Geometry and dynamics in moduli spaces

Lecture 12. Solution of Arnold’s Problem on interval exchange permutations

(after a joint work with V. Delecroix, E. Goujard, P. Zograf)

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> *There is an awful moment in popular books on cosmic theories (that breezily begin with plain straightforward chatty paragraphs) when there suddenly start to sprout mathematical formulas, which immediately blind one’s brain. We do not go as far as that here.*
>
> *V. Nabokov*

General answer for the asymptotic proportion of transitive and “cyclic” interval exchange permutations

Arnold's problem

2002-8. The (C, B, A)-permutation of the set $\{1, 2, \dots, n\}$ transports to the last place the subset $A = \{1, 2, \dots, a\}$ preceded by the transported set $B = \{a + 1, \dots, a + b\}$ while the starting position is occupied by $C = \{a + b + 1, \dots, n\}$.

Some of these $(n - 1)(n - 2)/2$ permutations permute *cyclically* (like the addition of a constant to the residues mod n), and some of these cyclic permutations are *transitive* (like the addition of the constant 1).

Find the proportion of both the cyclic and the transitive cyclic permutations among the (C, B, A) -permutations for large n.

More generally, starting from a permutation of k elements, one defines a permutation of the set $\{1, \dots, n\}$ from its decomposition into k segments $\{a_i + 1, a_{i+1} - 1\}$. The problem is to study the statistics of the Young diagrams formed by the cycle lengths of the resulting permutations, for the case of large n and random decompositions of n into k parts.

Cyclic versus transitive interval exchange permutations

Let π be an irreducible non-degenerate permutation. Denote by $p_{tr}(\pi)$ and by $p_{cyc}(\pi)$ the asymptotic proportions of respectively transitive and “cyclic” (in the sense of Arnold) permutations among all π-interval exchange permutations.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020)
The quantities $p_{cyc}(\pi)$ and $p_{tr}(\pi)$ satisfy the following relation:

$$p_{cyc}(\pi) = p_{tr}(\pi) \cdot \zeta(d)\,,$$

where d is the number of elements in π.

Remark. Note that $\zeta(d) \to 1$ as $d \to +\infty$, and the convergence is very rapid. The above formula shows, in particular, that for permutations of sufficiently large number of elements the quantities $p_{cyc}(\pi)$ and $p_{tr}(\pi)$ become basically indistinguishable: majority of “cyclic” (in the sense of Arnold) permutations are necessarily transitive.

Characters of the symmetric group

Recall that a representation ρ of the symmetric group S_n is a homomorphism $\rho : S_n \to \mathrm{GL}(V)$ where V is a finite dimensional complex vector space. The simplest example is given by the permutation action of S_n on coordinates in $\mathbb{C}^n$. This action leaves invariant the 1-dimensional subspace generated by the sum $\vec{e}_1 + \vec{e}_2 + \ldots + \vec{e}_n$ of the vectors of the basis and the $(n-1)$-dimensional subspace $W_n := \{\sum x_i \vec{e}_i : \sum x_i = 0\}$, where $\vec{e}_i$ denotes the elements of the standard basis of $\mathbb{C}^n$. The representation $\mathbf{St}_n$ induced on W_n is irreducible (i.e. it does not contain non-trivial invariant subspaces).

Now define the characters of the exterior powers of the representation $\mathbf{St}_n$

$$\chi_j(g) := \mathrm{tr}(g, \pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0 \leq j \leq n-1)\,.$$

Answer to Arnold's problem for general permutations

Let $\mathcal{H}^{comp}(m_1,\dots,m_n)$ be the component of the stratum associated to the canonical suspension over π. Let $\operatorname{Vol}\mathcal{H}^{comp}(m_1,\dots,m_n)$ and $\operatorname{Vol}_1\mathcal{H}^{comp}(m_1,\dots,m_n)$ be respectively the Masur–Veech volume of this component and the contribution of 1-cylinder square-tiled surfaces to this volume.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) *A random π-interval exchange permutation is cyclic with probability*

$$p_{cyc}(\pi) = \frac{\operatorname{Vol}_1\mathcal{H}^{comp}(m_1,\dots,m_n)}{\operatorname{Vol}\mathcal{H}^{comp}(m_1,\dots,m_n)}\,,$$

Let $d = \dim\mathcal{H}(m_1,\dots,m_n) = 1 + \sum_{j=1}^{n}(m_j+1)$ be the number of elements in π. Let $\nu \in S_{d-1}$ be any permutation which decomposes into cycles of lengths $(m_1+1),\dots,(m_n+1)$, and let μ_k be the multiplicity of the entry k in the multiset $\{m_1,\dots,m_n\}$. Then

$$\operatorname{Vol}_1\mathcal{H}(m_1,\dots,m_n) = \frac{2\zeta(d)}{(d-2)!}\cdot\prod_k\frac{1}{(k+1)^{\mu_k}}\cdot\sum_{j=0}^{d-1} j!\,(d-1-j)!\,\chi_j(\nu)\,.$$

Case of hyperelliptic components

When the permutation π corresponds to a hyperelliptic connected component, the formulae become much more explicit.

Theorem (J. Athreya, A. Eskin, A. Zorich, 2016)

$$\operatorname{Vol}\mathcal{H}^{hyp}(2g-2)=\frac{2\pi^{2g}}{(2g+1)!}\cdot\frac{(2g-3)!!}{(2g-2)!!}\,.$$

$$\operatorname{Vol}\mathcal{H}^{hyp}(g-1,g-1)=\frac{4\pi^{2g}}{(2g+2)!}\cdot\frac{(2g-2)!!}{(2g-1)!!}\,.$$

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020)

$$\frac{\operatorname{Vol}_1\mathcal{H}^{hyp}(2g-2)}{\operatorname{Vol}\mathcal{H}^{hyp}(2g-2)}=\frac{\zeta(2g)}{\pi^{2g}}\cdot 2g(2g+1)\cdot\frac{(2g-2)!!}{(2g-3)!!}\,.$$

$$\frac{\operatorname{Vol}_1\mathcal{H}^{hyp}(g-1,g-1)}{\operatorname{Vol}\mathcal{H}^{hyp}(g-1,g-1)}=\frac{\zeta(2g+1)}{2\pi^{2g}}\cdot(2g+1)(2g+2)\cdot\frac{(2g-1)!!}{(2g-2)!!}\,.$$

Case of even and odd components

Theorem (V. Delecroix, 2013) *For any stratum, for which all degrees of zeroes are even, one has*

$$\begin{aligned}\operatorname{Vol}_1 \mathcal{H}^{odd}(m_1,\dots,m_n) - \operatorname{Vol}_1 \mathcal{H}^{even}(m_1,\dots,m_n) \\ - (-1)^{\varphi} \operatorname{Vol}_1 \mathcal{H}^{hyp}(m_1,\dots,m_n) = \frac{\zeta(d)}{2^{g-2}} \cdot \prod_k \frac{1}{(k+1)^{\mu_k}}\,.\end{aligned}$$

Combining this formula with the formula for the total contribution

$$\begin{aligned}\operatorname{Vol}_1 \mathcal{H}(m_1,\dots,m_n) = \operatorname{Vol}_1 \mathcal{H}^{hyp}(m_1,\dots,m_n) \\ + \operatorname{Vol}_1 \mathcal{H}^{odd}(m_1,\dots,m_n) + \operatorname{Vol}_1 \mathcal{H}^{even}(m_1,\dots,m_n)\end{aligned}$$

and with a formula for $\operatorname{Vol}_1 \mathcal{H}^{hyp}(m_1,\dots,m_n)$ presented at the previous slide, we compute the individual contributions $\operatorname{Vol}_1 \mathcal{H}^{odd}(m_1,\dots,m_n)$ and $\operatorname{Vol}_1 \mathcal{H}^{even}(m_1,\dots,m_n)$ separately.

Correction due to hyperelliptic components

For strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(2k,2k)$ the above formula includes the contribution of hyperelliptic 1-cylinder square-tiled surfaces to the Masur-Veech volume. Abelian differentials in these hyperelliptic components also have certain parity φ of the spin structure. This parity is present in the sign $(-1)^\varphi$ of the contribution of the hyperelliptic component. This parity is computed using the formula below.

Theorem (M. Kontsevich, A. Zorich, 2003) *Parity of the spin structure determined by an Abelian differential from the hyperelliptic component* $\mathcal{H}^{hyp}(2g-2)$ *equals*

$$\varphi(\mathcal{H}^{hyp}(2g-2)) \equiv \left[\frac{g+1}{2}\right] \pmod 2 .$$

Parity of the spin structure of the hyperelliptic component $\mathcal{H}^{hyp}(g-1,g-1)$ *for odd genera* g *equals*

$$\varphi\left(\mathcal{H}^{hyp}(g-1,g-1)\right) \equiv \left(\frac{g+1}{2}\right) \pmod 2 \quad for\ \ odd\ \ g .$$

Principal and minimal strata

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) *The contribution of 1-cylinder square-tiled surfaces to the Masur–Veech volume of the principal and of the minimal stratum of Abelian differentials satisfy*

$$\operatorname{Vol}_1 \mathcal{H}(1^{2g-2}) = \frac{\zeta(4g-3)}{4g-2} \cdot \frac{4}{2^{2g-2}}$$

$$\operatorname{Vol}_1 \mathcal{H}(2g-2) = \frac{\zeta(2g)}{2g} \cdot \frac{4}{2g-1}$$

In particular, we recover the value

$$\operatorname{Vol}_1 \mathcal{H}(2) = \frac{\zeta(4)}{4} \cdot \frac{4}{3} = \frac{\zeta(4)}{3}\,,$$

which we have already seen from the direct computation.

Large genus asymptotics

Asymptotics of Masur–Veech volume in large genera

Let $\mathbf{m} = (m_1, \dots, m_n)$ be an unordered partition of an even number $2g-2$, $|\mathbf{m}| = m_1 + \dots + m_n = 2g-2$. Denote by Π_{2g-2} the set of all partitions. The following result was conjectured by A. Eskin and A. Zorich in 2003 and proved independently by A. Aggarwal and by D. Chen, M. Möller, A. Sauvaget, D. Zagier in 2020.

Asymptotics of Volumes (A. Aggarwal, 2020; D. Chen, M. Möller, A. Sauvaget, D. Zagier, 2020). *For any* $\mathbf{m} \in \Pi_{2g-2}$ *one has*

$$\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n) = \frac{4}{(m_1+1)\cdot\dots\cdot(m_n+1)} \cdot (1+\varepsilon(\mathbf{m}))\,,$$

$$\text{where } |\varepsilon(\mathbf{m})| \le \frac{\mathrm{const}}{\sqrt{g}}\,.$$

Let

$$\frac{\operatorname{Vol}\mathcal{H}^{even}(2k_1,\dots,2k_n)}{\operatorname{Vol}\mathcal{H}^{odd}(2k_1,\dots,2k_n)} = 1 + \delta(\mathbf{k})\,.$$

Then $\max_{\mathbf{k}\in\Pi_{g-1}} |\delta(\mathbf{k})| \to 0$ *as* $g \to +\infty$.

Volume contribution of 1-cylinder surfaces

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) *The contribution* $\operatorname{Vol}_1 \mathcal{H}(m_1, \dots, m_n)$ *of* 1*-cylinder square-tiled surfaces to the Masur–Veech volume of any stratum of Abelian differentials satisfies*

$$\frac{\zeta(d)}{d+1} \cdot \frac{4}{(m_1+1)\dots(m_n+1)} \le \operatorname{Vol}_1 \mathcal{H}(m_1, \dots, m_n) \le \frac{\zeta(d)}{d-\frac{10}{29}} \cdot \frac{4}{(m_1+1)\dots(m_n+1)}\,,$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n) = m_1 + \dots + m_n + n + 1$*.*

Let

$$\frac{\operatorname{Vol}_1 \mathcal{H}^{even}(2k_1, \dots, 2k_n)}{\operatorname{Vol}_1 \mathcal{H}^{odd}(2k_1, \dots, 2k_n)} = 1 + \Delta(k)\,.$$

Then $\max_{\mathbf{k}\in\Pi_{g-1}} |\Delta(\mathbf{k})| \to 0$ *as* $g \to +\infty$*.*

Arnold's problem for large number of intervals

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich) *The relative contribution $p_1(\mathcal{H}^{comp}(m_1,\dots,m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume of any nonhyperelliptic component of any stratum satisfies*

$$d \cdot p_1(\mathcal{H}^{comp}(m_1,\dots,m_n)) \to 1 \ \textit{as}\ \ g \to +\infty\,,$$

where $d = \dim_{\mathbb{C}} \mathcal{H}^{comp}(m_1,\dots,m_n) = m_1 + \dots + m_n + n + 1$ and convergence is uniform for all strata in genus g.

Let $\pi \in S_k$ be a non degenerate irreducible permutation in the complement of the Rauzy classes of hyperelliptic components. The probability $p_{tr}(\pi)$ that a random π-interval exchange transformation is transitive satisfies

$$k \cdot p_{tr}(\pi) \to 1 \ \textit{as}\ \ k \to +\infty\,,$$

where convergence is uniform for all permutations in S_k satisfying the abovementioned conditions.

Recall that for $\pi \in S_k$ as above one has $\frac{p_{cyc}(\pi)}{p_{tr}(\pi)} = \zeta(k)$ and that $\zeta(k) \to 1$ as $k \to +\infty$, where the convergence is very rapid.

Conjectural universality in large genera

Conjecture (V. Delecroix, E. Goujard, P. Zograf, A. Zorich). *Let* $x > 0$*. Let* $\mathcal{C}$ *be a non-hyperelliptic connected component of a stratum of Abelian differentials. Let* $p_k(\mathcal{C})$ *denote the probability that a random Abelian square-tiled surface in* $\mathcal{C}$ *has* k *cylinders. Then uniformly for* k *in* $\{0, 1, \ldots, \lfloor x \log(\dim_{\mathbb{C}} \mathcal{C}) \rfloor\}$ *and uniformly in* $\mathcal{C}$ *such that* $\dim \mathcal{C} \to \infty$

$$p_{k+1}^{Ab}(\mathcal{C}) = \frac{1}{\dim_{\mathbb{C}} \mathcal{C}} \cdot \frac{(\log \dim_{\mathbb{C}} \mathcal{C})^k}{k!} \cdot \left(\frac{1}{\Gamma\left(1 + \frac{k}{\log \dim_{\mathbb{C}} \mathcal{C}}\right)} + o(1) \right) .$$

In plain words, the Conjecture claims that the statistics $p_k^{Ab}(\mathcal{C})$ becomes practically indistinguishable from the statistics of the number of disjoint cycles in the cycle decomposition of a random permutation in $S_{\dim_{\mathbb{C}} \mathcal{C}}$, with respect to the uniform probability measure on the symmetric group of $\dim_{\mathbb{C}} \mathcal{C}$ elements.

Numerical evidence

The Conjecture is based on analyzing huge experimental data. We experimentally collected statistics of the number $K_{\mathcal{C}}(S)$ of maximal horizontal cylinders in cylinder decompositions of random square-tiled surfaces in about 30 connected components $\mathcal{C}$ of strata in genera from 40 to $10\,000$. In particular, the least squares linear approximation for components $\mathcal{C}$ of dimension $\dim_{\mathbb{C}} \mathcal{C}$ between 400 and $20\,000$ gives:

$$\begin{aligned}
\mathbb{E}(K_{\mathcal{C}}) \quad &\sim 0.999 \log \dim_{\mathbb{C}} \mathcal{C} + 0.581 \quad &&\approx 0.999 \log \dim_{\mathbb{C}} \mathcal{C} + \gamma + 0.004 \\
\mathbb{V}(K_{\mathcal{C}}) \quad &\sim 0.996 \log \dim_{\mathbb{C}} \mathcal{C} - 1.043 \quad &&\approx 0.996 \log \dim_{\mathbb{C}} \mathcal{C} + \gamma - \zeta(2) + 0.02
\end{aligned}$$

Visually the graphs of distributions $p_{\mathcal{C}}^{Ab}(k)$ and $\frac{s(\dim \mathcal{C}, k)}{(\dim \mathcal{C})!}$ are, basically, indistinguishable for large genera.

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\end{aligned}$$

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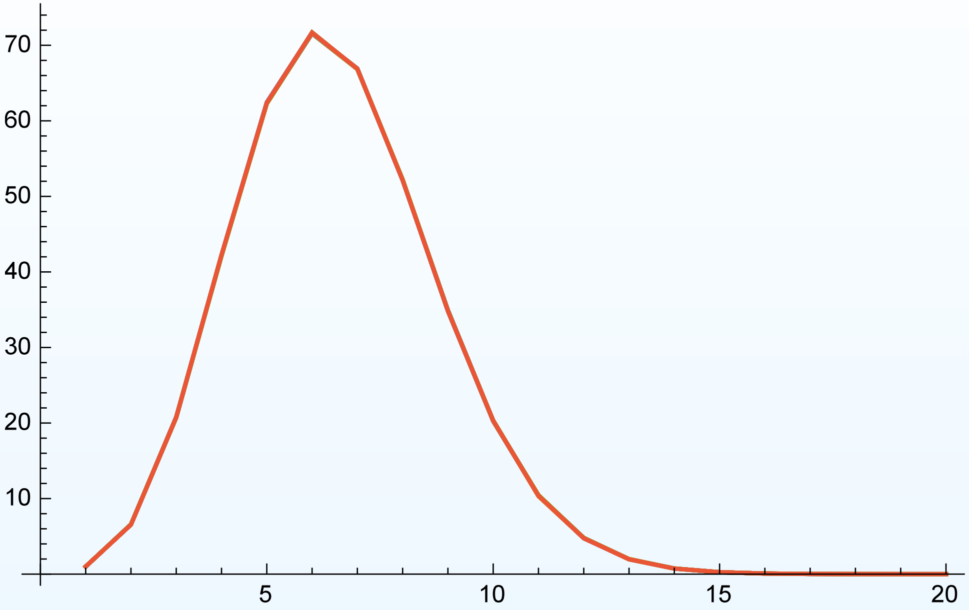
We have a rigorous (and quite involved) proof of the fact that the statistics of the number of cylinders in a random square-tiled surface in the principal stratum of quadratic differentials in genus g converges (in a very strong sense) to the statistics of the number of cycles in a random permutation of $3g-3$ elements, with respect to a very specific non-uniform measure on S_{3g-3} as $g \to +\infty$.

Conjectural large genus universality graphically



Statistics collected for the principal stratum in genus $g = 100$.

Conclusion. In order to answer Arnold's question for a concrete permutation, we need to know the Masur-Veech volume of the associated component of the stratum of Abelian differentials and, moreover, compute sophisticated contributions of k-cylinder square-tiled surfaces to this volume. Conjecturally, when the permutation becomes large enough, we need to know only the size d of this permutation (and verify that it is not hyperelliptic) and look at statistics of random permutations in S_d (for $k = 1$ we have proved this rigorously).