Forms and currents on Berkovich spaces

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Goal: propose a theory that allows for some calculus with differential forms, currents, etc. on Berkovich spaces

Motivation: non-archimedean (aspects of) Arakelov geometry

Put non-archimedean places on the same footing as archimedean ones

To fix ideas, let $\mathscr{X}$ be a projective flat scheme over $\mathbf{Z}$, normal, let $\overline{L}$ be a hermitian line bundle on $\mathscr{X}$.
Faltings used Arakelov geometry to define inductively the height of any subvariety $\mathscr{Y}$ of $\mathscr{X}$: if s is a nonzero section of L^m on $\mathscr{Y}$, then

$$h_L(\mathscr{Y}) = \frac{1}{m} h_L\big(\operatorname{div}(s)\big) + \int_{\mathscr{X}(\mathbf{C})} \log \|s\|^{-1/m} c_1(\overline{L})^d \delta_Y,$$

where $\operatorname{div}(s)$ is the divisor of s (a cycle) and $d = \dim \mathscr{Y}(\mathbf{C})$.
Here, $c_1(\overline{L})$ is the curvature form of the hermitian line bundle $\overline{L}$ on $X(\mathbf{C})$, and $c_1(\overline{L})^d \delta_Y$ is the measure on $\mathscr{Y}(\mathbf{C})$ given by its dth power.

Using Berkovich spaces, one can rewrite this formula so that for any subvariety Y of $X = \mathscr{X}_{\mathbf{Q}}$, and any nonzero section s of L^m on Y,

$$h_{\overline{L}}(\overline{Y}) = \frac{1}{m} h_{\overline{L}}(\overline{\operatorname{div}(s)}) + \sum_{p<\infty} \int_{X_p^{\mathrm{an}}} \log \|s\|^{-1/m} c_1(\overline{L})^d \delta_Y + \int_{Y(\mathbf{C})} \log \|s\|^{-1/m} c_1(\overline{L})^d \delta_Y.$$

The integral at p takes care of the vertical components at p of the divisor of s, viewed as a (meromorphic) regular section on the Zariski closure $\overline{Y}$ of Y in $\mathscr{X}$.

The point is the following: if $\mathscr{Y}$ is normal, any irreducible component of the special fiber $\mathscr{Y}_{\mathbf{F}_p}$ is the reduction of a unique point of the Berkovich space Y_p^{an}.

The measure $c_1(\overline{L})^d \delta_Y$ is a linear combination of Dirac masses at those points.

Let k be a complete valued field.
Let X be a proper k-analytic space, $\overline{L}$ a line bundle on X with a continuous metric.

Assume that this metric is **semipositive** in the sense of Zhang — a uniform limit of metrics associated to models $(\mathscr{X}_m, \mathscr{L}_m)$ such that $\mathscr{L}$ is nef on the special fiber.

THEOREME

The discrete measures $c_1(\overline{L_m})^d \delta_Y$ converge to a unique, well-defined measure $c_1(\overline{L})^d \delta_Y$.

Gubler: explicit computation of this measure for abelian varieties.

Applications of these measures

- equidistribution theorems of points of small height on Berkovich spaces (ACL, Favre/Rivera-Letelier, Baker/Rumely; Thuillier; Gubler, Yuan; Szpiro/Tucker...)
- application to the Bogomolov conjecture (Gubler, Yamaki)
- non-archimedean analogue of the Monge-Ampère problem (Kontsevich/Tschinkel; Yuan/Zhang, Liu, Boucksom/Favre/Jonsson...)
- (pluri)potential theory (Baker/Rumely; Thuillier)

What next?

Despite the notation chosen for the measures $c_1(\overline{L})^d\delta_Y$, there is no definition of an analytic object $c_1(\overline{L})$ on Berkovich spaces.

For **curves**, this is enough to define a complete analytic Arakelov theory (Thuillier).

In higher dimensions, the non-Archimedean Arakelov geometry of Bloch/Gillet/Soulé proposes a definition of a first Chern curvature form of $\overline{L}$, but it is of an algebraic nature, lying in a limit of Chow groups of models of X. Moreover, their definition requires resolution of singularities.

Differential forms/currents on Berkovich spaces

I want to explain a definition of differential forms/currents that seems to be suitable for an analytic Arakelov geometry in any dimension:

- the Poincaré-Lelong equation holds;
- the theory recovers the previous measures.

The **tools** we use are:

- A notion of superforms, supercurrents, and a $d/d^{\sharp}$-calculus on real spaces invented by **Aron Lagerberg** (Math. Zeit, 2011) to adapt ideas from complex pluripotential theory to convex function theory and tropical geometry;
- Analytic tropical geometry, in the spirit of **Walter Gubler**, but in a local setting.

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SUPERFORMS (AFTER LAGERBERG)

Let V be a finite dimensional affine space over $\mathbf{R}$.
A (smooth) (p,q)-superform over an open subset U of V is an element of

$$\mathscr{A}^{p,q}(U) = \mathscr{C}^\infty(U) \otimes \bigwedge^p \vec{V}^* \otimes \bigwedge^q \vec{V}^*.$$

After choosing an origin of V and a basis of $\vec{V}$, it can be written

$$\omega = \sum_{|I|=p, |J|=q} \omega_{IJ}(x)\, \mathrm{d}' x_I \otimes \mathrm{d}'' x_J,$$

for some smooth functions ω_{IJ} on U.

Bi-graded algebra $\mathscr{A}^{*,*}(U)$:

$$\omega \wedge \omega' = (-1)^{(p+q)(p'+q')} \omega' \wedge \omega.$$

Bi-graded differential calculus: d', d''

Involution J given by $\mathrm{J}\,\mathrm{d}' x_i = \mathrm{d}'' x_i$.
Symmetry condition for forms of type (p,p): $\mathrm{J}\omega = (-1)^p \omega$.
$\mathrm{d}'' = \mathrm{J}\,\mathrm{d}'\,\mathrm{J}$.

Logarithm map: $\tau\colon (\mathbf{C}^*)^n \to \mathbf{R}^n$, $(z_1,\dots,z_n) \mapsto (\log|z_1|,\dots,\log|z_n|)$

Via

$$\mathrm{d}' x_j \to \mathrm{d}\log\left|z_j\right|, \quad \mathrm{d}'' x_j \to \mathrm{d}\arg(z_j),$$

superforms in $\mathscr{A}^{p,q}(U)$ pull back to smooth differential forms forms on $\tau^{-1}(U)$ which are invariant under the action of the compact group $(\mathbf{S}_1)^n$,

(p,q)-superforms give rise to forms of degree $p+q$, but not of degree (p,q).

Symmetric forms of type (p,p) lift to real forms of degree (p,p).

Integration of superforms

Assume $V = \mathbf{R}^n$. If ω is a (n,n)-superform on an open subset $U \subset V$, one defines

$$\int_U \omega = \int_U \omega^\sharp(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n, \qquad \omega = \omega^\sharp\,\mathrm{d}'x_1 \wedge \mathrm{d}''x_1 \dots.$$

It **depends** on the choice of a coordinate system.

Volume-vector: element of

$$\left|\bigwedge^n \vec{V}\right| = \mathrm{Or}(\vec{V}) \times^{\{\pm 1\}} \bigwedge^n \vec{V}.$$

A volume-vector μ gives rise to a well-defined integral

$$\int_U \langle \omega, \mu \rangle,$$

for $\omega \in \mathscr{A}^{n,n}(U)$.

Let P be a piecewise linear (PL) subspace of V, of dimension p.
Calibration of P: choice, for every p-dimensional face F of P, of a volume vector $\mu_F \in \left|\bigwedge^p \vec{V}\right|$. — Up to refinement of a PL decomposition of P.
Once P has been calibrated, one can integrate (p,p)-superforms on P.

Boundary calibration: Let F be a $(p-1)$-dimensional face of P, choose an orientation ε_F of $\vec{F}$.
For each p-dimensional face G of P of which F is a boundary, endow $\vec{G}$ with the "outgoing normal" orientation. Then, μ_G is identified with a p-vector in $\bigwedge^p \vec{G}$. Set

$$\partial\mu_F = \sum_{G\supset F} \mu_G \qquad \text{(up to the chosen orientation of } F\text{)}.$$

One can integrate a $(p-1,p)$-form with respect to this "boundary calibration". Beware: its support can be larger than ∂P.

Let P be a calibrated PL subspace of dimension p. Let ω be a $(p-1,p)$-superform.

Stokes formula:

$$\int_P \langle \mathrm{d}'\omega, \mu\rangle = \int_P \langle \omega, \partial\mu\rangle.$$

Green formula: Let α and β be (a,a) and (b,b)-symmetric forms on P, with $a+b=p-1$; then

$$\int_P \langle \alpha\wedge \mathrm{d}'\mathrm{d}''\beta - \mathrm{d}'\mathrm{d}''\alpha\wedge\beta, \mu\rangle = \int_P \langle \alpha\wedge \mathrm{d}''\beta - \mathrm{d}''\alpha\wedge\beta, \partial\mu\rangle.$$

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MOMENTS, TROPICALIZATIONS OF BERKOVICH SPACES

Let k be a complete ultrametric field.
Let X be a k-analytic space.

Moment: morphism $f\colon X \to T \simeq \mathbf{G}_{\mathrm{m}}^n$ to a split torus.

Tropicalization of a moment: $f_{\mathrm{trop}}\colon X \to T_{\mathrm{trop}} \simeq \mathbf{R}^n$.

Crucial fact (Berkovich/quantifier elimination in ACVF): if X is compact, then $f_{\mathrm{trop}}(X)$ is a PL subspace of $\mathbf{R}^n$.

Tropical chart: $(f\colon X \to T, P)$ where f is a moment and P is a compact PL-subspace containing $f_{\mathrm{trop}}(X)$.

Presheaf: For any open $U \subset X$, let $\mathscr{A}^{p,q}_{\mathrm{pre}}(U)$ be the inductive limit of all spaces $\mathscr{A}^{p,q}(P)$ over all tropical charts $(f\colon U \to T, P)$ of U.

The associated sheaf $\mathscr{A}^{p,q}_X$ is the **sheaf of smooth forms** of type (p,q) on X.

Sheaf of bi-graded algebras $\mathscr{A}^{*,*}_X$:

$$\omega \wedge \omega' = (-1)^{(p+q)(p'+q')} \omega' \wedge \omega.$$

Bi-graded differential calculus: d', d''

Involution J given by $\mathrm{J}\,\mathrm{d}' x_i = \mathrm{d}'' x_i$.

Symmetry condition for forms of type (p,p): $\mathrm{J}\omega = (-1)^p \omega$.

$\mathrm{d}'' = \mathrm{J}\,\mathrm{d}'\,\mathrm{J}$.

Smooth functions

$\mathscr{A}_X^0 := \mathscr{A}_X^{0,0}$ is the sheaf of **smooth functions** on X.
A function f is smooth if it can be locally written as a $\mathscr{C}^\infty$-function in logarithms of absolute values of invertible functions.

Stone-Weierstrass

Assume X is good and Hausdorff. Let U be an open subset of X.
Any continuous function on X with compact support in U can be uniformly approximated by smooth functions with compact support contained in U.

Partitions of unity

If X is **good** and **paracompact**, then these sheafs $\mathscr{A}_X^{p,q}$ are fine.

Let X be a compact analytic space, let $f\colon X \to T$ be a moment on X. Let $p = \dim X$.
Then $P = f_{\text{trop}}(X)$ is a compact PL-subspace of T_{trop}. The following construction **calibrates** P; it is valid once chosen a fine enough PL-decomposition of P.

Let F be any p-dimensional face of P. Fix an affine morphism of tori $q\colon T \to \mathbf{G}_{\mathrm{m}}^p$ such that $q_{\text{trop}}\colon \langle F\rangle \to \mathbf{R}^p$ is an isomorphism and $q_{\text{trop}}(\mathring{F}) \cap (q\circ f)_{\text{trop}}(\partial X) = \varnothing$.
let $\sigma\colon \mathbf{R}^p \to T_{\text{trop}}$ be its unique section with image $\langle F\rangle$.

Skeleton of a torus: subspace $\mathrm{S}(\mathbf{G}_{\mathrm{m}}^p) \subset \mathbf{G}_{\mathrm{m}}^p$ such that the composition

$$\mathrm{S}(\mathbf{G}_{\mathrm{m}}^p) \subset \mathbf{G}_{\mathrm{m}}^p \xrightarrow{\text{trop}} \mathbf{R}^p$$

is a topological homeomorphism.
The point η_r corresponding to the point $r = (r_1, \ldots, r_p) \in \mathbf{R}^p$ is the following multiplicative seminorm on the algebra $k[T_1^{\pm 1}, \ldots, T_p^{\pm 1}]$:

$$\left\| f \right\|_r = \sup \left| f_I \right| e^{i_1 r_1 + \cdots + i_p r_p}, \qquad f = \sum f_I T_1^{i_1} \ldots T_p^{i_p}.$$

Let $q_{\mathrm{trop}}(\mathring{F})_{\mathrm{sk}}$ be the inverse image of $q_{\mathrm{trop}}(\mathring{F})$ in $\mathrm{S}(\mathbf{G}_{\mathrm{m}}^p)$.

PROPOSITION

The map $q \circ f \colon X \to \mathbf{G}_{\mathrm{m}}^p$ is finite and flat above each point of $q_{\mathrm{trop}}(\mathring{F})_{\mathrm{sk}}$, and has a degree $d_F > 0$.

X a compact analytic space of dimension p,
moment: $f\colon X \to T$,
$P = f_{\mathrm{trop}}(X)$, compact polytope.
F, p-dimensional face of P,
$q\colon T \to \mathbf{G}_{\mathrm{m}}^{p}$, affine morphism of tori such that $q_{\mathrm{trop}}\colon \langle F\rangle \simeq \mathbf{R}^{p}$ and $q_{\mathrm{trop}}(\mathring{F}) \cap (q\circ f)_{\mathrm{trop}}(\partial X) = \varnothing$.
$\sigma\colon \mathbf{R}^{p} \to T_{\mathrm{trop}}$, unique section of q with image $\langle F\rangle$.
$d_F > 0$, degree of the map $q\circ f\colon X \to \mathbf{G}_{\mathrm{m}}^{p}$ at each point of $q_{\mathrm{trop}}(\mathring{F})_{\mathrm{sk}}$.

Volume-vector of the face *F*:

$$\mu_F = d_F\,\sigma_*(\left|\mathrm{e}_1 \wedge \dots \mathrm{e}_p\right|).$$

It does not depend on the choice of q.

Canonical calibrations: integral of forms

X a compact analytic space of dimension p,
moment: $f\colon X \to T$,
$P = f_{\mathrm{trop}}(X)$, compact polytope.
The canonical calibration μ of P allows to define

- the integral of a (p,p)-form $\omega = f^*\alpha$, where α is a (p,p)-superform on P.
- the integral of a $(p-1,p)$-form $\omega = f^*\alpha$, where α is a $(p-1,p)$-superform on P.

The following is an analog of the balance condition (Speyer,...) in tropical geometry.

Theoreme (Harmony condition)

Let X be compact n-dimensional analytic space. Let $f\colon X \to T$ be a moment, μ the canonical calibration of $f_{\mathrm{trop}}(X)$.
Every $n-1$-dimensional face of $f_{\mathrm{trop}}(X)$ which is not contained in $f_{\mathrm{trop}}(\partial X)$ does not contribute to the calibration $\partial\mu$.

Integral of superforms on analytic spaces

Let X be a n-dimensional analytic space X and ω be a (n,n)-form on X.
Definition of $\int_X \omega$. — Locally, $\omega = f^*\alpha$ for some moment $f\colon X \to T$ and a (p,p)-superform α on a calibrated polytope in T_{trop}; just integrate α!

The integral $\int_{\partial X} \omega$ of a $(n-1,n)$-form ω is defined analogously.
The harmony condition implies:

Proposition

If the support of a $(n-1,n)$-form ω does not meet ∂X, then $\int_{\partial X}\omega = 0$.

Proposition (Stokes/Green formulas)

For $\omega \in \mathscr{A}^{n,n}(X)$, symmetric $\alpha \in \mathscr{A}^{p,p}(X)$, $\beta \in \mathscr{A}^{q,q}(X)$, with $p+q=n-1$:

$$\int_X \mathrm{d}'\omega = \int_{\partial X} \omega.$$

$$\int_X \alpha \wedge \mathrm{d}'\mathrm{d}''\beta - \mathrm{d}'\mathrm{d}''\alpha \wedge \beta = \int_{\partial X} \alpha \wedge \mathrm{d}''\beta - \mathrm{d}''\alpha \wedge \beta.$$

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Spaces of currents

Let X be a purely n-dimensional k-analytic space without boundary.
The space $\mathscr{D}^{p,q}(X)$ of **(p,q)-currents** is defined as the topological dual of $\mathscr{A}_c^{n-p,n-q}(X)$ (smooth superforms with compact support).

Say a sequence of (p,q)-differential forms converges to zero is locally they can ultimately be defined using the same moment and a sequence of superforms which converges to 0, as well as all the derivatives of their coefficients.

Differential operators, d', d'', defined by duality.

Examples

- Differential forms
- Integration current on a closed subvariety
- — on a closed analytic domain

Poincaré–Lelong equation

Let f be a regular meromorphic function on X.
By linearity, one defines a $(1,1)$-current $\delta_{\operatorname{div}(f)}$ — integration current over the cycle $\operatorname{div}(f)$.

Proposition (Poincaré–Lelong equation)

$$\mathrm{d}'\,\mathrm{d}''\log\left|f\right| = \delta_{\operatorname{div}(f)}.$$

The proof combines:

- the Green formula;
- for a tropicalized affinoid V and $f \in \mathscr{O}(V)$ (non zero-divisor), the "constancy", for t small enough, of the calibered polytopes of $V \cap \{\left|f\right| = t\}$.

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METRIZED LINE BUNDLES AND THEIR CURVATURE

Let L be a line bundle on X.

A (smooth/continuous/...) **metric** on L is the data, for any local section s of L of a nonnegative function $\|s\|$, such that

- for any local function f and any local section s, $\|fs\| = |f| \, \|s\|$;
- $\|s\|(x) = 0$ iff $s(x) = 0$;
- if s does not vanish, then $\log\|s\|$ is smooth/continuous/...

A smoothly metrized line bundle $\overline{L}$ has a **curvature form**, given locally as

$$c_1(\overline{L}) = \mathrm{d}'\mathrm{d}''\log\|s\|^{-1}, \quad \text{for any local non-vanishing section } s.$$

If the metric is only continuous, then $c_1(\overline{L})$ is a current.

Extension of the **Poincaré–Lelong equation:** for any regular meromorphic section s of L,

$$\mathrm{d}'\mathrm{d}''\log\|s\|^{-1} + \delta_{\mathrm{div}(s)} = c_1(\overline{L}).$$

Compatibility with intersection theory: if $\overline{L}$ is smooth and X is proper, then

$$\int_X c_1(\overline{L})^n = \deg_L(X).$$

Totally degenerate abelian varieties

Assume X is a totally degenerate abelian variety, with Tate uniformisation $\pi\colon \mathbf{G}_{\mathrm{m}}^n \to \mathbf{G}_{\mathrm{m}}^n/M = X$, where $M \simeq \mathbf{Z}^n$ is a lattice.
Any line bundle L can be described (Bosch–Lütkebohmert) as the quotient of the trivial line bundle on $\mathbf{G}_{\mathrm{m}}^n$ by an action of M of the form

$$m\cdot(x,a) = (mx, r(m)\langle x,\lambda(m)\rangle a), \quad x\in\mathbf{G}_{\mathrm{m}}^n, \quad a\in\mathbf{G}_{\mathrm{a}}, \quad m\in M,$$

where $\lambda\colon M\to\mathbf{Z}^n$ is a group morphism and $r\colon M\to\mathbf{G}_{\mathrm{m}}$ is a map such that

$$\langle m_1,\lambda(m_2)\rangle = r(m_1m_2)r(m_1)^{-1}r(m_2)^{-1}.$$

This allows to describe explicitly the cubical metric on L, at least after pull-back to $\mathbf{G}_{\mathrm{m}}^n$. **It is smooth!**
View M as a lattice in $\mathbf{R}^n$ via the tropicalization map $\mathbf{G}_{\mathrm{m}}^n\to\mathbf{R}^n$. The map $m\mapsto \log|\langle m,\lambda(m)|^{-1}$ extends to a quadratic form q_L on $\mathbf{R}^n$. Then,

$$\pi^* c_1(\overline{L}) = \mathrm{d}'\mathrm{d}''\pi^* q_L.$$

This allows to show that the (n,n)-form $c_1(\overline{L})^n$ is supported by the skeleton of X, and is invariant by translation.

FORMAL METRICS

Let $\mathscr{X}$ be a formal model of X, and $\mathscr{L}$ be a line bundle on $\mathscr{X}$ with generic fiber L.

This defines a metric on L, but this metric is not smooth.

Assume that $\mathscr{X}$ is proper and the reduction of $\mathscr{L}$ is nef. Then, locally around a point $x \in X$, it can be defined by a maximum of absolute values of invertible functions:

$$\|s\| = \max(|f_1|, \dots, |f_m|).$$

Then $c_1(\overline{L})$ is a closed positive (1,1)-current.

The **product of currents** $c_1(\overline{L})^n$ is defined by approximation, as in complex pluripotential theory (Bedford-Taylor, Demailly). It coincides with the measures evoked at the beginning of the talk.

For simplicity, take $\mathscr{X} = \mathbf{P}_2$ and $\mathscr{L} = \mathscr{O}(1)$. The "Weil metric" is given by

$$\|s_P\|([x_0 : x_1 : x_2]) = \frac{|P(x_0, x_1, x_2)|}{\max(|x_0|, |x_1|, |x_2|)},$$

for P homogeneous of degree 1.

The space $\mathbf{P}_2^{\mathrm{an}}$ has a distinguished point γ, corresponding to the Gauß norm on $k[T_1, T_2]$. It is the unique point of $\mathbf{P}_2^{\mathrm{an}}$ whose reduction $\tilde{\gamma}$ is the generic point of $\mathbf{P}_{2,\tilde{k}}$. Moreover,

$$c_1(\overline{\mathscr{O}(1)})^2 = \delta_\gamma.$$

The support of $c_1(\overline{\mathscr{O}(1)})$ is the set of points x of $\mathbf{P}_2^{\mathrm{an}}$ whose reduction $\tilde{x}$ is not closed.