Igusa zeta functions and diophantine geometry

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Motivic integration and its interactions with model theory and non-archimedean analysis
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Real semi-simple Lie groups

Let G be a semi-simple Lie group with trivial center, μ a Haar measure on G.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite dimensional faithful representation of G in a real vector space V. Let $\|\cdot\|$ be a norm on $\mathrm{End}(V)$.

For any $T > 0$, let $B_T = \{g \in G;\ \|\rho(g)\| \leq T\}$ — compact in G.

Theorem (Maucourant, 2004)

When $T \to \infty$:

1. ***volume estimate:*** $\mu(B_T) \sim cT^d \log(T)^e$ *for some real number c, some rational number d and some integer e;*
2. ***convergence of measures:*** *there exists a measure μ_∞ on $\mathrm{End}(V)$ such that for any function $f \in \mathscr{C}(\mathbb{P}\,\mathrm{End}(V))$,*

$$\frac{1}{\mu(B_T)} \int_{B_T} f(\rho(g))\, \mathrm{d}\mu(g) \to \int_{\mathbb{P}\,\mathrm{End}(V)} f(g)\, \mathrm{d}\mu_\infty(g).$$

THEOREM (MAUCOURANT)

$$\mu(B_T) \sim cT^d \log(T)^e$$

$$\mu(B_T)^{-1} \int_{B_T} f(\rho(g))\, \mathrm{d}\mu(g) \to \int_{\mathbb{P}\,\mathrm{End}(V)} f(g)\, \mathrm{d}\mu_\infty(g).$$

The numbers d and e are explicitly defined in terms of the relative root system of G and the weights of ρ.

One has $0 \leq e \leq \mathrm{rank}_{\mathbb{R}}(G)$.

The measure μ_∞ is supported by a submanifold of $\mathbb{P}\,\mathrm{End}(V)$ which is bi-invariant under G.

Principle of proof: $K\mathfrak{a}^+K$-decomposition and integration formula.

Let G be a semi-simple algebraic group over $\mathbb{Q}$.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a faithful representation of G in a finite dimensional $\mathbb{Q}$-vector space V (with a unique highest weight).

For any $p \in \{\text{prime numbers}\} \cup \{\infty\}$, let $\|\cdot\|_p$ be a p-adic norm on $\mathrm{End}(V) \otimes \mathbb{Q}_p$.

Compatibility assumption: there exists a basis (e_i) of $\mathrm{End}(V)$ such that for almost all p: for any $u \in \mathrm{End}(V) \otimes \mathbb{Q}_p$ with coordinates (u_i), one has $\|u\|_p = \max(|u_i|_p)$.

Adeles: $\mathbb{A}$ = restricted product $\prod'_p \mathbb{Q}_p$.

Then, $G(\mathbb{A})$ is a locally compact group — restricted product $\prod'_p G(\mathbb{Q}_p)$.

For any $T > 0$, let $B_T = \{g = (g_p) \in G(\mathbb{A})\,;\, \prod_p \left\|\rho(g_p)\right\|_p \leq T\}$ — **compact set in $G(\mathbb{A})$.**

$$B_T = \{g = (g_p) \in G(\mathbb{A})\,;\, \prod_p \|\rho(g_p)\|_p \leq T\}$$

Fix a Haar measure μ on $\mathbf{G}(\mathbb{A})$.

THEOREM (GORODNIK, MAUCOURANT, OH, 2007)

When $T \to \infty$:

1. ***volume estimate:*** $\mu(B_T) \sim cT^a \log(T)^b$ *for some positive real number c, some rational number a and some non-negative integer b;*
2. ***convergence of measures:*** *for any function* $f \in \mathscr{C}(\mathbb{P}(\mathrm{End}(V) \otimes \mathbb{A}))$,

$$\frac{1}{\mu(B_T)} \int_{B_T} f(\rho(g))\, \mathrm{d}\mu(g) \to \int_{\mathbb{P}(\mathrm{End}(V)\otimes\mathbb{A})} f(g)\, \mathrm{d}\mu_\infty(g).$$

THEOREM (GORODNIK, MAUCOURANT, OH)

$$\mu(B_T) \sim cT^a \log(T)^b$$

$$\frac{1}{\mu(B_T)} \int_{B_T} f(\rho(g))\, \mathrm{d}\mu(g) \to \int_{\mathbb{P}(\mathrm{End}(V)\otimes\mathbb{A})} f(g)\, \mathrm{d}\mu_\infty(g)$$

Again, a and b can be computed explicitly in terms of the weights of ρ, the root system of G and the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ they possess.

The measure μ_∞ is supported by $X_\rho(\mathbb{A})$, where X_ρ is the Zariski closure of $\rho(G)$ in $\mathbb{P}\,\mathrm{End}(V)$ — DE CONCINI–PROCESI's **wonderful compactification** of G.

MOTIVATION/CONSEQUENCE OF THESE ESTIMATES

These **volume estimates**, resp. **convergence of measures** are one step in understanding the number, resp. the distribution, of

- points in $\Gamma \cap B_T$, where Γ is a lattice of the Lie group G — **lattice points in balls**;
- points in $G(\mathbb{Q}) \cap B_T$ — **rational points of "bounded height"**.

When $T \to \infty$, and for adequate representations ρ, the obtained estimates are

1. $\#(\Gamma \cap B_T) \sim V(T)/\mu(G/\Gamma)$;
2. $\#(G(\mathbb{Q}) \cap B_T) \sim V(T)/\mu(G(\mathbb{A})/G(\mathbb{Q}))$ — with a deliberately ignored twist caused by automorphic characters.

Other actors of the play:

1. Duke, Rudnick, Sarnak; Eskin, McMullen;
2. Shalika, Tschinkel, Takloo-Biglash.

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2. CONJECTURES OF BATYREV, MANIN AND PEYRE
   - Rapid preview
   - Volumes

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Rational points of algebraic varieties

A basic problem in **diophantine geometry** consists in deciding whether **diophantine equations** have solutions or not, more generally, to tell as much as possible about the set of solutions.

From a **geometrical point of view**, describe the set of rational points of algebraic varieties defined over $\mathbb{Q}$, or the set of integral points of algebraic varieties over $\mathbb{Z}$.

We are interested in varieties whose rational points are dense for the Zariski topology. We thus have to sort them according to their "arithmetic complexity", that is, their **height**.

HEIGHTS

Essential example: a point $P \in \mathbb{P}^n(\mathbb{Q})$, with homogeneous coordinates $[x_0 : \cdots : x_n]$ coprime integers, has height $H(P) = \max(|x_0|, \dots, |x_n|)$.

Finiteness property (NORTHCOTT): for any $B > 0$, there are only finitely many points $P \in \mathbb{P}^n(\mathbb{Q})$ such that $H(P) \leq B$.

Question: How many, when $B \to \infty$?

Answer (SCHANUEL): $\sim \frac{2^n}{\zeta(n+1)} B^{n+1}$.

Analytical tool: the "height zeta function", *i.e.*, the generating series

$$Z_{\mathbb{P}^n}(s) = \sum_{P \in \mathbb{P}^n(\mathbb{Q})} H(P)^{-s}.$$

Understand abscissa of convergence, meromorphic continuation, location of poles,...

THE CONJECTURE OF BATYREV, MANIN, PEYRE

Question: What happens if one restricts to points lying in a subvariety X of $\mathbb{P}^n$?

Conjectural answer (MANIN): If X is smooth, anticanonically embedded, it should be $\approx B(\log B)^{t-1}$, where $t = \operatorname{rank}\operatorname{Pic}(X)$, provided:

- you allow to enlarge the ground field;
- you exclude from X some strict algebraic subvarieties.

Refinement (PEYRE): it might even be $\sim cB(\log B)^{t-1}$, with an arithmetical description of the constant c.

Many, often non-trivial, examples:

- flag varieties (LANGLANDS's theory of Eisenstein series);
- toric varieties; equivariant compactifications of vector spaces;
- wonderful compactifications of adjoint semi-simple groups;
- Del Pezzo surfaces of degree ≥ 4...

... but a counter-example (total space of the family of diagonal cubic surfaces in $\mathbb{P}^3$).

PEYRE'S CONSTANT — THE VOLUME PART

In all cases where the conjectures have been established for the variety X, the constant c in front of the asymptotic expansion can be expressed as the product of 4 factors:

- some (uninteresting) rational numbers;
- some rational number related to the position of the anticanonical class in the effective cone;
- the cardinality of a Galois cohomology group;
- the volume of (an adequate part of) the adelic space $X(\mathbb{A})$ with respect to a suitable measure.

Peyre's Tamagawa measure

Assume $K_X = \mathscr{O}_X(-d)$ for some integer d.

- for each prime $p \leq \infty$, p-adic measure defined by suitable local gauge forms: since $K_X \sim \mathscr{O}_X(-d)$, one can take a meromorphic differential form ω with $\operatorname{div}(\omega) = -dH_0 \cap X$ and set

$$\tau_p = |\omega|_p \left(\frac{|x_0|_p}{\max(|x_0|_p, \ldots, |x_n|_p)} \right)^d$$

- find suitable convergence factors λ_p — Peyre takes $\tau_p = L_p(1, \operatorname{Pic}(\overline{X}))^{-1}$;
- define τ as the absolutely convergent product $L^*(1, \operatorname{Pic}(\overline{X})) \prod_p (\lambda_p \tau_p)$.

EXAMPLE: PEYRE'S MEASURE FOR $\mathbb{P}^n$

For $X = \mathbb{P}^n$, $d = n+1$ — Homogeneous coordinates $[x_0 : \cdots : x_n]$

On the chart $x_0 = 1$ and $|x_j| \leq 1$ for all j, one has

$$\tau_p = \frac{|\mathrm{d}x_1 \cdots \mathrm{d}x_n|_p}{\max(|x_0|_p, \ldots, |x_n|_p)^d} = |\mathrm{d}x_1|_p \cdots |\mathrm{d}x_n|_p.$$

One has $\tau_\infty(\mathbb{P}^n(\mathbb{R})) = (n+1)2^n$.

For p prime, τ_p is nothing but the canonical measure on $\mathbb{P}^n(\mathbb{Q}_p) = \mathbb{P}^n(\mathbb{Z}_p)$, hence:

$$\tau_p(\mathbb{P}^n(\mathbb{Q}_p)) = p^{-n} \# \mathbb{P}^n(\mathbb{F}_p) = 1 + p^{-1} + \cdots + p^{-n} = \frac{1 - p^{-n-1}}{1 - p^{-1}}.$$

Convergence factors: $\lambda_p = L_p(s, \mathrm{Pic}(\overline{\mathbb{P}^n})) = (1 - p^{-s})^{-1}$; hence $\lambda_p \tau_p(\mathbb{P}^n(\mathbb{Q}_p)) = 1 - p^{-n-1}$, and finally:

$$\tau(X(\mathbb{A})) = L^*(1, \mathrm{Pic}(\overline{X})) \prod (\lambda_p \tau_p(\mathbb{P}^n(\mathbb{Q}_p))) = (n+1)2^n / \zeta(n+1).$$

This is essentially **SCHANUEL's constant!**

To study more general varieties, in any projective embedding, a convenient language is that of "adelic metrics" on line bundles:

Measures: If K_X is endowed with an adelic metric, one obtains a measure on $X(\mathbb{Q}_p)$ by glueing the local gauge forms $|\omega|_p / \|\omega\|_p$ where ω is a local non-vanishing top-diff. form.

Heights: for $\overline{L} = (L, (\|\cdot\|_p))$, define a height

$$H(P) = \prod_{p\leq\infty} \|\mathsf{f}\|_p^{-1}, \qquad P \in X(\mathbb{Q}), \quad 0 \neq \mathsf{f} \in L(P).$$

More generally, for $\mathsf{f} \in \Gamma(X, L)$, one can define a height function on the adelic space of $X_{\mathsf{f}} := X \setminus \operatorname{div}(\mathsf{f})$:

$$H_{\mathsf{f}}(P) = \prod_{p\leq\infty} \left\|\mathsf{f}(P_p)\right\|_p^{-1}, \qquad P = (P_p) \in X_{\mathsf{f}}(\mathbb{A}).$$

GOAL

Generalize the results recalled at the beginning ($X_{\mathfrak{f}}$ = semi-simple group) and understand the volume of height balls, or the measure-theoretical behaviour of these height balls, when $T \to \infty$.

This requires to define convergence factors for the (essentially affine) variety $X_{\mathfrak{f}}$.

Limit measure: of the type introduced by Peyre.

Also: real/p-adic case — in all the products above, take only the corresponding factor.

Limit measure is supported on $\operatorname{div}(f)(\mathbb{Q}_p)$.

Tool: analytic properties of the Mellin transforms

$$\int_{X_{\mathfrak{f}}(\mathbb{Q}_p)} \|\mathfrak{f}\|(P)^s \,\mathrm{d}\tau_p(P), \qquad \int_{X_{\mathfrak{f}}(\mathbb{A})} H_{\mathfrak{f}}(P)^{-s} \,\mathrm{d}\tau(P).$$

Basic remark: The first one is a kind of “global” local Igusa zeta function, and the second one is an adelic version.

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   - Measures: gauge forms vs. metrics

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Measures: gauge forms vs. metrics

F, local field with a fixed Haar measure.

Let X be a smooth projective variety over a local field F, purely of dimension n,

D divisor on X, $U = X \setminus |D|$.

How to define measures on U?

1. a gauge form $\omega \in K_X(U) = \Omega^n_X(U)$ defines a measure $|\omega|$;
2. given a metric on the line bundle K_X, one may take local forms ω and define τ_X by glueing the measures $|\omega| / \|\omega\|$.
3. metric on $K_X(D)$: take local forms ω and glue the measures $\omega / \|\omega \mathsf{f}_D\|$ to define $\tau_{(X,D)}$.

If X is an equivariant compactification of a group G, then $K_X = \mathscr{O}_X(-D)$ with $|D| = X \setminus G$,

pick $\omega \in K_X(G)$ a (left-)invariant differential form,

Then, $\operatorname{div}(\omega) = -D$ and $\|\omega \mathsf{f}_D\| = \text{cst} = 1$, hence $\tau_{(X,D)} = \frac{|\omega|}{\|\omega \mathsf{f}_D\|} = |\omega|$ is a Haar measure on G.

Height balls

L effective divisor with support $|D|$, metric on L, f_L canonical section of $\mathscr{O}_X(L)$;

for $T > 0$, the inequality $\|\mathsf{f}_L\| \leq 1/T$ defines a compact subset B_T in $U(F)$;

fix a metric on $K_X(D)$, gets a measure $\tau_{(X,D)}$ on $U(F)$,

volume of B_T: $V(T) = \tau_{(X,D)}(B_T)$.

Definition

Mellin transform:

$$Z(s) = \int_0^\infty t^{-s}\, \mathrm{d}V(t) = \int_{U(F)} \|\mathsf{f}_L\|^s \tau_{(X,D)}.$$

Tauberian theory relates analytic properties of $Z(s)$ to the asymptotic behaviour of $V(T)$.

Then, the detailed asymptotic behaviour of $V(T)$ can be used to study the convergence of the probability measures $V(T)^{-1}\tau_{(X,D)}|_{B_T}$.

IGUSA ZETA FUNCTION

$$Z(s) = \int_0^\infty t^{-s}\,\mathrm{d}V(t) = \int_{U(F)} \|\mathsf{f}_L\|^s \tau_{(X,D)}.$$

Simple remark: $Z(s)$ is a kind of Igusa zeta function; it should not be looked as in integral on $U(F)$ but computed using the projective compactification $X(F)$.

Geometric assumption: over $\overline{F}$, the irreducible components D_α of D are smooth, and intersect transversally.

To simplify the exposition, I pretend here that the irreducible components of D over F are geometrically irreducible.

Consider the corresponding stratification (D_A) of X, hence D_A is a smooth subvariety of X of codimension $\#A$ (or empty).

Decomposition of divisors: $D = \sum \rho_\alpha D_\alpha$, $L = \sum \lambda_\alpha D_\alpha$.

Local computation "at infinity"

We may use finitely many local charts on $X(F)$ to study the integral $Z(s)$.

The part "around" a point $x \in D_A^\circ(F)$ can be computed as

$$\int \prod_\alpha \|\mathsf{f}_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha}\, \mathrm{d}\tau_X(x) = \int \prod_{\alpha\in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha}\, \varphi(x;y;s) \prod_{\mathrm{d}} x_\alpha \mathrm{d}y.$$

Then, the analytic properties of $Z(s)$ are completely clear and can be expressed in terms of the combinatorics of the stratification (D_A).

For example, **abscissa of convergence** =

$$\max_{\substack{D_\alpha(F)\neq\emptyset \\ \lambda_\alpha>0}} \frac{\rho_\alpha - 1}{\lambda_\alpha} \quad ;$$

order of pole = numbers of α that achieve equality;

leading coefficient = sum of integrals over all minimal stratas A consisting of such αs.

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- Definition of a Tamagawa measure
- The adelic zeta function

LOCAL MEASURES AND CONVERGENCE FACTORS

Let X be a smooth projective variety over $\mathbb{Q}$,

D effective divisor on X, $U = X \setminus |D|$.

Fix an adelic metric on $K_X(D)$; this defines measures $\tau_{(X,D),p}$ on $U(\mathbb{Q}_p)$ for all p.

To define a measure on $U(\mathbb{A})$ from these τ_p, one needs convergence factors λ_p such that the infinite product

$$\prod_p \lambda_p \tau_p(U(\mathbb{Z}_p))$$

converges absolutely.

Examples:

- X equivariant compactification of a semi-simple algebraic group G, τ_p=Haar measure: one may take $\lambda_p = 1$;
- same, but G unipotent: $\lambda_p = 1$;
- same, but G is a torus, $\lambda_p = L_p(1, X^*(G_{\overline{\mathbb{Q}}}))$;
- if $D = \emptyset$, $\lambda_p = L_p(1, \mathrm{Pic}(X_{\overline{\mathbb{Q}}}))^{-1}$.

A choice of convergence factors

Same notations: X, D, U, τ_p.

Geometric assumption: $\mathrm{H}^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$.

Two free $\mathbb{Z}$-modules with a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action:

- $\Gamma(U_{\overline{\mathbb{Q}}}, \mathscr{O}_X^*)/\overline{\mathbb{Q}}^*$;
- $\mathrm{Pic}(U_{\overline{\mathbb{Q}}})/\mathrm{torsion}$.

Virtual $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module: $\mathrm{EP}(U) = \Gamma(U_{\overline{\mathbb{Q}}}, \mathscr{O}_X^*)/\overline{\mathbb{Q}}^* - \mathrm{Pic}(U_{\overline{\mathbb{Q}}})/\mathrm{torsion}$.

Theorem

One may take $\lambda_p = L_p(1, \mathrm{EP}(U))$ for all $p < \infty$.

Definition

Global measure on $U(\mathbb{A})$:

$$\tau_{(X,D)} = L^*(1, \mathrm{EP}(U))^{-1} \prod_{p<\infty} \left(L_p(1, \mathrm{EP}(U))\tau_{(X,D),p}\right)\tau_{(X,D),\infty}.$$

HEIGHT ON THE ADELIC SPACE $U(\mathbb{A})$

Let L be an effective divisor supported on $|D|$

f_L the canonical section of $\mathscr{O}_X(L)$

Choosing an adelic metric on $\mathscr{O}_X(L)$, one get a **height function on the adelic space** $U(\mathbb{A})$ defined by

$$H_L((x_p)) = \prod_{p\le\infty} \left\|\mathsf{f}_L(x_p)\right\|_p^{-1}.$$

PROPOSITION

The function H_L defines a continuous exhaustion of $U(\mathbb{A})$.

Height ball: compact subset $B_T = \{x \in U(\mathbb{A})\,;\, H_L(x) \le T\}$.

Volume and zeta function:

$$V(T) = \tau_{(X,D)}(B_T), \qquad Z(s) = \int_0^\infty t^{-s}\,\mathrm{d}V(t) = \int_{U(\mathbb{A})} H_L(x)^{-s}\,\mathrm{d}\tau_{(X,D)}(x).$$

Modulo absolute convergence, one has

$$Z(s) = L^*(1, \mathrm{EP}(U))^{-1} \prod_{p<\infty} \big(L_p(1, \mathrm{EP}(U)) Z_p(s)\big) Z_\infty(s),$$

where for $p \leq \infty$,

$$Z_p(s) = \int_{U(\mathbb{Q}_p)} \|\mathsf{f}_L(x)\|^s \, \mathrm{d}\tau_{(X,D),p}(x)$$

is the p-adic Igusa zeta function described previously.

Recall the decomposition $D = \sum \rho_\alpha D_\alpha$, $L = \sum \lambda_\alpha D_\alpha$, as well as the transversality assumption on the D_α. Then, choosing compatibly adelic metrics on $\mathscr{O}(D_\alpha)$, one has:

$$Z_p(s) = \int_{X(\mathbb{Q}_p)} \prod_\alpha \left\|\mathsf{f}_{D_\alpha}\right\|^{s\lambda_\alpha - \rho_\alpha} \mathrm{d}\tau_{X,p}(x).$$

The previous computation in charts shows that it converges absolutely for $\Re(s) > \max((\rho_\alpha - 1)/\lambda_\alpha)$.

Denef's formula

For almost all p, one can give a precise formula for $Z_p(s)$ in terms of the reduction mod. p of the whole situation. This done by adapting the method used by J. Denef to prove that the degrees of the local zeta functions are bounded when one makes the prime number p vary.

Proposition

For p large enough, and for any complex number s such that $\Re(s) > (\rho_\alpha - 1)/\lambda_\alpha$, one has

$$Z_p(s) = \sum_A p^{-\dim X} \# D_A^\circ(\mathbb{F}_p) \prod_{\alpha\in A} \frac{p-1}{p^{s\lambda_\alpha - \rho_\alpha + 1} - 1}.$$

This follows from the fact that the local computation in charts around $x \in D_A^\circ$ can be done using étale coordinates $((x_\alpha)_{\alpha\in A}, y)$ such that $\left\| \mathsf{f}_{D_\alpha} \right\| = |x_\alpha|$, etc., and from the explicit computation:

$$\int_{p\mathbb{Z}_p} |x|^s \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_{p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p} p^{-ns} \,\mathrm{d}x = \sum_{n=1}^{\infty} p^{-ns-n}\left(1 - \frac{1}{p}\right) = p^{-1} \frac{p-1}{p^{s+1}-1}.$$

Meromorphic continuation of an Euler product

Let $\sigma = \max(\rho_\alpha/\lambda_\alpha)$, let $A(L,D)$ be the set of α where equality is achieved

One can deduce from Denef's formula that for $\Re(s) > \sigma - \varepsilon$,

$$Z_p(s) = p^{-\dim X} \# U(\mathbb{F}_p) \prod_{\alpha \in A(L,D)} (1 + p^{-s\lambda_\alpha - \rho_\alpha + 1})$$

hence

1. $\prod L_p(1, \mathrm{EP}(U)) Z_p(s)$ converges absolutely for $\Re(s) > \sigma$;
2. $\prod L_p(1, \mathrm{EP}(U)) Z_p(s) \prod_{\alpha \in A(L,D)} (1 - p^{-s\lambda_\alpha + \rho_\alpha - 1})$ converges absolutely for $\Re(s) > \sigma - \varepsilon$.

Consequently, one obtains a meromorphic continuation of the form

$$Z(s) = L^*(1, \mathrm{EP}(U))^{-1} \prod_p \big(L_p(1, \mathrm{EP}(U)) Z_p(s) \big) = \varphi(s) \prod_{\alpha \in A(L,D)} \zeta(\lambda_\alpha (s - \sigma) + 1),$$

with

$$\varphi(1) = \prod_{\alpha \in A(L,D)} \zeta^*(1) \int_{X(\mathbb{A}_F)} \prod_{\alpha \notin A(L,D)} H_{D_\alpha}(x)^{\rho_\alpha - \sigma\lambda_\alpha} \, \mathrm{d}\tau_X(x).$$

Conclusion

Let E be the divisor $\sigma L - D$; it is effective and its support is contained in $|D|$.

Let $t = \#A(L,D)$.

Some more calculation implies:

$$\lim_{s\to\sigma} Z(s)(s-\sigma)^t \prod_{\alpha\in A(L,D)} \lambda_\alpha = \int_{X(\mathbb{A})} H_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

Using tauberian theorems, we deduce:

Theorem

When $T\to\infty$,

1. *one has the asymptotic expansion $V(T) = \tau_{(X,D)}(B_T) \sim B^\sigma (\log B)^{t-1} \left(\sigma(t-1)!\prod_{\alpha\in A(L,D)} \lambda_\alpha\right)^{-1} \int_{X(\mathbb{A})} H_E(x)^{-1}\,\mathrm{d}\tau_X(x)$;*
2. *the probability measures $V(T)^{-1}\tau_{(X,D)}|_{B_T}$ equidistribute to the measure*

$$\frac{1}{\int_{X(\mathbb{A})} H_E(x)^{-1}\,\mathrm{d}\tau_X(x)} H_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

Two more comments

1. The comparison of the geometric estimates we obtained with those of Maucourant *et al.* is an exercice for specialists of wonderful compactifications of algebraic groups.
2. The question, raised by A. Macintyre at the beginning of the conference, of quantifier elimination in adele rings prompted to me the analogy with results of Comte/Lion/Rolin concerning the behaviour of volumes of parametrized subsets of $\mathbb{R}^m$.
   Do they have an adelic analogue?