

Geometry and dynamics in moduli spaces

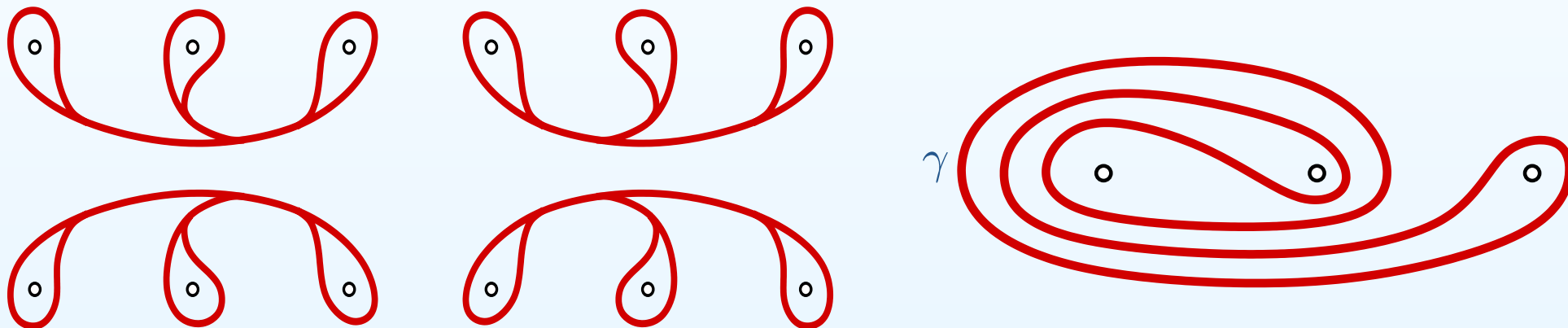
Solutions of some problems from homework assignment 2

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Université Paris Cité

April 9, 2026

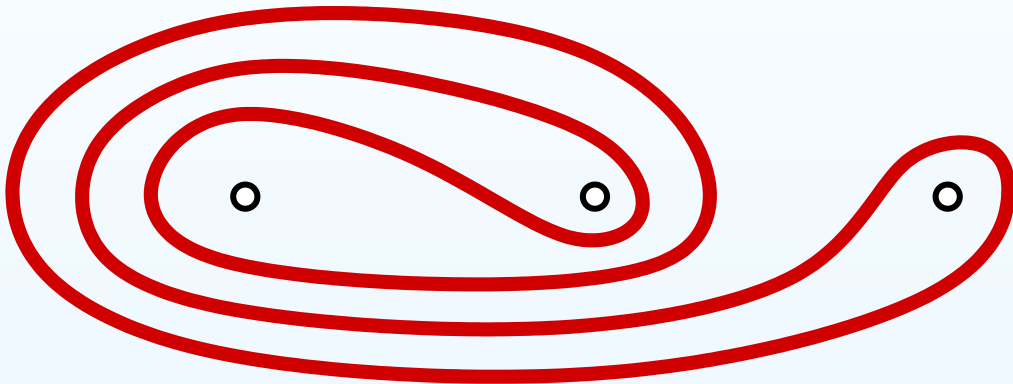
Playing with train tracks

The picture on the left presents the four “basic” train tracks τ_1, \dots, τ_4 on a four-punctured sphere. Find which of these four basic train-tracks (if any) carries the simple closed curve γ on top of the right picture. Indicate the resulting weights.



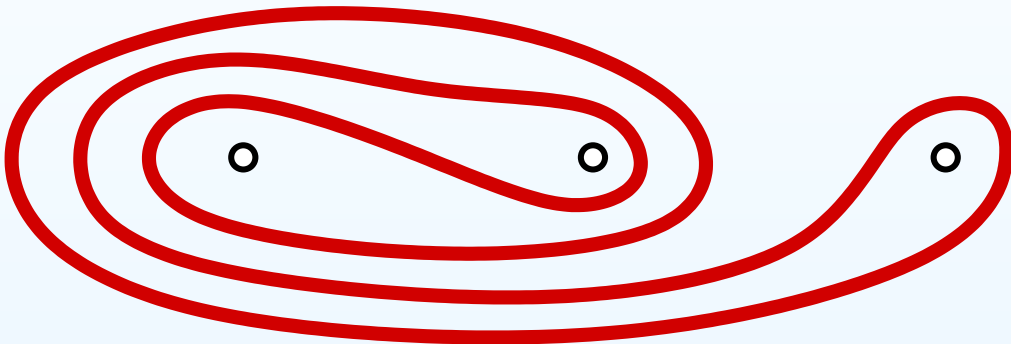
Train tracks carrying simple closed curves

Find which of the four basic train-tracks carries the simple closed curve γ on the right picture. Indicate the resulting weights.



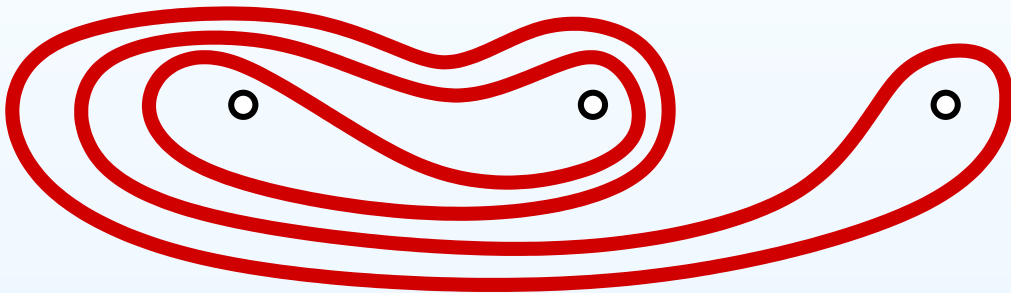
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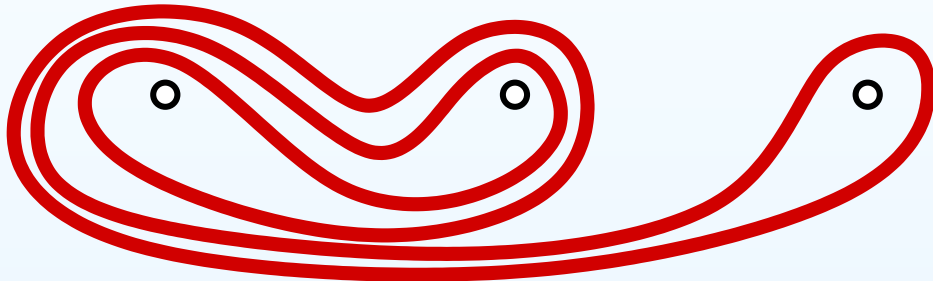
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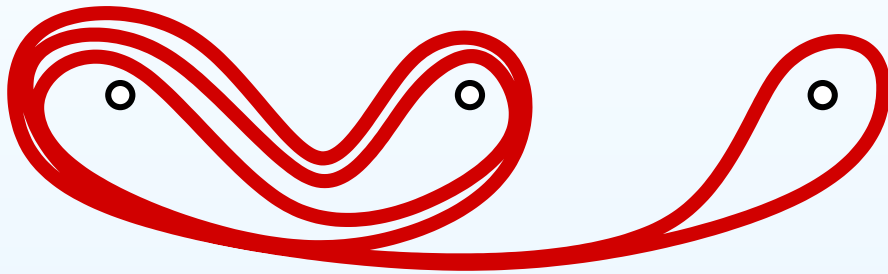
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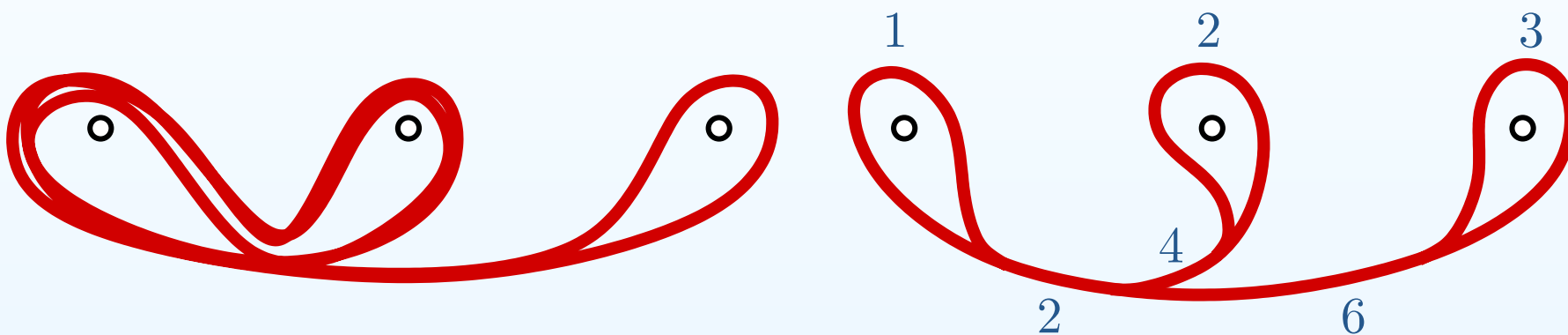
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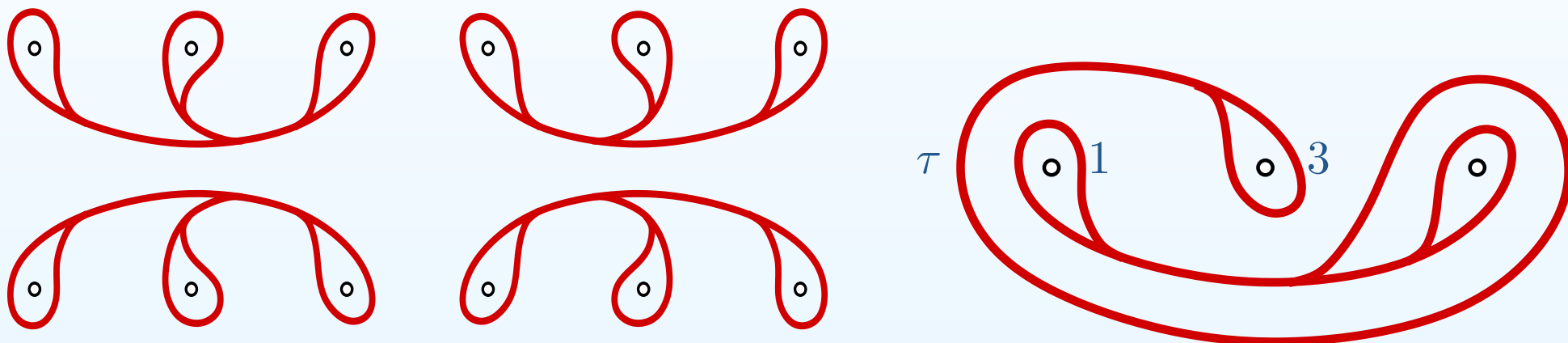
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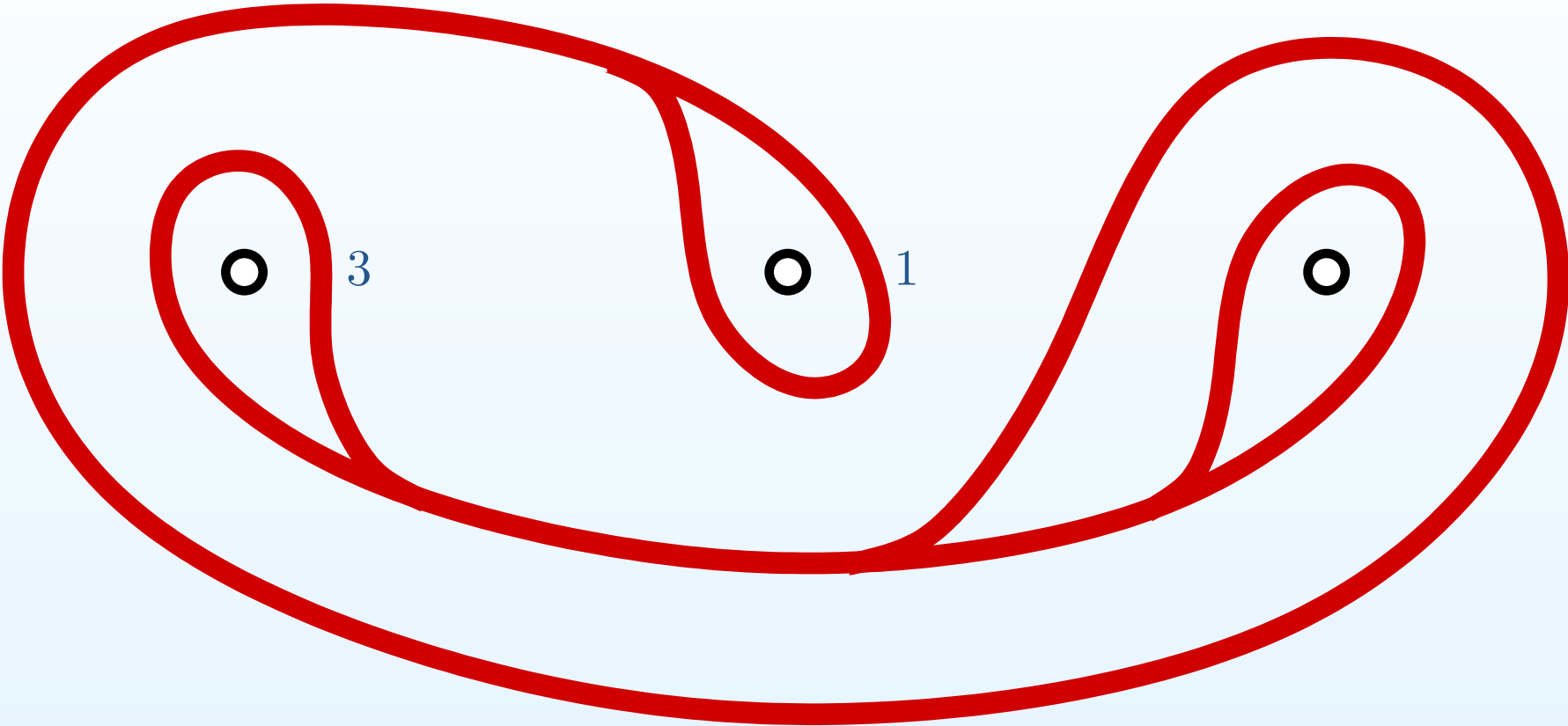


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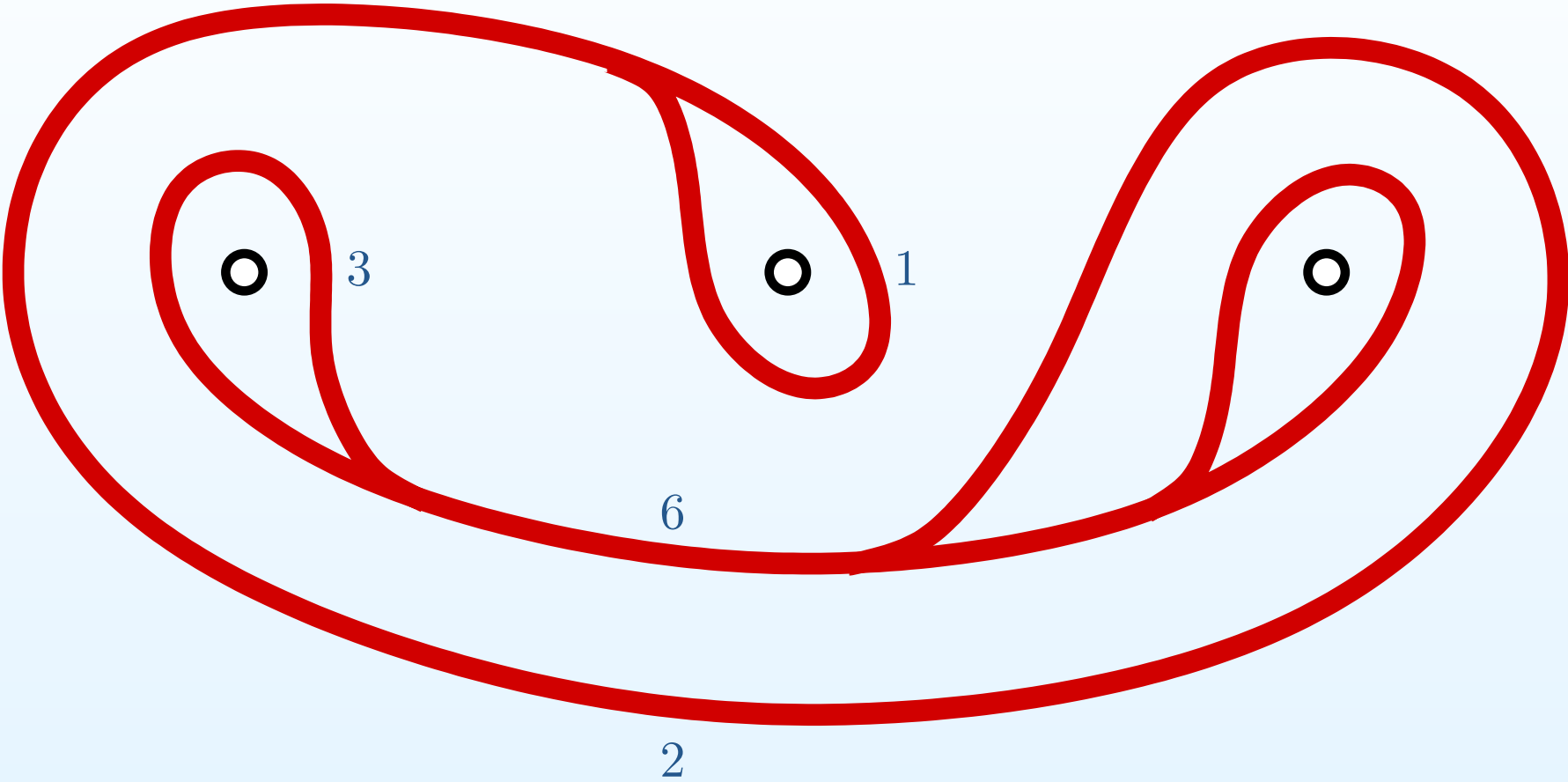
Find a sequence of “zips” and/or “unzips” which deform the given train track τ of the right picture to one of the four “basic” train tracks τ_1, \dots, τ_4 . Indicate the weights of all branches of the initial and of the final train tracks.



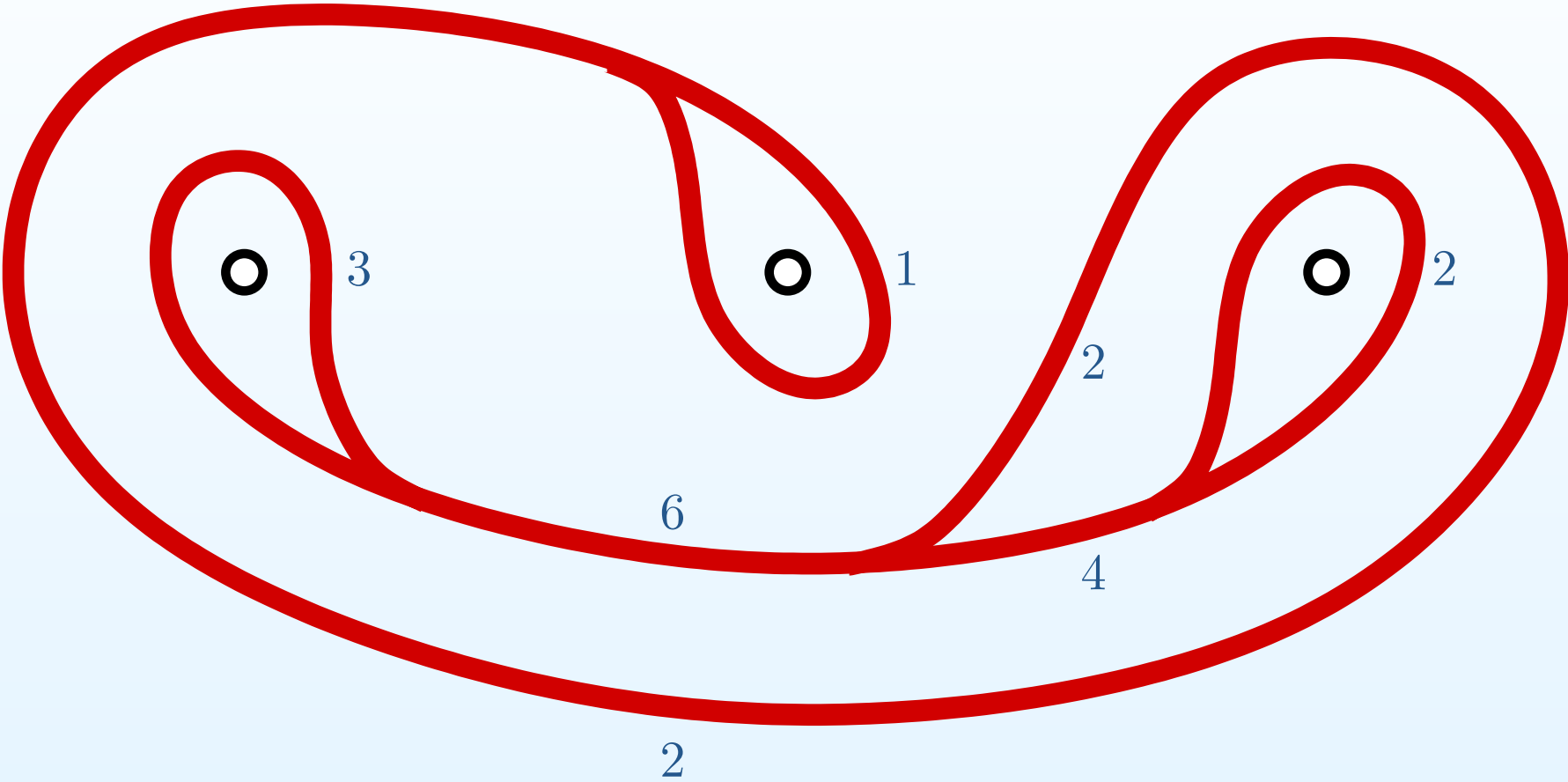
Zippering and unzipping train tracks



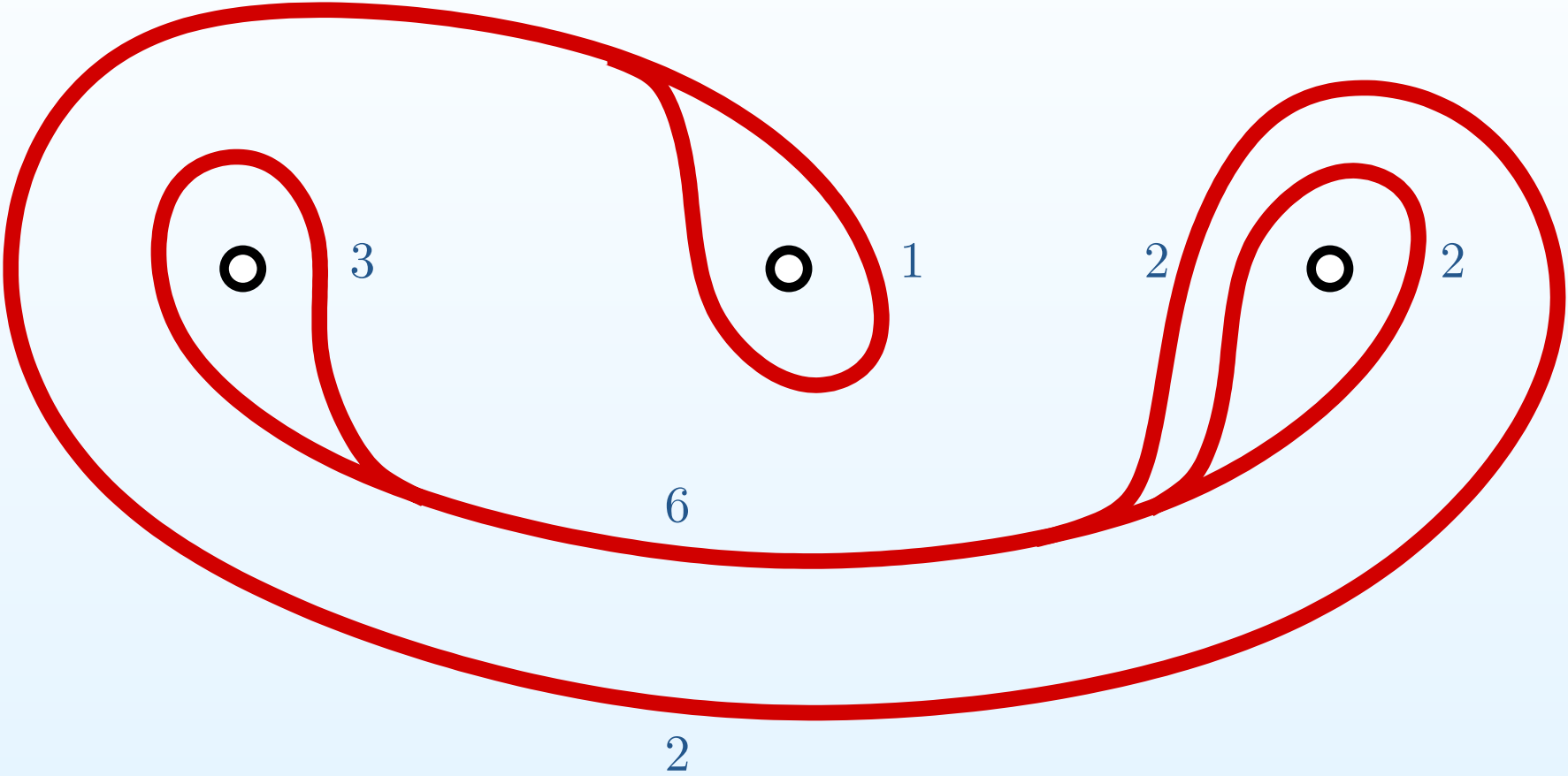
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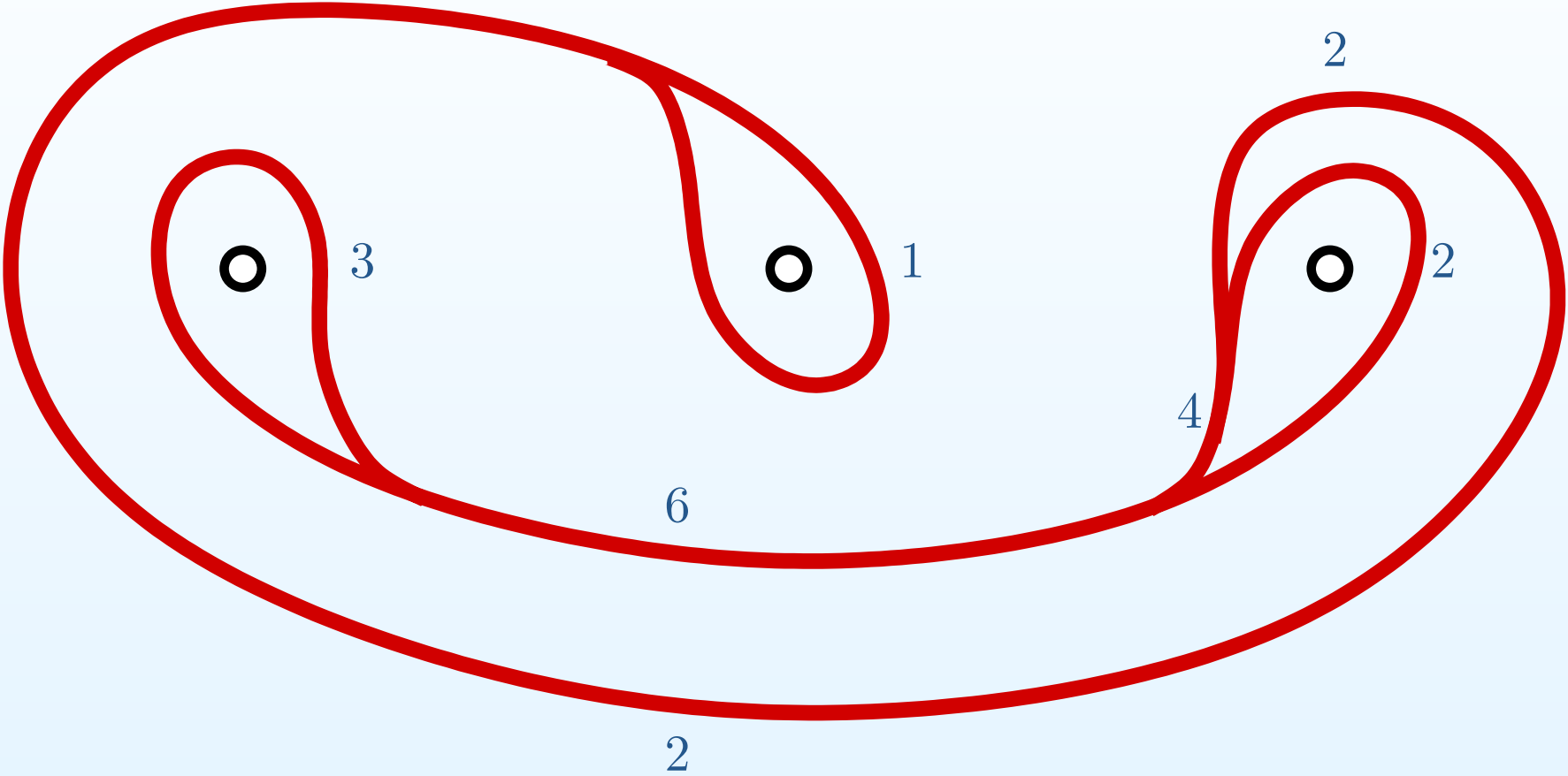
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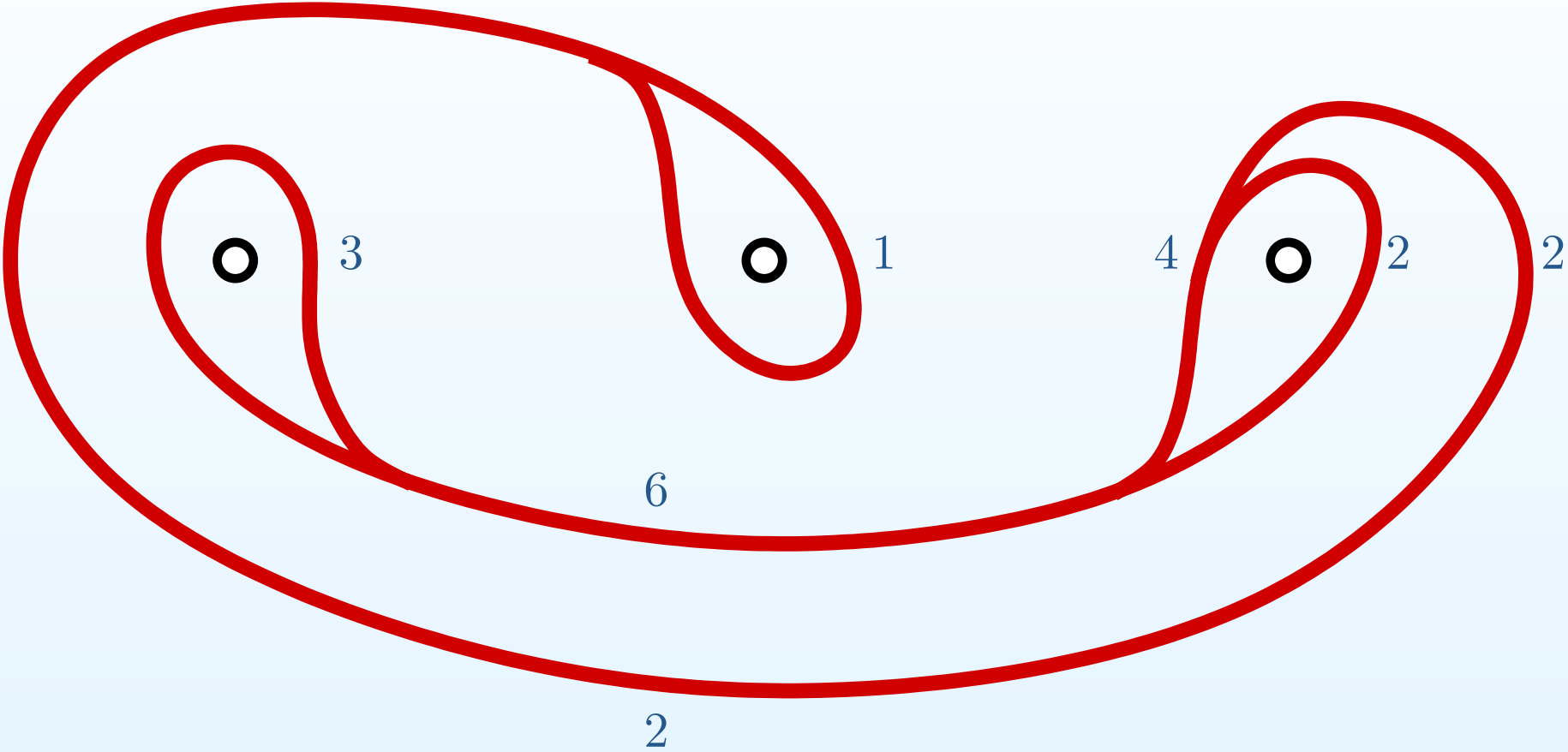
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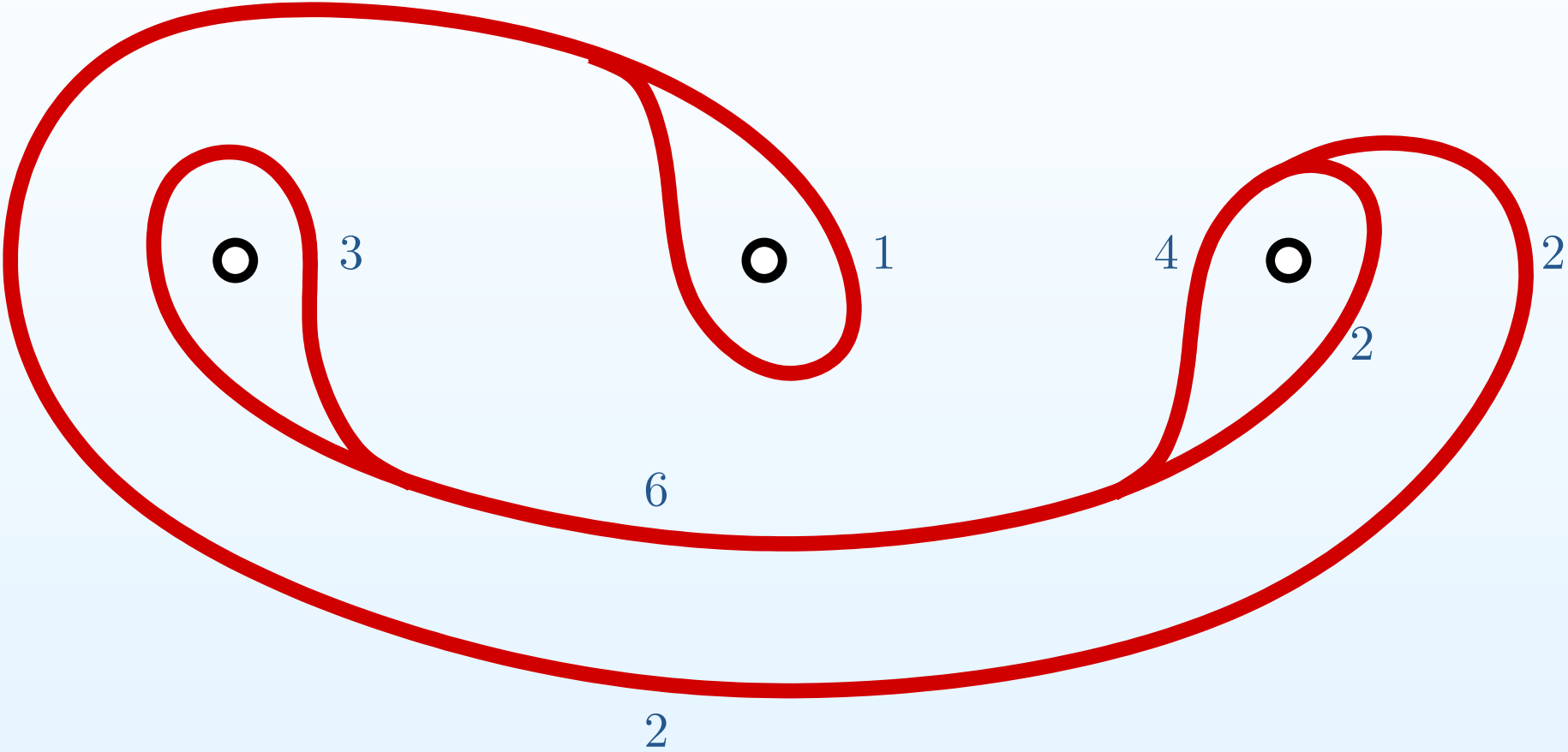
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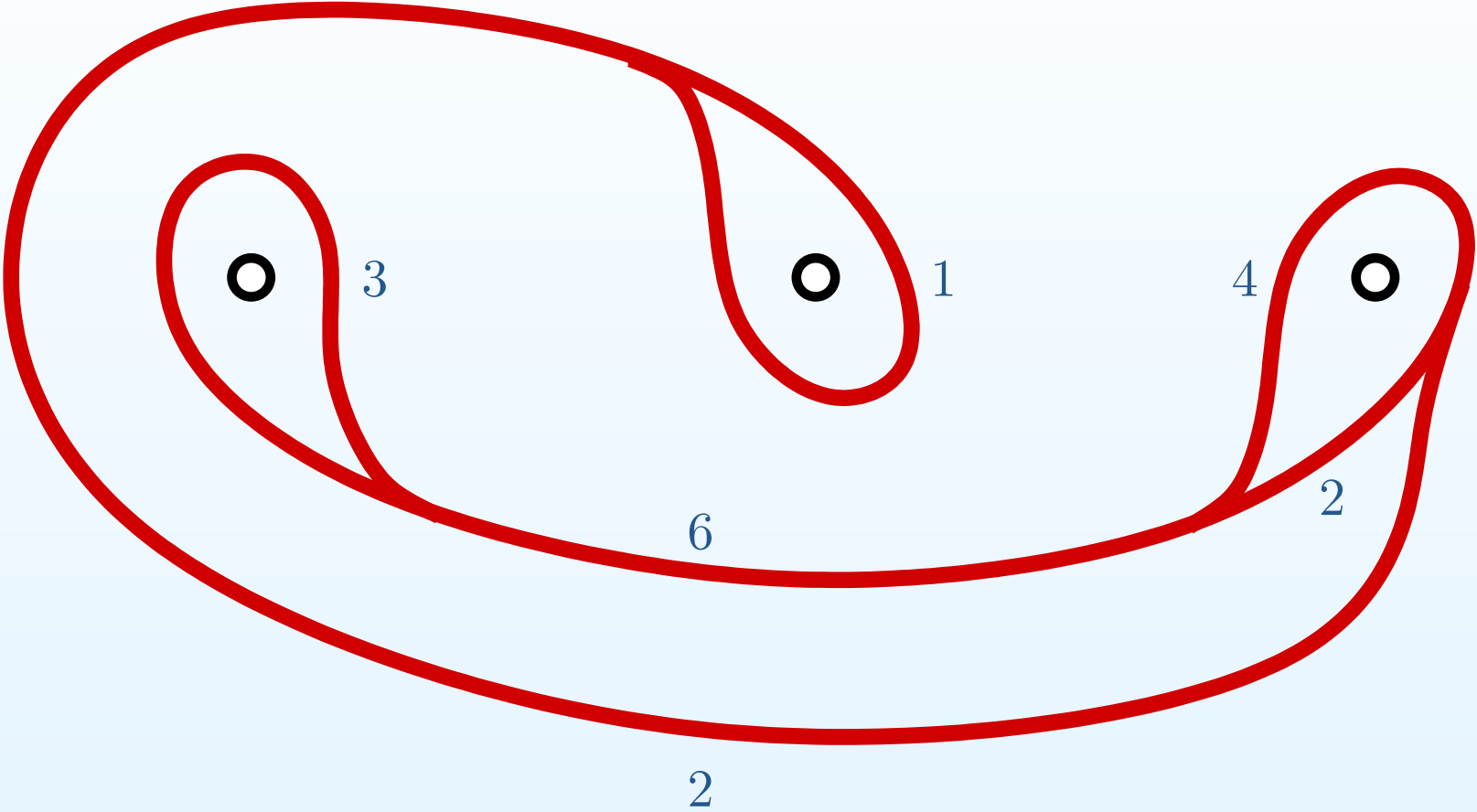
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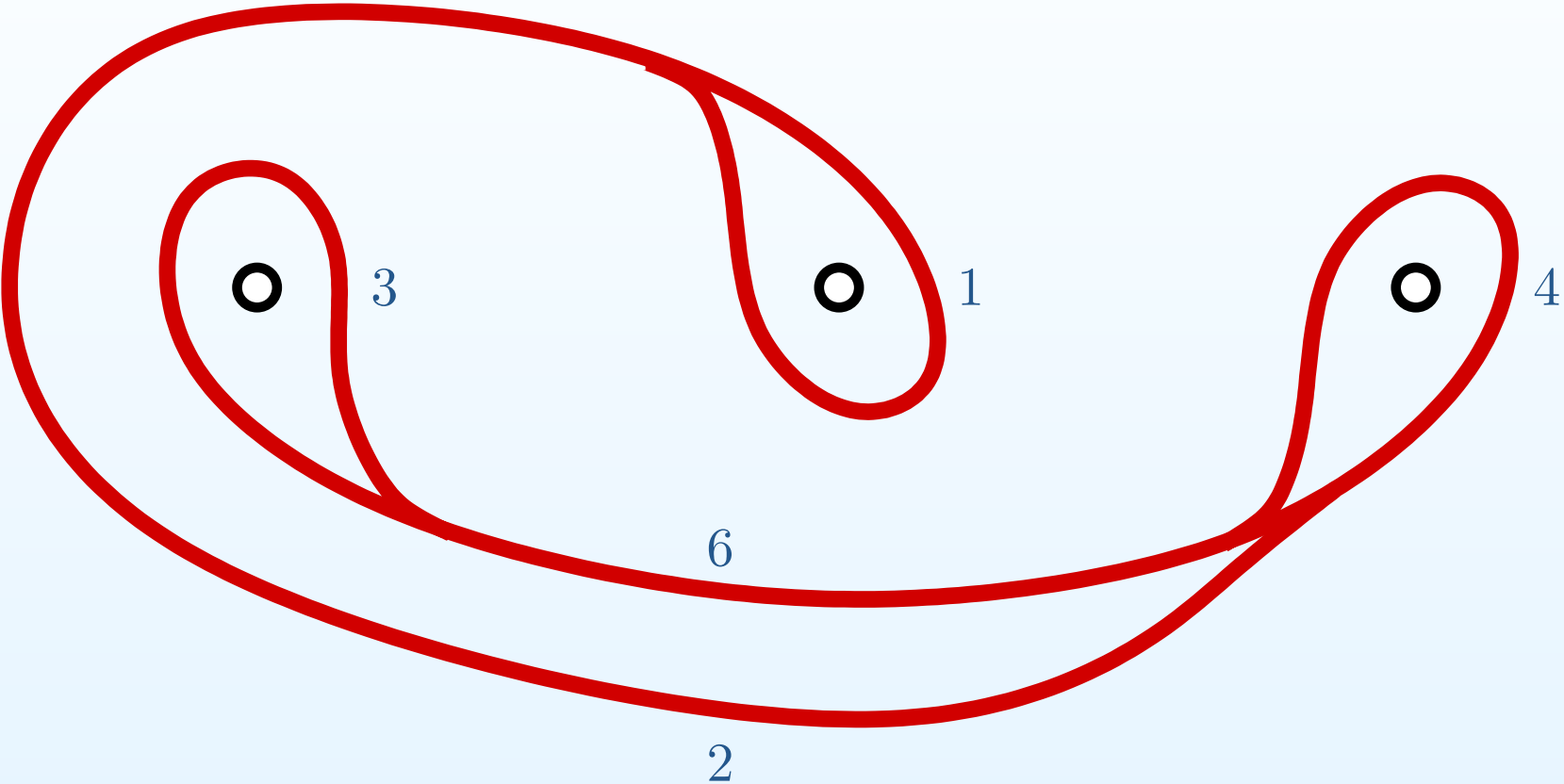
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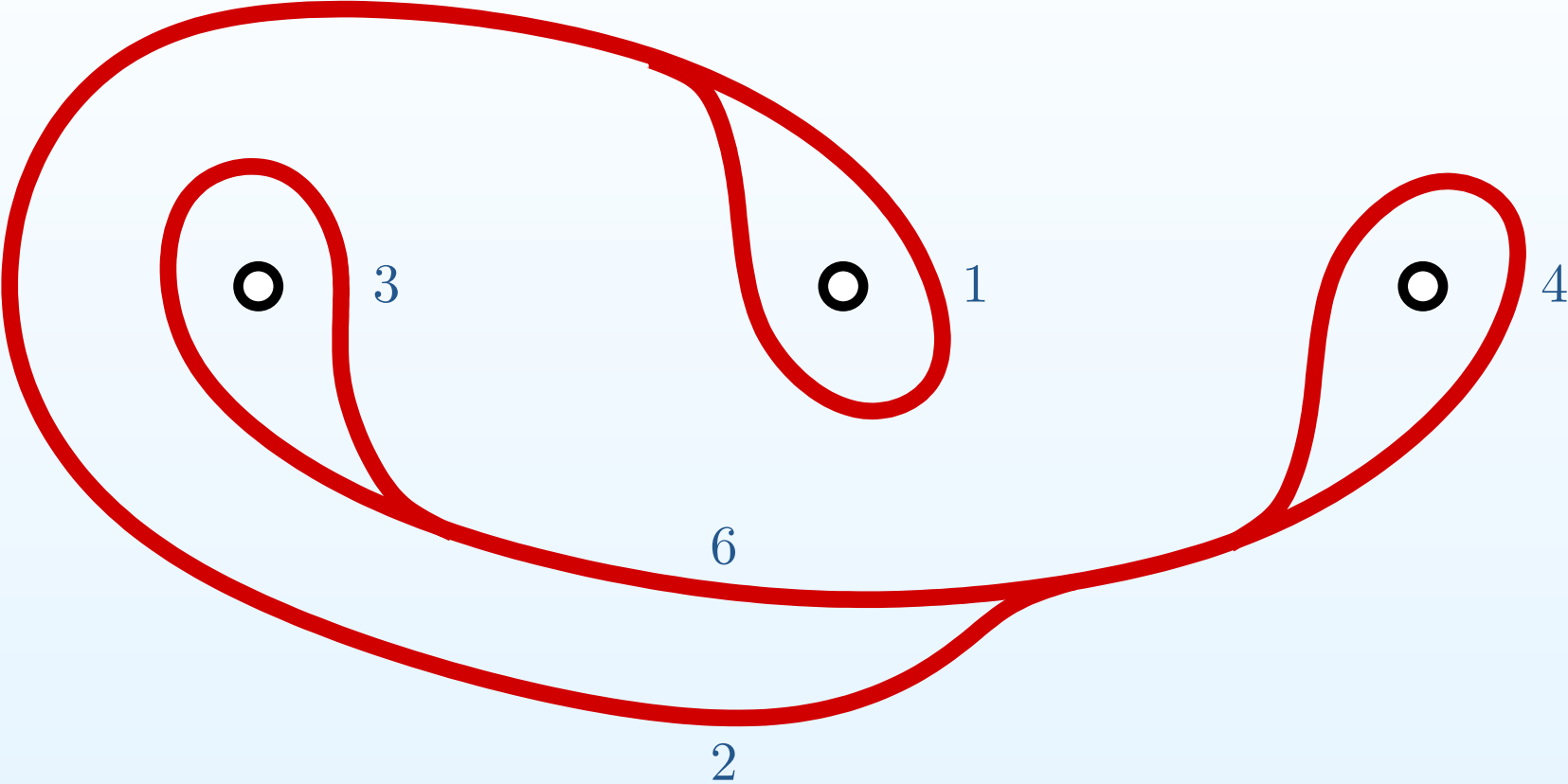
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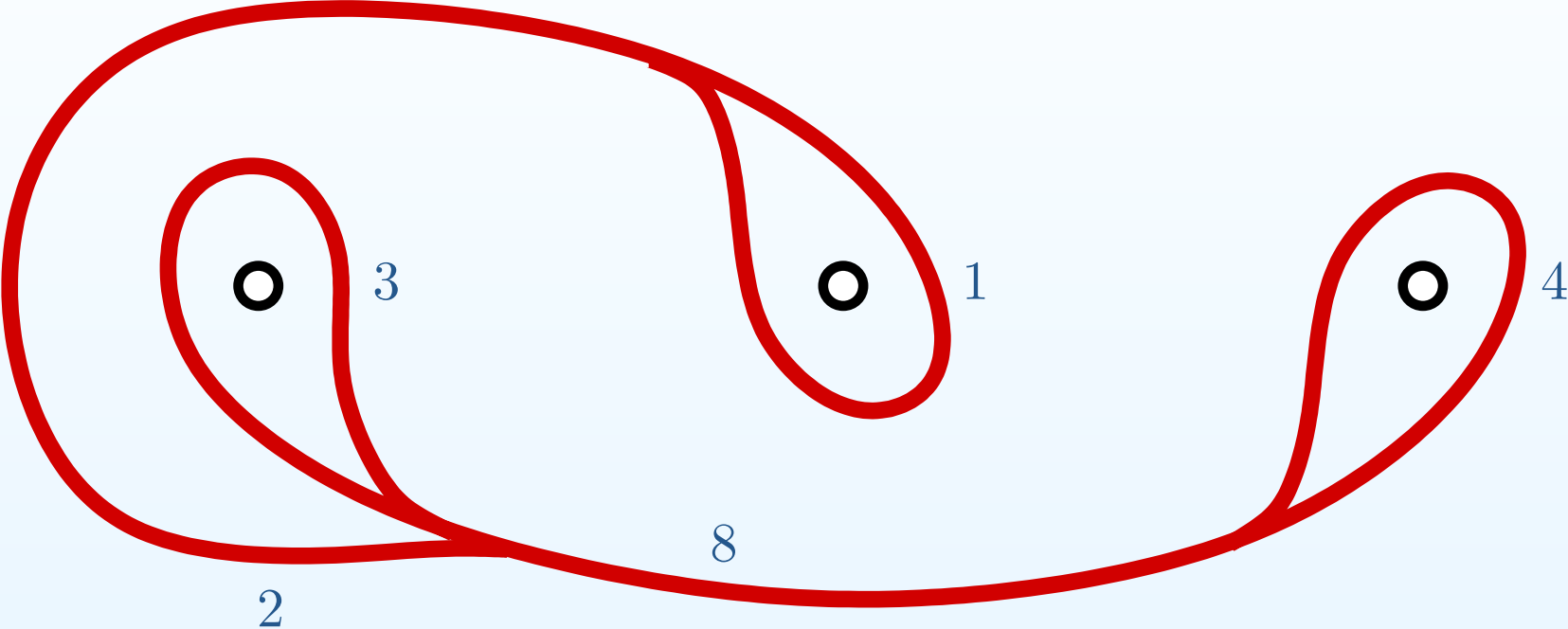
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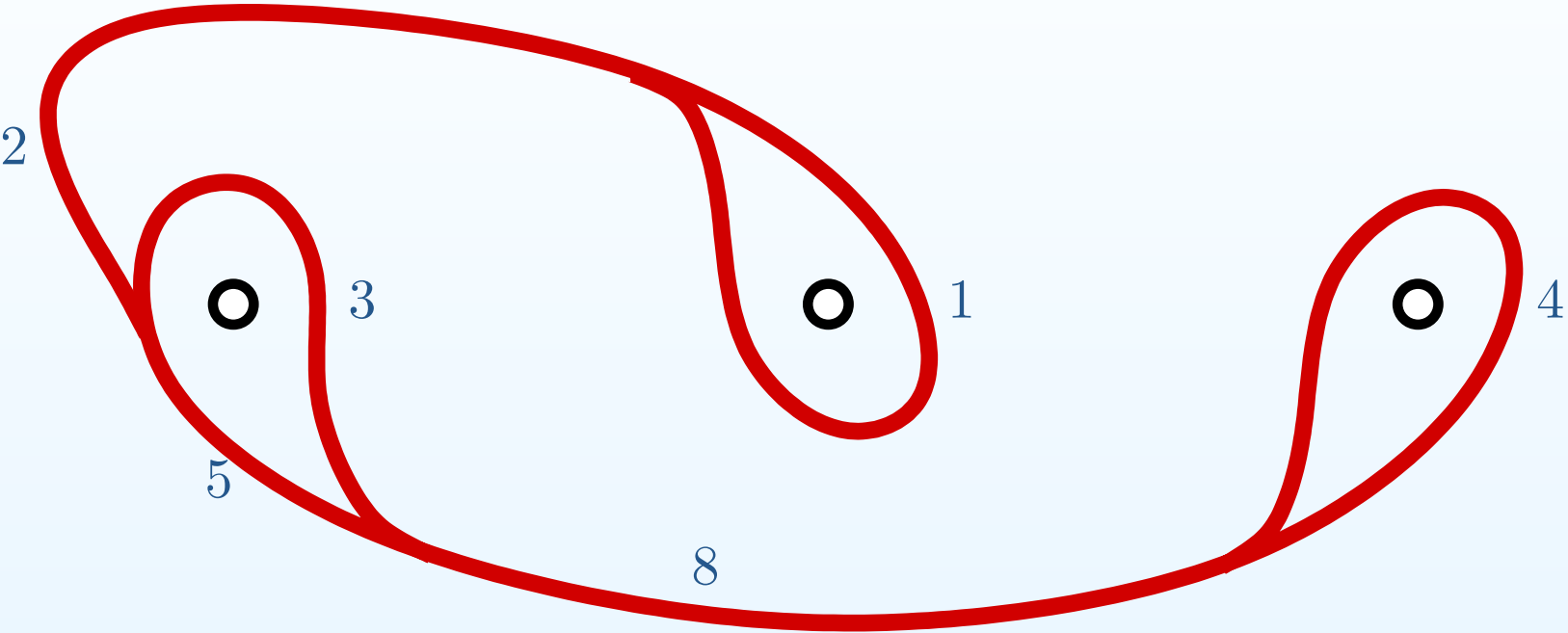
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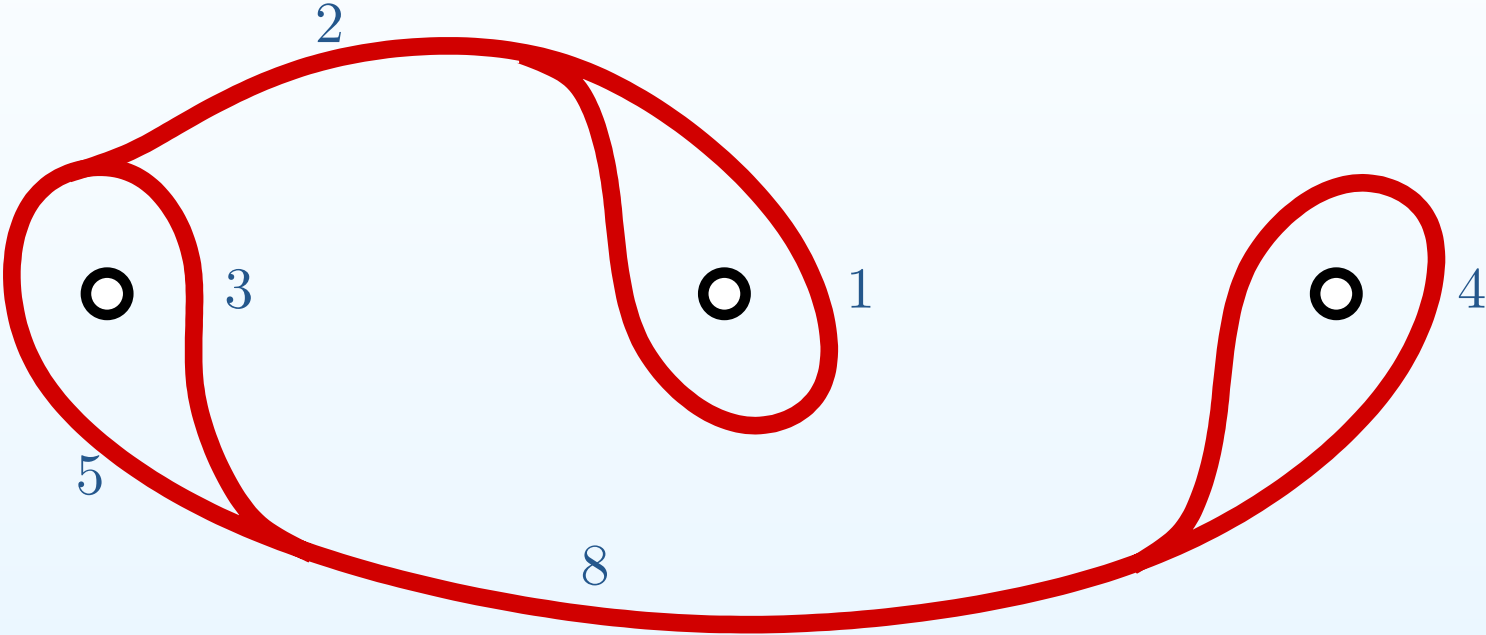
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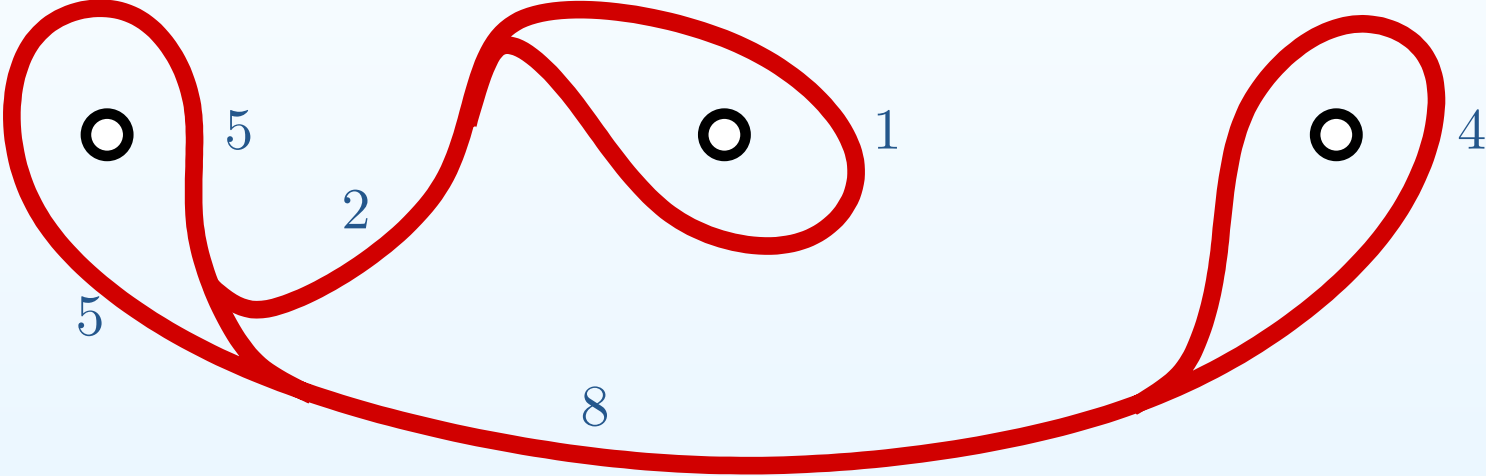
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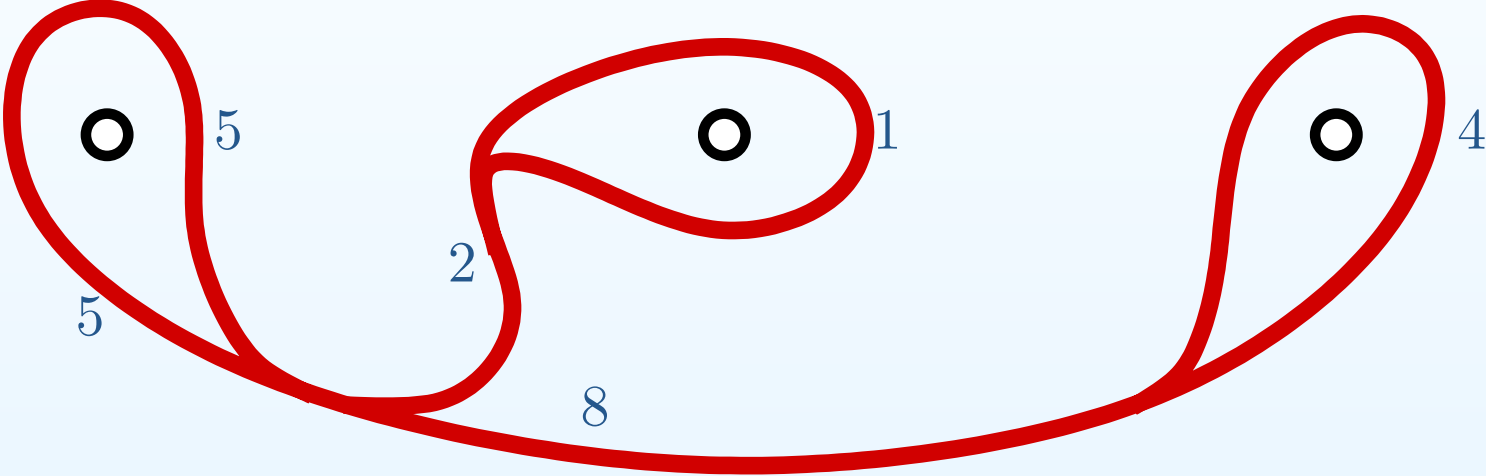
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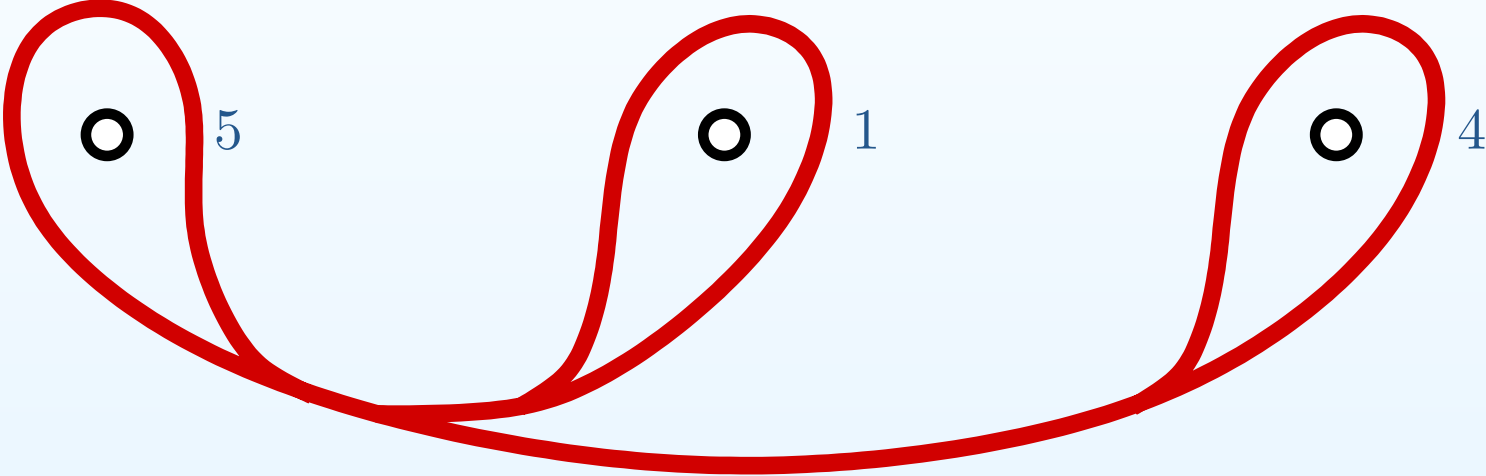
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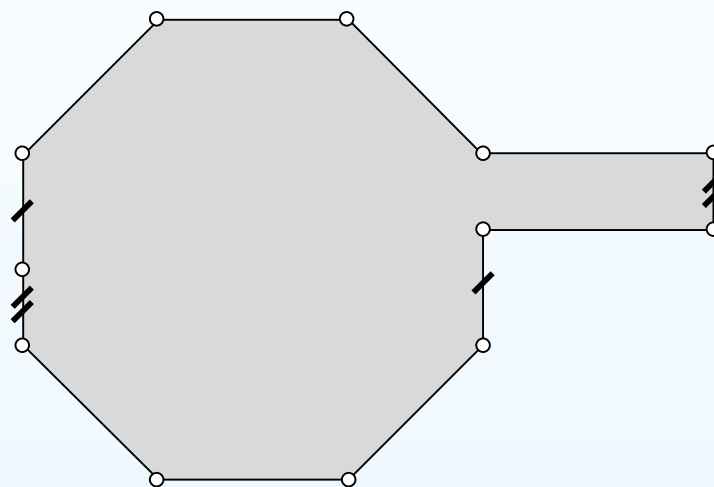
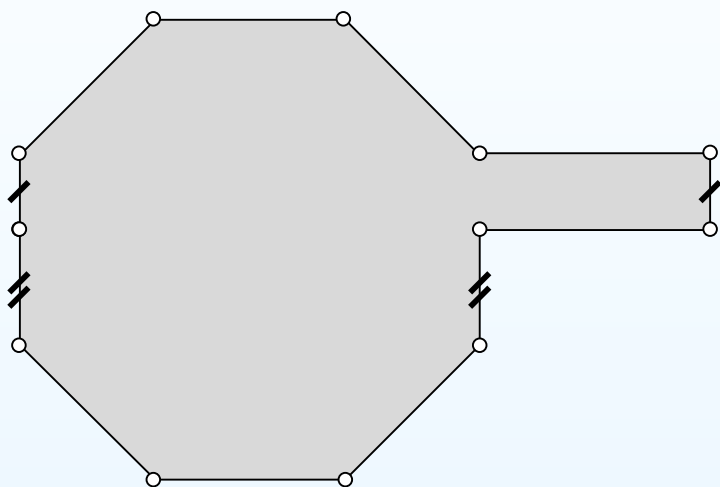


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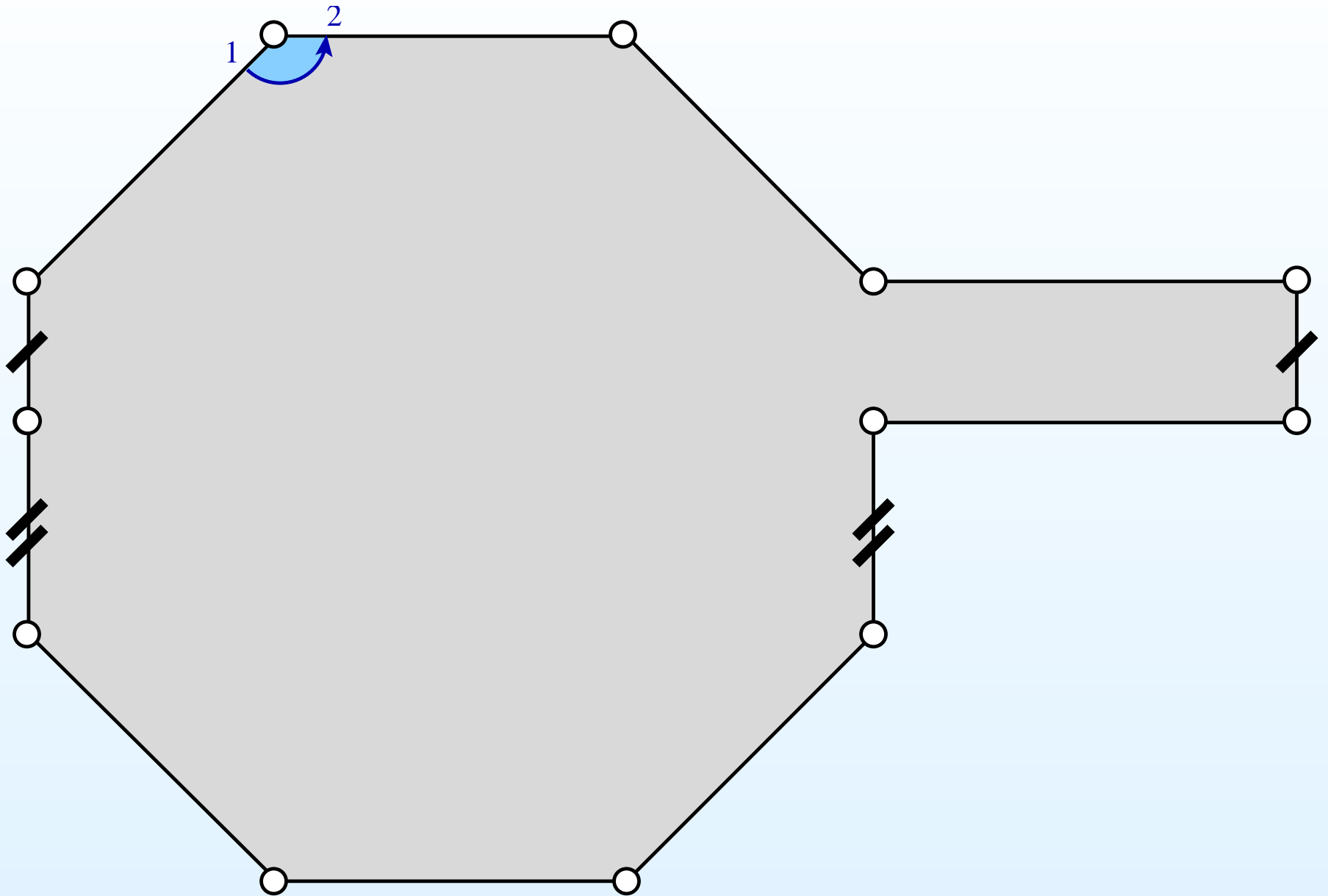
Parity of the spin-structure

- Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.

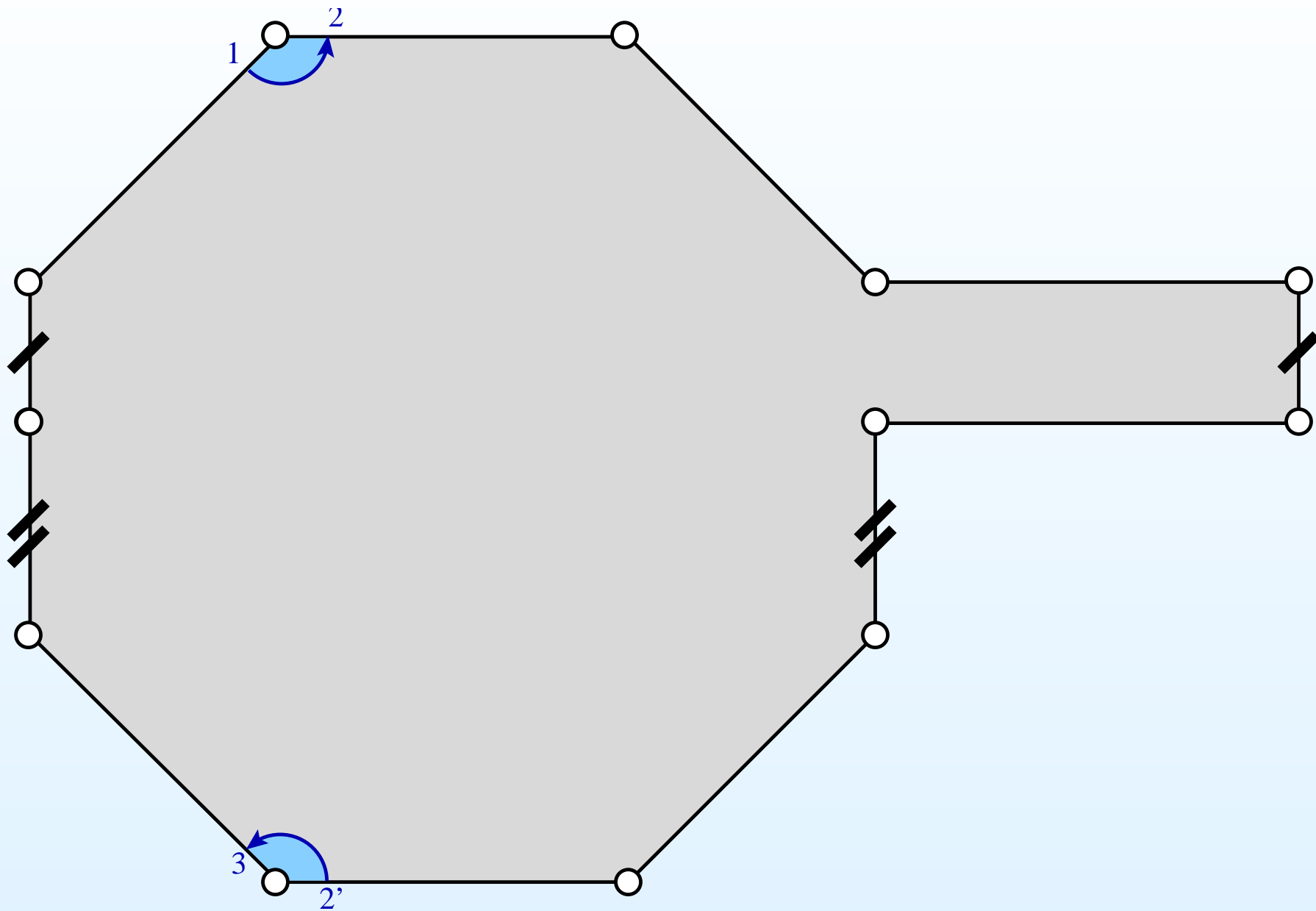


- Compute the parity of the spin structure for these surfaces (and notice that it is not the same).
- Determine which of the two surfaces is hyperelliptic.
- Find the hyperelliptic involution of this surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.

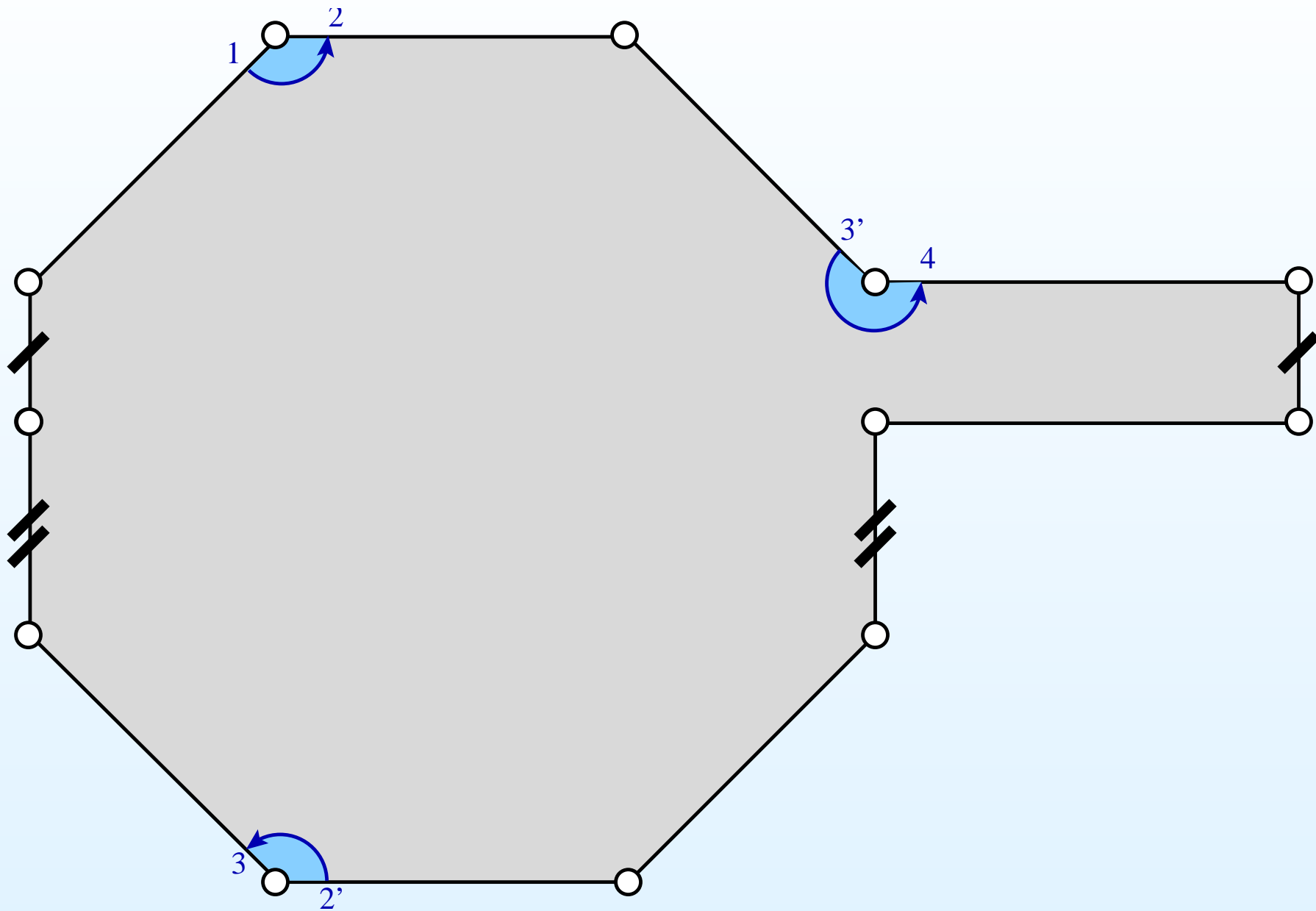
- Proof that the following flat surface belongs to the stratum $\mathcal{H}(4)$.



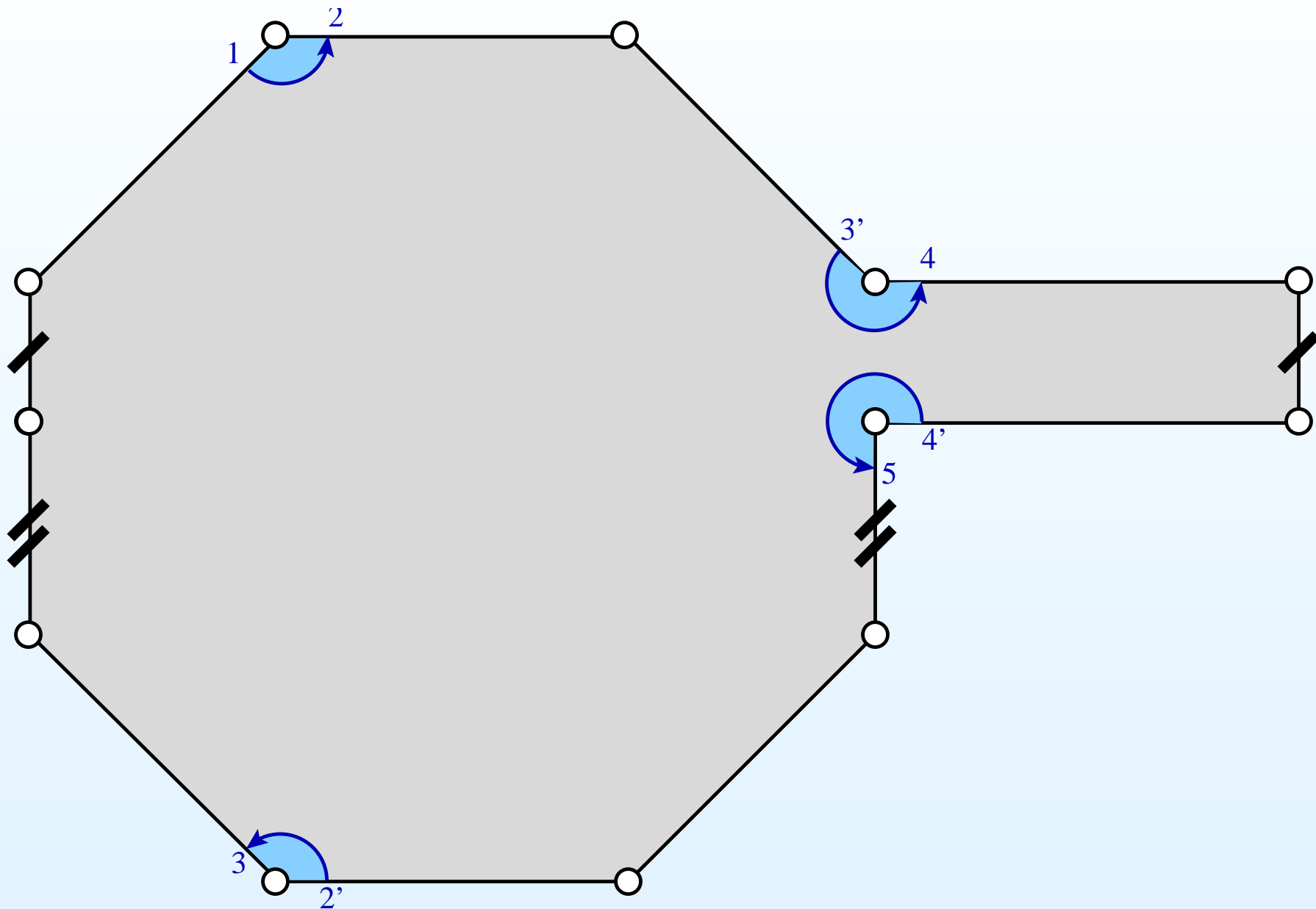
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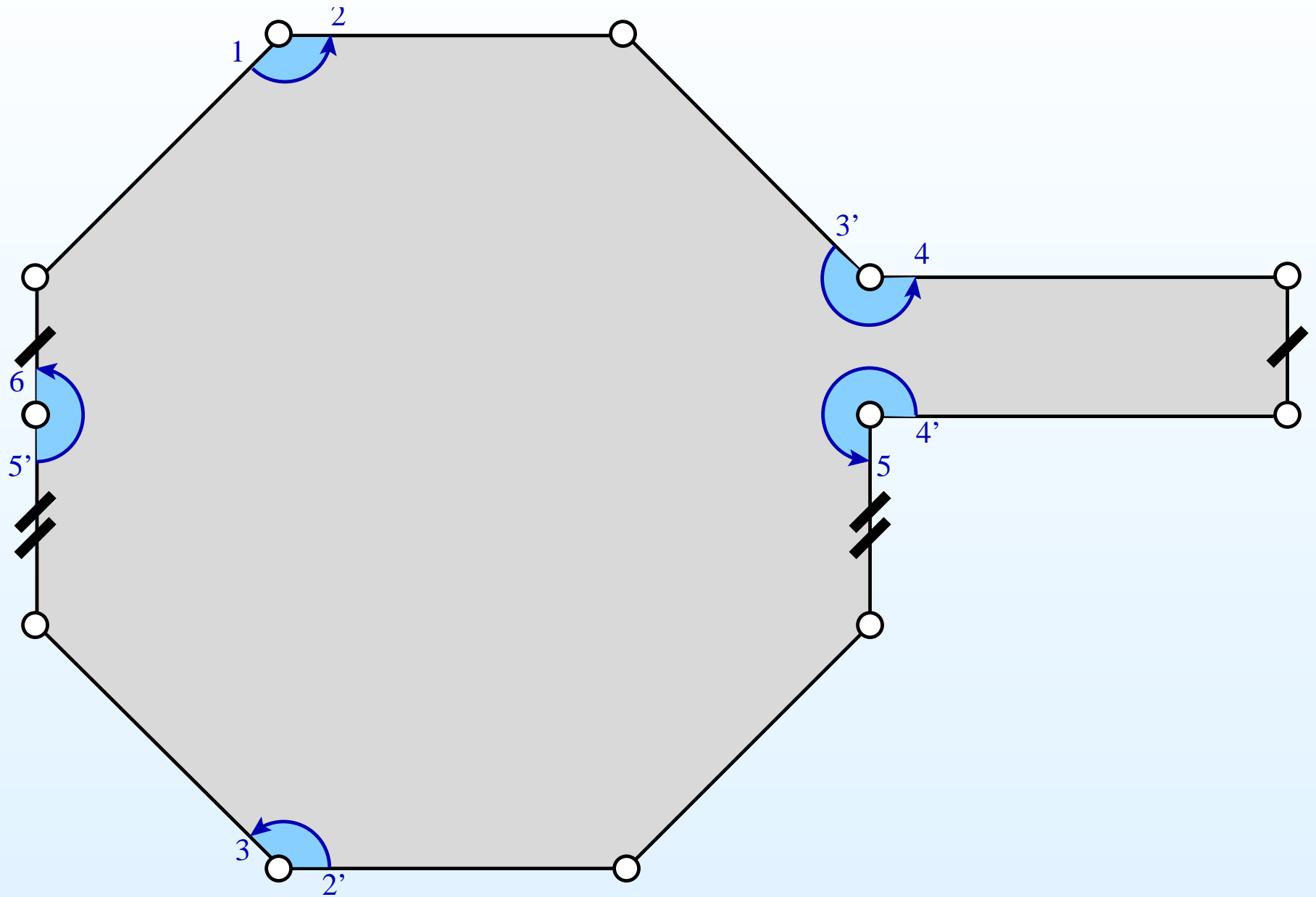
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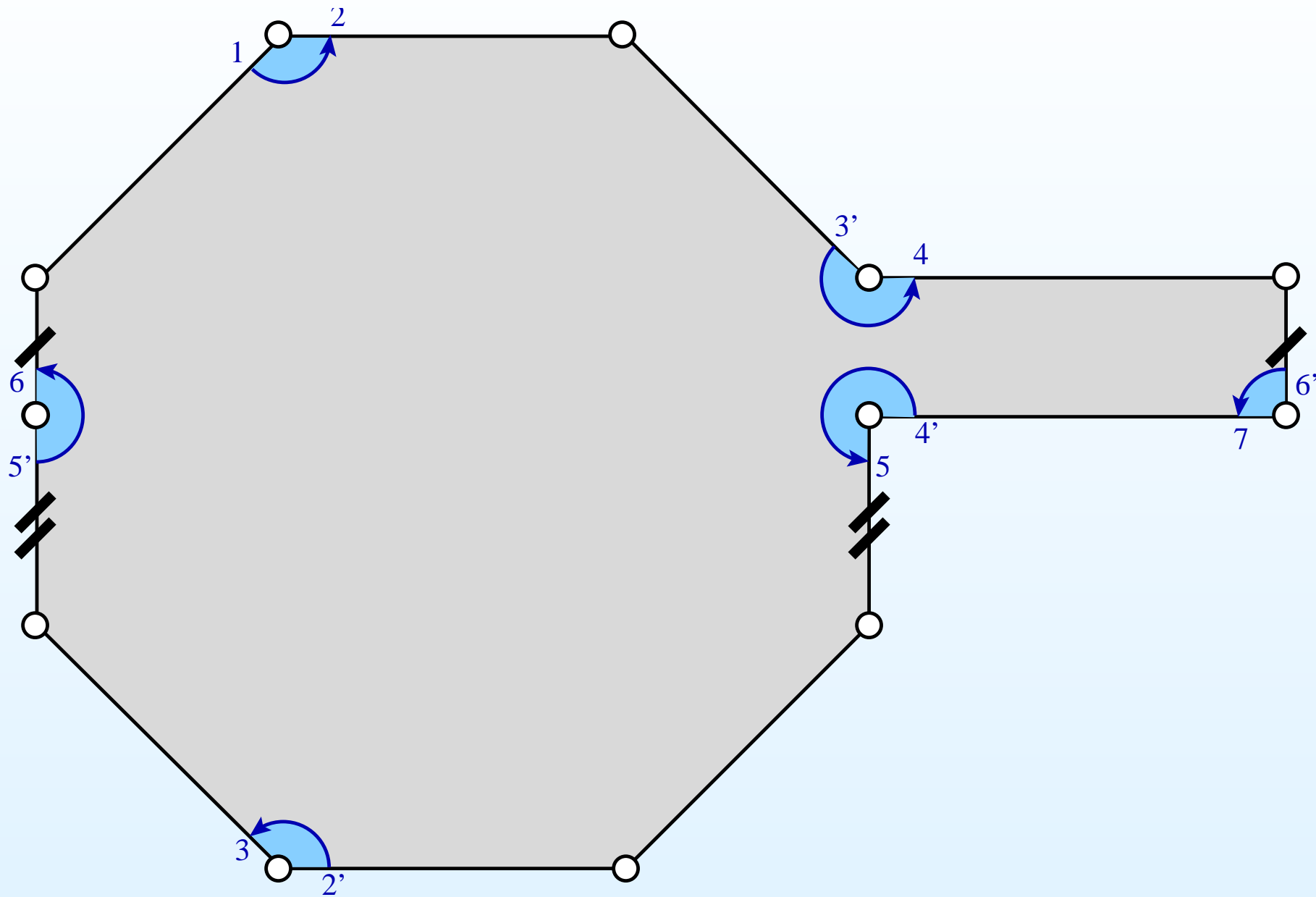
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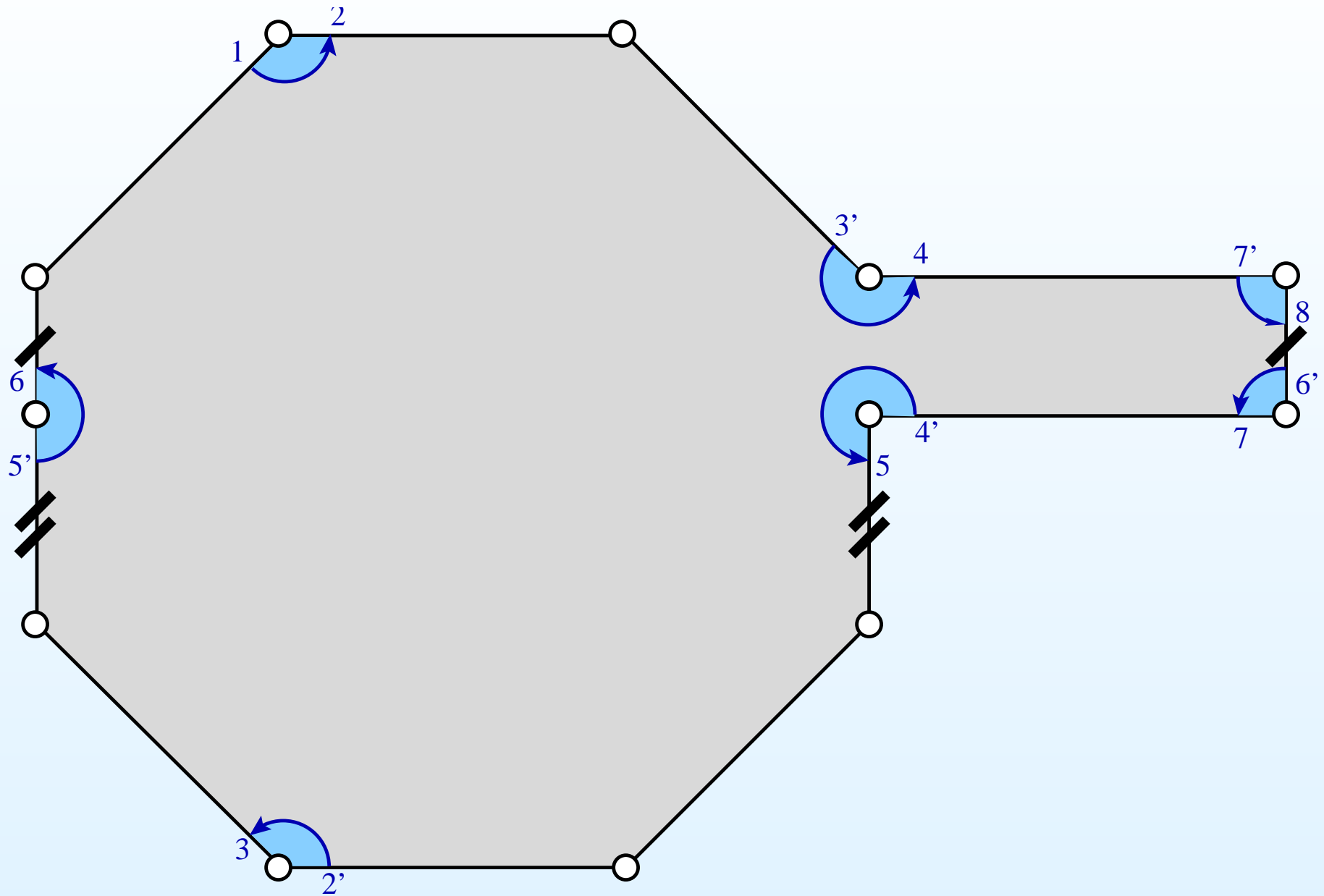
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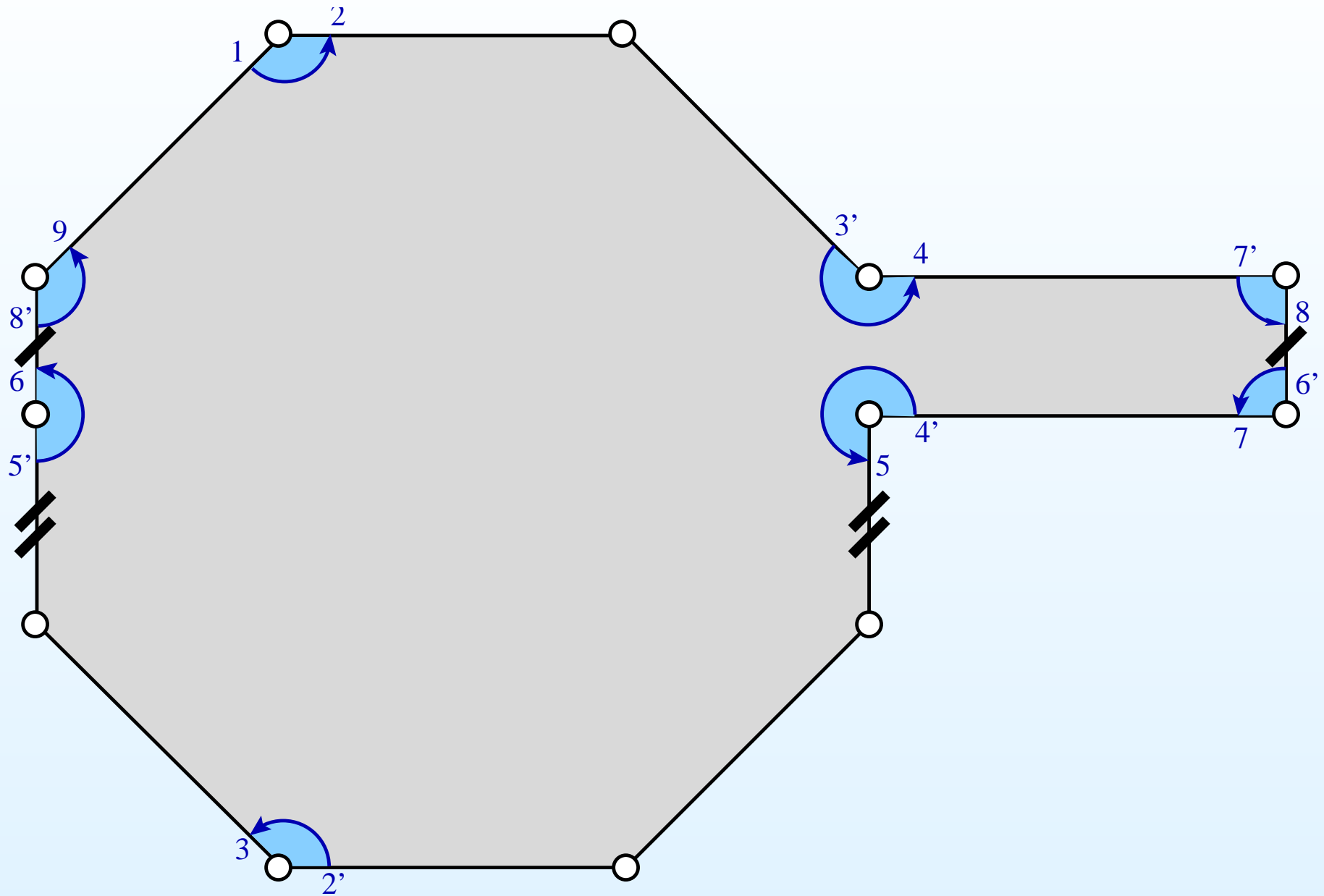
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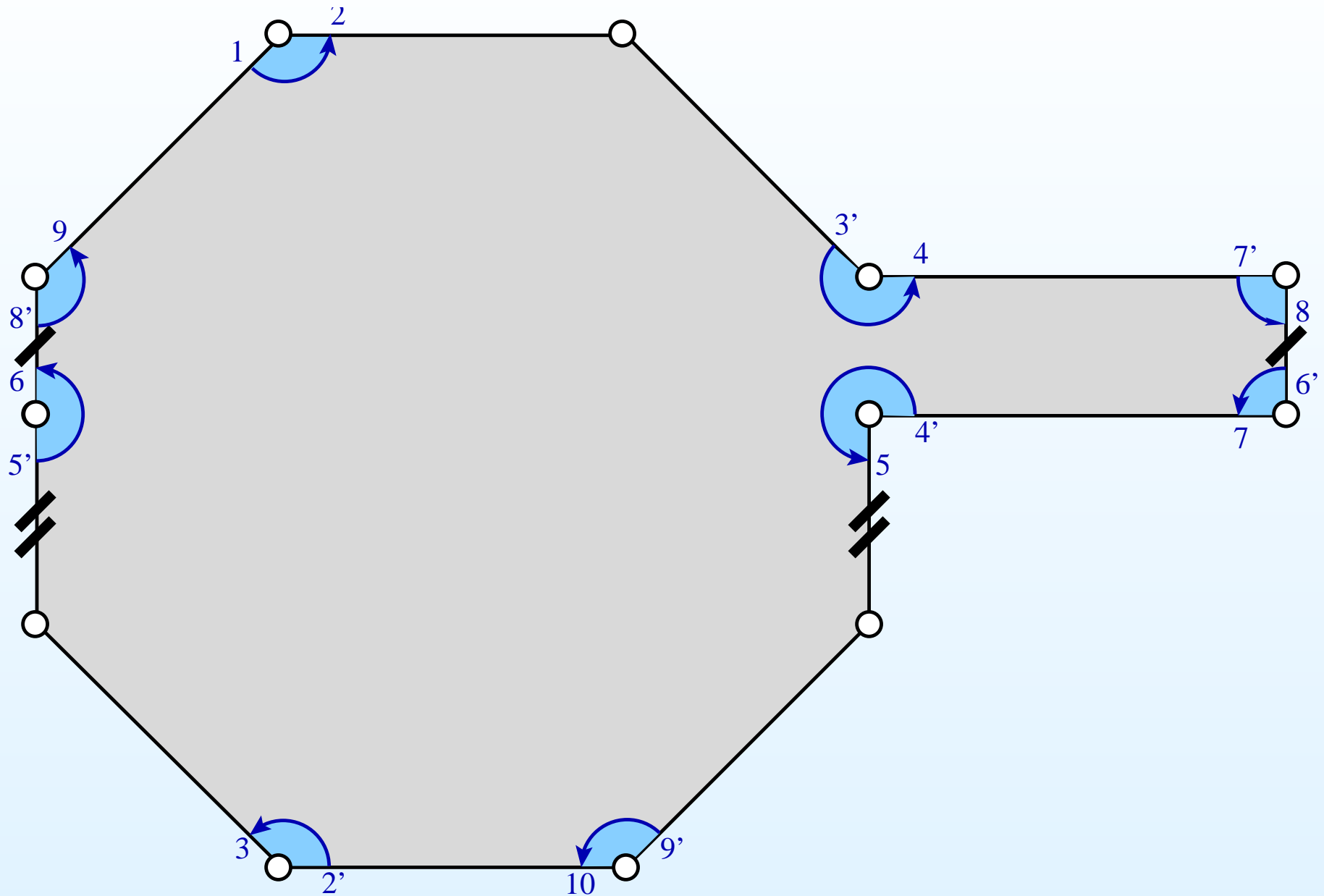
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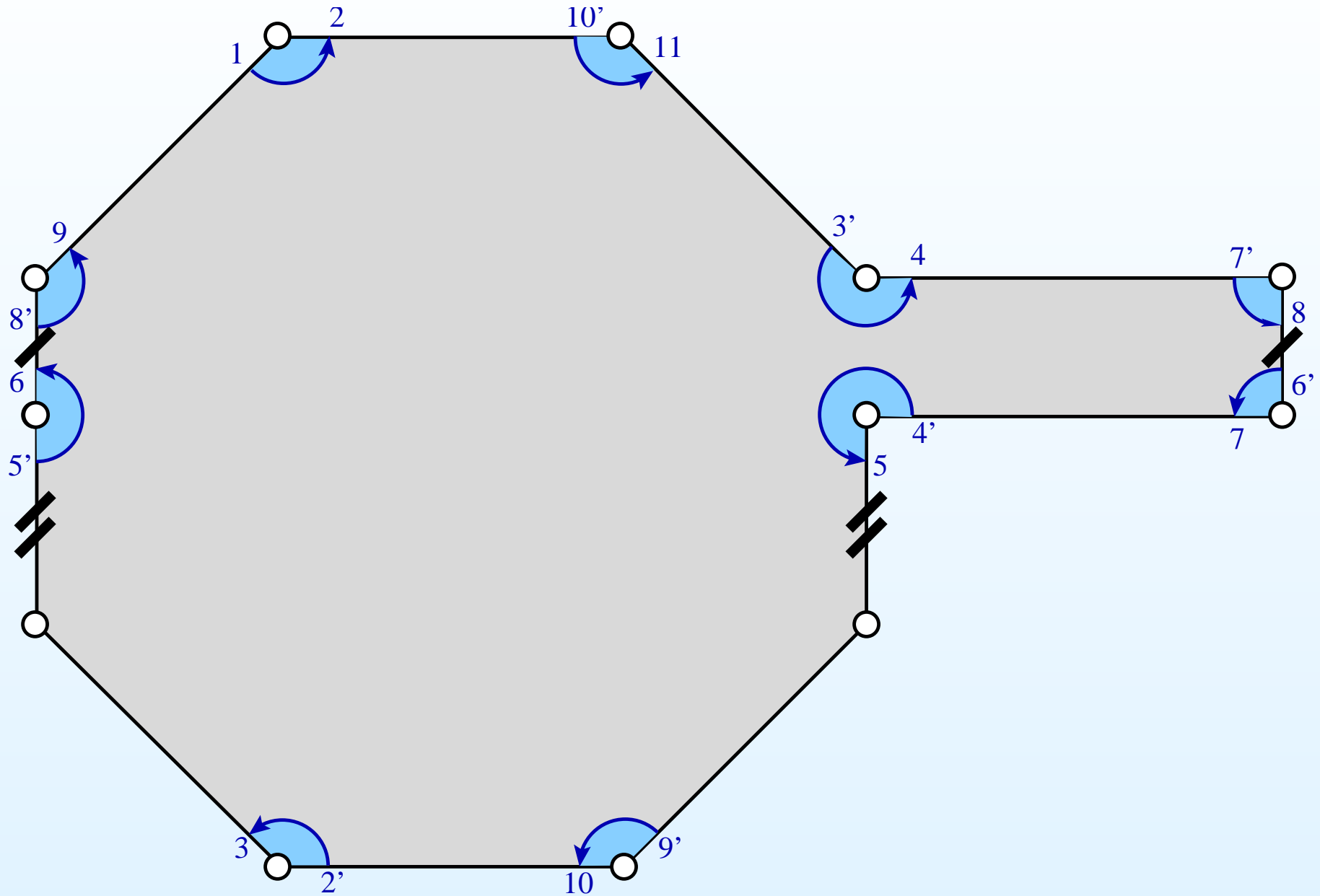
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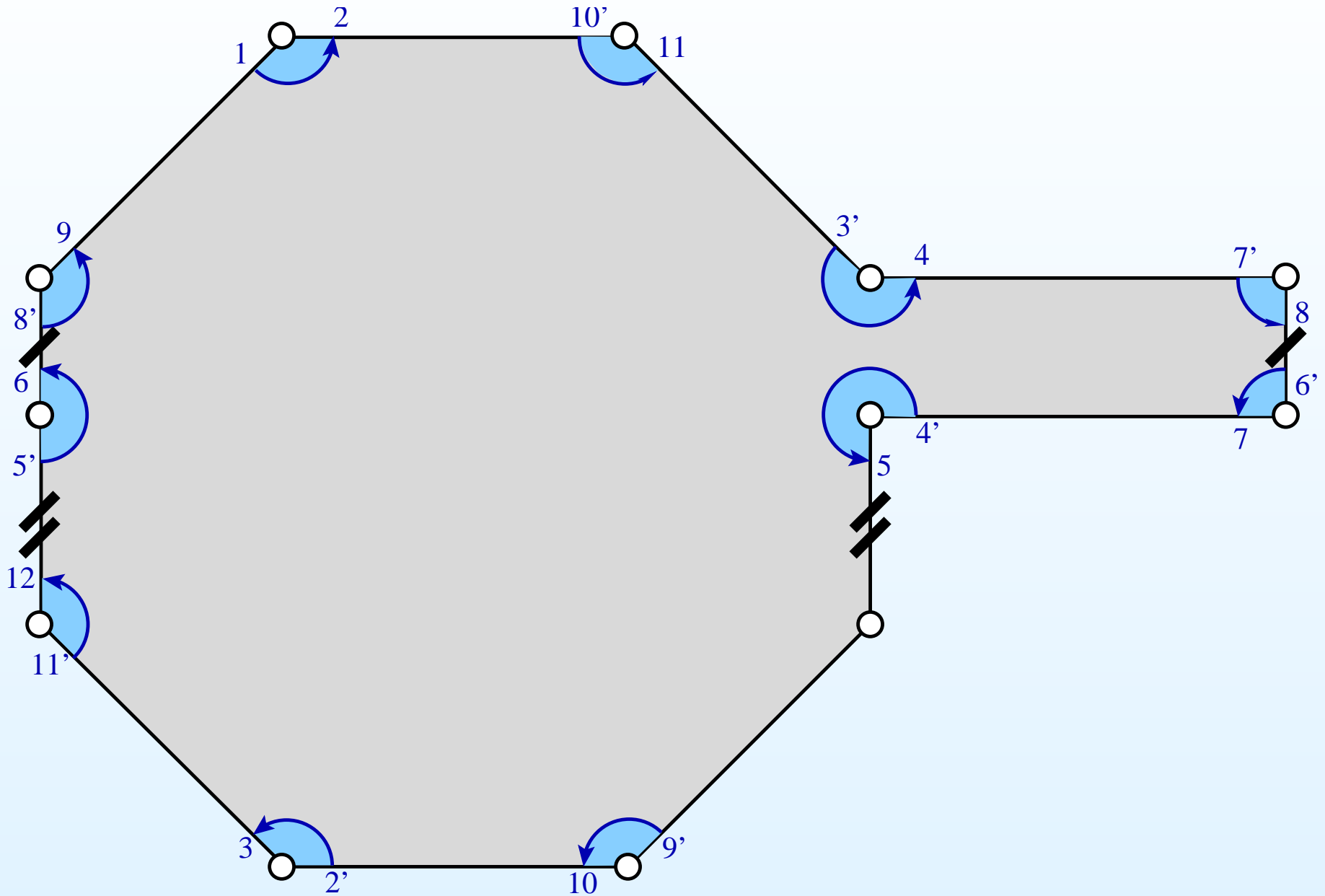
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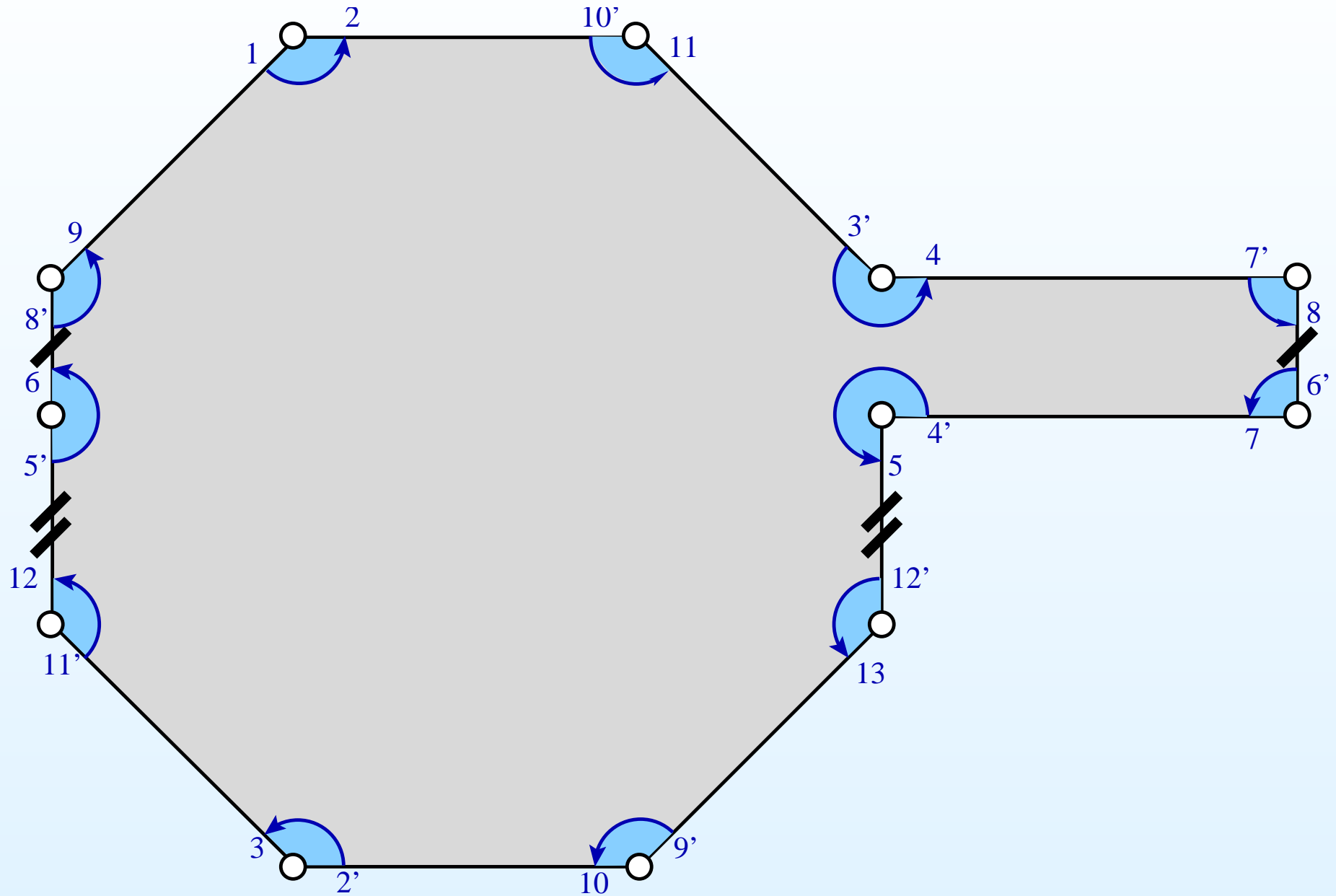
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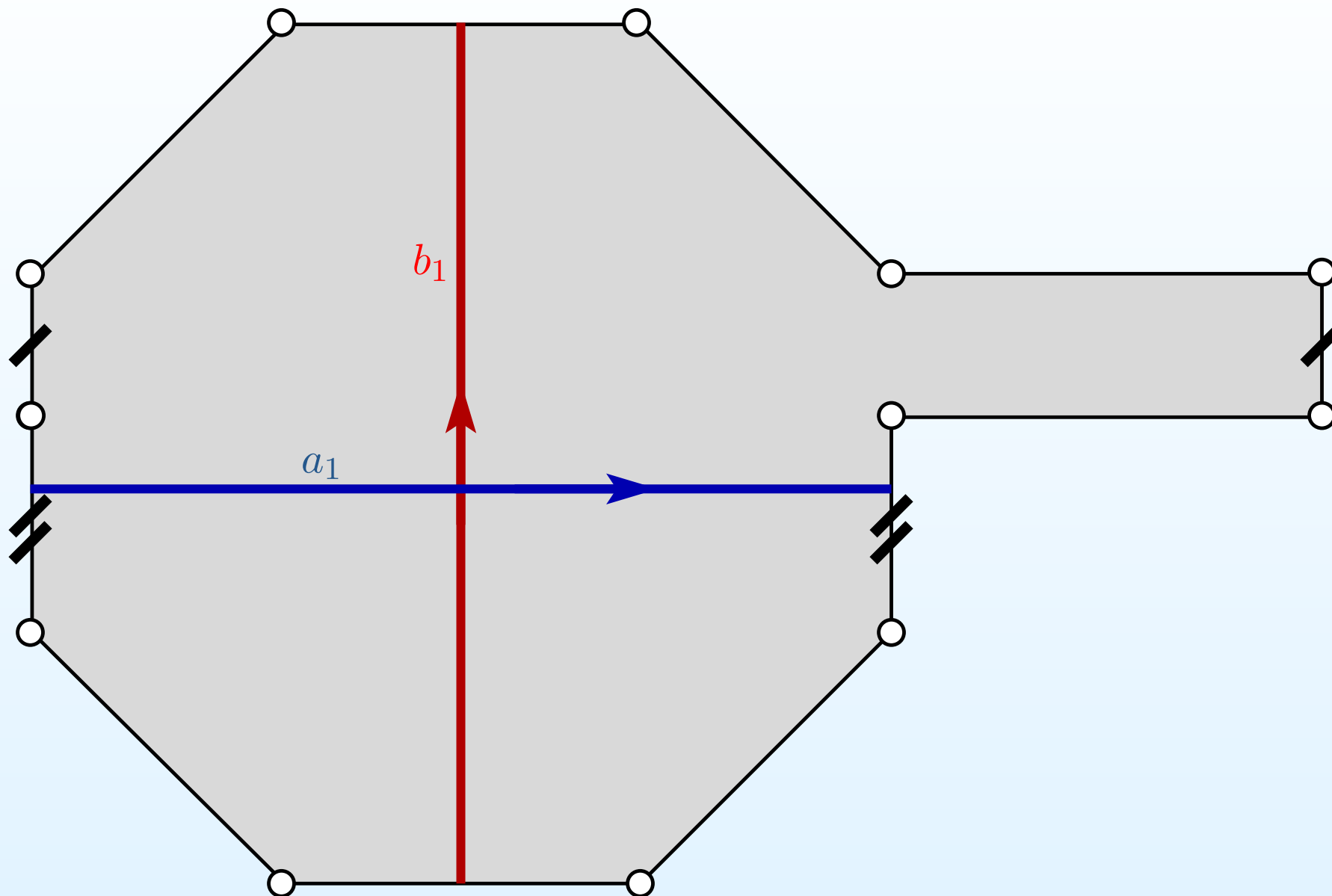
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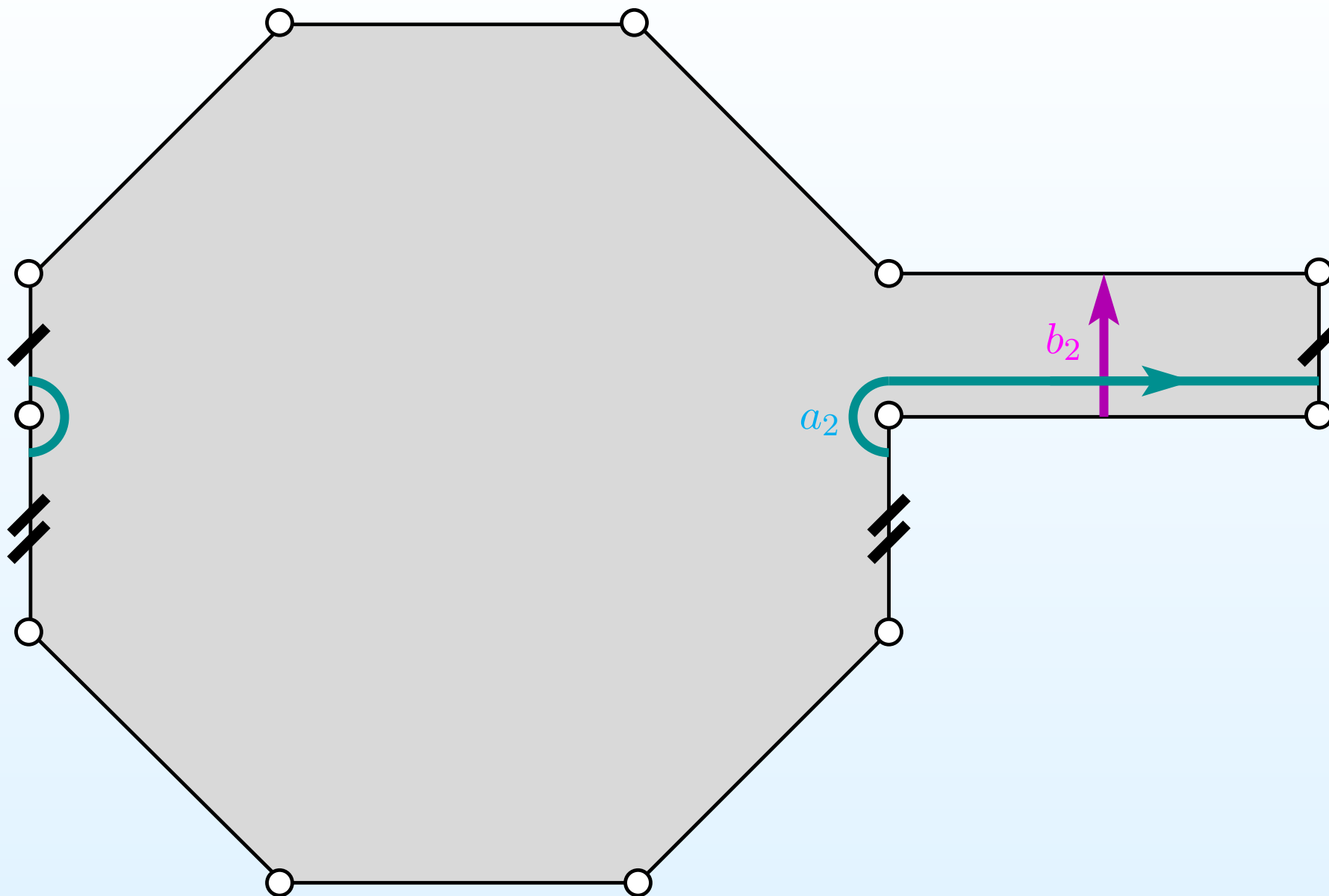
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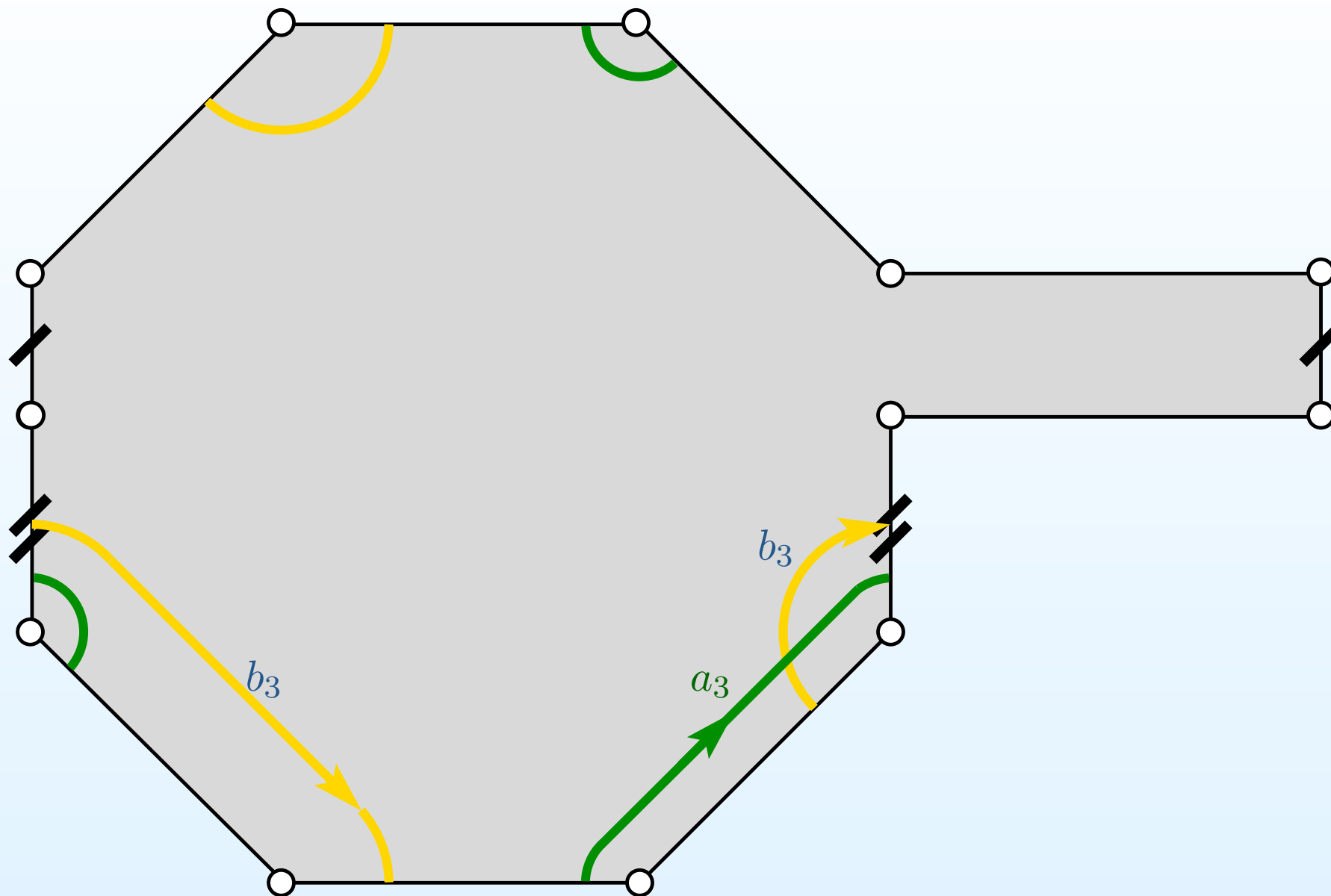
- Construction of a canonical basis of cycles.



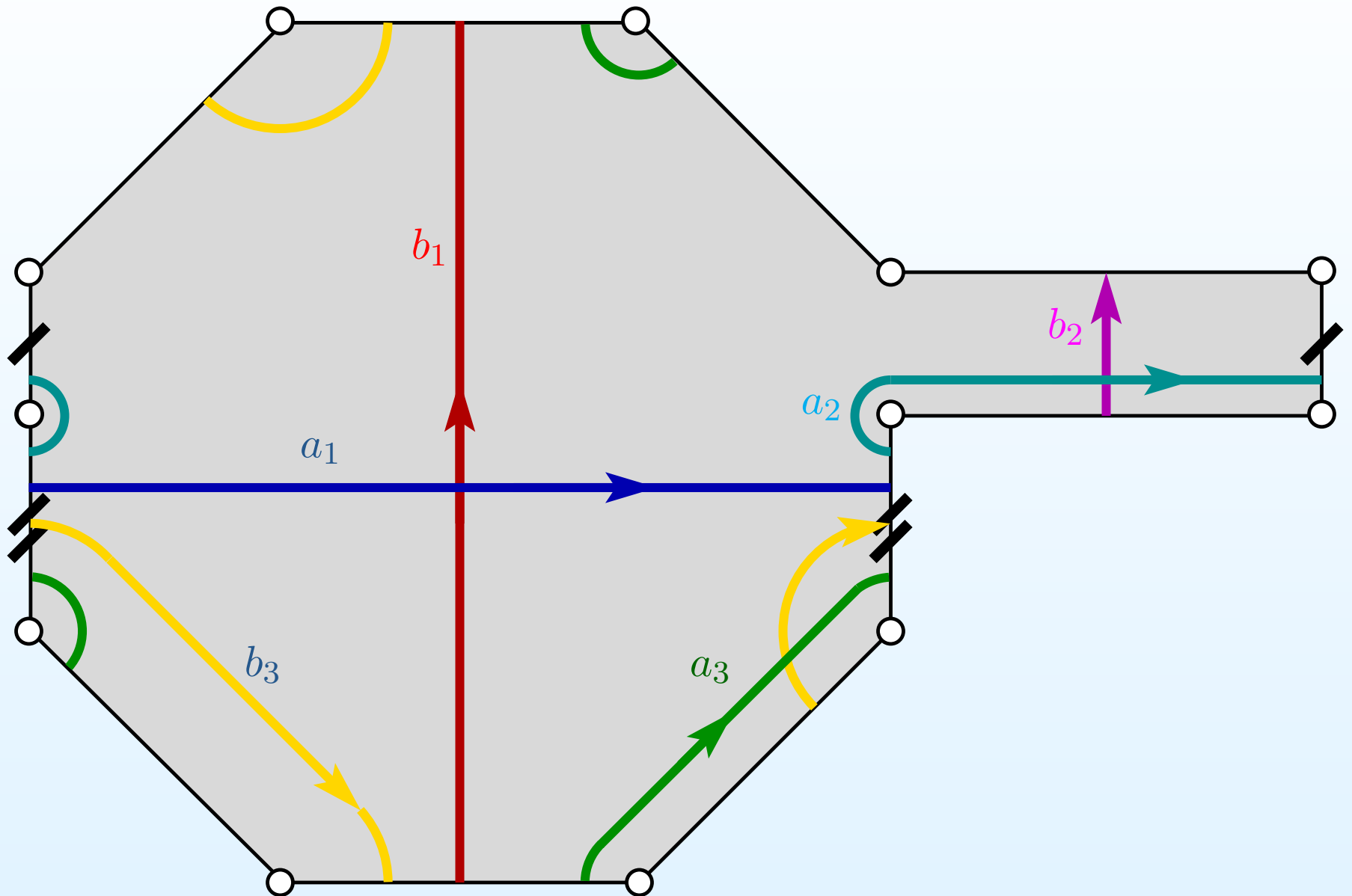
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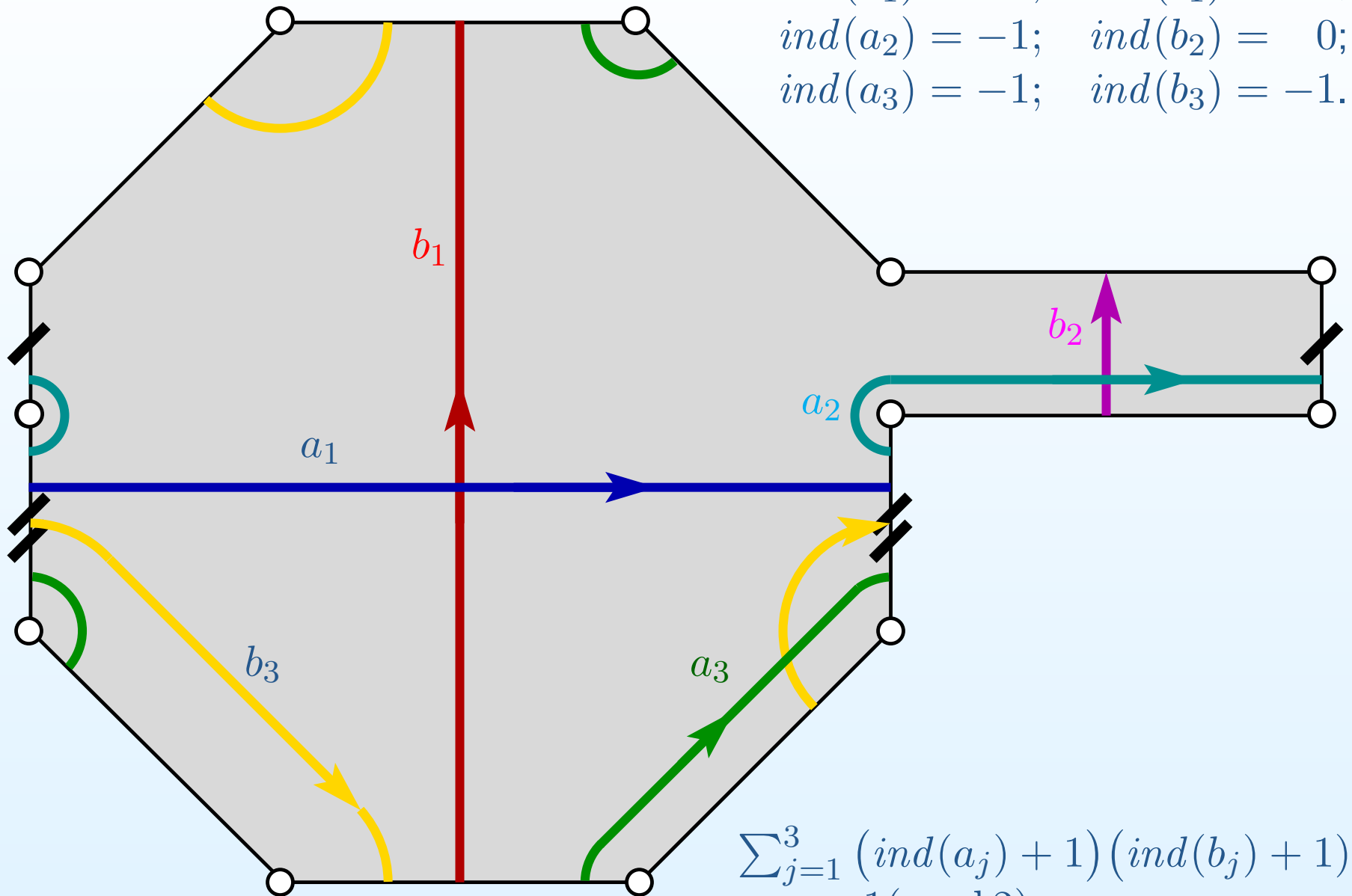


- Construction of a canonical basis of cycles.



- Evaluation of the parity of a spin-structure.

$$\begin{aligned} \text{ind}(a_1) &= 0; & \text{ind}(b_1) &= 0; \\ \text{ind}(a_2) &= -1; & \text{ind}(b_2) &= 0; \\ \text{ind}(a_3) &= -1; & \text{ind}(b_3) &= -1. \end{aligned}$$



$$\sum_{j=1}^3 (\text{ind}(a_j) + 1) (\text{ind}(b_j) + 1) = 1 \pmod{2}$$

Spin-structure of minimal hyperelliptic components

Theorem (M. Kontsevich, A. Zorich, 2003) *Parity of the spin structure determined by an Abelian differential from the hyperelliptic component $\mathcal{H}^{hyp}(2g - 2)$ equals*

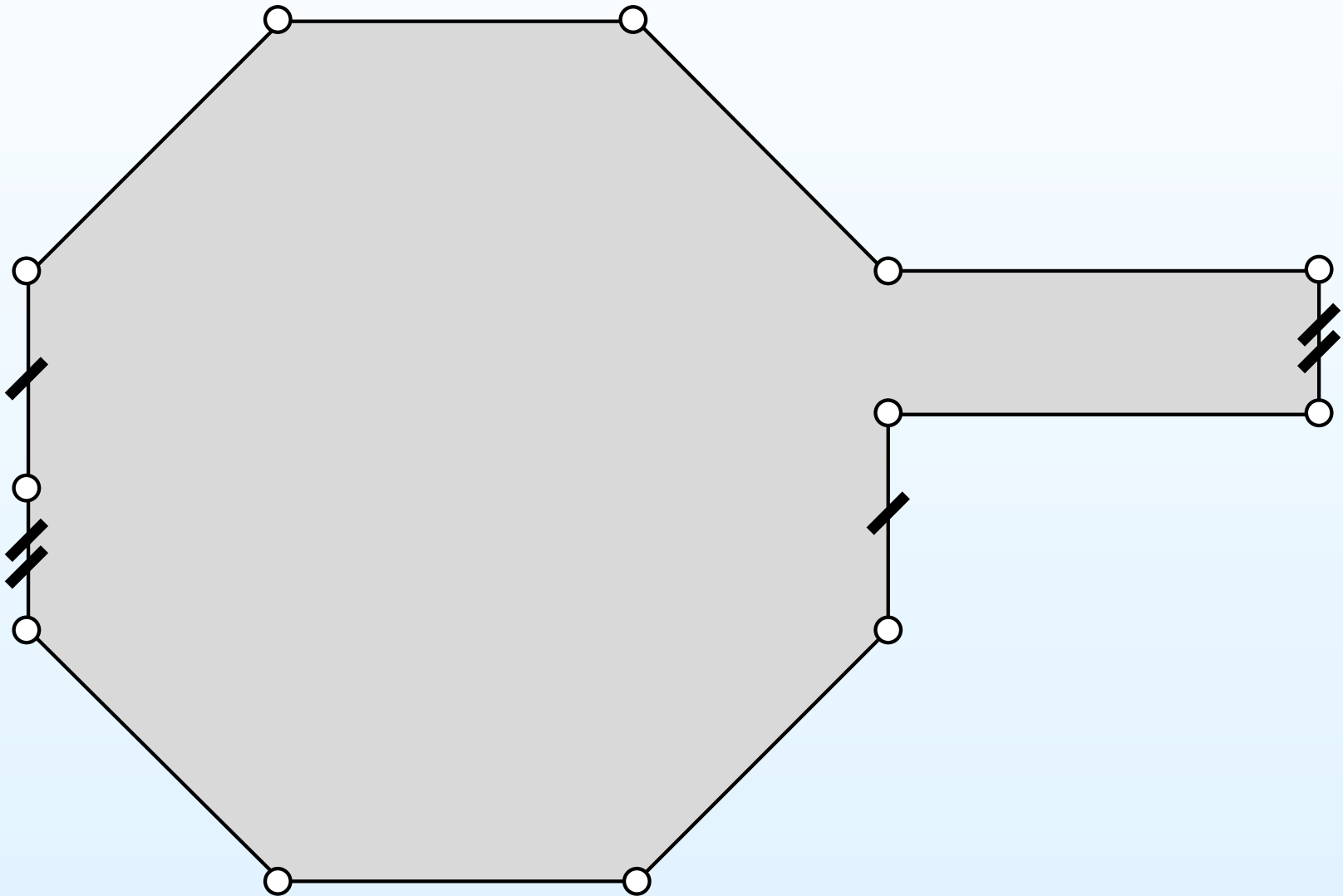
$$\varphi(\mathcal{H}^{hyp}(2g - 2)) \equiv \left[\frac{g + 1}{2} \right] \pmod{2}.$$

In particular,

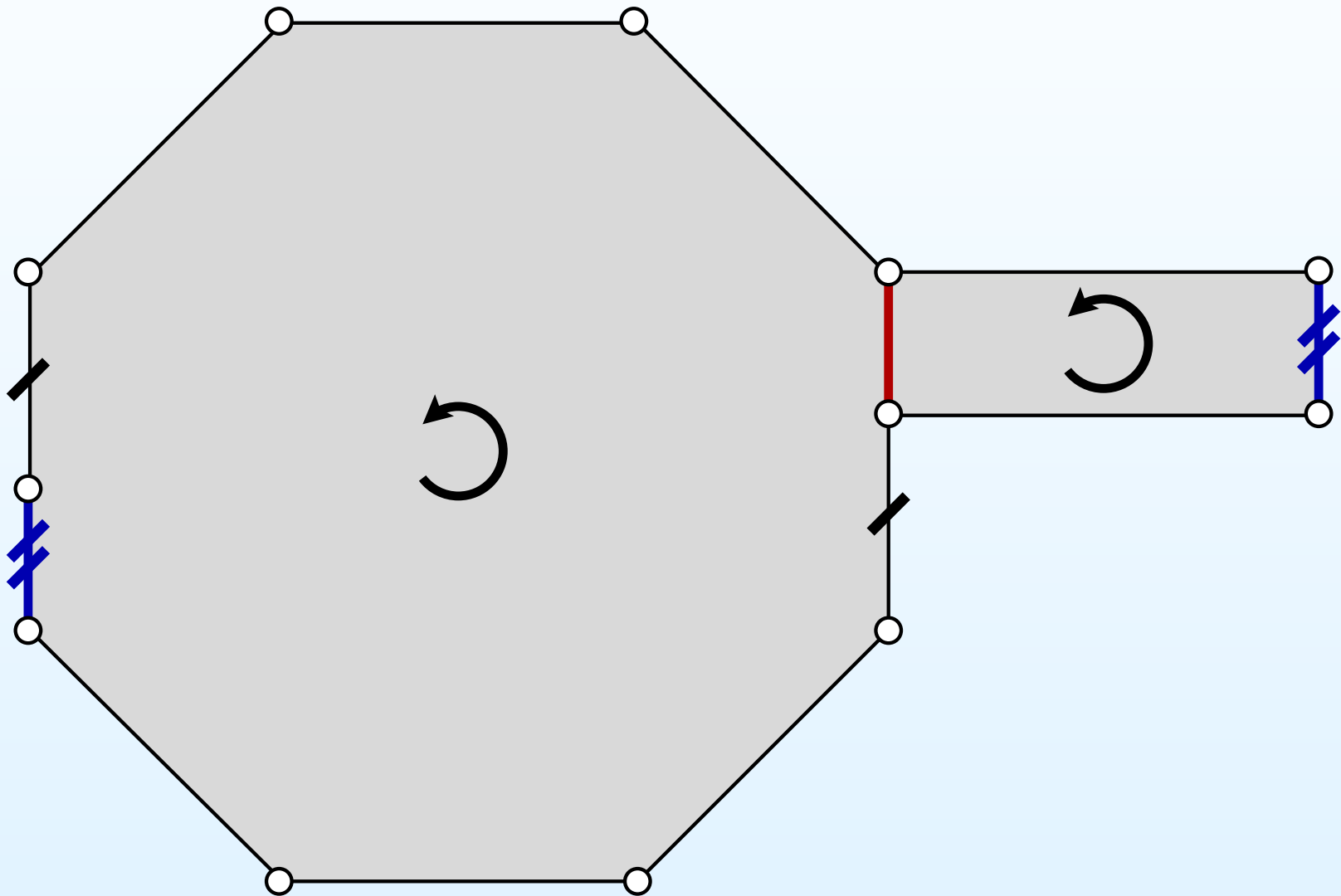
$$\varphi(\mathcal{H}^{hyp}(0)) = 1 \quad \varphi(\mathcal{H}^{hyp}(2)) = 1 \quad \varphi(\mathcal{H}^{hyp}(4)) = 0$$

and we conclude that the surface constructed above lives in $\mathcal{H}^{odd}(4)$.

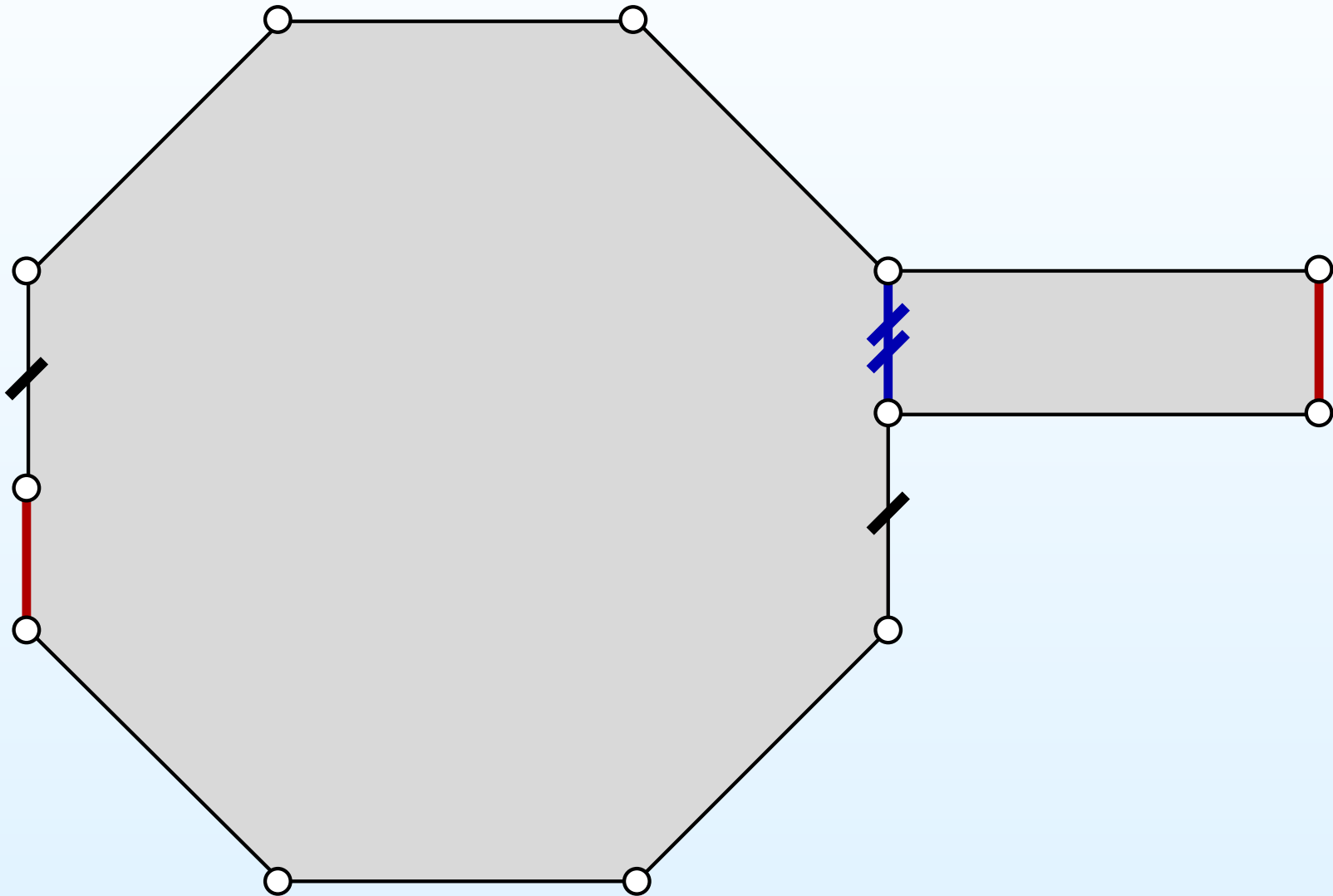
- Find the hyperelliptic involution of the remaining surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2 = 2 \cdot 3 + 2 = 8$ such points.



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