Diophantine Geometry and Analytic Spaces

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1. Introduction

Diophantine Geometry can be roughly defined as the *geometric* study of *diophantine equations*. Historically, and for most mathematicians, those equations are polynomial equations with integer coefficients and one seeks for integer, or rational, solutions; generalizations to number fields come naturally. However, it has been discovered in the XIX$^{\text{th}}$ century that number fields share striking similarities with finite extensions of the field $k(t)$ of rational functions with coefficients in a field k, the analogy being the best when k is a finite field. From this point of view, rings of integers of number fields are analogues of rings of regular functions on a regular curve. Namely, both rings are Dedekind (*i.e.*, integrally closed, one-dimensional, Noetherian) domains.

When one studies Diophantine Geometry over number fields, the geometric shape defined by the polynomial equations over the complex numbers plays an obvious important role. Be it sufficient to recall the statement of Mordell conjecture (proved by Faltings (1983)): a Diophantine equation whose associated complex shape is a compact Riemann surface of genus at least 2 has only finitely solutions. Over function fields, such a role can only be played by analytic geometry over non-archimedean fields, a much more recent theory than its complex counterpart.

The talk had been devoted to a survey of recent works in Diophantine Geometry over function fields, where analytic geometry over non-archimedean fields in the sense of Berkovich (1990) took a significant place. Since this topic was not the main one of the conference, the talk had been deliberately informal and the present notes aim at maintaining this character, in the hope that they will be useful for geometers of all obediences, be it Diophantine, tropical, complex, non-archimedean...

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2. The standard height function

In all the sequel, we fix a field F which can be, either the field $\mathbf{Q}$ of rational numbers (arithmetic case), or the field $k(T)$ of rational functions with coefficients in a given field k (geometric case). This terminology will be explained later. We let $\bar{F}$ be an algebraic closure of F.

The standard height function $h\colon \mathbf{P}^n(\bar{F}) \to \mathbf{R}$ is a function measuring the "complexity" of a point in projective space with homogeneous coordinates in $\bar{F}$.

We begin by describing it on the subset $\mathbf{P}^n(F)$ of F-rational points.

2.1. Arithmetic case. — Let $\mathbf{x}$ be a point in $\mathbf{P}^n(\mathbf{Q})$. We may assume that its homogeneous coordinates $[x_0 : \dots : x_n]$ are chosen so as to be coprime integers; they are then well defined up to a common sign and one defines

$$h(\mathbf{x}) = \log \max(|x_0|, \dots, |x_n|). \tag{2.2}$$

One first observes a *finiteness property:* for any real number B, there are only finitely many points $\mathbf{x} \in \mathbf{P}^n(\mathbf{Q})$ such that $h(\mathbf{x}) \leqslant B$. Indeed, this bound gives only finitely many possibilities for each coordinate.

The height function behaves well under morphisms. Let $f \colon \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ be a rational map given by homogeneous forms $f_0, \dots, f_m \in \mathbf{Q}[X_0, \dots, X_n]$ of degree d, without common factor. Its exceptional locus is the closed subspace in $\mathbf{P}^n$ defined by the simultaneous vanishing of all f_is. Then, one proves easily that there exists a constant c such that $h(f(\mathbf{x})) \leqslant dh(\mathbf{x}) + c$, for any point $\mathbf{x} \in \mathbf{P}^n(\mathbf{Q})$ such that $\mathbf{x} \notin E$. The converse inequality is more subtle and relies on the *Nullstellensatz: Let V be a closed subscheme of $\mathbf{P}^n$ such that $V \cap E = \varnothing$; then, there exists a constant c_X such that $h(f(\mathbf{x})) \geqslant dh(\mathbf{x}) - c_X$ for any $\mathbf{x} \in X(\mathbf{Q})$.*

To add on the subtlety behind these apparently simple estimates, let me remark that it is easy, given an explicit map f, to write down an explicit acceptable constant c; however, giving an explicit constant c_X requires a quite non-trivial statement called the *effective arithmetic Nullstellensatz.*

2.3. Geometric case. — Let now $\mathbf{x}$ be a point in $\mathbf{P}^n(k(T))$. Again, we may choose a system of homogeneous coordinates $[x_0 : \dots : x_n]$ of $\mathbf{x}$ consisting of polynomials in $k[T]$ without common factors. Such a system is unique up to multiplication by a common nonzero constant. Let us define the height of $\mathbf{x}$ by the formula

$$h(\mathbf{x}) = \max(\deg f_0, \dots, \deg f_n). \tag{2.4}$$

If the base field k is finite, then the height satisfies a similar finiteness property as in the arithmetic case: since there are only finitely many polynomials $f_i \in k[T]$ of given degree, the set of points $\mathbf{x} \in \mathbf{P}^n(k(T))$ such that $h(\mathbf{x}) \leqslant B$ is finite, for any B.

The height function has exactly the same properties with respect to morphism as in the arithmetic case.

2.5. Geometric interpretation (geometric case). — In the geometric case, the height can be given a *geometric interpretation*, free of homogeneous coordinates. Indeed, let C be the projective line over k. Any point $\mathbf{x} \in \mathbf{P}^n(F)$ can be interpreted as a morphism $\varphi_{\mathbf{x}} \colon C \to \mathbf{P}^n$ of k-schemes. When $\varphi_{\mathbf{x}}$ is generically one-to-one, then $h(\mathbf{x})$ can be computed as the degree of the the rational curve C, as embedded in $\mathbf{P}^n$ through $\varphi_{\mathbf{x}}$. In the general case, one has

$$h(\mathbf{x}) = \deg \varphi_{\mathbf{x}}^* \mathscr{O}(1), \tag{2.6}$$

that is, $h(\mathbf{x})$ is the degree of the pull-back to C of the tautological line bundle on $\mathbf{P}^n$.

2.7. Extension to the algebraic closure (geometric case). — The previous geometric interpretation suggests a way to define the height on the whole of $\mathbf{P}^n(\bar{F})$. Namely, let E be a finite extension of F; it is the field of rational functions on a projective regular curve C_E defined over a finite extension of k. Any point $\mathbf{x} \in \mathbf{P}^n(E)$ can be interpreted as a morphism $\varphi_{\mathbf{x}} \colon C_E \to \mathbf{P}^n$ and one sets

$$h(\mathbf{x}) = \frac{1}{[E:F]} \deg \varphi_{\mathbf{x}}^* \mathscr{O}(1). \tag{2.8}$$

One checks that the right-hand-side of this formula does not depend on the actual choice of a finite extension E such that $\mathbf{x} \in \mathbf{P}^n(E)$, thus defining a function $h \colon \mathbf{P}^n(\bar{F}) \to \mathbf{R}$.

2.9. Absolute values. — Using absolute values, one can give a general definition of the standard height function, valid for any finite extension of F.

Recall that an absolute value on a field F is a map $|\cdot| : F \to \mathbf{R}_{\geqslant 0}$ subject to the following axioms: $|0| = 0$, $|1| = 1$, $|ab| = |a|\,|b|$ and $|a+b| \leqslant |a| + |b|$ for any $a, b \in F$. Two absolute values $|\cdot|$ and $|\cdot|'$ are said to be equivalent if there exists a positive real number λ such that $|a|' = |a|^\lambda$ for any $a \in F$. The trivial absolute value on F is defined by $|a|_0 = 1$ for any $a \in F^*$.

Let M_F be the set of non-trivial absolute values of F, up to equivalence. Any class $v \in F$ possesses a preferred, normalized, representative, denoted $|\cdot|_v$, so that the *product formula* holds:

$$\prod_{v \in M_F} |a|_v = 1 \qquad \text{for any } a \in F^\times. \tag{2.10}$$

It connects the non-trivial absolute values (on the left-hand-side) and the trivial one (on the right-hand-side).

The field $F = \mathbf{Q}$ possesses the usual archimedean absolute value, denoted $|\cdot|_\infty$. Absolute values nonequivalent to that one are ultrametric and each of them is associated to a prime number p; the corresponding normalized p-adic absolute value is characterized by the equalities $|p|_p = 1/p$ and $|a|_p = 1$ for any integer a which is prime to p. Therefore, $M_{\mathbf{Q}} = \{\infty, 2, 3, 5, 7, \dots\}$. A similar description applies to number fields, the normalized ultrametric absolute values are in correspondence with the maximal ideals of the ring of integers, while the archimedean absolute values correspond to real or pair of conjugate complex embeddings of the field.

All absolute values of the field $F = k(T)$ are ultrametric. They correspond to the closed points of the projective line $\mathbf{P}^1_k$ (whose field of rational functions is precisely F). Similarly, if E is a finite extension of F, the set M_E is naturally in bijection with the set of closed points of the corresponding curve. The product formula is nothing but the formula that claims that the number of zeroes of a rational function on a curve is equal to the number of poles (in both cases, counted with multiplicity).

In this language, the height function on $\mathbf{P}^n(\bar{F})$ can be defined as

$$h(\mathbf{x}) = \frac{1}{[E:F]} \sum_{v \in M_E} \log \max(|x_0|_v, \dots, |x_n|_v), \tag{2.11}$$

where E is a finite extension of F and $\mathbf{x} = [x_0 : \dots : x_n] \in \mathbf{P}^n(E)$.

2.12. Properties. — In the arithmetic case, or, in the geometric case over a *finite* base field k, the height function satisfies an important *finiteness principle*, due to Northcott (1950): for any real number B and any positive integer d, the set of points $\mathbf{x} \in \mathbf{P}^n(\bar{F})$ such that $[F(\mathbf{x}) : F] \leqslant d$ and $h(\mathbf{x}) \leqslant B$ is finite. Obviously, this property does not hold in the geometric case, when the base field is infinite.

In all cases, the height function has a similar behavior with respect to morphisms. *Let* $f\colon \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ *be a rational map defined by homogeneous polynomials* $(f_0,\dots,f_m)$ *of degree* d*, without common factor; let* $E\subset \mathbf{P}^n$ *be the locus defined by* $f_0,\dots,f_m$*. Then, there exists a constant* c_f *such that* $h(f(\mathbf{x}))\leqslant dh(\mathbf{x})+c_f$ *for any* $\mathbf{x}\in\mathbf{P}^n(\bar F)$ *such that* $\mathbf{x}\notin E$*. Let* X *be a closed subscheme of* $\mathbf{P}^n$ *such that* $X\cap E=\varnothing$*; then, there exists a real number* c_X *such that* $h(f(\mathbf{f}))\geqslant dh(\mathbf{x})-c_X$ *for any* $\mathbf{x}\in X(\bar F)$*.*

3. Heights for line bundles, canonical heights

3.1. Heights for line bundles. — For applications, it is important to understand precisely the behavior of heights under morphisms. This is embodied in the following fact, called the *height machine.* Let $\mathscr{F}(X(\bar F);\mathbf{R})$ be the vector space of real valued functions on $X(\bar F)$, and let $\mathscr{F}_{\mathrm b}(X(\bar F);\mathbf{R})$ be its subspace of bounded functions. There exists a unique additive map

$$h\colon\ \mathrm{Pic}(X)\otimes_{\mathbf Z}\mathbf{R}\to\mathscr{F}(X(\bar F);\mathbf{R})/\mathscr{F}_{\mathrm b}(X(\bar F);\mathbf{R}),\quad L\mapsto h_L$$

such that for any closed embedding $f\colon X\hookrightarrow\mathbf{P}^n_F$ of X into a projective space,

$$h_{f^*\mathscr{O}(1)}\equiv h\circ f\pmod{\mathscr{F}_{\mathrm b}(X(\bar F);\mathbf{R})}.$$

Uniqueness comes from the fact that $\mathrm{Pic}(X)\otimes_{\mathbf Z}\mathbf{R}$ is generated by line bundles of the form $f^*\mathscr{O}(1)$, for some closed embedding f. The existence follows from basic properties of the height on projective spaces, namely its behavior under Segre and Veronese embeddings.

Moreover, the previous formula holds not only for embeddings f, but for any morphisms f. As a consequence, one get the desirable functoriality: *if* $f\colon Y\to X$ *is a morphism of projective algebraic varieties over* F *and* $L\in\mathrm{Pic}(X)\otimes\mathbf{R}$*, then* $h_L\circ f\equiv h_{f^*L}$ *(modulo bounded functions).*

Any function in the class h_L deserves to be called a *height function on* X *with respect to* L. However, it may be desirable to point out specific height functions with good properties. In the following paragraphs, we show some cases where this is indeed possible.

3.2. Algebraic dynamics (Tate, Silverman). — Let X be a projective variety over F, and assume that it carries a dynamical system $\varphi\colon X\to X$, and a real line bundle $L\in\mathrm{Pic}(X)\otimes\mathbf{R}$ such that that $\varphi^*L\simeq L^q$, for some real number $q>1$. Let h^0_L be some arbitrary representative of h_L; then, the following formula

$$\hat h_L(x)=\lim_{n\to\infty}q^{-n}h^0_L(\varphi^n(x))$$

defines a height function $\hat h_L$ on $X(\bar F)$ with respect to L, which is independent of the choice of h^0_L. Moreover, it satisfies the following functional equation

$$\hat h_L(\varphi(x))=q\hat h_L(x),\quad\text{for any } x\in X(\bar F).$$

In fact, it is the unique height function with respect to L which satisfies this functional equation. We cal it the *canonical height function.*

Abelian varieties furnish especially beautiful examples of this situation. In fact, if X is an Abelian variety (a projective group variety) over F and L is an ample symmetric line bundle on X (that is, $[-1]^*L\simeq L$, where $[-1]$ is the inversion on X), then $[n]^*L\simeq L^{n^2}$ for any integer n. The various canonical height functions, for all integers $n\geqslant 2$, coincide and are called the *Néron-Tate height* on X.

Similarly, projective spaces, for the maps $[x_0 : \dots : x_n] \mapsto [x_0^q : \dots : x_n^q]$ (for some integer $q \geqslant 2$) and any line bundle, and more generally toric varieties are also interesting examples.

There are also nice examples for some K3-surfaces initiated by Silverman. (There, it is useful to work with $\operatorname{Pic}(X) \otimes \mathbf{R}$, rather than $\operatorname{Pic}(X)$.)

3.3. Height functions for geometric ground fields. — Assume that $F = k(C)$ is the field of rational functions on a regular curve C which is projective, geometrically irreducible over a field k. Let X be a projective variety over F and L be a real line bundle on X. A projective k-variety $\mathscr{X}$ together with a flat morphism $\pi\colon \mathscr{X} \to C$ the generic fiber of which is X is called a *model* of X over C; any line bundle $\mathscr{L} \in \operatorname{Pic}(\mathscr{X}) \otimes \mathbf{R}$ which gives back L on X is called a model of L.

Now, let $x \in X(\bar{F})$; it is defined over a finite extension E of F which is the field of rational functions on a regular integral curve C', finite over C. By projectivity of X and regularity of C', the point x is the generic fiber of a morphism $\varepsilon_x\colon C' \to X$. Then, one can define

$$h_{\mathscr{L}}(x) = \frac{1}{[C' : C]} \deg_{C'} \varepsilon_x^* \mathscr{L}.$$

The function $h_{\mathscr{L}}$ is a height function with respect to L.

3.4. Arakelov geometry. — This point of view offers a sophisticated, but powerful, way to mimic the geometric case in order to obtain actual height functions in the arithmetic case. Let X be a projective variety over a number field F, let L be a real line bundle on X.

Let $\mathscr{X}$ be a model of X over the ring of integers $\mathfrak{o}_F$, let $\mathscr{L} \in \operatorname{Pic}(\mathscr{X}) \otimes \mathbf{R}$ be a model of L. If we observe the analogy between function fields and number fields under the point of view offered by the product formula, we see that $\mathfrak{o}_F$ behaves as the ring of regular function of an affine curve. Consequently, to get a height function, we need to compactify somehow the spectrum of $\mathfrak{o}_F$ taking into account the archimedean places of F. This is where Arakelov's ideas come in.

For any archimedean place v of F, set $\mathbf{C}_v = \overline{F_v} \simeq \mathbf{C}$ and let us endow the holomorphic line bundle L_v on the complex analytic space $X(\mathbf{C}_v)$ with a continuous hermitian metric. Such a metric is a way to define the *size* of sections of L_v. It can be defined as a continuous function on the total space of L_v inducing a hermitian norm on each fiber $L_{v,x} \simeq \mathbf{C}_v$ above a point $x \in X(\mathbf{C}_v)$. In other words, any holomorphic section s of L_v over an open subset U of $X(\mathbf{C}_v)$ is given a norm $\|s\|$, which is a continuous function $U \to \mathbf{R}_+$, in such a way that $\|fs\|(x) = |f(x)| \|s\|(x)$ for any $x \in U$ and any holomorphic function f on U, and that $\|s\|(x) \neq 0$ if $s(x) \neq 0$.

Let $\overline{\mathscr{L}} = (\mathscr{L}, (\|\cdot\|_v))$ be the datum of such a model $\mathscr{L}$ and of hermitian metrics at all archimedean places v of F. It is called an hermitian line bundle over $\mathscr{X}$. Algebraic operations on line bundles such as taking duals, or tensor products, can be done at the level of hermitian metrics, so that there is a group $\widehat{\operatorname{Pic}}(\mathscr{X})$ of isomorphism classes of hermitian line bundles on $\mathscr{X}$.

Now, for $x \in X(\bar{F})$, there is a finite extension E of F such that $x \in X(E)$, and a morphism $\varepsilon_x\colon \operatorname{Spec} \mathfrak{o}_E \to \mathscr{X}$ which extends x. We can then define

$$h_{\overline{\mathscr{L}}}(x) = \frac{1}{[E : F]} \widehat{\deg}\, \varepsilon_x^* \overline{\mathscr{L}},$$

where $\widehat{\deg}$ means the Arakelov degree, an analogue for hermitian line bundles over Spec $\mathfrak{o}_E$ of the geometric degree of line bundles over complete curves.

3.5. Adelic metrics. — One may push the analogy between number fields and function fields a bit further and do at non-archimedean places what Arakelov geometry does at archimedean places. This gives rise to the technique of adelic metrics, which works both in the geometric and in the arithmetic settings.

Let X be a projective variety over F. An adelic metric on L is a family $(\|\cdot\|_v)$ of continuous metrics on the line bundle L at all places v of F satisfying some "adelic" condition.

So let v be a place of F. First, complete F for the absolute value given by v, then take its algebraic closure; this field admits a unique absolute value extending v; take its completion for that absolute value. Let $\mathbf{C}_v$ be the field "of v-adic complex numbers" so obtained; it is complete and algebraically closed. A v-adic metric for the line bundle L can can be defined similarly as in the case of archimedean absolute values, as a continuous function on the total space of the line bundle restricting to the absolute value v on each fiber $L_x \simeq \mathbf{C}_v$ at each point $x \in X(\mathbf{C}_v)$.

If L is very ample and $(s_1, \dots, s_n)$ is a basis of $H^0(X, L)$, then there is a unique metric on L such that

$$\max(\|s_1\|_v(x), \dots, \|s_n\|_v(x)) = 1 \quad \text{for all } x \in X(\mathbf{C}_v).$$

Such a metric is called standard. A family of metrics on L will be called a standard adelic metric if it is defined by this formula for all places v of F.

More generally, a metric (or a family of metrics) on L will be called standard if one can write $L \simeq L_1 \otimes L_2^{-1}$ for two very ample line bundles L_1 and L_2, in such a way that the metric on L is the quotient of standard metrics on L_1 and L_2.

However, the field $\mathbf{C}_v$ is not locally compact, so that the resulting metrics lack good properties. Therefore, one imposes the further condition that the metric can be written as a standard metric times a function of the form e^{δ_v}, where δ_v is a continuous and bounded function on $X(\mathbf{C}_v)$. Considering families of v-adic metrics, one imposes that the function δ_v be identically 0 at almost all places v of F.

4. Level sets for the canonical height

We consider a polarized dynamical system (X, L, φ) over F, and its associated canonical height function $\hat{h}$ satisfying the functional equation $\hat{h}(\varphi(x)) = q\hat{h}(x)$ for any $x \in X(\bar{F})$, where $q > 1$. The most important case will be the one associated to Abelian varieties.

If Y is a subvariety of X and t is a real number, we let $Y(t)$ be the set of points $x \in Y(\bar{F})$ such that $\hat{h}(x) \leqslant t$.

4.1. Preperiodic points. — Let $x \in X(\bar{F})$. Its *orbit* under φ is the sequence $(x, \varphi(x), \varphi^{(2)}(x), \dots)$ obtained by iterating φ. One says that x is periodic if there exists $p \geqslant 1$ such that $x = \varphi^{(p)}(x)$. One says that x is preperiodic if its orbit is finite or, equivalently, if there are integers $n \geqslant 0$ and $p \geqslant 1$ such that $\varphi^{(n)}(x) = \varphi^{(n+p)}(x)$.

When X is an Abelian variety and φ is the multiplication by an integer $d \geqslant 2$, preperiodic points are exactly the torsion points of X. One direction is clear: if x has finite order, say m, then every multiple of x is killed by the multiplication by m; since there are finitely many points $a \in X(\bar{F})$ such that $[m]a = 0$, the orbit of x is finite. Conversely, if the orbit of x is finite, let n and $p \geqslant 1$ be integers such that $\varphi^{(n)}(x) = \varphi^{(n+p)}(x)$; this implies $[m^n]x = [m^{n+p}]x$, hence $[m^n(m^p - 1)]x = 0$, so that x is a torsion point.

The canonical height of a preperiodic point must be zero. Indeed, let x be preperiodic and let n and $p \geqslant 1$ be integers such that $\varphi^{(n)}(x) = \varphi^{(n+p)}(x)$. Computing the canonical height of both sides of the equality, we get $q^n \hat{h}(x) = q^{n+p}\hat{h}(x)$, hence $\hat{h}(x) = 0$ since $q > 1$, so that $q^n \neq q^{n+p}$.

4.2. Points of canonical height zero. — If F is a number field, or a function field over a finite field, the converse holds. Let $x \in X(\bar{F})$ be such that $\hat{h}(x) = 0$. Let E be a finite extension of F such that $x \in X(E)$. Any point in the orbit of x has canonical height zero; by Northcott's finiteness theorem, the orbit of x is finite. In fact, this statement was the main result of Northcott (1950)!

However, if F is a function field over an algebraically closed field k, Northcott's finiteness theorem is false and this property does not hold anymore. Indeed, we can consider a "constant" dynamical system (Y, M, ψ) defined over k and view it over F. Then, all points in $Y(\bar{k})$ have canonical height zero, but they are usually not preperiodic.

In fact, a theorem of Chatzidakis & Hrushovski (2008) shows that this obstruction is essentially the only one. This generalizes an old result of Lang & Néron (1959) for Abelian varieties. Because it is simpler to quote, let us only give the particular case due to Baker (2009). *Let $X = \mathbf{P}^1$ and $\varphi \in F(T)$ be a rational function of degree $q \geqslant 2$, then there exists a non-preperiodic point $x \in X(\bar{F})$ such that $\hat{h}(x) = 0$ if and only is φ is conjugate (by a homography in $\bar{F}$) to a rational function $\psi \in k(T)$.*

4.3. The geometry of points of canonical height zero. — In the 60s, motivated by the conjecture of Mordell and its extension by Zhang, Manin and Mumford had raised a question about torsion points of Abelian varieties lying in subvarieties.

Over a number field, this expectation proved to be a theorem, due to Raynaud (1983). Namely, assume that X is an Abelian variety over a number field F and let Y be a closed subvariety of X. Then the Zariski closure of $Y(0)$ is a finite union of translates of Abelian subvarieties by torsion points.

Over function fields, possible constant Abelian varieties within X create difficulties. To state the result, we must recall the existence of a maximal constant Abelian subvariety X' in X which is defined over k, and of a trace map $X' \otimes_k F \to X$. The result is that the Zariski closure of $Y(0)$ is a finite union of varieties Z such that the quotient of Z by its stabilizer G_Z is a translate of a subvariety of $(X/G_Z)' \otimes_k F$ defined over k by a torsion point of X/G_Z.

4.4. In the context of dynamical systems, the question of Manin & Mumford generalizes as follows. Let Y be a subvariety of X and let $Y(0)$ be the set of points $x \in X(\bar{K})$ such that $\hat{h}(x) = 0$. The idea is that $Y(0)$ should not be dense in Y unless this is somewhat explained by the geometry of Y with respect to φ, for example, unless Y is itself preperiodic. However, this basic expectation proved to be false, by a counterexample of Ghioca & Tucker. A subsequent paper by Ghioca *et al* (2011) tries to correct the basic prediction.

4.5. The conjecture of Bogomolov. — Still in conjunction with Mordell's conjecture, Bogomolov (1980) had raised the following strengthening of Manin–Mumford's question. Namely, if C is a curve over genus $\geqslant 2$ embedded in its Jacobian J, does there exists a positive real number ε such that $C(\varepsilon) = \{x \in C(\bar{F})\,;\, \hat{h}(x) \leqslant \varepsilon\}$ is finite?

This question has been generalized by Zhang (1995) to subvarieties of Abelian varieties over a number field: if Y is a subvariety of an Abelian variety X, does there exists a positive real number ε such that $Y(\varepsilon)$ is contained in a finite union of translates of Abelian subvarieties of X by torsion points. In other words, is it true that $Y(\varepsilon) \subset \overline{Y(0)}$ for small enough ε?

These two questions have been solved over a number fields, by Ullmo (1998) and Zhang (1998) respectively. They make a crucial use of equidistribution arguments. Soon after, David & Philippon (1998) gave another proof, still over number fields. The case of function fields is mostly open, the last part of this text will be devoted to explaining how Gubler had been able to use ideas of equidistribution to prove important cases in this setting.

5. Equidistribution (arithmetic case)

5.1. Equidistribution is a prevalent theme of analytic number theory: it is a (partially) quantitative way of describing how discrete objects collectively feature a continuous phenomenon. The most famous result is probably the equidistribution modulo 1 of multiples $n\alpha$ of some fixed irrational number α.

Here, we are interested in algebraic points $x \in X(\bar{F})$ of a variety X defined over F. To have some chance of getting some continuous phenomenon, we consider, no only the points themselves, but also their conjugates, that is, their full orbit of those points under the Galois group $\mathrm{Gal}(\bar{F}/F)$. The continuous phenomenon requires some topology, so we fix a place v of F and an embedding of $\bar{F}$ into the field $\mathbf{C}_v$.

Let x be any point in $X(\bar{F})$. Viewed from the field F, the point x is not discernible from any of its conjugates $x_1 = x, \dots, x_m$ which are obtained from x by letting the group of F-automorphisms of $\bar{F}$ act. So we define a probability measure $\mu(x)$ on $X(\mathbf{C}_v)$ by

$$\mu(x) = \frac{1}{m} \sum_{j=1}^{m} \delta_{x_j},$$

where δ_{x_j} is the Dirac measure at the point $x_j \in X(\bar{F}) \subset X(\mathbf{C}_v)$.

The first equidistribution result in this field is the following.

***Theorem 5.2* (Szpiro *et al* (1997)).** — *Assume that F is a number field and v is an archimedean place of F. Let X be an Abelian variety over F and let (x_n) be a sequence of points in $X(\bar{F})$ satisfying the following two assumptions:*

– *The Néron–Tate heights of x_n goes to 0 when $n \to \infty$;*

– *For any strict subvariety Y of X, the set of indices n such that $x_n \in Y$ is finite.*

Then the sequence of probability measures $(\mu(x_n))$ on the complex torus $X(\mathbf{C}_v)$ converges to the normalized Haar measure of $X(\mathbf{C}_v)$.

The proof uses Arakelov geometry and holds in a wider context than that of Abelian varieties. We shall see more about it shortly but I would like to describe how Ullmo (1998) and Zhang (1998) used those ideas to obtain a *proof* of Bogomolov's conjecture.

5.3. So assume that Y is a subvariety of X containing a sequence (x_n) of algebraic points such that $\hat{h}(x_n) \to 0$ and which is dense in Y for the Zariski topology. We want to show that Y is a translate of an Abelian subvariety of X by a torsion point. To that aim, we may mod out X and Y by the stabilizer of Y. The definition of the Néron–Tate height on X

comes from some ample line bundle; its Riemann form on $X(\mathbf{C}_v)$ is a positive differential form ω of bidegree $(1,1)$.

Now, a geometric result implies that there exists a positive integer m such that the map

$$\varphi\colon Y^m \to X^{m-1}, \quad (y_1,\dots,y_m) \mapsto (y_2 - y_1, \dots, y_m - y_{m-1})$$

is generically finite. From the sequence (x_n), one constructs a similar sequence (y_n) of points in $Y^m(\bar{F})$ whose height converge to zero and which are Zariski dense in Y^m; more precisely, for any strict subvariety Z of Y^m, the set of indices n such that $y_n \in Z(\bar{F})$ is finite. A variant of Theorem 5.2 (see also Theorem 5.9 below) implies that the sequence $\mu(y_n)$ of probability measures converges to the canonical probability measure on $Y^m(\mathbf{C}_v)$ given by the differential form $(\omega_1 + \cdots + \omega_m)^{md}$ on the smooth locus of Y^m. (Here, $d = \dim(Y)$ and ω_j means the differential form on Y^m coming from ω on the jth factor of Y^m.) Write $\mu(Y^m)$ for this measure; in fact, one has $\mu(Y^m) = \mu(Y)^m$. So we have the equidistribution property

$$\mu(y_n) \to \mu(Y)^m.$$

If we apply the map φ, we get automatically

$$\mu(\varphi(y_n)) \to \varphi_*\mu(Y)^m.$$

On the other hand, the sequence $(\varphi(y_n))$ also satisfies an equidistribution property, but the limit measure being $\mu(\varphi(Y))$. This implies an *equality* of probability measures

$$\varphi_*\mu(Y)^m = \mu(\varphi(Y))),$$

a geometric refinement of the initial fact that φ is generically finite with image $\varphi(Y)$.

However, both sides of this equality come from differential forms, and this equality implies that the differential forms $(\omega_1 + \cdots + \omega_m)^{md}$ and $\varphi^*(\omega_1 + \cdots + \omega_{m-1})^{md}$ on Y^m coincide up to a constant multiple.

The contradiction comes from the fact that $(\omega_1 + \cdots + \omega_m)$ is strictly positive everywhere (at least, on the smooth locus of Y^m) while $\varphi^*(\omega + 1 + \cdots + \omega_{m-1})^{md}$ vanishes where φ is not étale (that is, a local diffeomorphism), in particular on the diagonal of Y^m. To be resolved, this contradiction requires that $md = 0$, hence that Y is a point, necessarily a torsion point.

5.4. Heights for subvarieties. — One of the major ingredients in the proof of the equidistribution theorem is a notion of a height with respect to a metrized line bundle, not only for points, but for all subvarieties. The definition, first introduced by Faltings (1991), goes as follows.

We assume that X is a projective variety over a number field F and L is a line bundle on X. Let $\mathscr{X}$ be a projective flat model over the ring $\mathfrak{o}_F$ and $\mathscr{L}$ be line bundle on $\mathscr{X}$ which is a model of L. We also assume that L be induced with with smooth hermitian metrics at all archimedean places of F.

From these metrics, complex differential geometry defines differential forms $c_1(\overline{L}_v)$ on the complex analytic varieties $X(\mathbf{C}_v)$, for all archimedean places v of F. This form is called the *first Chern form*, or the *curvature form*, of the hermitian line bundle $\bar{L}_v$; it is a representative of the first Chern class of L in the De Rham cohomology of $X(\mathbf{C}_v)$. It is really a fundamental tool in complex algebraic geometry. For example, when X is smooth, say, a theorem of Kodaira asserts that L is ample if and only if it possesses a hermitian metric such that its curvature form is positive definite on each tangent space of X. I refer to Griffiths & Harris (1978) for more details.

Faltings's definition of the height of an irreducible closed subvariety $\mathscr{Y} \subset \mathscr{X}$ is by induction on its dimension.

If $\dim(\mathscr{Y}) = 0$, then $\mathscr{Y}$ is a closed point; then, its residue field $\kappa(\mathscr{Y})$ is a finite field and one defines

$$h_{\overline{\mathscr{L}}}(\mathscr{Y}) = \log \operatorname{Card}(\kappa(\mathscr{Y})). \tag{5.5}$$

Otherwise, one can consider a (nonzero) meromorphic section s of some power $\mathscr{L}^m$ of $\mathscr{L}$ on $\mathscr{Y}$. Its divisor $\operatorname{div}(s)$ is a formal linear combination of irreducible closed subschemes $\mathscr{Z}_j$ of $\mathscr{Y}$, with multiplicities a_j (the order of vanishing, or minus the order of the pole of s along $\mathscr{Z}_j$) and

$$h_{\overline{\mathscr{L}}}(\mathscr{Y}) = \frac{1}{m}\sum a_j h_{\overline{\mathscr{L}}}(\mathscr{Z}_j) + \sum_v \int_{Y(\mathbf{C}_v)} \log \|s\|^{-1/m} c_1(\overline{L}_v)^{\dim Y} \tag{5.6}$$

where $Y = \mathscr{Y} \otimes F$ and v runs over archimedean places of F. In fact, the right hand side of this formula does not depend on the choice of s.

One can prove that this new definition recovers the previous one for points. More precisely, let $y \in X(\bar{F})$, let $Y \in X$ be the corresponding closed point and let $\mathscr{Y}$ be its Zariski closure in $\mathscr{X}$.

$$h_{\overline{\mathscr{L}}}(\mathscr{Y}) = \deg(Y) h(y),$$

where $\deg(Y)$ is the degree of the closed point Y, or the degree of Y as a subvariety of X with respect to the line bundle L.

***Proposition 5.7* (Zhang (1995)).** — *Assume that L is ample, that $\mathscr{L}$ is relatively numerically effective and that the curvature forms $c_1(\overline{L}_v)$ are nonnegative for any archimedean place v of F.*

Then, Let (x_n) be a sequence of points in $X(\bar{F})$. Assume that for any strict subvariety Y of X, the set of indices n such that $x_n \in Y$ is finite. Then,

$$\liminf_n h_{\overline{\mathscr{L}}}(x_n) \geqslant \frac{h_{\overline{\mathscr{L}}}(\mathscr{X})}{\dim(\mathscr{X}) \deg_L(X)}. \tag{5.8}$$

This proposition follows easily from a (difficult) theorem in Arakelov geometry that implies the existence of global sections over $\mathscr{X}$ of large powers $\mathscr{L}^m$ which have controlled norms. Using those sections in the inductive definition of the height leads readily to the indicated inequality.

In presence of a sequence (x_n) for which the inequality is an equality, Szpiro *et al* (1997) proved that the probability measures $\mu(x_n)$ equidistribute towards the measure $\mu_{X,v} = c_1(\bar{L}_v)^{\dim(X)}/\deg_L(X)$ on $X(\mathbf{C}_v)$. The heart of the proof is to apply the fundamental inequality (5.8) for small perturbations of the hermitian metrics, as a *variational principle.* Since $X(\mathbf{C}_v)$ is compact and metrizable, the space of probability measures on $X(\mathbf{C}_v)$ is metrizable and compact, so we may assume that $\mu(x_n)$ converges to some limit μ and we need to prove that μ is proportional to $c_1(\bar{L}_v)^{\dim X}$.

Let us multiply the metric on $\bar{L}_v$ by some function of the form $e^{-\varepsilon\varphi}$, where φ is a smooth function on $X(\mathbf{C}_v)$. Then, the left hand side of the inequality (5.8) becomes

$$\lim_n \left(h_{\overline{\mathscr{L}}}(x_n) + \varepsilon \int_{X(\mathbf{C}_v)} \varphi \, \mathrm{d}\mu(x_n) \right) = \frac{h_{\overline{\mathscr{L}}}(\mathscr{X})}{\dim(\mathscr{X}) \deg_L(X)} + \varepsilon \int_{X(\mathbf{C}_v)} \varphi \, \mathrm{d}\mu,$$

while its right hand side is

$$\frac{h_{\overline{\mathscr{L}}}(\mathscr{X})}{\dim(\mathscr{X})\deg_L(X)} + \varepsilon \int_{X(\mathbf{C}_v)} \varphi \frac{c_1(\bar{L}_v)^{\dim X}}{\deg_L(X)} + \mathrm{O}(\varepsilon^2).$$

Consequently, when $\varepsilon \to 0$,

$$\varepsilon(\mu(\varphi) - \mu_{X,v}(\varphi)) \geqslant \mathrm{O}(\varepsilon^2).$$

For small positive ε, we get $\mu(\varphi) \geqslant \mu_{X,v}(\varphi)$, and we have the opposite inequality for small negative ε. Consequently, $\mu(\varphi) = \mu_{X,v}(\varphi)$, hence the equality $\mu = \mu_{X,v}$.

A subtle point of the proof lies in the possibility of applying Proposition 5.7 to the modified line bundle. When the curvature form $c_1(\overline{L})$ is strictly positive, then it remains so for small perturbations, hence the proof is legitimate. This is what happens in Theorem 5.2, and what is needed for the proof of Bogomolov's conjecture by Ullmo and Zhang.

Inspired by an inequality of Siu and the holomorphic Morse inequalities of Demailly, Yuan (2008) proved the following general equidistribution theorem.

***Theorem 5.9* (Yuan (2008)).** — *Assume that F is a number field and v is an archimedean place of F. Let X be an algebraic variety over F, let $\bar{L}$ be an ample line bundle on X with a semi-positive adelic metric. let (x_n) be a sequence of points in $X(\bar{F})$ satisfying the following two assumptions:*

- *The heights of x_n with respect to $\bar{L}$ converge to 0 when $n \to \infty$;*
- *For any strict subvariety Y of X, the set of indices n such that $x_n \in Y$ is finite.*

Then the sequence of probability measures $(\mu(x_n))$ on the complex space $X(\mathbf{C}_v)$ converges to the unique probability measure proportional to $c_1(\bar{L})^{\dim(X)}$.

I cannot say much more on this here, and I must refer the reader to the paper of Yuan (2008).

Observe anyway that under the indicated hypotheses, $c_1(\bar{L})$ is not necessarily a differential form, but only a positive current of bidegree $(1,1)$. Consequently, the definition of the measure $c_1(\bar{L})^{\dim X}$ requires some work. It goes back to fundamental work in pluripotential theory by Bedford & Taylor (1982) and Demailly (1985). In our setting, it can be defined by an approximation process, considering sequences smooth *positive* hermitian metrics on L which converge uniformly to the initial metric. See my survey Chambert-Loir (2011) for more details.

6. Measures on analytic spaces

6.1. Our setting is that of a global field F. Let X be a projective algebraic (irreducible) variety over F. For any place v of F, we will consider the analytic space X_v^{an} associated to v.

If v is archimedean, then $X_v^{\mathrm{an}} = X(\mathbf{C}_v)$ is the set of complex points of X, where $\mathbf{C}_v = \mathbf{C}$ is viewed as a F-algebra via the embedding corresponding to the place v.

When v is non-archimedean, then X_v^{an} is the analytic space over the complete algebraically closed field $\mathbf{C}_v$, as defined by Berkovich (1990). I must refer to the other contributions in this volume for background on Berkovich spaces, notably those of M. Baker and A. Ducros and I will content myself with the few following comments. First of all, X_v^{an} is a reasonable topological space: it is compact and locally pathwise connected; it is even locally contractible (this is a theorem of Berkovich when X is smooth, recently extended to the general case by Hrushovski–Loeser). Moreover, X_v^{an} contains the set $X(\mathbf{C}_v)$ as a dense subset, and the topology of X_v^{an} restricts to its natural (totally disconnected) topology on $X(\mathbf{C}_v)$. So since X_v^{an} is compact and locally connected, this is due to the presence of many more points, some of them will play a crucial role below.

6.2. To fix the ideas, assume that we are in the geometric case, so that $F = k(C)$ is the field of functions on a curve C. Let $\mathscr{X}$ be a model of X over the curve C and let $\mathscr{L}$ be a line bundle on $\mathscr{X}$; let L be its restriction to X.

For any place v of F, the model $\mathscr{L}$ gives rise to a "v-adic metric on L": any section s of L on an open subset U of X has a norm $\|s\|$ which is a continuous function on the corresponding subset U_v^{an} of X_v^{an}, and does not vanish if s does not vanish. Of course, it extends the metric we had defined earlier on $X(\mathbf{C}_v)$.

The first observation is that there exists a measure, which we write $c_1(\overline{L})_v^{\dim X}$, on X_v^{an} such that for any nonzero global section of L,

$$h_{\mathscr{L}}(\mathscr{X}) = h_{\mathscr{L}}(\overline{\operatorname{div}(s)}) + \sum_v \int_{X_v^{\mathrm{an}}} \log \|s\|_v^{-1} \, c_1(\bar{L})_v^{\dim X},$$

where $\overline{\operatorname{div}(s)}$ is the Zariski closure in $\mathscr{X}$ of the divisor of s. More generally, if Y is an integral subvariety of X, with Zariski closure $\mathscr{Y}$ in $\mathscr{X}$, and if s is any nonzero global section of $L|_Y$, then one can define a measure $c_1(\overline{L})_v^{\dim Y}\delta_Y$ on Y_v^{an} such that

$$h_{\mathscr{L}}(\mathscr{Y}) = h_{\mathscr{L}}(\overline{\operatorname{div}(s)}) + \sum_v \int_{X_v^{\mathrm{an}}} \log \|s\|_v^{-1} \, c_1(\bar{L})_v^{\dim Y}\delta_Y.$$

This measure is defined as follows, see Chambert-Loir (2006); the presentation given here, using algebraically closed valued fields is due to Gubler (2007*a*).

The analytic space $\mathscr{X}_v$ admits a canonical reduction X_v over the residue field of $\mathbf{C}_v$, which maps to the natural reduction of $\mathscr{X}$. Moreover, there is a reduction map $X_v^{\mathrm{an}} \to \mathsf{X}_v$ and the generic each irreducible component Z of X_v is the image of unique point z of X_v^{an}. By functoriality, one also has a line bundle L_v on $\mathscr{X}_v$. The measure $c_1(\bar{L})_v^{\dim X}$ is the following linear combination of Dirac measures:

$$c_1(\bar{L})_v^{\dim X} = \sum_{\mathsf{Z}} (c_1(\mathsf{L}_v)^{\dim X}|\mathsf{Z})\delta_z$$

where the coefficients $(c_1(\mathsf{L}_v)^{\dim X}|\mathsf{Z})$ are given by usual (numerical) intersection theory.

This measure is positive if $\mathscr{L}$ is relatively ample, and its total mass is equal to the degree of X with respect to L.

The definition of the measure $c_1(\bar{L})_v^{\dim Y}\delta_Y$ is analogous.

Up to its measure-theoretic formulation, the validity of the asserted formula for heights follows from work of Gubler (1998).

6.3. Zhang (1995) had defined a notion of semipositive metric, which are defined as uniform limits of metrics given by models $(\mathscr{X}, \mathscr{L})$, where $\mathscr{L}$ is vertically numerically effective—that is, gives a nonnegative degree to any subvariety. He also showed that semipositive metrized line bundles allow to define heights of subvarieties by approximation from the case of models/classical Arakelov geometry.

Adapting this construction I defined in Chambert-Loir (2006) the measures $c_1(\bar{L})_v^{\dim X}$ by approximation from the above definition in the case of models. In the end, the proof is very close to that of the existence of products of positive $(1,1)$-currents by Bedford &

Taylor (1982). (In fact, this article only considers projective varieties over a local p-adic field; the general case has been treated by Gubler (2007*a*), in a similar fashion.)

6.4. As we have shown in Chambert-Loir & Thuillier (2009), these measures can be used to recover the heights defined by Zhang. Namely, if $\bar{L}$ is a line bundle on X with a semi-positive adelic metric, Y is an integral subvariety of X, and s is a regular meromorphic section of $L|_Y$, then

$$h_{\bar{L}}(Y) = h_{\bar{L}}(\operatorname{div}(s)) + \sum_v \int_{X_v^{\mathrm{an}}} \log \|s\|_v^{-1} \, c_1(\bar{L})_v^{\dim Y} \delta_Y.$$

In the case of curves, and a few cases in higher dimensions, I showed in Chambert-Loir (2006) that they also give rise to equidistribution theorems totally analogous to the one of Szpiro *et al* (1997). The article of Yuan (2008) proved what can be considered the most general equidistribution theorem possible in this context. Namely, the v-adic analogue of Theorem 5.9 still holds. While that paper restricts to the case of number fields, its ideas have been transposed to the case of function fields by Faber (2009) and Gubler (2008).

6.5. In some cases, one can deduce from these equidistribution theorems explicit results in algebraic number theory. Let us give an example in the case of the line bundle $L = \mathscr{O}(1)$ on the projective line $X = \mathbf{P}^1$, with its metrization giving rise to the standard height. Fix an ultrametric place v of F. Then, the measure $c_1(\bar{L})_v$ on X_v^{an} is the Dirac measure at a particular point γ, called the Gauss-point because it corresponds to the Gauss-norm on the algebra $F_v[T]$ (viewed as the algebra of functions on the affine line $\mathbf{A}^1 = \mathbf{P}^1 \setminus \{\infty\}$). So in this case, the equidistribution theorem asserts that for any sequence (x_n) of distinct points on $X(\bar{F})$ such that $h(x_n) \to 0$, the measures $\mu(x_n)$ on X_v^{an} converge to the Dirac measure δ_γ.

This gives a strong constraint on such sequences. For example, it is impossible that all x_n be totally v-adic. Indeed, if x_n is totally v-adic, then the measure $\mu(x_n)$ is supported by the compact subset $X(F_v)$ of X_v^{an}. If all x_n were totally v-adic, the limit measure of $\mu(x_n)$ would be supported by $X(F_v)$, but the Gauss-point does *not* belong to $X(F_v)$. Similar results were proved by Baker & Hsia (2005).

7. Bogomolov's conjecture for totally degenerate abelian varieties

7.1. Gubler (2007*b*) had the idea of using these measures to attack the unsolved Bogomolov conjecture over function fields, using equidistribution theorems of points of small height at some place of the ground field to get a proof of the conjecture following the strategy of Ullmo (1998); Zhang (1998).

So let F be a function field and let v be a place of F. Let X be an Abelian variety over F, let $\bar{L}$ be an ample symmetric line bundle on X with its canonical adelic metric that gives rise to the Néron-Tate height $\hat{h}$. Let Y be a closed integral subvariety of X. We want to prove that $Y(\bar{F})$ does not contain a Zariski-dense sequence (y_n) of points such that $\hat{h}(y_n) \to 0$, unless Y is the translate of an Abelian subvariety by a torsion point. Assume the contrary.

We redo the same geometric reduction, assuming that the stabilizer of Y is trivial, and that the morphism $\varphi\colon Y^m \to X^{m-1}$ given by $(y_1, \dots, u_m) \mapsto (y_2 - y_1, \dots, y_m - y_{m-1})$ is generically finite, with image Z. As above, we construct a dense sequence (y_j) of small points in Y^m whose image $(\varphi(y_j))$ s a dense sequence of small points in Z. This gives two equidistribution theorems in the Berkovich spaces $(Y^m)_v^{\mathrm{an}}$ and Z_v^{an} at the chosen place v, with respect to canonical measures $\mu_v(Y^m) = c_1(\bar L|_{Y^m})_v^{m\dim Y}$ and $\mu_v(Z) = c_1(\bar L|_Z)^{\dim Z}$, where we write $\bar L|_{Y^m}$ and $\bar L|_Z$ for the metrized line bundles on Y^m and Z deduced from those naturally given by $\bar L$ on X^m and X^{m-1}. By construction, $\varphi_*\mu_v(Y^m) = \mu_v(Z)$.

To get a contradiction, we need to have more information about these measures.

7.2. If X has good reduction at v, the very definition of the measure $\mu_v(X)$ shows that it is the Dirac measure at a single point of X_v^{an}. Indeed, let $\mathscr{X}$ be the Néron model of X over the ring of integers $\mathfrak{o}_v$ of F_v; since X has good reduction, $\mathscr{X}$ is proper and smooth, and its special fiber is an Abelian variety. Then, one can show that the generic point of this fiber has a unique preimage ξ under the reduction map from the Berkovich space X_v^{an} to the special fiber. One has $\mu_v(X) = \deg_L(X)\delta_\xi$.

In the case where all of X, Y and Z have good reduction at v (this happens for almost all places v), the measures $\mu_v(Y^m)$ and $\mu_v(Z)$ are supported at a single point and the equality of measures $\varphi_*\mu_v(Y^m) = \mu_v(Z)$ gives no contradiction.

Also, if X has good reduction, the measures $\mu_v(Y^m)$ and $\mu_v(Z)$ will be supported at finitely many points and it will still be difficult to draw a contradiction.

7.3. Consequently, to succeed, this equidistribution approach needs to consider places of bad reduction of X. The case treated by Gubler (2007*b*) is the one of a *totally degenerate Abelian variety*, when the reduction is as worst as possible. (In some sense, archimedean places are places of bad reduction, and we shall see that they are at least twice as bad as the worst possible ultrametric places of bad reduction.)

Possibly after some finite extension of F, By theorems of Tate, Raynaud, Bosch, Lütkebohmert in Tate's setting of rigid analytic, extended to the Berkovich context in (Berkovich, 1990, §6.5), the analytic space X_v^{an} associated to the Abelian variety X can be written as the quotient of a torus $\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}}$ by a discrete subgroup Ω of rank g in $\mathbf{G}_{\mathrm{m}}^g(F_v)$. In fact, the torus $\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}}$ is the universal cover of the Berkovich space X_v^{an}. So these Abelian varieties have a uniformization, but only a partial one, because a complex Abelian variety is the quotient of $\mathbf{C}^g$ by a lattice of rank $2g$.

Here enters tropical geometry.

7.4. We first analyse the tropicalization of a torus. By definition, the Berkovich space of $\mathbf{G}_{\mathrm{m}}$ at the place v is the set of all multiplicative seminorms on the ring $F_v[T, T^{-1}]$ which extend the fixed absolute value on F_v. So there is a natural map from $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$ to the real line $\mathbf{R}$ that maps a semi-norm χ to the real number $-\log|\chi(T)|$. In fact, the semi-norm χ is viewed as a point x of $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$, and $|\chi(T)|$ is viewed as $|T(x)|$, so that a more natural way to write this map is $\tau\colon x \mapsto -\log|T(x)|$. An even more natural way would be to consider the map $x \mapsto |T(x)|$ from $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$ to $\mathbf{R}_+^*$, because it does not require the choice of a logarithm function.

This "tropicalization" map τ is continuous and surjective. It has a canonical section $\sigma\colon \mathbf{R} \to \mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$ for which $\sigma(t)$ is the Gauss-norm corresponding to the radius e^t:

$$|P(\sigma(t)| = \sup_{n\in\mathbf{Z}} |a_n|\, e^{nt}, \quad \text{if } P = \sum a_n T^n.$$

This section σ is a homeomorphism onto its image $S(\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}})$ which is called the *skeleton* of $\mathbf{G}_{\mathrm{m},v}^{\mathrm{an}}$.

In higher dimensions, we have a similar tropicalization map $\tau\colon \mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}} \to \mathbf{R}^g$ and a section σ whose image $S(\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}})$ is the skeleton of $\mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}}$.

In the case of a uniformized totally degenerate Abelian variety, one can tropicalize its universal cover and mod out by the image of the lattice Ω. This gives a diagram:

$$\begin{array}{ccc} \mathbf{G}_{\mathrm{m},v}^{g,\mathrm{an}} & \xrightarrow{\tau} & \mathbf{R}^g \\ \downarrow & & \downarrow \\ X_v^{\mathrm{an}} & \xrightarrow{\tau_X} & \mathbf{R}^g/\Lambda \end{array}$$

where $\Lambda = \tau(\Omega)$. Moreover, the section σ descends to a section σ_X of τ_X whose image $S(X_v^{\mathrm{an}})$ is called the skeleton of X_v^{an}. This is a real torus of dimension g in X_v^{an} onto which X_v^{an} retracts canonically.

The proof of the following two theorems is long and difficult and cannot be described here.

***Theorem 7.5* (Gubler (2007*a*), Corollary 9.9).** — *The canonical measure $c_1(\bar{L})_v^{\dim X}$ on X_v^{an} is the unique Haar measure supported by the real torus $S(X_v^{\mathrm{an}})$ of total mass $\deg_L(X)$.*

***Theorem 7.6* (Gubler (2007*a*), Theorem 1.3).** — *Let Y be an integral subvariety of X; let d be its dimension.*

The image $\tau(Y_v^{\mathrm{an}})$ is a union of simplices of $\mathbf{R}^g/\Lambda$ of dimension d.

Restricted to any of those simplices, the direct image $\tau_*(c_1(\bar{L}|_Y)_v^{\dim Y})$ on $\mathbf{R}^g/\Lambda$ of the canonical measure of Y is a positive multiple of the Lebesgue measure.

7.7. Given the last two theorems, Gubler (2007*b*) can complete the proof of the Bogomolov conjecture when the given Abelian variety has totally degenerate reduction at the place v.

Indeed, in the above situation of a generically finite map $\varphi\colon Y^m \to W \subset X^{m-1}$, one can push the equality of measures $\varphi_*\mu_v(Y^m) = \mu_v(W)$ to the tropicalization $(\mathbf{R}^g/\Lambda)^{m-1}$. Let $\nu_Y = \tau_*(\mu_v(Y))$, $\nu_W = \tau_*(\mu_v(W))$; these are measures on $(\mathbf{R}^g/\Lambda)$ and $(\mathbf{R}^g/\Lambda)^{m-1}$ respectively. Let ψ be the map $(\mathbf{R}^g/\Lambda)^m \to (\mathbf{R}^g/\Lambda)^{m-1}$ given by $(a_1,\dots,a_m) \mapsto (a_2 - a_1,\dots,a_m - a_{m-1})$. By naturality of tropicalization, one has $\tau\circ\varphi = \psi\circ\tau$, hence $\psi_*(\nu_Y^m) = \nu_W$.

Let δ be a simplex of dimension $\dim(Y)$ appearing in $\tau(Y)$. By Theorem 7.6, the restriction of the measure ν_Y to δ is a positive multiple of the Lebesgue measure. In particular, $\nu_Y(\delta) > 0$. Then δ^m is a simplex of $\tau(Y^m)$ whose image by ψ is $\psi(\delta^m)$. However, the definition of ψ shows that $\psi(\delta^m)$ has dimension $\leqslant m\dim(Y) - \dim(Y) < m\dim(Y) = \dim(W)$. Indeed, ψ is linear and the simplex δ embedded diagonally into δ^m maps to 0. By Theorem 7.6, ν_W is a sum of Lebesgue measures of $\dim(W)$-dimensional simplices, so that $\nu_W(\psi(\delta^m)) = 0$. Since $\psi_*(\nu_Y^m) = \nu_W$, it follows that $\nu_Y(\delta) = 0$. This contradiction concludes Gubler's proof of the Bogomolov conjecture when there is a place of totally degenerate reduction.

7.8. In our discussion of Manin–Mumford's conjecture over function fields, we had seen necessary to take care to constant Abelian subvarieties. They do not appear in Gubler's statement. Indeed, if an Abelian variety has totally degenerate reduction at some place, it cannot contain any constant Abelian subvariety. However, a general treatment of Bogomolov's conjecture over function fields would take them into account. A precise statement is given in the paper by Yamaki (2010), with partial generalizations of Gubler's result to cases where there is bad reduction, although not totally degenerate.

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