Invent. math. 148, 421–452 (2002)
Digital Object Identifier (DOI) 10.1007/s002220100200

Inventiones mathematicae

On the distribution of points of bounded height on equivariant compactifications of vector groups

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Oblatum 24-IV-2001 & 22-X-2001
Published online: 1 February 2002 –

Abstract. We prove asymptotic formulas for the number of rational points of bounded height on smooth equivariant compactifications of the affine space.

0. Introduction

A theorem of Northcott asserts that for any real number B there are only finitely many rational points in the projective space $\mathbf{P}^n$ with *height* smaller than B. An asymptotic formula for this number (as B tends to infinity) has been proved by Schanuel [19]. Naturally, it is interesting to consider more general projective varieties and there are indeed a number of results in this direction. The techniques employed can be grouped in three main classes:

– the classical circle method in analytic number theory permits to treat complete intersections of small degree in projective spaces of large dimension (cf. for example [5]);
– harmonic analysis on adelic points of reductive groups leads to results for toric varieties [2], flag varieties [12] and horospherical varieties [20];
– elementary (but nontrivial) methods for del Pezzo surfaces of degree 4 or 5, cf. de la Bretèche [7], and for cubic surfaces (Salberger, Swinnerton-Dyer, Heath-Brown, cf. [18] and the references therein).

This research has been stimulated by a conjecture put forward by Batyrev and Manin. They proposed in [1] an interpretation of the growth rate in terms of the mutual positions of the class of the line bundle giving the

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Mathematics Subject Classifications (2000): 11G50 (11G35, 14G05)

projective embedding, the anticanonical class and the cone of effective divisors in the Picard group of the variety. Peyre refined this conjecture in [17] by introducing an adelic Tamagawa-type number which appears as the leading constant in the expected asymptotic formula for the anticanonical embedding. Batyrev and the second author proposed an interpretation of the leading constant for arbitrary ample line bundles, see [3].

In this paper we consider a new class of varieties, namely *equivariant compactifications of vector groups*. On the one hand, we can make use of harmonic analysis on the adelic points of the group. On the other hand, such varieties have a rich geometry. In particular, in contrast to flag varieties and toric varieties, they admit geometric deformations, and in contrast to the complete intersections treated by the circle method their Picard group can have arbitrarily high rank. Their geometric classification is a difficult open problem already in dimension 3 (see [13]).

A basic example of such algebraic varieties is of course $\mathbf{P}^n$ endowed with the action of $\mathbf{G}_a^n \subset \mathbf{P}^n$ by translations. A class of examples is provided by the following geometric construction: Take $X_0 = \mathbf{P}^n$ endowed with the translation action of $\mathbf{G}_a^n$. Let Y be a smooth subscheme of X_0 which is contained in the hyperplane at infinity. Then, the blow-up $X = \mathrm{Bl}_Y(X_0)$ contains the isomorphic preimage of $\mathbf{G}_a^n$ and the action of $\mathbf{G}_a^n$ lifts to X. The rank of $\mathrm{Pic}(X)$ is equal to the number of irreducible components of Y. Using equivariant resolutions of singularities, one can produce even more complicated examples, although less explicitly.

Our first steps towards this paper are detailed in [8] and [9]. There we studied the cases $n = 2$ with Y a finite union of $\mathbf{Q}$-rational points and $n > 2$ with Y a smooth hypersurface contained in the hyperplane at infinity.

We now describe the main theorem of this article. *Let X be an equivariant compactification of the additive group $\mathbf{G}_a^n$ over a number field F.* Unless explicitly stated, we shall always *assume that X is smooth and projective.* The boundary divisor $D = X \setminus \mathbf{G}_a^n$ is a sum of irreducible components D_α ($\alpha \in \mathcal{A}$). We do not assume that they are geometrically irreducible.

The Picard group of X is free and has a canonical basis given by the classes of D_α. The cone of effective divisors consists of the divisors $\sum_{\alpha\in\mathcal{A}} d_\alpha D_\alpha$ with $d_\alpha \geqslant 0$ for all α. Denote by K_X^{-1} the anticanonical line bundle on X and by $\rho = (\rho_\alpha)$ its class in $\mathrm{Pic}(X)$.

Let $\lambda = (\lambda_\alpha)$ be a class contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)$ and $\mathcal{L}_\lambda$ the corresponding line bundle, equipped with a smooth adelic metric (see 3.1 for the definition). With the above notations, we have $\lambda_\alpha > 0$ for all α. Denote by $H_{\mathcal{L}_\lambda}$ the associated exponential height on $X(F)$. Let $a_\lambda = \max(\rho_\alpha/\lambda_\alpha)$ and let b_λ be the cardinality of

$$\mathcal{B}_\lambda = \{\alpha \in \mathcal{A}\,;\ \rho_\alpha = a_\lambda \lambda_\alpha\}.$$

Put

$$c_\lambda = \prod_{\alpha\in\mathcal{B}_\lambda} \lambda_\alpha^{-1}.$$

For example, one has $a_\rho = 1$ and $b_\rho = \operatorname{rk}\operatorname{Pic}(X)$. We denote by $\tau(\mathcal{K}_X)$ the Tamagawa number as defined by Peyre in [17].

Theorem 0.1. a) *The series*

$$Z_\lambda(s) = \sum_{x \in \mathbf{G}_a^n(F)} H_{\mathcal{L}_\lambda}(x)^{-s}$$

converges absolutely and uniformly for $\mathrm{Re}(s) > a_\lambda$ *and has a meromorphic continuation to* $\mathrm{Re}(s) > a_\lambda - \delta$ *for some* $\delta > 0$, *with a unique pole at* $s = a_\lambda$ *of order* b_λ. *Moreover,* Z_λ *has polynomial growth in vertical strips in this domain.*

b) *There exist positive real numbers* τ_λ, δ' *and a polynomial* $P_\lambda \in \mathbf{R}[x]$ *of degree* $b_\lambda - 1$ *with leading coefficient*

$$c_\lambda \tau_\lambda / (b_\lambda - 1)!$$

such that the number $N(\mathcal{L}_\lambda, B)$ *of* F*-rational points in* $\mathbf{G}_a^n$ *with height* $H_{\mathcal{L}_\lambda}$ *smaller than* B *satisfies*

$$N(\mathcal{L}_\lambda, B) = B^{a_\lambda} P_\lambda(\log B) + O(B^{a_\lambda - \delta'}),$$

for $B \to \infty$.

c) *For* $\lambda = \rho$ *we have* $\tau_\rho = \tau(\mathcal{K}_X)$.

Remark 0.2. Granted the smoothness assumption on the adelic metric, note that the normalization of the height in Theorem 0.1 can be arbitrary. By a theorem of Peyre [17, § 5], this means that the points of bounded anticanonical height are equidistributed with respect to the Tamagawa measure (compatible with the choice of the height function) in the adelic space $X(\mathbf{A}_F)$. Let us explain this briefly. Fix a smooth adelic metric on the anticanonical line bundle, this defines a height function H. Let $\mathrm{d}\tau_H$ be the renormalized Tamagawa measure on $X(\mathbf{A}_F)$. A smooth positive function f on $X(\mathbf{A}_F)$ determines another height function, namely $H' = fH$. Applied to such H', the main theorem implies that

$$\lim_{B \to +\infty} \frac{(r-1)!}{c_\rho} \frac{1}{B(\log B)^{r-1}} \sum_{\substack{x \in \mathbf{G}_a^n(F) \\ H(x) \leqslant B}} f(x) = \int_{X(\mathbf{A}_F)} f(\mathbf{x}) \mathrm{d}\tau_H(\mathbf{x}).$$

Remark 0.3. Theorem 0.1 implies that the open subset $\mathbf{G}_a^n$ does not contain any accumulating subvarieties.

The proof proceeds as follows.

First, we extend the height function to the adelic space $\mathbf{G}_a^n(\mathbf{A}_F)$. Next we apply the additive Poisson formula and find a representation of $Z_\lambda(s)$ as a sum over the characters of $\mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F)$ of the Fourier transforms of H.

The Fourier transforms of H decompose as products over all places v of "global" integrals on $X(F_v)$ which are reminiscent of Igusa zeta functions.

The most technical part of the paper is devoted to the evaluation of these products: meromorphic continuation and control of their growth in vertical strips. For this, we need to consider the special case where D has strict normal crossings: this means that over an algebraic closure of F, D is a sum of smooth irreducible divisors meetings transversally. Then, at almost all nonarchimedean places, we compute explicitly the local Fourier transforms in terms of the reduction of X modulo the corresponding prime. The obtained formulas resemble Denef's formula in [10] for Igusa's local zeta function. For the remaining nonarchimedean places we find estimates. This leads to a proof of the meromorphic continuation of the Fourier transforms at each character.

In the general case, when D is not assumed to have strict normal crossings, we introduce a proper modification $\pi\colon \tilde{X} \to X$ (a composition of equivariant blow-ups with smooth centers lying on D) such that $\tilde{X}$ is a smooth projective equivariant compactification of $\mathbf{G}_a^n$ whose boundary divisor $\tilde{D}$ has strict normal crossings. We then replace X by $\tilde{X}$ and λ by $\pi^*\lambda$ and perform all the computations on $\tilde{X}$. Of course, the outcome is independent of the chosen resolution: in Lemma 6.1 we prove that $a_{\pi^*\lambda} = a_\lambda$ and $b_{\pi^*\lambda} = b_\lambda$.

The Poisson formula, combined with invariance properties of the height, reduces the summation over $\mathbf{G}_a^n(F)$ to one over a lattice. The meromorphic continuation in part a) of the theorem follows from additional estimates for the Fourier transforms at the infinite places.

At this stage one has a meromorphic continuation of $Z_\lambda(s)$ to the domain $\mathrm{Re}(s) > a_\lambda - \delta$ for some $\delta > 0$, with a single pole at $s = a_\lambda$ whose order is less or equal than b_λ. It remains to check that the order of the pole is exactly b_λ; we need to prove that the limit

$$\lim_{s\to a_\lambda} Z_\lambda(s)(s-a_\lambda)^{b_\lambda}$$

is strictly positive.

For $\lambda = \rho$ this is more or less straightforward: the main term is given by the summand corresponding to the trivial character and the Tamagawa number defined by Peyre appears naturally in the limit. For other λ we use the Poisson formula again and relate the limit to an integral of the height over some subspace which is shown to be strictly positive.

Part b) follows by a Tauberian theorem.

Acknowledgements. The work of the second author was partially supported by the NSA.

1. Geometry

Let X be a smooth projective equivariant compactification of the additive group $\mathbf{G}_a^n$ of dimension n over a field F: X is a (smooth, projective) algebraic variety endowed with an action of $\mathbf{G}_a^n$ with a dense orbit isomorphic to $\mathbf{G}_a^n$. We will always assume that X is smooth and projective. Let $D = X \setminus \mathbf{G}_a^n$ denote the boundary divisor. We have a decomposition in irreducible components

$$D = \bigcup_{\alpha \in \mathcal{A}} D_\alpha$$

where $\mathcal{A}$ is a finite set and for each $\alpha \in \mathcal{A}$, D_α is an integral divisor in X. We do not assume that the D_α are geometrically irreducible, nor that they are smooth.

A similar description holds over any extension of F. In particular, let $\bar{F}$ be a separable closure of F and let

$$D_{\bar{F}} = \bigcup_{\alpha \in \mathcal{A}_{\bar{F}}} D_{\bar{F},\alpha}$$

be the decomposition of $D_{\bar{F}}$ in irreducible components. The natural action of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on $\bar{X}$ induces an action of Γ_F on $\mathcal{A}_{\bar{F}}$ such that for any $\alpha \subset \mathcal{A}_{\bar{F}}$, $g(D_\alpha) = D_{g(\alpha)}$.

Proposition 1.1. *One has natural isomorphisms of Γ_F-modules (resp. Γ_F-monoids)*

$$\bigoplus_{\alpha \in \mathcal{A}_{\bar{F}}} \mathbf{Z} D_{\bar{F},\alpha} \to \mathrm{Pic}(\bar{X}) \quad \textit{and} \quad \bigoplus_{\alpha \in \mathcal{A}_{\bar{F}}} \mathbf{N} D_{\bar{F},\alpha} \to \Lambda_{\mathrm{eff}}(\bar{X}),$$

where $\Lambda_{\mathrm{eff}}(X)$ is the monoid of classes of effective divisors of X.

Subsequently, we identify the divisors D_α (and the corresponding line bundles) with their classes in the Picard group.

Proof. These maps are equivariant under the action of Γ_F. Hence, it remains to show injectivity and surjectivity.

For this we may assume $\bar{F} = F$. Let $\mathcal{L}$ be a line bundle on X. As X is smooth and $\mathrm{Pic}(\mathbf{G}_a^n) = 0$, there is a divisor D in X not meeting $\mathbf{G}_a^n$ such that $\mathcal{L} \simeq \mathcal{O}_X(D)$. Such a divisor D is a sum $\sum_{\alpha \in A} n_\alpha D_\alpha$. Moreover, such a D is necessarily unique. If $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ for $D = \sum_\alpha n_\alpha D_\alpha$ and $D' = \sum_\alpha n'_\alpha D_\alpha$, then the canonical rational section $\mathsf{s}_D/\mathsf{s}_{D'}$ of $\mathcal{O}_X(D - D') = \mathcal{O}_X$ is a rational function on X without zeroes nor poles on $\mathbf{G}_a^n$. Rosenlicht's lemma (see Lemma 1.2 below) implies that $\mathsf{s}_D/\mathsf{s}_{D'}$ is constant, so that $D = D'$.

We remark that any effective cycle Z on X is rationally equivalent to a cycle that does not meet $\mathbf{G}_a^n$. Indeed, if t is the parameter of a subgroup of $\mathbf{G}_a^n$ isomorphic to $\mathbf{G}_a$, we can consider the specialization of the cycles $t + Z$ when $t \to \infty$.

Lemma 1.2 (Rosenlicht). *If $f \in F(X)$ has neither zeroes nor poles on $\mathbf{G}_a^n$, then $f \in F^*$.*

Corollary 1.3. 1) *The Γ_F-module* $\mathrm{Pic}(\bar{X})$ *is a permutation module. In particular,* $\mathrm{H}^1(\Gamma_F, \mathrm{Pic}(\bar{X})) = 0$.

2) $\mathrm{Pic}(X) = \mathrm{Pic}(\bar{X})^{\Gamma_F}$ *is a free* $\mathbf{Z}$*-module of finite rank equal to the number of* Γ_F*-orbits in* $\mathcal{A}_{\bar{F}}$.

For any $\alpha \in \mathcal{A}$, we shall denote by by F_α the algebraic closure of F in the function field of D_α. Over F_α, the irreducible components of D_α are geometrically irreducible. We also denote by ζ_{F_α} the Dedekind zeta function of the number field F_α.

Let f be a nonzero linear form on $\mathbf{G}_a^n$ viewed as an element of $F(X)$. Its divisor can be written as

$$\mathrm{div}(f) = E(f) - \sum_{\alpha \in \mathcal{A}} d_\alpha(f) D_\alpha$$

where $E(f)$ is the unique irreducible component of $\{f = 0\}$ that meets $\mathbf{G}_a^n$ and $d_\alpha(f)$ are integers. (The divisor $E(f)$ can also be seen as the closure in X of the hypersurface of $\mathbf{G}_a^n$ defined by f.)

Since $E(f)$ is rationally equivalent to $\sum_{\alpha \in A} d_\alpha(f) D_\alpha$, the preceding proposition implies the following lemma:

Lemma 1.4. *For any nonzero linear form f and any $\alpha \in \mathcal{A}$, one has* $d_\alpha(f) \geqslant 0$.

If $\mathbf{a} \in \mathbf{G}_a^n(F)$, let $f_{\mathbf{a}} = \langle \cdot, \mathbf{a} \rangle$ be the associated linear form and define

$$\mathcal{A}_0(\mathbf{a}) = \{\alpha\,;\, d_\alpha(f) = 0\} \quad \text{and} \quad \mathcal{A}_0(\mathbf{a}) = \{\alpha\,;\, d_\alpha(f) = 1\}. \tag{1.1}$$

Proposition 1.5. *Let X be a normal equivariant compactification of $\mathbf{G}_a^n$ over F. Every line bundle $\mathcal{L}$ on X admits a unique $\mathbf{G}_a^n$-linearization. If $\mathcal{L}$ is effective then* $\mathrm{H}^0(X, \mathcal{L})$ *has a unique line of $\mathbf{G}_a^n$-invariant sections.*

Proof. Since $\mathbf{G}_a^n$ has no nontrivial characters, it follows from Proposition 1.4 and from the proof of Proposition 1.5 in [16], Chapter 1, that any line bundle on X admits a unique $\mathbf{G}_a^n$-linearization.

Assume that $\mathrm{H}^0(X, \mathcal{L}) \neq 0$ and consider the induced action of $\mathbf{G}_a^n$ on the projectivization $\mathbf{P}(\mathrm{H}^0(X, \mathcal{L}))$. Borel's fixed point theorem ([6], Theorem 10.4) implies that there exists a nonzero section $\mathsf{s} \in \mathrm{H}^0(X, \mathcal{L})$ such that the line $F\mathsf{s}$ is fixed under this action. As $\mathbf{G}_a^n$ has no nontrivial characters, s itself is fixed. The divisor $\mathrm{div}(\mathsf{s})$ is $\mathbf{G}_a^n$-invariant; therefore, $\mathrm{div}(\mathsf{s})$ does not meet $\mathbf{G}_a^n$ and is necessarily a sum $\sum d_\alpha D_\alpha$ such that $\mathcal{L} \simeq \mathcal{O}_X(\sum d_\alpha D_\alpha)$. Because of Proposition 1.1, every other such section will be proportional to s.

Remark 1.6. We observe that if $D = \sum d_\alpha D_\alpha$ for integers $d_\alpha \geqslant 0$, then the canonical section s_D of $\mathcal{O}_X(D)$ is $\mathbf{G}_a^n$-invariant.

2. Vector fields

We now recall some facts concerning vector fields on equivariant compactifications of algebraic groups. Let G be a connected algebraic group over F and $\mathfrak{g}$ its Lie algebra of invariant vector fields. Let X be a smooth equivariant compactification of G. Denote by $D = X \setminus G$ the boundary. We assume that D is a divisor with strict normal crossings. Let $\mathcal{T}_X$ be the tangent bundle of X. Evaluating a vector field at the neutral element $\mathbf{1}$ of G induces a "restriction map"

$$\mathrm{H}^0(X, \mathcal{T}_X) \to \mathcal{T}_{X,\mathbf{1}} = \mathfrak{g}.$$

Conversely, given $\partial \in \mathfrak{g}$, there is a unique vector field ∂^X such that for any open subset U of X and any $f \in \mathcal{O}_X(U)$,

$$\partial^X(f)(x) = \partial_g f(g \cdot x)|_{g=\mathbf{1}}.$$

The map $\partial \mapsto \partial^X$ is a section of the restriction map.

Lemma 2.1. *For any $\partial \in \mathfrak{g}$ the restriction $\partial^X|_G$ is invariant under G.*

Proposition 2.2. *Let $x \in D$ and fix a local equation s_D of D in a neighborhood U of x. Then, for any $\partial \in \mathfrak{g}$, $\partial^X \log s_D = \frac{\partial^X(s_D)}{s_D}$ is a regular function in U.*

Proof. By purity, it suffices to prove this when D is smooth in a neighborhood of x (since X is smooth, a function which is regular on the complement to a codimension 2 subscheme is regular everywhere). Thus we can choose (étale) local coordinates $x_1, \dots, x_n$ in U (i.e., elements of $\mathcal{O}_X(U)$, $\Omega^1_X|_U$ is free over $\mathcal{O}_U$ with basis $dx_1, \dots, dx_n$) such that $s_D = x_1$. Moreover, we can write uniquely

$$\partial^X = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j}$$

for some functions $f_j \in \mathcal{O}_X(U)$. Let D_x be the irreducible component of D containing x. Necessarily, G stabilizes D_x so that the function $g \mapsto x_1(g \cdot x)$ is identically 0 in a Zariski neighborhood of $\mathbf{1}$ in G and a fortiori, $\partial^X(x_1)$ vanishes on U. By definition, $\partial^X(x_1) = f_1$ and hence f_1 is a multiple of x_1: there exists a unique $g_1 \in \mathcal{O}_X(U)$ such that $f_1 = x_1 g_1$ and

$$\partial^X \log x_1 = g_1 \in \mathcal{O}_X(U).$$

The lemma is proved.

Example 2.3. When G is $\mathbf{G}_a = \operatorname{Spec} F[x]$ or $\mathbf{G}_m = \operatorname{Spec} F[x, x^{-1}]$, the Lie algebra $\mathfrak{g}$ has a canonical basis ∂, given by the local parameter x at the neutral element (respectively 0 and 1) of G. If we embed G in $\mathbf{P}^1$ equivariantly, we get $\partial^X = \partial/\partial x$ for $G = \mathbf{G}_a$ and $x\partial/\partial x$ for $G = \mathbf{G}_m$. We see that it vanishes at infinity (being $\{\infty\}$ or $\{0; \infty\}$, accordingly). This is a general fact, as the following lemma shows.

Lemma 2.4 (Hassett/Tschinkel). *There exist integers $\rho_\alpha \geqslant 1$ such that*

$$\omega_X^{-1} \simeq \mathscr{O}_X\left(\sum \rho_\alpha D_\alpha\right).$$

If $G = \mathbf{G}_a^n$, then for each α, $\rho_\alpha \geqslant 2$.

Proof. We only prove that $\rho_\alpha \geqslant 1$, because the sharper bound which is valid in the case of $G = \mathbf{G}_a^n$ won't be used below. We refer to [13], Theorem 2.7 for its proof.

Let $\partial_1, \dots, \partial_n$ be a basis of $\mathfrak{g}$. Then, $\delta := \partial_1^X \wedge \dots \wedge \partial_n^X$ is a global section of the line bundle $\det \mathscr{T}_X = \omega_X^{-1}$. Moreover, δ does not vanish on G. Therefore, we can write $\operatorname{div}(\delta) = \sum \rho_\alpha D_\alpha$ for nonnegative integers ρ_α. Necessarily, $\omega_X^{-1} \simeq \mathscr{O}_X(\sum \rho_\alpha D_\alpha)$, hence we have to prove that these integers are positive.

Fix any x in the smooth part of $X \setminus G$, so that there exists a unique $\alpha \in \mathscr{A}$ such that $x \in D_\alpha$. Pick local coordinates $x_1, \dots, x_n$ in a neighborhood U of x in such a way that in U, D is defined by the equation $x_1 = 0$. Write $\partial_i^X = \sum f_{ij} \frac{\partial}{\partial x_j}$, with $f_{ij} \in \mathscr{O}_X(U)$. We have seen in Proposition 2.2 that for any i, $\partial_i^X(x_1) = f_{i1} \in (x_1)$. Hence, $\delta \in (x_1)\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}$.

3. Metrizations

Let F be a number field and $\mathfrak{o}_F$ its ring of integers. Denote by F_v the completion of F at a place v, by $\mathfrak{o}_v$ the ring of integers in F_v if v is nonarchimedean, by $\mathbf{A}_F$ the ring of adeles of F and by $\mathbf{A}_{\mathrm{fin}}$ the (restricted) product of F_v over the nonarchimedean places v. The valuation in F_v is normalized in such a way that for any Haar measure μ_v on F_v and any measurable subset $I \subset F_v$, $\mu_v(a\,I) = |a|_v\,\mu_v(I)$. In particular, it is the usual absolute value for $F_v = \mathbf{R}$, its square for $F_v = \mathbf{C}$, it satisfies $|p|_p = 1/p$ if $F_v = \mathbf{Q}_p$ and if $\mathscr{N}_v : F_v \to \mathbf{Q}_p$ is the norm map, $|x|_v = |\mathscr{N}_v(x)|_p$. Let X be an equivariant compactification of $\mathbf{G}_a^n$ as above and $\mathscr{L}$ a line bundle on X, endowed with its canonical linearization.

Definition 3.1. *A* smooth adelic metric *on $\mathscr{L}$ is a family of v-adic norms $\|\cdot\|_v$ on $\mathscr{L}$ for all places v of F satisfying the following properties:*

a) *if v is archimedean, then $\|\cdot\|_v$ is $\mathscr{C}^\infty$;*
b) *if v is nonarchimedean, then $\|\cdot\|_v$ is locally constant;*
(i.e., the norm of any local nonvanishing section is $\mathscr{C}^\infty$, resp. locally constant)
c) *there exists an open dense subset $U \subset \operatorname{Spec}(\mathfrak{o}_F)$, a flat projective U-scheme $\mathscr{X}_{/U}$ extending X together with an action of $\mathbf{G}^n_{a/U}$ extending the action of $\mathbf{G}_a^n$ on X and a linearized line bundle $\mathscr{L}$ on $\mathscr{X}_{/U}$ extending the linearized line bundle on X, such that for any place v lying over U, the v-adic metric on $\mathscr{L}$ is given by the integral model.*

Lemma 3.2. *Let v be a nonarchimedean valuation of F and $\|\cdot\|_v$ a locally constant v-adic norm on $\mathcal{L}$. Then the stabilizer of $(\mathcal{L}, \|\cdot\|_v)$, i.e., the set of $g \in \mathbf{G}_a^n(\mathfrak{o}_v)$ which act isometrically on $(\mathcal{L}, \|\cdot\|_v)$, is a compact open subgroup of $\mathbf{G}_a^n(\mathfrak{o}_v)$.*

Proof. First we assume that $\mathcal{L}$ is effective. By Proposition 1.5, we have a nonzero invariant global section s.

If m and p_2 denote the action and the second projection $G \times X \to X$, respectively, endow the trivial line bundle $m^*\mathcal{L} \otimes p_2^*\mathcal{L}^{-1}$ on $\mathbf{G}_a^n \times X$ with the tensor-product metric. This is a locally constant metric on the trivial line bundle. The function on $\mathbf{G}_a^n(F_v) \times \mathbf{G}_a^n(F_v)$ given by

$$(g, x) \mapsto \|\mathsf{s}(g + x)\|_v \, \|\mathsf{s}(x)\|_v^{-1} .$$

is the norm of the canonical basis 1 and therefore extends to a locally constant function on $\mathbf{G}_a^n(F_v) \times X(F_v)$. Its restriction to the compact subset $\mathbf{G}_a^n(\mathfrak{o}_v) \times X(F_v)$ is uniformly continuous. Since it is locally constant and equal to 1 on $\{1\} \times X(F_v)$, there exists a neighborhood of $\{1\} \times X(F_v)$ on which it equals 1. Such a neighborhood contains a neighborhood of the form $K_v \times X(F_v)$, where K_v is a compact open subgroup of $\mathbf{G}_a^n(\mathfrak{o}_v)$. This proves the lemma in the effective case.

In the general case, we write $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ for two effective line bundles (each having a nonzero global section). We can endow $\mathcal{L}_2$ with any locally constant v-adic metric and $\mathcal{L}_1$ with the unique (necessarily locally constant) v-adic metric such that the isomorphism $\mathcal{L}_1 \simeq \mathcal{L} \otimes \mathcal{L}_2$ is an isometry. By the previous case, there exist two compact open subgroups $K_{1,v}$ and $K_{2,v}$ contained in $\mathbf{G}_a^n(\mathfrak{o}_v)$ which act isometrically on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Their intersection acts isometrically on $\mathcal{L}$.

Proposition 3.3. *If $\mathcal{L}$ is endowed with a smooth adelic metric then for all but finitely many places v of F the stabilizer of $(\mathcal{L}, \|\cdot\|_v)$ is equal to $\mathbf{G}_a^n(\mathfrak{o}_v)$. Therefore, their product over all finite places of F is a compact open subgroup of $\mathbf{G}_a^n(\mathbf{A}_{fin})$.*

Proof. It suffices to note that if v lies over the open subset $U \subset \mathrm{Spec}(\mathfrak{o}_F)$ given by the definition of an adelic metric then the stabilizer of $(\mathcal{L}, \|\cdot\|_v)$ equals $\mathbf{G}_a^n(\mathfrak{o}_v)$.

From now on we choose adelic metrics on the line bundles $\mathcal{L}$ in such a way that the tensor product of two line bundles is endowed with the product of the metrics (this can be done by fixing smooth adelic metrics on a $\mathbf{Z}$-basis of $\mathrm{Pic}(X)$ and extending by linearity). We shall denote by $\mathbf{K}$ the compact open subgroup of $\mathbf{G}_a^n(\mathbf{A}_{\mathrm{fin}})$ stabilizing all these metrized line bundles on X. We also fix an open dense $U \subset \mathrm{Spec}(\mathfrak{o}_F)$, a flat and projective model $\mathcal{X}_{/U}$ over U and models of the line bundles $\mathcal{L}$ such that the chosen v-adic metrics for all line bundles on X are given by these integral models.

4. Heights

Let X be an algebraic variety over F and $(\mathcal{L}, \|\cdot\|_v)$ an adelically metrized line bundle on X. The associated *height function* is defined as

$$H_{\mathcal{L}} : X(F) \to \mathbf{R}_{>0} \qquad H_{\mathcal{L}}(x) = \prod_v H_{\mathcal{L},v}(x) := \prod_v \|\mathsf{s}\|_v (x)^{-1},$$

where s is any F-rational section of $\mathcal{L}$ not vanishing at x. The product formula ensures that $H_{\mathcal{L}}$ does not depend on the choice of the F-rational section s (though the local heights $H_{\mathcal{L},v}(x)$ do).

In Sect. 3 we have defined simultaneous metrizations of line bundles on equivariant compactifications of $\mathbf{G}_a^n$. This allows to define compatible systems of heights

$$H\colon X(F) \times \mathrm{Pic}(X)_{\mathbf{C}} \to \mathbf{C}.$$

Fix for any $\alpha \in \mathcal{A}$ some non-zero $\mathbf{G}_a^n$-invariant section s_α of $\mathcal{O}_X(D_\alpha)$. We can then extend the height pairing H to a pairing

$$H = \prod_v H_v : \mathbf{G}_a^n(\mathbf{A}_F) \times \mathrm{Pic}(X)_{\mathbf{C}} \to \mathbf{C} \tag{4.1}$$

by the mapping

$$(\mathbf{x}; \mathbf{s}) = \Big((\mathbf{x}_v); \sum s_\alpha D_\alpha\Big) \mapsto \prod_v \prod_\alpha \|\mathsf{s}_\alpha\|_v (\mathbf{x}_v)^{-s_\alpha}.$$

Recall that $\mathbf{K}$ denotes the compact open subgroup of $\mathbf{G}_a^n(\mathbf{A}_F)$ stabilizing all line bundles on X together with their chosen metrization. By construction, we have the following proposition.

Proposition 4.2. *The height pairing H defined in* (4.1) *is $\mathbf{K}$-invariant in the first component and (exponentially) linear in the second component.*

Proposition 4.3. *Assume that $\mathcal{L}$ is in the interior of the effective cone of* $\mathrm{Pic}(X)$ *(i.e., $\mathcal{L} \simeq \mathcal{O}_X(\sum d_\alpha D_\alpha)$ for some positive integers $d_\alpha > 0$). Then, for any real B, there are only finitely many $x \in \mathbf{G}_a^n(F)$ such that $H(x; \mathcal{L}) \leqslant B$.*

In view of Lemma 2.4, this applies to the anticanonical line bundle K_X^{-1}.

Proof. Let $\mathcal{M}$ be an ample line bundle on X and let ν be a sufficiently large integer such that $\mathcal{L}^\nu \otimes \mathcal{M}^{-1}$ is effective. It follows from the preceding section that $\mathcal{L}^\nu \otimes \mathcal{M}^{-1}$ has a section s which does not vanish on $\mathbf{G}_a^n$. This implies that the function $x \mapsto -\log H(x; \mathcal{L}^\nu \otimes \mathcal{M}^{-1})$ is bounded from above on $\mathbf{G}_a^n(F)$. Therefore, there exists a constant $C > 0$ such that for any $x \in \mathbf{G}_a^n(F)$,

$$H(x; \mathcal{L}) \geqslant C H(x; \mathcal{M})^{1/\nu}.$$

We may now apply Northcott's theorem and obtain the desired finiteness.

Remark 4.4. The same argument shows that the rational map given by the sections of a sufficiently high power of $\mathcal{L}$ is an embedding on $\mathbf{G}_a^n$.

The main tool in the study of asymptotics for the number of points of bounded height is the *height zeta function*

$$Z(\mathbf{s}) = \sum_{x \in \mathbf{G}_a^n(F)} H(x; \mathbf{s})^{-1}, \qquad \mathbf{s} = (s_\alpha) \in \mathrm{Pic}(X)_{\mathbf{C}}.$$

Proposition 4.5. *There exists an non-empty open subset Ω in* $\mathrm{Pic}(X)_{\mathbf{R}}$ *such that $Z(\mathbf{s})$ converges absolutely to a bounded holomorphic function in the tube domain $\Omega + i\,\mathrm{Pic}(X)_{\mathbf{R}}$ in the complex vector space* $\mathrm{Pic}(X)_{\mathbf{C}}$.

Proof. Fix a basis $(\mathcal{L}_j)_j$ of $\mathrm{Pic}(X)$ consisting of (classes of) line bundles lying in the interior of the effective cone of $\mathrm{Pic}(X)$. Let Ω_t denote the open set of all linear combinations $\sum t_j \mathcal{L}_j \in \mathrm{Pic}(X)_{\mathbf{R}}$ such that for any j, $t_j > t$. Fix some ample line bundle $\mathcal{M}$. It is well known that the height zeta function of X relative to $\mathcal{M}$ converges for $\mathrm{Re}(s)$ big enough, say $\mathrm{Re}(s) \geqslant \sigma_0$. In the proof of Proposition 4.3 we may choose some ν which works for any $\mathcal{L}_j$, so that for any j

$$H(x; \mathcal{L}_j)^{-1} \ll H(X; \mathcal{M})^{-1/\nu}.$$

Since $H(\cdot; \mathcal{L}_j)$ is bounded from below on $\mathbf{G}_a^n(F)$ it follows that for any $\mathcal{L} \in \Omega_1$

$$H(x; \mathcal{L})^{-1} \ll H(X; \mathcal{M})^{-1/\nu}$$

Therefore, the height zeta function converges absolutely and uniformly on the tube domain $\Omega_{\nu\sigma_0} + i\,\mathrm{Pic}(X)_{\mathbf{R}}$.

The following sections are devoted to the study of analytic properties of $Z(\mathbf{s})$.

5. The Poisson formula

We recall basic facts concerning harmonic analysis on the group $\mathbf{G}_a^n$ over the adeles $\mathbf{A} = \mathbf{A}_F$ (cf., for example, [21]). For any prime number p, we can view $\mathbf{Q}_p/\mathbf{Z}_p$ as the p-Sylow subgroup of $\mathbf{Q}/\mathbf{Z}$ and we can define a local character ψ_p of $\mathbf{G}_a(\mathbf{Q}_p)$ by setting

$$\psi_p \colon x_p \mapsto \exp(2\pi i x_p).$$

At the infinite place of $\mathbf{Q}$ we put

$$\psi_\infty \colon x_\infty \mapsto \exp(-2\pi i x_\infty),$$

(here x_∞ is viewed as an element in $\mathbf{R}/\mathbf{Z}$). The product of local characters gives a character ψ of $\mathbf{G}_a(\mathbf{A}_{\mathbf{Q}})$ and, by composition with the trace, a character of $\mathbf{G}_a(\mathbf{A}_F)$.

If $\mathbf{a} \in \mathbf{G}_a^n(\mathbf{A}_F)$, recall that $f_{\mathbf{a}} = \langle \cdot, \mathbf{a} \rangle$ is the corresponding linear form on $\mathbf{G}_a^n(\mathbf{A}_F)$ and let $\psi_{\mathbf{a}} = \psi \circ f_{\mathbf{a}}$. This defines a map $\mathbf{G}_a^n(\mathbf{A}_F) \to \mathbf{G}_a^n(\mathbf{A}_F)^*$. It is well known that this map is a Pontryagin duality. The subgroup $\mathbf{G}_a^n(F) \subset \mathbf{G}_a^n(\mathbf{A}_F)$ is discrete, cocompact and we have an induced Pontryagin duality

$$\mathbf{G}_a^n(\mathbf{A}_F) \to (\mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F))^*.$$

We fix selfdual Haar measures $\mathrm{d}x_v$ on $\mathbf{G}_a(F_v)$ for all v. We refer to [21] for an explicit normalization of these measures. We will use the fact that for all but finitely many v the volume of $\mathbf{G}_a(\mathfrak{o}_v)$ with respect to $\mathrm{d}x_v$ is equal to 1. Thus we have an induced selfdual Haar measure $\mathrm{d}x$ on $\mathbf{G}_a(\mathbf{A}_F)$ and the product measure $\mathrm{d}\mathbf{x}$ on $\mathbf{G}_a^n(\mathbf{A}_F)$. The Fourier transform (in the adelic component) of the height pairing on $\mathbf{G}_a^n(\mathbf{A}_F) \times \mathrm{Pic}(X)_{\mathbf{C}}$ is defined by

$$\hat{H}(\psi_{\mathbf{a}}; \mathbf{s}) = \int_{\mathbf{G}_a^n(\mathbf{A}_F)} H(\mathbf{x}; \mathbf{s})^{-1} \psi_{\mathbf{a}}(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

We will use the Poisson formula in the following form (cf. [15], p. 280).

Theorem 5.1. *Let Φ be a continuous function on $\mathbf{G}_a^n(\mathbf{A}_F)$ such that both Φ and its Fourier transform $\hat{\Phi}$ are integrable and such that the series*

$$\sum_{\mathbf{x} \in \mathbf{G}_a^n(F)} \Phi(\mathbf{x} + \mathbf{b})$$

converges absolutely and uniformly when $\mathbf{b}$ belongs to $\mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F)$. Then,

$$\sum_{\mathbf{x} \in \mathbf{G}_a^n(F)} \Phi(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbf{G}_a^n(F)} \hat{\Phi}(\psi_{\mathbf{a}}).$$

The following lemma (a slight strengthening of Proposition 4.5) verifies the two hypotheses of the Poisson formula 5.1 concerning H.

Lemma 5.2. *Let X be a smooth projective equivariant compactification of $\mathbf{G}_a^n$ and H the height pairing defined in Sect. 4. There exists a nonempty open subset $\Omega \subset \mathrm{Pic}(X)_{\mathbf{R}}$ such that for any $\mathbf{s} \in \Omega + i\,\mathrm{Pic}(X)_{\mathbf{R}}$ the series*

$$\sum_{x \in \mathbf{G}_a^n(F)} H(x + \mathbf{b}; \mathbf{s})^{-1}$$

converges absolutely, uniformly in $\mathbf{b} \in \mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F)$ and locally uniformly in $\mathbf{s}$. Moreover, for such $\mathbf{s}$, the function $H(\cdot; \mathbf{s})^{-1}$ is integrable over $\mathbf{G}_a^n(\mathbf{A}_F)$.

Proof. Since the natural action of $\mathbf{G}_a^n$ on $\mathrm{Pic}(X)$ is trivial, for any $\mathbf{b} \in \mathbf{G}_a^n(\mathbf{A}_F)$ the function $x \mapsto H(x + \mathbf{b}; \mathcal{L})$ is a height function for $\mathcal{L}$, induced by a "twisted" adelic metric. When $\mathbf{b}$ belongs to some compact subset K of $\mathbf{G}_a^n(\mathbf{A}_F)$ there exists a constant $C_K(\mathcal{L})$ such that for any $\mathbf{x} \in \mathbf{G}_a^n(\mathbf{A}_F)$ and any $\mathbf{b} \in K$,

$$C_K(\mathcal{L})^{-1} H(\mathbf{x}; \mathcal{L}) \leqslant H(\mathbf{x}+\mathbf{b}; \mathcal{L}) \leqslant C_K(\mathcal{L}) H(\mathbf{x}; \mathcal{L}).$$

Indeed, the quotient $H(\mathbf{x}+\mathbf{b}; \mathcal{L})/H(\mathbf{x}; \mathcal{L})$ defines a bounded continuous function of $(\mathbf{x}, \mathbf{b}) \in \mathbf{G}_a^n(\mathbf{A}_F) \times K$. (Only at a finite number of places do they differ, and at these places v, a comparison is provided by the compactness of $X(F_v)$.) As $\mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F)$ is compact, one may take for K any compact set containing a fundamental domain. This implies the uniform convergence of the series once $\mathbf{s}$ belongs to the tube domain of Proposition 4.5.

The integrability follows at once : for such $\mathbf{s}$,

$$\int_{\mathbf{G}_a^n(\mathbf{A}_F)} |H(\mathbf{x}; \mathbf{s})|^{-1}\, \mathrm{d}\mathbf{x} \leqslant \sum_{x \in \mathbf{G}_a^n(F)} \int_{x+K} |H(x+\mathbf{b}; \mathbf{s})|^{-1}\, \mathrm{d}\mathbf{b}$$
$$\leqslant C_K \operatorname{vol}(K) \sum_{x \in \mathbf{G}_a^n(F)} |H(x; \mathbf{s})|^{-1}.$$

The following proposition follows from the invariance of the height under the action of the compact $\mathbf{K}$.

Proposition 5.3. *For all characters $\psi_{\mathbf{a}}$ of $\mathbf{G}_a^n(\mathbf{A}_F)$ which are nontrivial on the compact subgroup $\mathbf{K}$ of $\mathbf{G}_a^n(\mathbf{A}_{fin})$ and all $\mathbf{s}$ such that $H(\cdot; \mathbf{s})^{-1}$ is integrable, we have*

$$\hat{H}(\mathbf{s}; \psi_{\mathbf{a}}) = 0.$$

Consequently, the set $\mathfrak{d}_X$ of all $\mathbf{a} \in \mathbf{G}_a^n(F)$ such that $\psi_{\mathbf{a}}$ is trivial on $\mathbf{K}$ is is a sub-$\mathfrak{o}_F$-module of $\mathbf{G}_a^n(F)$, commensurable with $\mathbf{G}_a^n(\mathfrak{o}_F)$. We have obtained a *formal* identity for the height zeta function:

$$Z(\mathbf{s}) = \sum_{\mathbf{a} \in \mathfrak{d}_X} \hat{H}(\mathbf{s}; \psi_{\mathbf{a}}). \tag{5.4}$$

The last four sections of the paper are concerned with analytic arguments leading to the evaluation of $\hat{H}$ and to the fact that Equation (5.4) holds for all $\mathbf{s}$ in some tube domain. It will be necessary to assume that the boundary divisor D has strict normal crossings, which means that over the separable closure $\bar{F}$ of F, D is a union of smooth irreducible components meeting transversally.

We sum up these sections in the following Proposition. Recall that for $\mathbf{a} \in \mathbf{G}_a^n(F)$, $\mathcal{A}_0(\mathbf{a})$ is the set of $\alpha \in \mathcal{A}$ such that $d_\alpha(f_{\mathbf{a}}) = 0$. In particular, $\mathcal{A}_0(0) = \mathcal{A}$. Also, for any $t \in \mathbf{R}$, we denote by Ω_t the open tube domain in $\operatorname{Pic}(X)_{\mathbf{C}}$ defined by the inequalities $s_\alpha > \rho_\alpha + t$. Finally, let $\|\cdot\|_\infty$ denote any norm on the real vector space $\mathbf{G}_a^n(F \otimes_{\mathbf{Q}} \mathbf{R}) \simeq \prod_{v|\infty} F_v^n$.

Proposition 5.5. *Assume that D has strict normal crossings.*

a) *For any* $\mathbf{s} \in \Omega_0$, $H(\cdot;\mathbf{s})$ *is integrable on* $\mathbf{G}_a^n(\mathbf{A}_F)$.

b) *For any* $\varepsilon > 0$ *and* $\mathbf{a} \in \mathfrak{d}_X$ *there exists a holomorphic bounded function* $\varphi(\mathbf{a};\cdot)$ *on* $\Omega_{-1/2+\varepsilon}$ *such that for any* $\mathbf{s} \in \Omega_0$

$$\hat{H}(\psi_{\mathbf{a}};\mathbf{s}) = \varphi(\mathbf{a};\mathbf{s}) \prod_{\alpha\in\mathcal{A}_0(\mathbf{a})} (s_\alpha - \rho_\alpha)^{-1}.$$

c) *For any* $N > 0$ *there exist constants* $N' > 0$ *and* $C(\varepsilon, N)$ *such that for any* $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$ *and any* $\mathbf{a} \in \mathfrak{d}_X$ *one has the estimate*

$$|\varphi(\mathbf{a};\mathbf{s})| \leqslant C(\varepsilon, N)(1 + \|\mathrm{Im}(\mathbf{s})\|)^{N'}(1 + \|\mathbf{a}\|_\infty)^{-N}.$$

6. Meromorphic continuation of the height zeta function

For the proof of our main theorem, we need to extend the estimates of Proposition 5.5 when D is not assumed to have strict normal crossings.

Lemma 6.1. *Proposition 5.5 remains true without the hypothesis that D has strict normal crossings.*

Proof. Equivariant resolution of singularities (in char. 0, see [11] or [4]) shows that there exists a composition of equivariant blow-up with smooth centers $\pi\colon \tilde{X} \to X$ such that

- $\tilde{X}$ is a smooth projective equivariant compactification of $\mathbf{G}_a^n$;
- π is equivariant (hence induces an isomorphism on the $\mathbf{G}_a^n$);
- the boundary divisor $\tilde{D} = \tilde{X} \setminus \mathbf{G}_a^n$ has strict normal crossings.

The map π induces a map from the set $\tilde{\mathcal{A}}$ of irreducible components of $\tilde{D}$ to $\mathcal{A}$. Moreover, π has a canonical section obtained by associating to a component of D its strict transform in $\tilde{X}$. This allows to view $\mathcal{A}$ as a subset of $\tilde{\mathcal{A}}$, the complementary subset $\mathcal{E}$ consisting of exceptional divisors. Via the identifications of $\mathrm{Pic}(X)$ with $\mathbf{Z}^{\mathcal{A}}$ and $\mathrm{Pic}(\tilde{X})$ with $\mathbf{Z}^{\tilde{\mathcal{A}}}$, the map π^* on line bundles induces a linear map $i\colon \mathbf{Z}^{\mathcal{A}} \to \mathbf{Z}^{\tilde{\mathcal{A}}}$ which is a section of the first projection $\mathbf{Z}^{\tilde{\mathcal{A}}} = \mathbf{Z}^{\mathcal{A}} \oplus \mathbf{Z}^{\mathcal{E}} \to \mathbf{Z}^{\mathcal{A}}$. (We also extend i by linearity to the vector spaces of complexified line bundles.)

Concerning metrizations, it is possible to endow line bundles on $\tilde{X}$ with adelic metrics which extend the metrics coming from X via π^*. Remark that with such a choice, the height function on $\mathbf{G}_a^n(\mathbf{A}_F)$ and its Fourier transform on $\tilde{X}$ extend those on X. For instance, for any $\mathbf{s} \in \mathrm{Pic}(X)_{\mathbf{C}}$ and any $\mathbf{x} \in \mathbf{G}_a^n(\mathbf{A}_F)$, one has

$$H(\mathbf{x};\mathbf{s}) = \tilde{H}(\mathbf{x}; i(\mathbf{s})).$$

As i maps $\mathbf{N}^{*\mathcal{A}}$ into $\mathbf{N}^{*\tilde{\mathcal{A}}}$, it follows in particular that $H(\cdot;\mathbf{s})$ is integrable for $\mathbf{s}\in\Omega_0$ and that for any $\mathbf{a}\in\mathbf{G}_a^n(\mathbf{A}_F)$ and any $\mathbf{s}\in\Omega_0$, one has

$$\hat{H}(\psi_{\mathbf{a}};\mathbf{s}) = \hat{\tilde{H}}\ (\psi_{\mathbf{a}}; i(\mathbf{s})).$$

Unless π is an isomorphism, π is not smooth and $\omega_{\tilde{X}}\otimes\pi^*\omega_X^{-1}$ has a nonzero coefficient at each exceptional divisor. For any effective λ, this implies that $i(\lambda)-a_\lambda\tilde{\rho}$ lies on the boundary of $\Lambda_{\mathrm{eff}}(\tilde{X})$. (Recall that the classes of ω_X^{-1} and $\omega_{\tilde{X}}^{-1}$ in $\mathbf{Z}^{\mathcal{A}}$ and $\mathbf{Z}^{\tilde{\mathcal{A}}}$ are denoted ρ and $\tilde{\rho}$.) It follows that $a_{i(\lambda)}=a_\lambda$ and that

$$\mathcal{B}(i(\lambda)) = \{\tilde{\alpha}\in\tilde{\mathcal{A}}\,;\ i(\lambda)=a_{i(\lambda)}\tilde{\rho}\}$$

can be identified with $\mathcal{B}(\lambda)$.

Consequently, any “exceptional” factor $(s_\alpha-\tilde{\rho}_\alpha)^{-1}$ for $\alpha\in\mathcal{E}$ extends to a holomorphic bounded function on $i(\Omega_{-1/2})$ This shows that Proposition 5.5 remains true without the hypothesis that D has strict normal crossings.

One has also the following result which allows to determine the leading constant in the asymptotics.

Proposition 6.2. *Denote by $\tau(K_X)$ the Tamagawa number of X defined by the adelic metrization on K_X. One then has the equality*

$$\varphi(0;\rho)=\tau(K_X).$$

In particular, it is a strictly positive real number.

Proof. Recall first Peyre’s definition [17] of the Tamagawa measure in this context, which extends Weil’s definition in the context of linear algebraic groups [22]. For any place v of F and any local non-vanishing differential form ω on $X(F_v)$, the self-dual measure on $\mathbf{G}_a^n(F_v)$ induces a local measure $|\omega|\,/\,\|\omega\|$ on $X(F_v)$. These local measures glue to define a global measure $\mathrm{d}\mu_v$ on $X(F_v)$.

Recall that Artin’s L-function of $\mathrm{Pic}(X_{\bar{F}})$ is equal to $\prod_\alpha\zeta_{F_\alpha}$. Using Deligne’s theorem on Weil’s conjectures (but it will also follow from our calculations), Peyre has shown that the Euler product

$$\prod_v\left(\mu_v(X(F_v))\prod_\alpha\zeta_{F_\alpha,v}(1)^{-1}\right)$$

converges absolutely. He defined the Tamagawa number of X as

$$\tau(K_X)=\prod_\alpha\mathrm{res}_{s=1}\zeta_{F_\alpha}(s)\times\prod_v\left(\mu_v(X(F_v))\prod_\alpha\zeta_{F_\alpha,v}(1)^{-1}\right).$$

(Note however that Peyre apparently doesn’t use the selfdual measure for his definition in *loc. cit.*, but inserts the appropriate correcting factor $\Delta_F^{-\dim X/2}$.)

On the open subset $\mathbf{G}_a^n(F_v)$, we can define the local measure with the differential form $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ induced by the canonical coordinates of $\mathbf{G}_a^n$. Let $c \in F^\times$ be such that $\omega = c \prod_\alpha \mathsf{s}_\alpha{}^{-\rho_\alpha}$. On $\mathbf{G}_a^n(F_v)$, we thus have

$$d\mu_v = |c|_v^{-1} \prod_\alpha \|\mathsf{s}_\alpha\|\,(x)^{-\rho_\alpha}\,\mathrm{d}x = |c|_v^{-1}\, H_v(x;\rho)\,\mathrm{d}x$$

so that, the boundary subset being of measure 0,

$$\begin{aligned}\mu_v(X(F_v)) &= \mu_v(\mathbf{G}_a^n(F_v)) \\ &= |c|_v^{-1} \int_{\mathbf{G}_a^n(F_v)} H_v(x;\rho)\,\mathrm{d}x \\ &= |c|_v^{-1}\, \hat{H}_v(\psi_0;\rho).\end{aligned}$$

Moreover, it follows from these results and the absolute convergence of the Euler products of Dedekind zeta functions for $\mathrm{Re}(s) > 1$ that for any $s > 1$, $H(\cdot; s\rho)$ is integrable on $X(\mathbf{A}_F)$ and that one has

$$\begin{aligned}\varphi(0;\rho) &= \lim_{s\to 1^+} \varphi(0; s\rho) \\ &= \lim_{s\to 1^+} \hat{H}(\psi_0; s\rho) \prod_{\alpha\in\mathcal{A}} (s\rho_\alpha - \rho_\alpha) \\ &= \prod_\alpha \Big(\lim_{s\to 1^+} \rho_\alpha(s-1)\zeta_{F_\alpha}(1+\rho_\alpha(s-1))\Big) \times \\ &\qquad \lim_{s\to 1^+} \prod_\alpha \zeta_{F_\alpha}(1+\rho_\alpha(s-1))^{-1} \prod_v \hat{H}_v(\psi_0; s\rho) \\ &= \prod_\alpha \mathrm{res}_{s=1}\zeta_{F_\alpha}(s) \times \\ &\qquad \lim_{s\to 1^+} \prod_v \hat{H}_v(\psi_0; s\rho) \prod_\alpha \zeta_{F_\alpha,v}(1+\rho_\alpha(s-1))^{-1}\end{aligned}$$

The estimates of Proposition 9.5 below allow us to interchange the limit $\lim_{s\to 1}$ and the infinite product $\prod_v$, so that

$$\begin{aligned}\varphi(0;\rho) &= \prod_\alpha \mathrm{res}_{s=1}\zeta_{F_\alpha}(s) \times \prod_v \left(\hat{H}_v(\psi_0;\rho) \prod_\alpha \zeta_{F_\alpha,v}(1)^{-1} \right) \\ &= \tau(K_X) \prod_v |c|_v = \tau(K_X),\end{aligned}$$

by the product formula.

It follows from these results that the Poisson formula (5.4) applies for any $\mathbf{s} \in \Omega_0$ and that

$$Z(\mathbf{s}) = \sum_{\mathbf{a}\in\mathfrak{d}_X} \hat{H}(\psi_{\mathbf{a}}; \mathbf{s}) = \sum_{\mathbf{a}\in\mathfrak{d}_X} \varphi(\mathbf{a}; \mathbf{s}) \prod_{\alpha\in\mathcal{A}_0(\mathbf{a})} (s_\alpha - \rho_\alpha)^{-1}.$$

Hence,

$$Z(\mathbf{s})\prod_{\alpha\in\mathcal{A}}(s_\alpha-\rho_\alpha)=\sum_{\mathbf{a}\in\mathfrak{d}_X}\varphi(\mathbf{a};\mathbf{s})\prod_{\substack{\alpha\in\mathcal{A}\\ \alpha\notin\mathcal{A}_0(\mathbf{a})}}(s_\alpha-\rho_\alpha).$$

This last series is a sum of holomorphic functions on $\Omega_{-1/2}$ and taking $N > n[F:\mathbf{Q}]$, the estimate in Proposition 5.5 implies that the series converges locally uniformly in that domain. Therefore, we have shown that the function

$$\mathbf{s}\mapsto Z(\mathbf{s})\prod_{\alpha\in\mathcal{A}}(s_\alpha-\rho_\alpha)$$

has a holomorphic continuation to $\Omega_{-1/2}$ with polynomial growth in vertical strips. The restriction of $Z(\mathbf{s})$ to a line $\mathbf{C}(\lambda_\alpha)$ (with $\lambda_\alpha > 0$ for all α) gives

$$Z(s\lambda)=h_\lambda(s)\times\prod_{\alpha\in\mathcal{A}}(s\lambda_\alpha-\rho_\alpha)^{-1}$$

where h_λ is a holomorphic function for $s\in\mathbf{C}$ such that

$$\mathrm{Re}(s)>\max\left(\left(\rho_\alpha-\frac{1}{2}\right)/\lambda_\alpha\right)=a_\lambda-\frac{1}{2\min\lambda_\alpha},$$

providing a meromorphic continuation of $Z(s\lambda)$ to this domain. The rightmost pole is at

$$a_\lambda=\max_\alpha(\rho_\alpha/\lambda_\alpha)$$

and its order is less or equal than b_λ.

Theorem 6.3. *Let X be a smooth projective equivariant compactification of $\mathbf{G}_a^n$ over a number field F. For any strictly effective $\lambda\in\mathrm{Pic}(X)$, the height zeta function $Z(s\lambda)$ converges absolutely for all s such that $\mathrm{Re}(s)>a_\lambda$, has a meromorphic extension to the left of a_λ, with a single pole at a_λ of order $\leqslant b_\lambda$.*

When $\lambda=\rho$, $a_\rho=1$, $b_\rho=\mathrm{rk}\,\mathrm{Pic}(X)=r$ and Proposition 6.2 implies the equality

$$\lim_{s\to 1}Z(s\rho)(s-1)^r=\tau(K_X)\prod_{\alpha\in\mathcal{A}}\rho_\alpha^{-1},$$

hence the order of the pole is exactly r. Using a standard Tauberian theorem, we deduce an asymptotic expansion for the number of rational points in $\mathbf{G}_a^n$ of bounded anticanonical height.

Theorem 6.4. *Let X be a smooth projective equivariant compactification of $\mathbf{G}_a^n$ over a number field F.*

There exists a real number $\delta > 0$ *and a polynomial* $P \in \mathbf{R}[X]$ *of degree* $r-1$ *such that the number of* F*-rational points on* $\mathbf{G}_a^n \subset X$ *of anticanonical height* $\leqslant B$ *satisfies*

$$N(K_X^{-1}, B) = BP(\log B) + O(B^{1-\delta}).$$

The leading coefficient of P *is given by*

$$\frac{1}{(r-1)!}\tau(K_X)\prod_{\alpha\in\mathcal{A}}\rho_\alpha^{-1}.$$

7. Asymptotics

Let $\lambda \in \Lambda_{\mathrm{eff}}(X)$ be an effective class and $\mathcal{L}_\lambda$ the corresponding line bundle on X equipped with a smooth adelic metric as in Sect. 3.

Recall from the previous section that the function $s \mapsto Z(s\lambda)$ is holomorphic for $\mathrm{Re}(s) > a_\lambda = \max(\rho_\alpha/\lambda_\alpha)$, admits a meromorphic continuation to the left of a_λ and has a pole of order $\leqslant b_\lambda$ at a_λ, where b_λ is the cardinality of

$$\mathcal{B}_\lambda = \{\alpha \in \mathcal{A}\,;\ \rho_\alpha = a_\lambda\lambda_\alpha\}.$$

(Geometrically, the integer b_λ is the codimension of the face of the effective cone $\Lambda_{\mathrm{eff}}(X)$ containing the class $a_\lambda\lambda - \rho$.)

In this section we prove that the order of the pole of the height zeta function $Z(s\lambda)$ at $s = a_\lambda$ is exactly b_λ and derive our main theorem.

Denote by $Z_\lambda(s)$ the sub-series

$$Z_\lambda(s) = \sum_{\substack{\mathbf{a}\text{ such that}\\ \mathcal{A}_0(\mathbf{a})\supset\mathcal{B}_\lambda}} \hat{H}(\mathbf{a}; s\lambda).$$

It follows from the calculations above that

$$\lim_{s\to a_\lambda} Z(s\lambda)(s-a_\lambda)^{b_\lambda} = \lim_{s\to a_\lambda} Z_\lambda(s)(s-a_\lambda)^{b_\lambda} \tag{7.1}$$

since all other terms converge to zero when $s \to a_\lambda$, with uniform convergence of the series.

The set V_λ of $\mathbf{a} \in \mathbf{G}_a^n(F)$ such that $\mathcal{A}_0(\mathbf{a})$ contains $\mathcal{B}_\lambda$ is a sub-vector space. Let $\mathbf{G}_\lambda \subset \mathbf{G}_a^n$ be the F-sub-vector space defined by the corresponding linear forms $\langle \mathbf{a}, \cdot\rangle$. Then, the autoduality on $\mathbf{G}_a^n(\mathbf{A}_F)$ identifies V_λ with the dual of $\mathbf{G}_\lambda(\mathbf{A}_F)\mathbf{G}_a^n(F)$. We apply the Poisson summation formula and obtain

$$\begin{aligned} Z_\lambda(s) &= \int_{G_\lambda(\mathbf{A}_F)\mathbf{G}_a^n(F)} H(\mathbf{x}; s\lambda)^{-1}\,\mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{x}\in(\mathbf{G}_a^n/\mathbf{G}_\lambda)(F)} \int_{\mathbf{G}_\lambda(\mathbf{A}_F)} H(\mathbf{x}+\mathbf{y}; s\lambda)^{-1}\,\mathrm{d}\mathbf{y}. \end{aligned}$$

(The justification of Poisson summation formula is as in Sect. 5.)

Proposition 7.2. *For each* $x \in (\mathbf{G}_a^n/\mathbf{G}_\lambda)(F)$ *there exists a strictly positive real* $\tau_\lambda(x) > 0$ *such that*

$$\lim_{s \to a_\lambda^+} (s - a_\lambda)^{b_\lambda} \int_{\mathbf{G}_\lambda(\mathbf{A}_F)} H(x + \mathbf{y}; s\lambda)^{-1}\, \mathrm{d}\mathbf{y} = \tau_\lambda(x).$$

Proof. It is sufficient to prove the proposition when $x = 0$ as the integrals for different values of x are comparable. The Zariski closure Y of $\mathbf{G}_\lambda$ in X need not be smooth. Nevertheless, we may introduce a proper modification $\pi : \tilde{X} \to X$ such that the Zariski closure $\tilde{Y} \subset \tilde{X}$ of $\mathbf{G}_\lambda$ is a smooth equivariant compactification whose boundary is a divisor with strict normal crossings obtained by intersecting the components of the boundary of $\tilde{X}$ with $\tilde{Y}$. The arguments of Lemma 6.1 show that we do not lose any generality by assuming this on X itself.

Lemma 7.3. *There exist integers* $\rho'_\alpha \leqslant \rho_\alpha$ *for* $\alpha \in \mathcal{A}$ *such that*

$$\omega_Y^{-1} = \sum \rho'_\alpha (D_\alpha \cap Y).$$

Moreover, $\rho'_\alpha = \rho_\alpha$ *if and only if* $\alpha \in \mathcal{B}(\lambda)$.

Proof. We prove this by induction on the codimension of Y in X. If $\mathbf{G}_\lambda = \operatorname{div}(f_{\mathbf{a}})$, the adjunction formula shows that

$$\omega_Y^{-1} = \omega_X^{-1}(-Y)|_Y = \sum (\rho_\alpha - d_\alpha(\mathbf{a}))(D_\alpha \cap Y).$$

For $\mathbf{a} \in V_\lambda$ the nonnegative integer $d_\alpha(\mathbf{a})$ is equal to 0 if and only if $\alpha \in \mathcal{B}_\lambda$. The lemma is proved.

End of the proof of Proposition 7.2. Let $\mathrm{d}\tau'$ be the (nonrenormalized) Tamagawa measure on $Y(\mathbf{A}_F)$: by definition, on $\mathbf{G}_\lambda(\mathbf{A}_F)$, $\mathrm{d}\tau = H(\mathbf{y}; \rho')^{-1}\mathrm{d}\mathbf{y}$. We have to estimate

$$\lim_{s \to a_\lambda} (s - a_\lambda)^{b_\lambda} \int_{Y(\mathbf{A}_F)} H(\mathbf{y}; s\lambda - \rho')^{-1}\, \mathrm{d}\tau'.$$

The integral in the limit is of the type studied in Proposition 5.5, but on the subgroup Y. Denote therefore by primes $'$ restrictions of objects from X to Y. Thanks to the previous lemma, one has $a_{\lambda'} = a_\lambda$, $\mathcal{B}(\lambda') = \mathcal{B}(\lambda)$ and and $b_{\lambda'} = b_\lambda$. Therefore, with notations from Proposition 5.5,

$$\begin{aligned}
&\lim_{s \to a_\lambda} (s - a_\lambda)^{b_\lambda} \int_{Y(\mathbf{A}_F)} H(\mathbf{y}; s\lambda - \rho')^{-1}\, \mathrm{d}\tau' \\
&= \lim_{s \to a_{\lambda'}} (s - a_{\lambda'})^{b_{\lambda'}} \prod_{\alpha \in \mathcal{B}(\lambda)} \zeta_{F_\alpha}(1 + s\lambda'_\alpha - \rho'_\alpha)\varphi'(s\lambda' - \rho', 0) \\
&= \varphi'(0, 0) > 0.
\end{aligned}$$

The proposition is proved.

We now can conclude as in the case of the anticanonical line bundle:

Theorem 7.4. *Let* $\lambda \in \mathrm{Pic}(X)$ *be the class of a line bundle contained in the interior of the effective cone* $\Lambda_{\mathrm{eff}}(X)$*, and equip the corresponding line bundle* $\mathcal{L}_\lambda$ *with a smooth adelic metric as in Sect. 3. There exist a polynomial* $P_\lambda \in \mathbf{R}[X]$ *of degree* $b_\lambda - 1$ *and a real number* $\varepsilon > 0$ *such that the number* $N(\mathcal{L}_\lambda, B)$ *of* F*-rational points on* $\mathbf{G}_a^n$ *of* $\mathcal{L}_\lambda$*-height* $\leqslant B$ *satisfies*

$$N(\mathcal{L}_\lambda, B) = B^{a_\lambda} P_\lambda(\log B) + O(B^{a_\lambda - \varepsilon}).$$

The leading term of P_λ is equal to

$$\frac{1}{(b_\lambda - 1)!} \prod_{\alpha \in \mathcal{B}_\lambda} \lambda_\alpha^{-1} \left(\sum_{x \in (\mathbf{G}_a^n/\mathbf{G}_\lambda)(F)} \tau_\lambda(x) \right).$$

It is compatible with the description by Batyrev and Tschinkel in [3].

8. General estimates

In the remaining sections we have to prove Proposition 5.5. First we define a set of bad places S: as in Sect. 3, fix a good model $\mathcal{X}_{/U}$ (flat projective U-scheme) over a dense open subset $U \subset \mathrm{Spec}(\mathfrak{o}_F)$. In particular, we assume that $\mathcal{X}_{/U}$ extends X as an equivariant compactification of $\mathbf{G}_a^n$ whose boundary consists of the schematic closures of the D_α. Moreover, we can assume that for any α, the chosen section s_α of $\mathcal{O}(D_\alpha)$ extends to $\mathcal{X}_{/U}$ and that its norm coincides with the one that can be defined with this model at all finite places dominating U. In particular, at such places v, the local height functions restricted to $\mathbf{G}_a^n(F_v)$ are invariant under $\mathbf{G}_a^n(\mathfrak{o}_v)$. We can also assume that for any finite place v dominating U, the (nonrenormalized) Tamagawa measure $\mathrm{d}\mu_v$ is given on $\mathbf{G}_a^n(F_v)$ by $H_v(x;\rho)\,\mathrm{d}\mathbf{x}$.

Let S be the set of places v of F such that either

- v is archimedean, or
- v doesn't dominate U, or
- the residual characteristic of v is 2 or 3, or
- the volume of $\mathbf{G}_a^n(\mathfrak{o}_v)$ with respect to $\mathrm{d}\mathbf{x}_v$ is not equal to 1 (equivalently, F is ramified at v), or
- over the local ring $\mathfrak{o}_v$, the union $\bigcup_\alpha D_\alpha$ is not a transverse union of smooth relative divisors over U.

For $\mathbf{a} \in \mathfrak{d}_X$, let $f_{\mathbf{a}}$ be the corresponding linear form on $\mathbf{G}_a^n$, considered as an element of $F(X)$ or of $F(\mathcal{X}_{/U})$ and $\mathrm{div}(f_{\mathbf{a}})$ its divisor. We denote by $S(\mathbf{a})$ the set of all valuations v such that either

- v is contained in S, or
- $\mathrm{div}(f_{\mathbf{a}})$ is not flat over U, *i.e.* if $\mathfrak{m}_v$ denotes the maximal ideal of v in $\mathfrak{o}_F$, $\mathbf{a}$ belongs to $\mathfrak{m}_v\mathfrak{d}_X$.

Finally, recall that for any $t \in \mathbf{R}$, Ω_t is the tube domain in $\mathrm{Pic}(X)_{\mathbf{C}}$ consisting of classes (s_α) such that for any α, $\mathrm{Re}(s_\alpha) > \rho_\alpha + t$.

We prove estimates for finite products $\prod_{v \in S(\mathbf{a})} \hat{H}_v(\psi_{\mathbf{a}}; \mathbf{s})$ in Sect. 8. Then, in Sects. 9 and 10 we compute explicitly the local Fourier transforms $\hat{H}_v(\psi_{\mathbf{a}}; \mathbf{s})$ for all $v \notin S(\mathbf{a})$.

In the following four sections we temporarily assume that the irreducible components of the boundary of X are geometrically irreducible. In Sect. 11, we shall explain how our results extend to the general case, (still assuming that D has strict normal crossings).

Proposition 8.1. *Let Σ be any finite set of places of F containing the archimedean places. For every $\varepsilon > 0$ and any $N > 0$ there exists a constant $C(\Sigma, \varepsilon, N)$ such that for all $\mathbf{a} \in \mathfrak{d}_X$ and all $\mathbf{s} = (s_\alpha) \in \Omega_{-1+\varepsilon}$ one has the estimate*

$$\left| \prod_{v \in \Sigma} \hat{H}_v(\psi_{\mathbf{a}}; \mathbf{s}) \right| \leqslant C(\Sigma, \varepsilon, N) \frac{(1 + \|\mathbf{s}\|)^{N[F:\mathbf{Q}]}}{\prod_{v | \infty} (1 + \|\mathbf{a}\|_v)^N},$$

where $\|\mathbf{s}\| = \|\mathrm{Re}(\mathbf{s})\| + \|\mathrm{Im}(\mathbf{s})\|$.

We subdivide the proof of this proposition into a sequence of lemmas.

Lemma 8.2. *The function $H_v(\cdot; \mathbf{s})^{-1}$ is integrable on $\mathbf{G}_a^n(F_v)$ if and only if $\mathbf{s} \in \Omega_{-1}$ (*i.e.*, $\mathrm{Re}(s_\alpha) > \rho_\alpha - 1$ for all $\alpha \in \mathcal{A}$). Moreover, for all $\varepsilon > 0$ and all nonarchimedean places v, there exists a constant $C_v(\varepsilon)$ such that for all $\mathbf{s} \in \Omega_{-1+\varepsilon}$ and all $\mathbf{a} \in \mathfrak{d}_X$ one has the estimate*

$$\left| \hat{H}_v(\psi_{\mathbf{a}}; \mathbf{s}) \right| \leqslant C_v(\varepsilon).$$

Proof. Without loss of generality, we can assume that $\psi_{\mathbf{a}}(\mathbf{x}) \equiv 1$. Now, using an analytic partition of unity on $X(F_v)$ and the assumption that the boundary $D = \sum_{\alpha \in \mathcal{A}} D_\alpha$ is a divisor with strict normal crossings, we see that it suffices to compute the integral over a relatively compact neighborhood of the origin in F_v^n (denoted by $\mathcal{B}$) on which we have a set coordinates $x_1, \ldots, x_n$ so that in $\mathcal{B}$ the divisor D is defined by the equations $x_1 \cdots x_a = 0$ for some $a \in \{1, \ldots, n\}$. In $\mathcal{B}$ there exist continuous bounded functions h_α (for $\alpha \in \mathcal{A}$) such that

$$\int_{\mathcal{B}} H_v(\mathbf{x}; \mathbf{s})^{-1} \, \mathrm{d}\mathbf{x}_v = \int_{\mathcal{B}} \prod_{i=1}^{a} |x_i|_v^{s_{\alpha(i)} - \rho_{\alpha(i)}} \exp\left(\sum s_\alpha h_\alpha(x) \right) \mathrm{d}\mu_v.$$

The integral over $\mathcal{B}$ is comparable to an integral of the same type with functions h_α replaced by 0. The lemma is now a consequence of the following well-known lemma.

Lemma 8.3. *Let K be a local field. The function $x \mapsto |x|^s$ is integrable on the unit ball in K if and only if* $\mathrm{Re}(s) > -1$.

Proof. We may assume $s \in \mathbf{R}$. Choose $c \in \,]0;1[$ such that the annulus $c < |x| \leqslant 1$ has positive measure in K and let $I_0 = \int_{c<|x|\leqslant 1} |x|^s \,\mathrm{d}x$. Then, we have

$$I_n = \int_{c^{n+1}<|x|\leqslant c^n} |x|^s \,\mathrm{d}x = c^{n(s+1)} I_0.$$

It follows that the integral over K converges if and only if the geometric series $\sum c^{n(s+1)}$ converges, that is if $s+1>0$.

Proposition 8.4. *Let v be an archimedean place of F and let $\varepsilon_v = [F_v : \mathbf{R}]$. For any $N > 0$ and any $\varepsilon > 0$, there exists a constant $C_v(\varepsilon, N)$ such that for any $\mathbf{s} \in \Omega_{-1+\varepsilon}$ and all $\mathbf{a} \in \mathfrak{d}_X$ (and $\neq 0$), we have*

$$\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s}) \leqslant C_v(\varepsilon, N) \frac{(1+\|\mathbf{s}\|)^{\varepsilon_v N}}{(1+\|\mathbf{a}\|_v)^N}.$$

Proof. Let us assume for the moment that $F_v = \mathbf{R}$. We shall write $\mathbf{a}_v = (a_1, \dots, a_n)$, $\mathbf{x}_v = (x_1, \dots, x_n)$ and $\mathrm{d}\mathbf{x}_v = \mathrm{d}\mathbf{x}$. Then, in the domain Ω_{-1} one has

$$\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s}) = \int_{\mathbf{R}^n} H_v(\mathbf{x};\mathbf{s})^{-1} \exp(-2i\pi\langle \mathbf{a}, \mathbf{x}\rangle)\, \mathrm{d}\mathbf{x}.$$

Now integrate by parts: for any $j \in \{1, \dots, n\}$ we have

$$2i\pi a_j \hat{H}_v(\psi_{\mathbf{a}};\mathbf{s}) = \int_{\mathbf{R}^n} \frac{\partial}{\partial x_j} H_v(\mathbf{x};\mathbf{s})^{-1} \exp(-2i\pi\langle \mathbf{a}, \mathbf{x}\rangle)\, \mathrm{d}\mathbf{x}$$

and by induction

$$(2i\pi a_j)^N \hat{H}_v(\psi_{\mathbf{a}};\mathbf{s}) = \int_{\mathbf{R}^n} \left(\frac{\partial^N}{\partial x_j^N} H_v(\cdot;\mathbf{s})^{-1} \right)(\mathbf{x}) \exp(-2i\pi\langle \mathbf{a}, \mathbf{x}\rangle)\, \mathrm{d}\mathbf{x}.$$

For any α, let us define $h_\alpha = \log \left\| \mathsf{s}_{D_\alpha} \right\|$; it is a $\mathcal{C}^\infty$ function on $\mathbf{G}_a^n(\mathbf{R})$.

Let x be a point of $X(\mathbf{R})$ and let A be the set of $\alpha \in \mathcal{A}$ such that $x \in D_\alpha$. If t_α is a local equation of D_α in a neighborhood U of some $\mathbf{R}$-point of D_α, then there is a $\mathcal{C}^\infty$ function φ_α on U such that for $\mathbf{x} \in U \cap \mathbf{G}_a^n(\mathbf{R})$ we have

$$h_\alpha(\mathbf{x}) = \log|t_\alpha(\mathbf{x})| + \varphi_\alpha(\mathbf{x}).$$

It then follows from Propositions 2.1 and 2.2 that for each α,

$$\frac{\partial}{\partial x_j} h_\alpha(\mathbf{x}) = \frac{1}{2}\frac{\partial}{\partial x_j} \log|t_\alpha(\mathbf{x})| + \frac{\partial}{\partial x_j}\varphi_\alpha(\mathbf{x})$$

extends to a $\mathcal{C}^\infty$ function on $X(\mathbf{R})$.

From the equality

$$H_v(\mathbf{x};\mathbf{s})^{-1} = \prod_{\alpha\in\mathcal{A}} \exp(-s_\alpha h_\alpha(\mathbf{x}))$$

we deduce by induction the existence of an isobaric polynomial $P_N \in \mathbf{R}[X_\alpha^{(1)},\dots,X_\alpha^{(N)}]$ of weighted-degree N (with the convention that each $X_\alpha^{(p)}$ has weight p) and such that

$$(\partial/\partial x_j)^N H_v(\mathbf{x};\mathbf{s})^{-1} = H_v(\mathbf{x};\mathbf{s})^{-1} P_N\big(s_\alpha\partial_j h_\alpha, s_\alpha\partial_j^2 h_\alpha,\dots,s_\alpha\partial_j^N h_\alpha\big).$$

This implies that there exists a constant $C_v(\varepsilon,N,j)$ such that

$$\left|(2i\pi a_j)^N\right|\left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s})\right| \leqslant C_v(\varepsilon,N,j)(1+\|\mathbf{s}\|)^N \int_{\mathbf{R}^n} |H_v(\mathbf{x};\mathbf{s})|^{-1}\ \mathrm{d}\mathbf{x}.$$

Choosing j such that $|a_j|$ is maximal gives $|a_j| \geqslant \|\mathbf{a}\|/\sqrt{n}$, hence an upper bound

$$\left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s})\right| \leqslant C'_v(\varepsilon,N)\frac{(1+\|\mathbf{s}\|)^N}{(1+\|\mathbf{a}\|)^N}\hat{H}_v(\mathrm{Re}(\mathbf{s});\psi_0),$$

where ψ_0 is the trivial character and $C'_v(\varepsilon,N)$ some positive constant. To conclude the proof it suffices to remark that for any $\varepsilon>0$, $\hat{H}_v$ is bounded on the set $\Omega_{-1+\varepsilon}$ (but the bound depends on ε).

The case $F_v=\mathbf{C}$ is treated using a similar integration by parts. (The exponent 2 on the numerator comes from the fact that for a complex place v, $\|\cdot\|_v$ is the square of a norm.)

9. Nonarchimedean computation at the trivial character

In this section we consider only $v\notin S$. Let $\mathfrak{m}_v\subset\mathfrak{o}_v$ be the maximal ideal, $k_v=\mathfrak{o}_v/\mathfrak{m}_v$ and $q=\#k_v$. Recall that we have fixed a good model $\mathcal{X}_{/U}$ over some $U\subset\operatorname{Spec}\mathfrak{o}_F$. To simplify notations we will write $\mathbf{x}=\mathbf{x}_v$, $\mathrm{d}\mathbf{x}=\mathrm{d}\mathbf{x}_v$ etc.

The following formula is an analogue of Denef's formula in [10, Thm 3.1] for Igusa's local zeta function.

Theorem 9.1. *For all $v\notin S$ and all $\mathbf{s}\in\Omega_{-1}\subset\mathrm{Pic}(X)_{\mathbf{C}}$ we have*

$$\hat{H}_v(\psi_0;\mathbf{s}) = q^{-\dim X}\sum_{A\subset\mathcal{A}} \#D_A^\circ(k_v)\prod_{\alpha\in A}\frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1}.$$

Remark 9.2. For $\mathbf{s}=\rho$ we get $\#\mathcal{X}_{/U}(k_v)/q^{\dim X}$, the expected local density at v.

Proof. We split the integral along residue classes mod $\mathfrak{m}_v$. Let $\tilde{x} \in \mathcal{X}(k_v)$ and $A = \{\alpha \in \mathcal{A}\,;\ \tilde{x} \in D_\alpha\}$ so that $\tilde{x} \in D_A^\circ$.

We can introduce local (étale) coordinates x_α ($\alpha \in A$) and y_β ($\beta \in B$, $\#A + \#B = \dim X$) around $\tilde{x}$ such that locally, the divisor D_α is defined by the vanishing of x_α. Then, the local Tamagawa measure identifies with the measure $\prod \mathrm{d}x_\alpha \times \prod \mathrm{d}y_\beta$ on $\mathfrak{m}_v^A \times \mathfrak{m}_v^B$. If $\mathrm{d}\mathbf{x}$ denotes the fixed measure on $\mathbf{G}_a^n(F_v)$, one has the equality of measures on $\mathbf{G}_a^n(F_v) \cap \mathrm{red}^{-1}(\tilde{x})$:

$$\mathrm{d}\mathbf{x} = H_v(\mathbf{x};\rho)\,\mathrm{d}\mu_v = \prod_{\alpha\in A} q^{\rho_\alpha v(x_\alpha)} \prod \mathrm{d}x_\alpha \times \prod \mathrm{d}y_\beta.$$

Consequently,

$$\begin{aligned}\int_{\mathrm{red}^{-1}(\tilde{x})} H_v(\mathbf{x};\mathbf{s})^{-1}\,\mathrm{d}\mathbf{x} &= \int_{\mathfrak{m}_v^A\times\mathfrak{m}_v^B} q^{-\sum_{\alpha\in A}(s_\alpha-\rho_\alpha)v(x_\alpha)} \prod_{\alpha\in A}\mathrm{d}x_\alpha \prod_{\beta\in B}\mathrm{d}y_\beta \\ &= \frac{1}{q^{\#B}} \prod_{\alpha\in A}\int_{\mathfrak{m}_v} q^{-(s_\alpha-\rho_\alpha)v(x)}\,\mathrm{d}x \\ &= \frac{1}{q^{\dim X}} \prod_{\alpha\in A} \frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1}\end{aligned}$$

where the last equality follows from

$$\begin{aligned}\int_{\mathfrak{m}_v} q^{-sv(x)}\,\mathrm{d}x &= \sum_{n=1}^{\infty} q^{-sn}\,\mathrm{vol}\left(\mathfrak{m}_v^n \setminus \mathfrak{m}_v^{n+1}\right) \\ &= \sum_{n=1}^{\infty} q^{-sn} q^{-n}\left(1-\frac{1}{q}\right) \\ &= \frac{1}{q}\frac{q-1}{q^{1+s}-1}.\end{aligned} \tag{9.3}$$

Summing these equalities for $\tilde{x} \in \mathcal{X}_{/U}(k_v)$ gives the desired formula.

To estimate further $\hat{H}_v(\psi_0;\mathbf{s})$ and the product over all places, we need to give an estimate for the number of k_v-points in D_α which is uniform in v.

Lemma 9.4. *There exists a constant $C(X)$ such that for all $v \notin S$ and all $A \subset \mathcal{A}$ we have the estimates:*

- *if* $\#A = 1$, $\left|\#D_A(k_v) - q_v^{\dim X-1}\right| \leqslant C(X) q_v^{\dim X-3/2}$;
- *if* $\#A \geqslant 2$, $\#D_A(k_v) \leqslant C(X) q_v^{\dim X-\#A}$.

Proof. We use the fact that for any projective variety Y of dimension $\dim Y$ and degree $\deg Y$ the number of k_v-points is estimated as

$$\#Y(k_v) \leqslant (\deg Y)\#\mathbf{P}^{\dim Y}(k_v),$$

(see, e.g., Lemma 3.9 in [9]). Since X is projective, all D_A can be realized as subvarieties in some projective space. This proves the second part. To prove the first part, we apply Lang-Weil's estimate [14] to the geometrically irreducible smooth U-scheme D_α.

Proposition 9.5. *For all $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for any $\mathbf{s} \in \Omega_{-\frac{1}{2}+\varepsilon}$ and any finite place $v \notin S$,*

$$\left| \hat{H}_v(\psi_0; \mathbf{s}) \prod_{\alpha \in \mathcal{A}} (1 - q^{-(1+s_\alpha-\rho_\alpha)}) - 1 \right| \leqslant C(\varepsilon) q^{-1-\varepsilon}.$$

Proof. Using the uniform estimates from Lemma 9.4 we see that in the formula for $\hat{H}_v$, each term with $\#A \geqslant 2$ is $O(q^{-(\frac{1}{2}+\varepsilon)\#A}) = O(q^{-1-2\varepsilon})$, with uniform constants in O. Turning to the remaining terms, we get

$$\begin{aligned}
&1 + \sum_{\alpha \in \mathcal{A}} \left(\frac{1}{q} + O(1/q^2) \right) \frac{q-1}{q^{1+s_\alpha-\rho_\alpha} - 1} \\
&= 1 + \sum_{\alpha \in \mathcal{A}} q^{-(1+s_\alpha-\rho_\alpha)} \left(1 - \frac{1}{q} \right) (1 - q^{-(1+s_\alpha-\rho_\alpha)})^{-1} + O(q^{-3/2}) \\
&= 1 + \sum_{\alpha \in \mathcal{A}} \frac{q^{-(1+s_\alpha-\rho_\alpha)}}{1 - q^{-(1+s_\alpha-\rho_\alpha)}} + O(q^{-3/2}) \\
&= \prod_{\alpha \in \mathcal{A}} (1 - q^{-(1+s_\alpha-\rho_\alpha)})^{-1} (1 + O(q^{-1-2\varepsilon})) + O(q^{-3/2}).
\end{aligned}$$

Finally, we have the desired estimate.

For X as above and $\mathbf{s} = (s_\alpha) \in \mathrm{Pic}(X)_{\mathbf{C}}$, the (multi-variable) Artin L-function is given by

$$L(\mathrm{Pic}(X); \mathbf{s}) = \prod_{\alpha \in \mathcal{A}} \zeta_F(s_\alpha) = \prod_{v \text{ finite}} \prod_{\alpha \in \mathcal{A}} (1 - q^{-s_\alpha})^{-1}.$$

From standard properties of Dedekind zeta functions, we conclude that $L(\mathrm{Pic}(X); \mathbf{s})$ admits a meromorphic continuation and that it has polynomial growth in vertical strips. For $\mathbf{s} = (s, \ldots, s)$ we get the usual Artin L-function of $\mathrm{Pic}(X)$—here a power of the Dedekind zeta function—which has been used in the regularization of the Tamagawa measure in Proposition 6.2.

Corollary 9.6. *For all $\varepsilon > 0$ there exists a holomorphic bounded function $\varphi(0; \cdot)$ on $\Omega_{-1/2+\varepsilon}$ such that for any $\mathbf{s} \in \Omega_0$ one has*

$$\hat{H}(\psi_0; \mathbf{s}) = \varphi(0; \mathbf{s}) \prod_{\alpha \in \mathcal{A}} (s_\alpha - \rho_\alpha)^{-1}.$$

Moreover, there exist constants $N > 0$ and $C(\varepsilon)$ such that for any $\mathbf{s}$ in $\Omega_{-1/2+\varepsilon}$,

$$|\varphi(0; \mathbf{s})| \leqslant C(\varepsilon)(1 + \|\mathrm{Im}(\mathbf{s})\|)^N.$$

Proof. For any place v of F, let

$$f_v(\mathbf{s}) = \hat{H}_v(\psi_0;\mathbf{s}) L_v(\mathrm{Pic}(X);\mathbf{s}-\rho+1)^{-1}.$$

The previous proposition shows that in $\Omega_{-1/2+\varepsilon}$, the Euler product $\prod_v f_v$ converges absolutely to a holomorphic and bounded function f. For any $\mathbf{s} \in \Omega_0$, one has

$$\begin{aligned}\hat{H}(\psi_0;\mathbf{s}) &= \prod_v \hat{H}_v(\psi_0;\mathbf{s}) = \prod_v f_v(\mathbf{s}) \prod_v L_v(\mathrm{Pic}(X);\mathbf{s}-\rho+1)\\ &= f(\mathbf{s}) L(\mathrm{Pic}(X);\mathbf{s}).\end{aligned}$$

It suffices to define

$$\varphi(0;\mathbf{s}) = f(\mathbf{s}) L(\mathrm{Pic}(X);\mathbf{s}-\rho+1) \prod_{\alpha\in\mathcal{A}} (s_\alpha - \rho_\alpha).$$

The polynomial growth of $\varphi(0;\cdot)$ in vertical strips follows from the boundedness of f in $\Omega_{-1/2+\varepsilon}$ and from the fact that Dedekind zeta functions have polynomial growth in such vertical strips.

10. Other characters

Our aim here is to compute as explicitely as possible the Fourier transforms of local heights with character $\psi_{\mathbf{a}}$. In general, this will be only possible up to some error term.

The calculations in the preceding section allow us to strengthen Proposition 8.1. For $\mathbf{a} \in \mathbf{G}_a^n(F)$, we denote by $\|\mathbf{a}\|_\infty$ a norm of $\mathbf{a}$ in the real vector space defined by extension of scalars via the diagonal embedding $F \hookrightarrow F \otimes_{\mathbf{Q}} \mathbf{R} = \prod_{v|\infty} F_v$.

Proposition 10.1. *For any $\varepsilon > 0$, there exist an integer $\kappa \geqslant 0$ and for any $N \geqslant 0$, a constant $C(\varepsilon, N)$ such that for any $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$ and all $\mathbf{a} \in \mathfrak{d}_X$ we have*

$$\prod_{v\in S(\mathbf{a})} \left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s})\right| \leqslant C(\varepsilon,N)(1+\|\mathbf{s}\|)^N (1+\|\mathbf{a}\|_\infty)^{\kappa-N}.$$

Proof. For $v \in S$ the local integrals converge absolutely in the domain under consideration and are bounded as in Proposition 8.1. For $v \notin S$, we have shown that there exists a constant c such that for all $\mathbf{a} \in \mathfrak{d}_X$ and all $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$ one has the estimate

$$\left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s}) - 1\right| \leqslant \frac{c}{q^\varepsilon}.$$

This implies that $\left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s})\right|$ is bounded independently of $\mathbf{a}$, $v \in S(\mathbf{a}) \setminus S$ and $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$. For any nonzero $a \in \mathfrak{o}_F$, there is a trivial estimate

$$\sum_{\mathfrak{p}\supset(a)} 1 \ll \sum_{\mathfrak{p}\supset(a)} \log \mathcal{N}(\mathfrak{p}) \ll \log \mathcal{N}(a),$$

which implies that for some constant κ,

$$\prod_{v\in S(\mathbf{a})\setminus S} \left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s})\right| \ll \prod_{v|\infty} (1+\|\mathbf{a}\|_v)^{\kappa}.$$

Using Proposition 8.1, we have for all $N > 0$,

$$\begin{aligned}\prod_{v\in S(\mathbf{a})} \left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s})\right| &\ll C(S,\varepsilon,N)(1+\|s\|)^{N[F:\mathbf{Q}]} \prod_{v|\infty} (1+\|\mathbf{a}\|_v)^{\kappa-N} \\ &\leqslant C'(S,\varepsilon,N)(1+\|s\|)^{N[F:\mathbf{Q}]}(1+\|\mathbf{a}\|_\infty)^{(\kappa-N)[F:\mathbf{Q}]}.\end{aligned}$$

Replacing N by $N[F:\mathbf{Q}]$ and κ by $\kappa[F:\mathbf{Q}]$ concludes the proof of the proposition.

For the explicit calculation at good places, let us recall some notations: $\mathfrak{m}_v \subset \mathfrak{o}_v$ is the maximal ideal, $\pi = \pi_v$ a uniformizing element, k_v the residue field, $q = q_v = \#k_v$, $\psi = \psi_v = \psi_{\mathbf{a},v}$, $\mathbf{x} = \mathbf{x}_v$, $\mathbf{a} = \mathbf{a}_v$, $\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}_v$. As in Sect. 1, let $f = f_{\mathbf{a}}$ be a linear form on $\mathbf{G}_a^n$ and write $\operatorname{div}(f) = E - \sum_\alpha d_\alpha D_\alpha$. We have also defined $\mathcal{A}_0(\mathbf{a}) = \{\alpha\,;\ d_\alpha(f_{\mathbf{a}}) = 0\}$ and $\mathcal{A}_1(\mathbf{a}) = \{\alpha\,;\ d_\alpha(f_{\mathbf{a}}) = 1\}$.

Proposition 10.2. *There exists a constant $C(\varepsilon)$ independent of $\mathbf{a} \in \mathfrak{d}_X$ such that for any $v \notin S(\mathbf{a})$,*

$$\left|\hat{H}_v(\psi_{\mathbf{a}};\mathbf{s}) \prod_{\alpha\in\mathcal{A}_0(\mathbf{a})} (1-q^{-(1+s_\alpha-\rho_\alpha)}) - 1\right| \leqslant C(\varepsilon) q^{-1-\varepsilon}.$$

Similarly to the proof of Proposition 9.5 in the preceding section, this proposition is proved by computing the integral on residue classes.

Let $\tilde{x} \in X(k_v)$ and $A = \{\alpha\,;\ \tilde{x} \in D_\alpha\}$. We now consider three cases:

Case 1. $A = \varnothing$. — The sum of all these contribution is equal to the integral over $\mathbf{G}_a^n(\mathfrak{o}_v)$:

$$\int_{\mathbf{G}_a(\mathfrak{o}_v)} H_v(\mathbf{x};\mathbf{s})^{-1} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathfrak{o}_v^n} \psi(\langle \mathbf{a},\mathbf{x}\rangle)\,\mathrm{d}\mathbf{x} = 1.$$

Case 2. $A = \{\alpha\}$ and $\tilde{x} \notin E$. — We can introduce analytic coordinates x_α and y_β around $\tilde{x}$ such that locally $f(x) = u x_\alpha^{-d_\alpha}$ with $u \in \mathfrak{o}_v^*$. Then, we compute the integral of $H_v(\mathbf{x};\mathbf{s})^{-1}\psi_{\mathbf{a}}(\mathbf{x})$ as

$$\int_{\mathrm{red}^{-1}(\tilde{x})} = \int_{\mathfrak{m}_v\times\mathfrak{m}_v^{n-1}} q^{-(s_\alpha-\rho_\alpha)v(x_\alpha)}\psi\left(ux_\alpha^{-d_\alpha}\right)\mathrm{d}x_\alpha\mathrm{d}\mathbf{y}$$
$$= \frac{1}{q^{n-1}}\sum_{n_\alpha\geqslant 1} q^{-(1+s_\alpha-\rho_\alpha)n_\alpha}\int_{\mathfrak{o}_v^*}\psi\left(u\pi^{-n_\alpha d_\alpha}u_\alpha^{-d_\alpha}\right)\mathrm{d}u_\alpha.$$

Lemma 10.3. *For all integers $d\geqslant 0$ and $n\geqslant 1$ and all $u\in\mathfrak{o}_v^*$,*

$$\int_{\mathfrak{o}_v^*}\psi(u\pi^{-nd}t^d)\,\mathrm{d}t = \begin{cases} 1-1/q & \text{if } d=0;\\ -1/q & \text{if } n=d=1;\\ 0 & \text{else.}\end{cases}$$

Proof. If $d=1$, the computation runs as follows

$$\int_{\mathfrak{o}_v^*}\psi(u\pi^{-n}t)\,\mathrm{d}t = \int_{\mathfrak{o}_v}\psi(u\pi^{-n}t)\,\mathrm{d}t - \frac{1}{q}\int_{\mathfrak{o}_v}\psi(u\pi^{-n+1}t)\,\mathrm{d}t$$
$$= \begin{cases} 0 & \text{if } n\geqslant 2;\\ -1/q & \text{if } n=1.\end{cases}$$

For $d\geqslant 2$, let $r=v_p(d)$. Since we assumed F_v to be unramified over $\mathbf{Q}_p$, $r=v_\pi(d)$. We will integrate over disks $D(\xi,\pi^e)\subset\mathfrak{o}_v^*$ for $e\geqslant 1$ suitably chosen. Indeed, if $v\in\mathfrak{o}_v$ and $t=\xi+\pi^e v$,

$$t^d = \xi^d + d\pi^e\xi^{d-1}v \pmod{\pi^{2e}}$$

hence, if e is chosen such that

$$e-nd+r<0 \quad\text{and}\quad 2e-nd\geqslant 0,$$

$$\int_{D(\xi,\pi^e)}\psi(u\pi^{-nd}t^d)\,\mathrm{d}t = q_v^{-e}\psi(u\pi^{-nd}\xi^d)\int_{\mathfrak{o}_v}\psi(d\pi^{e-nd}u\xi^{d-1}v)\,\mathrm{d}v = 0.$$

We can find such an e if and only if $2(nd-r-1)\geqslant nd$, *i.e.* $nd\geqslant 2r+2$. If $r=0$, this is true since $d\geqslant 2$. If $r\geqslant 1$, one has $nd\geqslant p^r\geqslant 2r+2$ since we assumed $p\geqslant 5$.

This lemma implies the following trichotomy:

$$\int_{\mathrm{red}^{-1}(\tilde{x})} H(\mathbf{x};\mathbf{s})^{-1}\psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{q-1}{q^n}\frac{1}{q^{1+s_\alpha-\rho_\alpha}-1} \qquad \text{if } d_\alpha=0;$$
$$= -\frac{1}{q^n}q^{-(1+s_\alpha-\rho_\alpha)} \qquad \text{if } d_\alpha=1;$$
$$=0 \qquad \text{if } d_\alpha\geqslant 2.$$

Case 3. $\#A \geqslant 2$ or $\#A = 1$ and $\tilde{x} \in E$. — Under some transversality assumption, we could compute explicitely the integral as before. We shall however content ourselves with the estimate obtained by replacing ψ by 1 in the integral.

The total contribution of these points will therefore be smaller than

$$q^{-\dim X} \sum_{\#A \geqslant 2} \#D_A^{\circ}(k_v) \prod_{\alpha \in A} \frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1} + q^{-\dim X} \sum_{A=\{\alpha\}} \#(D_\alpha \cap E)(k_v) \frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1}. \qquad (10.4)$$

Finally, the Fourier transform is estimated as follows:

$$\hat{H}_v(\psi_{\mathbf{a}}; \mathbf{s}) = 1 + q^{-n} \sum_{\alpha \in \mathcal{A}_0(\mathbf{a})} \#D_\alpha^{\circ}(k_v) \frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1} - q^{-n} \sum_{\alpha \in \mathcal{A}_1(\mathbf{a})} \#D_\alpha^{\circ}(k_v) \frac{1}{q^{1+s_\alpha-\rho_\alpha}} + ET$$

with an "error term" ET smaller than the expression in (10.4). It is now a simple matter to rewrite this estimate as in the statement of 10.2. □

We deduce from these estimates that $\hat{H}(\psi_{\mathbf{a}}; \mathbf{s})$ has a meromorphic continuation:

Corollary 10.5. *For any $\varepsilon > 0$ and $\mathbf{a} \in \eth_X \setminus \{0\}$ there exists a holomorphic bounded function $\varphi(\mathbf{a}; \cdot)$ on $\Omega_{-1/2+\varepsilon}$ such that for any $\mathbf{s} \in \Omega_0$*

$$\hat{H}(\psi_{\mathbf{a}}; \mathbf{s}) = \prod_v \hat{H}_v(\psi_{\mathbf{a}}; \mathbf{s}) = \varphi(\psi_{\mathbf{a}}; \mathbf{s}) \prod_{\alpha \in \mathcal{A}_0(\mathbf{a})} \zeta_F(1 + s_\alpha - \rho_\alpha).$$

Moreover, for any $N > 0$ there exist constants $N' > 0$ and $C(\varepsilon, N)$ such that for any $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$, one has the estimate

$$\left|\varphi(\psi_{\mathbf{a}}; \mathbf{s})\right| \leqslant C(\varepsilon, N)(1 + \|\mathrm{Im}(\mathbf{s})\|)^{N'}(1 + \|\mathbf{a}\|_\infty)^{-N}.$$

11. The nonsplit case

In this section we extend the previous calculations of the Fourier transform to the nonsplit case, *i.e.* when the geometric irreducible components of $X \setminus \mathbf{G}_a^n$ are no longer assumed to be defined over F.

Lemma 11.1. *Let $x \in X(F)$ and $A = \{\alpha \in \mathcal{A}; x \in D_\alpha\}$. Fix for any orbit $\bar{\alpha} \in A/\Gamma_F$ some element α and let F_α be the field of definition of D_α. Then there exist an open neighborhood U of x, étale coordinates around x*

over $\bar{F}$, x_α ($\alpha \in A$) and y_β such that x_α is a local equation of D_α and such that the induced morphism $U_{\bar{F}} \to \mathbf{A}^n_{/\bar{F}}$ descends to an étale morphism

$$U \to \prod_{\bar{\alpha} \in A/\Gamma_F} \mathrm{Res}_{F_\alpha/F} \mathbf{A}^1 \times \mathbf{A}^{n-a} \qquad (a = \# A).$$

An analogous result holds over the local fields F_v and also on $\mathfrak{o}_v$, v being any finite place of F such that $v \notin S$.

(We have denoted by Res the functor of restriction of scalars *à la* Weil.)

Proof. Chose some element α in each orbit $\bar{\alpha}$ and fix a local equation x_α for the corresponding D_α which is defined over the number field F_α. Then if $\alpha' = g\alpha$ (for some $g \in \Gamma_F$) is another element in the orbit of α, set $x_{\alpha'} = g \cdot x_\alpha$. This is well defined since we assumed that the equation x_α is invariant under Γ_{F_α}.

Finally, add F-rational local étale coordinates corresponding to a basis of the subspace in $\Omega^1_{X,x}$ which is complementary to the span of dx_α for $\alpha \in A$.

Let v be a place of F. The above lemma allows us to identify a neighborhood of x in $X(F_v)$ (for the analytic topology) with a neighborhood of 0 in the product $\prod_{\alpha \in A/\Gamma_v} F_{v,\alpha} \times F_v^{n-a}$, the local heights $\prod_{\alpha\in\alpha} H_{\alpha,v}(\xi)$ being replaced by

$$\mathcal{N}_{F_{v,\alpha}/F_v}(\xi) \times h_{\alpha,v}(\xi)$$

where $h_{\alpha,v}$ is a smooth function.

Similarly, for good places v we identify $\mathrm{red}^{-1}(\tilde{x})$ with $\prod \mathfrak{m}_{v,\alpha} \times \mathfrak{m}^{n-a}$ and the functions $h_{\alpha,v}$ are equal to 1.

The assertions of Sect. 8 still hold in this more general case and the proofs require only minor modifications. However the calculations of Sects. 9 and 10 have to be redone.

Theorem 11.2 (cf. Thm. 9.1). *One has*

$$\hat{H}_v(\psi_0; \mathbf{s}) = q_v^{-\dim X} \sum_{A \subset \mathcal{A}/\Gamma_v} \# D_A^\circ(k_v) \prod_{\alpha \in A/\Gamma_v} \frac{q_v^{f_\alpha} - 1}{q_v^{f_\alpha(1+s_\alpha-\rho_\alpha)} - 1}$$

where f_α is degree of $F_{v,\alpha}$ over F_v.

Corollary 11.3 (cf. Prop. 9.5). *One has*

$$\hat{H}_v(\psi_0; \mathbf{s}) = \prod_{\alpha \in \mathcal{A}/\Gamma_v} \left(1 - q_v^{-f_\alpha(1+s_\alpha-\rho_\alpha)}\right)^{-1} \left(1 + O\left(q_v^{-1-\varepsilon}\right)\right).$$

In the general case, the multi-variable Artin L-function of $\mathrm{Pic}(\bar{X})$ is defined as

$$L(\mathrm{Pic}(\bar{X}); \mathbf{s}) = \prod_{\alpha \in \mathcal{A}/\Gamma_F} \zeta_{F_\alpha}(s_\alpha).$$

Its restriction to the line $(s, \dots, s)$ is the usual Artin L-function of $\mathrm{Pic}(\bar{X})$. It has a pole of order $\#(\mathcal{A}/\Gamma_F) = \mathrm{rk}(\mathrm{Pic}\, X)$ at $s = 1$.

Corollary 11.4 (cf. Cor. 9.6). *For all $\varepsilon > 0$, there exists a holomorphic bounded function φ on $\Omega_{-1/2+\varepsilon}$ such that for any $\mathbf{s} \in \Omega_0$ one has*

$$\hat{H}(\psi_0; \mathbf{s}) = \varphi(\mathbf{s}) L(\mathrm{Pic}(\bar{X}); \mathbf{s} - \rho + 1)$$

and the Tamagawa measure of $X(\mathbf{A}_F)$ is equal to

$$\tau(K_X) = \varphi(\rho)/L^*(\mathrm{Pic}(\bar{X}); \mathbf{1}).$$

At nontrivial characters, the calculations are modified analogously and we have

$$\hat{H}(\psi_{\mathbf{a}}; \mathbf{s}) = \varphi(\psi_{\mathbf{a}}; \mathbf{s}) \prod_{\alpha \in \mathcal{A}_0(\mathbf{a})/\Gamma_F} \zeta_{F_\alpha}(s_\alpha - \rho_\alpha + 1)$$

for some holomorphic function $\varphi(\mathbf{a}; \cdot)$ as in Corollary 10.5.

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