THE ABSOLUTELY CONTINUOUS SPECTRUM OF THE ALMOST MATHIEU OPERATOR

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Abstract. We prove that the spectrum of the almost Mathieu operator is absolutely continuous if and only if the coupling is subcritical. This settles Problem 6 of Barry Simon's list of Schrödinger operator problems for the twenty-first century.

1. Introduction

This work is concerned with the almost Mathieu operator $H = H_{\lambda,\alpha,\theta}$ defined on $\ell^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(2\pi[\theta + n\alpha])u_n \tag{1.1}$$

where $\lambda \neq 0$ is the coupling, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency and $\theta \in \mathbb{R}$ is the phase. This is the most studied quasiperiodic Schrödinger operator, arising naturally as a physical model (see [L3] for a recent historical account and for the physics background).

We are interested on the decomposition of the spectral measures in atomic (corresponding to point spectrum), singular continuous and absolutely continuous parts. Our main result is the following.

Main Theorem 1. *The spectral measures of the almost Mathieu operator are absolutely continuous if and only if* $|\lambda| < 1$.

1.1. Background.

Singularity of the spectral measures for $|\lambda| \geq 1$ had been previously established (it follows from [LS], [L1], [AK]). Thus the Main Theorem reduces to showing absolute continuity of the spectral measures for $|\lambda| < 1$, which is Problem 6 of Barry Simon's list [S2].

We recall the history of this problem following [J]. Aubry-André conjectured the following dependence on λ of the nature of the spectral measures:

(1) (Supercritical regime) For $|\lambda| > 1$, spectral measures are pure point,
(2) (Subcritical regime) For $|\lambda| < 1$, spectral measures are absolutely continuous.

A measure-theoretical version of this conjecture was proved by Jitomirskaya [J]: it holds for almost every α and θ.

The description of the supercritical regime turns out to be wrong as stated. More precisely, for generic α there can never be point spectrum [G], [AS], whatever λ and θ are chosen, and for every α there is a generic set of θ for which there is similarly no point spectrum [JS]. Thus the result of [J] is essentially the best possible in the supercritical regime (one can still look for more optimal conditions

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on the parameters, which can be sometimes useful for other purposes, see [AJ1] and [AJ2]).

There was some hope that the description of the subcritical regime was actually correct as stated, since the work of Last [L2], Gesztesy-Simon [GS] (see also Last-Simon [LS]) established that there are absolutely continuous components (of some spectral measures) for every α and θ (belief in the conjecture was however not unanimous, due to lack of any further evidence for generic α).

Two key advances happened recently. In [AJ2], the problem was settled for almost every α and every θ, and soon later, in [AD] it was settled for every α (to be precise, for every α that can not be dealt by [J], [AJ1]) and almost every θ. Those two results are based on quite independent methods.

1.2. **Outline.** Our proof of the complete conjecture splits into two parts that do not interact. The arithmetic properties of α, more precisely whether it is "well approximated by rational numbers" or not, will decide which of the two methods will be applied.

Let p_n/q_n be the continued fraction approximants to α and let

$$\beta = \beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}. \tag{1.2}$$

For our problem, the key distinction is whether $\beta = 0$ (the *subexponential regime*) or $\beta > 0$ (the *exponential regime*).

1.2.1. *The subexponential regime.* In [E], Eliasson introduced a sophisticated KAM scheme that allowed him to study the entire spectrum of one-dimensional quasiperiodic Schrödinger operators in the *perturbative regime*. Applied to the almost Mathieu operator, his results imply that, for frequencies satisfying the usual Diophantine condition $\alpha \in \mathrm{DC}$, that is $\ln q_{n+1} = O(\ln q_n)$, and for $|\lambda|$ sufficiently small (depending on α), the spectral measures of the almost Mathieu operator are absolutely continuous.

In [AJ2], a non-perturbative method was introduced that, when applied to the almost Mathieu operator, gives sharp estimates through the whole subcritical regime for $\alpha \in \mathrm{DC}$. Absolute continuity of the spectral measures was then concluded by showing that, after an appropriate "change of coordinates", the smallness requirements of the KAM scheme of Eliasson were satisfied.

In order to extend the conclusions of [AJ2] to the subexponential regime $\ln q_{n+1} = o(q_n)$, there are two main difficulties. The first is that some key estimates of [AJ2] break down in this setting (essentially for not achieving exponential decay of Fourier coefficients which is needed to address the entire $\beta = 0$ regime). The second is that the "easy path" consisting of reducing to a KAM scheme is out of reach. Indeed, the expected limit of the KAM method is the Brjuno condition $\sum \frac{\ln q_{n+1}}{q_n} < \infty$ on α, but this is still stronger than $\ln q_{n+1} = o(q_n)$. Thus a novel, more robust, approach to absolute continuity of the spectral measures will need to be implemented.

We notice that the discussion in the subexponential regime yields significant information which goes beyond the absolute continuity of the spectral measures (see for instance Remark 3.2 for an example), and can also be applied to the more general context considered in [AJ2] (see §3.2).

1.2.2. *The exponential regime.* In the exponential regime, our approach will be to show that each exponentially close rational approximation p_n/q_n gives a lower bound on the mass of the absolutely continuous component of a spectral measure, and that this lower bound converges to the total mass of the spectral measure.

In [AD], this approach was used to prove absolute continuity of the integrated density of states, which is the average of the spectral measures over different θ (absolute continuity of the spectral measures for almost every θ is obtained as a consequence of this result, by applying [BJ] and [K]). The key point of [AD] was to compare averages of the spectral measures (restricted to a large part of the spectrum) over long sequences $\{\theta + jq_n\alpha\}_{j=0}^{b_n-1}$ with the corresponding objects for the periodic operator obtained by replacing α with p_n/q_n. In such approach, we clearly lose control of individual phases, and one can not hope to recover a result for every phase by an abstract scheme such as Kotani's.

Here we will describe a key novel mechanism of "cancellation" among different phases (which we hope will find wider applicability). We show that an abnormally small (compared with the total mass) absolutely continuous component for any θ implies the existence of an abnormally large absolutely continuous component for some $\theta + jq_n\alpha$. The latter possibility giving a contradiction, we conclude that all spectral measures have approximately the correct size.

Remark 1.1. Let us mention that the description of the critical regime at this point is quite accurate but not complete. One conjectures (it is explicit in [J]) that for $|\lambda| = 1$, for every α and θ the spectral measures are singular continuous. This is proved for every α in the exponential regime and every θ (Gordon's Lemma, [G], [AS]), almost every α and θ ([GJLS]), and it is currently known to hold for every α and almost every θ ([AK]). See also [A] for a recent discussion including further evidence for the conjecture.

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2. Preliminaries

2.1. Cocycles.

Let $\alpha \in \mathbb{R}$, $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$. We call (α, A) a *(complex) cocycle*. The *Lyapunov exponent* is given by the formula

$$L(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n(x)\| dx, \tag{2.1}$$

where A_n is defined by

$$A_n(x) = A(x + (n-1)\alpha) \cdots A(x). \tag{2.2}$$

It turns out (since irrational rotations are uniquely ergodic), that

$$L(\alpha, A) = \lim_{n\to\infty} \sup_{x\in\mathbb{R}/\mathbb{Z}} \frac{1}{n} \ln \|A_n(x)\| \tag{2.3}$$

if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We say that (α, A) is *uniformly hyperbolic* if there exists a continuous splitting $\mathbb{C}^2 = E^s(x) \oplus E^u(x)$, $x \in \mathbb{R}/\mathbb{Z}$ such that for some $C > 0$, $c > 0$, and for every $n \geq 0$, $\|A_n(x) \cdot w\| \leq Ce^{-cn}\|w\|$, $w \in E^s(x)$ and $\|A_n(x)^{-1} \cdot w\| \leq Ce^{-cn}\|w\|$, $w \in E^u(x + n\alpha)$. In this case, of course $L(\alpha, A) > 0$. We say that (α, A) is *bounded* if $\sup_{n\geq 0} \sup_{x\in\mathbb{R}/\mathbb{Z}} \|A_n(x)\| < \infty$.

Given two cocycles $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$, a *(complex) conjugacy* between them is a continuous $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ such that $A^{(2)}(x) = B(x+\alpha)A^{(1)}(x)B(x)^{-1}$ holds. The Lyapunov exponent is clearly invariant under conjugacies.

We assume now that (α, A) is a *real* cocycle, that is, $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. The notion of real conjugacy (between real cocycles) is the same as before except that we now ask for $B \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$. Equivalently, one looks for $B \in C^0(\mathbb{R}/2\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ satisfying $B(x+1) = \pm B(x)$. Real conjugacies still preserve the Lyapunov exponent.

We say that (α, A) is reducible if it it (real) conjugate to a constant cocycle.

The fundamental group of $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to $\mathbb{Z}$. Let

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{2.4}$$

Any $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is homotopic to $x \mapsto R_{nx}$ for some $n \in \mathbb{Z}$ called the degree of A and denoted $\deg A = n$.

Assume now that $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is homotopic to the identity. Then there exists $\psi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and $u : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+$ such that

$$A(x) \cdot \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = u(x,y) \begin{pmatrix} \cos 2\pi(y + \psi(x,y)) \\ \sin 2\pi(y + \psi(x,y)) \end{pmatrix}. \tag{2.5}$$

The function ψ is called a *lift* of A. Let μ be any probability on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ which is invariant by the continuous map $T : (x,y) \mapsto (x+\alpha, y+\psi(x,y))$, projecting over Lebesgue measure on the first coordinate (for instance, take μ as any accumulation point of $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \nu$ where ν is Lebesgue measure on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$). Then the number

$$\rho(\alpha, A) = \int \psi d\mu \mod \mathbb{Z} \tag{2.6}$$

does not depend on the choices of ψ and μ, and is called the *fibered rotation number* of (α, A), see [JM] and [H].

The fibered rotation number is invariant under real conjugacies which are homotopic to the identity. In general, if $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$ are real conjugate, $B(x+\alpha)A^{(2)}(x)B(x)^{-1} = A^{(1)}(x)$, and $B : \mathbb{R}/2\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ has degree k (that is, it is homotopic to $x \mapsto R_{kx/2}$) then $\rho(\alpha, A^{(1)}) = \rho(\alpha, A^{(2)}) + k\alpha/2$.

2.2. $\mathrm{SL}(2,\mathbb{R})$ **action.** Recall the usual action of $\mathrm{SL}(2,\mathbb{C})$ on $\overline{\mathbb{C}}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$.

In the following we restrict to matrices $A \in \mathrm{SL}(2,\mathbb{R})$. Such matrices preserve $\mathbb{H} = \{z \in \mathbb{C}, \Im z > 0\}$. The Hilbert-Schmidt norm of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\|A\|_{\mathrm{HS}} = (a^2+b^2+c^2+d^2)^{1/2}$. Let $\phi(z) = \frac{1+|z|^2}{2\Im z}$ for $z \in \mathbb{H}$. Then $\|A\|_{\mathrm{HS}}^2 = 2\phi(A \cdot i)$.

One easily checks that $\|R_\theta A\|_{\mathrm{HS}} = \|AR_\theta\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}$, so $\phi(R_\theta z) = \phi(z)$.

We notice that $\phi(z) \geq 1$, $\phi(i) = 1$ and $|\ln\phi(z) - \ln\phi(w)| \leq \mathrm{dist}_{\mathbb{H}}(z,w)$ where $\mathrm{dist}_{\mathbb{H}}$ is the hyperbolic metric on $\mathbb{H}$, normalized so that $\mathrm{dist}_{\mathbb{H}}(ai, i) = |\ln a|$ for $a > 0$.

2.3. **Almost Mathieu operator.** We consider now almost Mathieu operators $\{H_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{R}}$. The definition is the same as in the introduction, though we will allow α to be a rational number p/q. The spectrum $\Sigma = \Sigma_{\lambda,\alpha,\theta}$ does not depend on θ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We let $\Sigma_{\lambda,\alpha}$ be this θ-independent set for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and we let $\Sigma_{\lambda,p/q} = \cup_\theta \Sigma_{\lambda,p/q,\theta}$ in the rational case. It is the set of E such that $(\alpha, S_{\lambda,E})$ is not uniformly hyperbolic, with $S_{\lambda,E}$ given by

$$S_{\lambda,E}(x) = \begin{pmatrix} E - 2\lambda\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.7}$$

The Lyapunov exponent is defined by $L_{\lambda,\alpha}(E) = L(\alpha, S_{\lambda,E})$.

Theorem 2.1 ([BJ], Corollary 2)**.** *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda \in \mathbb{R}$, $E \in \Sigma_{\lambda,\alpha}$, we have $L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}$.*

2.3.1. *Classical Aubry duality.* Let $\hat{H}_{\lambda,\alpha,\theta} = \lambda H_{\lambda^{-1},\alpha,\theta}$. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then (see [GJLS]) the spectrum of $\hat{H}_{\lambda,\alpha,\theta}$ is $\Sigma_{\lambda,\alpha}$. This reflects an important symmetry in the theory of the almost Mathieu operators, known as *Aubry duality*.

Classical Aubry duality expresses an algebraic relation between the families of operators $\{H_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ and $\{\hat{H}_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ which corresponds eigenvectors with Bloch waves. In our notation, it is just the computational fact that if $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is an ℓ^2 function whose Fourier series satisfies $\hat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$, then $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$ satisfies $S_{\lambda,E}(x) \cdot U(x) = e^{2\pi i\theta}U(x+\alpha)$.

2.3.2. *The spectral measure.* Fixing a phase θ and $f \in l^2(\mathbb{Z})$, we let $\mu^f = \mu^f_{\lambda,\alpha,\theta}$ be the spectral measure of $H = H_{\lambda,\alpha,\theta}$ corresponding to f. It is defined so that

$$\langle (H-E)^{-1}f, f\rangle = \int_{\mathbb{R}} \frac{1}{E'-E} d\mu^f(E') \tag{2.8}$$

holds for E in the resolvent set $\mathbb{C} \setminus \Sigma$.

We set $\mu = \mu^{e_{-1}} + \mu^{e_0}$ (where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis of $l^2(\mathbb{Z})$). It is well known that $\{e_{-1}, e_0\}$ form a generating basis of $l^2(\mathbb{Z})$ [CL], that is, there is no proper subset of $l^2(\mathbb{Z})$ which is invariant by H and contains $\{e_{-1}, e_0\}$. In particular the support of μ is Σ and if μ is absolutely continuous then all μ^f, $f \in l^2$, are absolutely continuous. From now on, we restrict our consideration to μ which we will call just *the* spectral measure.

2.4. **The m-functions.** The spectral measure $\mu = \mu_{\lambda,\alpha,\theta}$ can be studied through its Borel transform $M = M_{\lambda,\alpha,\theta}$,

$$M(z) = \int \frac{1}{E'-z} d\mu(E'). \tag{2.9}$$

It maps the upper-half plane $\mathbb{H}$ into itself.

For $z \in \mathbb{H}$, there are non-zero solutions $u^\pm$ of $Hu^\pm = zu^\pm$ which are ℓ^2 at $\pm\infty$, defined up to normalization. Let

$$m^\pm = \mp \frac{u_0^\pm}{u_{-1}^\pm}. \tag{2.10}$$

Then m^+ and m^- map $\mathbb{H}$ holomorphically into itself. Moreover, as discussed in [JL2],

$$M = \frac{m^+m^- - 1}{m^+ + m^-}. \tag{2.11}$$

The connection with the cocycle acting on $\overline{\mathbb{C}}$ arises since

$$S_{\lambda,z}(\theta) \cdot \mp m^\pm(\theta) = \mp m^\pm(\theta+\alpha). \tag{2.12}$$

Since the holomorphic function $m^\pm$ maps the upper-half plane into itself, the non-tangential limits $\lim_{\epsilon\to 0} m^\pm(E+i\epsilon)$ exist for almost every $E \in \mathbb{R}$, and define a measurable function of $\mathbb{R}$ which we still denote $m^\pm(E)$.

Theorem 2.2. *For every θ, for almost every E such that $L_{\lambda,\alpha}(E)=0$, we have $m^+_{\lambda,\alpha,\theta}(E) = -\overline{m^-_{\lambda,\alpha,\theta}}(E)$.*

Proof. It is a key result of Kotani Theory [S1] that the conclusion holds for almost every θ. The point here is to extend this to every θ. Fix some arbitrary θ, and let $\theta_n \to \theta$ be some sequence such that the conclusion holds for θ_n. Let $K = \{E \in \mathbb{R},\, L_{\lambda,\alpha}(E)=0\}$.

Let $T : \overline{\mathbb{H}} \to \overline{\mathbb{D}}$ be the conformal map taking $(-1,0,1)$ to $(-1,-i,1)$, $t_n^\pm(z) = T(m^\pm(\theta_n, T^{-1}(z)))$ and $t^\pm(z) = T(m^\pm(\theta, T^{-1}(z)))$. Notice that $t_n^\pm \to t^\pm$ uniformly on compacts of $\mathbb{D}$. Let $\eta_n^\pm = t_n^\pm dx$ and $\eta^\pm = t^\pm dx$, where dx is normalized Lebesgue measure on $\partial\mathbb{D}$. By the Poisson formula, $\eta_n^\pm \to \eta^\pm$ weakly. Since $|t_n^\pm| \leq 1$ and $|t^\pm| \leq 1$, we conclude that $\eta_n^\pm|K \to \eta^\pm|K$. By the hypothesis on θ_n, $t_n^+ = \overline{t_n^-}$ almost everywhere in K. Thus $\eta_n^+|K = \overline{\eta_n^-}|K$ and passing to the limit, $\eta^+|K = \overline{\eta^-}|K$. We conclude that $t^+ = -\overline{t^-}$ almost everywhere in K, which implies the result. □

Corollary 2.3. *Let $0<\lambda<1$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then for every $\theta \in \mathbb{R}/\mathbb{Z}$ there exists a measurable function $m_{\lambda,\alpha,\theta} : \Sigma_{\lambda,\alpha} \to \mathbb{H}$ such that $S_{\lambda,E}(\theta)\cdot m_{\lambda,\alpha,\theta}(E) = m_{\lambda,\alpha,\theta+\alpha}(E)$ and*

$$\frac{d}{dE}\mu_{\lambda,\alpha,\theta}(E) = \frac{1}{\pi}\phi(m_{\lambda,\alpha,\theta}(E)). \tag{2.13}$$

Proof. Let us show that $m = m^+|\Sigma$ has all the properties. Equivariance is obvious. We need to show that $m^+ \in \mathbb{H}$ for almost every $E \in \Sigma$, and that $\frac{d}{dE}\mu = \frac{1}{\pi}\phi(m^+)$.

First notice that $m^+ = -\overline{m^-}$ for almost every $E \in \Sigma$, by Theorems 2.1 and 2.2.

To show that $m^+ \in \mathbb{H}$ for almost every $E \in \Sigma$, it is enough to show that the set of E such that $m^+ = -\overline{m^-} \in \mathbb{R}\cup\{\infty\}$ has zero Lebesgue measure. Otherwise there would be a positive Lebesgue measure set of $E \in \mathbb{R}$ such that the non-tangential limit of m^+ is ∞ or such that the non-tangential limit of $m^+ + m^-$ is 0, both cases giving a contradiction (using that if the non-tangential limit of either m^+ or $m^+ + m^-$ is constant in a set of positive Lebesgue measure then m^+ or $m^+ + m^-$ is constant everywhere).

If E is such that $m^+ = -\overline{m^-} \in \mathbb{H}$ we have

$$\frac{d}{dE}\mu_{\lambda,\alpha,\theta}(E) = \frac{1}{\pi}\lim_{\epsilon\to 0}\Im M_{\lambda,\alpha,\theta}(E+i\epsilon) = \frac{1}{\pi}\phi(m^+_{\lambda,\alpha,\theta}(E)).^1 \tag{2.14}$$

□

2.5. **Integrated density of states.** The integrated density of states is the function $N = N_{\lambda,\alpha} : \mathbb{R} \to [0,1]$ such that

$$N(E) = \int_{\mathbb{R}/\mathbb{Z}} \mu_\theta(-\infty, E]d\theta, \tag{2.15}$$

[1] This implies that $\frac{d}{dE}\mu \geq \frac{1}{\pi}$. Notice that even if the relation $m^+ = -\overline{m^-} \in \mathbb{H}$ (almost everywhere with $L=0$) was only known for a dense subset of θ, the estimate $\frac{d}{dE}\mu \geq \frac{1}{\pi}$ (almost everywhere with $L=0$) could then be concluded for every θ. Let us point out that we have not used the θ-independence of absolutely continuous spectrum obtained in [LS].

which is a continuous non-decreasing surjective function. The Thouless formula relates the Lyapunov exponent to the integrated density of states

$$L(E) = \int_{\mathbb{R}} \ln |E' - E| dN(E'). \tag{2.16}$$

There is also a relation to the fibered rotation number

$$N(E) = 1 - 2\rho(\alpha, S_{\lambda,E}) \tag{2.17}$$

where $\rho(\alpha, S_{\lambda,E}) \in [0, 1/2]$.

2.6. **Periodic case.** Let $\alpha = p/q$, and let $A = S_{\lambda,E}$. The spectrum $\Sigma_{\lambda,p/q,\theta}$ is the set of all E such that $|\mathrm{tr} A_q(\theta)| \leq 2$, where $A = S_{\lambda,E}$. The set of E such that $|\mathrm{tr} A_q(\theta)| < 2$ the union of q intervals, and the closure of each interval is called a band. We order the bands from left to right. Inside a band, $\mathrm{tr} A_q(\theta)$ is a monotonic function onto $[-2, 2]$.

We define $N_{\lambda,p/q,\theta} = \frac{1}{q} \sum_{i=0}^{q-1} \mu_{\lambda,p/q,\theta+i\alpha}(-\infty, E]$. Inside the i-th band, we have the formulas

$$qN_{\lambda,p/q,\theta}(E) = k - 1 + (-1)^{q+k-1} 2\rho(\theta, E) + \frac{1 - (-1)^{q+k-1}}{2} \tag{2.18}$$

where $0 < \rho(\theta, E) < 1/2$ is such that $\mathrm{tr} A_q(\theta) = 2 \cos 2\pi\rho(\theta, E)$.

In the interior of a band, $\mu_{\lambda,p/q,\theta}$ has a smooth density. Since $|\mathrm{tr} A_q(\theta)| < 2$, there is a well defined fixed point $m_{\lambda,\alpha,\theta}(E)$ of $A_q(\theta)$ in $\mathbb{H}$. Then

$$\frac{d}{dE} \mu_{\lambda,p/q,\theta}(E) = \frac{1}{\pi} \phi(m(\theta)). \tag{2.19}$$

2.7. **Bounded eigenfunctions and absolutely continuous spectrum.**

Theorem 2.4. *Let $\mathcal{B}$ be the set of $E \in \mathbb{R}$ such that the cocycle $(\alpha, S_{\lambda,E})$ is bounded. Then $\mu_{\lambda,\alpha,\theta}|\mathcal{B}$ is absolutely continuous for all $\theta \in \mathbb{R}$.*

This well known result follows from [GP]. We will actually need an explicit estimate, contained in [JL1], [JL2] (we give a proof since we found no reference for the exact statement we need).

Lemma 2.5. *We have $\mu(E - i\epsilon, E + i\epsilon) \leq C\epsilon \sup_{0 \leq s \leq C\epsilon^{-1}} \|A_s\|_0^2$, where $C > 0$ is a universal constant.*

Proof. We have $\Im M = \frac{\Im m^+ \Im m^-}{|m^+ + m^-|^2} \left(\frac{1+|m^+|^2}{\Im m^+} + \frac{1+|m^-|^2}{\Im m^-} \right)$. Since $\Im m^+, \Im m^- > 0$, $\frac{\Im m^+ \Im m^-}{|m^+ + m^-|^2} \leq \frac{1}{2}$ and

$$\Im M \leq \frac{1}{2} \left(\frac{1 + |m^+|^2}{\Im m^+} + \frac{1 + |m^-|^2}{\Im m^-} \right). \tag{2.20}$$

Clearly $\Im M(E + i\epsilon) \geq \frac{1}{2\epsilon} \mu(E - \epsilon, E + \epsilon)$. so

$$\frac{1}{2\epsilon} \mu(E - \epsilon, E + \epsilon) \leq \max \frac{1 + |m^\pm(E + i\epsilon)|^2}{\Im m^\pm(E + i\epsilon)}. \tag{2.21}$$

We want thus to estimate

$$\frac{1 + |m(E + i\epsilon)|^2}{\Im m(E + i\epsilon)} \leq C \sup_{0 \leq s \leq C\epsilon^{-1}} \|A_s\|_0^2 \tag{2.22}$$

for $m = m^+$, $m = m^-$. By symmetry, we will only consider the case $m = m^+$. Let $m_\beta = R_{-\beta} \cdot m$. Those are so-called m-functions for the corresponding half-line problem with appropriate boundary conditions, see [JL2], §2. Assume now that ϵ^{-1} is an integer (the general statement reduces to this case). By Proposition 3.9 of [LS] (a consequence of Theorem 1.1 of [JL1]), such m-functions satisfy the bound

$$\Im m_\beta(E+i\epsilon) \leq (5+\sqrt{24}) \sum_{s=0}^{1+\epsilon^{-1}} \|A_s\|_0^2. \tag{2.23}$$

We notice that the quantity $\frac{1+|z|^2}{\Im z} = 2\phi(z)$ is invariant under R_β. By choosing β appropriately so to maximize $\Im m_\beta$, m_β becomes purely imaginary with $\Im m_\beta \geq 1$, and $\frac{1+|m|^2}{\Im m} \leq 2\Im m_\beta$. Then (2.22) follows from (2.23). □

2.8. **Corona estimates.** Given a non-zero vector $U \in \mathbb{C}^2$, it is easy to find a matrix with first column U that belongs to $\mathrm{SL}(2,\mathbb{C})$. We just have to solve an equation of the type $ad-bc=1$, and it is trivial to get estimates on the size of the solutions. If U depends holomorphically of a parameter, to obtain a holomorphic solution of the same problem with good estimates is much more challenging, and it is related to the famous Corona Theorem of Carleson [C].

The Corona Theorem states that if $d \geq 1$ and $a_i : \mathbb{D} \to \mathbb{C}$, $1 \leq i \leq d$ are bounded holomorphic functions such that $\max_i |a_i| \geq \epsilon$ pointwise then there exist bounded holomorphic functions $b_i : \mathbb{D} \to \mathbb{C}$, $1 \leq i \leq d$ such that $\sum a_i b_i = 1$.

After the work of Wolff, good estimates on the solutions b_i were obtained. For instance, Uchiyama [U] (see Trent [T] for a published generalization) showed that if $\delta \leq (\sum |a_i|^2)^{1/2} \leq 1$ pointwise then the b_i can be chosen such that $(\sum |b_i|^2)^{1/2} \leq C\delta^{-2}(1-\ln\delta)$, with C independent of d. (Let us point out that [C] gives an upper bound of the form $C_d\delta^{-C_d}$ with C_d depending on d, that would be enough for our purposes.)

If instead of functions of the disk one consider functions of an annulus $\{x \in \mathbb{C}/\mathbb{Z}, |\Im z| < a\}$, the conclusion of the Corona Theorem (with the Uchiyama estimates) is still valid, and is a consequence of the disk version (because the annulus is uniformized by the disk and has amenable fundamental group).

The following is an equivalent convenient formulation of the case $d=2$ of Uchiyama's Theorem for the annulus.

Theorem 2.6. *Let $U : \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2$ be an analytic function. Assume that $\delta_1 \leq \|U(x)\| \leq \delta_2^{-1}$ for $|\Im x| < a$. Then there exists $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ with first column U and such that $\|B\|_a \leq C\delta_1^{-2}\delta_2^{-1}(1-\ln\delta_1\delta_2)$.*

3. The subexponential regime

We will follow the basic scheme of [AJ2], based on quantitative duality. In this approach, almost localization estimates for the dual operator yield information on the Fourier series of a "conjugacy" to constant. The almost localization estimate gives exponential decay away from "resonant" sites, but this does not ensure convergence for all energies (for generic energies, it is actually divergent). Still, the estimates yield a good control of the dynamics.

Some of the estimates in [AJ2] lose exponential control of the decay of Fourier coefficients, and hence are too weak to deal with the small denominators arising in the regime $\beta = 0$ (the fight between the decay of Fourier coefficients and the small denominators happens when we need to solve the cohomological equation with small error). This is overcome by the systematic use of estimates in a definite strip for the truncated "conjugacies". We then need to relate the control of the dynamics with absolutely continuous spectrum (as described in the introduction, [AJ2] invokes the KAM approach at this point, which we can not do). We have good estimates on cocycle growth in terms of the resonant character of the dual phase, and bounds on cocycle growth yield upper bounds on the spectral measures. We still need estimates connecting the "parametrization by dual phase" with the "parametrization by energy"[2], which is done through a third parametrization, by fibered rotation number.

Another interpretation of the proof is that we give some Hölder control (in certain scales, we do not actually show full Hölder continuity here) on the spectral measures, while showing that the support of the singular part has Hausdorff dimension zero (with good coverings at the right scales to match the other estimate).

3.1. **Strong localization estimates.** Let $\alpha \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\epsilon_0 > 0$. We say that k is an ϵ_0-resonance if $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-|k|\epsilon_0}$ and $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j| \leq |k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}$.

Remark 3.1. In particular, there exists always at least one resonance, 0. If $\beta = 0$, $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-|k|\epsilon_0}$ implies $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j| \leq |k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}$ for k large.

We order the ϵ_0-resonances $|n_1| \leq |n_2| \leq \ldots$. We say that θ is ϵ_0-resonant if the set of resonances is infinite.

Definition 3.1. We say that $\{\hat{H}_{\lambda,\alpha,\theta}\}_{\theta \in \mathbb{R}}$ (see §2.3.1) satisfies a strong localization estimate if there exists $C_0 > 0$, $\epsilon_0 > 0$, $\epsilon_1 > 0$ such that for every eigenfunction $\hat{H}\hat{u} = E\hat{u}$ satisfying $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, and for every $C_0|n_j| < k < C_0^{-1}|n_{j+1}|$ we have $|\hat{u}_k| \leq C_0 e^{-\epsilon_1 |k|}$.

Theorem 3.1 ([AJ2], Theorem 5.1)**.** *If $\beta = 0$ and $0 < \lambda < 1$ then $\{\hat{H}_{\lambda,\alpha,\theta}\}_{\theta \in \mathbb{R}}$ satisfies a strong localization estimate.*

3.2. **A generalization.** This section can be ignored if one is only interested in the proof of the Main Theorem.

Let $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be analytic and let $H = H_{v,\alpha,\theta} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the quasiperiodic Schrödinger operator given by $(Hu)_n = u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha) u_n$. The almost Mathieu operator corresponds to the special case $v(\theta) = 2\lambda \cos 2\pi\theta$ for some $\lambda \neq 0$.

As for the almost Mathieu case, the spectral properties of $\{H_{v,\alpha,\theta}\}_{\theta \in \mathbb{R}}$ are intimately connected with the Schrödinger cocycles $\{(\alpha, S_{v,E}\}_{E \in \mathbb{R}}$, where $S_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}$, and several key notions have identical development, including spectral measures §2.3.2, m-functions §2.4 (except for Corollary 2.3 which needs to be reformulated), integrated density of states §2.5 and bounded eigenfunctions §2.7.

Most importantly, classical Aubry duality (§2.3.1) can be extended to this setting: the operators $\hat{H}_{v,\alpha,\theta}$ given by $(\hat{H}\hat{u})_n = \sum \hat{v}_k \hat{u}_{n-k} + 2\cos(2\pi(\theta + n\alpha))\hat{u}_n$, where

[2]Each energy usually (almost everywhere) corresponds to finitely many dual phases, but we have not been able to rule out (and it is not even heuristically clear that this should be the case, see footnote 11 of [AJ2]) that for some exceptional set of energies there could be uncountably many ones. This is closely related to the coexistence of both point and singular continuous spectrum for the dual model. Happily for us, the exceptional set is very small (with Hausdorff dimension zero).

$v(x) = \sum \hat{v}_k e^{2\pi ikx}$ have the property that if $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is an ℓ^2 function such that $\hat{H}_{v,\alpha,\theta}\hat{u} = E\hat{u}$, then $S_{v,E}(x) \cdot U(x) = e^{2\pi i\theta}U(x+\alpha)$, where $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$.

Let us say that v is *small* if the family $\{\hat{H}_{v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ is almost localized (the definition of almost localization being the same as in the almost Mathieu case). In particular, $v(x) = 2\lambda\cos 2\pi x$ is small if $0 < \lambda < 1$. In general, this notation is justified by Theorem 5.1 of [AJ2] which shows that if $0 < \lambda < \lambda_0(v)$ then $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ is almost localized in the whole subexponential regime.

We will actually prove the following more general result in the subexponential regime.

Theorem 3.2. *If v is small and $\beta = 0$ then the spectral measures of $H_{v,\alpha,\theta}$ are absolutely continous.*

All the discussion below applies essentially unchanged to operators $H_{v,\alpha,\theta}$ with small v and $\beta = 0$. Besides replacing mentions of λ by v and of the bound $0 < \lambda < 1$ by the condition that v is small, all the few places where modifications are necessary will be explicitly pointed out in a footnote.

3.3. **Localization and reducibility.** Until the end of this section we fix $0 < \lambda < 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\beta = 0$. For an energy $E \in \Sigma$, it is shown in Theorem 3.3 of [AJ2] that there exists some $\theta \in \mathbb{R}$ and $\hat{u} = (\hat{u}_i)_{i\in\mathbb{Z}}$ such that $\hat{H}\hat{u} = E\hat{u}$, $\hat{u}_0 = 1$, $|\hat{u}_i| \leq 1$. Until the end of this section, whenever $E \in \Sigma$ is fixed, we will choose some arbitrary θ and $\hat{u}$ with those properties, and we will denote $A = S_{\lambda,E}$.

By the strong localization estimate, if θ is non-resonant then $\hat{u}$ is localized, that is, it is the Fourier series of an analytic function. Classical Aubry duality (§2.3.1) yields a connection between localization and reducibility (see for instance Theorem 2.5 of [AJ2][3]):

Theorem 3.3. *If θ is non-resonant then (α, A) is reducible.*

3.4. **Bounds on growth.** The starting information on the cocycle growth is given by Theorem 2.1, that $L(\alpha, A) = 0$.[4] In our context this means that for any $\delta > 0$ there exists $c_\delta > 0$, $C_\delta > 0$ such that

$$\sup_{|\Im x|<c_\delta} \|A_k(x)\| \leq C_\delta e^{-\delta k}. \tag{3.1}$$

The constants c_δ and C_δ do not depend on E, only on λ and α.[5] All further constants may depend on α and λ (respectively v). In the following C is big and c is small.

For a bounded analytic function f defined on a strip $\{|\Im z| < \epsilon\}$ we let $\|f\|_\epsilon = \sup_{|\Im z|<\epsilon} |f(z)|$. If f is a bounded continuous function on $\mathbb{R}$, we let $\|f\|_0 = \sup_{x\in\mathbb{R}} |f(x)|$.

Our goal in this section is to prove:

[3] Their argument only needs the arithmetical properties of α to solve the cohomological equation $\phi(x+\alpha) - \phi(x) = b(x) - \int_0^1 b(x)dx$ with b analytic, and this can be always done when $\beta = 0$.

[4] For the generalization, one applies Theorem 6.2 of [AJ2] whose proof is unchanged in the $\beta = 0$ regime.

[5] In the case of the almost Mathieu operator it is possible to show that we can take $c_\delta = -\frac{1}{2\pi}\ln\lambda$. For the generalization, it is possible to show that it is enough to choose c_δ such that v holomorphic in a neighborhood of $\{|\Im x| \leq c_\delta\}$ and $c_\delta \leq \frac{1}{2\pi}\epsilon_1$ where ϵ_1 is the one in the strong localization estimate.

Theorem 3.4. *We have* $\|A_n\|_c \leq Cn^C$.

Given Fourier coefficients $\hat{w} = (\hat{w}_k)_{k\in\mathbb{Z}}$ and an interval $I \subset \mathbb{Z}$, we let $w^I = \sum_{k\in I} \hat{w}_k e^{2\pi ikx}$. The length of the interval $I = [a, b]$ is $|I| = b - a$.

We will say that a trigonometrical polynomial $p : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ has essential degree at most k if its Fourier coefficients outside an interval I of length k are vanishing.

Let p_n/q_n be the approximants of α. We recall the basic properties:

$$\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} = \inf_{1\leq k\leq q_{n+1}-1} \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}, \tag{3.2}$$

$$1 \geq q_{n+1}\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq 1/2. \tag{3.3}$$

The condition $\beta = 0$ implies

$$q_{n+1} \leq e^{o(q_n)}. \tag{3.4}$$

Theorem 3.5 ([AJ2], Theorem 6.1)**.** *Let* $1 \leq r \leq [q_{n+1}/q_n]$. *If* p *has essential degree* $k = rq_n - 1$ *and* $x_0 \in \mathbb{R}/\mathbb{Z}$ *then*

$$\|p\|_0 \leq Cq_{n+1}^{Cr} \sup_{0\leq j\leq k} |p(x_0 + j\alpha)|. \tag{3.5}$$

In particular, under the condition $\beta = 0$

$$\|p\|_0 \leq Ce^{o(k)} \sup_{0\leq j\leq k} |p(x + j\alpha)|. \tag{3.6}$$

Lemma 3.6. *We have* $o(|n_{j+1}|) \geq \ln \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq c|n_j|$.

Proof. This follows immediately from $\beta = 0$. $\square$

Choose $C|n_j| < n < C^{-1}|n_{j+1}|$ of the form $n = rq_k - 1 < q_{k+1}$, let $I = [-[n/2], n - [n/2]]$ and define $u(x) = u^I(x)$. Let $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$. Then

$$A(x) \cdot U(x) - e^{2\pi i\theta}U(x + \alpha) = e^{4\pi i\theta} \begin{pmatrix} h(x) \\ 0 \end{pmatrix}, \tag{3.7}$$

where

$$\hat{h}_k = \chi_I(k) 2\cos 2\pi(\theta + k\alpha)\hat{u}_k + \sum_{j\in\{-1,1\}} \chi_I(k - j)\hat{u}_{k-j}, \tag{3.8}$$ [6]

where χ_I is the characteristic function of I. Since $\hat{H}\hat{u} = E\hat{u}$, we also have

$$-\hat{h}_k = \chi_{\mathbb{Z}\setminus I}(k) 2\cos 2\pi(\theta + k\alpha)\hat{u}_k + \sum_{j\in\{-1,1\}} \chi_{\mathbb{Z}\setminus I}(k - j)\hat{u}_{k-j}. \tag{3.9}$$ [7]

The estimates $|\hat{u}_k| < Ce^{-c|k|}$ for $C^{-1}n < |k| < Cn$, $|\hat{u}_k| \leq 1$ for all k then imply that $|\hat{h}_k| \leq Ce^{-cn}e^{-ck}$, that is $\|h\|_c \leq Ce^{-cn}$.

In the following, δ and δ_0 will be suitably small constants (much smaller than the c that appeared so far).

Theorem 3.7. *We have* $\inf_{|\Im x|<\delta_0} \|U(x)\| \geq ce^{-\delta n}$.

Proof. Otherwise, by (3.1), $|u(x + j\alpha)| \leq ce^{-\delta n/2}$ for some x with $\Im x = t$, $|t| < \delta_0$ and $0 \leq j \leq n$. Then $\|u_t\|_0 \leq ce^{-\delta n/5}$ by Theorem 3.5, where $u_t(x) = u(x + ti)$. This contradicts $\int u_t(x)dx = 1$. $\square$

[6]For the generalization one has $\hat{h}_k = \chi_I(k) 2\cos 2\pi(\theta + k\alpha)\hat{u}_k + \sum \chi_I(k - j)\hat{v}_j\hat{u}_{k-j}$.

[7]For the generalization one has $-\hat{h}_k = \chi_{\mathbb{Z}\setminus I}(k) 2\cos 2\pi(\theta + k\alpha)\hat{u}_k + \sum \chi_{\mathbb{Z}\setminus I}(k - j)\hat{v}_j\hat{u}_{k-j}$.

Let $B(x) \in \mathrm{SL}(2,\mathbb{C})$ be the matrix whose first column $U(x)$ given by Theorem 2.6. Then

$$B(x+\alpha)^{-1}A(x)B(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_3(x) & \beta_4(x) \end{pmatrix}. \tag{3.10}$$

where $\|b(x)\|_{\delta_0} \le Ce^{3\delta n}$, and $\|\beta_1(x)\|_{\delta_0}, \|\beta_3(x)\|_{\delta_0}, \|\beta_4(x)\|_{\delta_0} \le Ce^{-cn}$. Taking $\Phi(x)$ the product of $B(x)^{-1}$ and a constant diagonal matrix, $\Phi(x) = DB(x)^{-1}$, where $D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$, with $d^2 = \max\{\|\beta_3\|_{\delta_0}^{1/2}, e^{-cn}\}$, we get

$$\Phi(x+\alpha)A(x)\Phi(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + Q(x), \tag{3.11}$$

where $\sup_{|\Im x|<\delta_0} \|Q(x)\| \le Ce^{-cn}$ and $\sup_{|\Im x|<\delta_0} \|\Phi(x)\| \le Ce^{cn}$. Thus

$$\sup_{0\le s\le ce^{cn}} \|A_s\|_{\delta_0} \le Ce^{cn}. \tag{3.12}$$

Proof of Theorem 3.4. Let $m \ge C$. By Lemma 3.6 we can choose $C \ln m \le n \le C \ln m$ so that $C|n_j| < n < C^{-1}|n_{j+1}|$ and $n = rq_k - 1 < q_{k+1}$ for some j and k. By (3.12), $\|A_m\|_c \le Cm^C$. □

3.5. **Triangularization in a definite strip.**

Theorem 3.8. *Fix some $n = |n_j|$ and let $N = |n_{j+1}|$ if defined, otherwise let $N = \infty$. Then there exists $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ analytic with $\|B\|_c \le e^{o(n)}$ such that*

$$B(x+\alpha)A(x)B(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_3(x) & \beta_4(x) \end{pmatrix}, \tag{3.13}$$

with $\|\beta_1\|_c, \|\beta_3\|_c, \|\beta_4\|_c \le e^{-cN}$ and $\|b\|_c \le e^{-cn}$. In particular

$$\|A_s\|_c \le Ce^{o(n)}, \quad 0 \le s \le e^{cn}. \tag{3.14}$$

Proof. Let $u(x) = u^I(x)$ for $I = [-cN, cN]$. Let $rq_k > Cn_j$ be minimal with $rq_k - 1 < q_{k+1}$ and let $J = [-[rq_k/2], rq_k - 1 - [rq_k/2]]$. Define $U(x)$ as before, and define also $U^J(x)$. Then our previous estimate Theorem 3.7 can be improved to $\inf_{|\Im x|<c} \|U^J(x)\| \ge e^{-o(n)}$. The estimate is better since we can use Theorem 3.4 instead of the weaker estimate (3.1). Since $\|U - U^J\|_c \le e^{-cn}$, we get

$$\inf_{|\Im x|<c} \|U(x)\| \ge e^{-o(n)}. \tag{3.15}$$

Moreover, we have $A(x) \cdot U(x) = e^{2\pi i\theta} U(x+\alpha) + \begin{pmatrix} h(x) \\ 0 \end{pmatrix}$ with $\|h\|_c \le e^{-cN}$. Taking $\tilde{B}$ given by Theorem 2.6, we get

$$\tilde{B}(x+\alpha)A(x)\tilde{B}(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \tilde{\beta}_1(x) & \tilde{b}(x) \\ \tilde{\beta}_3(x) & \tilde{\beta}_4(x) \end{pmatrix}, \tag{3.16}$$

with $\|\tilde{\beta}_1\|_c, \|\tilde{\beta}_3\|_c, \|\tilde{\beta}_4\|_c \le e^{-cN}$ and $\|\tilde{b}\|_c \le e^{o(n)}$. If $n \le C$ we are done, otherwise let $b^{(1)}(x)$ be obtained by truncating the Fourier series of $\tilde{b}$, so that it has the Fourier coefficients with $|k| \le n-1$. We solve exactly

$$W(x+\alpha)\begin{pmatrix} e^{2\pi i\theta} & b^{(1)}(x) \\ 0 & e^{-2\pi i\theta} \end{pmatrix} W(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \tag{3.17}$$

with $W(x) = \begin{pmatrix} 1 & -w(x) \\ 0 & 1 \end{pmatrix}$, that is $b^{(1)}(x) - e^{-2\pi i\theta} w(x+\alpha) + e^{2\pi i\theta} w(x) = 0$, or in terms of Fourier coefficients, $\hat{w}_k = -\hat{b}_k \frac{e^{-2\pi i\theta}}{1-e^{-2\pi i(2\theta - k\alpha)}}$. So we get $\|W\|_c \leq e^{o(n)}$. Let $B(x) = W(x)\tilde{B}(x)$. Noticing that $\|\tilde{b} - b^{(1)}\|_c \leq e^{-cn}$, we obtain the estimates on the coefficients of $B(x+\alpha)A(x)B(x)^{-1}$. The second statement follows immediately from the first. □

3.6. **Lower bounds on the integrated density of states.**

Theorem 3.9. *Let $n = |n_j|$ and let $N = |n_{j+1}|$ if defined, otherwise let $N = \infty$. Let $Ce^{-cN} \leq \epsilon \leq e^{-o(n)}$. Then there exists $W : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ analytic with $\|W\|_{cn^{-C}} \leq C\epsilon^{-1/4}$ such that $Q(x) = W(x+\alpha)A(x)W^{-1}$ satisfies*

$$\|Q\|_0 \leq 1 + C\epsilon^{1/2} \tag{3.18}$$

Proof. Let B be given by Theorem 3.8. Let $D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$ where $d = \|B\|_c \epsilon^{1/4}$. Let $W(x) = DB(x)$. If $\epsilon \leq e^{-o(n)}$ we have $\|W\|_0 \leq C\epsilon^{-1/4}$. Moreover

$$W(x+\alpha)A(x)W(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} q_1(x) & q_2(x) \\ q_3(x) & q_4(x) \end{pmatrix}, \tag{3.19}$$

with $\|q_1\|_0, \|q_3\|_0, \|q_4\|_0 \leq C\epsilon^{-1/2}e^{-cN}$ and $\|q_2\|_0 \leq C\epsilon^{1/2}$. If $\epsilon \geq Ce^{-cN}$ then $\|Q\|_0 \leq 1 + C\epsilon^{1/2}$. □

Corollary 3.10. *The integrated density of states is $1/2$-Hölder for every δ.*

Proof. By (2.16), $L(E+i\epsilon) \geq c(N(E+\epsilon) - N(E-\epsilon))$ for every $\epsilon > 0$. So it is enough to show that for $0 < \epsilon < c$, $L(E+i\epsilon) < C\epsilon^{1/2}$. The condition $0 < \epsilon < c$ implies that ϵ belongs to the range specified by Theorem 3.9 for some $n = |n_j|$. Let W and Q be given by Theorem 3.9. Then $L(E+i\epsilon) = L(\alpha, \tilde{A}) \leq \ln \|\tilde{A}\|_0$ where $\tilde{A}(x) = Q(x) + W(x+\alpha)\begin{pmatrix} i\epsilon & 0 \\ 0 & 0 \end{pmatrix} W(x)^{-1}$. Clearly $\ln \|\tilde{A}\|_0 \leq \|Q\|_0 - 1 + \|W\|_0^2 \epsilon \leq C\epsilon^{1/2}$, so the result follows. □

Lemma 3.11. *If $E \in \Sigma$ then for $0 < \epsilon < 1$, $N(E+\epsilon) - N(E-\epsilon) \geq c\epsilon^{3/2}$.*

Proof. Let $\delta = c\epsilon^{3/2}$. Since $L(E) = 0$, by Thouless formula we have

$$L(E+i\delta) = \int \frac{1}{2} \ln 1 + \frac{\delta^2}{|E-E'|^2} dN(E'). \tag{3.20}$$

We split the integral in $I_1 = \int_{|E-E'|>1}$, $I_2 = \int_{\epsilon<|E-E'|<1}$, $I_3 = \int_{\epsilon^4<|E-E'|<\epsilon}$ and $I_4 = \int_{|E-E'|<\epsilon^4}$. We clearly have $I_1 \leq c^2\epsilon^3$. By Corollary 3.10, it easily follows that $I_4 = \sum_{k\geq 4} \int_{\epsilon^k > |E-E'| > \epsilon^{k+1}} 1 + \frac{\delta^2}{|E-E'|^2} dN(E') \leq C \sum_{k\geq 4} \epsilon^{k(1/2)} \ln 1 + c^2\epsilon^{1-2k} \leq C\epsilon^{7/4}$.

Using Corollary 3.10, we can also estimate, with $m = [-\ln \epsilon]$,

$$\begin{aligned} I_2 &\leq \sum_{k=0}^{m} \int_{e^{-k-1}}^{e^{-k}} 1 + \frac{\delta^2}{|E-E'|^2} dN(E') \\ &\leq C \sum_{k=0}^{m} c^2\epsilon^3 e^{2k+2} e^{-k/2} \leq Cc^2 e^{-3m} e^{(3/2)m} \leq Cc^2\delta. \end{aligned} \tag{3.21}$$

It follows that $I_3 \geq L(E+i\delta) - c\delta$. It is well known that $L(E+i\delta) \geq \delta/10$ for $0<\delta<1$ (see Theorem 1.7 of [DS]). Thus $I_3 \geq \delta/20$. Since $I_2 \leq C(N(E+\epsilon) - N(E-\epsilon)) \ln \epsilon^{-1}$, the result follows. $\square$

3.7. **Real conjugacies.** Again, fix $n = |n_j|$, $N = |n_{j+1}|$ and let $u(x) = u^I(x)$, $I = [-cN, cN]$. Let $U(x)$ be defined as before, and let $\tilde{U}(x) = e^{\pi i n_j x} U(x)$. Let $\tilde{\theta} = \theta - n_j \alpha/2$. Let $B(x)$ be the matrix with columns $\tilde{U}(x)$ and $\overline{\tilde{U}(x)}$. Let $L^{-1} = \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}$. Notice that

$$A(x) \cdot \tilde{U}(x) = e^{2\pi i \tilde{\theta}} \tilde{U}(x+\alpha) + \begin{pmatrix} h(x) \\ 0 \end{pmatrix} \tag{3.22}$$

with $\|h(x)\|_c \leq e^{-cN}$. Notice that in the considerations below, we must pass to a double cover where the dynamics is like $x \mapsto x + \alpha/2$, but the condition $\beta = 0$ is independent of working with α or $\alpha/2$.

Theorem 3.12. *We have*

$$\inf_{x \in \mathbb{R}/\mathbb{Z}} |\det B(x)| \geq cL^{-C}. \tag{3.23}$$

Proof. Recall the estimate

$$\inf_{x \in \mathbb{R}/\mathbb{Z}} \|U(x)\| \geq e^{-o(n)}. \tag{3.24}$$

Minimize over $\lambda \in \mathbb{C}$, $x \in \mathbb{R}/2\mathbb{Z}$ the quantity $\|\overline{\tilde{U}(x)} - \lambda \tilde{U}(x)\|$. This gives some λ_0, x_0. If the result does not hold then

$$\|e^{-2\pi i j \tilde{\theta}} \overline{\tilde{U}(x_0 + j\alpha)} - e^{2\pi i j \tilde{\theta}} \lambda_0 \tilde{U}(x_0 + j\alpha)\| \leq cL^{-C}, \quad 0 \leq j \leq CL^C. \tag{3.25}$$

This implies that $\|\overline{\tilde{U}(x_0 + j\alpha)} - \lambda_0 \tilde{U}(x_0 + j\alpha)\| \leq cL^{-c}$ for $0 \leq j \leq cL^{1-c}$, and as before (first truncating the Fourier series at scale $C \ln L$) $\sup_{x \in \mathbb{R}/\mathbb{Z}} \|\overline{\tilde{U}(x)} - \lambda_0 \tilde{U}(x)\| \leq cL^{-c}$. But taking $j = [L/4]$ in (3.25), we get $\|i\overline{\tilde{U}(x)} + i\lambda_0 \tilde{U}(x)\| \leq cL^{-c}$, so that $\|\tilde{U}(x)\| \leq cL^{-c}$. This contradicts $\|\tilde{U}(x)\| = \|U(x)\| \geq ce^{-o(n)}$. $\square$

Take now $S = \Re \tilde{U}$, $T = \Im \tilde{U}$, and let $\tilde{W}$ be the matrix with columns S and $\pm T$, so to have $\det \tilde{W} > 0$. Then

$$A(x) \cdot \tilde{W}(x) = \tilde{W}(x+\alpha) \cdot R_{\mp \tilde{\theta}} + O(e^{-cN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{3.26}$$

Let $W(x) = |\det B(x)/2|^{-1/2} \tilde{W}(x)$ so to have $\det W = 1$. Then

$$A(x) \cdot W(x) = \frac{|\det B(x+\alpha)|^{1/2}}{|\det B(x)|^{1/2}} W(x+\alpha) \cdot R_{\mp \tilde{\theta}} + O(e^{-cN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{3.27}$$

Since $\det W(x) = 1$, this gives

$$A(x) \cdot W(x) = W(x+\alpha) \cdot R_{\mp \tilde{\theta}} + O(e^{-cN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{3.28}$$

Assume first that n_j is even so that $W(x+1) = W(x)$ and everything is defined in $\mathbb{R}/\mathbb{Z}$. Letting $\Phi(x) = W(x+\alpha)^{-1} A(x) W(x)$, we get $\|\rho(\alpha, \Phi) \pm \tilde{\theta}\|_{\mathbb{R}/\mathbb{Z}} \leq Ce^{-cN}$. Assume now that n_j is odd, so that $W(x+1) = -W(x)$. Letting $\Phi(x) = R_{(x+\alpha)/2} W(x+\alpha)^{-1} A(x) W(x) R_{-x/2}$, we get Φ defined in $\mathbb{R}/\mathbb{Z}$ with

$$\Phi(x) = R_{\frac{\alpha}{2} \mp \tilde{\theta}} + O(e^{-cN}), \quad |\Im x| < cL^{-C}. \tag{3.29}$$

Then $\|\rho(\alpha, \Phi) - \frac{\alpha}{2} \pm \tilde{\theta}\|_{\mathbb{R}/\mathbb{Z}} \leq Ce^{-cN}$.

In either case, for some k with $||k|-n| \leq 1$

$$|\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\rho(\alpha,\Phi) - k\alpha\|_{\mathbb{R}/\mathbb{Z}}| \leq \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}/10. \tag{3.30}$$

To estimate the topological degree of W, it is enough to estimate the degree of $\frac{M(x)}{\|M(x)\|}$ for $M = S$ or for $M = T$. Notice that $\| \int e^{-\pi i n_j x}(S(x) + iT(x))dx\| \geq 1$, and select $M = S$ or $M = T$ so that $\int \|M(x)\| \geq 1/2$. We have of course $A(x) \cdot M(x) = M(x+\alpha) + O(e^{-cn})$, $|\Im x| < c$, which allows us to estimate

$$\inf_{x\in\mathbb{R}/\mathbb{Z}} \|M(x)\| \geq ce^{-o(n)}. \tag{3.31}$$

as before. Truncating the Fourier series of M keeping the $|k| < Cn$ the resulting $\tilde{M}(x)$ is such that $\|\tilde{M}(x) - M(x)\| \leq \|M(x)\|/2$, so we just have to estimate the degree of $\frac{\tilde{M}(x)}{\|\tilde{M}(x)\|}$. We do this by counting the number of zeroes of the coordinates of $\tilde{M}(x)$, and we get $|\deg W| \leq Cn$. Then

$$|\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\rho(\alpha, A) - m\alpha\|_{\mathbb{R}/\mathbb{Z}}| \leq \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}/10, \text{ for some } |m| \leq Cn. \tag{3.32}$$

This implies the following result.

Lemma 3.13. *If θ has a resonance n_j then there exists $|m| \leq C|n_j|$ such that $\|2\rho(\alpha, A) - m\alpha\|_{\mathbb{R}/\mathbb{Z}} < 2e^{-\epsilon_0 m}$.*

3.8. **Proof of the Main Theorem in the case $\beta = 0$.** Let $\mathcal{B}$ be the set of $E \in \Sigma$ such that $(\alpha, S_{\lambda,E})$ is bounded, and $\mathcal{R}$ be the set of $E \in \Sigma$ such that $(\alpha, S_{\lambda,E})$ is reducible. By Theorem 2.4, it is enough to prove that for every $\xi \in \mathbb{R}$, $\mu = \mu_{\lambda,\alpha,\xi}$ is such that $\mu(\Sigma \setminus \mathcal{B}) = 0$.

Notice that $\mathcal{R} \setminus \mathcal{B}$ has only E such that $(\alpha, S_{v,E})$ is analytically reducible to parabolic. It follows that $\mathcal{R}\setminus\mathcal{B}$ is countable: indeed for any such E there exists $k \in \mathbb{Z}$ such that $2\rho(\alpha, S_{\lambda,E}) = k\alpha$ in $\mathbb{R}/\mathbb{Z}$. If $E \in \mathcal{R}$, any non-zero solution $H_{\lambda,\alpha,\xi}u = Eu$ satisfies $\inf_{n\in\mathbb{Z}} |u_n|^2 + |u_{n+1}|^2 > 0$. In particular there are no eigenvalues in $\mathcal{R}$, and $\mu(\mathcal{R} \setminus \mathcal{B}) = 0$. Thus it is enough to prove that $\mu(\Sigma \setminus \mathcal{R}) = 0$.

Let $K_m \subset \Sigma$, $m \geq 0$ be the set of E such that there exists $\theta \in \mathbb{R}$ and a bounded normalized solution $\hat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ with a resonance $2^m \leq |n_j| < 2^{m+1}$. We are going to show that $\sum \mu(\overline{K_m}) < \infty$. By Theorem 3.3, $\Sigma \setminus \mathcal{R} \subset \limsup K_m$. By the Borel-Cantelli Lemma, $\sum \mu(\overline{K_m}) < \infty$ implies that $\mu(\Sigma \setminus \mathcal{R}) = 0$.

To every $E \in K_m$, let $J_m(E)$ be an open $\epsilon_m = Ce^{-c2^m}$ neighborhood of E. This is chosen so to have $\sup_{0\leq s\leq 10\epsilon_m^{-1}} \|A_s\|_0 \leq e^{o(2^m)}$ by (3.14). By Lemma 2.5,

$$\mu(J_m(E)) \leq Ce^{o(2^m)}|J_m(E)|, \tag{3.33}$$

where $|\cdot|$ is used for Lebesgue measure. Take a finite subcover $\overline{K_m} \subset \cup_{j=0}^r J_m(E_j)$. Refining this subcover if necessary, we may assume that every $x \in \mathbb{R}$ is contained in at most 2 different $J_m(E_j)$.

By Lemma 3.11, $|N(J_m(E))| \geq c|J_m(E)|^2$. By Lemma 3.13, if $E \in K_m$ then $\|N(E) - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq Ce^{-c2^m}$ for some $|k| < C2^m$. This shows that $N(K_m)$ can be covered by $C2^m$ intervals T_s of length Ce^{-c2^m}. Since $|T_s| < C|N(J_m(E))|$ for any s, $E \in K_m$, there are at most $2C+4$ intervals $J_m(E_j)$ such that $N(J_m(E_j))$ intersects T_s. We conclude that there are at most $C2^m$ intervals $J_m(E_j)$. Then

$$\mu(K_m) \leq \sum_{j=0}^r \mu(J_m(E_j)) \leq C2^m e^{o(2^m)} e^{-c2^m}, \tag{3.34}$$

which gives $\sum_m \mu(\overline{K_m}) \leq C$.

Remark 3.2. In fact this argument shows that the set of energies for which the cocycle is unbounded has Hausdorff dimension zero. By Theorem 2.5, this set contains the set of energies where the spectral measures (and the integrated density of states) are not Lipschitz.

4. The $\beta > 0$ regime

Our work on the case $\beta > 0$ starts with the idea of [AD] to prove absolute continuity of the integrated density of states. There it is shown that for a rational approximant, a large set of the spectrum can be selected where we have the pointwise estimate

$$\frac{d}{dE}N_{\lambda,p/q} \leq (1+o(1))\frac{d}{dE}N_{\lambda,\alpha}, \tag{4.1}$$

as long as α is exponentially close to p/q, which easily implies that the absolutely continuous part of $dN_{\lambda,\alpha}$ has mass close to 1.

It seems a very hard problem to obtain pointwise estimates for the spectral measures themselves, and for the moment we can not prevent

$$\frac{d}{dE}\mu_{\lambda,p/q,\theta} \geq (1+\epsilon)\frac{d}{dE}\mu_{\lambda,\alpha,\theta}. \tag{4.2}$$

However, we are able to show that this can not happen for too many energies. The idea is to show that such bad situation leads to improved estimates

$$\frac{d}{dE}\mu_{\lambda,p/q,\theta'} \leq (1-\delta)\frac{d}{dE}\mu_{\lambda,\alpha,\theta'}, \tag{4.3}$$

for some other θ'. Then, integrating on E, we conclude that if for some phase θ the total mass of the absolutely continuous part is less than $1-\epsilon$ then we can find another θ' for which the total mass of the absolutely continuous part is greater than $1+\delta$, which is clearly impossible.

4.1. **Proof of the Main Theorem assuming $\beta > 0$.** Throughout this section, we fix λ, α and θ, and we assume $\beta > 0$.

Let $Y \subset \Sigma_{\lambda,\alpha}$ and $\tilde{m}$ be as in Theorem 2.3. It is enough to prove that

$$\int_Y \frac{d}{dE}\phi(\tilde{m}(\theta,E))dE = 2\pi. \tag{4.4}$$

The hypothesis implies that

$$\left|\alpha - \frac{p}{q}\right| < e^{-(\beta-o(1))q} \tag{4.5}$$

for arbitrarily large q. Fix some p/q with this property and q large.

For a fixed energy E, write $A = A^{(\lambda,E)}$, $A_n = A_n^{(\lambda,p/q,E)}$ and $\tilde{A}_n = A_n^{(\lambda,\alpha,E)}$.

Let $c = \min\{\beta/2, -\ln\lambda/2\}$.

Let $X_{\lambda,p/q,\theta}$ be the set of E such that $\mathrm{tr}A_q(\theta) = 2\cos 2\pi\rho(\theta)$ with $1/q^2 < \rho(\theta) < 1/2 - 1/q^2$.

The following plays the role of Lemma 3.1 of [AD].

Lemma 4.1. *We have $\mu_{\lambda,p/q,\theta}(\Sigma_{\lambda,p/q,\theta} \setminus X_{\lambda,p/q,\theta}) \leq 4/q$.*

Proof. We have $dN_{\lambda,p/q,\theta} = \frac{1}{q}\sum_{k=0}^{q-1}\mu_{\lambda,p/q,\theta+k\alpha}$ and $dN_{\lambda,p/q,\theta}(\Sigma_{\lambda,p/q,\theta} \setminus X_{\lambda,p/q,\theta}) = 4/q^2$. □

Lemma 4.2 ([AD], Lemma 3.2)**.** *We have*

$$|\Sigma_{\lambda,p/q} \setminus \Sigma_{\lambda,\alpha}| \le e^{-(c-o(1))q}. \tag{4.6}$$

In particular,

$$|X_{\lambda,p/q,\theta} \setminus \Sigma_{\lambda,\alpha}| \le e^{-(c-o(1))q}. \tag{4.7}$$

The following is the appropriate version of Lemma 3.3 of [AD] that we need. Though the claim is formally different, since our set $X_{\lambda,p/q,\theta}$ is somewhat larger than the set $X_{\lambda,p/q}$ given in [AD], the proof is exactly the same.

If E belongs to the interior of $\Sigma_{\lambda,p/q,\theta}$, let $m(\theta,E)$ be the fixed point of $A_q(\theta)$ in $\mathbb{H}$, as in §2.6.

Lemma 4.3. *We have*

$$\sup_{E\in X_{\lambda,p/q,\theta}} \sup_{x\in\mathbb{R}} \ln \phi(m(x,E)) = o(q). \tag{4.8}$$

The analogous of Lemma 3.4 of [AD] holds again with same proof.

Lemma 4.4. *We have* $\mu_{\lambda,p/q,\theta}(X_{\lambda,p/q,\theta} \setminus Y) = o(1)$.

It follows from Lemmas 4.1 and 4.4 that

$$\int_{X_{\lambda,p/q,\theta}\cap Y} \phi(m(\theta,E))dE \ge 2\pi - o(1). \tag{4.9}$$

We will need to estimate how well the cocycle can be compared with the rational case. By Lemma 4.4, we have $A(\theta+k\alpha) = B(\theta+(k+1)p/q)R_{\psi(\theta+kp/q)}B(\theta+kp/q)^{-1}$ with $\ln\|B(\theta)\| = o(q)$, by taking $B(\theta+kp/q)\cdot i = m(\theta+kp/q,E)$. We have $\prod_{i=q-1}^{0} R_{\psi(\theta+ip/q)} = B(\theta)^{-1}A_q(\theta)B(\theta) = R_{\pm\rho(\theta)}$. The following is the appropriate version of Lemma 4.1 of [AD]. Let $b = [e^{c/10q}]$.

Lemma 4.5. *For* $0 \le k < b$ *we have* $\|B(\theta)^{-1}\tilde{A}_{kq}(\theta)B(\theta) - R_{\pm k\rho(\theta)}\| = O(e^{-cq/4})$.

Proof. Write

$$\tilde{A}_k(\theta) = \prod_{i=k-1}^{0} A(\theta+i\alpha) = \prod_{i=k-1}^{0} B(\theta+(i+1)p/q)Q_iB(\theta+ip/q)^{-1}. \tag{4.10}$$

Then $\|Q_i - R_{\psi(\theta+ip/q)}\| = O(e^{-(3c/2-o(1))q})$ for $0 \le i < bq$. Thus $\tilde{A}_{kq}(\theta) = B(\theta)QB(\theta)^{-1}$ where $Q = \prod_{i=kq-1}^{0} Q_i$ satisfies $\|Q - R_{\pm k\rho(\theta)}\| = O(e^{-(c-o(1))q})$. □

We now diverge from [AD]. The cancellation mechanism will evolve along the next four lemmas. The basis is an equality for periodic elliptic matrices in $\mathrm{SL}(2,\mathbb{R})$:

Lemma 4.6. *Let* c/d *be a rational number which is not an integer multiple of* $1/2$*, and let* $z_0 \in \mathbb{H}$*. If* $B_0 \in \mathrm{SL}(2,\mathbb{R})$

$$\frac{1}{d}\sum_{k=0}^{d-1} \phi(B_0 \cdot R_{ck/d} \cdot z_0) = \phi(z_0)\phi(B_0 \cdot i). \tag{4.11}$$

Proof. Let $Z \in \mathrm{SL}(2,\mathbb{R})$ be a matrix taking i to z_0. We want to estimate

$$\frac{1}{d}\sum_{k=0}^{d-1} \|B_0 \cdot R_{ck/d} \cdot Z\|_{\mathrm{HS}}^2 = \frac{1}{2}\|Z\|_{\mathrm{HS}}^2\|B_0\|_{\mathrm{HS}}^2. \tag{4.12}$$

By considering rotations, this is the same as showing
(4.13)

$$\frac{1}{d}\sum_{k=0}^{d-1}\|\begin{pmatrix}\nu & 0\\ 0 & \nu^{-1}\end{pmatrix}\cdot R_{x+ck/d}\cdot\begin{pmatrix}\nu' & 0\\ 0 & \nu'^{-1}\end{pmatrix}\|_{\mathrm{HS}}^2=\frac{1}{2}\|\begin{pmatrix}\nu' & 0\\ 0 & \nu'^{-1}\end{pmatrix}\|_{\mathrm{HS}}^2\|\begin{pmatrix}\nu & 0\\ 0 & \nu^{-1}\end{pmatrix}\|_{\mathrm{HS}}^2,$$

for any x, ν and ν'. A direct computation gives

$$\begin{aligned}(4.14)\qquad &\frac{1}{d}\sum_{k=0}^{d-1}\|\begin{pmatrix}\nu & 0\\ 0 & \nu^{-1}\end{pmatrix}\cdot R_{x+ck/d}\cdot\begin{pmatrix}\nu' & 0\\ 0 & \nu'^{-1}\end{pmatrix}\|_{\mathrm{HS}}^2=\\ &\frac{1}{d}\sum_{k=0}^{d-1}\nu^2\nu'^2+\nu^{-2}\nu'^{-2}+(\nu^2-\nu^{-2})(\nu'^2-\nu'^{-2})(\cos^2 2\pi(x+kc/d)-1)\\ &=\frac{1}{2}(\nu^2+\nu^{-2})(\nu'^2+\nu'^{-2}),\end{aligned}$$

since

$$(4.15)\qquad \frac{1}{d}\sum_{k=0}^{d-1}\cos^2 2\pi(x+ck/d)=\frac{1}{d}\sum_{k=0}^{d-1}\frac{1+\cos 4\pi(x+ck/d)}{2}=1/2.$$

□

Next we obtain an estimate on general elliptic $\mathrm{SL}(2,\mathbb{R})$ matrices:

Lemma 4.7. *For every $\epsilon>0$ there exists $\delta>0$ such that if $B_0\in\mathrm{SL}(2,\mathbb{R})$ with $\|B_0\|<e^{\delta q}$, $e^{-\delta q}<\rho<1/2-e^{-\delta q}$, $z_0\in\mathbb{H}$ with $\phi(z_0)<e^{\delta q}$ and $b_0>e^{\epsilon q}$ then as q grows we have*

$$(4.16)\qquad \frac{1}{b_0}\sum_{k=0}^{b_0-1}\phi(B_0\cdot R_{\pm k\rho}\cdot z_0)>(1-o(1))\phi(z_0)\phi(B_0\cdot i).$$

Proof. Let c/d, $d\le b_0^{1/2}$ maximal, be an approximant of ρ. It is enough to show that

$$(4.17)\qquad \frac{1}{d}\sum_{k=0}^{d-1}\phi(B_0\cdot R_{\pm k\rho}\cdot z_0)>(1-o(1))\phi(z_0)\phi(B_0\cdot i).$$

Consider the points $w_k'=B_0\cdot R_{\pm kc/d}\cdot z_0$, $w_k=B_0\cdot R_{\pm k\rho}\cdot z_0$. Then $\mathrm{dist}_{\mathbb{H}}(w_k,w_k')=o(1)$. It follows that it is enough to show that

$$(4.18)\qquad \frac{1}{d}\sum_{k=0}^{d-1}\phi(w_k')>(1-o(1))\phi(z_0)\phi(B_0\cdot i).$$

This follows from Lemma 4.6. □

We now apply the previous estimates to the cocycle, using that it is well shadowed by rotations by Lemma 4.5.

Lemma 4.8 (Cancellation along orbits, fixed energy)**.** *Let $z\in\mathbb{H}$, $E\in X_{\lambda,p/q,\theta}$. If $1\le\kappa\le 2$ is such that*

$$(4.19)\qquad |\ln\phi(z)-\ln\phi(m(\theta,E))|\ge\ln\kappa$$

then

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(\tilde{A}_{kq}(\theta)\cdot z)\geq(\frac{1+\kappa^2}{2\kappa}-o(1))\phi(m(\theta,E)). \tag{4.20}$$

Proof. There are two cases to consider. If $\phi(z)$ is not $e^{o(q)}$, then by the previous lemma, $\phi(\tilde{A}_{kq}(\theta)\cdot z)$ is not $e^{o(q)}$ for $0\leq k\leq b-1$ and

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(\tilde{A}_{kq}(\theta)\cdot z)\geq 2\phi(m(\theta,E)). \tag{4.21}$$

Assume now that $\phi(z)=e^{o(q)}$. Set $B_0=B(\theta)$, $z_0=B(\theta)^{-1}\cdot z$, $\rho=\rho(\theta)$, $b_0=b$ in the previous lemma. Then

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(B(\theta)\cdot R_{\pm k\rho(\theta)}B(\theta)^{-1}\cdot z)>(1-o(1))\phi(z_0)\phi(B(\theta)\cdot i). \tag{4.22}$$

We have $B(\theta)\cdot i=m(\theta,E)$. Moreover $\mathrm{dist}_{\mathbb{H}}(z,m(\theta,E))\geq\ln\kappa$ (since $\ln\phi$ is 1-Lipschitz in the hyperbolic metric), so we also have $\mathrm{dist}_{\mathbb{H}}(z_0,i)\geq\ln\kappa$. It follows that $\phi(z_0)\geq(1+\kappa^2)/2\kappa$, so we have

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(B(\theta)\cdot R_{\pm k\rho(\theta)}B(\theta)^{-1}\cdot z)>(1-o(1))\frac{1+\kappa^2}{2\kappa}\phi(m(\theta,E)). \tag{4.23}$$

We now notice that $\mathrm{dist}_{\mathbb{H}}(B(\theta)\cdot R_{\pm k\rho(\theta)}B(\theta)^{-1}\cdot z,\tilde{A}_{kq}\cdot z)=o(1)$ by Lemma 4.5 and $\phi(z)=e^{o(q)}$, so $(1-o(1))\phi(B(\theta)\cdot R_{k\rho(\theta)}B(\theta)^{-1}\cdot z)\leq\phi(\tilde{A}_{kq}\cdot z)$, which gives

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(\tilde{A}_{kq}\cdot z)>(1-o(1))\frac{1+\kappa^2}{2\kappa}\phi(m(\theta,E)). \tag{4.24}$$

□

This estimate can be applied to the case $z=\tilde{m}(\theta)$ and integrated to yield:

Lemma 4.9 (Cancellation along orbits, integrated version)**.** *For every $\epsilon>0$ there exists $\delta>0$ such that if*

$$\int_{X_{\lambda,p/q,\theta}\cap Y}\phi(\tilde{m}(\theta,E))<(1-\epsilon-o(1))2\pi \tag{4.25}$$

then

$$\frac{1}{b}\sum_{k=0}^{b-1}\int_{X_{\lambda,p/q,\theta}\cap Y}\phi(\tilde{m}(\theta+k\alpha,E))>(1+\delta-o(1))2\pi. \tag{4.26}$$

Proof. Let $W\subset X_{\lambda,p/q,\theta}\cap Y$ be the set such that $\phi(\tilde{m}(\theta,E))<(1-\epsilon/2)\phi(m(\theta,E))$. Then by (4.9)

$$\int_W\phi(m(\theta,E))>\epsilon\pi-o(1). \tag{4.27}$$

Applying the previous lemma with $z=\tilde{m}(\theta,E)$ we get

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(\tilde{m}(\theta+k\alpha,E))>(1+\delta-o(1))\phi(m(\theta,E)),\quad E\in X_{\lambda,p/q,\theta}\cap Y\setminus W, \tag{4.28}$$

$$\frac{1}{b}\sum_{k=0}^{b-1}\phi(\tilde m(\theta+k\alpha,E))>(1+\delta-o(1))\phi(m(\theta,E)),\quad E\in W, \tag{4.29}$$

Integrating and using (4.9) we get

$$\frac{1}{b}\sum_{k=0}^{b-1}\int_{X_{\lambda,p/q,\theta}\cap Y}\phi(\tilde m(\theta+k\alpha,E))>(1-o(1))2\pi+(\delta-o(1))(\epsilon\pi-o(1)). \tag{4.30}$$

□

The conclusion of this lemma being obviously impossible for q large, we must have

$$\int_{X_{\lambda,p/q,\theta}\cap Y}\phi(\tilde m(\theta,E))>(1-o(1))2\pi, \tag{4.31}$$

which implies (4.4) as q grows.

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