$$

the right-hand side being independent of the choice of a nonzero element $m \in M$. This independence follows from the fact that for any nonzero $a \in \mathfrak{o}_E$, the norm of the ideal (a) coincides with the absolute value of the norm of a. In fact, it is an avatar of the product formula that was used to define the height on projective spaces.

3.5. Adelic metrics. One may push the analogy between number fields and function fields a bit further and do at non-Archimedean places what Arakelov geometry does at Archimedean places. This gives rise to the technique of adelic metrics, which works both in the geometric and in the arithmetic settings.

Let X be a projective variety over F. An adelic metric on L is a family $(\|\cdot\|_v)$ of continuous metrics on the line bundle L at all places v of F satisfying some "adelic" condition.

So let v be a place of F. First, complete F for the absolute value given by v, then take its algebraic closure; this field admits a unique absolute value extending v; take its completion for that absolute value. Let $\mathbf{C}_v$ be the field "of v-adic complex numbers" so obtained; it is complete and algebraically closed. A v-adic metric for the line bundle L can be defined similarly as in the case of Archimedean absolute values, as a continuous function on the total space of the line bundle restricting to the absolute value v on each fiber $L_x \simeq \mathbf{C}_v$ at each point $x \in X(\mathbf{C}_v)$.

If L is very ample and $(s_1, \dots, s_n)$ is a basis of $H^0(X, L)$, then there is a unique metric on L such that

$$\max(\|s_1\|_v(x), \dots, \|s_n\|_v(x)) = 1 \quad \text{for all } x \in X(\mathbf{C}_v).$$

Such a metric is called standard. A family of metrics on L will be called a standard adelic metric if it is defined by this formula for all places v of F.

More generally, a metric (or a family of metrics) on L will be called standard if one can write $L \simeq L_1 \otimes L_2^{-1}$ for two very ample line bundles L_1 and L_2, in such a way that the metric on L is the quotient of standard metrics on L_1 and L_2.

However, the field $\mathbf{C}_v$ is not locally compact, so that the resulting metrics lack good properties. Therefore, one imposes the further condition that the metric can be written as a standard metric times a function of the form e^{δ_v}, where δ_v is a continuous and bounded function on $X(\mathbf{C}_v)$. Considering families of v-adic metrics, one imposes that the function δ_v be identically 0 for all but finitely many places v of F.

4. Level sets for the canonical height

We consider a polarized dynamical system (X, L, φ) over F with weight $q > 1$, as in §3.2. Let $\hat{h}$ be its associated canonical height function, satisfying the functional equation $\hat{h}\big(\varphi(x)\big) = q\hat{h}(x)$ for any $x \in X(\overline{F})$. The most important case will be the one associated to Abelian varieties.

If Y is a subvariety of X and t is a real number, we let $Y(t)$ be the set of points $x \in Y(\overline{F})$ such that $\hat{h}(x) \le t$.

4.1. Preperiodic points. Let $x \in X(\overline{F})$. Its *orbit* under φ is the sequence of points of X obtained by iterating φ, namely $(x, \varphi(x), \varphi^{(2)}(x), \dots)$. One says that x is periodic if there exists $p \ge 1$ such that $x = \varphi^{(p)}(x)$. One says that x is preperiodic if its orbit is finite or, equivalently, if there are integers $n \ge 0$ and $p \ge 1$ such that $\varphi^{(n)}(x) = \varphi^{(n+p)}(x)$.

When X is an Abelian variety and φ is the multiplication by an integer $d \ge 2$, preperiodic points are exactly the torsion points of X. One direction is clear: if x has finite order, say m, then every multiple of x is killed by the multiplication by m; since there are finitely many points $a \in X(\overline{F})$ such that $[m]a = 0$, the orbit of x is finite. Conversely, if the orbit of x is finite, let n and $p \ge 1$ be integers such that $\varphi^{(n)}(x) = \varphi^{(n+p)}(x)$; this implies $[d^n]x = [d^{n+p}]x$, hence $[d^n(d^p - 1)]x = 0$, so that x is a torsion point.

The canonical height of a preperiodic point must be zero. Indeed, let x be preperiodic and let n and $p \ge 1$ be integers such that $\varphi^{(n)}(x) = \varphi^{(n+p)}(x)$. Computing the canonical height of both sides of the equality, we get $q^n\hat{h}(x) = q^{n+p}\hat{h}(x)$, hence $\hat{h}(x) = 0$ since $q > 1$, so that $q^n \ne q^{n+p}$.

4.2. Points of canonical height zero. If F is a number field, or a function field over a finite field, the converse holds. Let $x \in X(\overline{F})$ be such that $\hat{h}(x) = 0$. Let E be a finite extension of F such that $x \in X(E)$. Any point in the orbit of x

has canonical height zero; by Northcott's finiteness theorem, the orbit of x is finite. In fact, this statement was the main result of Northcott [**29**]!

However, if F is a function field over an algebraically closed field k, Northcott's finiteness theorem is false and this property does not hold anymore. Indeed, we can consider a "constant" dynamical system (Y, M, ψ) defined over k and view it over F. Then, all points in $Y(\bar{k})$ have canonical height zero, unless k is the algebraic closure of a finite field, they are usually not preperiodic.

In fact, a theorem of Chatzidakis and Hrushovski [**11**] shows that this obstruction is essentially the only one. This generalizes an old result of Lang and Néron [**28**] for Abelian varieties. Because it is simpler to quote, let us only give the particular case due to [**2**]. *Let* $X = \mathbf{P}^1$ *and* $\varphi \in F(t)$ *be a rational function of degree* $q \geq 2$*, then there exists a nonpreperiodic point* $x \in X(\overline{F})$ *such that* $\hat{h}(x) = 0$ *if and only* φ *is conjugate (by a homography in* $\overline{F}$*) to a rational function* $\psi \in k(t)$*.*

4.3. The geometry of points of canonical height zero. In the 60s, motivated by the conjecture of Mordell and its extension by Lang, Manin and Mumford had conjectured that an integral subvariety of a complex Abelian variety cannot contain a dense set of torsion points, unless the subvariety is the translate of an Abelian subvariety by a torsion point.

This expectation proved to be a theorem, due to Raynaud [**31**] (there are many other proofs now). We quote it in the slightly different, but equivalent, form: *Assume that* X *is an Abelian variety over an algebraically closed field of characteristic zero and let* Y *be a closed subvariety of* X*. Then the Zariski closure of* $Y(0)$ *is a finite union of translates of Abelian subvarieties by torsion points.* In fact, the proof relies on techniques from arithmetic geometry, and its crucial part assumes that X and Y are defined over a number field.

Similarly, the study of the analogue of Manin and Mumford's question over algebraically closed fields of positive characteristic would reduce to the case of function fields. But there, eventual constant Abelian varieties within X create difficulties. The precise theorem has been first proved by Scanlon [**33, 34**]. Before the author states it, let him recall the existence of the *Chow trace* of X, a "maximal" Abelian variety X' defined over k together with a morphism $X' \otimes_k F \to X$. Then, *if* Y *is a subvariety of* X*, the Zariski closure of* $Y(0)$ *is a finite union of varieties* Z *such that the quotient of* Z *by its stabilizer* G_Z *is a translate of a subvariety of* $(X/G_Z)' \otimes_k F$ *defined over* k *by a torsion point of* X/G_Z*.*

4.4. In the context of dynamical systems, the question of Manin and Mumford generalizes as follows. Let Y be a subvariety of X and let $Y(0)$ be the set of points $x \in X(\bar{K})$ such that $\hat{h}(x) = 0$. Is it true that $Y(0)$ is not dense in Y, unless this is somewhat explained by the geometry of Y with respect to φ, for example, unless Y is itself preperiodic? However, the answer to this basic expectation was proved to be *false*, by an example of Ghioca and Tucker. A subsequent paper by Ghioca, Tucker, and Zhang [**21**] tries to correct the basic prediction.

4.5. The conjecture of Bogomolov. Still in conjunction with Mordell's conjecture, Bogomolov [**7**] had strengthtend Manin – Mumford's question by requiring to prove that over number field, if C is a curve over genus ≥ 2 embedded in its Jacobian J, there exists a positive real number ε such that $C(\varepsilon) = \{x \in C(\overline{F}) ; \hat{h}(x) \leq \varepsilon\}$ is finite.

This conjecture has been generalized by Zhang [**40**] to subvarieties of Abelian varieties over a number field: if Y is a subvariety of an Abelian variety X, does there exists a positive real number ε such that $Y(\varepsilon)$ is contained in a finite union of translates of Abelian subvarieties of X by torsion points, contained in Y? In other words, is it true that $Y(\varepsilon) \subset \overline{Y(0)}$ for small enough ε?

These two questions have been solved positively by Ullmo [**37**] and Zhang [**41**] respectively. They make a crucial use of equidistribution arguments that will be explained below. Soon after, David and Philippon [**15**] gave another proof.

The analogous case of function fields is mostly open, the last part of this text will be devoted to explaining how Gubler had been able to use ideas of equidistribution to prove important cases in this setting. Note that over a function field of characteristic zero, the case of a curve embedded in its Jacobian (Bogomolov's original on) has been settled positively by Faber [**17**] and Cinkir [**12**] using a formula of Zhang [**42**] for the height of the Gross – Schoen cycle.

5. Equidistribution (arithmetic case)

5.1. Equidistribution is a prevalent theme of analytic number theory: it is a (partially) quantitative way of describing how discrete objects collectively model a continuous phenomenon. The most famous result is probably the equidistribution modulo 1 of multiples $n\alpha$ of some fixed irrational number α, due to Weyl.

Here, we are interested in algebraic points $x \in X(\overline{F})$ of a variety X defined over F. To have some chance of getting some continuous phenomenon, we consider, not only the points themselves, but also their conjugates, that is, the full orbit of those points under the Galois group $\operatorname{Gal}(\overline{F}/F)$. The continuous phenomenon requires some topology, so we fix a place v of F and an embedding of $\overline{F}$ into the field $\mathbf{C}_v$.

Let x be any point in $X(\overline{F})$. Viewed from the field F, the point x is not discernible from any of its conjugates $x_1 = x, \dots, x_m$ which are obtained from x by letting the group of F-automorphisms of $\overline{F}$ act. So we define a probability measure $\mu(x)$ on $X(\mathbf{C}_v)$ by

$$\mu(x) = \frac{1}{m} \sum_{j=1}^{m} \delta_{x_j},$$

where δ_{x_j} is the Dirac measure at the point $x_j \in X(\overline{F}) \subset X(\mathbf{C}_v)$.

The first equidistribution result in this field is the following.

Theorem 5.2 (Szpiro, Ullmo, and Zhang [**36**]). *Assume that F is a number field and v is an Archimedean place of F. Let X be an Abelian variety over F and let (x_n) be a sequence of points in $X(\overline{F})$ satisfying the following two assumptions:*

- *The Néron – Tate height of x_n goes to 0 when $n \to \infty$;*
- *For any subvariety Y of X such that $Y \neq X$, the set of indices n such that $x_n \in Y$ is finite.*

Then the sequence of probability measures $\big(\mu(x_n)\big)$ on the complex torus $X(\mathbf{C}_v)$ converges vaguely to the normalized Haar measure of $X(\mathbf{C}_v)$.

The proof uses Arakelov geometry and holds in a wider context than that of Abelian varieties. We shall see more about it shortly but I would like to describe how Ullmo [**37**] and Zhang [**41**] used those ideas to obtain a *proof* of Bogomolov's conjecture.

5.3. So assume that Y is a subvariety of X, with $Y \neq X$, containing a sequence (x_n) of algebraic points such that $\hat{h}(x_n) \to 0$ and which is dense in Y for the Zariski topology. We want to show that Y is a translate of an Abelian subvariety of X by a torsion point. To that aim, we may mod out X and Y by the stabilizer of Y. The definition of the Néron – Tate height on X comes from some ample line bundle; its Riemann form on $X(\mathbf{C}_v)$ is a positive differential form ω of bidegree $(1, 1)$.

Now, a geometric result implies that there exists a positive integer m such that the map

$$\varphi \colon Y^m \to X^{m-1}, \quad (y_1, \dots, y_m) \mapsto (y_2 - y_1, \dots, y_m - y_{m-1})$$

is generically finite. From the sequence (x_n), one constructs a similar sequence (y_n) of points in $Y^m(\overline{F})$ whose heights converge to zero and which are Zariski dense in Y^m; more precisely, for any subvariety Z of Y^m such that $Z \neq Y^m$, the set of indices n such that $y_n \in Z(\overline{F})$ is finite. A variant of Theorem 5.2 (see also Theorem 5.6 below) implies that the sequence $\mu(y_n)$ of probability measures converges to the canonical probability measure on $Y^m(\mathbf{C}_v)$ given by the differential form $(\omega_1 + \dots + \omega_m)^{md}$ on the smooth locus of Y^m. (Here, $d = \dim(Y)$ and ω_j means the differential form on Y^m coming from ω on the jth factor of Y^m.) Write $\mu(Y^m)$ for this measure; in fact, one has $\mu(Y^m) = \mu(Y)^m$. So we have the equidistribution property

$$\mu(y_n) \to \mu(Y)^m.$$

If we apply the map φ, we get automatically

$$\mu(\varphi(y_n)) \to \varphi_* \mu(Y)^m.$$

On the other hand, the sequence $\big(\varphi(y_n)\big)$ also satisfies an equidistribution property, but the limit measure being $\mu\big(\varphi(Y^m)\big)$. This implies an *equality* of probability measures

$$\varphi_* \mu(Y)^m = \mu\big(\varphi(Y^m))\big),$$

a geometric refinement of the initial fact that φ is generically finite with image $\varphi(Y^m)$.

However, both sides of this equality come from differential forms, and this equality implies that the differential forms $(\omega_1 + \dots + \omega_m)^{md}$ and $\varphi^*(\omega_1 + \dots + \omega_{m-1})^{md}$ on Y^m coincide up to a constant multiple.

The contradiction comes from the fact that $(\omega_1 + \dots + \omega_m)$ is strictly positive everywhere (at least, on the smooth locus of Y^m) while $\varphi^*(\omega_1 + \dots + \omega_{m-1})^{md}$ vanishes where φ is not étale (that is, not a local diffeomorphism), in particular on the diagonal of Y^m. To be resolved, this contradiction requires that $md = 0$, hence that Y is a point, necessarily a torsion point.

5.4. Heights for subvarieties. One of the major ingredients in the proof of the equidistribution theorem is a notion of a height with respect to a metrized line bundle, not only for points, but for all subvarieties. The definition, first introduced by Faltings [**20**], goes as follows.

We assume that X is a projective variety over a number field F and L is a line bundle on X. Let $\mathscr{X}$ be a projective flat model over the ring $\mathfrak{o}_F$ and $\mathscr{L}$ be line bundle on $\mathscr{X}$ which is a model of L. We also assume that L is endowed with smooth Hermitian metrics at all Archimedean places of F.

From these metrics, complex differential geometry defines differential forms $c_1(\bar{L}_v)$ on the complex analytic varieties $X(\mathbf{C}_v)$, for all Archimedean places v of F.

This form is called the *first Chern form*, or the *curvature form*, of the Hermitian line bundle $\bar{L}_v$; it is a representative of the first Chern class of L in the de Rham cohomology of $X(\mathbf{C}_v)$. It is really a fundamental tool in complex algebraic geometry. For example, when X is smooth, say, the Kodaira embedding theorem asserts that L is ample if and only if it possesses a Hermitian metric such that its curvature form is positive definite on each tangent space of X. I refer to §1.4 of [**22**] for more details.

Faltings's definition of the height of an irreducible closed subvariety $\mathscr{Y} \subset \mathscr{X}$ is by induction on its dimension.

If $\dim(\mathscr{Y}) = 0$, then $\mathscr{Y}$ is a closed point; then, its residue field $\kappa(\mathscr{Y})$ is a finite field and one defines

$$h_{\overline{\mathscr{L}}}(\mathscr{Y}) = \log \operatorname{Card}\bigl(\kappa(\mathscr{Y})\bigr). \tag{5.1}$$

Otherwise, one can consider a (nonzero) meromorphic section s of some power $\mathscr{L}^m$ of $\mathscr{L}$ on $\mathscr{Y}$. Its divisor $\operatorname{div}(s)$ is a formal linear combination of irreducible closed subschemes $\mathscr{Z}_j$ of $\mathscr{Y}$, with multiplicities a_j (the order of vanishing, or minus the order of the pole of s along $\mathscr{Z}_j$) and

$$h_{\overline{\mathscr{L}}}(\mathscr{Y}) = \frac{1}{m} \sum a_j h_{\overline{\mathscr{L}}}(\mathscr{Z}_j) + \sum_v \int_{Y(\mathbf{C}_v)} \log \|s\|^{-1/m} c_1(\bar{L}_v)^{\dim Y} \tag{5.2}$$

where $Y = \mathscr{Y} \otimes F$ and v runs over Archimedean places of F. In fact, the right-hand side of this formula does not depend on the choice of s.

One can prove that this new definition recovers the previous one for points. More precisely, let $y \in X(\overline{F})$, let $Y \in X$ be the corresponding closed point and let $\mathscr{Y}$ be its Zariski closure in $\mathscr{X}$. Then,

$$h_{\overline{\mathscr{L}}}(\mathscr{Y}) = \deg(Y) h(y),$$

where $\deg(Y)$ is the degree of the closed point Y, or the degree of Y as a subvariety of X with respect to the line bundle L.

Proposition 5.5 (Zhang [**40**]). *Assume that L is ample, that $\mathscr{L}$ is relatively numerically effective and that the curvature forms $c_1(\bar{L}_v)$ are nonnegative for any Archimedean place v of F.*

Let (x_n) be a sequence of points in $X(\overline{F})$. Assume that for any subvariety Y of X such that $Y \neq X$, the set of indices n such that $x_n \in Y$ is finite. Then,

$$\liminf_n h_{\overline{\mathscr{L}}}(x_n) \geq \frac{h_{\overline{\mathscr{L}}}(\mathscr{X})}{\dim(\mathscr{X}) \deg_L(X)}. \tag{5.3}$$

This proposition follows easily from a (difficult) theorem in Arakelov geometry that implies the existence of global sections over $\mathscr{X}$ of large powers $\mathscr{L}^m$ which have controlled norms. Using those sections in the inductive definition of the height leads readily to the indicated inequality.

In presence of a sequence (x_n) such that $h_{\overline{\mathscr{L}}}(x_n)$ converges to the right-hand side of Inequality 5.3, Szpiro, Ullmo, and Zhang [**36**] proved that the probability measures $\mu(x_n)$ equidistribute towards the measure $\mu_{X,v} = c_1(\bar{L}_v)^{\dim(X)} / \deg_L(X)$ on $X(\mathbf{C}_v)$. The heart of the proof is to apply the fundamental inequality (5.3) for small perturbations of the Hermitian metrics, as a *variational principle.* Since $X(\mathbf{C}_v)$ is compact and metrizable, the space of probability measures on $X(\mathbf{C}_v)$ is metrizable and compact, so we may assume that $\mu(x_n)$ converges to some limit μ and we need to prove that μ is proportional to $c_1(\bar{L}_v)^{\dim X}$.

Let us multiply the metric on $\bar{L}_v$ by some function of the form $\mathrm{e}^{-\varepsilon\varphi}$, where φ is a smooth function on $X(\mathbf{C}_v)$. Then, the left-hand side of the inequality (5.3) becomes

$$\lim_n \left(h_{\overline{\mathscr{L}}}(x_n) + \varepsilon \int_{X(\mathbf{C}_v)} \varphi \, \mathrm{d}\mu(x_n) \right) = \frac{h_{\overline{\mathscr{L}}}(\mathscr{X})}{\dim(\mathscr{X}) \deg_L(X)} + \varepsilon \int_{X(\mathbf{C}_v)} \varphi \, \mathrm{d}\mu,$$

while its right-hand side is

$$\frac{h_{\overline{\mathscr{L}}}(\mathscr{X})}{\dim(\mathscr{X}) \deg_L(X)} + \varepsilon \int_{X(\mathbf{C}_v)} \varphi \frac{c_1(\bar{L}_v)^{\dim X}}{\deg_L(X)} + \mathrm{O}(\varepsilon^2).$$

Consequently, when $\varepsilon \to 0$,

$$\varepsilon\big(\mu(\varphi) - \mu_{X,v}(\varphi)\big) \geq \mathrm{O}(\varepsilon^2).$$

For small positive ε, we get $\mu(\varphi) \geq \mu_{X,v}(\varphi)$, and we have the opposite inequality for small negative ε. Consequently, $\mu(\varphi) = \mu_{X,v}(\varphi)$, hence the equality $\mu = \mu_{X,v}$.

A subtle point of the proof lies in the possibility of applying Proposition 5.5 to the modified line bundle. When the curvature form $c_1(\overline{L})$ is strictly positive, then it remains so for small perturbations, hence the proof is legitimate. This is what happens in Theorem 5.2, and what is needed for the proof of Bogomolov's conjecture by Ullmo and Zhang.

Inspired by an inequality of Siu and the holomorphic Morse inequalities of Demailly, Yuan [**39**] proved the following general equidistribution theorem.

THEOREM 5.6 (Yuan [**39**]). *Assume that F is a number field and v is an Archimedean place of F. Let X be an algebraic variety over F, let $\bar{L}$ be an ample line bundle on X with a semi-positive adelic metric. Let (x_n) be a sequence of points in $X(\overline{F})$ satisfying the following two assumptions*:

- *The heights of x_n with respect to $\bar{L}$ converge to 0 when $n \to \infty$;*
- *For any subvariety Y of X such that $Y \neq X$, the set of indices n such that $x_n \in Y$ is finite.*

Then the sequence of probability measures $(\mu(x_n))$ on the complex space $X(\mathbf{C}_v)$ converges to the unique probability measure proportional to $c_1(\bar{L})^{\dim(X)}$.

I cannot say much more on this here, and I must refer the reader to the paper of Yuan [**39**].

Observe anyway that under the indicated hypotheses, $c_1(\bar{L})$ is not necessarily a differential form, but only a positive current of bidegree $(1,1)$. Consequently, the definition of the measure $c_1(\bar{L})^{\dim X}$ requires some work. It goes back to fundamental work in pluripotential theory by Bedford and Taylor [**4**] and Demailly [**16**]. In our setting, it can be defined by an approximation process, considering sequences of smooth *positive* Hermitian metrics on L which converge uniformly to the initial metric. See my survey [**9**] for more details.

6. Measures on analytic spaces

6.1. Our setting is that of a global field F. Let X be a projective algebraic (irreducible) variety over F. For any place v of F, we will consider the analytic space X_v^{an} associated to v.

If v is Archimedean, then $X_v^{\mathrm{an}} = X(\mathbf{C}_v)$ is the set of complex points of X, where $\mathbf{C}_v = \mathbf{C}$ is viewed as an F-algebra via the embedding corresponding to the place v.

When v is non-Archimedean, then X_v^{an} is the analytic space over the complete algebraically closed field $\mathbf{C}_v$, as defined by Berkovich [**5**]. I must refer to the other contributions in this volume for background on Berkovich spaces, as well as to those of Baker [**1**] and Conrad [**13**] in the proceedings of the 2007 Arizona conference edited by Savitt and Thakur. Here, I will content myself with the few following comments. First of all, X_v^{an} is a reasonable topological space: it is compact and locally pathwise connected. It is even locally contractible; this is the main theorem of Berkovich [**6**] when X is smooth, recently extended to the general case by Hrushovski and Loeser [**27**]. Moreover, X_v^{an} contains the set $X(\mathbf{C}_v)$ as a dense subset, and the topology of X_v^{an} restricts to its natural (totally disconnected) topology on $X(\mathbf{C}_v)$. So X_v^{an} has many other points than those of $X(\mathbf{C}_v)$, some which will play a crucial role below.

6.2. To fix ideas, assume that we are in the geometric case, so that $F = k(C)$ is the field of functions on a curve C. Let $\mathscr{X}$ be a projective model of X over the curve C and let $\mathscr{L}$ be a line bundle on $\mathscr{X}$; let L be its restriction to X.

For any place v of F, the model $\mathscr{L}$ gives rise to a "v-adic metric on L." This notion is similar to what was discuted in the case of $X(\mathbf{C}_v)$; in particular, any section s of L on an open subset U of X has a norm $\|s\|$ which is a continuous function on the corresponding subset U_v^{an} of X_v^{an}, and does not vanish if s does not vanish. The construction of the metric is also similar to that of standard metrics. Assume first that $\mathscr{L}$ be very ample; then, the metric on L is the unique metric such that for any generating set $(s_1, \dots, s_n)$ of the module $\Gamma(\mathscr{X}, \mathscr{L})$ of integral global sections, one has

$$\max(\|s_1\|(x), \dots, \|s_n\|(x)) = 1 \quad \text{for any } x \in X_v^{\mathrm{an}}.$$

In general, one can at least write $\mathscr{L}$ as the difference $\mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$ of two very ample line bundles on $\mathscr{X}$, and the metric on L is the quotient of the to metrics given by the models $\mathscr{L}_1$ and $\mathscr{L}_2$.

I claim that there exists a measure, written $c_1(\bar{L})_v^{\dim X}$, on X_v^{an} such that for any nonzero global section of L,

$$h_{\mathscr{L}}(\mathscr{X}) = h_{\mathscr{L}}\big(\overline{\operatorname{div}(s)}\big) + \sum_v \int_{X_v^{\mathrm{an}}} \log\|s\|_v^{-1} c_1(\bar{L})_v^{\dim X},$$

where $\overline{\operatorname{div}(s)}$ is the Zariski closure in $\mathscr{X}$ of the divisor of s. More generally, if Y is an integral subvariety of X, with Zariski closure $\mathscr{Y}$ in $\mathscr{X}$, and if s is any nonzero global section of $L|_Y$, one can define a measure $c_1(\bar{L})_v^{\dim Y} \delta_Y$ on Y_v^{an} such that

$$h_{\mathscr{L}}(\mathscr{Y}) = h_{\mathscr{L}}\big(\overline{\operatorname{div}(s)}\big) + \sum_v \int_{X_v^{\mathrm{an}}} \log\|s\|_v^{-1} c_1(\bar{L})_v^{\dim Y} \delta_Y.$$

This measure is defined as follows, see [**8**]; the presentation given here, using algebraically closed valued fields, is due to Gubler [**24**].

The analytic space $\mathscr{X}_v$ admits a canonical reduction X_v over the residue field of $\mathbf{C}_v$, which maps to the natural reduction of $\mathscr{X}$. Moreover, there is a reduction map $X_v^{\mathrm{an}} \to \mathsf{X}_v$ and the generic point of an irreducible component Z of X_v is the image of a unique point z of X_v^{an}. By functoriality, one also has a line bundle L_v on $\mathscr{X}_v$. The measure $c_1(\bar{L})_v^{\dim X}$ is the following linear combination of Dirac measures:

$$c_1(\bar{L})_v^{\dim X} = \sum_{\mathsf{Z}} (c_1(\mathsf{L}_v)^{\dim X} | \mathsf{Z}) \delta_z$$

where the coefficients $(c_1(\mathsf{L}_v)^{\dim X} | \mathsf{Z})$ are given by usual (numerical) intersection theory and the sum is over the irreducible components Z of X_v.

This measure is positive if $\mathscr{L}$ is relatively ample, and its total mass is equal to the degree of X with respect to L.

The definition of the measure $c_1(\bar{L})_v^{\dim Y} \delta_Y$ is analogous.

Up to its measure-theoretic formulation, the validity of the asserted formula for heights follows from work of Gubler [**23**].

6.3. Zhang [**40**] had defined a notion of semipositive metrics, which are defined as uniform limits of metrics given by models $(\mathscr{X}, \mathscr{L})$, where $\mathscr{L}$ is relatively numerically effective — that is, gives a nonnegative degree to any vertical subvariety. He also showed that semipositive metrized line bundles allow to define heights of subvarieties by approximation from the case of models/classical Arakelov geometry.

Adapting this construction I defined in [**8**] the measures $c_1(\bar{L})_v^{\dim X}$ by approximation from the above definition in the case of models. In the end, the proof is very close to that of the existence of products of positive $(1,1)$-currents by Bedford and Taylor [**4**]. (In fact, this article only considers projective varieties over a local p-adic field; the general case has been treated by Gubler [**24**], in a similar fashion.)

6.4. As we have shown in [**10**], these measures can be used to recover the heights defined by Zhang [**40**]. Namely, if $\bar{L}$ is a line bundle on X with a semipositive adelic metric, Y is an integral subvariety of X, Zhang defined the height $h_{\bar{L}}(Y)$ of Y with respect to $\bar{L}$. For any regular meromorphic section s of $L|_Y$, one has

$$h_{\bar{L}}(Y) = h_{\bar{L}}(\operatorname{div}(s)) + \sum_v \int_{X_v^{\mathrm{an}}} \log\|s\|_v^{-1} c_1(\bar{L})_v^{\dim Y} \delta_Y.$$

In the case of curves, and a few cases in higher dimensions, I showed in [**8**] that they also give rise to equidistribution theorems totally analogous to the one of Spiro, Ullmo, and Zhang [**36**]. The article of Yuan [**39**] proved what can be considered the most general equidistribution theorem possible in this context. Namely, the non-Archimedean analogue of Theorem 5.6 still holds. While that paper restricts to the case of number fields, its ideas have been transposed to the case of function fields by Faber [**18**] and Gubler [**26**].

I also refer to [**30**] for a general discussion of convergence of measures on the Berkovich projective space, as well as for a non-Archimedean analogue of Weyl's equidistribution criterion.

6.5. In some cases, one can deduce from these equidistribution theorems explicit results in algebraic number theory. Let us give an example in the case of the line bundle $L = \mathscr{O}(1)$ on the projective line $X = \mathbf{P}^1$, with its metrization giving rise to the standard height. Fix an ultrametric place v of F. Then, the measure $c_1(\bar{L})_v$ on X_v^{an} is the Dirac measure at a particular point γ, called the Gauss-point because it corresponds to the Gauss-norm on the algebra $F_v[T]$ (viewed as the algebra of functions on the affine line $\mathbf{A}^1 = \mathbf{P}^1 \setminus \{\infty\}$). So in this case, the equidistribution theorem asserts that for any sequence (x_n) of distinct points on $X(\overline{F})$ such that $h(x_n) \to 0$, the measures $\mu(x_n)$ on X_v^{an} converge to the Dirac measure δ_γ.

This gives a strong constraint on such sequences. For example, it is impossible that all x_n be totally v-adic (an algebraic point is "totally v-adic" if all of its conjugates are defined over F_v). Indeed, if x_n is totally v-adic, then the measure $\mu(x_n)$ is supported by the compact subset $X(F_v)$ of X_v^{an}. If all x_n were totally v-adic, the limit measure of $\mu(x_n)$ would be supported by $X(F_v)$, but the Gauss-point does *not* belong to $X(F_v)$. Similar results were proved by Baker et Hsia [**3**].

7. Bogomolov's conjecture for totally degenerate abelian varieties

7.1. Gubler [**25**] had the idea of using these measures to attack the unsolved Bogomolov conjecture over function fields, using equidistribution theorems for points of small height at some place of the ground field to get a proof of the conjecture following the strategy of Ullmo [**37**] and Zhang [**41**].

So let F be a function field and let v be a place of F. Let X be an Abelian variety over F, let $\bar{L}$ be an ample symmetric line bundle on X with its canonical adelic metric that gives rise to the Néron-Tate height $\hat{h}$. Let Y be a closed integral subvariety of X which is not the translate of an Abelian subvariety by a torsion point. We want to prove that for some positive ε, $Y(\varepsilon)$ is not Zariski-dense. Assume the contrary. We would then want to construct a sequence (y_n) in $Y(\overline{F})$ satisfying the assumptions of the equidistribution theorem, namely $\hat{h}(y_n) \to 0$ and for any subvariety Z of Y such that $Z \neq Y$, the set of integers n such that $y_n \in Z$ is finite. However, the set of subvarieties of Y may be uncountable, hence such a sequence may not exist. Anyway, one can construct a *net* (y_n), that is a family of points indexed by a filtered ordered set N, such that $h(y_n) \to 0$ and for any subvariety $Z \subsetneq Y$, the set of indices n such that $y_n \in Z$ is bounded in N. The statement and the proof of the equidistribution theorem adapt readily to this case.

We redo the same geometric reduction, assuming that the stabilizer of Y is trivial, and that the morphism $\varphi \colon Y^m \to X^{m-1}$ given by $(y_1, \dots, u_m) \mapsto (y_2 - y_1, \dots, y_m - y_{m-1})$ is generically finite, with image Z. As above, we construct a generic net (y_n) of small points in Y^m whose image $\big(\varphi(y_n)\big)$ s a generic net of small points in Z. This gives two equidistribution theorems in the Berkovich spaces $(Y^m)_v^{\text{an}}$ and Z_v^{an} at the chosen place v, with respect to canonical measures $\mu_v(Y^m) = c_1(\bar{L}|_{Y^m})_v^{m \dim Y}$ and $\mu_v(Z) = c_1(\bar{L}|_Z)^{\dim Z}$, where we write $\bar{L}|_{Y^m}$ and $\bar{L}|_Z$ for the metrized line bundles on Y^m and Z deduced from those naturally given by $\bar{L}$ on X^m and X^{m-1}. By construction, $\varphi_* \mu_v(Y^m) = \mu_v(Z)$.

To get a contradiction, we need to have more information about these measures.

7.2. If X has good reduction at v, the very definition of the measure $\mu_v(X)$ shows that it is the Dirac measure at a single point of X_v^{an}. Indeed, let $\mathscr{X}$ be the Néron model of X over the ring of integers $\mathfrak{o}_v$ of F_v; since X has good reduction, $\mathscr{X}$ is proper and smooth, and its special fiber is an Abelian variety. Then, one can show that the generic point of this fiber has a unique preimage ξ under the reduction map from the Berkovich space X_v^{an} to the special fiber. One has $\mu_v(X) = \deg_L(X)\delta_\xi$.

In the case where all of X, Y and Z have good reduction at v (this happens for almost all places v), the measures $\mu_v(Y^m)$ and $\mu_v(Z)$ are supported at a single point and the equality of measures $\varphi_* \mu_v(Y^m) = \mu_v(Z)$ gives no contradiction.

Also, if X has good reduction, the measures $\mu_v(Y^m)$ and $\mu_v(Z)$ will be supported at finitely many points and it will still be difficult to draw a contradiction.

7.3. Consequently, to succeed, this equidistribution approach needs to consider places of bad reduction of X. The case treated by Gubler [**25**] is the one of *totally degenerate Abelian varieties*, those being as far as possible from Abelian varieties of good reduction. Recall that any Abelian variety over F has a canonical model over the local ring $\mathfrak{o}_{F,v}$ at the place v, called its Néron model. Possibly after a finite extension of the ground field, the connected component of the identity of the special fiber of the Néron model is an extension of torus $\mathbf{G}_{\mathrm{m}}^a$ by an Abelian variety; totally degenerate Abelian varieties are those for which this torus has dimension $\dim(X)$.

Assume this is the case and set $g = \dim(X)$. Possibly after some finite extension of F, By theorems of Tate, Raynaud, Bosch, Lütkebohmert in Tate's setting of rigid analytic spaces, extended to the Berkovich context in [**5**, §6.5], the analytic space X_v^{an} associated to the Abelian variety X can be written as the quotient of a torus $\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}}$ by a discrete subgroup Ω of rank g in $\mathbf{G}_{\mathrm{m}}^g(F_v)$. In fact, the torus $\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}}$ is the universal cover of the Berkovich space X_v^{an}.

In particular, the topological fundamental group of the analytic space associated to our totally degenerate Abelian variety X is isomorphic to $\mathbf{Z}^g$. This does not reflect however the richness of étale covers of Abelian varieties — the fundamental group of a complex Abelian varieties of dimension g his $\mathbf{Z}^{2g}$, while the ℓ-adic fundamental group would be $\mathbf{Z}_\ell^{2g}$ (provided ℓ is distinct from the characteristic of the ground field). This indicates that, in some sense, the reduction at Archimedean places is at least twice as bad as the worst possible ultrametric places of bad reduction.

Here enters tropical geometry.

7.4. We first analyze the tropicalization of a torus. By definition, the Berkovich space of $\mathbf{G}_{\mathrm{m}}$ at the place v is the set of all multiplicative seminorms on the ring $F_v[T, T^{-1}]$ which extend the fixed absolute value on F_v. So there is a natural map from $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$ to the real line $\mathbf{R}$ that maps a semi-norm χ to the real number $-\log|\chi(T)|$. In fact, the semi-norm χ is viewed as a point x of $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$, and $|\chi(T)|$ is viewed as $|T(x)|$, so that a more natural way to write this map is $\tau\colon x \mapsto -\log|T(x)|$. An even more natural way would be to consider the map $x \mapsto |T(x)|$ from $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$ to $\mathbf{R}_+^*$, because it does not require the choice of a logarithm function.

This "tropicalization" map τ is continuous and surjective. It has a canonical section $\sigma\colon \mathbf{R} \to \mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$ for which $\sigma(t)$ is the Gauss-norm corresponding to the radius e^t:

$$|P(\sigma(t)| = \sup_{n\in\mathbf{Z}} |a_n| \mathrm{e}^{nt}, \quad \text{if } P = \sum a_n T^n.$$

This section σ is a homeomorphism onto its image $S(\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}})$ which is called the *skeleton* of $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$.

In higher dimensions, we have a similar coordinate-wise tropicalization map $\tau\colon \mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}} \to \mathbf{R}^g$ and a section σ whose image $S(\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}})$ is the skeleton of $\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}}$.

In the case of a uniformized totally degenerate Abelian variety, one can tropicalize its universal cover and mod out by the image of the lattice Ω. This gives a diagram:

$$\begin{tikzcd}
\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}} \arrow[r, "\tau"] \arrow[d] & \mathbf{R}^g \arrow[d] \\
X_v^{\mathrm{an}} \arrow[r, "\tau_X"] & \mathbf{R}^g/\Lambda
\end{tikzcd}$$

where $\Lambda = \tau(\Omega)$. Moreover, the section σ descends to a section σ_X of τ_X whose image $S(X_v^{\mathrm{an}})$ is called the skeleton of X_v^{an}. This is a real torus of dimension g in X_v^{an} onto which X_v^{an} retracts canonically.

The proof of the following two theorems is long and difficult and cannot be described here.

Theorem 7.5 ([**24**, Corollary 9.9]). *The canonical measure $c_1(\bar{L})_v^{\dim X}$ on X_v^{an} is the unique Haar measure supported by the real torus $S(X_v^{\mathrm{an}})$ of total mass $\deg_L(X)$.*

Theorem 7.6 ([**24**, Theorem 1.3]). *Let Y be an integral subvariety of X; let d be its dimension.*

The image $\tau_X(Y_v^{\mathrm{an}})$ is a union of simplices of $\mathbf{R}^g/\Lambda$ of dimension d.

Restricted to any of those simplices, the direct image $(\tau_X)_*(c_1(\bar{L}|_Y)_v^{\dim Y})$ on $\mathbf{R}^g/\Lambda$ of the canonical measure of Y is a positive multiple of the Lebesgue measure.

7.7. Given the last two theorems, Gubler [**25**] can complete the proof of the Bogomolov conjecture when the given Abelian variety has totally degenerate reduction at the place v.

Indeed, in the above situation of a generically finite map $\varphi\colon Y^m \to W \subset X^{m-1}$, one can push the equality of measures $\varphi_*\mu_v(Y^m) = \mu_v(W)$ to the tropicalization $(\mathbf{R}^g/\Lambda)^{m-1}$. Let $\nu_Y = (\tau_X)_*(\mu_v(Y))$, $\nu_W = (\tau_{X^{m-1}})_*\big(\mu_v(W)\big)$; these are measures on $(\mathbf{R}^g/\Lambda)$ and $(\mathbf{R}^g/\Lambda)^{m-1}$ respectively. Let ψ be the map $(\mathbf{R}^g/\Lambda)^m \to (\mathbf{R}^g/\Lambda)^{m-1}$ given by $(a_1,\dots,a_m) \mapsto (a_2-a_1,\dots,a_m-a_{m-1})$. By naturality of tropicalization, one has $\tau\circ\varphi = \psi\circ\tau$, hence $\psi_*(\nu_Y^m) = \nu_W$.

Let δ be a simplex of dimension $\dim(Y)$ appearing in $\tau_X(Y)$. By Theorem 7.6, the restriction of the measure ν_Y to δ is a positive multiple of the Lebesgue measure. In particular, $\nu_Y(\delta) > 0$. Then δ^m is a simplex of $\tau_{X^m}(Y^m)$ whose image by ψ is $\psi(\delta^m)$. However, the definition of ψ shows that $\psi(\delta^m)$ has dimension $\leq m\dim(Y) - \dim(Y) < m\dim(Y) = \dim(W)$. Indeed, ψ is linear and the simplex δ embedded diagonally into δ^m maps to 0. By Theorem 7.6, ν_W is a sum of Lebesgue measures of $\dim(W)$-dimensional simplices, so that $\nu_W(\psi(\delta^m)) = 0$. Since $\psi_*(\nu_Y^m) = \nu_W$, it follows that $\nu_Y(\delta) = 0$. This contradiction concludes Gubler's proof of the Bogomolov conjecture when there is a place of totally degenerate reduction.

7.8. In our discussion of Manin−Mumford's conjecture over function fields, it was necessary to take care of constant Abelian subvarieties. They do not appear in Gubler's statement. Indeed, if an Abelian variety has totally degenerate reduction at some place, it cannot contain any constant Abelian subvariety. However, a general treatment of Bogomolov's conjecture over function fields would take them into account. A precise statement is given in the paper by Yamaki [**38**], with partial generalizations of Gubler's result to cases where there is bad reduction, although not totally degenerate.

Acknowledgements

The author thanks Matt Baker, Xander Faber, and Joe Rabinoff for their comments during the talk, and for having sent me their notes. The author especially thanks Xander Faber for his efficient cheerleading, without which this survey would probably never have appeared. The author also thanks the two referees for their many valuable suggestions.

At the time of the conference, the author was affiliated with Université de Rennes 1 and a member of Institut universitaire de France.

References

[1] M. Baker, *An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves*, p-adic geometry, Univ. Lecture Ser., vol. 45, Amer. Math. Soc., Providence, RI, 2008, pp. 123–174. MR2482347 (2010g:14029)

[2] M. Baker, *A finiteness theorem for canonical heights attached to rational maps over function fields*, J. Reine Angew. Math. **626** (2009), 205–233, DOI 10.1515/CRELLE.2009.008. MR2492995 (2011c:14075)

[3] M. H. Baker and L.-C. Hsia, *Canonical heights, transfinite diameters, and polynomial dynamics*, J. Reine Angew. Math. **585** (2005), 61–92, DOI 10.1515/crll.2005.2005.585.61. MR2164622 (2006i:11071)

[4] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), no. 1-2, 1–40, DOI 10.1007/BF02392348. MR674165 (84d:32024)

[5] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. MR1070709 (91k:32038)

[6] V. G. Berkovich, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84, DOI 10.1007/s002220050323. MR1702143 (2000i:14028)

[7] F. A. Bogomolov, *Points of finite order on abelian varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 4, 782–804, 973 (Russian). MR587337 (81m:14031)

[8] A. Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235, DOI 10.1515/CRELLE.2006.049 (French). MR2244803 (2008b:14040)

[9] *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume II*, London Mathematical Society Lecture Note Series, vol. 384, Cambridge University Press, Cambridge, 2011. Edited by Raf Cluckers, Johannes Nicaise and Julien Sebag. MR2905861

[10] A. Chambert-Loir and A. Thuillier, *Mesures de Mahler et équidistribution logarithmique*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 3, 977–1014 (French, with English and French summaries). MR2543659 (2011a:37189)

[11] Z. Chatzidakis and E. Hrushovski, *Difference fields and descent in algebraic dynamics. I*, J. Inst. Math. Jussieu **7** (2008), no. 4, 653–686, DOI 10.1017/S1474748008000273. MR2469450 (2009m:03051a)

[12] Z. Cinkir, *Zhang's conjecture and the effective Bogomolov conjecture over function fields*, Invent. Math. **183** (2011), no. 3, 517–562, DOI 10.1007/s00222-010-0282-7. MR2772087 (2012c:14057)

[13] B. Conrad, *Several approaches to non-Archimedean geometry*, p-adic geometry, Univ. Lecture Ser., vol. 45, Amer. Math. Soc., Providence, RI, 2008, pp. 9–63. MR2482345 (2011a:14047)

[14] C. D'Andrea, T. Krick, and M. Sombra, *Height of varieties in multiprojective spaces and arithmetic Nullstellensätze*, available at `arXiv:1103.4561`.

[15] S. David and P. Philippon, *Minorations des hauteurs normalisées des sous-variétés de variétés abéliennes*, Number theory (Tiruchirapalli, 1996), Contemp. Math., vol. 210, Amer. Math. Soc., Providence, RI, 1998, pp. 333–364, DOI 10.1090/conm/210/02795 (French, with English and French summaries). MR1478502 (98j:11044)

[16] J.-P. Demailly, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) **19** (1985), 124 (French, with English summary). MR813252 (87g:32030)

[17] X. W. C. Faber, *The geometric Bogomolov conjecture for curves of small genus*, Experiment. Math. **18** (2009), no. 3, 347–367. MR2555704 (2010k:11103)

[18] X. W. C. Faber, *Equidistribution of dynamically small subvarieties over the function field of a curve*, Acta Arith. **137** (2009), no. 4, 345–389, DOI 10.4064/aa137-4-4. MR2506588 (2010j:37149)

[19] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366, DOI 10.1007/BF01388432 (German). MR718935 (85g:11026a)

[20] G. Faltings, *Diophantine approximation on abelian varieties*, Ann. of Math. (2) **133** (1991), no. 3, 549–576, DOI 10.2307/2944319. MR1109353 (93d:11066)

[21] D. Ghioca, T. J. Tucker, and S. Zhang, *Towards a dynamical Manin-Mumford conjecture*, Int. Math. Res. Not. IMRN **22** (2011), 5109–5122, DOI 10.1093/imrn/rnq283. MR2854724 (2012i:37142)

[22] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics. MR507725 (80b:14001)

[23] W. Gubler, *Local heights of subvarieties over non-Archimedean fields*, J. Reine Angew. Math. **498** (1998), 61–113, DOI 10.1515/crll.1998.054. MR1629925 (99j:14022)

[24] W. Gubler, *Tropical varieties for non-Archimedean analytic spaces*, Invent. Math. **169** (2007), no. 2, 321–376, DOI 10.1007/s00222-007-0048-z. MR2318559 (2008k:14085)

[25] W. Gubler, *The Bogomolov conjecture for totally degenerate abelian varieties*, Invent. Math. **169** (2007), no. 2, 377–400, DOI 10.1007/s00222-007-0049-y. MR2318560 (2008f:14060)

[26] W. Gubler, *Equidistribution over function fields*, Manuscripta Math. **127** (2008), no. 4, 485–510, DOI 10.1007/s00229-008-0198-3. MR2457191 (2010a:14052)

[27] E. Hrushovski and F. Loeser, *Non-archimedean tame topology and stably dominated types*, available at `arXiv:1009.0252`.

[28] S. Lang and A. Néron, *Rational points of abelian varieties over function fields*, Amer. J. Math. **81** (1959), 95–118. MR0102520 (21 #1311)

[29] D. G. Northcott, *Periodic points on an algebraic variety*, Ann. of Math. (2) **51** (1950), 167–177. MR0034607 (11,615c)

[30] C. Petsche, *A criterion for weak convergence on Berkovich projective space*, Math. Ann. **348** (2010), no. 2, 449–465, DOI 10.1007/s00208-010-0485-7. MR2672310 (2011g:37252)

[31] M. Raynaud, *Sous-variétés d'une variété abélienne et points de torsion*, Arithmetic and geometry, Vol. I, Progr. Math., vol. 35, Birkhäuser Boston, Boston, MA, 1983, pp. 327–352 (French). MR717600 (85k:14022)

[32] M. Baker, B. Conrad, S. Dasgupta, K. S. Kedlaya, and J. Teitelbaum, *p-adic geometry*, University Lecture Series, vol. 45, American Mathematical Society, Providence, RI, 2008. Lectures from the 10th Arizona Winter School held at the University of Arizona, Tucson, AZ, March 10–14, 2007; Edited by David Savitt and Dinesh S. Thakur. MR2482343 (2010a:14001)

[33] T. Scanlon, *Diophantine geometry from model theory*, Bull. Symbolic Logic **7** (2001), no. 1, 37–57, DOI 10.2307/2687822. MR1836475 (2002e:03059)

[34] T. Scanlon, *A positive characteristic Manin-Mumford theorem*, Compos. Math. **141** (2005), no. 6, 1351–1364, DOI 10.1112/S0010437X05001879. MR2185637 (2006k:03065)

[35] J. H. Silverman, *Rational points on K3 surfaces: a new canonical height*, Invent. Math. **105** (1991), no. 2, 347–373, DOI 10.1007/BF01232270. MR1115546 (92k:14025)

[36] L. Szpiro, E. Ullmo, and S. Zhang, *Équirépartition des petits points*, Invent. Math. **127** (1997), no. 2, 337–347, DOI 10.1007/s002220050123 (French). MR1427622 (98i:14027)

[37] E. Ullmo, *Positivité et discrétion des points algébriques des courbes*, Ann. of Math. (2) **147** (1998), no. 1, 167–179, DOI 10.2307/120987 (French). MR1609514 (99e:14031)

[38] K. Yamaki, *Geometric Bogomolov conjecture for abelian varieties and some results for those with some degeneration* (*with an appendix by Walter Gubler*: *the minimal dimension of a canonical measure*), Manuscripta Math., to appear, DOI 10.1007/s00229-012-0599-1.

[39] X. Yuan, *Big line bundles over arithmetic varieties*, Invent. Math. **173** (2008), no. 3, 603–649, DOI 10.1007/s00222-008-0127-9. MR2425137 (2010b:14049)

[40] S. Zhang, *Small points and adelic metrics*, J. Algebraic Geom. **4** (1995), no. 2, 281–300. MR1311351 (96e:14025)

[41] S.-W. Zhang, *Equidistribution of small points on abelian varieties*, Ann. of Math. (2) **147** (1998), no. 1, 159–165, DOI 10.2307/120986. MR1609518 (99e:14032)

[42] S.-W. Zhang, *Gross-Schoen cycles and dualising sheaves*, Invent. Math. **179** (2010), no. 1, 1–73, DOI 10.1007/s00222-009-0209-3. MR2563759 (2012a:14016)

Département de Mathématiques, Bâtiment 425, Faculté des sciences d'Orsay, Université Paris-Sud 11, 91405 Orsay Cedex, France

E-mail address: `Antoine.Chambert-Loir@math.u-psud.fr`