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Points of Bounded Height on Equivariant Compactifications of Vector Groups, II

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We prove asymptotic formulas for the number of rational points of bounded height on certain equivariant compactifications of the affine space.

INTRODUCTION

In the past decade there has been much interest in establishing asymptotics for the number of points of bounded height on algebraic varieties defined over number field. Manin and Batyrev [1] have formulated conjectures describing such asymptotics in geometrical terms. These conjectures have been further refined by Peyre in [8].

More precisely, let X be a smooth projective algebraic variety defined over a number field F and $H\colon X(F) \to \mathbf{R}_{>0}$ an exponential height function on the set of rational points of X defined by some metrized ample line bundle $\mathscr{L}$. One wants to relate the asymptotic behaviour of the counting function

$$N(U, \mathscr{L}, B) = \#\{x \in U(F);\, H(x) \leqslant B\}$$

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to geometric invariants of X, such as the cone of effective line bundles and the (anti-) canonical line bundle of X. Here, U is a sufficiently small Zariski dense open subset; it is the complement of possible "accumulating subvarieties." If X is a Fano variety and $\mathscr{L} = K_X^{-1}$, one expects that

$$N(U, K_X^{-1}, B) \sim \frac{\Theta(X)}{(r-1)!} B(\log B)^{r-1}$$

where $r = \mathrm{rkPic}(X)$ and $\Theta(X)$ is the product of three numbers, a Tamagawa number which measures the volume of the closure of rational points in the adelic points $\overline{X(F)} \subset X(\mathbf{A}_F)$ with respect to the metrization, a rational number defined in terms of the cone of effective divisors, and the order of the non-trivial part of the Brauer group of X.

Such a description cannot hold universally (see the example by Batyrev and Tschinkel [2]), but there are two classes of algebraic varieties where it does hold, those for which the *circle method* in analytic number theory applies and those possessing many symmetries, such as an *action* (with a dense orbit) of a linear algebraic group. The circle method is concerned with complete intersections of small degree and small codimension in projective space. They have moduli, but only a few projective embeddings; the Picard group is $\mathbf{Z}$. As a reference, let us mention the papers by Birch [4] and Schmidt [9]. The other approach leads, via harmonic analysis on the adelic points of the corresponding group, to a proof of conjectured asymptotic formulas for toric varieties (see [3]) or for generalized flag varieties (using Langlands' work on Eisenstein's series, see [6]). These have Picard groups of higher ranks but no deformations due to the *rigidity* of reductive groups.

In this paper we treat certain equivariant compactifications of vector groups. In a previous paper [5] we had established asymptotic formulas for blowups of $\mathbf{P}^2$ in any number of points *on a line*. Here we work out the case of blow ups of a projective space $\mathbf{P}^n$ of dimension at least 3 in a smooth codimension 2 subvariety contained in a hyperplane. It should be clear to the reader that these varieties admit deformations (they are parametrized by an open subset of an appropriate Hilbert scheme).

More precisely, let $f \in \mathbf{Z}[x_1, ..., x_n]$ be a homogeneous polynomial of degree d and $X \to \mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[x_0, ..., x_n])$ the blowup of the ideal generated by (x_0, f). Suppose that the hypersurface defined by f in $\mathbf{P}^{n-1}_{\mathbf{C}}$ is smooth and let $U \simeq \mathbf{A}^n$ be the inverse image in X of $\mathbf{A}^n \subset \mathbf{P}^n$. Then, $\mathbf{X}_{\mathbf{C}}$ is a smooth projective variety, with Picard group $\mathbf{Z}^2$ and trivial Brauer group (recall that the Brauer group is a birational invariant for smooth projective varieties). Moreover, $X_{\mathbf{C}}$ is an equivariant compactification of

$\mathbf{G}_a^n$. There is a natural metrization on K_X^{-1} (recalled below) which allows us to define the height function and the *height zeta function*,

$$Z(U, K_X^{-1}, s) = \sum_{x \in U(\mathbf{Q})} H_{K_X^{-1}}(x)^{-s}.$$

The series converges absolutely for $\mathrm{Re}(s) \gg 0$. Our main theorem is:

THEOREM 1. *There exists a function h which is holomorphic in the domain $\mathrm{Re}(s) > 1 - 1/2n$ such that*

$$Z(U, K_X^{-1}, s) = \frac{h(s)}{(s-1)^2} \qquad \textit{and} \qquad h(1) = \Theta(X) \neq 0.$$

A standard Tauberian theorem implies that X satisfies Peyre's refinement of Manin's conjecture:

COROLLARY 2. *We have the asymptotic formula*

$$N(U, K_X^{-1}, B) \sim \Theta(X)\, B \log(B)$$

as B tends to infinity.

In fact, we will prove asymptotics for every $\mathscr{L}$ on X such that its class is contained in the interior of the effective cone $\Lambda_{\mathrm{eff}}(X)$. Moreover, we will prove estimates for the growth of $Z(s)$ in vertical strips in the neighbourhood of $\mathrm{Re}(s) = 1$. It is well known that this implies a more precise asymptotic expansion for the counting function $N(U, \mathscr{L}, B)$; see Theorem 4.4 and its corollary at the end of the paper.

1. GEOMETRY AND HEIGHTS

Let $f \in \mathbf{Z}[x_1, ..., x_n]$ be a homogeneous polynomial of degree d with co-prime coefficients and $\pi: X \to \mathbf{P}^n$ the blowup of the ideal (x_0, f) in $\mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[x_0, ..., x_n])$. We denote by Z_f the hypersurface defined by f in $\mathbf{P}^{n-1}$. Throughout the paper we assume that $Z_{f, \mathbf{C}}$ is smooth, irreducible, and doesn't contain any hyperplane. In other words, $n \geqslant 3$ and $d \geqslant 2$. The universal property of blowups implies that the scheme X is an equivariant compactification of the additive group $\mathbf{G}_a^n = \mathrm{Spec}(\mathbf{Z}[x_1, ..., x_n])$.

Denote by D_1 the exceptional divisor in X and by D_0 the strict transform of the divisor $x_0 = 0$ in $\mathbf{P}^n$. Let $U \simeq \mathbf{G}_a^n$ be the inverse image of $\mathbf{G}_a^n$ under π. We identify rational points in U with their image in the affine space $\mathbf{G}_a^n \subset \mathbf{P}^n$.

If $\mathbf{s} \in \mathbf{C}^2$, denote $D(\mathbf{s}) = s_0[D_0] + s_1[D_1] \in \operatorname{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}$.

The following proposition summarizes the geometric facts needed in the following.

PROPOSITION 1.1. *The classes of the divisors D_0 and D_1 form a basis of* Pic(X). *For $\mathbf{s} = (s_0, s_1) \in \mathbf{Z}^2$, the divisor class $D(\mathbf{s})$ is effective iff $s_0 \geqslant 0$ and $s_1 \geqslant 0$. The variety $X_{\mathbf{Q}}$ is smooth; its anticanonical line bundle has class $D(n+1, n)$.*

Proof. See [5, Proposition 1.3 and Proposition 1.6] or [7, Chap. II, Sect. 8]. ∎

We now define height functions on X. We denote by Val($\mathbf{Q}$) the set of places of $\mathbf{Q}$; it is naturally identified with the set $\{2, 3, 5, ..., \infty\}$ consisting of infinity and of all prime numbers. If p is a prime number and $\mathbf{x} \in \mathbf{G}_a^n(\mathbf{Q}_p)$, let $\|\mathbf{x}\|_p = \max(|x_1|_p, ..., |x_n|_p)$ and define the functions $H_{D_1, p}$ and $H_{D_0, p}$ by

$$H_{D_1, p}(\mathbf{x})^{-1} = \max\left(\frac{1}{\max(1, \|\mathbf{x}\|_p)}, \frac{|f(\mathbf{x})|_p}{\max(1, \|\mathbf{x}\|_p)^d}\right) \tag{1.2}$$

$$H_{D_0, p}(\mathbf{x})^{-1} = \frac{H_{D_1, p}(\mathbf{x})}{\max(1, \|\mathbf{x}\|_p)}. \tag{1.3}$$

At the archimedian place of $\mathbf{Q}$ define the local height functions by replacing maximums by the square root of the sum of squares. For any place v of $\mathbf{Q}$ and any $\mathbf{s} = (s_0, s_1) \in \mathbf{C}^2$ we set

$$H_v(\mathbf{s}; \mathbf{x}) = H_{D_0, v}(\mathbf{x})^{s_0} H_{D_1, v}(\mathbf{x})^{s_1}. \tag{1.4}$$

Finally, we define a global height pairing

$$H\colon \operatorname{Pic}(X)_{\mathbf{C}} \times \mathbf{G}_a^n(\mathbf{A}_{\mathbf{Q}}) \to \mathbf{C}^*, \qquad H(\mathbf{s}; \mathbf{x}) = \prod_{v \in \mathrm{Val}(\mathbf{Q})} H_v(\mathbf{s}; \mathbf{x}_v). \tag{1.5}$$

PROPOSITION 1.6. *If $\mathscr{L} \in \operatorname{Pic}(X)$, the function $\mathbf{x} \mapsto H(\mathscr{L}; \mathbf{x})$ on $\mathbf{G}_a^n(\mathbf{Q})$ is an exponential height in the sense of Weil.*

Proof. See [5, (1.12), (1.13), and (2.2)]. ∎

The height zeta function is then defined by the series

$$Z(\mathbf{s}) = \sum_{\mathbf{x} \in \mathbf{G}_a^n(\mathbf{Q})} H(\mathbf{s}; \mathbf{x})^{-1}. \tag{1.7}$$

It converges *a priori* for all $\mathbf{s} \in \mathbf{C}^2$ such that $D(\operatorname{Re}(s))$ is sufficiently ample.

Let $\psi = \prod_v \psi_v \colon \mathbf{G}_a(\mathbf{A}_\mathbf{Q}) \to \mathbf{C}^*$ be the standard additive character of $\mathbf{A}_\mathbf{Q}$; it is trivial on $\mathbf{Q}$. If $\mathbf{a} \in \mathbf{Q}^n$, we define

$$\psi_\mathbf{a}(\mathbf{x}) = \psi(\langle \mathbf{a}, \mathbf{x} \rangle).$$

We use the standard self-dual Haar measure $d\mathbf{x}$ on $\mathbf{G}_a^n(\mathbf{A}_\mathbf{Q})$. For any $\mathbf{a} \in \mathbf{Q}^n$, define the Fourier transform

$$\hat{H}(\mathbf{s}; \psi_\mathbf{a}) = \int_{\mathbf{G}_a^n(\mathbf{A}_\mathbf{Q})} H(\mathbf{s}; \mathbf{x})^{-1} \psi_\mathbf{a}(\mathbf{x}) \, d\mathbf{x}.$$

It is the product of the local Fourier transforms $\hat{H}_v(\mathbf{s}; \psi_\mathbf{a})$.

For $\mathbf{s} \in \mathbf{C}^2$ such that both sides converge absolutely, we have the identity

$$Z(\mathbf{s}) = \sum_{\mathbf{a} \in \mathbf{Z}^n} \hat{H}(\mathbf{s}; \psi_\mathbf{a}). \tag{1.8}$$

This is a consequence of the usual Poisson formula and the invariance of the height pairing under the standard compact subgroup of $\mathbf{G}_a^n(\mathbf{A}_\mathbf{Q})$; see [5, the end of Section 2].

In the following sections we determine the domain of absolute convergence of the right-hand side and prove that $Z(\mathbf{s})$ admits a meromorphic continuation beyond this domain.

2. THE LOCAL FOURIER TRANSFORM AT THE TRIVIAL CHARACTER

We denote by S the minimal set of primes such that $Z_f \subset \mathbf{P}^{n-1}_\mathbf{Z}$ is smooth over Spec $\mathbf{Z}[S^{-1}]$. Let p be a prime number. When no confusion can arise we shall omit the index p from norms and absolute values.

2.1. *Decomposition of the Domain.* We define subsets of $\mathbf{Q}_p^n$ as follows:

— $U(0) = \mathbf{Z}_p^n$;

— if $0 < \beta < \alpha$, $U_1(\alpha, \beta)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| = p^{d\alpha - \beta}$;

— if $\alpha \geqslant 1$, $U_1(\alpha)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| \leqslant p^{(d-1)\alpha}$;

— if $\alpha \geqslant 1$, $U(\alpha)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| = p^{d\alpha}$.

The local height function is constant on each of these subsets. Namely,

- if $\mathbf{x} \in U(0)$, $H_{D_0, p} = H_{D_1, p} = 1$;
- if $\mathbf{x} \in U_1(\alpha, \beta)$, $H_{D_0, p} = p^{\alpha - \beta}$ and $H_{D_1, p} = p^{\beta}$;
- on $U_1(\alpha)$, $H_{D_0, p} = 1$ and $H_{D_1, p} = p^{\alpha}$;
- finally, if $\mathbf{x} \in U(\alpha)$, $H_{D_0, p} = p^{\alpha}$ and $H_{D_1, p} = 1$.

2.2. *Volumes.* Denote

$$\tau_p(f) = \left(1 - \frac{1}{p}\right) \frac{\# Z_f(\mathbf{F}_p)}{p^{n-2}}.$$

The Weil conjectures proved by Deligne imply that $\tau_p(f) = 1 + O(1/p)$. In a much more elementary way, it follows from Lemma 3.9 below that $\tau_p(f)$ is bounded as p varies.

LEMMA 2.3. *For $p \notin S$, we have*

$$\operatorname{vol}(U(0)) = 1 \tag{2.3a}$$

$$\operatorname{vol}(U_1(\alpha, \beta)) = \frac{p-1}{p} \tau_p(f)\, p^{n\alpha - \beta} \tag{2.3b}$$

$$\operatorname{vol}(U_1(\alpha)) = \tau_p(f)\, p^{(n-1)\alpha} \tag{2.3c}$$

$$\operatorname{vol}(U(\alpha)) = (1 - p^{-n} - p^{-1}\tau_p(f))\, p^{n\alpha}. \tag{2.3d}$$

Proof. For $\beta \geqslant 1$, let $\Omega(\beta)$ be the set of $\mathbf{x} \in \mathbf{Z}_p^n$ such that $\|\mathbf{x}\| = 1$ and $|f(\mathbf{x})| \leqslant p^{-\beta}$. By definition,

$$\operatorname{vol}(\Omega(\beta)) = p^{-n\beta} p^{\beta - 1} (p-1)\ \# Z_f(\mathbf{Z}/p^{\beta}\mathbf{Z}).$$

As Z_f is smooth of pure dimension $n-2$ over $\mathbf{Z}_p$, Hensel's lemma implies that

$$\# Z_f(\mathbf{Z}/p^{\beta}\mathbf{Z}) = p^{(\beta - 1)(n-2)} \# Z_f(\mathbf{F}_p).$$

Consequently,

$$\operatorname{vol}(\Omega(\beta)) = (p-1)\, p^{-\beta - 1} \frac{\# Z_f(\mathbf{F}_p)}{p^{n-2}} = \tau_p(f)\, p^{-\beta}.$$

As $U_1(\alpha) = p^{-\alpha}\Omega(\alpha)$, we have

$$\operatorname{vol}(U_1(\alpha)) = \tau_p(f)\, p^{(n-1)\alpha}.$$

Now,

$$U_1(\alpha, \beta) = p^{-\alpha} U_1(0, \beta) = p^{-\alpha}(\Omega(\beta) - \Omega(\beta + 1)),$$

therefore

$$\operatorname{vol}(U_1(\alpha,\beta)) = \frac{p-1}{p}\,\tau_p(f)\,p^{n\alpha-\beta}.$$

Finally, $U(\alpha) = p^{-\alpha}(\mathbf{Z}_p^n \backslash (p\mathbf{Z}_p^n \cup \Omega(1)))$, hence

$$\operatorname{vol}(U(\alpha)) = (1 - p^{-n} - p^{-1}\tau_p(f))\,p^{n\alpha}. \qquad \blacksquare$$

PROPOSITION 2.4. *Assume that $p \notin S$. Then,*

$$\hat{H}_p(\mathbf{s};\psi_0) = \hat{H}_{\mathbf{P}^n,p}(s_0) + \tau_p(f)\,\frac{p^{s_0-n} - p^{s_1-n}}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)},$$

where

$$\hat{H}_{\mathbf{P}^n,p}(s_0) = \frac{1-p^{-s_0}}{1-p^{n-s_0}}$$

denotes the Fourier transform (with respect to the trivial character ψ_0) of the local height function of $\mathbf{P}^n$ for the tautological line bundle at s_0.

Proof. By definition,

$$\begin{aligned}
\hat{H}_p(\mathbf{s};\psi_0) &= \int_{\mathbf{Q}_p^n} H(\mathbf{s};\mathbf{x})^{-1}\,\mathrm{d}\mathbf{x} \\
&= \int_{U(0)} + \sum_{1\leqslant\beta<\alpha}\int_{U_1(\alpha,\beta)} + \sum_{1\leqslant\alpha}\int_{U_1(\alpha)} + \sum_{1\leqslant\alpha}\int_{U(\alpha)}.
\end{aligned}$$

(For brevity, the integrand $H(\mathbf{s};\mathbf{x})^{-1}$ is omitted in integral signs.) We compute these sums separately. The integral over $U(0)$ is equal to 1. Then

$$\begin{aligned}
\sum_{1\leqslant\beta<\alpha}\int_{U_1(\alpha,\beta)} &= \frac{p-1}{p}\,\tau_p(f)\sum_{1\leqslant\beta<\alpha} p^{-\alpha s_0}p^{-\beta(s_1-s_0)}p^{\alpha n-\beta} \\
&= \frac{p-1}{p}\,\tau_p(f)\sum_{\beta=1}^{\infty} p^{-\beta(s_1-s_0+1)}\sum_{\alpha=\beta+1}^{\infty} p^{-\alpha(s_0-n)} \\
&= \frac{p-1}{p}\,\tau_p(f)\sum_{\beta=1}^{\infty} p^{-\beta(s_1-s_0+1)}p^{-\beta(s_0-n)}\,\frac{1}{p^{s_0-n}-1} \\
&= \frac{p-1}{p}\,\tau_p(f)\,\frac{1}{p^{s_0-n}-1}\sum_{\beta=1}^{\infty} p^{-\beta(s_1-n+1)} \\
&= \frac{p-1}{p}\,\tau_p(f)\,\frac{1}{p^{s_0-n}-1}\,\frac{1}{p^{s_1-n+1}-1}.
\end{aligned}$$

Concerning the integrals over $U_1(\alpha)$, we have

$$\sum_{1 \leqslant \alpha} \int_{U_1(\alpha)} = \tau_p(f) \sum_{\alpha=1}^{\infty} p^{-\alpha s_1} p^{(n-1)\alpha} = \tau_p(f) \frac{1}{p^{s_1-n+1}-1}.$$

Finally,

$$\begin{aligned}\sum_{1 \leqslant \alpha} \int_{U(\alpha)} &= (1-p^{-n}-p^{-1}\tau_p(f)) \sum_{\alpha=1}^{\infty} p^{-s_0\alpha} p^{n\alpha} \\ &= (1-p^{-n}-p^{-1}\tau_p(f)) \frac{1}{p^{s_0-n}-1}.\end{aligned}$$

Adding all these terms gives

$$\begin{aligned}\hat{H}_p(\mathbf{s};\psi_0) = 1 &+ (1-p^{-1})\,\tau_p(f) \frac{1}{p^{s_0-n}-1} \frac{1}{p^{s_1-n+1}-1} \\ &+ \tau_p(f) \frac{1}{p^{s_1-n+1}-1} + (1-p^{-n}-p^{-1}\tau_p(f)) \frac{1}{p^{s_0-n}-1} \\ = 1 &+ (1-p^{-n}) \frac{1}{p^{s_0-n}-1} \\ &+ \tau_p(f) \left((1-p^{-1}) \frac{1}{p^{s_0-n}-1} \frac{1}{p^{s_1-n+1}-1} + \frac{1}{p^{s_1-n+1}-1} \right. \\ &\left. - p^{-1} \frac{1}{p^{s_0-n}-1} \right) \\ &= \hat{H}_{\mathbf{P}^n,p}(s_0) + p^{-1}\tau_p(f) \frac{p-1+p^{s_0-n+1}-p-p^{s_1-n+1}+1}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)} \\ &= \hat{H}_{\mathbf{P}^n,p}(s_0) + p^{-1}\tau_p(f) \frac{p^{s_0-n+1}-p^{s_1-n+1}}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)} \\ &= \hat{H}_{\mathbf{P}^n,p}(s_0) + \tau_p(f)\, p^{n-1} \frac{p^{-s_1}-p^{-s_0}}{(1-p^{n-s_0})(1-p^{n-1-s_1})}.\end{aligned}$$

3. THE LOCAL FOURIER TRANSFORM AT A NONTRIVIAL CHARACTER

In this subsection we evaluate the local Fourier transform at p for a nontrivial character $\psi_{\mathbf{a}}$. Let $S(\mathbf{a})$ be the union of S and of the set of primes p such that $\mathbf{a} \in p\mathbf{Z}^n$. We assume that $p \notin S(\mathbf{a})$.

Recall that $Z_f \subset \mathbf{P}^{n-1}_{\mathbf{Z}}$ denotes the subscheme defined by f and define $Z_{f,\mathbf{a}} = Z_f \cap H_{\mathbf{a}}$, where $H_{\mathbf{a}}$ is the hyperplane of $\mathbf{P}^{n-1}$ defined by $\mathbf{a}$. Finally, let $Z^t_{f,\mathbf{a}}$ (resp. $Z^{nt}_{f,\mathbf{a}}$) be the locus of points in $Z_{f,\mathbf{a}}$ where the intersection $Z_f \cap H_{\mathbf{a}}$ is *transverse* (resp., is *not transverse*). By assumption, Z_f and $H_{\mathbf{a}}$ are smooth over $\mathbf{Z}_p$.

Let $I(\alpha, \beta)$ be the integral of $\psi_{\mathbf{a}}$ over the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| \leqslant p^{d\alpha - \beta}$. Then, according to our partition of $\mathbf{Q}_p^n$, we have

$$\begin{aligned}
\hat{H}_p(\mathbf{s}; \psi_{\mathbf{a}}) &= 1 + \sum_{\alpha=1}^{\infty} \sum_{\beta=0}^{\alpha-1} p^{-\alpha s_0} p^{-\beta(s_1 - s_0)} \int_{\substack{\|\mathbf{x}\| = p^\alpha \\ |f(\mathbf{x})| = p^{d\alpha-\beta}}} \psi_{\mathbf{a}} \\
&\quad + \sum_{\alpha=1}^{\infty} p^{-\alpha s_1} \int_{\substack{\|\mathbf{x}\| = p^\alpha \\ |f(\mathbf{x})| \leqslant p^{\alpha(d-1)}}} \psi_{\mathbf{a}} \\
&= 1 + \sum_{\alpha=1}^{\infty} \sum_{\beta=0}^{\alpha-1} p^{-\alpha s_0} p^{-\beta(s_1-s_0)} (I(\alpha, \beta) - I(\alpha, \beta+1)) \\
&\quad + \sum_{\alpha=1}^{\infty} p^{-\alpha s_1} I(\alpha, \alpha) \\
&= 1 + \sum_{\alpha=1}^{\infty} p^{-\alpha s_0} I(\alpha, 0) \\
&\quad - (p^{s_1 - s_0} - 1) \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\alpha} p^{-\alpha s_0} p^{-\beta(s_1-s_0)} I(\alpha, \beta).
\end{aligned}$$

LEMMA 3.1. *If $t \in \mathbf{Q}_p$, the mean value over $\mathbf{Z}_p^*$ of $\psi(t\,\cdot)$ is equal to*

$$\frac{\int_{\mathbf{Z}_p^*} \psi(tu)\, du}{\int_{\mathbf{Z}_p^*} du} = \begin{cases} 1 & \text{if } t \in \mathbf{Z}_p; \\ -1/(p-1) & \text{if } v_p(t) = -1; \\ 0 & \text{if } v_p(t) \leqslant -2. \end{cases}$$

Proof. Indeed, we have

$$\begin{aligned}
\int_{\mathbf{Z}_p^*} \psi(tu)\, du &= \int_{\mathbf{Z}_p} \psi(tu)\, du - \int_{p\mathbf{Z}_p} \psi(tu)\, du \\
&= \int_{\mathbf{Z}_p} \psi(tu)\, du - \frac{1}{p} \int_{\mathbf{Z}_p} \psi(ptu)\, du
\end{aligned}$$

The integral of a nontrivial character over a compact group is 0, hence this integral equals 0 if $t \notin p^{-1}\mathbf{Z}_p$, equals $-1/p$ if $t \in p^{-1}\mathbf{Z}_p \backslash \mathbf{Z}_p$, and equals $1-1/p$ if $t \in \mathbf{Z}_p$. This proves the lemma. ∎

Using the change of variables $\mathbf{x} = p^{-\alpha}\mathbf{y}u$ with $\|\mathbf{y}\| = 1$ and some fixed $u \in \mathbf{Z}_p^*$, we have

$$I(\alpha, \beta) = p^{n\alpha} \int_{\substack{\|y\|=1 \\ |f(y)| \leqslant p^{-\beta}}} \psi(p^{-\alpha}\langle \mathbf{a}, \mathbf{y}\rangle\, u)\, d\mathbf{y}.$$

We can now integrate over all $u \in \mathbf{Z}_p^*$, use Lemma 3.1, and obtain

$$I(\alpha, \beta) = p^{n\alpha} \int_{\substack{\|y\|=1 \\ |f(\mathbf{y})| \leqslant p^{-\beta}}} \frac{\int_{\mathbf{Z}_p^*} \psi(p^{-\alpha}\langle \mathbf{a}, \mathbf{y}\rangle\, u)}{\int_{\mathbf{Z}_p^*} du}\, d\mathbf{y}$$

$$= p^{n\alpha} \int_{\substack{\|y\|=1 \\ |f(\mathbf{y})| \leqslant p^{-\beta}}} \left\{ \begin{matrix} 1 & \text{if} \quad \langle \mathbf{a}, \mathbf{y}\rangle \in p^{\alpha}\mathbf{Z}_p \\ -1/(p-1) & \text{if} \quad \langle \mathbf{a}, \mathbf{y}\rangle \in p^{\alpha-1}\mathbf{Z}_p - p^{\alpha}\mathbf{Z}_p \\ 0 & \text{else} \end{matrix} \right\} d\mathbf{y}$$

$$= p^{n\alpha} \int_{\substack{|f(\mathbf{y})| \leqslant p^{-\beta} \\ \|y\|=1 \\ |\langle \mathbf{a}, \mathbf{y}\rangle| \leqslant p^{-\alpha}}} \frac{p}{p-1}\, d\mathbf{y} - \int_{\substack{|f(\mathbf{y})| \leqslant p^{-\beta} \\ \|y\|=1 \\ |\langle \mathbf{a}, \mathbf{y}\rangle| \leqslant p^{1-\alpha}}} \frac{1}{p-1}\, d\mathbf{y}.$$

This implies the following formula:

$$I(\alpha, \beta) = p^{n\alpha} \left(\frac{p}{p-1} \operatorname{vol}(\|\mathbf{x}\| = 1;\, p^{\beta} \mid f(\mathbf{x});\, p^{\alpha} \mid \langle \mathbf{a}, \mathbf{x}\rangle) \right.$$
$$\left. - \frac{1}{p-1} \operatorname{vol}(\|\mathbf{x}\| = 1;\, p^{\beta} \mid f(\mathbf{x});\, p^{\alpha-1} \mid \langle \mathbf{a}, \mathbf{x}\rangle) \right). \tag{3.2}$$

LEMMA 3.3. *If* $1 \leqslant \beta \leqslant \alpha$, *one has*

$$\operatorname{vol}(\|\mathbf{x}\| = 1;\, p^{\beta} \mid f(\mathbf{x});\, p^{\alpha} \mid \langle \mathbf{a}, \mathbf{x}\rangle) = p^{-\alpha} p^{(2-n)\beta} \left(1 - \frac{1}{p}\right) \# Z_{f, \mathbf{a}}(\mathbf{Z}/p^{\beta}\mathbf{Z}).$$

If $0 = \beta < \alpha$, *then*

$$\operatorname{vol}(\|\mathbf{x}\| = 1;\, p^{\alpha} \mid \langle \mathbf{a}, \mathbf{x}\rangle) = (1 - p^{1-n})\, p^{-\alpha}.$$

Proof. Denote by $L(\alpha, \beta)$ the set of $\mathbf{x} \in \mathbf{Z}_p^n$ such that

$$\|\mathbf{x}\| = 1, \qquad p^\beta \mid f(\mathbf{x}), \qquad \text{and} \qquad p^\alpha \mid \langle \mathbf{a}, \mathbf{x} \rangle.$$

To compute the volume of $L(\alpha, \beta)$, we shall split it under all residue classes modulo p^β.

Fix $\xi \in \mathbf{Z}_p^n$ such that $\|\xi\| = 1$, $p^\beta \mid f(\xi)$, and $p^\beta \mid \langle a, \xi \rangle$. We compute the volume of the intersection of $L(\alpha, \beta)$ with the residue class of ξ modulo p^β. If $\mathbf{x} = \xi + u$, the equations for u are written

$$p^\beta \mid u \qquad \text{and} \qquad \langle \mathbf{a}, u \rangle = -\langle \mathbf{a}, \xi \rangle \qquad (\text{mod } p^\alpha).$$

Hence, the intersection of $L(\alpha, \beta)$ with the residue class of ξ has volume

$$p^{-\alpha}(p^{-\beta})^{n-1}.$$

As there are

$$(p^\beta - p^{\beta-1}) \, \# Z_{f, \mathbf{a}}(\mathbf{Z}/p^\beta \mathbf{Z})$$

such residue classes modulo p^β, the lemma is proved. ∎

In particular,

$$I(\alpha, \beta) = 0 \qquad \text{if} \quad 1 \leqslant \beta < \alpha. \tag{3.4}$$

Moreover, if $\alpha \geqslant 2$,

$$\begin{aligned}
&\mathrm{vol}(\|\mathbf{x}\| = 1;\ p^\alpha \mid f(\mathbf{x});\ p^{\alpha-1} \mid \langle \mathbf{a}, \mathbf{x} \rangle) \\
&\quad = \frac{1}{p} \mathrm{vol}(\|\mathbf{x}\| = 1;\ p^{\alpha-1} \mid f(\mathbf{x});\ p^{\alpha-1} \mid \langle \mathbf{a}, \mathbf{x} \rangle) \\
&\quad = \frac{1}{p} p^{(1-\alpha)(n-1)} \left(1 - \frac{1}{p}\right) \# Z_{f, \mathbf{a}}(\mathbf{Z}/p^{\alpha-1} \mathbf{Z}).
\end{aligned}$$

Therefore, when $\alpha \geqslant 2$, Eq. (3.2) gives

$$I(\alpha, \alpha) = p^\alpha \# Z_{f, \mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}) - p^{n-2} p^{\alpha-1} \# Z_{f, \mathbf{a}}(\mathbf{Z}/p^{\alpha-1} \mathbf{Z}).$$

If $\alpha = 1$, one has

$$\mathrm{vol}(\|\mathbf{x}\| = 1;\ p \mid f(\mathbf{x})) = \left(1 - \frac{1}{p}\right) p^{1-n} \# Z_f(\mathbf{Z}/p\mathbf{Z})$$

and

$$I(1, 1) = p \# Z_{f, \mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) - \# Z_f(\mathbf{Z}/p\mathbf{Z}).$$

We had computed in [5, Proof of Lemma 3.5] the integral

$$I(\alpha, 0) = \int_{\|\mathbf{x}\| = p^\alpha} \psi_{\mathbf{a}} = \begin{cases} -1 & \text{if} \quad \alpha = 1; \\ 0 & \text{if} \quad \alpha \geqslant 2. \end{cases}$$

Now we replace all terms $I(\alpha, \beta)$ in the equation preceding Lemma 3.1 by their value and obtain the formula

$$\hat{H}(\mathbf{s}; \psi_{\mathbf{a}}) = 1 - p^{-s_0} + (p^{s_1 - s_0} - 1)\, p^{-s_1} \# Z_f(\mathbf{F}_p)$$
$$- (p^{s_1 - s_0} - 1)(1 - p^{n - s_1 - 2}) \sum_{\alpha = 1}^{\infty} p^{-\alpha(z_1 - 1)} \# Z_{f, \mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}).$$

LEMMA 3.6. *For all* $\alpha \geqslant 1$,

$$\# Z_{f, \mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}) \leqslant p^{(n-3)(\alpha - 1)} \# Z^t_{f, \mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) + p^{(n-2)(\alpha - 1)} \# Z^{nt}_{f, \mathbf{a}}(\mathbf{Z}/p\mathbf{Z}).$$

Proof. The inequality is trivially true for $\alpha = 1$. We prove it for any α by induction: to lift a point in $Z_{f, \mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z})$ to a point in $Z_{f, \mathbf{a}}(\mathbf{Z}/p^{\alpha+1} \mathbf{Z})$, one needs to solve two equations in $\mathbf{u} \in \mathbf{F}_p^n$;

$$\langle \nabla f(\mathbf{x}), \mathbf{u} \rangle \equiv p^{-\alpha} f(\mathbf{x}), \qquad \langle \mathbf{a}, \mathbf{u} \rangle \equiv p^{-\alpha} \langle \mathbf{a}, \mathbf{x} \rangle \qquad (\text{mod } p).$$

A point in $Z_{f, \mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z})$ which reduces to a point in $Z^t_{f, \mathbf{a}}$ modulo p has p^{n-3} lifts in $Z_{f, \mathbf{a}}(\mathbf{Z}/p^{\alpha+1} \mathbf{Z})$. On the other hand, a point reducing to a point in $Z^{nt}_{f, \mathbf{a}}$ has p^{n-2} or 0 lifts according to the two linear equations being compatible or not. This implies the lemma. ∎

PROPOSTION 3.7. *The subvariety* $Z_{f, \mathbf{a}}$ *has dimension* $n - 3$. *If not empty,* $Z^{nt}_{f, \mathbf{a}}$ *is a closed subscheme of* $Z_{f, \mathbf{a}}$ *of dimension* 0 *and of bounded degree. There exists a constant* C, *independent of* $\mathbf{a}$ *and* p, *such that*

$$\# Z^t_{f, \mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) \leqslant C p^{n-3}, \qquad \# Z^{nt}_{f, \mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) \leqslant C.$$

As a corollary, one gets:

COROLLARY 3.8. *There exists a constant* C *such that for all* α *and* $p \notin S(\mathbf{a})$,

$$\# Z_{f, \mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}) \leqslant C p^{(n-3)\alpha} + C p^{(n-2)(\alpha - 1)}.$$

Proof of Proposition 3.7. The set $Z_{f, \mathbf{a}}$ is defined by the two equations $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle = 0$. Fix the coordinates $x_1, ..., x_n$ so that $\mathbf{a}$ is the first vector. Up to a constant, one may write

$$f(\mathbf{x}) = x_1^d + g_1(x_2, ..., x_n)\, x_1^{d-1} + \cdots + g_{d-1} x_1 + g_d(x_2, ..., x_n)$$

for some homogeneous polynomials g_i of degree i. Then, denoting $\mathbf{x}=(x_1, \mathbf{x}')$, $Z^{nt}_{f,\mathbf{a}}$ is defined by the equations

$$x_1 = g_d(\mathbf{x}') = \partial_2 g_d(\mathbf{x}') = \cdots = \partial_n g_d(\mathbf{x}') = 0.$$

On $Z_{f,\mathbf{a}}$, $\partial_1 f(0, \mathbf{x}') = g_{d-1}(\mathbf{x}')$, and on $Z^{nt}_{f,\mathbf{a}} \subset Z_{f,\mathbf{a}}$, $\partial_i f(0, \mathbf{x}') = \partial_i g_d(\mathbf{x}')$. As Z_f is smooth, $g_{d-1}(\mathbf{x}')$ doesn't vanish on $Z^{nt}_{f,\mathbf{a}}$ which must therefore be either empty or of dimension 0. Its degree cannot exceed $d(d-1)^{n-1}$. The bound on the number of $\mathbf{F}_p$-rational points is a consequence of the following (certainly well-known) easy lemma. ∎

Lemma 3.9. *Let $k=\mathbf{F}_q$ be a finite field and X a closed subscheme of $\mathbf{P}^n_k$ of dimension d. Then*

$$\# X(\mathbf{F}_q) \leqslant \mathbf{P}^d(\mathbf{F}_q) \deg X.$$

Proof. We use induction on d. For $d=0$ the result is clear. Then, one can assume that X is reduced, irreducible, and not contained in any hyperplane. For any k-rational hyperplane $H \subset \mathbf{P}^n$, $X \cap H$ is a closed subscheme of H of dimension $d-1$ and of degree $\leqslant \deg X$. By induction,

$$\#(X \cap H)(\mathbf{F}_q) \leqslant \# \mathbf{P}^{d-1}(\mathbf{F}_q) \deg X.$$

Finally, any point of $X(\mathbf{F}_q)$ is contained in exactly $\# \mathbf{P}^{n-1}(\mathbf{F}_q)$ rational hyperplanes in $\mathbf{P}^n$, so that

$$\# X(\mathbf{F}_q)\ \# \mathbf{P}^{n-1}(\mathbf{F}_q) \leqslant \mathbf{P}^{d-1}(\mathbf{F}_q)\ \# \mathbf{P}^n(\mathbf{F}_q) \deg X.$$

As $n \geqslant d$, this implies

$$\# X(\mathbf{F}_q) \leqslant \frac{q^{n+1}-1}{q^n-1} \frac{q^d-1}{q-1} \deg X \leqslant \mathbf{P}^d(\mathbf{F}_q) \deg X. \quad ∎$$

4. THE HEIGHT ZETA FUNCTION

From now on we fix some $\varepsilon > 0$ and consider only $\mathbf{s}$ in the subset Ω of $\mathbf{C}^2$ defined by the inequalities $\mathrm{Re}(s_0) > n + \frac{1}{2} + \varepsilon$ and $\mathrm{Re}(s_1) > n - \frac{1}{2} + \varepsilon$.

Proposition 4.1. *There exists a holomorphic function g on Ω which has polynomial growth in vertical strips such that*

$$\hat{H}(\mathbf{s}; \psi_0) = g(\mathbf{s}) \frac{1}{(s_0 - n - 1)(s_1 - n)}.$$

Moreover, $g(n+1, n)=\tau(X)$, the Tamagawa number of X corresponding to our chosen metrization of $K_X^{-1}=(n+1)\,D_0+nD_1$.

Proof. Indeed, we see from Proposition 2.4 and the estimate $\tau_p(f)=1+O(1/p)$ that, for $p\notin S$,

$$\hat{H}_p(\mathbf{s};\psi_0)(1-p^{n-s_0})(1-p^{n-1-s_1})=1+O(p^{-3/2}),$$

the O being uniform in p. Consequently, if ζ denotes Riemann's zeta function, one has for $\mathrm{Re}(s_0)>n+1$ and $\mathrm{Re}(s_1)>n$ the formula

$$\hat{H}(\mathbf{s};\psi_0)=\zeta(s_0-n)\,\zeta(s_1-n+1)\,\hat{H}_\infty(\mathbf{s};\psi_0)\times\prod_{p\nmid\infty}\hat{H}_p(\mathbf{s};\psi_0)(1-p^{n-s_0})(1-p^{n-1-s_1}).$$

The infinite product on the right-hand side converges absolutely for $\mathbf{s}\in\Omega$ to a holomorphic bounded function on Ω. The existence of a holomorphic function g as in the proposition is now proved. Its value at $(s_0, s_1)=(n+1, n)$ is equal to

$$(\mathrm{res}_{s=1}\,\zeta(s))^2\,\hat{H}_\infty((n+1, n);\psi_0)\prod_{p\nmid\infty}\hat{H}_p((n+1, n);\psi_0)(1-p^{-1})^2,$$

which is exactly the definition of $\tau(X)$ by Peyre (cf. [8]; note that $\mathrm{Pic}(X)=\mathbf{Z}^2$ as a Galois module, hence its Artin L-function is nothing but $\zeta(s)^2$). The growth of g in vertical strips follows from standard estimates for the Riemann zeta function. ∎

LEMMA 4.2. *There exists a constant $C>0$ such that for all $\mathbf{a}\in\mathbf{Z}^n\setminus\{0\}$, all $p\notin S(\mathbf{a})$, and all $(s_0, s_1)\in\Omega$, one has*

$$|\hat{H}_p(\mathbf{s};\psi_{\mathbf{a}})-1|\leqslant Cp^{-3/2}.$$

Proof. Recall the formula (3.5),

$$\hat{H}_p(\mathbf{s};\psi_{\mathbf{a}})-1=-p^{-s_0}+(p^{-s_0}-p^{-s_1})\,p^{n-2}\left(1-\frac{1}{p}\right)^{-1}\tau_p(f)-(p^{s_1-s_0}-1)(1-p^{n-s_1-2})\sum_{\alpha=1}^{\infty}p^{-\alpha(s_1-1)}\#Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha\mathbf{Z}).$$

We have to estimate all of the terms on the right-hand side. The first one is $p^{-s_0}=O(p^{-2})$. Then, as $\tau_p(f)$ is bounded, the second one is

$$O(p^{n-2-\mathrm{Re}(s_0)})+O(p^{n-2-\mathrm{Re}(s_1)})=O(p^{-3/2}).$$

For the last term T_3 we use Corollary 3.8 so that, denoting $\sigma_1 = \mathrm{Re}(s_1)$;

$$\sum_{\alpha=1}^{\infty} p^{-\alpha(s_1-1)} \# Z_{f,\mathbf{a}}(\mathbf{Z}/p^{\alpha}\mathbf{Z})$$

$$\leqslant C \sum_{\alpha=1}^{\infty} p^{-\alpha(\sigma_1-1)} p^{(n-3)\alpha} + C \sum_{\alpha=1}^{\infty} p^{-\alpha(\sigma_1-1)} p^{(n-2)(\alpha-1)}$$

$$\leqslant C \frac{1}{p^{\sigma_1-n+2}-1} + Cp^{2-n} \frac{1}{p^{\sigma_1-n+1}-1}.$$

Moreover,

$$|1-p^{n-s_1-2}| \leqslant 2$$

so that

$$|T_3| \ll \frac{p^{\sigma_1-\sigma_0}+1}{p^{\sigma_1-n+2}-1} + 2Cp^{2-n} \frac{p^{\sigma_1-\sigma_0}+1}{p^{\sigma_1-n+1}-1}$$

$$\ll (p^{n-2-\sigma_0} + p^{n-2-\sigma_1}) + p^{1-n}(p^{n-1-\sigma_0} + p^{n-1-\sigma_1})$$

$$\ll p^{-3/2}$$

as $n \geqslant 2$. The lemma is proved. ∎

PROPOSITION 4.3. *For each* $\mathbf{a} \in \mathbf{Z}^n \setminus \{0\}$, $\hat{H}(\mathbf{s}; \psi_{\mathbf{a}})$ *is a holomorphic function on* Ω. *Moreover, there exist constants* $C > 0$ *and* ν (*which are independent of* $\mathbf{s}$ *and* $\mathbf{a}$) *such that*

$$|\hat{H}(\mathbf{s}; \psi_{\mathbf{a}})| \leqslant C(1 + \|\mathrm{Im}(s)\|)^{\nu} (1 + \|\mathbf{a}\|)^{-n-1}.$$

Proof. Write

$$\hat{H}(\mathbf{s}; \psi_{\mathbf{a}}) = \prod_{p \notin S(\mathbf{a})} \hat{H}_p \times \prod_{p \in S(\mathbf{a})} \hat{H}_p \times \hat{H}_{\infty}.$$

The convergence of the first infinite product to a bounded holomorphic function follows from the preceding lemma. As in Lemma 3.7 of [5], there exists a constant $\kappa > 0$ such that

$$\left| \prod_{p \in S(\mathbf{a})} \hat{H}_p(\mathbf{s}; \psi_{\mathbf{a}}) \right| \ll (1 + \|\mathbf{a}\|)^{\kappa}.$$

Now we use the rapid decrease of $\hat{H}_\infty$ as a function of $\mathbf{a}$

$$|\hat{H}_\infty(\mathbf{s};\psi_{\mathbf{a}})| \ll (1+\|\mathrm{Im}(s)\|)^{n+\kappa+1}(1+\|\mathbf{a}\|)^{-n-\kappa-1}$$

established in Proposition 2.13 of [5] to conclude the proof. ▮

THEOREM 4.4. *The height zeta function converges in the domain* $\mathrm{Re}(s_0)>n+1$, $\mathrm{Re}(s_1)>n$. *Moreover, there exists a holomorphic function g in the domain* $\mathrm{Re}(s_0)>n+\frac{1}{2}$, $\mathrm{Re}(s_1)>n-\frac{1}{2}$ *such that*

$$Z(\mathbf{s}) = g(\mathbf{s})\,\frac{1}{(s_0-n-1)(s_1-n)}.$$

The function g has polynomial growth in vertical strips and $g(n+1,n) = \tau(X)$.

Specializing to $\mathbf{s}=s(n+1,n)$ and using a standard Tauberian theorem, one obtains the following corollary.

COROLLARY 4.5. *There exist a polynomial* P_X *of degree* 1 *and a real number* $\alpha>0$ *such that the number of points in* $U(\mathbf{Q})\subset X(\mathbf{Q})$ *of anticanonical height* $\leqslant B$ *satisfies*

$$N(U, K_X^{-1}, B) = BP(\log B) + O(B^{1-\alpha}).$$

Moreover, if $\tau(X)$ *denotes the Tamagawa number, then the leading coefficient of* P_X *is equal to*

$$\frac{\tau(X)}{(n+1)\,n},$$

as predicted by Peyre's refinement of Manin's conjecture.

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