Some applications of potential theory to number theoretical problems on analytic curves

Antoine Chambert-Loir

Institut de recherche mathématique de Rennes, Université de Rennes 1
and
Institute for Advanced Study, Princeton

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What is Potential Theory? — Classical Physics

- Newton's gravitation field is the gradient of a function (Lagrange), which Green and Gauß called a **potential**;
- also holds for electrostatics;
- the equilibrium state is the solution of a **Laplace equation** (with boundary conditions);
- the potential can be deduced by integrating the fundamental solution of the Laplacian (**kernel**):
  - Newtonian potential, $1/\|x-y\|^{n-2}$ in dimension $n \geqslant 3$,
  - logarithmic potential, $-\log\|x-y\|$ in dimension $n=2$;
- **Robin's problem:** compute the mass distribution giving rise to the equilibrium potential.

Logarithmic potential theory and Complex function theory

We assume that the dimension $n = 2$.
Identifying $\mathbf{R}^2$ with $\mathbf{C}$, it appears that **two-dimensional potential theory** is very closely connected to **one-dimensional complex analysis**.
For example, the real part of a holomorphic function is harmonic, and conversely on a simply connected open set.

Applications (see, e.g., Ransford's book):

- uniform approximation (theorems of Bernstein–Walsh, Mergelyan...);
- Schwarz's lemma and its generalizations;
- complex dynamics...

Generalizations to Riemann surfaces.

Logarithmic potential theory

M = compact connected Riemann surface; $o \in M$,
$K \subset M$ **non-polar** compact subset such that $o \notin K$.

Green function: there exists a unique function $g_K : M \setminus \{o\} \to \mathbf{R}$ with the properties:

- g_K is harmonic on $M \setminus (\{o\} \cup K)$;
- $g_K(p) + \log|z(p) - z(o)|$ extends continuously at o (where z is a local parameter around o on M);
- $g_K(p) \to 0$ when $p \to \partial K$ (up to a polar subset of ∂K).

Equilibrium measure: Laplacian of g_K (in the sense of distributions) — probability measure μ_K supported by ∂K.

Robin constant (relative to the local parameter z):

$$-\log \operatorname{cap}_z(K) = \lim_{p \to o} g_K(p) + \log|z(p) - z(o)|.$$

Capacitary norm on $\mathrm{T}_o M$: defined by $\left\| \frac{\partial}{\partial z} \right\| = \operatorname{cap}_z(K)$.

Logarithmic potential theory and Classical number theory

In the 1920s, surprising connections with number theory were shown:
Define the Fekete–Szegő set of a subset $A \subset \mathbf{C}$ as the set $\mathrm{FS}(A)$ of **algebraic integers** all of whose conjugates belong to A.

Theorem (Fekete–Szegő)

Let K be a compact subset of $\mathbf{C}$, invariant under complex conjugation.

- *If* $\mathrm{cap}_{z^{-1}}(K) < 1$, $\mathrm{FS}(K)$ *is finite;*
- *If* $\mathrm{cap}_{z^{-1}}(K) \geqslant 1$, *then for any open neighborhood U of K,* $\mathrm{FS}(U)$ *is infinite.*

Logarithmic potential theory and more recent number theory

If a is an algebraic number, let $\mu(a)$ be the Galois invariant probability measure on $\mathbf{C}$ supported by the conjugates of a: if the minimal polynomial of a is $c\prod_{j=1}^{d}(X-a_j)$, then

$$\delta(a) = \frac{1}{d}\sum_{j=1}^{d}\delta_{a_j} \qquad (\delta_z = \text{Dirac mass at } z).$$

Theorem (Serre, cf. Bilu, Rumely)

Let K be a compact subset of $\mathbf{C}$, invariant under complex conjugation such that $\operatorname{cap}_{z^{-1}}(K) = 1$.
The equilibrium measure μ_K of K is the unique probability measure on K which is a limit of measures of the form $\delta(a)$, for algebraic integers a.

Height of an algebraic number

Let a be an algebraic number, with minimal polynomial $P = c\prod_{j=1}^{d}(X - a_j)$.

$$\text{height}(a) = \frac{1}{d}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| \frac{\mathrm{d}\theta}{2\pi} \qquad \text{Mahler measure}$$

$$= \frac{1}{d}\log c + \frac{1}{d}\sum_{j=1}^{d}\log^+(\left|a_j\right|) \qquad \text{(Jensen's formula)}$$

$$= \frac{1}{d}\log c + \int_{\mathbf{C}} \log^+(|z|)\mathrm{d}\delta_a(z)$$

and using all absolute values of $\overline{\mathbf{Q}}$,

$$= \sum_{p\leqslant\infty}\frac{1}{d}\sum_{j=1}^{d}\log^+(\left|a_j\right|_p).$$

Observe that for $K = \{|z| \leqslant 1\}$, $g_K = \log^+$ and $\mu_K = \mathrm{d}\theta/2\pi$.

Generalizations

- **working on algebraic varieties**
  - usual Diophantine geometry setup
  - Arakelov geometry defines the height of subvarieties (Faltings, Bost-Gillet-Soulé)
- **putting all places on equal foot:** non-archimedean potential theory
  - Rumely's theory (end of 1980s);
  - using Berkovich's spaces (Baker/Rumely, Favre/Rivera-Letelier, Thuillier)
- **beyond potential theory**
  - Néron functions in Diophantine geometry;
  - Green currents from Arakelov geometry;
  - function theory, Laplace operators, measures,... on Berkovich spaces.

Metrics *vs.* Néron functions

X projective variety over $\mathbf{Q}$, D effective Cartier divisor on X
L the line bundle $\mathscr{O}_X(D)$, s_D the canonical section of L

Néron function for D: function λ_D on $X(\mathbf{C}_p) \setminus |D|$ such that

$$\lambda_{D,p}(x) + \log|f|_p(x)$$

extends continuously for any local equation f of D.

Green function for D: function on $X(\mathbf{C}) \setminus |D|$ such that

$$\mathrm{dd}^c g_D + \delta_D = \omega_D, \quad \text{a smooth } (1,1)\text{-form on } X(\mathbf{C}).$$

Continuous/smooth Metric for L: consistent way of defining the norm of local non-vanishing sections of L as continuous/smooth positive functions.

Equivalent concepts: use the formulae

$$\lambda_{D,p}(x) = -\log\|s_D\|_p(x), \qquad g_D(x) = -\log\|s_D\|_\infty(x).$$

ω_D is the **Chern form** of the metrized line bundle $\overline{L}$.

Metrized line bundles and local heights — archimedean picture

X projective complex variety purely of dimension m
$\widehat{D}_j = (D_j, g_{D_j})$, $0 \leqslant j \leqslant m$, divisors and Green functions, intersecting properly.

Multiplying m of the Chern forms ω_{D_j} furnishes a differential form of type (m, m) on $X(\mathbf{C})$, hence a signed measure.

Inductive definition of the **local height pairing:**

$$(\widehat{D}_0 \cdots \widehat{D}_m | X)_\infty = (\widehat{D}_0 \cdots \widehat{D}_{m-1} | D_m)_\infty + \int_{X(\mathbf{C})} g_{D_m} \omega_{D_0} \ldots \omega_{D_{m-1}}$$

Remarks:

- the integral converges;
- multilinear;
- independent on the order of the $\widehat{D}_j$.

Metrized line bundles and local heights — non-archimedean picture

X projective variety purely of dimension m over a p-adic field
$\widehat{D}_j = (D_j, g_{D_j})$, $0 \leqslant j \leqslant m$, divisors and "smooth" Néron functions, intersecting properly.

Local height pairing defined using **arithmetic intersection theory** on $\mathbf{Z}_p$-schemes.

There are no Chern forms anymore, but one may **define** measures on the Berkovich space X_p so that the inductive formula holds:

$$(\widehat{D}_0 \cdots \widehat{D}_m | X)_p = (\widehat{D}_0 \cdots \widehat{D}_{m-1} | D_m) + \int_{X_p} g_{D_m} \omega_{D_0} \dots \omega_{D_{m-1}}$$

Metrized line bundle and heights — admissible metrics

Both in the archimedean and non-archimedean cases, it is necessary to enlarge the settings of smooth metrized line bundles.
One good notion is that of an **admissible metrized line bundle** (Zhang, 1995).
Semipositive metrics: uniform limits of smooth metrics with positive first Chern forms, resp. given by a numerically effective model.
Admissible metric: quotient of two semipositive metrics.

For those metrics, one may define **local heights** and **measures** (ACL, 2006) by approximation.
The inductive formula still holds ("Mahler formula", ACL & Thuillier, 2009).

Metrized line bundle and heights — global picture

X projective variety over $\mathbf{Q}$, purely of dimension m
$\widehat{D}_j = (D_j, (g_{D_j,p})_p)$, $0 \leqslant j \leqslant m$, divisors and associated Green functions at all places, intersecting properly.
"Adelic condition": for almost all p, the $g_{D_j,p}$ are defined using a fixed $\mathbf{Z}$-scheme.

Global height pairing (in)finite sum over all primes:

$$(\widehat{D}_0 \dots \widehat{D}_m | X) = \sum_{p \leqslant \infty} (\widehat{D}_0 \dots \widehat{D}_m | X)_p.$$

Properties:

- multilinear;
- independent on the order of the $\widehat{D}_j$;
- vanishes for $\widehat{D}_0 = \widehat{\operatorname{div}}(f) = (\operatorname{div}(f), (\log |f|_p^{-1}))$: metrized line bundles define global heights.

Example: algebraic curves and potential theory

X algebraic curve over $\mathbf{C}$.
$o \in X$, z a local parameter at o.
K compact subset of X such that $o \notin K$.
$\widehat{o} = (o, g_K)$.
Robin constants are local heights:

$$\begin{aligned}
((\widehat{o} - \widehat{\mathrm{div}}(z)) \cdot \widehat{o} | X)_\infty &= (\widehat{o} - \widehat{\mathrm{div}}(z) | o)_\infty + \int_{X_\infty} g_K \mu_K \\
&= \lim_{q \to o} (\widehat{o} - \widehat{\mathrm{div}}(z) | q)_\infty \\
&= \lim_{q \to o} g_K(q) + \log |z(q)| \\
&= -\log \mathrm{cap}_z(K).
\end{aligned}$$

Similar computation at all places: **the self-intersection $(\widehat{o} \cdot \widehat{o})$ is the opposite of Rumely's adelic logarithmic capacity.**

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Bounding heights from below

X projective variety over $\mathbf{Q}$, purely of dimension m.
$\overline{L}$ line bundle on X plus metrics at all places.
Successive minima: $e_0 \geqslant \cdots \geqslant e_m$, with

$$e_j(\overline{L}) = \sup_{S, \operatorname{codim}(S) > j} \inf_{x \notin S} (\widehat{c}_1(\overline{L}) | x).$$

Theorem (Zhang)

If L is ample and $\overline{L}$ is semipositive, then

$$e_0(\overline{L}) \geqslant \frac{(\widehat{c}_1(\overline{L})^{m+1} | X)}{(m+1)(c_1(L)^m | X)} \geqslant \frac{1}{m+1} \left(e_0(\overline{L}) + \cdots + e_m(\overline{L}) \right).$$

The first inequality is a direct application of the analogue of Hilbert-Samuel theorem in Arakelov geometry.
The second inequality corresponds to the analogue of the Nakai-Moishezon criterion for ampleness.

Application to the theorem of Fekete–Szegő

The first minimum $e_0(\overline{L})$ is the threshold at which Zariski dense sets of points of small heights exist: the set of points $x \in X(\overline{\mathbf{Q}})$ such that $(\widehat{c}_1(\overline{L})|x) < \alpha$ is:

- not Zariski-dense if $\alpha < e_0(\overline{L})$;
- Zariski-dense if $\alpha > e_0(\overline{L})$.

Hence, the inequality $e_0(\overline{L}) \geqslant \dots$ is a "negative Fekete–Szegő": if $(\widehat{c}_1(\overline{L})^{m+1}|X) > 0$, then only finitely many points have small height.

When moreover $e_m(\overline{L}) \geqslant 0$, the second inequality implies a "positive Fekete–Szegő": if $(\widehat{c}_1(\overline{L})^{m+1}|X) \leqslant 0$, then $e_0(\overline{L}) \leqslant 0$, so both vanish — there exist points of arbitrary small positive height.

Algebraic dynamics

Let F be a number field and $\varphi \in F(T)$ be a rational function of degree $d \geqslant 2$.

Dynamical height: defined by a metrized line bundle $\overline{L}_\varphi$ characterized by $\varphi^* \overline{L}_\varphi \simeq \overline{L}_\varphi^d$.
(This metric is defined through a limit process, in the spirit of Tate's construction of the Néron–Tate height, and requires the formalism of admissible metrics.)

Functional equation:

$$(\widehat{c}_1(\overline{L}_\varphi)|\varphi(x)) = d(\widehat{c}_1(\overline{L}_\varphi)|x); \qquad (\widehat{c}_1(\overline{L}_\varphi)^2|\mathbf{P}^1_F) = 0.$$

In that case, $e_0(\overline{L}_\varphi) = e_1(\overline{L}_\varphi) = 0$.

Vanishing: $(\widehat{c}_1(\overline{L}_\varphi)|x) = 0 \Leftrightarrow x$ preperiodic

Algebraic dynamics (followed)

Petsche/Szpiro/Tucker's **dynamical pairing:**

$$\langle \varphi, \psi \rangle = (\widehat{c}_1(\overline{L}_\varphi)\widehat{c}_1(\overline{L}_\psi)|\mathbf{P}^1_F).$$

Controls the essential minimum of $(\widehat{c}_1(\overline{L}_\varphi)|x) + (\widehat{c}_1(\overline{L}_\psi)|x)$. Is positive unless strong coincidences:

$$\begin{aligned}(\widehat{c}_1(\overline{L}_\varphi)\widehat{c}_1(\overline{L}_\psi)|\mathbf{P}^1_F) &= -\frac{1}{2}\left((\widehat{c}_1(\overline{L}_\varphi) - \widehat{c}_1(\overline{L}_\psi))^2|\mathbf{P}^1_F\right) \\ &= -\frac{1}{2}\sum_v \int_{\mathbf{P}^1_{F_v}} u_v \,\mathrm{dd}^c u_v \\ &= \frac{1}{2}\sum_v \|u_v\|^2_{\mathrm{Dir}},\end{aligned}$$

where $u_v = \log(\|s\|_{\varphi,v} / \|s\|_{\psi,v})$ is the difference of the metrics of $\overline{L}_\varphi$ and $\overline{L}_\psi$ at place v, and $\|\cdot\|_{\mathrm{Dir}}$ is the **Dirichlet semi-norm.**

Algebraic dynamics (followed)

Dynamical pairing: $\langle \varphi, \psi \rangle = \frac{1}{2} \sum_v \|u_v\|^2_{\mathrm{Dir}}$.

Nonnegative; vanishes if and (almost) only if $u_v \equiv 0$ for all v.

This condition implies, **e.g.**, that all Julia sets coincide, that φ and ψ have the same preperiodic points, etc. (PhD Thesis of **Arman Mimar**, 1997)

Positivity is one aspect of the Hodge index theorem in Arakelov geometry, the other being the positivity of Néron–Tate height.

Other approach by Petsche/Szpiro/Tucker
Generalizations to any field by Baker/DeMarco, and in any dimension by Yuan/Zhang (2009).

Equidistribution

Theorem (Szpiro/Ullmo/Zhang,..., Yuan)

Assume that $(\widehat{c}_1(\overline{L})^{m+1}|X) = 0$ and that there exists a sequence (x_n) of points of $X(\overline{\mathbf{Q}})$ such that $(\widehat{c}_1(\overline{L})|x_n) \to 0$. Then for any place p, the measures $\delta(x_n)$ converge to the measure $c_1(\overline{L})^m$ on X_p.

These measures are products of Chern forms if $p = \infty$, live on Berkovich spaces otherwise.

The proof relies on a variational argument: apply Zhang's inequality to small perturbations of $\overline{L}$, at least if $\overline{L}$ is ample. For the general case, one needs to apply an estimate of arithmetic volumes due to Yuan (arithmetic analogue of a holomorphic Morse inequality proved by Demailly and Siu).

Theorem of Jentzsch–Szegő

Let $f = \sum_{j=0}^{\infty} a_j z^j \in \mathbf{C}[[z]]$ be a power series in one variable.
Radius of convergence $R \in (0, \infty)$.
Truncations $f_n = \sum_{j=0}^{n} a_j z^j$.

Theorem (Jentzsch)

Any point of the circle $C_R : \{|z| = R\}$ is a limit point of zeroes of truncations f_n.

Probability measure $\nu_n = \frac{1}{n} f_n^* \delta_0$ given by the zeroes of f_n.

Theorem (Szegő)

In a subsequence such that $|a_n|^{1/n} \to 1/R$, ν_n converges to the invariant probability measure on the circle C_R.

A general theorem

Let M be an analytic curve, $o \in M$, local parameter z at o.
Let $K \subset M$ be a compact subset such that $o \notin K$.
Let (f_n) be a sequence of regular functions on $M \setminus o$,
$k_n = -\operatorname{ord}_o(f_n)$, order of the pole.
Leading coefficient: $\operatorname{lc}_z(f_n) = \lim_{p \to o} |f_n|(p) |z(p) - z(o)|^{k_n}$.
Measures: $\nu_n = \frac{1}{k_n} f_n^* \delta_0$.

Theorem

Make the three assumptions:

- $\limsup_n \frac{1}{k_n} \log \|f_n\|_K \leqslant 0$;
- $\liminf_n \frac{1}{k_n} \log \operatorname{lc}_z(f_n) - \log \operatorname{cap}_z(K) \geqslant 0$;
- *for any compact subset* $E \subset \mathring{K}$, $\nu_n(E) \to 0$.

Then, $\nu_n \to \mu_K$.

Statement and proof inspired by Andrieveskii/Blatt's treatment of the Jentzsch–Szegő theorem.

Irreducibility of truncations

Let F be a p-adic field
$f = \sum_j a_j z^j \in F[[z]]$ a power series with radius of convergence $R \in (0, \infty)$;
f_n the truncation of f in degree n

Classical examples (Schur,...) over $\mathbf{Q}$ where all f_n are irreducible, for example $f = e^z$.

Corollary

Let d be a positive integer. In a sequence such that $|a_n|^{1/n} \to 1/R$, the number of irreducible factors of f_n whose degree is $\leqslant d$ is $o(n)$.

Irreducibility of truncations — Sketch of proof

Theorem

Let d be a positive integer. In a sequence such that $|a_n|^{1/n} \to 1/R$, the number of irreducible factors of f_n with degree $\leqslant d$ is $o(n)$.

Sketch of proof:

- The general Jentzsch–Szegő applies and shows that the measures ν_n of zeroes of f_n converge to the equilibrium measure of the disk D_R.
- That measure is the Dirac measure at the Berkovich point of the affine line corresponding to the Gauß norm of the disk D_R.
- Let F_d be the **finite** extension of F generated by numbers of degree $\leqslant d$.
- If irreducible factors of f_n with degree $\leqslant d$ weren't negligible, the measures ν_n would give some mass on the compact subset $\mathbf{P}^1(F_d)$ of $\mathbf{P}^1$.

Borel, Dwork, Pólya, Bertrandias

Let F be a number field.
For any place v of F, let K_v be a bounded subset of $\mathbf{C}_v$.
Let $f \in F[[z^{-1}]]$.

Theorem

Assume that

- *there exists a finite set S of places such that the coefficients of f belong to $\mathfrak{o}_{F,S}$;*
- *for any place v, f defines a meromorphic function on $\mathbf{C}_v \setminus K_v$;*

If, moreover, $-\sum_v \log \operatorname{cap}(K_v) > 0$, then f is the expansion of a rational function.

History: Borel, 1896 ; Pólya, 1925 ; Dwork, 1955 ; Bertrandias, 1960s ; Cantor, Rumely; Harbater, 1980s.

Formal functions on an algebraic curve

Let M be an algebraic curve over a number field F, $o \in M(F)$, z a local parameter at o.
For any place v of F, let K_v be a compact/affinoid subspace of $M(\mathbf{C}_v)$ such that $o \notin K_v$, let $U_v = M(\mathbf{C}_v) \setminus K_v$.
Let $f \in \widehat{\mathscr{O}}_{M,o}$ be a formal function on M at o.

Theorem (with J.-B. Bost)

Assume that

- *there exists a finite set S of places such that $f \in \mathfrak{o}_{F,S}[[z]]$;*
- *for any place v, f defines a meromorphic function on Ω_v.*

If, moreover, $\widehat{\deg}\,\mathrm{T}_o M = -\sum_v \log \mathrm{cap}_z(K_v) > 0$, then f is the expansion of a rational function on M.

Formal functions on an algebraic curve

Proof in two steps:

1. Algebraicity of f: techniques from Diophantine approximation.
2. Rationality of f: application of the Hodge index theorem in Arakelov geometry.

The theorem of Borel–Dwork–Pólya–Bertrandias then appears as an analogue of the following theorem in algebraic geometry:

Theorem (Hartshorne; Hironaka/Matsumura)

Let X be a smooth projective connected complex surface.
Let D be a divisor on X.
Let f be a (formal) meromorphic function on a neighborhood of D.
If the self-intersection $(D, D) = \deg \mathcal{N}_D X > 0$, then f extends uniquely to a rational function on X.