CONVERGENCE OF AN EXACT QUANTIZATION SCHEME

ARTUR AVILA

ABSTRACT. It has been shown by Voros [V1] that the spectrum of the one-dimensional homogeneous anharmonic oscillator (Schrödinger operator with potential q^{2M}, $M > 1$) is a fixed point of an explicit non-linear transformation. We show that this fixed point is globally and exponentially attractive in spaces of properly normalized sequences.

1. Introduction

Let $0 < \theta < \pi$ be a constant. For $E, E' > 0$, define

$$\theta(E', E) = \tan^{-1} \frac{\sin\theta}{E'E^{-1} + \cos\theta}. \tag{1.1}$$

Let $X = (X_k)_{k=1}^{\infty}$, $Y = (Y_j)_{j=1}^{\infty}$ be sequences of positive real numbers and define $\phi = (\phi_j)_{j=1}^{\infty}$ by

$$\phi_j(X, Y) = \frac{1}{\pi} \sum_k \theta(X_k, Y_j). \tag{1.2}$$

Let $Q = (Q_i)_{i=1}^{\infty}$ be a constant vector, and consider the operator $T \equiv T_{\theta,Q}$ given implicitly by $\phi(X, T(X)) = Q$. Of course $T(X)$ is only defined for certain sequences X. We remark that T is dilatation equivariant ($T(\lambda X) = \lambda T(X)$ for $\lambda > 0$) and positive in the sense that if $0 < X_k \leq X'_k$ for all $k > 0$ and if $T(X) = Y$ and $T(X') = Y'$ are defined then $Y_k \leq Y'_k$ for all $k > 0$.

In this paper we will be interested in the description of the dynamics of T acting on certain spaces of normalized sequences, under appropriate conditions on Q.

1.1. Relation to exact anharmonic quantization.

We now describe the physical motivation of the problem (for futher details and references, see [V1], and for more recent related work, see [V2]). Let us consider the one-dimensional anharmonic oscillator with even homogeneous polynomial potential, that is, the Schrödinger operator

$$(Hu)(q) = -\frac{d^2u}{dq^2} + q^{2M}u(q), \quad M = 2, 3, \ldots, \tag{1.3}$$

acting on $L^2(\mathbb{R})$. This operator has a *discrete spectrum*

$$0 < E_0 < E_1 < \ldots, \tag{1.4}$$

where $\lim E_j = \infty$.

Let

$$\theta = \frac{M-1}{M+1}\pi, \tag{1.5}$$

$$\alpha_\theta = \frac{\pi + \theta}{\pi} = \frac{2M}{M+1}. \tag{1.6}$$

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It is known that E_k has polynomial growth, more precisely:

Proposition 1.1 (see [V1], §2.1)**.** *The spectrum (1.4) of the operator (1.3) satisfies*

$$\nu = \lim_{k\to\infty} k^{-\alpha_\theta} E_k, \tag{1.7}$$

where α_θ is given by (1.6) and ν is positive and finite.

The semiclassical analysis provide much more information then what is contained in the above proposition, for instance, ν can be explicitely computed

$$\nu = \left(2\pi^{1/2} M\Gamma\left(\frac{3}{2}+\frac{1}{2M}\right)\Gamma\left(\frac{1}{2M}\right)^{-1}\right)^{\alpha_\theta}, \tag{1.8}$$

and higher order terms for the asymptotic development of E_k are also available (though the resulting series does not converge), let us only remark for motivation that

$$E_k = \nu k^{\alpha_\theta} + O(k^{\alpha_\theta - 1}). \tag{1.9}$$

It is convenient to split the spectrum according to parity

$$P_i^{\mathrm{even}} = E_{2i-2}, \quad P_i^{\mathrm{odd}} = E_{2i-1}, \quad i \geq 1. \tag{1.10}$$

It has been shown by Voros that P^{even} and P^{odd} are fixed points of of operators $T_{\theta,Q^{\mathrm{even}}}$ and $T_{\theta,Q^{\mathrm{odd}}}$ respectively, where

$$Q_k^{\mathrm{even}} = k - \frac{3}{4} + \frac{M-1}{4(M+1)}, \quad Q_k^{\mathrm{odd}} = k - \frac{1}{4} - \frac{M-1}{4(M+1)}. \tag{1.11}$$

Proposition 1.2 (see [V1], §3.1)**.** *The even and odd parts of the spectrum of the operator (1.3) satisfy equations*

$$T_{\theta,Q^{\mathrm{even}}}(P^{\mathrm{even}}) = P^{\mathrm{even}}, \quad T_{\theta,Q^{\mathrm{odd}}}(P^{\mathrm{odd}}) = P^{\mathrm{odd}}, \tag{1.12}$$

where θ, Q^{even} and Q^{odd} are as above.

Due to dilatation equivariance of T, the fixed point equation does not determine the spectrum completely. Numerical evidence was obtained (see [V1], §7.3) that indicated that it *does* determine the spectrum once one normalizes appropriately at $k \to \infty$, and that the operator T provides an exponentially convergent iterative scheme for determination of the spectrum.

Our main theorem will confirm those hopes. We show that there is only one fixed point for T subject to growth condition $2^{\alpha_\theta}\nu k^{\alpha_\theta} + o(k^{\alpha_\theta})$ (which thus coincides with the spectrum P), and that this is a globally attractive fixed point in the space of sequences with such growth. We also analyze the action of T on sequences whose growth is more accurately described in terms of polynomial error terms, including the type $2^{\alpha_\theta}\nu k^{\alpha_\theta} + O(k^{\alpha_\theta-\epsilon})$, $0 < \epsilon \leq 1$ (this is natural in view of the asymptotic estimate (1.9)), and we show that the fixed point is indeed exponentially attractive among such sequences.

Theorem 1.3. *Let $M > 1$, and let T denote the exact quantization operator related to the even or odd spectrum P of the operator (1.3). If $X = (X_k)_{k=1}^\infty$ satisfies $X_k = 2^{\alpha_\theta}\nu e^{o(1)} k^{\alpha_\theta}$ then $X^{(n)} \equiv T^n(X)$ converges pointwise to P, and indeed*

$$\lim_{n\to\infty} \sup_k k^{-\alpha_\theta} |X_k^{(n)} - P_k| = 0. \tag{1.13}$$

If moreover $X_k = P_k + O(k^{\alpha_\theta - \epsilon})$ with $0 < \epsilon < 2$ then

$$\sup_k k^{-\alpha_\theta+\epsilon} |X_k^{(n)} - P_k| \leq C\lambda^n, \tag{1.14}$$

where $C = C(X,\epsilon) > 0$ and $\lambda = \lambda(\epsilon) < 1$.

The operator T actually comes about as a (non-linear) quantization of a semiclassical Bohr-Sommerfeld linear operator. The main steps of our analysis involves showing that T behaves as a perturbation of the linear operator. The asymptotic limit $k \to \infty$ is given to certain accuracy by the semiclassical linear operator, which can be shown to have the required properties. We use two obvious features of T to show that the quantization does not destroy those properties. The first one is positivity, and the second one is equivariance by dilatation. Those properties are present both at the infinitesimal analysis (they are used in perturbative estimates of the operator norm of the derivative DT) as in the global analysis (where they are used in a key precompactness argument).

2. Proof of Theorem 1.3

2.1. **Setting and notations.** We will actually prove a slightly more general result, Theorem 2.1, about the operators $T_{\theta,Q}$. This result implies Theorem 1.3 immediately, using Propositions 1.1 and 1.2. The remaining analysis is completely self-contained.

We will need to make no restriction on $0 < \theta < \pi$. We will make two assumptions on the sequence Q_k:

$$Q_k = k + O(1) \tag{2.1}$$

$$Q_k > \left(k - \frac{1}{2}\right) \frac{\theta}{\pi}. \tag{2.2}$$

The first condition comes from the physical problem, and can be relaxed to $Q_k = e^{o(1)} k$ without any changes in our analysis. Notice that for any X,

$$\sum_{j=1}^{k} \phi_j(X, X) > \frac{\theta k^2}{2\pi} \quad k \geq 1, \tag{2.3}$$

in particular, if $\sum_{j=1}^{k} Q_j \leq (2\pi)^{-1} \theta k^2$ for some k then there is no fixed point for T, so some condition (possibly weaker) in the line of our second condition is necessary for our results to hold.

It will be convenient to work in logarithmic coordinates for computations. All variables in capital letters will denote positive real numbers (or vectors of positive real numbers). The corresponding non-capital letters will be reserved for their logarithms.

2.2. **Some spaces of sequences.** Let $u(\epsilon)$ be the space of $v = (v_i)_{i=1}^{\infty}$ of the form

$$v_k = O(k^{-\epsilon}), \tag{2.4}$$

with the norm

$$\|v\|_\epsilon = \sup k^\epsilon |v_k|. \tag{2.5}$$

Let $u^0(\epsilon)$ be the subspace of $u(\epsilon)$ consisting of v of the form

$$v_k = o(k^{-\epsilon}). \tag{2.6}$$

Given some vector x, we define affine spaces

$$u(x, \epsilon) = x + u(\epsilon) \tag{2.7}$$

and

$$u^0(x, \epsilon) = x + u^0(\epsilon). \tag{2.8}$$

We will use the special notation

$$u(\alpha, \epsilon) = u((\alpha \ln k)_{k=1}^{\infty}, \epsilon), \quad \alpha > 0, \epsilon \geq 0, \tag{2.9}$$

$$u^0(\alpha, \epsilon) = u^0((\alpha \ln k)_{k=1}^{\infty}, \epsilon), \quad \alpha > 0, \epsilon \geq 0. \tag{2.10}$$

Notice that if $x \in u(\alpha, \epsilon)$ then $u(x, \epsilon') = u(\alpha, \epsilon')$ provided $\epsilon' \leq \epsilon$.

The several affine spaces u parametrize by exponentiation spaces U, for instance

$$U(\alpha, \epsilon) = \{(X_k)_{k=1}^{\infty}, X_k > 0, X_k = k^{\alpha} + O(k^{\alpha-\epsilon})\}, \tag{2.11}$$

$$U^0(\alpha, 0) = \{(X_k)_{k=1}^{\infty}, X_k > 0, X_k = k^{\alpha} + o(k^{\alpha})\}. \tag{2.12}$$

We can now state our main result:

Theorem 2.1. *There exists a unique $\alpha_\theta > 0$ for which there exists a fixed point $X \in U(\alpha_\theta, 0)$ for T. Moreover,*

1. *The space $U^0(\alpha_\theta, 0)$ is invariant for T,*
2. *There exists a fixed point $P \in U(\alpha_\theta, 1)$,*
3. *P is a global attractor in $U^0(\alpha_\theta, 0)$, that is, for any $X \in U^0(\alpha_\theta, 0)$,*

$$\lim_{n\to\infty} \|T^n(x) - p\|_0 = 0, \tag{2.13}$$

4. *The spaces $U(P, \epsilon)$ are invariant for $0 \leq \epsilon < \alpha_\theta + 1$,*
5. *P is a global exponential attractor in $U(P, \epsilon)$, $0 < \epsilon < 2$, that is, for any $X \in U(P, \epsilon)$,*

$$\|T^n(x) - p\|_\epsilon \leq C\lambda^n, \tag{2.14}$$

where $C = C(\epsilon, \|x - p\|_\epsilon) > 0$ and $\lambda = \lambda(\epsilon) < 1$.

The proof of this result will take the remaining of this section.

2.3. **Lipschitz continuity in** $U(X, 0)$**.** Let us write $X \leq X'$ if $X_k \leq X'_k$ for all k. Then $X \leq X'$ and $Y \geq Y'$ implies $\phi(X, Y) \geq \phi(X', Y')$, which implies the positivity of T we stated before: $X \leq X'$ implies $T(X) \leq T(X')$. In particular, $T(X) \leq X$ if $\phi(X, X) \geq Q$ and $T(X) \geq X$ if $\phi(X, X) \leq Q$.

This also gives us a way to show that T is defined at some X: if $\phi(X, \underline{Y}) \leq Q \leq \phi(X, \overline{Y})$ then $T(X) = Y$ is defined and $\underline{Y} \leq Y \leq \overline{Y}$.

Lemma 2.2. *Assume that $T(X) = Y$ is defined. Then T is defined on $U(X, 0)$ and $T(U(X, 0)) = U(Y, 0)$. Moreover, $T : U(X, 0) \to U(Y, 0)$ is 1-Lipschitz.*

Proof. If $C^{-1}X \leq X' \leq CX$ then $\phi(X', C^{-1}Y) \leq \phi(C^{-1}X, C^{-1}Y) = Q = \phi(CX, CY) \leq \phi(X', CY)$. □

2.4. **The derivative.** Let

$$P(E, E') = \frac{EE'}{E^2 + 2\cos\theta EE' + E'^2}. \tag{2.15}$$

Notice that

$$\frac{d\phi_j}{dx_k}(X, Y) = \frac{-\sin\theta}{\pi} P(X_k, Y_j), \tag{2.16}$$

$$\frac{d\phi_j}{dy_j}(X, Y) = \sum_k \frac{\sin\theta}{\pi} P(X_k, Y_j), \tag{2.17}$$

and of course

$$\frac{d\phi_j}{dy_k}(X, Y) = 0, \quad j \neq k. \tag{2.18}$$

We can now use write a nice formal expression for the derivative of T with respect to logarithmic coordinates. If $T(x) = y$ is defined, let $DT(x) = (D_{ij}T(x))_{i,j\geq 1}$ be the infinite matrix

$$D_{ij}T(x) = \frac{P(X_j, Y_i)}{\sum_k P(X_k, Y_i)}. \tag{2.19}$$

This matrix is stochastic and positive, that is all entries are positive numbers and the sum of the entries in each row is 1. In particular, the operator norm of DT acting on bounded sequences is equal to 1.

Lemma 2.3. *Let $\mathcal{L}(u(0), u(0))$ be the space of bounded linear transformations on $u(0)$ with the operator norm. If $T(X) = Y$ is defined then $DT : u(x, 0) \to \mathcal{L}(u(0), u(0))$ is 4-Lipschitz.*

Proof. It follows immediately from the fact that T is 1-Lipschitz in $u(x, 0)$ that if $\|x' - x''\|_0 \leq C$ then for all $i, j > 0$,

$$e^{-4C} \leq \frac{D_{ij}T(x')}{D_{ij}T(x'')} \leq e^{4C}, \tag{2.20}$$

which easily implies the result. □

Notice that the previous proof implies that

$$\|T(x + v) - T(x) - DT(x)v\|_0 \leq 4\|v\|_0^2, \tag{2.21}$$

so DT is the actual derivative of $T : u(x, 0) \to u(T(x), 0)$.

2.5. **Weak contraction of DT in $u^0(x, 0)$.**

Lemma 2.4. *Let $T(X) = Y$ be defined. If $0 \neq v \in u^0(0)$ then $\|DT(x)v\|_0 < \|v\|_0$.*

Proof. This is automatic since DT is a stationary positive matrix. □

Corollary 2.5. *If $T(X) = Y$ is defined and $0 \neq v \in u^0(0)$ then $\|T(x + v) - T(x)\|_0 < \|v\|_0$.*

Proof. Integrate the previous estimate. □

Corollary 2.6. *There exists at most one fixed point in each $U^0(x, 0)$.*

2.6. **The drift.** Let us define the drift

$$D_\alpha = \frac{1}{\pi} \int_0^\infty \tan^{-1} \frac{\sin\theta}{s^\alpha + \cos\theta} ds. \tag{2.22}$$

Lemma 2.7. *The operator T is defined in $U(\alpha, 0)$ if and only if $\alpha > 1$, and in this case the spaces $U(\alpha, 0)$ are invariant. Moreover, if $X \in U(\alpha, 0)$, then letting $T^n(X) = X^{(n)}$ we have*

$$\liminf_{k\to\infty} x_k - \alpha \ln k \leq \liminf_{k\to\infty} x_k^{(n)} - \alpha \ln k + n\alpha \ln D_\alpha, \tag{2.23}$$

$$\limsup_{k\to\infty} x_k^{(n)} - \alpha \ln k + n\alpha \ln D_\alpha \leq \limsup_{k\to\infty} x_k - \alpha \ln k. \tag{2.24}$$

Proof. Let $T(X) = Y$, with $x_k \leq \alpha \ln k + C + o(1)$. Then a simple computation gives

$$\phi_j(X, Y) \geq e^{-C\alpha^{-1}+o(1)} D_\alpha Y_j^{1/\alpha}, \tag{2.25}$$

and since $\phi_j(X, Y) = j + O(1)$, we have $Y_j \leq e^{C+o(1)} D_\alpha^{-\alpha} j^\alpha$. Analogously, if $x_k \geq \alpha \ln k + C + o(1)$ then

$$\phi_j(X, Y) \leq e^{-C\alpha^{-1}+o(1)} D_\alpha Y_j^{1/\alpha}, \tag{2.26}$$

and since $\phi_j(X, Y) = j + O(1)$, we have $Y_j \geq e^{C+o(1)} D_\alpha^{-\alpha} j^\alpha$. □

In particular, if $X \in U(\alpha)$ then the iterates of X drift (pointwise) towards either 0 or ∞ unless $D_\alpha = 1$. Notice that

$$\frac{d}{d\alpha} D_\alpha = \frac{-1}{\pi} \int_0^\infty \frac{\sin\theta \ln s}{s^\alpha + 2\cos\theta + s^{-\alpha}} ds = \frac{-1}{\pi} \int_1^\infty \frac{\sin\theta \ln s(1-s^{-2})}{s^\alpha + 2\cos\theta + s^{-\alpha}} ds < 0, \tag{2.27}$$

$$\lim_{\alpha \to 1} D_\alpha = \infty, \quad \lim_{\alpha \to \infty} D_\alpha = \frac{\theta}{\pi}, \tag{2.28}$$

thus there exists a unique $\alpha_\theta > 1$ such that $D_{\alpha_\theta} = 1$. From now on, α_θ will denote this precise value.

Remark 2.1. One can actually compute explicitly

$$D_\alpha = \frac{\sin\left(\frac{\theta}{\alpha}\right)}{\sin\left(\frac{\pi}{\alpha}\right)}, \tag{2.29}$$

so $D_{\alpha_\theta} = 1$ implies $\alpha_\theta = 1 + \frac{\theta}{\pi}$.

Corollary 2.8. *The space $U^0(\alpha_\theta, 0)$ is invariant.*

2.7. **Construction of invariant sets.** Let U be one of the spaces defined. We say that $K \subset U$ is uniformly bounded in U if there exists $\underline{X} \leq \overline{X}$ in U such that for all $Y \in K$, $\underline{X} \leq Y \leq \overline{X}$. Notice that the notion of uniformly bounded in $U^0(\alpha_\theta, 0)$ coincides with precompactness, while the notion of uniformly bounded in $U(\alpha_\theta, 0)$ coincides with "bounded diameter".

Lemma 2.9. *In this setting*

(1) *There exists $\overline{X} \in U(\alpha_\theta, 1)$ with $T(\overline{X}) \leq \overline{X}$, and $\overline{X}$ can be chosen arbitrarily big,*
(2) *There exists $\underline{X} \in U(\alpha_\theta, 1)$ with $T(\underline{X}) \geq \underline{X}$ and $\underline{X}$ can be chosen arbitrarily small.*

Proof. (Here, more precisely in the proof of (2), is the only time we will use the condition (2.2) on Q_k.)

(1) Let $Q_k < k + K$. The required $\overline{X}$ is given by $\overline{X}_k = (k+A)^{\alpha_\theta}$ for all A sufficiently big. To see this, we must estimate, for A sufficiently big

$$\phi_j(\overline{X}, \overline{X}) > j + K \tag{2.30}$$

for all j.

One can approximate

$$\phi_j(\overline{X}, \overline{X}) = \frac{1}{\pi} \int_A^\infty \tan^{-1} \frac{\sin\theta}{s^{\alpha_\theta}(j+A)^{-\alpha_\theta} + \cos\theta} ds + O(1) \tag{2.31}$$

where the $O(1)$ term does not depend on A. Of course

$$\frac{1}{\pi} \int_A^\infty \tan^{-1} \frac{\sin\theta}{s^{\alpha_\theta}(j+A)^{-\alpha_\theta} + \cos\theta} ds = (j+A) \frac{1}{\pi} \int_{\frac{A}{j+A}}^\infty \tan^{-1} \frac{\sin\theta}{s^{\alpha_\theta} + \cos\theta} ds. \tag{2.32}$$

This last term can be rewritten (using the condition on α_θ) as

$$(j+A) - (j+A) \frac{1}{\pi} \int_0^{\frac{A}{j+A}} \tan^{-1} \frac{\sin\theta}{s^{\alpha_\theta} + \cos\theta} ds. \tag{2.33}$$

Let us show the inequality (which trivially implies the required bound)

$$j + \left(1 - \frac{\theta}{\pi}\right) A \leq (j+A) - (j+A) \frac{1}{\pi} \int_0^{\frac{A}{j+A}} \tan^{-1} \frac{\sin\theta}{s^{\alpha_\theta} + \cos\theta} ds \leq j + A, \tag{2.34}$$

or equivalently, with $B = (j + A)/A$,

$$1 - \frac{\theta}{\pi} \leq 1 - B\frac{1}{\pi}\int_0^{B^{-1}} \tan^{-1}\frac{\sin\theta}{s^{\alpha_\theta} + \cos\theta}ds \leq 1. \tag{2.35}$$

The right inequality being trivial, we estimate the left one

$$\frac{\theta}{\pi} \geq B\frac{1}{\pi}\int_0^{B^{-1}} \tan^{-1}\frac{\sin\theta}{s^{\alpha_\theta} + \cos\theta}ds, \tag{2.36}$$

which is obvious since the integrand is a decreasing function of s which tends to θ when s tends to 0.

(2) Let $Q_k > k - K$. The required $\underline{X}$ is given by

$$\underline{X}_k = N^{k-N^2}, \qquad k < N^2, \tag{2.37}$$

$$\underline{X}_k = (k - N^2 + N)^{\alpha_\theta}, \qquad k \geq N^2, \tag{2.38}$$

for N sufficiently big. We can estimate, as before, for $j \geq N^2$

$$\phi_j(\underline{X}, \underline{X}) = \frac{1}{\pi}\sum_{k=1}^{\infty}\theta(\underline{X}_k, \underline{X}_j) = \frac{1}{\pi}\sum_{k=1}^{N^2-1}\theta(\underline{X}_k, \underline{X}_j) + \frac{1}{\pi}\sum_{k=N^2}^{\infty}\theta(\underline{X}_k, \underline{X}_j) < j - K, \tag{2.39}$$

since

$$\frac{1}{\pi}\sum_{k=1}^{N^2-1}\theta(\underline{X}_k, \underline{X}_j) \leq (N^2 - 1)\frac{\theta}{\pi}, \tag{2.40}$$

$$\frac{1}{\pi}\sum_{k=N^2}^{\infty}\theta(\underline{X}_k, \underline{X}_j) \leq j - N^2 + N + O(1) \tag{2.41}$$

(the $O(1)$ independent of N and j). For $1 \leq j < N^2$ we estimate

$$\phi_j(\underline{X}, \underline{X}) = \frac{1}{\pi}\sum_{k=1}^{\infty}\theta(\underline{X}_k, \underline{X}_j) = \frac{1}{\pi}\sum_{k=1}^{j}\theta(\underline{X}_k, \underline{X}_j) + \frac{1}{\pi}\sum_{k=j+1}^{\infty}\theta(\underline{X}_k, \underline{X}_j) < j - K \tag{2.42}$$

(implying the result), since

$$\frac{1}{\pi}\sum_{k=1}^{j}\theta(\underline{X}_k, \underline{X}_j) \leq j\frac{\theta}{\pi} - \frac{\theta}{2\pi} \tag{2.43}$$

and

$$\frac{1}{\pi}\sum_{k=j+1}^{\infty}\theta(\underline{X}_k, \underline{X}_j) = o(1). \tag{2.44}$$

(the $o(1)$ in terms of N and independent of j). □

Corollary 2.10. *There exists a fixed point $P \in U(\alpha_\theta, 1)$. For any initial condition $Y \in U(\alpha_\theta, 1)$, $T^n(Y)$ converges to P in $U^0(\alpha_\theta, 0)$.*

Proof. The previous lemma gives us $\overline{X}, \underline{X} \in U(\alpha_\theta, 1)$ with $\underline{X} \leq Y \leq \overline{X}$ and with $\underline{X} \leq T(\underline{X}) \leq T(\overline{X}) \leq \overline{X}$. It follows that $T^n(\overline{X})$ decreases pointwise to some vector $\underline{X} \leq P \leq \overline{X}$. This vector is obviously a fixed point of T. This proves existence of the fixed point. Analogously, $T^n(\underline{X})$ increases pointwise to some fixed point, which must be the same by uniqueness. In particular, $T^n(Y)$ converges to P. □

Lemma 2.11. *Let $Y \in U^0(\alpha_\theta, 0)$. Then $T^n(Y) \to P$ in the $U^0(\alpha_\theta, 0)$ metric.*

Proof. We must show that for any Y, for any $\epsilon > 0$, there exists n_0 such that for $n > n_0$, $\|T^n(y) - p\|_0 < \epsilon$. Using the previous construction, we obtain vectors $\underline{X}, \overline{X} \in U(\alpha_\theta, 1)$ with $(1-\epsilon/3)\underline{X} \le Y \le (1+\epsilon/3)\overline{X}$. Let n_0 be such that

$$T^{n_0}(\overline{X}) \le (1+\epsilon/3)P, \tag{2.45}$$

$$T^{n_0}(\underline{X}) \ge (1-\epsilon/3)P. \tag{2.46}$$

It follows that for any $n > n_0$,

$$(1-\epsilon/3)^2 P \le T^n(Y) \le (1+\epsilon/3)^2 P \tag{2.47}$$

which gives the desired estimate. □

Corollary 2.12. *Let $K \subset U^0(\alpha_\theta, 0)$ be uniformly bounded. Then $\cup_{n=0}^{\infty} T^n(K)$ is uniformly bounded as well.*

Proof. Let $\underline{X} \le Y \le \overline{X}$ for all $Y \in K$. Then $T^n(\underline{X}), T^n(\overline{X}) \to P$ implies that $\{T^n(\underline{X}), T^n(\overline{X})\}_{n \ge 0}$ is precompact, so uniformly bounded, by say $\underline{X}'$, $\overline{X}'$. By positivity of T, for any $Y \in K$ one has $\underline{X}' \le T^n(Y) \le \overline{X}'$ as well. □

2.8. **Strong contraction of DT.** Let

$$S_\epsilon = \int_0^\infty \frac{s^{-\epsilon}}{s^{\alpha_\theta} + 2\cos\theta + s^{-\alpha_\theta}} ds. \tag{2.48}$$

Notice that

$$S_\epsilon = \int_1^\infty \frac{s^{-\epsilon} + s^{\epsilon-2}}{s^{\alpha_\theta} + 2\cos\theta + s^{-\alpha_\theta}} ds, \tag{2.49}$$

so that if $|\epsilon - 1| \ge \alpha_\theta$ then $S_\epsilon = \infty$ and if $|\epsilon - 1| < \alpha_\theta$ then $S_\epsilon = S_{2-\epsilon}$ is a strictly increasing function of $|\epsilon - 1|$.

Remark 2.2. It is possible to compute explicitly

$$S_\epsilon = \frac{\pi}{\alpha_\theta \sin\theta} \frac{\sin\left((1-\epsilon)\theta{\alpha_\theta}^{-1}\right)}{\sin\left((1-\epsilon)\pi{\alpha_\theta}^{-1}\right)}, \quad 0 < |\epsilon - 1| < \alpha_0, \tag{2.50}$$

while $S_1 = \lim_{\epsilon \to 1} S_\epsilon = \frac{\theta}{\alpha_\theta \sin\theta}$. In particular, using that $\alpha_\theta = 1 + \frac{\theta}{\pi}$ (Remark 2.1) we get $S_1 = \frac{\pi}{\alpha_\theta \sin\theta}$.

Lemma 2.13. *Let K be a uniformly bounded set in $U^0(\alpha_\theta, 0)$. If $|\epsilon - 1| \ge \alpha_\theta$ then for every $X \in K$ we have that $DT(X)$ is not a bounded operator in $u(\epsilon)$. If $|\epsilon - 1| < \alpha_\theta$ then there exists a norm $\|\cdot\|_c$ in $u(\epsilon)$ (equivalent to $\|\cdot\|_\epsilon$) and a constant C_ϵ such that $\|DT(X)v\|_c \le C_\epsilon \|v\|_c$ for $v \in u(\epsilon)$. Moreover, $C_\epsilon < 1$ for $|\epsilon - 1| < 2$.*

Proof. For $X \in K$, X and $T(X) = Y$ satisfy uniformly $x_k, y_k = \alpha_\theta \ln k + o(1)$. Let $v_k = k^{-\epsilon}$ and let $w = DT(X)v$. We have

$$w_j = \left(\sum_k \frac{X_k Y_j}{X_k^2 + 2\cos\theta X_k Y_j + Y_j^2} v_k\right) \left(\sum_k \frac{X_k Y_j}{X_k^2 + 2\cos\theta X_k Y_j + Y_j^2}\right)^{-1}. \tag{2.51}$$

which can be estimated as

$$w_j = \left(\sum_k \frac{e^{o_{\min\{j,k\}}(1)} k^{\alpha_\theta} j^{\alpha_\theta}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} k^{-\epsilon}\right) \left(\sum_k \frac{e^{o_{\min\{j,k\}}(1)} k^{\alpha_\theta} j^{\alpha_\theta}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}}\right)^{-1}. \tag{2.52}$$

We easily estimate

$$\sum_k e^{o_{\min\{j,k\}}(1)} \frac{k^{\alpha_\theta} j^{\alpha_\theta}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} = e^{o_j(1)} j S_0. \tag{2.53}$$

We can write now

$$\begin{aligned} j^\epsilon S_0 w_j &= \sum_k \frac{e^{o_{\min\{j,k\}}(1)} k^{\alpha_\theta-\epsilon} j^{\alpha_\theta-1+\epsilon}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} \\ &= \sum_{k\le \ln j} \frac{e^{o_k(1)} k^{\alpha_\theta-\epsilon} j^{\alpha_\theta-1+\epsilon}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} + \sum_{k> \ln j} \frac{e^{o_j(1)} k^{\alpha_\theta-\epsilon} j^{\alpha_\theta-1+\epsilon}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}}. \end{aligned} \tag{2.54}$$

Moreover,

$$\sum_{k\le \ln j} e^{o_k(1)} \frac{k^{\alpha_\theta-\epsilon} j^{\alpha_\theta-1+\epsilon}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} = o_j(1), \tag{2.55}$$

provided $\epsilon < \alpha_\theta + 1$ (for $\epsilon \ge \alpha_\theta + 1$ the sum is not even $O_j(1)$), and

$$\sum_{k> \ln j} \frac{e^{o_j(1)} k^{\alpha_\theta-\epsilon} j^{\alpha_\theta-1+\epsilon}}{k^{2\alpha_\theta} + 2\cos\theta k^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} = e^{o_j(1)} \int_0^\infty \frac{t^{\alpha_\theta-\epsilon} j^{\alpha_\theta}}{t^{2\alpha_\theta} + 2\cos\theta t^{\alpha_\theta} j^{\alpha_\theta} + j^{2\alpha_\theta}} dt = e^{o_j(1)} S_\epsilon, \tag{2.56}$$

provided that $|\epsilon - 1| < \alpha_\theta$ (if $|\epsilon - 1| \ge \alpha_\theta$ the sum is not even $O_j(1)$). We can now conclude, for $|\epsilon - 1| < \alpha_\theta$,

$$w_j j^\epsilon = e^{o_j(1)} \frac{S_\epsilon}{S_0} \tag{2.57}$$

and for $|\epsilon - 1| \ge \alpha_\theta$,

$$\lim_{j\to\infty} w_j j^\epsilon = \infty. \tag{2.58}$$

In particular, $DT(X)$ is a bounded operator in $u(\epsilon)$ if and only if $|\epsilon - 1| < \alpha_\theta$, in which case the bound is uniform on $X \in K$. Moreover, if $0 < \epsilon < 2$ then there exists $S_\epsilon S_0^{-1} < \hat{C}_\epsilon < 1$ and $N > 0$ (independent of $X \in K$) such that for $j > N$,

$$w_j j^\epsilon < \hat{C}_\epsilon. \tag{2.59}$$

Let us now fix N as above. Let $v'_k = \min\{N^{-\epsilon}, k^{-\epsilon}\}$, and $w' = DT(X)v'$. By Lemma 2.4, we have

$$\|w'\|_0 < \|v'\|_0 = N^{-\epsilon}, \tag{2.60}$$

where the inequality is uniform on $X \in K$ (using for instance Lemma 2.3), so there exists $\tilde{C}_\epsilon < 1$ independent of $X \in K$ with

$$\sup_{k\le N} w'_k \le \tilde{C}_\epsilon N^{-\epsilon}. \tag{2.61}$$

Let

$$\|u\|_c = \sup_k \frac{|u_k|}{|v'_k|}, \tag{2.62}$$

which is equivalent to the usual norm on $u(\epsilon)$, since

$$\|u\|_\epsilon \le \|u\|_c \le N^\epsilon \|u\|_\epsilon. \tag{2.63}$$

Clearly $\|DT(X)u\|_c \le C_\epsilon \|u\|_c$ with $C_\epsilon = \max\{\hat{C}_\epsilon, \tilde{C}_\epsilon\}$. □

Remark 2.3. Let $v_k = k^{-\epsilon}$ and $v^{(n)} = DT^n(P)v$. A lower bound for the spectral radius of $DT(P)$ in $u(\epsilon)$ is given by

$$\limsup_{n\to\infty} \|DT^n(P)v\|^{1/n} \geq \lim_{n\to\infty}\lim_{k\to\infty}(k^\epsilon v_k^{(n)})^{1/n} = \frac{S_\epsilon}{S_0}. \tag{2.64}$$

This achieves a minimum at $\epsilon = 1$ and one actually has $S_1 S_0^{-1} = \alpha_\theta - 1$ (see Remark 2.2). Notice that as $\theta \to \pi$ (which happens when $M \to \infty$ for the anharmonic oscillator), $\alpha_\theta = 1 + \frac{\theta}{\pi} \to 2$ so the contraction factor becomes weak. This should be compared to numerical estimates in [V1], §7.3.

Corollary 2.14. *Let $X \in U^0(\alpha_\theta, 0)$. If $0 < \epsilon < \alpha_\theta + 1$ then $T(U(X,\epsilon)) = U(T(X),\epsilon)$.*

Proof. Integrate the previous estimate. □

Corollary 2.15. *If $0 < \epsilon < 2$ then P is a global exponential attractor in $U(P,\epsilon)$.*

Proof. Let $\underline{X} \leq P \leq \overline{X} \in U(P,\epsilon)$, and let $K = \{\underline{X} \leq Y \leq \overline{X}\}$. Then $\cup_{n=0}^\infty T^n(K)$ is uniformly bounded in $U^0(\alpha_\theta, 0)$ (Lemma 2.12), and by Lemma 2.13 there exists $C < 1$ and a norm $\|\cdot\|_c$ in $u(\epsilon)$ such that if $X \in K$ then $\|DT^n(X)v\|_c \leq C^n\|v\|_c$. Integrating this inequality we see that if $Y \in K$ then $\|T^n(y) - p\|_c \leq C^n \|y - p\|_c$. □

Theorem 2.1 follows from Corollaries 2.10, 2.14, 2.15 and Lemmas 2.7 and 2.11.

Remark 2.4. Let us remark that while the operator T in $U(\alpha_\theta, 0)$ has a line of fixed points λP, $\lambda > 0$ (where P is the fixed point in $U^0(\alpha_\theta, 0)$), this line is not a global attractor in the $U(\alpha_\theta, 0)$ metric. Indeed it is easy to see that if $1 = n_1 < n_2 < ...$ is a sequence that grows sufficiently fast and

$$v_k = -1, n_{2j-1} \leq k < n_{2j}, \quad v_k = 1, n_{2j} \leq k < n_{2j+1}, \tag{2.65}$$

then letting $x = p + v$, $x^{(n)} = T^n(x)$ we have

$$\inf_{\lambda > 0} \|x^{(n)} - \lambda p\|_0 = \|x^{(n)} - p\|_0 = 1, \tag{2.66}$$

for all $n \geq 0$, and we do not even have pointwise convergence:

$$\liminf_{n\to\infty} x_k^{(n)} - p_k = -1, \quad \limsup_{n\to\infty} x_k^{(n)} - p_k = 1, \tag{2.67}$$

for all $k \geq 1$.

Remark 2.5. A construction similar to the previous remark shows that P is far from being exponentially attractive in the $U^0(\alpha_\theta, 0)$ metric: for any decreasing sequence $a_1 > a_2 > ...$ with $\lim_{k\to\infty} a_k = 0$, there exists $X \in U^0(\alpha_\theta, 0)$ such that $\|T^n(x) - p\|_0 > a_n$.

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Collège de France – 3 Rue d'Ulm, 75005 Paris – France.
E-mail address: avila@impa.br