SMOOTH SIEGEL DISKS VIA SEMICONTINUITY: A REMARK ON A PROOF OF BUFF AND CHERITAT

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Abstract. Recently, Xavier Buff and Arnaud Cheritat have provided an elegant proof of the existence of quadratic Siegel disks with smooth boundary. In this short note, we show how results of Yoccoz and Risler can be used to conclude the same result. Our proof is a small modification of the argument given by Buff and Cheritat.

1. Introduction

Recently, in [BC1], Xavier Buff and Arnaud Cheritat gave a new proof of the following unpublished result of Perez-Marco: *there exists a quadratic map with a Siegel disk whose boundary is a smooth (C^∞) Jordan curve.* Their proof involves both techniques of renormalization of [Y] and estimates for parabolic explosion.

Our aim in this note is to show that the same result follows easily from renormalization theory via two known results (of Yoccoz and Risler) by some general abstract reasoning (which is really just a small modification of [BC1]).

We would like to note that the method of parabolic explosion (coupled with renormalization) allows much greater control of the dynamics. In particular, in [BC1] it is also possible to conclude that the Siegel disks are accumulated by small cycles, see also [BC2] for even more dramatic applications. However, we find it worthwhile to investigate what is really needed for the argument, and we hope that the present treatment could be used in situations which are more general than the quadratic setting: indeed the proof works for families of rational or entire maps which do not have non-Brjuno Siegel disks, such as the families $z \to e^{2\pi i\alpha} z(1+z/d)^d$, $d \geq 2$, and $z \to e^{2\pi i\alpha} ze^z$ considered by Lukas Geyer in [G]. We remark that it is conjectured that Siegel disks of rational maps are always Brjuno.

In the last section we discuss the application of the method to the case of Herman rings. This application has been pointed out to us by Xavier Buff, who had obtained this result earlier by other methods.

2. Main result

2.1. **Siegel disks.** Let $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$. Let r_α be the conformal radius of the Siegel disk Δ_α of P_α if it exists, and let $r_\alpha = 0$ otherwise. If $r_\alpha > 0$, let $L_\alpha : \mathbb{D}_{r_\alpha} \to \Delta_\alpha$ (where $\mathbb{D}_r = \{|z| < r\}$) be the uniformization map satisfying $L_\alpha(0) = 0$ and $DL_\alpha(0) = 1$. The function L_α satisfies $P_\alpha(L_\alpha(z)) = L_\alpha(e^{2\pi i\alpha} z)$.

Let F_r be the space of holomorphic functions $f : \mathbb{D}_r \to \mathbb{C}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}_r$. Let E_r be a complete metric space of functions $f : \mathbb{D}_r \to \mathbb{C}$. We assume that for $r' > r$ we have $F_{r'} \subset E_r$ and that the inclusion is continuous. For instance, E_r can be taken as the Fréchet space of C^∞ functions $f : \overline{\mathbb{D}}_r \to \mathbb{C}$. The requirements also allows one to consider (subspaces of) certain spaces of quasianalytic functions, as the Banach space of C^∞ functions $f : \overline{\mathbb{D}}_r \to \mathbb{C}$ such that $f|\mathbb{D}_r$ is holomorphic and $\sup_{r \geq 2} \sup_{x \in \partial \mathbb{D}_r} \frac{|\partial^r f(x)|}{(r \ln r)^r} < \infty$.

Theorem 2.1. *Let $r_{\alpha_0} > 0$. For every $\delta > 0$, $0 < r < r_{\alpha_0}$, there exists $\alpha \in \mathbb{R}$ such that $|\alpha - \alpha_0| < \delta$, $r_\alpha = r$, $L_\alpha|\mathbb{D}_r \in E_r$, and $\mathrm{dist}_{E_r}(L_{\alpha_0}|\mathbb{D}_r, L_\alpha|\mathbb{D}_r) < \delta$.*

This theorem implies the existence of smooth Siegel disks in the family P_α.

In order to prove Theorem 2.1, we will use properties of the function $\alpha \to r_\alpha$. Two of them are elementary:

(P1) $r_\alpha = 0$ for a dense set of α. Indeed if $\alpha = \frac{p}{q} \in \mathbb{Q}$ and $r_\alpha > 0$ then P_α^q would have to be the identity on a neighborhood of 0, but P_α^q is a monic polynomial of degree 2^q.

(P2) The function $\alpha \to r_\alpha$ is upper semicontinuous. Indeed, if $\alpha_n \to \alpha$ and $\inf_n r_{\alpha_n} \geq r' > 0$ then Hurwitz Theorem $L_{\alpha_n}|\mathbb{D}_{r'}$ converges, in the topology of $F_{r'}$ (and hence also in the topology of E_r, $r < r'$) to a univalent function, which must coincide with $L_\alpha|\mathbb{D}_{r'}$.

We also need two non-elementary properties, which depend on renormalization theory through results of Yoccoz and Risler. Let us say that a function $h : \mathbb{R} \to \mathbb{R}$ is weakly lower semicontinuous at $c \in \mathbb{R}$, we have

$$\min\{\limsup_{y \to c+} h(y), \limsup_{y \to c-} h(y)\} \geq h(c). \tag{2.1}$$

(P3) r_α is weakly lower semicontinuous when α is non-Brjuno[1]. Indeed Yoccoz's Theorem [Y] implies that $r_\alpha = 0$ for non-Brjuno numbers, so by (P2) and $r_\alpha \geq 0$, $\alpha \in \mathbb{R}$, we see that r_α is even continuous at non-Brjuno numbers.[2]

(P4) r_α is weakly lower semicontinuous when α is Brjuno. Indeed, by a result of Risler, if α is Brjuno then there exists a set $\mathcal{B}_s$ restricted to which the function $\alpha \to r_\alpha$ is continuous (see Proposition 10 of [R]), and α is a Lebesgue density point of $\mathcal{B}_s$ (see Proposition 1 of [R] for other properties of the sets $\mathcal{B}_s$).[3]

The properties (P2-4) will be exploited through the following:

Lemma 2.2. *If $h : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous and weakly lower semicontinuous at every $c \in \mathbb{R}$, then h satisfies the Intermediate Value Theorem.*

Proof. Let $a < b$ such that $h(a) \neq h(b)$. To fix ideas, assume $h(a) < h(b)$. Let $h(a) < x < h(b)$ and let $c = \inf\{a \leq y \leq b,\ h(y) \geq x\}$. By upper semicontinuity, $h(c) \geq x$, so $a < c \leq b$. If $h(c) > x$, by (2.1) there exists $a < y < c$ such that $h(y) > x$, contradicting the definition of c. Thus $h(c) = x$ as required. □

Together with (P1), this yields:

Corollary 2.3. *If $r_\beta > 0$ then for every $0 < r < r_\beta$ and $\epsilon > 0$ there exists $\beta' \in \mathbb{R}$ such that $|\beta' - \beta| < \epsilon$ and $r_{\beta'} = r$.*

Proof of Theorem 2.1. Let β_i, ϵ_i be defined inductively as follows. Let $\beta_0 = \alpha_0$ and $\epsilon_0 = \delta$. Assuming β_i, ϵ_i defined, let $\epsilon_{i+1} < \frac{\epsilon_i}{10}$ be such that $r_\beta < r_{\beta_i} + 2^{-i}$

[1] Recall that $\alpha \in \mathbb{R}$ is called a Brjuno number if it is irrational and $\sum \frac{\ln q_{n+1}}{q_n} < \infty$, where q_n is the (increasing) sequence of denominators of the best rational approximations of α.

[2] This is the only place we shall use special properties of the quadratic family.

[3] It would be interesting to investigate if the estimates of Risler are enough to conclude that r_α is weakly lower semicontinuous also at non-Brjuno α, as this would remove the necessity of the step (P3) and make the whole argument much more general.

whenever $|\beta - \beta_i| < \epsilon_{i+1}$ (this is possible by upper semicontinuity). Then let β_{i+1} be such that $|\beta_{i+1} - \beta_i| < \frac{\epsilon_{i+1}}{10}$, $r_{\beta_{i+1}} = \frac{r + r_{\beta_i}}{2}$, and $\mathrm{dist}_{E_r}(L_{\beta_{i+1}}|\mathbb{D}_r, L_{\beta_i}|\mathbb{D}_r) < \frac{\epsilon_{i+1}}{10}$ (this is possible by Corollary 2.3 and the proof of (P2) above). It is easy to check that $\alpha = \lim \beta_i$ has the required properties. □

Remark 2.1. One easily checks that this proof yields a Cantor set of α satisfying the conclusions of Theorem 2.1.

2.2. **Herman rings.** Fix $a > 3$ and let $Q_\lambda(z) = e^{2\pi i \lambda} z^2 \frac{z+a}{1+az}$. Then $Q_\lambda(z)$, $\lambda \in \mathbb{R}$, is a diffeomorphism of S^1, and we let α_λ be its rotation number. Let r_λ be half the modulus of the Herman ring Ξ_λ of Q_λ if it exists, and let $r_\lambda = 0$ otherwise. If $r_\lambda > 0$, let $T_\lambda : A_{r_\lambda} \to \Xi_\lambda$ (where $A_r = \{-r < \ln |z| < r\}$) be the uniformization map satisfying $T_\lambda(1) = 1$ and $DT_\lambda(1) > 0$. It follows that $T_\lambda|S^1$ is a linearizing map for $Q_\lambda|S^1$, that is, $T_\lambda(e^{2\pi i \alpha_\lambda} z) = Q_\lambda(T_\lambda(z))$.

Let F_r be the space of holomorphic functions $f : A_r \to \mathbb{C}$ with the topology of uniform convergence on compact subsets of A_r. For $r > 0$, let E_r be a complete metric space of functions $f : A_r \to \mathbb{C}$, and let E_0 be a complete metric space of functions $f : S^1 \to \mathbb{R}$. For $r' > r$, we assume that $F_{r'} \subset E_r$ and that the inclusion is continuous.

Theorem 2.4. *Let $r_{\lambda_0} > 0$. For every $\delta > 0$, $0 < r < r_{\lambda_0}$, there exists $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < \delta$, $r_\lambda = r$, $T_\lambda|A_r \in E_r$, and $\mathrm{dist}_{E_r}(T_{\lambda_0}|A_r, T_\lambda|A_r) < \delta$.*

Theorem 2.5. *Let $r_{\lambda_0} > 0$. For every $\delta > 0$, there exists $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < \delta$, $r_\lambda = 0$, and there exists a linearizing map $T : S^1 \to S^1$ such that $T \in E_0$, $\mathrm{dist}_{E_0}(T_{\lambda_0}|S^1, T) < \delta$.*

Theorem 2.4 implies the existence of smooth Herman rings in the family Q_λ, while Theorem 2.5 implies the existence of values of α such that $Q_\lambda|S^1$ is smoothly, but not analytically, linearizable (when $Q_\lambda|S^1$ is analytically linearizable one automatically has $r_\lambda > 0$).

The proof of both theorems is the same as of Theorem 2.1 and we shall not repeat it here. We only need to replace Yoccoz's Theorem used in (P3) by a result of Geyer [G], since Proposition 10 of Risler also applies for Herman rings to yield (P4). In particular, the proof works also for the family $Q_\lambda(z) = e^{2\pi i \lambda} z e^{a(z-1/z)}$, where $0 < |a| < 1/2$ is a fixed parameter (this is the complexification of Arnold's standard family).

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References

[BC1] X. Buff and A. Cheritat. Quadratic Siegel disks with smooth boundaries. Preprint (http://fermat.ups-tlse.fr/ buff/).

[BC2] X. Buff and A. Cheritat. On the size of quadratic Siegel disks: Part I. Preprint (www.arXiv.org).

[G] L. Geyer. Siegel discs, Herman rings and the Arnold family. Trans. Amer. Math. Soc. 353 (2001), no. 9, 3661–3683.

[R] E. Risler. Linéarisation des perturbations holomorphes des rotations er applications. Mém. Soc. Math. Fr. 77 (1999).

[Y] J.-C. Yoccoz. Petits diviseurs en dimension 1. Astérisque 231 (1995).

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