Formal functions on arithmetic surfaces

A generalization of the criterion of Borel-Dwork-Pólya-Bertrandias

Antoine Chambert-Loir
Institut de recherche mathématique de Rennes, Université de Rennes 1
Joint work with Jean-Benoît Bost
Département de mathématiques, Université de Paris-Sud Orsay

Workshop on Arithmetic Geometry :
Diophantine approximation and Arakelov theory
Toronto, October 20–24, 2008

Contents

Rationality of formal power series

Let a formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ be given.

How to decide if it is the Taylor expansion of a rational function ?

We search for a criterion of an arithmetic nature :

1. we assume that the coefficients a_n lie in a number field ;
2. we impose arithmetic conditions on the coefficients (*e.g.*, that they are algebraic integers) ;
3. geometric conditions on the analytic behaviour (radius of convergence, of meromorphy), possibly at all places of the number field.

The theorem of Borel-Dwork-Pólya-Bertrandias

Theorem

Let F be a number field, let $f \in F[\![x^{-1}]\!]$.

Assumptions :

1. *the coefficients of f are S-integral for a finite set S of finite primes in F;*
2. *for any place $v \in S$, f defines a meromorphic function on the complement of a bounded subset K_v of $\mathbf{C}_v$.*
3. *the product of the* ***transfinite diameters*** *of the K_v is smaller than 1.*

Then f is a rational function.

Transfinite diameter

The **transfinite diameter** of a bounded metric space (X, d) is the limit $\mathrm{tdiam}(X)$ of the decreasing sequence (d_n) defined by

$$d_n = \left(\sup_{(x_1,\dots,x_n)\in X^n} \prod_{i\neq j} d(x_i, x_j) \right)^{1/n(n-1)} .$$

Examples :

- **disk** in a valued field : tdiam = radius ;
- **lemniscate** : $\{|P(z)| \leqslant 1\}$ in a valued field, $P = a_d z^d + \dots$: $\mathrm{tdiam} = |a_d|^{-1/d}$;
- **segment in C** : tdiam = length/4;
- **compact subset in C** : tdiam = capacity with respect to the point at infinity.

1. É. Borel, 1894, *Sur une application d'un théorème de M. Hadamard*. $f \in \mathbf{Z}[\![1/x]\!]$, $K_v = D(0,r)$, $r < 1$.
2. G. Pólya, 1928, *Über notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe*.
3. B. Dwork, 1960, *On the rationality of the zeta function of an algebraic variety*.
   $\Rightarrow$ If V is an algebraic variety over a finite field $\mathbf{F}_q$, then its zeta function $\zeta_V(t) = \exp\left(\sum_{n\geqslant 1} \#V(\mathbf{F}_{q^n})\frac{t^n}{n}\right)$ is a rational function.
4. F. Bertrandias, 1963, see Y. Amice's book *Les nombres p-adiques*.
5. D. Cantor, 1980, R. Rumely, 1989.
6. J-P. Bézivin, P. Robba, 1989 : New proof of the theorem of Lindemann-Weierstrass.

1. If f is holomorphic, this is easy. For example consider $f \in \mathbf{Z}[\![x]\!]$, holomorphic on $D(0,r)$, with $r > 1$.
   Write $f = \sum a_n x^n$.
   By Cauchy inequalities, $|a_n| \leqslant M/r^n$ goes to 0.
   Since $a_n \in \mathbf{Z}$, $a_n = 0$ for $n \gg 0$: f is a polynomial.
2. In some cases, one can prove that f is rational when $\prod \operatorname{tdiam}(K_v) = 1$, if one assumes moreover that f is algebraic.
   D. Harbater : $K_v = D(0, r_v)$.
3. However : $f = \sqrt{1 - 4/x} \in \mathbf{Z}[\![x]\!]$. Two ramification points, in 0 and 4, one can extends f to a holomorphic function outside the interval $K = [0, 4]$. One has $\operatorname{tdiam}(K) = 1$ but f is not rational.

The theorem of BDPB

Proofs

1. *Original proof* : establish the vanishing of appropriate **Hankel/Kronecker determinants.**
   Using analytical properties of f, show that they satisfy $\prod_v |D|_v < 1$.
   By the product formula, $D = 0$.
2. In his book *G-functions and geometry* (1989), Y. André gives a similar criterion for algebraicity/rationality, based on classical **diophantine approximation** techniques.
3. Small variation : In my *Bourbaki Seminar* (2001) about the work of Chudnovsky, André and Bost, proof of those criteria using the formalism of Arakelov geometry and Bost's **slope inequality**.

"*Even if all rationality proofs rely on proving the vanishing of the* $K_N(a)$ (Kronecker determinants) , *the form of the criteria may well disguise that fact.*" A. van der Poorten

Our main theorem

Consider an algebraic curve C over a number field F, a point $o \in C(F)$ and a formal function $f \in \widehat{\mathcal{O}_{C,o}}$ at the point o.

Assume that at all places v of F, f is induced by an meromorphic function on a v-adic open subset Ω_v of $C \otimes F_v$.

Conclusion : if the **global capacity** of the family (Ω_v) is at most 1, then f comes from a meromorphic function on C.

The measure of the size of the Ω_v relies on **capacity theory** for algebraic curves — generalization of the product of the transfinite diameters.

Proof in two steps :

1. f is algebraic. This part uses diophantine approximation (in the dialect of Arakelov geometry and slopes).
2. f is rational. This part uses the Hodge index theorem in Arakelov geometry.

Contents

The complex case

Let M be a Riemann surface, compact, connected
$K \subset M$ a compact subset, $\Omega = \complement K$, $P \in \Omega$.

There is a unique **potential** $g_K : \Omega \to \mathbf{R}$ such that

1. g_K is harmonic on $\Omega \setminus \{P\}$;
2. for nearly every $x \in \partial K$, $\lim_{z \to x} g_K(z) = 0$;
3. if t is a local coordinate around P, then

$$g_K = -\log(c_K \, |t - t(P)|) + o(1).$$

The real number c_K is the **capacity** of K with respect to P; it depends on the choice of t.

On the complex line $T_P M$, the hermitian norm defined by

$$\left\| \frac{\partial}{\partial t} \right\| = c_K$$

does not depend on any choice : **capacitary norm** on $T_P M$.

The p-adic case (after Rumely)

Let M be a projective smooth connected curve over a p-adic field F, $K \subset M$ an affinoid, $\Omega = \complement K$, $P \in \Omega$.

There is a full **potential theory** on M (R. Rumely, A. Thuillier).

Easier definition (inspired by Rumely's theory and a theorem of Fresnel/Matignon) relying on the following Theorem :

Theorem

If Ω is connected, there is a rational function $f \in F(M)$ regular outside P such that $K = \{x\,,\ |f(x)| \leqslant 1\}$.

Let d be the order of f at the point P;
then $g_K := \frac{1}{d} \max(\log|f|\,, 0)$ is the analogue of the **potential.**
If t is a local coordinate around P, define the **capacity** c_K such that

$$g_K = -\log(c_K\,|t - t(P)|) + o(1)$$

and the **capacitary norm** on $T_P M$ by $\left\| \dfrac{\partial}{\partial t} \right\| = c_K$.

Global capacity

Let M be an algebraic curve over a number field F, $P \in M(F)$.

For all places v choose a compact/affinoid subset K_v of the analytic curve over F_v such that $P \notin K_v$.

Assume that there is a rational function on M defining K_v for almost all finite places v.

Definition

The global capacity is the real number defined by

$$\log \operatorname{cap}((K_v); P) = \frac{1}{[F : \mathbf{Q}]} \sum_v \log \left\| \frac{\partial}{\partial t} \right\|_v$$
$$= -\widehat{\deg}(T_P M, \|\cdot\|^{cap}).$$

(The product formula implies that it does not depend on a chosen local parameter $t \in \mathcal{O}_{M,P}$ on M.)

Statement of the main rationality theorem

Keep $M, P, (K_v)$ as above.

Theorem

Let $f \in \widehat{\mathcal{O}_{M,P}}$ be a ***formal function*** *around P.*
Assume

1. *"the denominators of the coefficients of the Taylor expansion of f are divisible by only finitely many prime numbers";*
2. *for any place v, f defines a v-adic meromorphic function outside K_v;*
3. $\mathrm{cap}((K_v); P) < 1$.

Then f comes from a rational function on M.

Examples

The simplest example is that of the **projective line.** Our theorem generalizes the classical results of Borel-Dwork-Pólya-Bertrandias.

- **Borel-Dwork** : $F = \mathbf{Q}$, $M = \mathbf{P}^1$, $P = 0$, $t =$ usual coordinate. Take $K_v = D(0, R_v)$, with $R_v = 1$ for almost all v. Then $c_{K_v} = 1/R_v$ and $\mathrm{cap}((K_v); P) = (\prod R_v)^{-1}$.
- **Pólya-Bertrandias** : $M = \mathbf{P}^1$ (say over $\mathbf{Q}$), $P = \infty$, $t = 1/$usual coordinate.
  Let K_v be an compact/affinoid subset of $\mathbf{P}^1(\mathbf{C}_v)$, then c_{K_v} is its transfinite diameter.

The theorem extends readily to more general subsets than affinoid provided that the product of their "outer capacities" is less than 1.

Contents

Diophantine approximation *vs.* Slope inequalities

We prove that f is algebraic using **diophantine approximation.** In principle, we find a "small" polynomial P in two variables such that $P(x, f(x))$ vanishes with so high order at P that it must vanish everywhere.

We use however the method of **slopes inequalities** in Arakelov geometry due to J.-B. Bost.

The general idea consists in proving the **algebraicity of a suitable formal subscheme:**

- in (Bost, 2001) : formal leaf of a foliation
- here: formal graph of f in $X = \times \mathbf{P}^1$ at the point $(P, f(P))$.

Slope inequality

We need to **algebraize the formal graph** $(\hat{V}, o)$ of f in $X = M \times \mathbf{P}^1$ at $o = (P, f(P))$.

Let $\overline{L}$ be an ample line bundle on X with an adelic metric.

Spaces of sections $E_D = \Gamma(X, L^{\otimes D})$ with L^2/sup norms.

Algebraicity of $\hat{V}$ means non-injectivity of the restriction map $E_D \to \Gamma(\hat{V}, L^D)$. **Assume injectivity.**

The filtration of $\Gamma(\hat{V}, L^D)$ by $\Gamma(\hat{V}, \mathfrak{m}_o^i L^D)$ induces a filtration of E_D by subspaces E_D^i.

The ith jet is a morphism

$$\varphi_D^i \colon E_D^i \to \Gamma(o, \mathfrak{m}_o^i L^D) = \mathrm{S}^i(\Omega_o^1 \hat{V}) \otimes L^{\otimes D}|_o$$

with kernel E_D^{i+1}.

Slope inequality :

$$\widehat{\deg}\, \overline{E_D} \leqslant \sum_i \operatorname{rk} \varphi_D^i \left(\widehat{\deg}(\mathrm{S}^i T_P^* M) + D\, \widehat{\deg}(L|_o) + h(\varphi_D^i) \right)$$

Estimates

Slope inequality

$$\widehat{\deg}\,\overline{E_D} \leqslant \sum_i \operatorname{rk}\varphi_D^i \left(\widehat{\deg}(S^i T_P^* M) + D\,\widehat{\deg}(L|_O) + h(\varphi_D^i)\right)$$

The contradiction is obtained by inserting in this inequality general **estimates in Arakelov Geometry :**

- (rough) **arithmetic Hilbert-Samuel formula** :

$$\widehat{\deg}\,\overline{E_D} \geqslant -c_1 D^3$$

- **arithmetic degrees of symmetric powers** :

$$\widehat{\deg}(S^i T_P^* M) = i\,\widehat{\deg}\,T_P^* M$$

- **height of evaluation morphisms** (Schwarz lemma) :

$$h(\varphi_D^i) + i\,\widehat{\deg}\,T_P^* M \leqslant i \log \operatorname{cap}((K_v); P)$$

The Schwarz lemma

The height of the evaluation morphisms are estimated using the **Schwarz lemma,** a central analytic tool in classical diophantine approximation.

Here, it goes as follows : Assume $K_v = \{x\,,\, |f(x)|_v \leqslant 1\}$, f having a pole of order d at P, $\Omega_v = \complement K_v$, t a local parameter at P.

Let φ be holomorphic and bounded on Ω, vanishing at order i at P; write $\varphi = a_i t^i + \dots$ around P.

The function $\theta(x) = f(x)^i \varphi(x)^d$ is regular on $\Omega_v \setminus \{P\}$, at P, and bounded by $\|\varphi\|_\Omega^d$ at the "boundary" of Ω.
By the **maximum principle** :

$$|\theta(P)|_v = c_K^{-di}\,|a_i|^d \leqslant \|\varphi\|_\Omega^d\,,$$

hence $|a_i|_v \leqslant c_K^i\,\|\varphi\|_\Omega$.

A general algebraization criterion

General result of algebraization of a formal smooth curve $(\hat{V}, o)$ in a quasi-projective variety over a number field.

Theorem

If $\hat{V}$ is A-analytic and $\widehat{\deg}(T_o\hat{V}, (\|\cdot\|_v^{\mathrm{can}})) > 0$, then $\hat{V}$ is algebraic.

Key concepts :

1. **canonical v-adic semi-norms** on $T_o\hat{V}$ defined using the norms of the maps φ_D^i. Namely, for $u \in T_o\hat{V}$, one sets

$$\|u\|_v^{\mathrm{can}} = \limsup_{\frac{i}{D}\to\infty} \sup_{\substack{s\in E_D^i \\ \|s\|_v\leqslant 1}} \| \underbrace{\langle \varphi_D^i(s), u^{\otimes i}\rangle}_{\in L_o^{\otimes D}} \|^{1/i}$$

2. **generalized Arakelov degree** $\widehat{\deg}(T_o\hat{V}, (\|\cdot\|_v^{\mathrm{can}}))$
3. global notion of **A-analyticity** based on the v-adic **sizes** of $\hat{V}$

Contents

The picture

We know that f is algebraic.

Let M' be the normalization the Zariski closure of its graph in $M \times \mathbf{P}^1$. It is an algebraic curve with a morphism $\pi \colon M' \to M$ and **sections** defined on Ω_v.

Under the assumption that the global capacity $\operatorname{cap}((K_v); P) < 1$, we need to prove that π is an isomorphism.

The point P corresponds to a line bundle $\mathcal{O}(P)$, which one endows with the metrics defined by the potentials g_{K_v} — "v-adic Green functions".

Its first arithmetic Chern class $\widehat{c_1}(\mathcal{O}(P), \|\cdot\|^{\mathrm{cap}})$ can be defined as an object of $\widehat{\mathrm{CH}}^1(\mathcal{M})$, Bost's Arakelov-Chow group with L^2_1-regularity on an adequate model $\mathcal{M}$ of M.

Positivity of an Arakelov divisor

Lemma

The Arakelov divisor class $\widehat{c_1}(\mathcal{O}(P), \|\cdot\|^{cap})$ is nef and big.

By the very construction of potential Green functions, this class intersects nonnegatively :

- vertical divisors in $\mathcal{M}$ (at finite or infinite places);
- horizontal effective divisors in $\mathcal{M}$ whose trace on M is not equal to P.

Consequently, **this Arakelov divisor class is nef if and only if its self intersection is nonnegative.**

In fact, under the assumptions of the theorem, one has the equality

$$\widehat{c_1}(\mathcal{O}(P), \|\cdot\|^{\mathrm{cap}})^2 = \widehat{\deg}(T_P M, \|\cdot\|^{\mathrm{cap}}) = -\log \mathrm{cap}((K_v); P) > 0,$$

so that **this class is big**.

Construction of positive classes on M'

Let $\widehat{P} = \widehat{c_1}(\mathcal{O}(P), \|\cdot\|^{\text{cap}})$.

We decompose $\pi^*\mathcal{O}(P)$ as $\mathcal{O}(E_1) \otimes \mathcal{O}(E_2)$,
where $E_1 = f(P)$ and E_2 is an effective divisor on M' disjoint from E_2.

Similarly, we decompose $\pi^{-1}(\Omega_v) = \Omega_{1,v} \sqcup \Omega_{2,v}$ where $\Omega_{1,v} = f_v(\Omega_v)$.

This induces a decomposition

$$\pi^*\widehat{P} = \widehat{E_1} + \widehat{E_2},$$

in the Arakelov-Chow group $\widehat{\mathrm{CH}}_1(\mathcal{M}')$ of an adequate model of $\mathcal{M}'$.

This decomposition is orthogonal reflecting the fact that the potential Green functions of E_1 and E_2 have disjoint supports.

Moreover, $\pi^*\widehat{P}$ is big and nef.

Application of the Hodge index theorem

Recall :

- $\pi^*\widehat{P} = \widehat{E_1} + \widehat{E_2}$ is big and nef
- $\widehat{c}_1(\widehat{E_1})\widehat{c}_1(\widehat{E_2}) = 0.$

Since $\widehat{P}$ is nef and $\widehat{E_1}$ is effective,

$$0 \leqslant \widehat{c}_1(\widehat{P}) \cdot \pi_*\widehat{c}_1(\widehat{E_1}) = \pi^*\widehat{c}_1(\widehat{P}) \cdot \widehat{c}_1(\widehat{E_1}) = \widehat{c}_1(\widehat{E_1})^2.$$

Similarly $\widehat{c}_1(\widehat{E_2})^2 \geqslant 0.$

Consequently, the quadratic form

$$q(x,y) = (x\widehat{c}_1(\widehat{E_1}) + y\widehat{c}_1(\widehat{E_2}))^2 = x^2\widehat{c}_1(\widehat{E_1})^2 + y^2\widehat{c}_1(\widehat{E_2})^2$$

is nonnegative and the **Hodge index theorem** in Arakelov geometry implies that $\widehat{c}_1(\widehat{E_1})$ and $\widehat{c}_1(\widehat{E_2})$ are proportional.

Since they are orthogonal and their sum is big, one of them is zero, hence $\widehat{E_2} = 0.$

This implies $\pi^*\widehat{P} = \widehat{E_1}$, hence $\deg(\pi) = 1.$

Contents

Modular forms and overcovergent modular forms

A **modular form** of level N, weight k is a rule

$$(E/S, \alpha_N) \hookrightarrow f(E/S, \alpha_N) \in \omega_{E/S}^{\otimes k}$$

where S is a scheme, E/S is an elliptic curve, α_N a level N structure, which commutes with base change.

An **overconvergent modular form** of growth condition r is a similar rule which is only defined on the p-adic subset of the modular curve obtained by removing supersingular disks of radius $|r|$ around singular moduli.

Using the fact that the Eisenstein series E_{p-1} lifts the Hasse invariant if $p > 3$, N. Katz defines them in *Anvers III* as a rule

$$(E/S, \alpha_N, Y) \hookrightarrow f(E/S, \alpha_N, Y) \in \omega_{E/S}^{\otimes k}$$

where E/S is an elliptic curve, α_N a level N structure, and $Y \in \omega_{E/S}^{1-p}$ satisfies $YE_{p-1}(E/S) = r$, which commutes with base change.

Application to modular forms

Theorem

Let F be a number field, $\mathfrak{o}_F$ its ring of integers.
Let $f \in \mathfrak{o}_F[\![q]\!]$ and assume :

- *for any embedding $\sigma : F \hookrightarrow \mathbf{C}$, the radius of convergence of $f^\sigma \in \mathbf{C}[\![q]\!]$ is at least 1;*
- *at some finite place v, f extends to an* ***overconvergent*** *modular form of level N, weight k.*

Then f is algebraic.
If, moreover, the growth condition r of f satisfies $|r|_p < e^{-2\pi[F:\mathbf{Q}]/N}$, then f is a true modular form.

The proof is an application of our general criteria, together with two computations :

- injectivity radius of the uniformization map (mod. parabolic elements) aroung the cusp at infinity;
- capacity of the supersingular disks.